SYMMETRY GROUPS

IN

ELEMENTARY PARTICLE PHYSICS

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PREFACE

This thesis comprises work done by the author during the period 1964-1969 under the supervision of Professor Alladi Ramakrishnan, Director, MATSCIENCE, The Institute of Mathematical Sciences, Madras.

The thesis consists of the work done by the author in the field of group theory and applications to particle physics. It is divided into four parts, Part I dealing with the Clebsch-Gordan programme of arbitrary simple groups, Part II with the origin of unitary symmetry in strong interactions, Part III with the applications of symmetry principles to particle interactions to get sum rules which can be tested against experiments and Part IV with the generalized Clifford algebra of Yamazaki and its irreducible representations.

Twenty papers which form the subject matter of the thesis have been published or in the course of publication in established journals. Collaboration with my guide or some of my colleagues was necessiated by the nature and range of the problems dealt with and due acknowledgment is made in the chapters. The author is grateful to Professor Alladi Ramakrishnan for his guidance and encouragement throughout the course of this work.

He is indebted to Matscience for offering excellent facilities that enabled this work to be carried out.

It is indeed a pleasure to record and acknowledge the benefit of various discussions with my colleagues and the visiting scientists at the Institute of Mathematical Sciences on the work presented here.

MATSCIENCE,
The Institute of Mathematical Sciences, MADRAS-20.

January, 1970.

T. S. Santhanam
(T.S.Santhanam)

Dedicated to the memory of

MY FATHER

C O N T E N T S

Part III

APPLICATIONS OF SYMMETRY PRINCIPLES TO PARTICLE REACTIONS

CHAPTER XVI

CHAPTER 1

INTRODUCTION

The study of symmetry groups has become important and almost indispensable in Elementary Particle Physics since the discovery of strange particles. While the importance of the study of symmetry groups in Nuclear Physics has been realised since the pioneering work of Wigner, Racah and others, the application of symmetry groups to Elementary Particle Physics is of recent origin.

The main progress in modern physics after 1950 consists in the introduction of internal quantum numbers which have no dynamical interpretation like spin, momentum and energy, but was found necessary to explain certain conservation laws observed in the interactions between elementary particles. Though there has been no successful dynamical theory so far in explaining the complexities of interactions, certain general principles known as 'symmetry principles' have been known to govern the interactions. This assumed a pre-eminent role with Gell-Mann's formulation of SU(3) symmetry and its spectacular successes in the classification of elementary particles.

The proliferation of unstable and semi-stable systems called 'resonances'* almost amounting to two hundred in number in

*Width of 100 MeV would imply a life time $t \sim \frac{\hbar}{\Gamma} \sim 10^{-24}$ Sec.

recent years has necessitated in grouping them on some 'common grounds' of similar physical properties like spin, parity, baryon number etc. and almost similar properties like that of mass. In other words, the particles may be classified into smaller groups with very similar properties and so the task is reduced to the discussion of fewer entities. Of course, there could be 'small' deviations from this perfect structure which can be treated as perturbations.

A major step in this direction is the application of group theory particularly SU(3) and its generalizations to elementary particle interactions[1]. It is therefore desirable that the Racah algebra of symmetry groups of various types is developed as extensively as the Racah algebra of the group SU(2) (angular momentum). But as has been stressed by Wigner, Racah and others, several problems have to be solved before starting with such a programme.

This thesis is primarily concerned with these problems. It is devided into four parts. Part I consisting of four chapters (Chapter II to Chapter V) deals with the Clebsch-Gordan programme of arbitrary simple groups, in particular the problem of internal and external multiplicities. Part II comprising of two chapters (Chapter VI and Chapter VII) deals with the origin of unitary symmetry in strong interactions. Part III compmising of seven chapters (Chapter VIII to Chapter XIV) deals with several applications of symmetry groups to particle interactions. Part IV

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1) See for instance 'The Eightfold way', Eds.M.Gell-Mann and Y.Neeman, Benjamin Publishers, Inc., N.Y., 1964.
'Symmetry Groups in Nuclear and Particle Physics', Ed. F.J.Dyson, Benjamin Publishers, N.Y., 1966.

comprising of two chapters (Chapter XV and XVI) deals with the irreducible representations of the generalised clifford algebra of Yamazaki.

PART 1:- Clebsch-Gordan programme of arbitrary simple groups.

The first problem in this programme, which is closely connected with the labelling of the irreducible representations (IR) of the symmetry group (G), is the construction of invariants or casimir operators of G . This problem has already been solved[2]. The second problem concerns with the determination of a complete set of operators whose eigenvalues uniquely characterize an I.R. A given (IR) is specified by the eigenvalues of the casimir operators or equivalently by the component of the highest weight. From the highest weight, the other weights can be computed using shift or ladder operators. The main difficulty here is that the weights other than the highest one are not simple, but of multiplicity greater than one. ~~This multiplicity of weight in an I.R. of multiplicity greater than one.~~ This multiplicity of weight in an I.R. of G is called the 'internal' or 'inner multiplicity' structure[3].

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2) G.Racah, Group Theory and Spectroscopy, CERN, Reprint 61-8 (1961)45. L.C.Biedenharn, J.Math.Phys. 4, 436 (1963), Lectures on Theoretical Physics, W.E.Brittin,B.W.Downs and J.Down Eds. Interscience Publishers, Inc., New York (1963), Vol.5, p.346-352. B.Gruber and L.O'Raifeartaigh, J.Math.Phys. 5, 1796 (1964). L.O'Raifeartaigh, Lectures on 'Local Lie Groups and their representations', Matscience Report 25, (The Institute of Mathematical Sciences, Madras, India). M.Umezawa, Nucl.Phys. 48, 111 (1963), 53, 54 (1964), 57 65 (1964). M.Umezawa, MONIBKL. NEDERL.AKADEMIE VAN WETENSCHAPPEN, Amsterdam Series B, 69, No.5. 1966. M.Micu, Nucl.Phys. 60, 353 (1964). A.M.Perelomov and V.S.Popov, Soviet Phys.JETP Letts. 1,6 (1965). T.S.Santhanam, J.Math.Phys. 7, 1886 (1966).

3) The terminology is due to A.J.Macfarlane, L.O'Raifeartaigh and P.S.Rao, J.Math.Phys. 8, 536 (1967).

The third problem is that of the Clebsch-Gordan series and coefficients of G. Here again there is a problem in the direct product of two I.R's of G, which is in general reducible, a given I.R. may occur more than once and this we call the external multiplicity problem. Not much work has been carried out in the problem of Clebsch-Gordan coefficients of G except in some very special cases.

In Chapter II, some remarks are made on the construction of invariants of compact, local semi-simple Lie groups[4]. For three dimensional orthogonal grouo O(3), casimir considered the operator

$$I = J_x^2 + J_y^2 + J_z^2$$

where J_x, J_y and J_z are the generators of O(3). This operator commutes with J_x, J_y and J_z and its eigenvalues characteristic an I.R. of O(3). The generalization of G for any semisimple groupx was given by casimir, who introduced the operator

$$I = g^{\mu\nu} x_\mu x_\nu ,$$

$$g^{\mu\nu} = c^{\mu\sigma_1}_{\sigma_2} \; c^{\nu\sigma_2}_{\sigma_1}$$

4) T.S.Santhanam, J. Math. Phys. 7, 1886 (1966).

where C's are the structure constants and the X's are the generators of the group. A possible generalization of I[5] was given by Racah[2] who considered the operator

$$C_n = C^{\sigma_n}_{\sigma_1 \alpha} \, C^{\sigma_1}_{\sigma_2 \beta} \, \ldots \, C^{\sigma_{n-1}}_{\sigma_n \gamma} \quad X^{\alpha} X^{\beta} \ldots X^{\gamma}$$

and it is easy to verify that each of these operators commutes with every X. But, these are again not all the invariants of the group as Racah himself has recognized since it is found, for example, that for I.R's contragradient to each other and inequivalent, they have the same eigenvalues. Many[2] have suggested that if we replace the adjoint representation by the self-representation in C_n, then we can get all the invariants. In chapter I, it is suggested that one can still deal with adjoint representations, provided there is some mechanism by which symmetric coefficients are introduced. Some remarks have been made on the geometrical significance of the casimir operators in the adjoint space.

In Chapter II, new algebraic techniques[5] based on the work of Antoine and Speiser[6] have been presented on the computation of inner multiplicity structure of the group SU(3). The formula is just a simple improvement over the well known Kostant's formula[7]. The only thing is that the techniques developed in the evaluation of partition function are considerably simpler and great

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5) B.Gruber and T.S.Santhanam, Nuovo Cimento 45, 1046 (1966).
6) J.P.Antoine and D.Speiser, J.Math.Phys. 5, 1226 (1965), 5, 1560 (1965).
7) B.Kostant, Trans. Amer. Math. Soç. 93, 53 (1959).

simplification is achieved by limiting to only the dominant weights, in which case, only few Weyl reflections contribute.

In Chapter IV, the same method is applied[8] to the most complicated second rank group G(2) and explicit analytical formulae are given for the multiplicity of weights[9].

In Chapter V, the method of generating functions has been developed[10] to evaluate recursion relations for the partition functions of the classical groups. The recursion relation is particularly elegant for the group SU(ℓ +1).

In Appendix 1, many definitions of roots, simple roots, weights, dominant weights, highest weight which have been used in the text have been summarized[11].

In Appendix 2, many theorems on simple roots which form the main core in the evluation of 'inner' multiplicaties have been summarized and the material is collected from 'Lie algebra', N.Jacobson, Interscience Publishers, New York.

In Appendix 3, a simple derivation of Kostant's formula due to Steinberg is given.

In Appendix 4, a complete discussion on the evaluation of 'external' multiplicity is given.

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8) D.Radhakrishnan and T.S.Santhanam, J.Math.Phys. 8, 2206 (1967).

9) Dr.J.G.Belinfante informs me that he has programmed Kostant's formula for a computer (private communication). Our aim, however, is to get explicit analytical expressions.

10) T.S.Santhanam, preprint, ~~submitted to the~~ J. Math. Phys. (in press)

11) See for instance, T.S.Santhanam 'Group Theory and Unitary symmetry', Matscience Report 61, The Institute of Mathematical Sciences, Madras, India.

PART II:- Self consistent models and the origin of unitary symmetry in strong interactions.

Among the strongly interacting particles, we find multiplets of particles having the same spin and parity, but with slightly unequal masses. It is conventional to identity such a multiplet structure with the existence of an internal symmetry group, the multiplets constituting the various I.R's of the group. It is now well established that there are regularities in the particle (resonance) spectrum which go beyond charge independence in the sense that the multiplets can be further grouped into supermultiplets with the same spin, parity, baryon number and comparable masses which constitute I.R's of the group SU(3)[12]. In this case departure from complete symmetry are not yet well understood. All along, the symmetry group was given to start with and particles and resonances were accommodated with various I.R's of the symmetry group. The calculations have been carried out assuming the perturbations to be small and therefore neglected. But as to which multiplets should occur, the theory is silent. The Sakata model described the particles p,n and Λ to belong to the fundamental representation of U(3). However, it did not yield the correct multiplicity structure to the other particles. The Gell-Mann-Neeman version of SU(3) started with the eight dimensional representation of SU(3) directly. There are at least two shortcomings to this

12) See for example, 'The Eightfold way', Eds. M.Gell-Mann and Y.Neeman, W.A.Benjamin Inc., N.Y. (1964).

point of view. First, it does not tell which of the smaller representations actually occur. Secondly, one has to coin reasons why certain representations do not make their presence. In the literature such questions have been raised and to an extent explained[13].

There is a different line of approach which makes the connection more perspicuous[14]. In a dynamical scheme, when the particles and resonances appear in the direct channel of a two ~~particles~~ particle scattering process as a result of the exchange of these and other particles in the cross-channels, there are certain <u>self consistency</u> conditions imposed on the number of particles and their coupling strengths and the multiplets that can be exchanged to give an attractive force are not then arbitrary. There is then the possibility of looking for the dynamical origin of symmetries, starting from the existence of (mass-spin-parity degenerate) ~~multiple~~ multiplets of interacting particles and requiring self consistency. Suppose, we do not assume the existence of a symmetry group, <u>a priori</u>, but we assert that not only are the masses and spins of the various members of the multiplet equal, but also the total squared transition matrix elements into members of other multiplets. Then the propagators of each of the particles belonging to the multiplet are the same. Does this imply that there exists an underlying symmetry group and if so is it unique ?

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13) M.Gell-Mann, Physics 1, 63-75 (1964).

14) E.C.G.Sudarshan, Syracuse preprint 1206-SU-07-NYO-3399-07 'Symmetry in Particle Physics', 1964.
R.E.Cutkosky, Brandeis Lectures (1965).

In Part II of the this thesis, consisting of two chapters VI and VII, we address ourselves to this problem and we show in Chapter VI, that within a suitable dynamical framework, the answer is 'yes'[15]. The principle of the equality of propagators, we call it the 'Smushkevich principle'. We show that under certain dynamical assumptions the special unitary groups are singled out.

In Chapter VII, we show that the symmetry breaking can be suitably incorporated in this scheme[16].

In Appendix 5, we give certain identities of the recoupling coefficients. In the Appendix 6, we give Sakurai's demonstration[17] that the equality of masses does imply some symmetry group. In the Appendix 7, 8, and 9 we prove certain identities used in the text. In Appendix 10, we give a counter example of the case $(3 \otimes 3 \otimes 3)$ where the Smushkevich conditions are not powerful enough to single out a unique solution.

PART III:- <u>Application of symmetry groups to particle interactions.</u>

Symmetry is broken in re~~lat~~alistic situations. Many methods have been discussed in the literature on the symmetry breaking mechanism. The symmetry breaking we introduce is different from the other methods known. We believe that symmetry breaking manifests in mixing the various I.R's of the symmetry group, a fact

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15) E.C.G.Sudarshan, L.O'Raifeartaigh and T.S.Santhanam, Phys. Rev. 1092 (1964).
16) P.Narayanaswamy and T.S.Santhanam, Nuovo Cimento (in press).
17) J.J.Sakurai, Phys. Rev. Lett. 10, 446 (1963).

very analogous to the d-state admixture to the otherwise symmetric s-state ground state wave function of the deuteron. This has become particularly useful in the problem of identifying the Roper resonance (1400 MeV) which has all the quantum numbers same as the nucleon. In Chapter VIII, we study the consequences of representation mixing[18] in SU(3) and several sum rules are presented and some of them can be tested against experiments. In Chapter IX, the problem of representation mixing is studied in the framework of static SU(6) theory especially to the p-wave non-leptonic decays of hyperon[19]. In Chapter X, the same theory is applied to the leptonic decays and particularly an interesting relation between G_A and $(D/F)_{Ax}$ is derived[20].

In Chapter XI, the predictions of the higher symmetry groups like $S\left[U(3) \otimes U(3)\right]$ collinear and $SU(6)_W$ on the radiative decays of mesons are presented[21].

In Chapter XII, the algebras formed by the integrated currents constructed out of unrenormalized Heisenberg fields of strongly interacting particles are discussed[22]. In particular, eight dimensional baryonic fields are used in constructing the currents. <u>While the current algebra of Gell-Mann is independent of</u> <u>the explicit form of currents,</u> and therefore could

18) Alladi Ramakrishnan, T.S.Santhanam and A.Sundaram (Preprint).
19) T.S.Santhanam, Physics Letters, <u>21</u>, 234 (1964).
20) T.S.Santhanam, I.C.T.P. preprint IC/66/33 (unpublished).
21) H.Ruegg, W.Ruhl and T.S.Santhanam, Helv.Act.Physica, <u>40</u>, 9(1967).
22) P.Narayanaswamy, T.Pradhan and T.S.Santhanam, I.C.T.P. preprint (1966) unpublished.

have been postulated directly, nevertheless, it is equally interesting to see the models which reproduce the algebra. In Chapter XIII, the implications of the current algebra SU(2) x SU(2) on the electromagnetic form factors are studied[23]. In Chapter XIV, the Stueckelberg formulation of vector meson fields is used to study the A_1-π mixing and to reproduce some current algebra sum rules[24].

PART IV:- Clifford algebra and its generalizations.

In Part IV of this thesis consisting of the last two chapters, XV and XVI which have been included for reasons of completness, is described a completely new development in the study of unitary groups summarizing a programme of work at Matscience. It is in pursuance of establishing the hitherto unobserved connection between the unitary groups and the generalized clifford algebra initiated by Ramakrishnan[26]. The representations of this generalized clifford algebra have been recently obtained by A.O.Morris[27]. However, it was found soon that there exists a distinct method due to Rasevskii[28] to get the irreducible representations of clifford algebra. In

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23) T.S.Santhanam, A.Sundaram and K.Venkatesan preprint.
24) T.S.Santhanam, Nuovo Cimento, 57A, (1968) 440.
25) Alladi Ramakrishnan, J.Math.Anal. and Appl. 20, (1967) 9-16.
26) K.Yamazaki, J. Fac. Sci., University of Tokyo, Set I, 10, (1964) 147-195.
27) A.O.Morris, Quart.J.Math., Oxford (2) 18 (1967) 7-12.
28) P.K.Rasevskii, Am. Math. Soc. Transl. Series 2, Vol.6, (1957) 1.

Part IV of the thesis we address ourselves to the problems connected with the generalized clifford algebra and in particular in Chapter XV, we use the method of Rasevskii to obtain the irreducible representations of the generalized clifford algebra[29]. In Chapter XVI, an application of the theory of spinors in n-dimensions to the study of the relativistic wave equations of massless particles is given[30].

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29) A.Ramakrishnan, T.S.Santhanam and P.S.Chandrasekharan, to be published in J. Math. and Physical Sci., I.I.T., Madras. India.

A.Ramakrishnan, T.S.Santhanam, P.S.Chandrasekharan and A. Sundaram, J.Math. Anal. and Applications (in print).

A.Ramakrishnan, T.S.Santhanam and P.S.Chandrasekharan, Proc. of Matscience Symposia in Theoretical Physics and Mathematics, Vol.10, Plenum Press, New York, (to be published).

A.Ramakrishnan, P.S.Chandrasekharan, N.R.Ranganathan ~~and~~ T.S.Santhanam and R.Vasudevan, J.Math.Anal. and Application (to be published).

30) T.S.Santhanam and P.S.Chandrasekharan, Prog.Theor.Phys. Vol.41, (in press).

PART 1.

CLEBSCH-GORDAN PROGRAMME OF ARBITRARY SIMPLE GROUPS

CHAPTER II.

SOME REMARKS ON THE CONSTRUCTION OF INVARIANTS OF SEMISIMPLE LOCAL LIE GROUPS.

ABSTRACT.

A general form of the ℓ-invariants of compact semisimple local Lie Groups of rank ℓ as the spurs of the powers of the " Velocity Potential" operator is suggested. The possible generalization of these invariants beyond those of the adjoint group has been discussed.

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CHAPTER II.

REMARKS ON THE CONSTRUCTION OF INVARIANTS OF SEMI-SIMPLE LOCAL LIE GROUPS*

1. Introduction:- The first problem that one faces in the "Clebsch-Gordan" programme of arbitrary compact groups is to get a complete set of operators whose eigenvalues uniquely characterise an irreducible representation (I.R). These operators are the casimir operators which are functions of the generators of the group commuting with all the generators. For the three dimensional orthogonal group O(3), Casimir considered the operator

$$G = J_x^2 + J_y^2 + J_z^2 , \qquad \ldots . \qquad (1)$$

where J_x, J_y and J_z are the generators of O(3). This operator is known to commute with J_x, J_y and J_z. If the representation is irreducible, then Schur's lemma asserts that

$$G = \lambda I, \qquad \ldots . \qquad (2)$$

where $\lambda = j\,(j+1)$, (j = integral or half integral). We also know that any I.R. can also be characteised by the components of its highest weight and there is one-to-one correspondence between the components of the highest weight and the eigenvalues

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*T.S.Santhanam, J. Math. Phys. 7, 1886 (1966).

of the Casimir operators for an I.R. The generalization of G for any semi-simple group was given by Casimir, who introduced the operator

$$G = g^{\mu\nu} x_\mu x_\nu ,$$
$$g^{\mu\nu} = c^{\mu\sigma_1}_{\sigma_2} c^{\nu\sigma_2}_{\sigma_1} , \tag{3}$$

where the c's are the structure constants and x's are the generators of the group.

A possible generalization of G was given by Racah[1] who considered the operator

$$C_n = c^{\sigma_n}_{\sigma_1\alpha} c^{\sigma_1}_{\sigma_2\beta} \cdots\cdots c^{\sigma_{n-1}}_{\sigma_n\gamma} \underbrace{X^\alpha X^\beta \cdots\cdots X^\gamma}_{\leftarrow n \text{ terms} \rightarrow} \tag{4}$$

and it is easy to verify that each of these operators commutes with every X_ρ. But these are again not all the invariants of the group as Racah himself has recognized, since it is found, for example, that for I.R.'s contragradient to each other and unequivalent, they have the same eigenvalues. So a possibility of generalizing (4) presents

(1) G. Racah, Group Theory and Spectroscopy, CERN, Reprint 61 - 8 (1961) p.45.

itself. This is seen by noting that if we denote by $\overset{A}{X}_\alpha$ the adjoint representation of the group, we have

$$\left(\overset{A}{X}_\alpha\right)^\lambda_\mu = C^\lambda_{\mu\alpha} , \tag{5}$$

where λ, μ are regarded as matrix indices so that

$$C_n = Sp\left(\overset{A}{X}_\alpha \overset{A}{X}_\beta \ldots \overset{A}{X}_\gamma\right) X^\alpha X^\beta \ldots X^\gamma \tag{6}$$

The question then arises whether one can use in (6) an arbitrary representation $\hat{X}$ instead of the adjoint representation. The problem of generalizing C_n has been solved recently in References 2),3),4) and 5). We shall follow the notation used in Ref. 4.

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2) L.C.Biedenharn, J.Math.Phys. 4, 436 (1963),
Phys.Lett. 3, 69 (1962)

See also L.C.Biedenharn, 'Lectures in Theoretical Physics, W.E.Brittin, B.W.Downs and J.Downs, Eds. (Interscience Publishers, Inc., New York, 1963) Vol.5, p.346-352.

(3) H.Umezawa, Nucl. Phys. 48, 111 (1963),
53, 54 (1964),
57, 65 (1964).

See also, H.Umezawa, KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN - AMSTERDAM.
Reprint from Proceedings Series B, 69, No.5, 1966.

4) B.Gruber and L.O'Raifeartaigh, J.Math.Phys. 5, 1796 (1964).
See also L.O'Raifeartaigh, 'LECTURES ON LOCAL LIE GROUPS AND THEIR REPRESENTATIONS', Matscience Report 25 (The Institute of Mathematical Sciences, Madras, India).
L.O'Raifeartaigh, Symposia on Theoretical Physics, Edited by Alladi Ramakrishnan (Plenum Press, New York, 1966), Vol.2, pp 15.

5) M.Micu, Nucl. Phys. 60, 353 (1964)
A.M.Perelomov and V.S.Popov, Soviet Physics JETP Letters 1, 6 (1965)

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Suppose in Eq. (6), we replace the adjoint representation $\overset{A}{X}$ by an arbitrary representation $\hat{X}$. Let

$$I_n = Sp\left(\hat{X}_\alpha \hat{X}_\beta \cdots \hat{X}_\gamma\right) X^\alpha X^\beta \cdots X^\gamma \tag{7}$$

It has been proved in Ref.(4) that

$$\left[I_n, X_\alpha\right] = 0 \quad , \tag{8}$$

and also the completeness of these invariants has been established. In particular, the invariants for the classical groups A_ℓ, B_ℓ, C_ℓ and D_ℓ have been obtained using the self representation. These results are summarized in Table I.

TABLE I

INDEPENDENT INVARIANTS

Group	Description (Linear Realization)	Order of Invariants	Representation used to form it
A_ℓ	All unimodular $(\ell + 1) \times (\ell + 1)$ matrices	$2, 3, \ldots \ell+1$	Self
B_ℓ	$(2\ell + 1) \times (2\ell + 1)$ matrices	$2, 4, 6, \ldots 2\ell$	Self
C_ℓ	All sympletic $(2\ell \times 2\ell)$ matrices	$2, 4, 6, \ldots 2\ell$	Self
D_ℓ	All orthogonal $(2\ell \times 2\ell)$ matrices	$2, 4, 6, \ldots 2\ell - 2$ ℓ	Self and one of the two fundamental spinor representations.

Biedenharn [2], on the other hand, has used the fact that in the case of unitary groups, there exists not only the group algebra of the commutators

$$[X_\alpha , X_\beta] = C^{\gamma}_{\alpha\beta} X_\gamma , \qquad (9)$$

which is independent of the representation; but also the algebra of the anti-commutator for the special case of the self representation

$$\{ \overset{S}{X}_\alpha , \overset{S}{X}_\beta \} = d^{\gamma}_{\alpha\beta} \overset{S}{X}_\gamma \qquad (10)$$

Of course, one knows, that the anticommutator depends on the choice of the representation. The symmetric coefficients $d^{\gamma}_{\alpha\beta}$ have been used by Biedenharn[2] to construct all the invariants of the unitary group $U(n)$. Subsequently, the method has been extended to the case of $O(2\ell + 1)$ and $Sp_{2\ell}$ by Micu[5], where it has also been pointed out that for the orthogonal group in even dimensions, an invariant cannot be constructed in a similar way.

2. <u>Vector Potential Operators:</u>- The question naturally arises whether these invariants have any geometrical meaning. Do they specify any particular property of the parametric space ? In the case of the rotation group $O(3)$, the invariant $J_x^2 + J_y^2 + J_z^2$ can be interpreted as the square of the norm under rotations in the three dimensional space spanned by J_x, J_y and J_z. If so, how can one interpret the cubic and higher order invariants geometrically ? In literature,

such questions have been posed[6]. We now give a method of constructing the invariants using the "Velocity Potential Operators".

The "Velocity potential" is defined to be :-

$$U_\alpha^i(x) = \left[\frac{\partial \phi_i(x,y)}{\partial y_\alpha}\right]_{y=0} , \tag{11}$$

where the ϕ's are the transformation functions of the Lie group. In fact, as is well known[7], the whole analysis and the classification of continuous groups are accomplished by the study of U. The functions ϕ are analytic and the usual composition laws of the group (like the closedness, associativity, existence of the unit element and inverse) can be transformed to show that ϕ can be expanded in the form

$$\phi_\alpha(x,y) = x_\alpha + y_\alpha + a^\alpha_{\beta\gamma}\, x_\beta y_\gamma + f^\alpha x^2 y + \cdots\cdots$$

$$\alpha, \beta \ldots = 1, 2, \ldots \quad n = \text{Number of parameters} \tag{12}$$

The infinitesimal generators of the group X are defined by:-

$$X_\rho = \sum_i U_\rho^i(x) \frac{\partial}{\partial x_i} , \tag{13}$$

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6) See, for instance, L.P.Eisenhart, continuous groups of Transformations (Dover Publications, Inc., New York, 1961) pp. 155.

7) cf. Pontrjagin L.S., Topological Groups, Princeton 1946, (Princeton Math. Series).

and they obey the commutation relation

$$\left[X_\rho , X_\sigma\right] = C^{\mu}_{\rho\sigma} X_\mu \tag{14}$$

It is the famous proof of Lie that showed that these structure constants $C^{\mu}_{\rho\sigma}$ are nothing but the antisymmetric part of the second order coefficient $a^{\mu}_{\rho\sigma}$ occurring in the expansion Eq.(12) for the ϕ 's. In other words

$$C^{\alpha}_{\beta\gamma} = a^{\alpha}_{\beta\gamma} - a^{\alpha}_{\gamma\beta} \tag{15}$$

These concepts are indeed well known and are introduced just for continuity and notation. The U Operator for the group O(3), for example, looks like

$$U(x) = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}$$

$$x_i = (x, y, z) \tag{16}$$

However, in this case, the adjoint and self representations are both three dimensional. In terms of the natural basis of matrices, the matrix $U(x)$ can be written as:-

$$U(x) = \sum_i x^i \otimes X_i \quad , \tag{17}$$

where the X_i's are the natural basis of 3 x 3 antisymmetric matrices and the product is the direct product. The dimension of the parametric space is always equal to the number of generators, so that Eq.(17) is always defined.

3. <u>Invariants of the Adjoint Group</u>:-

The adjoint group P of a group G is defined through the homomorphism of G on the group of matrices. So, to every element $x \in G$, there corresponds a matrix $p \in P$. The adjoint of the infinitesimal group is called the infinitesimal adjoint group.

Let us start with the Casimir Operator:-

$$
\begin{aligned}
I_2 &= g_{\alpha\beta}\, X^{\alpha} X^{\beta} \\
&= C^{\sigma_1}_{\sigma_2 \alpha}\, C^{\sigma_2}_{\sigma_1 \beta}\, X^{\alpha} X^{\beta} \\
&= sp\,(\overset{A}{X}_{\alpha} \overset{A}{X}_{\beta})\, X^{\alpha} X^{\beta} .
\end{aligned}
\tag{18}
$$

If we define

$$
\begin{aligned}
\eta^{\sigma_1}_{\sigma_2} &= C^{\sigma_1}_{\sigma_2 \alpha}\, X^{\alpha} \\
&= (\overset{A}{X}_{\alpha} \otimes X^{\alpha})^{\sigma_1}_{\sigma_2} ,
\end{aligned}
\tag{19}
$$

then

$$
\begin{aligned}
I_2 &= sp\,(\eta)^2 \\
&= sp\,(\overset{A}{X}_{\alpha} \otimes X^{\alpha})^2
\end{aligned}
\tag{20}
$$

so that the n^{th} - order invariant may just be written as:-

$$I_n = Sp\,(\eta)^n = Sp\left(\overset{A}{X}_\alpha \otimes X^\alpha\right)^n$$

(21)

These are, of course, known as the invariants of the adjoint group[6]. The invariants are defined as the coefficients ψ in the expansion of the characteristic equation

$$\Delta\,(x, \rho) = \left\| \eta^\alpha_\beta(x) - \rho\,\delta^\alpha_\beta \right\| = 0$$

(22)

$\Delta(x, \rho)$ is called the characteristic matrix. The parameter ρ is supposed to define the invariant directions. Since the rank of $\| \eta^\alpha_\beta(x) \|$ is less than r (the number of parameters of the group which is the same as the dimension of the parametric space)[6], we can expand the characteristic matrix as:-

$$(-1)^r\,\Delta\,(x, \rho) = \rho^r - \psi_1(x)\,\rho^{r-1} + \psi_2(x)\,\rho^{r-2} + \cdots$$
$$+ \cdots$$
$$+ (-1)^{r-1}\,\psi_{r-1}(x)\,\rho$$

(23)

The $\psi_q(x)$ is the sum of the principal minors of $\|\eta^{\alpha}_{\beta}(x)\|$ of order q. Hence, if the rank of the matrix is q, the functions ψ_s for $s > q$ are zero and the characteristic Eq.(22) admits zero as a root of order $(r - q)$ at least. In fact, it is easy to see that

$$\psi_1 = Sp(\eta),$$

$$\psi_2 = \frac{1}{2}\left\{\left[Sp(\eta)\right]^2 - Sp(\eta)^2\right\}, \tag{24}$$

and so on. The theorem of Killing states that the ψ's are the invariants of the adjoint group[6]. Also, it has been shown that there are only ℓ independent ψ's where ℓ is the rank of the group. The matrix η is just the operator $(\overset{A}{X}_{\alpha} \otimes X^{\alpha})$ defined in the paper of Gruber and Raifeartaigh[4]. These are the velocity potential operators for the group of infinitesimal generators X of the group. For the case of O(3), the operator η is obtained by replacing, in the velocity potential $U(x)$ of the group, the components x of the parametric space by the infinitesimal generators X.

We get,

$$\eta(x) = \sum_i \overset{A}{X}_i \otimes X^i$$

$$\doteq U(x)$$

$$= \begin{bmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix} \tag{25}$$

One important caution is that though in the case of $O(3)$, $\eta(x)$ is the same for both self and adjoint representations (as they are identical in this case), in general Eq.(25) is defined through only the adjoint representation. The invariants of the group are then

$$I_n = \mathrm{Sp}\left[U(x)\right]^n \tag{26}$$

It is easy to show that for $O(3)$,

$I_4 = f(I_2)$ and odd order invariants are automatically zero.

Therefore, the method consists in first replacing the x's in the U matrix by the infinitesimal generators of the group. Then take the spurs of the powers of this new matrix. It is clear that the number of x's is indeed equal to the order of the group. It is also easy to check that $\mathrm{Sp}(U)^n$ is the same even if one permutes the x's in U. Of course, the choice of $U(x)$ strongly indicates that the corresponding group function is

$$\phi'_\alpha(x, y) = C^\alpha_{\beta\gamma}\, x_\beta\, y_\gamma . \tag{26'}$$

so that

$$\begin{aligned} U_i^\alpha(x) &= \left[\frac{\partial \phi'_\alpha(x,y)}{\partial y_i}\right]_{y=0} \\ &= C^\alpha_{\beta i}\, x_\beta \\ &= \eta_i^\alpha(x) , \end{aligned} \tag{27}$$

and hence

$$I_2 = Sp\,[U(x)]^2$$
$$= U^\alpha_i(x)\, U^i_\alpha(x)$$
$$= g_{\beta\gamma}\, x_\beta\, x_\gamma \qquad (28)$$

The form of $U^\alpha_\beta(x)$ and hence that of $\phi'_\alpha(x,\gamma)$ immediately tells us that we are in fact dealing with the invariants of the adjoint group.

4. Generalization beyond the Adjoint Group:

The next question is how to generalize these invariants beyond the adjoint group. One way suggested in the work of Gruber and Raifeartiagh[4] is to generalize η as

$$\eta = \sum_i \hat{X}_i \otimes x^i , \qquad (29)$$

where $\hat{X}$ is an arbitrary representation. In particular, they have used the self representation in constructing the invariants of the classical groups A_ℓ, B_ℓ, C_ℓ and D_ℓ except that in the last case, one needs in addition to the self representation, at least one of fundamental spinor representations. It is tempting to look, therefore, what happens to the " velocity potential operator" and consequently the group function when one replaces the operator $\left(\overset{A}{X}_i \otimes X^i\right)$ by $\left(\hat{X}_i \otimes X^i\right)$. It is very hard to answer these questions directly. However, the following things could be remarked. It is clear from the work of Biedenharn[2] that we need certain symmetric coefficients to get the invariants of the group U(n). If we want to generalize the invariants beyond the adjoint group, we can still retain the form.

$$I_n = Sp\,(U)^n . \tag{30}$$

But now, the U(x)'s are defined through the relation

$$U_i^\alpha(x) = \delta_i^\alpha + a_{\beta i}^\alpha x_\beta , \tag{31}$$

which implies that the group function $\phi_\alpha(x,y)$ is

$$\phi_\alpha(x,y) = x_\alpha + y_\alpha + a_{\beta\gamma}^\alpha x_\beta y_\gamma \tag{32}$$

In Eq.(31), the <u>a's are not the structure constants</u>, they are the second order coefficients in the expansion of the group function. Usually, in the normal parameter system, we make the symmetric part of a vanish so that the a's occurring in the expansion can be replaced by the structure constants. Suppose we retain both the symmetric and antisymmetric parts in Eq.(32), we have,

$$a^{\alpha}_{\beta\gamma} = \frac{1}{2}\left(d^{\alpha}_{\beta\gamma} + c^{\alpha}_{\beta\gamma}\right),$$

$$d^{\alpha}_{\beta\gamma} = a^{\alpha}_{\beta\gamma} + a^{\alpha}_{\gamma\beta},$$

$$c^{\alpha}_{\beta\gamma} = a^{\alpha}_{\beta\gamma} - a^{\alpha}_{\gamma\beta},$$

(33)

while $C^{\alpha}_{\beta\gamma}$, by Lie's theorem is the same as the structure constants, one can only speculate that the $d^{\alpha}_{\beta\gamma}$ may be the symmetric structure constants occurring in the anti-commutator of the x's,

$$\left\{ \overset{S}{X}_{\alpha} , \overset{S}{X}_{\beta} \right\} = d^{\gamma}_{\alpha\beta} \overset{S}{X}_{\gamma} ,$$

(34)

where we have used $\overset{S}{X}$ to denote the self-representation. We again emphasize that Eq.(34) is very sensitive to the choice of the representation. In many cases, the anti-commutator may not even close and we have not established that the d's in Eq.(34) are always the same d's occurring in Eq.(33). In the case of U(n), one does know that Eq.(34) is certainly true and this fact has been used by Biedenharn[2)] to write the invariants of the group U(n). Perhaps, the study of the symmetric spaces may throw more light on Eq.(34).

C H A P T E R III

SU(3): COMPACT FORMULA FOR $D(m') \otimes D(m)$ AND FOR MULTIPLICITY $M^{m'}(m'')$ OF $m'' \in D(m')$

A B S T R A C T

A simple formula for the multiplicity $M^{m'}(m'')$ of an arbitrary weight m'' belonging to the irreducible representation with m' as its highest weight is derived. It is then used to derive a compact and simple formula for the decomposition $D(m') \otimes D(m)$.

SU(3): COMPACT FORMULA FOR $D(m') \otimes D(m)$ AND FOR MULTIPLICITY $M^{m'}(m'')$ OF $m'' \in D(m')$ *

Introduction:

A difficulty that one confronts in the Clebsch-Gordan programme of the group SU(3) is the multiple occurrence of a given weight in an I.R. (This problem is often called the problem of "internal multiplicity" structure). In the literature, of course, there exists the Kostant's formula[1] to compute the weight multiplicities. However, it is too complicated for the practical calculation of multiplicities, because it involves, along with the summation over the Weyl group, the function P(u) which is equal to the number of decompositions of a given vector u into the sum of positive roots of the algebra. There is also the Freudenthal's recursion formula[2] for the weight multiplicities, which is equally complicated.

Recently, Antoine and Speciser [3],[4],[5] have given a

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* B.Gruber and T.S.Santhanam, Nuovo Cimento 45, 1046 (1966)

1) B.Kostant, Trans Amer.Math.Soc. 93, 53 (1959).
See also N.Jacobson, Lie Algebras, Interscience Publishers (1961) p.261.

2) N.Jacobson, ibid, p.247.

3) J.P.Antonine and D.Speiser, J.Math.Phys. 5, 1226 (1965), 5, 1560 (1965).

4) D.Speiser, Helv.Physica Acta 38, 73 (1965). Volume dedicated to Prof.E.C.G.Stueckelberg on this 60th birthday.

5) D.Speiser, Group Theoretical Concepts and Methods in Elementary Particle Physics, Gordan and Breach (New York, 1962) p.201.

geometrical method for computing the internal multiplicities in a very simple way. They have proved that if the Weyl's character formula[6] is re-expressed as a product, instead of as quotient, certain simplifications occur as well as that the method offers a very neat geometrical picture. In the first section, a brief discussion of their method is included. However, their geometrical method is again cumbersome for higher rank groups, while an principle the weight multiplicity is calculable.

We have developed an algebraic procedure of computing the weight multiplicities along the same lines of Antoine and Speiser. The generalization to higher rank groups then becomes straightforward. We derive an explicit expression for the internal multiplicity for the case of A_2 algebra (the corresponding group being SU(3)). Using this algebraic formula, we derive a compact formula for the decomposition of the direct products of I.R's into irreducible components. The case of SU(3) is particularly simple although not trivial like SU(2) (where the internal multiplicity is unity throughout). In the next chapter, we shall discuss the more difficult case of G(2).

Notations and Definitions.

We shall summarize the necessary and relevant definitions often used in the text of this and the next Chapters.

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6) H. Weyl, The classical Groups, Chapter VII
Princeton University Press (1946).

A vector V is positive of its first non-vanishing component is positive.

The vector V is greater than W, if the vector $(V - W)$ is positive.

The vectors connected by Weyl reflections are equivalent.

A vector greater than all its equivalents is called dominant.

For a semi-simple group, it is known[7] that if α is a root, then $(-\alpha)$ is also a root. Then the roots fall into two classes, positive and negative. We denote the positive roots by α and negative roots by β $(= -\alpha)$. A quantity of great interest is the vector $R_0 = \frac{1}{2}\sum_{i=1}^{m} \alpha_i$ (half the sum of positive roots). For a group of rank ℓ, there exists ℓ positive roots, called the positive primitive roots (some people call them as elementary or simple) such that any positive root φ_i is given by

$$\varphi_i = \sum_{j=1}^{\ell} c_{ij}\,\alpha_j\,, \quad i = 1, 2, \ldots m.$$

$$c_{ij} \geq 0\,, \quad \varphi_i \neq 0$$

7) G. Racah, Group Theory and Spectroscopy, CERN Report 61-8 (1961) p.45

We shall see later, the role played by these primitive roots.

Suppose we have a coordinate system with basis vectors $P_1 \dots P_\ell$. It defines an affine coordinate system, if all vectors belong to g^{c*} (lattice generated by ℓ basis vectors) if every $V \in g^c$ takes the form

$$V = \sum p_i P_i \qquad \text{with integers } p_i .$$

The fundamental domain D_o is defined by

$$p_i \geq 0 \quad \text{for any} \quad V \in D_o$$

For such a system, Weyl[1] has proved that

$$R_o = \frac{1}{2}\left(P_1 + \dots + P_\ell\right)$$

so that R_o lies inside D_o.

<u>Character Formula</u>

Weyl[6] has shown that the character of an IR of a semi-simple group may be written in the form :-

$$\chi = \frac{X}{\Delta}, \tag{3.1}$$

---

*If the group is semi-simple, its centre is a discrete group. Therefore, its image in a Euclidean space E_ℓ is a point lattice g_c generated by ℓ basis vectors.

where the characteristic $\quad X = X(K_0) = \sum_{S \in W} \delta_S \exp i(SK_0, \phi)$

$$\cancel{X_{eta}} \quad \frac{\cancel{X}}{\cancel{\Delta}} \quad K_0 \in g^c \cap D_0 ,$$

and $\Delta = X(R_0)$ = characteristic of the Identity Representation

$$= \sum_{S \in W} \delta_S \exp i(SR_0, \phi) \tag{3.2}$$

Here W denotes the Weyl group. ϕ are the group parameters $\delta_S = \pm 1$ according as the Weyl reflection is even or odd respectively. There exists another, but equivalent expression for the character of an IR with m as its highest weight.

$$\chi^{m}(\phi) = \sum_{m'' \in D(m)} \gamma_{m''} \exp i(m'', \phi) \tag{3.3}$$

where the summation is over all the weights m" of I.R. D(m). $\gamma_{m''}$ denotes the multiplicity of m". Antoine and Speiser[3] have expressed ($\frac{1}{\Delta}$) which when <u>multiplied</u> by X yields the character as

$$\frac{1}{\Delta} = e^{-i(R_0, \phi)} \sum_{k=0}^{\infty} Z^{k} , \tag{3.4}$$

with

$$Z = \sum_{S \in W}' (-\delta_S) \exp i\,(SR_0 - R_0, \phi)$$

One can then interpret Eq.(3.4) for $\frac{1}{\Delta}$ in the same way as one did for the character χ (Eq.(3.3)).

In a term $\gamma_P\, e^{i(P,\phi)}$, the multiplicity γ_P at the point P is the value of the funtion $\frac{1}{\Delta}$ at the point P. Divergence of the sum $\sum Z^k$ only means that the multiplicity goes on increasing. However, it has been shown[1] that <u>only values of $\frac{1}{\Delta}$ lying in a bounded domain of g^c are used.</u> To find $\frac{1}{\Delta}$ we have yet another formula[3]

$$\frac{1}{\Delta} = \exp(-R_0, \phi) \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \exp i \left[\sum k_i \beta_i , \phi\right] \tag{3.5}$$

Here, the β_i's are the negative roots.

Then quantity $\sum_{i=1}^{m} k_i \beta_i,\ k_i \geq 0$, clearly represents an arbitrary point of the lattice constructed on $\beta_1, \beta_2, \ldots, \beta_m$, with non-negative, integer coefficients. The sum over k's represents all points of one of the 2^m "octants" of this lattice in E_m. $\frac{1}{\Delta}$ is the same shifted by the Vector $(-R_0)$. We have then

$$\frac{1}{\Delta} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \exp i \left[\sum_{i=1}^{m} k_i \beta_i - R_0 , \phi \right] \tag{3.6}$$

And ~~have then~~ so the character formula becomes

$$\chi = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left\{ \sum_{S \in W} \delta_S \exp i \left[\sum_{i=1}^{m} k_i \beta_i + S K_0 - R_0 , \phi \right] \right\}$$

$$\delta_S = \pm 1 \tag{3.7}$$

The main result of Antoine and Speiser[1] is that it is quite enough to know the part χ_0 of χ contained in D_0; the other parts then will be obtained using the group W

$$\chi_0 = \chi , \tag{3.8}$$

with the condition

$$\sum_{i=1}^{m} k_i \beta_i + S K_0 - R_0 \quad \in \quad D_0 \tag{3.8a}$$

Since we are restricting any point to belong to D_0, it follows from our analysis that the multiplicity $\overline{M}(\gamma)$ of a vector γ in $\frac{1}{\Delta}$ is just the number of ways of expressing

$$\gamma = -R_0 + \sum_{i=1}^{m} k_i \beta_i \tag{3.9}$$

in terms of

$$\gamma = -R_0 + \sum_{i=1}^{l} k_i' \beta_i \tag{3.10}$$

Thus, $\overline{M}(\gamma)$ is the number of non-negative integral solutions of the Diophantine equations

$$\sum_{i=1}^{m} k_i \beta_i = \sum_{i=1}^{l} k_i' \beta_i$$

$$l \leq m \tag{3.11}$$

It should be remarked that this is exactly the partition function P(u) that enters the Kostant's formula[8]. Once $\overline{M}(\gamma)$ is known the multiplicity $M^{m'}(m'')$ is calculated from the Kostant's formula

$$M^{m'}(m'') = \sum_{s \in W} \delta_s \, \overline{M} \left[m'' - S(m' + R_0) \right]. \tag{3.12}$$

We exploit in our derivation that since the Weyl group W is known, it is sufficient to know $\overline{M}$ when m" is dominant. In this chapter we shall demonstrate this procedure for the case of SU(3).

A_2 algebra.

The roots can be well described in a three dimensional space as the vectors $e_i - e_j$, $i,j = 1,2,3$, where the e's are the three unit vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ and the positive roots are $\alpha_1 = (e_2 - e_3)$, $\alpha_2 = (e_1 - e_2)$ and $\alpha_3 = (\alpha_1 + \alpha_2)$. α_1 and α_2 are the positive primitive roots. We also have $R_0 = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3)$

$$= (\alpha_1 + \alpha_2) = \alpha_3 = (1,0,-1).$$

It is equally convenient to describe the weights also as vectors in a three dimensional space, with a condition on the

8) See Jan. Tarski, J. Math. Phys. 4, 569 (1963) for more details on the Partition function.

components

$$\sum_{i=1}^{3} m_i = 0$$

This is then a plane in the same space as roots.

The condition that $\frac{2(m,\alpha)}{(\alpha,\alpha)}$ = integer where m is a weight and α is a root becomes in this case

$$\frac{2\ m.(e_i - e_j)}{|e_i - e_j|^2} = (m_i - m_j)$$

$$(m_i - m_j) \quad = \text{integer}$$

Thus the differences of the components of the m's are integers. This along with $\sum_{i=1}^{3} m_i = 0$ yields that

$$m_i = \frac{\text{integer}}{3} .$$

Thus the conditions on the compnents of the weight are

$$\sum_{i=1}^{3} m_i = 0$$

$$m_i - m_j = (\text{integer})$$

$$m_i = \frac{1}{3}(\text{integer}) .$$

(3.13)

Weyl Group.

Under a reflection in the plane perpendicular to $(e_i - e_j)$ we have

$$m \longrightarrow m' = m - 2\,\frac{m.(e_i - e_j)}{|e_i - e_j|^2}\,(e_i - e_j)$$

$$= m - (m_i - m_j)\,(e_i - e_j)$$

$$= m \quad (m_i \longleftrightarrow m_j)$$

Thus the Weyl group W in this case consists of all permutations of the components of m and it therefore of order 3 ! . The dominant weights (highest among equivalents) have to satisfy the condition

$$m_1 \geq m_2 \geq m_3 \tag{3.14}$$

Let $\overline{M}(\gamma) = \overline{M}(k_1', k_2')$ denote the multiplicity of a vector

$$\gamma = -R_0 + k_1' \beta_1 + k_2' \beta_2 ,$$

where β_1, β_2 are the two negative primitive roots. Then $\overline{M}(\gamma)$ is given by the number of times γ can be written as

$$\gamma = -R_0 + k_1 \beta_1 + k_2 \beta_2 + k_3 \beta_3$$

with different coefficients, $k_i \geq 0$.

Then the internal multiplicity $M^{m'}(m'')$ of a weight m'' belonging to an IR with m' as its highest weight is given by Eq.(3.12). The problem of obtaining $\overline{M}(k_1', k_2')$ for A_2 then reduces to finding the number of ways of expressing the vector

$$k_1' \beta_1 + k_2' \beta_2$$

as

$$k_1 \beta_1 + k_2 \beta_2 + k_3 \beta_3 .$$

$$\beta_3 = \beta_1 + \beta_2$$

In other words, $\overline{M}(k_1', k_2')$ is nothing but the number of positive integral solutions of the Diophantine equation[9)]

$$k_1' = k_1 + k_3$$
$$k_2' = k_2 + k_3 \tag{3.15}$$

for given (k_1', k_2'). The above equations may be rewritten as

$$k_1 = k_1' - k_3$$
$$k_2 = k_2' - k_3 \tag{3.15a}$$

The condition that $k_i' \geq 0$, $k_i \geq 0$ immediately yields that

$$0 \leq k_3 \leq \min(k_1', k_2') \tag{3.16}$$

so that

$$\overline{M}(k_1', k_2') = \min(k_1'+1, k_2'+1) \tag{3.16a}$$

9) See McMahon Combinatory Analysis, Vol. II, Sec. VIII Chelsea Publishing Company N.Y. (1960).

once the expression for $\overline{M}(k_1', k_2')$ is known, $M^{m'}(m'')$ is immediately known.

In the particular case of A_2, an alternate method for getting $\overline{M}(k_1', k_2')$ has been worked out[10]. This essentially consists in using certain boundary conditions on $M^{m'}(m'')$ to get an expression for $\overline{M}$.

If we introduce the vectors

$$e_S = (m' + R_0) - S\,(m' + R_0), \tag{3.17}$$

it is easy to check that out of the vectors e_S (or equivalently, out of the elements of the Weyl group W) at most three can contribute to the multiplicity of the weight $m'' \in D(m')$, namely the system of vectors

$$\Sigma = \begin{cases} e_1 = (0, 0, 0) \\ e_{S_1} = [(0,\ m_2' - m_1' + 1,\ -(m_2' - m_3' + 1)] \\ e_{S_2} = [m_1' - m_2' + 1,\ -(m_1' - m_2' + 1),\ 0] \end{cases} \tag{3.18}$$

The other three vectors (corresponding to the other three elements of W) lead necessarily to negative integer coefficients. Then

---

10) B.Gruber and T.S.Santhanam, Nuovo cimento 45, 1046 (1966).

one gets

$$
\begin{aligned}
M^{m'}(m'') = \; & \bar{M}\,(m'' - (m' + R_0)) \\
& - \bar{M}\,(m'' - (m' + R_0) + e_{s_1}) \\
& - \bar{M}\,(m'' - (m' + R_0) + e_{s_2}) \\
= \; & \bar{M}\,(m_1' - m_1'' \,;\; m_3'' - m_3') \\
& - \bar{M}\,(m_1' - m_1'' \,;\; m_3'' - m_2' - 1) \\
& - \bar{M}\,(m_2' - m_1'' - 1 \,;\; m_3'' - m_3') .
\end{aligned}
$$

(3.19)

This is just the Kostant's formula that can at most contribute to the multiplicity of weights. Now we use the following boundary conditions on $\bar{M}$ to get an expression for $\bar{M}$

(1) Multiplicity of Equivalent weights is the same i.e.
$M^{m'}(sm'') = M^{m'}(m'')$

(2) The highest weight of a representation is non-degenerate
i.e. $M^{m'}(m') = 1$.

Also from the definition of $\overline{M}(k_1', k_2')$ follows

$$\overline{M}(0, k_2') = \overline{M}(k_1', 0) = 1,$$

$$\overline{M}(k_1', k_2') = \overline{M}(k_2', k_1')$$

and $$\overline{M}(k_1', k_2') = 0 \quad \text{if} \quad k_1' < 0 \quad \text{and/or} \quad k_2' < 0$$

(3.20)

Using these boundary conditions in the Kostant's formula, one gets

$$\overline{M}(k_1', k_1' + k_2') = \overline{M}(k_1' - 1; k_1') + 1,$$

(3.21)

and

$$\begin{aligned} &\overline{M}(k_1' + k_2'; k_1' + k_2') \\ &= \overline{M}(k_1' - 1; k_1' + k_2') \\ &\quad + \overline{M}(k_2' - 1; k_1' + k_2') \\ &\quad + 1 \end{aligned}$$

(3.22)

From the first equation, it follows immediately that :-

$$\overline{M}(k_1', k_2') = \min(k_1' + 1, k_2' + 1)$$

$$k_1' \geq 0, \quad k_2' \geq 0$$

To get the extrenal multiplicity we then proceed as follows[10) knowing $M^{m'}(m'')$ and the formula

$$\chi(m')\,\chi(m+R_0) = \sum_{m''} \chi(m''+R_0) \tag{3.23}$$

the decomposition of the direct product

$$D(m') \otimes D(m) = \sum_{m'''} n(m''')\,D(m''') , \tag{3.24}$$

can be written as

$$\sum_{m'''} n(m''')\;D\;(m'''+R_0)$$
$$= \sum_{m'' \in D(m')} \left\{ M^{m'}(m'')\,\delta_p\,D\,(m+R_0+m'') \right\},$$

(3.25)

where

(1) all $(m + R_0 + m'')$ have to be made dominant (if not already so). $\delta_p = 1$ if this can be achieved by an even permutation of the components, $\delta_p = -1$ otherwise

(2) All terms of the sum are omitted for which two components (or all three) of $(m + R_0 + m'')$ are equal.

Equation (23) can be written down more explicitly as:-

$$\chi(m')\ \chi(m+R_0) =$$

$$\left\{ \sum_{m''=m'}^{m_1', m_3', m_2'} (\beta_2) + \sum_{m''=m'+\beta_1}^{m_2', m_1', m_3'} (\beta_1) \right\}$$

$$\sum_{m''}^{m_3'', m_2'', m_1''} (-R_0) \left\{ \overset{m}{M}(m'')\ \delta_P\ D\ (m + R_0 + m'') \right\}$$

$$\beta_1 = (-1, 1, 0)$$

$$\beta_2 = (0, -1, 1).$$

(3.26)

The bounds of these sums are determined by Weyl reflections. The methods that have been used can be generalized to any classical groups since they are algebraic in nature. For G(2) we shall demonstrate this in the next Chapter.

CHAPTER IV

INTERNAL MULTIPLICITY STRUCTURE AND CLEBSCH-GORDAN SERIES FOR THE EXCEPTIONAL GROUP G(2).

ABSTRACT

An explicit algebraic expression is obtained for the multiplicity $\overline{M}(\gamma)$ of a vector γ belonging to the fundamental domain of the group G(2). Using this, the internal multiplicity $M^{m}(m')$ of a weight m' belonging to the Irreducible Representation $D(m)$ with the highest weight m is calculated through Kostant's formula for the dominant weights. The Clebsch-Gordan decomposition of the direct product of the two Irreducible Representations is then obtained.

INTERNAL MULTIPLICITY STRUCTURE AND CLEBSCH-GORDAN SERIES FOR THE EXCEPTIONAL GROUP G(2)*

Introduction

It is well known that the group G(2), which is a subgroup of O(7) has been extensively used in Nuclear Physics[1] and in Elementary Particle Physics[2] for classifying levels and for studying interactions among particles. It is desirable, therefore, that the Racah algebra of G(2) be developed as in the familiar theory of angular momentum. The problem of finding the invariants has been solved[3]. Any irreducible representation (IR) is ~~has~~ specified by the eigenvalues of the Casimir operators, or, equivalently, by the components of the highest weight.

The next problem is the determination of the internal and external multiplicity structures of the I.R's of the group. By Biedenharn's theorem[5], the external multiplicity of an IR D", occurring in the direct product of two IR's D and D', m is closely connected to the internal multiplicity of the weights in D

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*D.Radhakrishnan and T.S.Santhanam, J.Math.Phys. 8, 2206 (1967).

1) G.Racah, Phys. Rev. 76, 1352 (1949).

2) R.E.Behrends et al. Rev. Mod. Phys. 34, 1 (1962).

3) See Chapter II, for details.

4) We use the terminology introduced by A.J.Macfarlane, L.O'Raifeartaigh and P.S.Rao, J.Math.Phys. 8, 536 (1967).

5) L.C.Biedenharn, Phys. Lett. 3, 254 (1963).
G.E.Baird and L.C.Biedenharn, J.Math.Phys. 5, 1730 (1964).

or D'. Though the internal multiplicity structure is known through Kostant's formula[6], practical computations with it are very tedious. It rurns out that it is sufficient to know the multiplicity structure of $\frac{1}{\Delta}$ [7]. Knowing this, the multiplicity $M^m(m')$ of a weight m' contained in an IR with highest weight m can be calculated[8].

Recently, an algebráic method of getting $M^m(m')$ has been worked out[8] for the case of SU(3). In the present chapter we derive an expression for the internal multiplicity $M^m(m')$, for the group G(2). The problem is more complicated in view of the fact that there are six negative roots and two (negative) primitive roots.

II. <u>The Group G(2).</u>

The root diagram can be conveniently regarded as consisting of all vectors of the form $e_i - e_j$ and $e_i - 2e_j + e_k$, $(i,j,k = 1,2,3)$, which all belong to the hyperplane

$$\sum_{i=1}^{3} x_i = 0$$

The negative primitive roots are

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6) N.Jacobson, Lie Algebras (Interscience, New York, 1962, p.261).
7) J.P.Antoine and D.Speiser, J.Math.Phys. <u>5</u>, 1226 (1964).
8) B.Gruber and T.S.Santhanam, Nuovo Cimento, <u>45A</u>, 1046 (1966).

(a) $\beta_1 = (0, -1, 1) = e_3 - e_2$

$\beta_2 = (-1, 2, -1) = -e_1 + 2e_2 - e_3$

The weight space is three dimensional with a subsidiary condition

$$\sum_{i=1}^{3} m_i = 0 ,$$

where the m_i's are the components of the weight m. Using the theorem that $2(m,\alpha)/(\alpha,\alpha)$ = integer, where m is a weight and α is a root, it is clear that the components of m are integers.

Let us now discuss the Weyl group. Reflecting the weight (m_1, m_2, m_3) in the plane perpendicular to $e_i - e_j$, we see that $m_i \leftrightarrow m_j$ i.e., the components of m are permuted. Next, consider the reflection in the plane perpendicular to $e_i - 2e_j + e_k$ It can be seen that the effect of this is to permute the components of m with a <u>total</u> change of sign. Thus, we have considered all possible reflections perpendicular to the roots and seen that they permute the components of m or permute the components of m with an over all change in sign. The Weyl group is, therefore, of order 12. From these results, it follows that if $m = (m_1, m_2, m_3)$ is to be a dominant weight, then

(a) $\quad m_1 \geqslant m_2 \geqslant m_3$,

(b) $\quad m_1 \geqslant 0 \ , \ m_2 \leq 0 \ , \ m_3 \leq 0$ (1)

Proof. Assume (a) is not true, i.e., $m_r < m_{r+1}$ ($r = 1,2,3$). Applying such a Weyl reflection to m which exchanges m_r and m_{r+1}, we get a weight m' such that the first non-vanishing component is positive, thus leading to m' being higher than m. Hence $m_r \geqslant m_{r+1}$ which proves (a).

To prove (b), we note that condition (a) along with

$$\sum_{i=1}^{3} m_i = 0$$

leads immediately to $m_1 \geqslant 0$ and $m_3 \leq 0$. We need to prove only that $m_2 \leq 0$. Assume the contrary, i.e., $m_2 > 0$. Applying such a reflection which gives a weight m' with $-m_3$ as its first component, so that $m' - m$ has as its first component m_2, which is positive, we are led to a contradiction. Hence, $m_2 \leq 0$.

III. Multiplicity structure $\overline{M}(k_1, k_2)$

In order to find the multiplicity of the dominant weights, let us first calculate the multiplicities $\overline{M}$ of the vectors in using the expression[7]

$$\frac{1}{\Delta} = \sum_{a_1=0}^{\infty} \cdots \sum_{a_m=0}^{\infty} \exp i \left[\sum_{j=1}^{m} a_j \beta_j - R_0 , \varphi \right], \tag{2}$$

where a_i's are non-negative integers, the β_j's are all the negative roots and R_0 is half the sum of all positive roots. The multiplicity $\overline{M}$ of a particular vector γ of $\frac{1}{\Delta}$ (which belongs to the fundamental domain of a group of rank ℓ).

$$\gamma = k_1 \beta_1 + \cdots + k_\ell \beta_\ell - R_0 , \tag{3}$$

where $(\beta_1, \ldots, \beta_\ell)$ are the negative primitive roots ($\ell \leq n$) and $(k_1, \ldots, k_\ell)$ are non-negative integers, is then given by the number of ways γ can be written as a sum over all the negative roots

$$\gamma = \sum_{i=1}^{n} a_i \beta_i - R_0 \tag{4}$$

The multiplicities of the dominant weights m', $M^m(m')$, can then be obtained from Kostant's formula[4]

$$M^m(m') = \sum_{S \in W} \delta_S \, \overline{M} \left[m' - S(m + R_0) \right]$$

$$= \sum_{S \in W} \delta_S \, \overline{M} \, (k_1^S, k_2^S) , \tag{5}$$

where the summation extends over the elements of the Weyl group W and $\delta_s = \pm 1$ according as S is an even or odd reflection, respectively.

The problem of obtaining $M(k_1, k_2)$ for $G(2)$ then reduces to finding the number of ways $(k_1 \beta_1 + k_2 \beta_2)$ can be expressed as $(a_1 \beta_1 + \ldots\ldots\ldots\ldots + a_6 \beta_6)$ for given k_1 and k_2 i.e.

$$
\begin{aligned}
k_1 \beta_1 + k_2 \beta_2 = \; & a_1 \beta_1 + a_2 \beta_2 + a_3 (\beta_1 + \beta_2) \\
& + a_4 (2\beta_1 + \beta_2) + a_5 (3\beta_1 + \beta_2) \\
& + a_6 (3\beta_1 + 2\beta_2) \qquad (6)
\end{aligned}
$$

so that

$$
\begin{aligned}
k_1 &= a_1 + a_3 + 2a_4 + 3a_5 + 3a_6 , \\
k_2 &= a_2 + a_3 + a_4 + a_5 + 2a_6 . \qquad (7)
\end{aligned}
$$

We have to find all possible values allowed for $(a_1, \ldots\ldots, a_6)$ for given (k_1, k_2). These equation are known as Diophantine equations[9] and we have solved them using the theory of partitions. We shall now go to the details of finding the number of solutions of the Diophantine equations[10]. The crucial point in the analysis is that the k's and a's are non-negative integers. Otherwise, the number of solutions of the diophantine equations (7) is trivially infinite. The number of solutions of Eqs.(7) is just the number of distinct values allowed for the set $(a_1, \ldots\ldots, a_6)$ for given (k_1, k_2). To find this we proceed as follows:

First set $a_4 = a_5 = a_6 = 0$, then Eq.(7) reduces to

$$k_1 = a_1 + a_3 ,$$
$$k_2 = a_2 + a_3 .$$

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9) P.A.Macmahon, Combinatory Analysis, Vol.II, Sec.VIII, Chelsea Publishing Company, N.Y.(1960). The number of solutions of the Diophantine equations (7) can be given by the method of generating series. Now Eq.(7) can be written as a matrix equation

$$(k_1, k_2) = C \quad (a_1, \ldots\ldots, a_6)$$

where C is a (6 x 2) matrix. The number of solutions of Eq. (7) is then obtained as the coefficient $x_1^{k_1} \; x_2^{k_2}$ of the generating function

$$f(x_1, x_2) = \prod_{i=1}^{6} \left(1 - x_1^{C_{i1}} \; x_2^{C_{i2}} \right)^{-1}$$

where the C_{ij} are the elements of the matrix C. See Chapter V for greater details.

10) See ref. (4) for all details about the conditions for D' to dominate D. In this paper a complete list of references to earlier literature may be found.

which we rewrite as

$$a_1 = k_1 - a_3$$
$$a_2 = k_2 - a_3 .$$

It may be recalled that these are just the equations one gets in the inner multiplicity problem for the group SU(3)*. For fixed (k_1, k_2), a_3 can have the range

$$0 \leq a_3 \leq \min (k_1, k_2)$$

and so the number of values allowed for a_3 is given by

$$\min \; (k_1 + 1, \; k_2 + 1) \tag{8}$$

As the next step we set $a_3, a_4 \neq 0$, $a_5 = a_6 = 0$, and so Eq.(7) reduces to

$$k_1 = a_1 + a_3 + 2a_4$$
$$k_2 = a_2 + a_3 + a_4$$

which we rewrite as

$$a_1 = k_1 - a_3 - 2a_4$$
$$a_2 = k_2 - a_3 - a_4$$

The allowed non-zero values for a_4 are given by (for fixed a_3)

$$\min \left\{ \left[\frac{k_1 - a_3}{2}\right], \; (k_2 - a_3) \right\} \tag{9}$$

---

*See Chapter III.

where we have denoted by the square bracket the integral part of the expression. Of course, a_3 can have its range such that $a_4 \neq 0$. Two cases of ~~mm~~ inequality arises. when $k_1 \leq k_2$ which implies $\left[\frac{k_1}{2}\right] \leq k_2$, then the number of values of a_4 are

$$\sum_{a_3=0} \left[\frac{k_1 - a_3}{2}\right] \tag{9a}$$

and in order that $a_4 \neq 0$, we insist that $a_4 \geq 1$ and therefore the second equation implies

$$a_3 + 1 \leq k_2 \tag{9b}$$

Of course, if $k_1 \leq k_2$ then it is ~~time~~ true that the maximum value allowed for a_3 from (9a) $(k_1 - 2) \leq k_2 - 2$ and hence the subsidiary condition (9b) is automatically satisfied by the natural boundary of Eq.(9a).

If on the other hand $k_2 \leq \overset{k_1}{\cancel{\left[\frac{k_1}{2}\right]}}$ then the following case arises

$$\left[\frac{k_1}{2}\right] \leq k_2 < k_1$$

In this case the allowed range of values for a_3 is

$$0 \leq a_3 \leq k_2 ,$$

and the allowed values for $a_4 \neq 0$ is given by

$$\sum \left[\frac{k_1 - a_3}{2} \right]$$

but now the subsidiary condition (making $a_4 \neq 0$) $a_3 + 1 \leq k_2$, has to be carefully taken since the sum implies

$$a_3 \leq k_1 - 2$$

while the subsidiary condition implies

$$a_3 \leq k_2 - 1$$

Since

$$(k_2 - 1) \leq k_1 - 2 \quad \text{hence if} \quad k_2 < k_1$$

the subsidiary condition restricts the sum to a smaller number of values. Hence it should be imposed. We can do this by first allowing a_3 to range through all values allowed by the sum as this facilitates the evaluation of the sum and then subtract out those values which are not permitted by the subsidiary condition, i.e. values of a_3 beyond $k_2 - 1$ upto $k_1 - 2$.

As the next step we proceed to set $a_3, a_4, a_5 \neq 0$ and $a_6 = 0$ Eq.(7) reduces to

$$a_1 = k_1 - a_3 - 2a_4 - 3a_5$$

$$a_2 = k_2 - a_3 - a_4 - a_5$$

Now one has to consider the following inequalities

$$k_1 \leq k_2 , \quad \left[\frac{k_1}{2}\right] \leq k_2 < k_1 , \quad \left[\frac{k_1}{3}\right] \leq k_2 < \left[\frac{k_1}{2}\right]$$

and depending on these limits we have to determine whether the first or second equation has a say. Number of solutions for a_5 is then

$$\sum_{a_3, a_4} \min \left\{ \left[\frac{k_1 - a_3 - 2a_4}{3}\right] , (k_2 - a_3 - a_4) \right\} \tag{10}$$

For instance when $k_1 \leq k_2$ implying $\left[\frac{k_1}{3}\right] < k_2$ then the number of values allowed for a_5 is then

$$\sum_{a_3, a_4} \left[\frac{k_1 - a_3 - 2a_4}{3}\right]$$

To ensure that $a_5 \neq 0$, we put this condition on the other equation

$$a_3 + a_4 + 1 \leq k_2 .$$

On the other hand when $k_2 < \left[\frac{k_1}{3}\right]$ then the number of values allowed for a_5 becomes

$$\sum_{a_3, a_4} (k_2 - a_3 - a_4)$$

but now the condition that $a_5 \neq 0$ becomes

$$a_3 + 2a_4 + 3 \leq k_1$$

The next step is then to set $a_3, \dots, a_6 \neq 0$ so that the number of allowed non-zero values of a_6 is then

$$\sum_{a_3, a_4, a_5} \min \left\{ \left[\frac{k_1 - a_3 - 2a_4 - 3a_5}{3} \right], \left[\frac{k_2 - a_3 - a_4 - a_5}{2} \right] \right\}$$

when $\left[\frac{k_1}{3}\right] \leq \left[\frac{k_2}{2}\right]$, then the first term is minimum and ~~to~~ to ensure that $a_6 \neq 0$, we insist $a_3 + a_4 + a_5 + 2 \leq k_2$.
when $\left[\frac{k_1}{3}\right] > \left[\frac{k_2}{2}\right]$, then the second term is decisive and in this case to ensure that $a_6 \neq 0$ we set

$$a_3 + 2a_4 + 3a_5 + 3 \leq k_1 .$$

Thus the number of solution of the Diophantine equation (7) is given by the allowed distinct values of the set $(a_1, \dots, a_6)$

and is thus given by (for given (k_1, k_2))

$$\bar{M}(k_1,k_2) = \min(k_1+1,\ k_2+1)$$

$$+ \sum_{a_3} \min\left\{ \left[\frac{k_1-a_3}{2}\right], (k_2-a_3) \right\}$$

$$+ \sum_{a_3,a_4} \min\left\{ \left[\frac{k_1-a_3-2a_4}{3}\right], (k_2-a_3-a_4) \right\}$$

$$+ \sum_{a_3,a_4,a_5} \min\left\{ \left[\frac{k_1-a_3-2a_4-3a_5}{3}\right], \left[\frac{k_2-a_3-a_4-a_5}{2}\right] \right.$$

(11)

The evaluation of the sums of integral parts is straightforward and we summarize them as follows

$$\left.\begin{aligned} \sum \left[\frac{k_1-i}{2}\right] &= \frac{k_1^2-1}{4} \quad \text{for } k_1 \text{ odd}, \\ &= \frac{k_1^2}{4} \quad \text{for } k_1 \text{ even} \end{aligned}\right\}$$

$$k_1 \geq 2$$

(12)

If we denote by

$$f(k) = \sum_{i,j} \left[\frac{k-i-2j}{3}\right]$$

then can be easily verified by actual computation that $f(k)$ obeys the difference equation

$$f(k) - f(k-6) = \frac{1}{2}\left[(k-4)(k-1)+4\right]_{k \geq 3}$$

This difference equation has the solution

$$f(k) = \frac{1}{36}\left(k^3 + \frac{3}{2}k^2 - 3k + d\right)$$

where d is as yet not known, to be fixed by boundary condition for each mod 6 of k. We find that

$$d = \left(0, \frac{1}{2}, -8, \frac{9}{2}, -4, -\frac{7}{2}\right)$$

for k = (0,1,2,3,4,5) mod 6 respectively. Thus

$$\sum_{i,j} \left[\frac{k-i-2j}{3}\right] = \frac{1}{36}\left(k^3 + \frac{3}{2}k^2 - 3k + d\right) \qquad (13)$$

$$\sum \left[\frac{k_1 - i - 2j - 3k}{3}\right] = \overline{M}\,(k_1 - 3,\ k_2),$$

(14)

$$\sum \left[\frac{k_2 - i - j - k}{2}\right] = \frac{1}{48}\, k_2 \left\{ (k_2-2)^3 + 10\,(k_2-2)^2 + 30\,(k_2-2) + 24 \right\}$$

for k_2 even ≥ 2,

$$= \frac{1}{48}\,(k_2-1) \left\{ (k_2-2)^3 + 11\,(k_2-2)^2 + 39\,(k_2-2) + 45 \right\}$$

for k_2 odd ≥ 3

(15)

The important point is that the subsidiary conditions have to be imposed wherever they are applicable. In these cases, we first allow all values in the sum and subtract those which are forbidden by the subsidiary conditions.

We find the following expressions for $\overline{M}(k_1, k_2)$ in the various limits and inequalities between k_1, k_2.

One finds

$$\overline{M}(k_1, k_2)_{k_1 \le k_2} = (1+k_1) + \sum \left[\frac{k_1 - i}{2}\right] + \sum_{i,j} \left[\frac{k_1 - i - 2j}{3}\right] + \sum_{i,j,k} \left[\frac{k_1 - i - 2j - 3k}{3}\right]$$

$$= (1+k_1) + \frac{k_1^2 - 1}{4} \quad \text{for } k_1 \text{ odd} \ge 3$$

or

$$\frac{k_1^2}{4} \quad \text{for } k_1 \text{ even} \ge 2$$

$$+ \frac{1}{36}\left(k_1^3 + \frac{3}{2}k_1^2 + 3k_1 + d\right) \tag{16}$$

$$+ \overline{M}(k_1 - 3, k_2).$$

$$\overline{M}(k_1, k_2)_{k_1 \ge 3k_2} = (1+k_2) + \sum (k_2 - i) + \sum_{i,j} (k_2 - i - j)$$

$$+ \sum_{i,j,k} \left[\frac{k_2 - i - j - k}{2} \right]$$

$$= \frac{1}{48} (k_2 + 2)(k_2^3 + 10 k_2^2 + 30 k_2 + 24) \quad \text{for even } k_2$$

$$= \frac{1}{48} (k_2 + 1)\left(k_2^3 + 11 k_2^2 + 39 k_2 + 45\right) \quad \text{for odd } k_2.$$

(17)

$$\underset{2k_2 \le k_1 < 3k_2}{\overline{M}(k_1, k_2)} = (1 + k_2) + \sum (k_2 - i)$$

$$+ \sum_{i+j+1 \le k_2} \left[\frac{k_1 - i - 2j}{3} \right]$$

$$+ \sum_{i+2j+3k+3 \le k_1} \left[\frac{k_2 - i - j - k}{2} \right]$$

$$= \frac{1}{2} (k_2 + 1)(k_2 + 2)$$

$$+ f(k_1) - f(k_1 - 2k_2 + 1)$$

$$+ \frac{1}{48} k_2 \left\{ (k_2 - 2)^3 + 10 (k_2 - 2)^2 + 30 (k_2 - 2) + 24 \right\}$$

for even $k_2 \geq 2$

$$+ \frac{1}{48}(k_2-1)\left\{(k_2-2)^3 + 11(k_2-2)^2 + 39(k_2-2) + 45\right\}$$

for odd $k_2 \geqslant 3$,

$$- \frac{1}{48} K \left\{(K-2)^3 + 10(K-2)^2 + 30(K-2) + 24\right\}$$

for even $K \geqslant 2$

$$- \frac{1}{48}(K-1)\left\{(K-2)^3 + 11(K-2)^2 + 39(K-2) + 45\right\}$$

for odd $K \geqslant 3$,

(18)

Here,
$$K = k_2 - 1 - \left[\frac{k_1-3}{3}\right]$$

$$\underset{3k_2 \leq 2k_1 < 4k_2}{\overline{M}(k_1, k_2)} = (1+k_2) + \sum_{i+1 \leq k_2} \left[\frac{k_1-i}{2}\right]$$

$$+ \sum_{i+j+1 \leq k_2} \left[\frac{k_1-i-2j}{3}\right]$$

$$+ \sum_{i+2j+3k+3 \leq k_1} \left[\frac{k_2-i-j-k}{2}\right].$$

$$= (1+k_2) + \frac{k_1^2-1}{4} \quad \text{for odd } k_1 \geq 3$$

$$+ \frac{k_1^2}{4} \quad \text{for even } k_1 \geq 2$$

$$- \left\{ \frac{(k_1-k_2)^2-1}{4} \right\} \quad \text{for odd } (k_1-k_2) \geq 3$$

$$- \frac{(k_1-k_2)^2}{4} \quad \text{for even} (k_1-k_2) \geq 2$$

$$+ f(k_1) - f(k_1-2k_2+1)$$

$$+ \frac{1}{48} k_2 \left\{ (k_2-2)^3 + 10(k_2-2)^2 + 30(k_2-2) + 24 \right\} \quad \text{for even } k_2 \geq 2$$

$$+ \frac{1}{48}(k_2-1) \left\{ (k_2-2)^3 + 11(k_2-2)^2 + 39(k_2-2) + 45 \right\} \quad \text{for odd } k_2 \geq 3$$

$$- \frac{1}{48} K \left\{ (K-2)^3 + 10(K-2)^2 + 30(K-2) + 24 \right\} \quad \text{for even } K \geq 2$$

$$- \frac{1}{48}(K-1) \left\{ (K-2)^3 + 11(K-2)^2 + 39(K-2) + 45 \right\} \quad \text{for odd } K \geq 3$$

(19)

$$\overline{M}(k_1, k_2)_{2k_2 < 2k_1 < 3k_2} = (1 + k_2) + \sum_{i+1 \le k_2} \left[\frac{k_1 - i}{2}\right]$$

$$+ \sum_{i+j+1 \le k_2} \left[\frac{k_1 - i - 2j}{3}\right]$$

$$+ \sum_{i+j+k+2 \le k_2} \left[\frac{k_1 - i - 2j - 3k}{3}\right]$$

$$= (1 + k_2) + \frac{k_1^2 - 1}{4} \quad \text{for odd } k_1 \geq 3$$

$$+ \frac{k_1^2}{4} \quad \text{for even } k_1 \geq 2$$

$$- \left\{\frac{(k_1 - k_2)^2 - 1}{4}\right\} \quad \text{for odd} (k_1 - k_2) \geq 3$$

$$- \frac{(k_1 - k_2)^2}{4} \quad \text{for even } (k_1 - k_2) \geq 2$$

$$+ f(k_1) - f(k_1 - 2k_2 + 1)$$

$$+ \overline{M}(k_1 - 3, k_2') \text{ with } k_2' \geq k_1 - 3$$

$$- \overline{M}(k_1 - k_2 - 2; k_2'') \text{ with } k_2'' \geq (k_1 - k_2 - 2)$$

(20)

Eq. (16) is a difference equation and can be solved for each modulus 6 of k_1. However, Eq.(16) itself is sufficient to determine $\overline{M}(k_1, k_2)$ $(k_1 \leq k_2)$ straightaway.

IV. Multiplicity Structure $M^m(m')$.

Kostant's formula Eq.(5) can now be used to find the multiplicity structure $M^m(m')$, since $\overline{M}$ can be evaluated using our formulae 16-20. The next problem is then of the Weyl group which is of order twelve. Great simplicity is achieved by first setting m' to be dominant as the multiplicity of the other weights can be easily known from this. This makes only a few reflections to contribute to Eq.(5), as the other elements of the Weyl group make the argument of $\overline{M}$ negative. This we shall see as follows. Consider the argument of $\overline{M}$, i.e $m' - S(m + R_0)$. Since $\overline{M}$ is the number of ways of weight m' ($= m + k_1 \beta_1 + k_2 \beta_2$) can be expressed as

$$m' = -R_0 + \sum_{i=1}^{6} a_i \beta_i + S(m+R_0) \qquad R_0 = (3,-1,-2) \tag{21}$$

we have

$$\gamma = m' - S(m+R_0) = -R_0 + \sum_{i=1}^{6} a_i \beta_i = -R_0 + k_1^S \beta_1 + k_2^S \beta_2 = \left(k_1^S, k_2^S\right) \tag{22}$$

Thus

$$(m' + R_0) - S\,(m + R_0) = k_1^S \beta_1 + k_2^S \beta_2$$

$$= \left(-k_2^S \,,\, -k_1^S + k_2^S \,,\, k_1^S - k_2^S\right)$$

We can therefore express γ in the (k_1^S, k_2^S) notation for all the twelve elements of the Weyl group. We have earlier seen that the Weyl group consists of all six permutations of the components of a weight <u>and</u> all six permutations of the components of the weight with a <u>total</u> change in sign. We denote these elements by S_{123}, S_{132}, S_{213}, S_{312}, S_{231}, S_{321} and $\bar{S}_{123}$, $\bar{S}_{132}$, $\bar{S}_{213}$, $\bar{S}_{231}$, $\bar{S}_{312}$, $\bar{S}_{321}$ respectively and δ_S in Eq.(5) is ± 1 depending on ~~the~~ whether the permutation is even or odd. In the (k_1^S, k_2^S) notation, Eq. (5) becomes

$$\begin{aligned}
M^m(m') = {} & \bar{M}\left(m_3' - m_1' + m_1 - m_3 \,;\, m_1 - m_1'\right) \\
& - \bar{M}\left(m_3' - m_1' + m_1 - m_2 - 1 \,;\, m_1 - m_1'\right) \\
& - \bar{M}\left(m_3' - m_1' + m_2 - m_3 - 4 \,,\, m_2 - m_1' - 4\right) \\
& + \bar{M}\left(m_3' - m_1' + m_2 - m_1 - 9 \,,\, m_2 - m_1' - 4\right) \\
& + \bar{M}\left(m_3' - m_1' + m_3 - m_2 - 6 \,,\, m_3 - m_1' - 5\right) \\
& - \bar{M}\left(m_3' - m_1 + m_3 - m_1 - 10 ,\, m_3 - m_1' - 5\right)
\end{aligned}$$

$$+ \ \overline{M}\left(m_3' - m_1' + m_3 - m_1 - 10 \ , \ -(m_1' + m_1 + 6)\right)$$
$$- \ \overline{M}\left(m_3' - m_1' + m_2 - m_1 - 8 \ , \ -(m_1' + m_1 + 6)\right)$$
$$- \ \overline{M}\left(m_3' - m_1' + m_3 - m_2 - 6 \ , \ -(m_1' + m_2 + 2)\right)$$
$$+ \ \overline{M}\left(m_3' - m_1' + m_2 - m_3 - 4 \ , \ -(m_1' + m_3 + 1)\right)$$
$$+ \ \overline{M}\left(m_3' - m_1' + m_1 - m_2 - 1 \ , \ -(m_1' + m_2 + 2)\right)$$
$$- \ \overline{M}\left(m_3' - m_1' + m_1 - m_3, \ -(m_1' + m_3 + 1)\right) \tag{23}$$

Suppose now m' is dominant. Then both m and m' satisfy conditions (a) and (b) of Sec. II. i.e.,

$$m_1' \geq m_2' \geq m_3' \ , \quad m_1' + m_2' + m_3' = 0,$$
$$m_1' \geq 0, \ m_2' \leq 0, \ m_3' \leq 0$$
$$m_1 \geq m_2 \geq m_3 \ , \quad m_1 + m_2 + m_3 = 0,$$
$$m_1 \geq 0, \ m_2 \leq 0, \ m_3 \leq 0.$$

It is then easy to see that

$$m_2 - m_1' - 4 < 0$$
$$m_3 - m_1' - 5 < 0$$
$$-(m_1' + m_1 + 6) < 0$$
$$m_3' - m_1' + m_3 - m_2 - 6 < 0 \tag{24}$$

and since $\overline{M}(k_1, k_2) = 0$ when k_1 or $k_2 < 0$, it follows that only five terms from Eq.(23) are non-vanishing when m' is dominant. Thus Eq.(23) becomes

$$\begin{aligned} M^m(m') = \; & \overline{M}\left(m_3' - m_1' + m_1 - m_3 \,,\; m_1 - m_1'\right) \\ & - \overline{M}\left(m_3' - m_1' + m_1 - m_2 \,,\; m_1 - m_1'\right) \\ & + \overline{M}\left(m_3' - m_1' + m_2 - m_3 - 4 \,,\; -(m_1' + m_3 + 1)\right) \\ & + \overline{M}\left(m_3' - m_1' + m_1 - m_2 - 1 \,,\; -(m_1' + m_2 + 2)\right) \\ & - \overline{M}\left(m_3' - m_1' + m_1 - m_3 \,,\; -(m_1' + m_3 + 1)\right) \end{aligned} \qquad (25)$$

Equation (25) along with Eqs. (16)-(20) give $M^m(m')$ for any dominant weight m'. The multiplicity of any other weight can be found using the Weyl reflections. In Eqs. (16)-(20), the intervals for k_1 and k_2 depend sensitively on the coefficients of the a's in the diophantine equations (7). These coefficients are entries of the Cartan matrix* and thus are characteristic of the group in question.

V. External Multiplicity Structure.

It is well-known from the work of Biedenharn[5] that, if $D(\Lambda)$ and $D'(\Lambda')$ are two IR's of a group L with Λ and Λ' as their highest weights respectively, and if D' dominates[10] D,

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* See Appendix 3.

then the product $D' \times D$ contains IR's for which $(\Lambda' + m)$ are highest weights, where m stands for all weights contained in D. The multiplicity of the representation $(\Lambda' + m)$ in the reduction of $D' \times D$ is the same as the internal multiplicity of the weight m in the representation D. The conditions for D' to dominate D for $G(2)$ are[4] $\lambda_1' \geqslant 2\lambda_1 + 3\lambda_2$, $\lambda_2' \geqslant \lambda_1 + 2\lambda_2$ where (λ_1', λ_2'), (λ_1, λ_2) are the components of Λ' and Λ in the familiar two component notation. More explictly, Biedenharn's theorem can be stated in terms of characters:*

$$\chi^{D \times D'}(\phi) = \sum_{m} \gamma_m \, \chi^{(\Lambda' + m)}(\phi) \tag{26}$$

The assumption that D' dominates D is needed to make $(\Lambda' + m)$ satisfy the conditions for it to be dominant so that it can be the highest weight of some representation in the reduction.

The important point is that the representation with $(\Lambda' + m)$ as highest weight occurs γ_m times where γ_m is the internal multiplicity of m in $D(\Lambda)$. γ_m can be immediately computed for any m in $D(\Lambda)$ using our results in Sec. IV. Thus knowing $M^m(m')$ and equation (5) the Clebsch-Gordan reduction of the product of two (IR'S) can be immediately written down.

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* See Appendix 4.

We give a few examples of multiplicities of some weights using the results obtained by us.

Consider the I.R $D^7(1,0)$, defined in the conventional $D^N(\lambda_1,\lambda_2)$ notation, where the highest weight (λ_1,λ_2) is given as λ_1 times one fundamental weight and λ_2 times the other. The connection with the three component form is given by

$$\begin{aligned} m_1 &= \lambda_1 + 2\lambda_2 \\ m_2 &= -\lambda_2 \\ m_3 &= -\lambda_1 - \lambda_2 . \end{aligned}$$

We calculate the internal multiplicity of the dominant weight $(0,0)$. From Eq.(25), we find that

$$M^{(1,0)}(0,0) = \overline{M}(2,1) - \overline{M}(0,1) - \overline{M}(2,0)$$

Now using Eqs. (16-20), we find that

$$\overline{M}(2,1) = 3,\ \overline{M}(0,1) = 1,\ \overline{M}(2,0) = 1.$$

so that

$$M^{(1,0)}(0,0) = 1.$$

Similarly, for the internal multiplicity of the dominant weight $(0,0)$ in the representation $D^{14}(0,1)$, we get

$$\begin{aligned} M^{(0,1)}(0,0) &= \overline{M}(3,2) - \overline{M}(2,2) - \overline{M}(3,0) \\ &= 7-4-1 = 2. \end{aligned}$$

Let us now consider the direct product $D^{14}(0,1) \times D^{1547}(3,2)$. It can be seen that $D^{1547}(3,2)$ dominates $D^{14}(0,1)$. The various weights of $D^{14}(0,1)$ are

$$(0,1),\ (3,-1),\ (1,0),\ (-1,1),\ (2,-1)$$
$$(-3,2),\ (3,-2),\ (-2,1),\ (1,-1),\ (-1,0)$$
$$(-3,1),\ (0,-1),\ (0,0).$$

Using Biedenharn's Theorem, Eq.(23), we see that

$$\begin{aligned} & D^{14}(0,1) \times D^{1547}(3,2) \\ = {} & D^{4096}(3,3) + D^{3003}(6,1) + D^{2926}(4,2) + D^{2079}(2,3) \\ & + D^{1728}(5,1) + D^{748}(0,4) + D^{714}(6,0) + D^{896}(1,3) \\ & + D^{729}(2,2) + D^{924}(4,1) + D^{273}(0,3) + D^{448}(3,1) \\ & + 2.\ D^{1547}(3,2). \end{aligned}$$

It should be noted that the occurance of $D^{1547}(3,2)$ twice in the above reduction is precisely due to the appeance of the weight $(0,0)$ twice in $D^{14}(0,1)$.

CHAPTER V.

GENERATING FUNCTIONS OF CLASSICAL GROUPS AND EVALUATION OF PARTITION FUNCTIONS

ABSTRACT

The generating functions of classical groups are used to set up recursion relations for their partition functions. These are then used to find the internal multiplicity structure of the weights using Kostant's formula.

GENERATING FUNCTIONS OF CLASSICAL GROUPS AND EVALUATION OF PARTITION FUNCTIONS*

Introduction

The Clebsch-Gordan (C.G.) programme of classical groups suffers from two major difficulties. Unlike the rotation group in three dimensions for which the C.G. programme is well known, many other classical groups do not possess the properties of simple reducibility and the equivalence of an irreducible representation (I.R.) and its conjugate. Here, we mean by the lack of simple reducibility, the multiple occurrance of an I.R. in the product of two I.R's. This multiplicity is called the external multiplicity[1]. However, many relations have been worked out[2),3)], which relate this external multiplicity to the multiple occurrance of a given weight in an I.R., a feature not shared by the I.R's of O(3), is called the internal multiplicity structure.

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* T.S.Santhanam, communicated to J.Math. Phys.

1) The terminology is due to - A.J.Macfarlane, L.O'Raifeartaigh and P.S.Rao, J.Math.Phys. 8, 536 (1967).
2) L.C.Biedenharn, Phys. Letts. 3, 254 (1963).
3) G.E.Baird and L.C.Biedenharn, J.Math.Phys. 5, 1730 (1964). See Appendix 4.
3) G.Racah, Lectures on group theoretical concepts and methods in Elementary Particle Physics, ed.F.Gursey (Gordon and Breach Science Publishers, New York, 1964)
   D.R.Speiser, Lectures on theory of compact Lie groups in group theoretical concepts and Methods in Elementary Particle Physics, Loc.cit.

At present the internal multiplicity structure can be worked out using Kostant's formula[4]. There exist, however, many other methods (for instance, the recursion method of Freudenthal[5]), although in practice, Kostant's formula is the most useful. Kostant's formula involves the partition function of expressing a non-negative integral linear combination of positive roots in terms of a non-negative integral linear combination of primitive roots. These partition functions have been known so far only for rank two and three groups[6].

Recently we have developed a method of obtaining the partition functions for A_ℓ ($\sim SU(\ell + 1)$) using the generating functions. In this, we set up recursion relations for the partition functions, which are then used in conjunction with Kostant's formula to compute the internal multiplicities. Of course, the calculation gets more and more involved as one goes to large ℓ . However, the method is precise.

In this chapter, we work out the generating functions for A_ℓ , B_ℓ , C_ℓ , D_ℓ and G_2. ~~Zk~~ We also obtain recursion relations for the internal multiplicity.

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4) N.Jacobson, Lie Algebras, p.261 (Interscience Publishers 1962).
5) N.Jacobson, Lie Algebras, loc cit. p.247.
6) J.Tarski, J.Math.Phys. 4, 569 (1963).

In section 2, the general discussion of Kostant's formula is given. We discuss the cases of $A_\ell \sim SU(\ell + 1)$, $B_\ell \sim O(2\ell + 1)$, $C_\ell \sim (Sp_{2\ell})$, $D_\ell \sim O(2\ell)$ and G_2 in sections (3)-(7). The discussion includes the Weyl group, the structure of positive and primitive (simple) roots and the Diophantine equations. Explicit formulae are obtained and possible recursion relations for the partition functions are given. In Sec.(8), the connection between internal and external multiplicity structures is discussed. In Sec.(9), the conclusions are given. Many of the properties of the classical groups (structure of positive and primitive roots and so on) are contained in many places. We have taken them from the papers of Dynkin[7].

2. Kostant's formula

The inner multiplicity $M^m(m')$ of a weight m' belonging to the irreducible representation $D(m)$ of highest weight m is given by Kostant's formula[4] which is

$$M^m(m') = \sum_{s \in W} \delta_s \, P\left[s(m+R_0) - (m'+R_0) \right], \qquad (2.1)$$

7) E.B.Dynkin, Amer.Math.Soc. Translations Series 2, Vol.6 (1967).

where W is the Weyl group and R_0 is half the sum of positive roots $\delta_S = \pm 1$ according as whether the reflection is even or odd respectively. P(M) is the partition function for the weight M. This is the number of ways the weight M can be written as a sum over all the positive roots

$$M = \sum_{i=1}^{n} a_i \varphi_i , \tag{2.2}$$

with different non-negative integers a_i. On the other hand, Antoine and Speiser[8] have shown that the vector

$$S\,(m+R_0) - (m'+R_0)$$

can be expressed for a fixed $S \in W$ uniquely in terms of the primitive roots as

$$S\,(m+R_0) - (m'+R_0) = \sum_{i=1}^{l} k_i \beta_i \tag{2.3}$$

l being the rank of the group. From (2.2) and (2.3), it is clear that P(M) is the number of ways we can write

$$\sum_{i=1}^{l} k_i \beta_i = \sum_{\mu=1}^{n} a_\mu \varphi_\mu \tag{2.4}$$

$$k_i \geq 0 , \quad a_\mu \geq 0$$

k_i and a_μ are integers

8) J.P.Antoine and D.Speiser, J.Math.Phys. 5, 1226 and 1560 (1964).

for given k_i. It can be shown that $P(k_1, \ldots, k_\ell)$ is the multiplicity $\overline{M}(\gamma)$ of a vector γ of $\frac{1}{\Delta}$ [8] where the $\frac{1}{\Delta}$ is related to the character by Weyl's formula

$$\chi^{m}(\xi) = \frac{X(m+R_0)}{\Delta} \qquad \Delta = X(R_0) \tag{2.5}$$

$X(m+R_0)$ is the alternating elementary sum

$$X(m+R_0) = \sum_{S \in W} \delta_S \exp\left[S(m+R_0), \xi \right], \tag{2.6}$$

where ξ are the coordinates of the toroid (the group parameters). Hence (2.1) can be written as

$$M^{m}(m') = \sum_{S \in W} \delta_S \, \overline{M}\,(k_1^S, \ldots, k_\ell^S)$$

If we can calculate the partition function $\overline{M}\,(k_1^S, \ldots, k_\ell^S)$ then $M^m(m')$ can be computed in ~~the~~ principle. In the following few sections, we shall explicitly calculate $\overline{M}\,(k_1^S, \ldots, k_\ell^S)$ for various classical groups.

3. A_ℓ ($\sim$ SU(ℓ + 1))

The roots of this algebra are given by $e_i - e_j$, $i,j = 1,\ldots,(\ell + 1)$. The e_i form an orthogonal basis in (ℓ + 1) dimensional space in which the roots and weights are defined. There are $\ell(\ell + 1)$ roots. The $\frac{1}{2}\ell(\ell + 1)$ positive roots are then obtained as $(e_i - e_j)$ $i<j$. The primitive (simple) roots in this case are $\beta_i = e_i - e_{i+1}$, $i = 1,\ldots \ell$. Equation (2.4) then can be written as

$$k_i = C_{i\mu}\, a_\mu \qquad \begin{array}{l} i = 1,\ldots,\ell \\ \mu = 1,\ldots,\frac{1}{2}\ell(\ell+1) \end{array} \tag{3.1}$$

Where C is the $\left(\frac{1}{2}\ell(\ell+1)\times\ell\right)$ diemensional rectnagular matrix

$$\begin{array}{cc} & \mu \to 1,\ldots\ldots,\frac{1}{2}\ell(\ell+1) \\ C_{i\mu} = \; i \to 1,\ldots,\ell & \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 \cdots & 0 & 1 & 0 \cdots & 0 & 1 & 0 \cdots & 0 & \cdots & & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & & & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & & & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & & & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & & & 1 \\ & & & & & & & & & & & 1 \end{array}\right) \end{array} \tag{3.2}$$

It can easily be seen that only for the case of $\ell = \frac{1}{2}$, the matrix C is a non-singular square matrix so that there is a unique solution i.e. $\overline{M}(k_1) = 1$. However, in general C is a rectangular matrix and so given the vector κ and the matrix C, the number of a's is trivially infinite and it is only because we have the restriction that the elements of the matrix C are non-negative integers the very question of the number of solutions (number of a's, the compnents of the vector a are again non-negative integers) makes a meaning after all*). We recognise, that the number of solutions of Eq.(2.4) is given by the coefficient of $x_1^{k_1} x_2^{k_2} \cdots\cdots x_\ell^{k_\ell}$ of the generating functions. To solve the Biophantine equations (3.1), (actually we mean finding the number of solutions for given k and C) we now use the method of generating functions. Let $f(x_1, \ldots, x_\ell)$ be the generating function defined by

$$f_\ell(x_1, \ldots x_\ell) = \prod_{i=1}^{\frac{1}{2}\ell(\ell+1)} \frac{1}{\left(1 - x_1^{c_{1i}} x_2^{c_{2i}} \cdots x_\ell^{c_{\ell i}}\right)} \tag{3.3}$$

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*) I am grateful to Professor Ramakrishnan for focussing my attention to this general problem. There is a discussion about such a matrix equation in the book on 'Linear Differential Operators' by Cornelus, Lanczas, D.Van Nostrand Company Limited (London) (1961), p.115. However, the general problem of finding the number of solutions seems to remain open, although the generating function method we have developed in principle gives a solution to this problem.

$x_1, \ldots\ldots, x_\ell$ are chosen arbitrary parameters with modulus less than one. $\overline{M}(k_1, \ldots, k_\ell)$ is now given by the coefficient of $x_1^{k_1} \cdots\cdots x_\ell^{k_\ell}$ in $f_\ell(x_1, \ldots\ldots, x_\ell)$. This can be checked by actually expanding $f_\ell(x_1, \ldots, x_\ell)$ in power series. Since the matrix C is known, we can write the following important relation

$$f_\ell(x_1, \ldots, x_\ell) = \left\{ \prod_{i=1}^{\ell} \left(1 - \prod_{j=\ell-i+1}^{\ell} x_j \right) \right\}^{-1} f_{\ell-1}(x_1, \ldots\ldots, x_{\ell-1}) \tag{3.4}$$

Now, we can expand (3.4) in power series. $\overline{M}(k_1, \ldots\ldots, k_\ell)$ is the coefficient of $x_1^{k_1} \cdots\cdots x_\ell^{k_\ell}$ in (3.4). If $\overline{M}(k_1, \ldots, k_{\ell-1})$ is the coefficient of $x_1^{k_1} \cdots\cdots x_{\ell-1}^{k_{\ell-1}}$ in $f_{\ell-1}(x_1, \ldots\ldots, x_{\ell-1})$ then it is easily seen that

$$\overline{M}(k_1, \ldots, k_\ell) = \sum_{r_{\ell-1}=0}^{\infty} \cdots \sum_{r_2=0}^{\infty} \sum_{r_1=0}^{\infty} \overline{M}(k_1 - r_1,\ k_2 - r_1 - r_2; \ldots \ldots k_{\ell-1} - r_1 - r_2 - \cdots - r_{\ell-1}) \tag{3.5}$$

with

$$0 \le r_1 \le k_1$$
$$0 \le r_1 + r_2 \le k_2$$
$$\cdots$$
$$0 \le r_1 + r_2 + \cdots + r_{\ell-1} \le k_{\ell-1},$$

and

$$r_1 + r_2 + \cdots + r_\ell = k_\ell ,$$

so that

$$0 \leq r_1 + r_2 + \cdots + r_{\ell-1} \leq \min(k_\ell, k_{\ell-1})$$

Define a new set of variables

$$i_1 = r_1 , \quad i_2 = r_1 + r_2 , \quad \ldots \quad i_{\ell-1} = r_1 + r_2 + \cdots + r_{\ell-1} \tag{3.6}$$

then

$$\overline{M}(k_1, \ldots, k_\ell) = \sum_{i_{\ell-1}=i_{\ell-2}}^{\min(k_{\ell-1}, k_\ell)} \sum_{i_{\ell-2}=i_{\ell-3}}^{k_{\ell-2}} \cdots \sum_{i_2=i_1}^{k_2} \sum_{i_1=0}^{k_1} \overline{M}(k_1 - i_1 ; k_2 - i_2 ; \ldots , k_{\ell-1} - i_{\ell-1}) . \tag{3.7}$$

Eq.(3.7) is exactly the recursion relation we want since it facilitates the computation of the partition function for any A_ℓ (ℓ arbitrary) in terms of the simple partition function

A_2, viz.

$$\overline{M}(k_1, k_2) = \sum_{0}^{\min(k_1,k_2)} 1$$

$$= 1 + \min(k_1, k_2) \qquad (3.8)$$

which has been obtained earlier[9]. The weight space is again $(\ell + 1)$ dimensional with the condition on the components of a weight m,

$$\sum_{i=1}^{\ell+1} m_i = 0$$

Using Weyl's theorems, it can be proved that the components are (integer)/ $(\ell + 1)$. The Weyl group in this case permute the components of m and is of order $(\ell + 1)!$ The dominant weights satisfy

$$m_1 \geq m_2 \geq \cdots \geq m_{\ell+1},$$
$$\sum_{i=1}^{\ell+1} m_i = 0 \qquad (3.9)$$

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9) B.Gruber and T.S.Santhanam, Nuovo Cimento 45A, 1046 (1966):

These properties of the dominant weight will be used in picking up the non-vanishing contribution to $M^m(m')$.

4. $B_\ell (\sim O_{2\ell+1})$.

The roots of this algebra are $\pm(e_i \pm e_j)$, $\pm e_i$, $i=1,\ldots\ldots\ell$

There are $2\ell^2$ of them. The ℓ^2 positive roots may be obtained as $e_i - e_j$, $e_i + e_j$ and e_i $(i<j)$. The simple roots in this case ~~given~~ are given by $\beta_{i-1} = e_{i-1} - e_i$, $\beta_\ell = e_\ell$.

Equation (24) then takes the form

$$k_i = C_{i\mu}\, a_\mu ,\qquad \begin{array}{l} i=1,\ldots\ldots,\ell \\ \mu = 1,\ldots\ldots\ell^2 \end{array} \tag{4.1}$$

where C is the $(\ell^2 \times \ell)$ dimensional rectangular matrix

$$\begin{array}{cc} & \mu \longrightarrow 1,\ldots\ldots \ell^2 \\ C_{i\mu} = \begin{array}{c} \ell \\ \vdots \\ 1, \\ \uparrow \\ i \end{array} & \left(\begin{array}{c|cccc|cccc|ccc} & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & & 0 \\ & 1 & 1 & & 2 & 1 & 1 & & 1 & & & 0 \\ & 1 & 1 & & 2 & 1 & 1 & & 2 & & & 0 \\ C^{A_\ell} & \vdots & & & \vdots & \vdots & & & \vdots & & & \vdots \\ \xleftrightarrow{\quad} & & & & & & & & & & & \\ \tfrac{1}{2}\ell(\ell+1) & \vdots & & & \vdots & \vdots & & & \vdots & & & \vdots \\ & 1 & 1 & & 2 & 1 & 1 & & 2 & & & 0 \\ & 1 & 2 & & 2 & 1 & 2 & & 2 & & & 1 \\ & 2 & 2 & & 2 & 2 & 2 & & 2 & & & 2 \end{array} \right) \end{array} \tag{4.2}$$

The generating function in this case is

$$f_{\ell}^{B_\ell}(x_1,\ldots,x_\ell) = \prod_{i=1}^{\ell^2} \frac{1}{\left(1 - x_1^{c_{1i}} x_2^{c_{2i}} \ldots x_\ell^{c_{\ell i}}\right)} \qquad (4.3)$$

It can be easily checked that unlike the case of A_ℓ, there is no simple recursion relation between $f_\ell^{B_\ell}$ and $f_{\ell-1}^{B_{\ell-1}}$. However, the following very interesting relation can be obtained, which of course is obvious from the structure of the C-matrix Eq.(4.2)

$$f_{\ell}^{B_\ell}(x_1,\ldots,x_\ell) = \frac{f_\ell^{A_\ell}(x_1,\ldots,x_\ell)}{\prod_{i=2}^{\ell}\prod_{j=0}^{\ell-i}\left(1 - \prod_{k=i-1}^{\ell} x_k \prod_{r=\ell-j}^{\ell} x_r\right)} \qquad (4.4)$$

It is therefore clear that for large values of ℓ the recursion relation Eq.(4.4) is not simple. For $\ell = 2$, Eq.(4.4) read as

$$f_2^{B_2}(x_1,x_2) = \frac{f_2^{A_2}(x_1,x_2)}{\left(1 - x_1 x_2^2\right)} . \qquad (4.5)$$

So that the recursion relation for $\overline{M}$ is

$$\overline{M}^{B_2}(k_1,k_2) = \sum_i \overline{M}^{A_2}(k_1-i,\, k_2-2i) \tag{4.6}$$

which is the relation obtained by Gruber and Zaccaria earlier[10].

The weight space is ℓ dimensional and the components may be integers or half integers. The Weyl group in this case consists of all possible permutation of the components of m together with all possible changes of sign and is therefore of order $2^{\ell}\,\ell!$. The dominant weights satisfy

$$m_1 \geqslant m_2 \geqslant \cdots\cdots \geqslant m_\ell \geqslant 0 \tag{4.7}$$

5. C_ℓ ($\sim$ Sp(2ℓ)).

The roots of this algebra are $\pm(e_i \pm e_j)$, $\pm 2e_i$, $i=1,\ldots \ell$. It should be stressed that the factor 2 in the second class of roots is very important and makes this algebra different from B_ℓ. There are $2\ell^2$ roots. The ℓ^2 positive roots are given by $(e_i - e_j)$, $e_i + e_j$, $2e_i$, $i<j$. The simple roots in this

10) B.Gruber and F.Zaccaria, to appear in Suppl. Il Nuovo Cimento.

case are $\beta_{i-1} = e_{i-1} - e_i \; (i=1,\dots l)$, $\beta_l = 2e_l$. Eq.(2.4) is then

$$k_i = C_{i\mu}\, a_\mu , \qquad i = 1, \dots l, \quad \mu = 1, \dots l^2 \tag{5.1}$$

where C is the $l^2 \times l$ dimensional rectangular matrix

$$C_{i\mu} = \begin{array}{ccc} & & \mu \rightarrow 1, \dots l^2 \\ \begin{array}{c} l \\ \vdots \\ \vdots \\ 2 \\ \uparrow \\ i \end{array} & \begin{array}{c} C^{A_l} \\ \xleftrightarrow{\frac{1}{2} l(l+1)} \end{array} & \left(\begin{array}{ccc|ccc|cc} 1 & 1 \dots & 2 & 0 & 0 \dots & 0 & \dots & 0 \\ 1 & 1 & 2 & 1 & 1 & 2 & & 0 \\ 1 & 1 & 2 & 1 & 1 & 2 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 2 & 1 & 2 & 2 & & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & & 1 \end{array} \right) \end{array} \tag{5.2}$$

The generating function is of the same type of $f_l^{B_l}(x_1, \dots x_l)$ but the elements of C are different in view of Eq.(5.2). Again in this case, there is no simple recursion relation between $f_{\ell}^{C_{\ell}}$ and $f_{\ell-1}^{C_{\ell-1}}$. However, the following relation can be easily verified.

$$f_{\ell}^{C_{\ell}}(x_1,\ldots,x_{\ell}) = \frac{f_{\ell}^{A_{\ell}}(x_1,\ldots,x_{\ell})}{\prod_{i=1}^{\ell}\prod_{j=1}^{\ell-i}\left(1-\prod_{k=i}^{\ell}\prod_{r=\ell-j}^{\ell-1} x_r\right)} \tag{5.3}$$

For the special case of $\ell = 2$, the above relation reads as

$$f_2^{C_2}(x_1,x_2) = \frac{f_2^{A_2}(x_1,x_2)}{(1-x_1^2x_2)} \tag{5.4}$$

so that the relation (4.6) is derived with $k_1 \longleftrightarrow k_2$

$$\overline{M}^{C_2}(k_1,k_2) = \sum_i \overline{M}\,(k_1-2i;\,k_2-i) \tag{5.5}$$

This is not surprising because of the known isomorphism between C_2 and B_2.

The weight space is again ℓ -dimensional and the components of the weight one integers. The Weyl group is the same as that for B_{ℓ} and is of order $2^{\ell}\ell!$. This consists of all the permutations of the components of the weight and all changes in sign. The dominant weight satisfies

$$m_1 \geqslant m_2 \geqslant \cdots \geqslant m_\ell \geqslant 0 \tag{5.6}$$

6. D_ℓ ($\sim O(2\ell)$).

The roots are given by $\pm(e_i \pm e_j)$, $i,j = 1,\dots,\ell$ and there are $2(\ell^2-\ell)$ of them. The $\ell(\ell-1)$ positive roots are then $e_i + e_j$ and $e_i - e_j$ $i<j$. The simple roots are $\beta_{i-1} = e_{i-1} - e_i$, $i=1,\dots\ell$ and $\beta_\ell = e_{\ell-1} + e_\ell$ Eq.(2.4) is then

$$k_i = C_{i\mu}\, a_\mu \qquad i = 1,\dots,\ell \qquad \mu = 1,\dots\, \ell(\ell-1) \tag{6.1}$$

where C is the $\ell(\ell-1) \times \ell$ dimensional rectangular matrix

$$\mu \longrightarrow 1,\ \dots\ \ell(\ell-1)$$

$$C_{i\mu} = \begin{array}{c} \\ i \to 1,\dots,\ell \end{array} \left(\begin{array}{c|ccc|ccc|cc|cc} & 1 & 0 \cdots & 0 & 1 & 1 \cdots & 1 & 0 \cdots & 0 & & 0 \\ & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & & 0 \\ & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 2 & \cdots & 0 \\ C^{A'_\ell} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \xleftrightarrow[\frac{1}{2}\ell(\ell+1)-1]{} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 2 & & 1 \\ & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & & 2 \\ & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & & 1 \\ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & 1 \end{array} \right) \tag{6.2}$$

where $C^{A_\ell'}$ denotes the matrix C^{A_ℓ} with the column $(0,0,\ldots 0,1,1)$ missing. In this case also, there is the following recursion relation

$$f_\ell^{D_\ell}(x_1,\ldots,x_\ell) = \frac{f_\ell^{A_\ell}(x_1,\ldots x_\ell)\,[1-x_{\ell-1}x_\ell]}{\left\{\prod_{k=2}^{\ell-2}\left(1-\prod_{r=\ell-k}^{\ell-2} x_r x_\ell\right)\right\}\left\{\prod_{r=0}^{\ell-4}\prod_{k=0}^{\ell-r-4}\left(1-\prod_{s=r+1}^{\ell} x_s \prod_{t=\ell-k-2-r}^{\ell-2} x_t\right)\right\}} \tag{6.3}$$

For $\ell = 2$, the above relation gives

$$f_2^{D_2}(x_1,x_2) = f_2^{A_2}(x_1,x_2)\,[1-x_1x_2] = \frac{1}{(1-x_1)(1-x_2)} \tag{6.4}$$

and so $\overline{M}(k_1, k_2) = 1$ for all k_1, k_2. This of course is a known result. For $\ell = 3$, this yields

$$f_3^{D_3}(x_1,x_2,x_3) = \frac{f_3^{A_3}(x_1,x_2,x_3)\,[1-x_2x_3]}{(1-x_1x_3)} \, . \tag{6.5}$$

so that

$$\overline{M}(k_1, k_2, k_3) = \sum_{i=0}^{\min(k_1,k_3)} \left[\overline{M}^{A_3}(k_1-i;\, k_2;\, k_3-i) - \overline{M}^{A_3}(k_1-i,\, k_2-1,\, k_3-i-1) \right]$$

(6.6)

The weight space is l dimensional. The components of the weight must be integers of half-integers. The Weyl group in this case consists of all permutations of the components of the weight (corresponding to the reflection perpendicular to the roots $e_i - e_j$) and all changes of sign <u>in pairs</u> (corresponding to the reflection perpendicular to the roots $e_i + e_j$, and is of order $2^{l-1} l!$.

The condition for a weight to be dominant is

$$m_1 \geq m_2 \geq \dots \geq m_{l-1} \geq |m_l|$$

7. G_2.

The roots for this exceptional group are $\pm(e_i - e_j)$, $\pm e_i$, $i,j = 1,2,3$; $e_3 = -(e_1+e_2)$. The six positive roots are $(e_1 - e_2)$, $(e_1 - e_3)$, $(e_2 - e_3)$, e_1, e_2, $-e_3 = (e_1+e_2)$

The simple roots are $\beta_1 = e_1 - e_2$ and $\beta_2 = e_2$. Eq.(2.4) then becomes

$$k_i = c_{i\mu}\, a_\mu \qquad i = 1, 2, \quad \mu = 1, 2, \dots 6 \tag{7.1}$$

where the (6 x 2) rectangular matrix C is

$$c_{i\mu} = \begin{array}{c} \\ i \end{array} \begin{array}{c} \mu = 1, \dots, 6 \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 & 3 \end{pmatrix} \end{array} \tag{7.2}$$

The ~~general~~ generating function is then

$$f^{G_2}(x_1, x_2) = (1-x_1)^{-1}(1-x_2)^{-1}(1-x_1x_2)^{-1}(1-x_1x_2^2)^{-1}(1-x_1x_2^3)^{-1}(1-x_1^2x_2^3)^{-1} \tag{7.3}$$

and so one immediately sees the following relations

$$\begin{aligned} f^{G_2}(x_1, x_2) &= \frac{f_2^{A_2}(x_1, x_2)}{(1-x_1x_2^2)(1-x_1x_2^3)(1-x_1^2x_2^3)} \\ &= \frac{f_2^{B_2}(x_1, x_2)}{(1-x_1x_2^3)(1-x_1^2x_2^3)} \end{aligned} \tag{7.4}$$

It follows therefore[10),11)]

$$\overline{M}^{G_2}(k_1,k_2) = \sum_{i,j,k} \overline{M}\,(k_1-i-j-2k;\ k_2-2i-3j-3k) \qquad (7.5)$$

The above sum has been explicitly carried out in ref.[11)] for various inequalities of k_1 and k_2. From (7.4) it also follows that

$$\overline{M}^{G_2}(k_1,k_2) = \sum_{i,j} \overline{M}^{B_2}(k_1-i-2j;\ k_2-3i-3j) \qquad (7.6)$$

The weight space in this case is again three dimensional like A_2 with the component of a weight satisfying

$$m_1 + m_2 + m_3 = 0.$$

The components of the weights are integers. The Weyl group is of order 12 and consists of the six permutations of (m_1,m_2,m_3) corresponding to the reflection perpendicular to the roots (e_1-e_2), (e_2-e_3), (e_1-e_3) and six permutations with a total change in sign corresponding to the roots e_i. The dominant weight satisfies

10) B.Gruber and F.Zaccaria, to appear in Suppl. Il Nuovo Cimento.
11) D.Radhakrishnan and T.S.Santhanam, J.Math.Phys. 8, 2206 (1967).

$$m_1 \geq m_2 \geq m_3, \quad m_1 \geq 0, \; m_2 \leq 0, \; m_3 \leq 0. \tag{7.7}$$

8. External Multiplicity

In the case of rotation groups in three dimensions, an I.R's is characterised by the eigenvalue j of the single Casimir operator J^2, which is integral or half integral. One is then familiar with the C.G. series

$$D^{j_1} \otimes D^{j_2} = \sum_{j = j_1 + j_2}^{|j_1 - j_2|} \oplus D^{j} \tag{8.1}$$

where D^j denotes an I.R. with the highest weight j. If $j_1 > j_2$ (in which case we shall say that the representation D^{j_1} dominates D^{j_2}), the right hand side of (8.1) can be interpreted as those I.R's whose highest weights are obtained by adding to the highest weight of the dominat I.R. i.e. D^{j_1}, all the weights of the I.R. D^{j_2} (from j_2 to $-j_2$). This is the main con-~~s~~tent of Biedenharn's theorem[2]. The conditions for one I.R. to dominate another I.R. have been worked out[1]. The general idea follows from the two equivalent formulae for the character

$$\chi^{m}(\xi) = \sum_{m' \in D(m)} M^{m}(m') \exp i(m', \xi) \tag{8.2}$$

where the $\chi^{m}(\xi)$ is the character of an I.R. with the highest weight m and ξ are the group parameters. The other is Weyl's formula

$$\chi^{m}(\xi) = \frac{X(m+R_0)}{X(R_0)} \tag{8.3}$$

where

$$X(m+R_0) = \sum_{S \in W} \delta_S \exp i\left[S(m+R_0), \xi\right]$$

Suppose, we are interested in the product of I.R's $D(\Lambda_1)$ and $D(\Lambda_2)$ with Λ_1 and Λ_2 as their highest weights respectively. Then

$$\chi(\Lambda_1)\,\chi(\Lambda_2) = \frac{\sum_{S \in W} \delta_S \exp i\left[S(\Lambda_1+R_0), \xi\right]}{\sum_{S \in W} \delta_S \exp i\left[S R_0, \xi\right]}$$

$$\cdot \sum_{m' \in D(\Lambda_2)} M^{\Lambda_2}(m') \exp i(m', \xi) . \tag{8.4}$$

where we have used Eq.(8.2) for $\chi(\Lambda_2)$ and (8.3) for $\chi(\Lambda_1)$ Eq.(8.4) can now be regrouped to be written as

$$\chi(\Lambda_1)\,\chi(\Lambda_2) = \frac{\sum\limits_{\substack{S \in W \\ m' \in D(\Lambda_2)}} \delta_S\, M^{\Lambda_2}(m')\, \exp\, i\left[S(\Lambda_1 + m' + R_0), \xi\right]}{\sum\limits_{S \in W} \delta_S\, \exp\, i\left[S R_0, \xi\right]} \tag{8.5}$$

where we have used the property

$$S(P) + S(Q) = S(P+Q) \tag{8.6}$$

Eq.(8.5) can now be interpreted as follows. In the product $D(\Lambda_1) \times D(\Lambda_2)$ where $D(\Lambda_1)$ dominates $D(\Lambda_2)$, only these I.R's with the highest weight $\Lambda_1 + m'$ occur $m' \in D(\Lambda_2)$ in the reduction. These I.R's occur with the multiplicity $M^{\Lambda_2}(m')$ i.e. multiplicity of the weight m' in the I.R. with highest weight Λ_2. The condition of dominance of one I.R. over the other is needed to the make $(\Lambda_1 + m')$ dominant. These have been more general formulae of G.Racah and D.Speiser[3] which do not involve the condition that one I.R. dominates the other. For our purpose, Eq.(8.5) is quite sufficient. Thus, we realize that the external multiplicity is very closely related to the internal multiplicity structure.

9. Conclusion.

We have constructed generating function for the various classical groups. A_ℓ , B_ℓ , C_ℓ ; D_ℓ and G_2 . These are then used to set up recursion relations for the partition function which enter Kostant's formula for the ~~inner~~ multiplicity structure. The essential idea of the whole analysis is the realization that the number of solutions of the matrix equation $\kappa = Ca$ (for given κ and C) where the matrix C is in general a rectangular matrix with non-negative integer coefficients and the components of the vectors κ and a are again non-negative integers is given by the coefficient of $x_1^{k_1} \ldots x_\ell^{k_\ell}$ of the generating function. In many cases the explicit ~~x~~ evaluation of the number of solutions is not possible and so we have set up recursion relations. While in the case of A_ℓ , the recursion relation is between the partition functions of A_ℓ , and $A_{\ell-1}$, in the cases of B_ℓ , C_ℓ and D_ℓ the recursion relations for their partition functions are among these and of A_ℓ . For G(2), there are two recursion relations one with A_2 and the other with B_2. We have also discussed the connection between the internal and external multiplicity structures.

TABLE OF SIMPLE AND POSITIVE ROOTS OF CLASSICAL GROUPS

$A_\ell \sim SU(\ell+1)$:

System of roots : $(e_i - e_j)$, $i,j = 1,\dots \ell+1$

System of positive roots: $(e_i - e_j)$, $i < j$, $i,j = 1,\dots \ell+1$

System of simple roots: $(e_i - e_{i+1})$, $i = 1,\dots \ell$

$B_\ell \sim O(2\ell+1)$:

System of roots : $\left.\begin{matrix} \pm e_i \\ \pm (e_i \pm e_j) \end{matrix}\right\}$ $i,j = 1,\dots \ell$

System of positive roots: $\left.\begin{matrix} + e_i \\ + (e_i \pm e_j) \end{matrix}\right\}$ $i,j = 1,\dots \ell$

System of simple roots: $e_i - e_{i+1}$ $i,j = 1,\dots \ell-1$

and e_ℓ

$C_\ell \sim Sp(2\ell)$:

System of roots: $\left.\begin{matrix} \pm 2e_i \\ \pm (e_i \pm e_j) \end{matrix}\right\}$ $i,j = 1,\dots \ell$

System of positive roots: $\left.\begin{matrix} 2e_i \\ (e_i \pm e_j) \end{matrix}\right\}$ $i,j = 1,\dots \ell$

System of simple roots: $\left.\begin{matrix} e_i - e_{i+1} \\ 2e_\ell \end{matrix}\right\}$ $i,j = 1,\dots \ell - 1$

$D_\ell \sim O(2\ell)$:

System of roots: $\pm(e_i \pm e_j)$, k, i,j = 1,... ℓ

System of positive roots: $(e_i \pm e_j)$, i,j = 1,... ℓ

System of simple roots: $e_i - e_{i+1}$, i = 1,... ℓ-1

$e_{\ell-1} + e_\ell$

Exceptional Group G(2):

System of roots: $\pm e_i$, $e_i - e_j$ i, j = 1,2,3

System of positive roots: e_i, $e_i - e_j$ i,j = 1,2,3

System of simple roots: $(e_1 - e_2)$, e_2

The e_i's are unit vectors in ℓ or $\ell + 1$ dimensional vector space. We shall not bother to write the table for the other exceptional groups as we have not worked the inner multiplicity structure of these groups. The system of roots of these groups can be found in Dynkin's article.

APPENDIX 1

CONCEPTS OF ROOTS, SIMPLE ROOTS, WEIGHTS, DOMINANT WEIGHTS AND HIGHEST WEIGHT*

The Standard Form of a Semi-simple Lie Algebra.

Let g be a Lie algebra of dimension r . Consider the eigen value problem of the operator $A(X)$ defined by $A(X) = [A, X] = \rho X$. If the secular equation of the operator has r distinct roots, then we have r linearly independent eigen vectors which can be used as a basis for the vector space underlying g . If, however, the secular equation has degenerate roots, r linearly independent vectors may not exist. Hence, a coordinate system for g cannot be arrived at by the above mentioned method. But for semi-simple Lie algebras we have the following.

THEOREM (Cartan): For a semi-simple Lie algebra g if we choose A so that the secular equation of $A(X)$ has the maximum number of distinct roots (which we can), the only degenerate root is $\rho = 0$ and if ℓ is the multiplicity of the root, there exist corresponding to this root, ℓ linearly independent eigen-vectors any two of which commute.

The number ℓ is called rank of g .

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* T.S.Santhanam, 'Group Theory and Unitary Symmetry', MATSCIENCE REPORT 61, The Institute of Mathematical Sciences, Madras and references quoted there.

We shall choose as basis the ℓ linearly independent eigenvectors (say) $H_1, \ldots\ldots, H_\ell$ corresponding to the degenerate root $\rho = 0$ together with the $(r-\ell)$ linearly independent eigenvectors $E_\alpha, E_\beta \ldots\ldots$ corresponding to the distinct roots $\alpha, \beta, \ldots\ldots,$

The commutational relations for $H_1, \ldots\ldots, H_\ell$; $E_\alpha, E_\beta, \ldots$ can be obtained to be

$$\left[H_i, H_j\right] = 0 \tag{1}$$

$$\left[H_i, E_\alpha\right] = \alpha_i E_\alpha, \tag{2}$$

$$\left[E_\alpha, E_\beta\right] = N_{\alpha\beta} E_{\alpha+\beta} \text{ if } (\alpha+\beta) \text{ is not a vanishing root,} \tag{3}$$

$$\left[E_\alpha, E_{-\alpha}\right] = \alpha^i H_i. \tag{4}$$

The structure constants are then,

$$C^\tau_{ij} = 0, \quad C^\tau_{i\alpha} = \alpha_i \delta^\tau_\alpha, \quad C^{\alpha+\beta}_{\alpha\beta} = N_{\alpha\beta},$$

$$C^\tau_{\alpha\beta} = 0 \text{ if } \tau \neq \alpha + \beta.$$

Further,

$$\left[A, H_i\right] = 0 \tag{5}$$

$$\left[A, E_\alpha\right] = \alpha E_\alpha. \tag{6}$$

As A is an eigenvector of $[A, X] = \rho X$,

$$A = \lambda^i H_i . \tag{7}$$

From (6), (7) and (2), it follows that

$$\alpha = \lambda^i \alpha_i . \tag{8}$$

The Concept of Root:-

The form (8) is called a root of the semi-simple Lie algebra $\mathfrak{g}$. It can be thought of as a vector in a ℓ-dimensional vector space.

A root is said to be positive if its first non-vanishing component is positive (in an arbitrary basis). A root is called simple (sometimes the terminology primitive or elementary is also used in the literature) if it is a positive root and in addition cannot be decomposed into the sum of two positive roots.

THEOREM (1): For a simple group of rank ℓ there exist ℓ simple roots and they are all linearly independent (we shall call the set of simple roots the π-system).

(2) Any non-simple root can be expressed as a linear combination of the simple roots $\sum_{\alpha \in \pi} R_i \alpha_i$ where R_i are all positive or all negative integers.

(3) If α is a root, then $-\alpha$ is also a root for any simple group.

(4) If α and β are two roots then

$$\frac{2\,(\alpha\beta)}{(\alpha\alpha)} = \text{integer}$$

and $\beta - \frac{2(\alpha\beta)}{(\alpha\alpha)}\,\alpha$ is also a root. Here $(\alpha\beta)$ denotes their scalar product. If φ is the angle between α and β, then from Theorem (4) above follows that

$$\cos^2 \varphi = \frac{1}{4}\; m\, n,$$

and

$$\frac{\alpha^2}{\beta^2} = \frac{m}{n}.$$

Here m and ~~km~~ n are integers. This would mean that the angle φ can assume only certain values (implying thereby some kind of a quantization of the angle). In particular, this is true for the simple roots. The allowed angles are 90°, 120°, 135° and 150° and the ratio between their lengths become

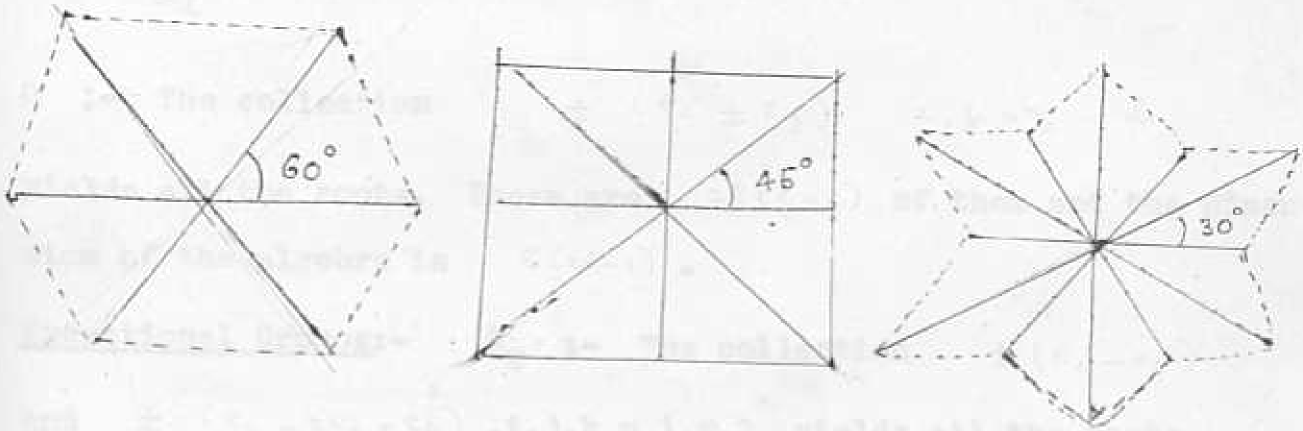
$$\frac{\alpha^2}{\beta^2} = \begin{array}{lll} 1 & \text{if} & \varphi = 120^\circ \\ 2 & \text{if} & \varphi = 135^\circ \\ 3 & \text{if} & \varphi = 150^\circ . \end{array}$$

If $\varphi = 90^\circ$, ~~x~~ then the ratio of lengths is undetermined.

Classical Groups:-

The realization of A_ℓ is the group of unitary, unimodular matrices in the complex space of $(\ell + 1)$ dimensions $Su(\ell+1)$. The realization of B_ℓ and D_ℓ are the real orthogonal groups in $(2\ell+1)$ and 2ℓ dimensions respectively. The realization of C_ℓ is the group of unitary matrices in complex 2ℓ dimensions satisfying the condition $U^T J U = J$ where J is a non-singular antisymmetric matrix. In other words, the realization of C_ℓ is the sympletic group in complex 2ℓ dimensions.

It should be kept in mind that not all the roots are simple. If the order of the group is N (denoting the total number of elements) ℓ of the elements commute among themselves (ℓ fold degeneracy). Out of the rest $(N-\ell)$ elements, each gives rise to a root vector. However, since both α and $-\alpha$ are roots, the distinct roots are only $\frac{N-\ell}{2}$ in number. Out of these ℓ, we have seen, are simple. Therefore, there are $\frac{N-3\ell}{2}$ non-simple roots. The entire root diagram could be constructed (the root diagram is two dimensional when $\ell = 2$ for example). The root diagrams for A_2, B_2, C_2 and G_2 are shown in the fig.



A_2 $N = 8$ B_2 $N = 10$ G_2 $N = 14$

In general the entire root diagram is obtained in the following way.

Classical Groups.

A_ℓ the collection of $\ell(\ell+1)$ differences $\{(e_i - e_j)\}$ $i,j = 1,\ldots \ell+1$ of $(\ell+1)$ unit vectors yields all the roots. The dimension of the algebra is $(\ell+1)^2 - 1$.

B_ℓ :- The roots are obtained from $\pm e_i$, $\pm(e_i \pm e_j)$ $i,j = 1,\ldots\ldots \ell$

The dimension of the algebra is $\ell(2\ell+1)$.

C_ℓ :- The collection $\pm 2e_i, \pm(e_i \pm e_j)$ $_{i,j=1,\dots \ell}$ yields the roots of C_ℓ.

D :- The collection $\pm(e_i \pm e_j)$ $i,j=1,\dots \ell$ yields all the roots. There are $2\ell(\ell-1)$ of them and the dimension of the algebra is $\ell(2\ell-1)$.

Exceptional Groups:- G_2 :- The collection $\pm(e_i - e_j)$ and $\pm(e_i - 2e_j + e_k)$,i,j,k = 1,2,3 yields all the roots. The order of the group is 14.

F_4 :- The diagram of B_4 with 16 more vectors $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ (Total 48 vectors and dimension is 52).

E_6 :- The diagram A_5, the vectors $\pm\sqrt{2}\, e_8$ and

$\frac{1}{2}(\pm e_1 \pm \dots \pm e_6)$, $\pm e_7/\sqrt{2}$

Constitute the root diagram of E_6. Here we take four positive and four negative in the first fraction. The total number of vectors are 72 and the dimension is 78.

E_7 :- The diagram A_7 and the vectors $\frac{1}{2}(\pm e_1 \pm \dots \pm e_8)$

Where ~~thm~~ we take four positive and four negative signs. This constitutes the root diagram of E_7. The number of vectors is 126 and the dimension is 133.

E_8 :- The diagram D_8 and the vectors $\frac{1}{2}(\pm e_1 \pm \dots \pm e_8)$ with each sign occurring an even number of times forms the root diagram of E_8. There are 240 vectors and the dimension of the algebra is 248.

Representation of Lie Group and Lie Algebras

Let G be a Lie group. If to each element of G, we can associate a linear operator $R(g)$ of a certain n dimensional vector space V such that if $g_1 \cdot g_2 = g_3 \in G$, then $R(g_1)\, R(g_2) = R(g_2)$ and the association $g \longrightarrow R(g)$ is further continuous, then R is a n-dimensional representation of G.

Let $\mathfrak{g}$ be the Lie algebra. If to each element ξ of $\mathfrak{g}$ we can associate an operator $A(\xi)$ acting on V such that

$$A(\xi + \eta) = A(\xi) + A(\eta)$$

$$A(C\xi) = C\, A(\xi)$$

$$A\left([\xi, \eta]\right) = [A(\xi), A(\eta)]$$

then A is said to be a n-dimensional representation of $\mathfrak{g}$.

THEOREM 1:- Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then any representation of G is a representation of $\mathfrak{g}$ and vice versa.

THEOREM 2: The commutation relation of the Lie algebra (hence that of the Lie group) is true for any representation. Two representation $A_1(\xi)$ and $A_2(\xi)$ are said to be equivalent, if there exists a nonsingular operator U such that

$$U\ A_1(\xi)\ U^{-1} = A_2(\xi)$$

for any ξ.

A representation $\xi \longrightarrow A(\xi)$ is <u>reducible</u>, if the operators $A(\xi)$ acting on the vector space V leave a proper sub-space of V invariant.

If a representation $A(\xi)$ is reducible, then, it could be brought, by equivalence, to the standard matrix form

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

A representation which could not be brought to this form by equivalence is called an <u>irreducible representation.</u>

A representation $\xi \longrightarrow A(\xi)$ is <u>decomposable</u> if the operators $A(\xi)$ leave two mutually orthogonal subspaces which together span the whole space V. If a representation A is decomposable then there is an equivalent representation in which A could be brought to the form $\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$.

THEOREM 3:- Every representation of a compact Lie group (see chavalley 'Lie Groups' for definition) is finite dimensional and is equivalent to a unitary representation. Thus $R(g)$ takes the form

$$R(g) = \exp\ i\ \epsilon^{\alpha} X_{\alpha}$$

where ϵ^{α}'s are real and X_{α} is hermitian.

THEOREM 4:- For a unitary group, if a representation is reducible, then it is fully reducible to the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

The concept of Weight:-

Consider a n-dimensional matrix representation of a semi-simple Lie algebra $\mathfrak{g}$. The representation is completely specified by r-matrices (r being the dimension of $\mathfrak{g}$) D_{ρ} $\rho = 1 \ldots r$ which satisfy the equation

$$\left[D_{\rho}, D_{\sigma} \right] = C^{\lambda}_{\rho\sigma} D_{\lambda}$$

where $C^{\lambda}_{\rho\sigma}$ are the structure constants of $\mathfrak{g}$. Let us express the representation with respect to the standard Cartan form. Let $H'_1, \dots H'_\ell, E'_\alpha, \dots E'_\gamma$ be the matrices in the representation corresponding to the basis $H_1, \dots H_\ell, E_\alpha, E_\gamma$ of $\mathfrak{g}$. Let ~~us~~ u be the simultaneous eigenvector of the diagonal matrices $H'_1, \dots H'_\ell$ so that

$$H'_i u = m_i u .$$

Then the l-components $(m_1, \dots m_\ell)$ can be thought of as the components of a l-dimensional vector m which is called the weight vector. It should be noticed that while the root vectors characterize the infinitesimal Lie group, the weight vectors characterize the representation.

THEOREM 1: Every representation has at least one weight (see Racah's Princeton notes for proof).

THEOREM 2: A vector u of weight m which is a linear combination of vector u_k of weights m_k, m_k = m for each k, must vanish. (The corresponding theorem in matrices is that the eigenvectors corresponding to two distinct eigenvalues of a hermitian matrix are orthogonal).

THEOREM 3: There exists at most n linearly independent weights corresponding to a representation.

THEOREM 4: If u is a vector of weight m, then E_α is an eigenvector with weight $(m + \alpha)$.

THEOREM 5: If a representation is irreducible, then all the H_i's (we drop the primes for convenience and these denote the matrix representation) can be simultaneously diagonalized.

THEOREM 6: If m is a weight and α is a root then

$$\frac{2\,(m\alpha)}{(m\alpha)} = \text{integer}$$

and $m - \frac{2(m\alpha)}{(\alpha\alpha)}\,\alpha$ is a weight. (Note: There is no theorem analogus to that of the roots that if m_1 and m_2 are weights, then

$$\frac{2\,(m_1 m_2)}{(m_1, m_1)} \quad \text{is an integer)}$$

THEOREM: The set of all weights is invariant under the Weyl group S of trans-formations generated by reflections with respect to the hyperplanes passing through the origin and perpendicular to the roots.

DEFINITIONS: A weight is said to be positive, if its first non-vanishing component (in an arbitrary basis) is positive. One weight is said to be higher than the other, if their difference is positive. Thus weights are equivalent if they are connected by a transformation belonging to S.

A weight higher than all its equivalents is said to be dominant. A weight is called simple if it belongs to only one eigenvector. The highest among the dominant weights is called the highest weight.

THEOREM: An irreducible representation is uniquely characterized by its highest weight which is simple.

THEOREM: Two irreducible representation are equivalent if their highest weights are equal.

THEOREM: For a semi-simple Lie algebra of rank ℓ, there are weights (called fundamental dominant weights) such that any dominant weight is a non-negative integral linear combination of them.

THEOREM: There are ℓ fundamental irreducible representations $A_1, \ldots\ldots A_\ell$, which have the fundamental weights as their highest weights. The dimension of the representation with highest weight Λ is given by

$$d = \prod_{\alpha \in \Sigma_+} 1 + \frac{(\Lambda\, \alpha)}{(g\alpha)}$$

where

$$g = \frac{1}{2} \sum_{\beta \in \Sigma_+} \beta$$

Σ_+ is the system of all positive roots.

<u>LEMMA</u>: Any weight m of the I.R. of the Lie group G with highest weight Λ can be written in the form

$$m = \Lambda - \sum_{\alpha} C_\alpha \, r(\alpha)$$

where the $r(\alpha)$ are the positive roots of G and the C_α are non-negative integers.

<u>Proof.</u> In any I.R, the highest weight state $|\Lambda\rangle$ in the only state ~~such~~ such that

$$E_\alpha \, |\Lambda\rangle = 0 \quad \text{for all positive } \alpha \, .$$

Hence given any state $|m\rangle$, either $m = \Lambda$ or else there is at least one E_α with positive α such that

$$E_\alpha \, |m\rangle = |m + r(\alpha)\rangle \, .$$

Similarly either $|m + r(\alpha)\rangle = |\Lambda\rangle$, or else there is at least one E_β with positive β such that

$$E_\beta \, E_\alpha |m\rangle = E_\beta \, |m + r(\alpha)\rangle = |m + r(\alpha) + r(\beta)\rangle \, .$$

If we proceed in this way, the fact that all I.R's are finite dimensional, implies that we eventually reach

$$m + r(\alpha) + r(\beta) + \dots + r(\gamma) \qquad = E_{\gamma} \dots E_{\beta} E_{\alpha} |m\rangle$$

such that

$$E_{\delta} | m + r(\alpha) + r(\beta) + \dots + r(\gamma) \rangle = 0$$

for all E_{δ} with positive δ. In this case, we have

$$\Lambda = m + r(\alpha) + r(\beta) + \dots + r(\gamma)$$

and since any given $r(\tau)$ can occur on the right C_{τ} times $C_{\tau} = 0,1,2,\dots$

$$m = \Lambda - \sum_{\alpha} C_{\alpha} \; r(\alpha).$$

This proves the lemma.

APPENDIX 2.

THE PROPERTIES OF SIMPLE ROOTS.

We have seen that the simple roots play a very important role in the discussion of multiplicity structures, we summarize here some of the interesting properties of the simple roots.

We recall the definition of simple roots. For a group of rank ℓ, there exists ℓ independent roots $\alpha_1, \ldots \alpha_\ell$ (constituting a basis in ℓ-dimensional vector space) such that any root ρ can be expressed as

$$\rho = \sum_{i=1}^{\ell} \lambda_i \alpha_i$$

where the coefficients λ_i are integers and either all $\lambda_i \geqslant 0$ or all $\lambda_i \leqslant 0$. The system of roots $(\alpha_1, \ldots \alpha_\ell)$ is called the simple roots, (or primitive roots or elementary roots) and we denote it by π. We can prove the following[1]:

(i) If $\alpha, \beta \in \pi$, $\alpha \neq \beta$, then $\alpha - \beta$ is not a root.
This follows from the definition

(ii) If α, $\beta \in \pi$, and $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$.

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1) N. Jacobson, Lie Algebra, Interscience Publishers 1962, Chapter IV, page 120.

(iii) The set π constitutes a basis for a vector space in ℓ -dimensions. If ρ is any psotive root then

$$\rho = \sum_{\alpha \in \pi} k_\alpha \alpha$$

where the k_α are non-negative integers.

(iv) If ρ is a positive root and $\rho \notin \pi$ then there ~~ask~~ exists an $\alpha \in \pi$ such that $\rho - \alpha$ is a positive root.

DEFINITION. If $\pi = (\alpha_1, \ldots \alpha_\ell)$ is a simple system of roots, the matrix

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

is called a Cartan matrix of the Lie algebra. The diagonal entries of the matrix are $A_{ii} = 2$ and off diagonal elements are negative because of condition (ii).

If $i \neq j$, the α_i and α_j are linearly independent so that if θ_{ij} is the angle between α_i and α_j then $0 \leq \cos^2\theta_{ij} < 1$ hence $0 \leq A_{ij} A_{ji} < 4$. This implies that either both A_{ij} and A_{ji} are zero or one is -1 while the other is -1, -2, or -3.

The determinant of the Cartan matrix is a non-zero multiple of that (α_i,α_j). Hence

$$\det\left[A_{ij}\right] \neq 0$$

If $\beta = \sum k_i\alpha_i$ is a root, then we define the level

$$|\beta| = \sum |k_i|$$

The level is a positive integer and the positive roots of level one are just the $\alpha_i \in \pi$. <u>The set of roots is determined by the simple system π and the Cartan matrix.</u> In other words, the sequences $(k_1, k_2, \cdots k_\ell)$ such that $\sum k_i\alpha_i$ are roots can be determined from the matrix $\left[A_{ij}\right]$. We shall just give an example. The Cartan matrix for G_2 is ~~an~~

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

that is,

$$\frac{2(\alpha_1,\alpha_2)}{(\alpha_1,\alpha_1)} = -1, \quad \frac{2(\alpha_1,\alpha_2)}{(\alpha_2,\alpha_2)} = -3.$$

Since $\alpha_1 - \alpha_2$ is not a root, these relations would imply that α_1 string containing α_2 and α_2 string containing α_1 are, respectively

$$\alpha_2 : \quad \alpha_2, \alpha_2 + \alpha_1$$

$$\alpha_1 : \quad \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2 + \alpha_1 + 3\alpha_2$$

$\alpha_2 + 2\alpha_1$ is not a root since

$$\frac{2(\alpha_2 + \alpha_1, \alpha_1)}{(\alpha_1, \alpha_1)} = -2 + 2 + 0$$

which means that the chain $\alpha_2 + \alpha_1$ must stop. On the otherhand,

$$\frac{2(\alpha_1 + \alpha_2, \alpha_2)}{\alpha_2, \alpha_2} = -3 + 2 = -1 < 0 \quad \text{thereby}$$

implying that

$$\alpha_1 + \alpha_2 + \alpha_2 = \alpha_1 + 2\alpha_2 \text{ is a root.}$$

Since

$$\frac{2(\alpha_1 + 2\alpha_2, \alpha_2)}{(\alpha_2, \alpha_2)} = -3 + 4 > 0$$

this implies that the α_1-chain must step at $\alpha_1 + 2\alpha_2$. Thus the

Only positive root of level there is $\alpha_1 + 2\alpha_2$. Since $2(\alpha_1+\alpha_2)$ is not root, only positive hot of level four is $\alpha_1 + 3\alpha_2$. On the other hand $2\alpha_1 + 3\alpha_2$ can be verified to be a root since

$$\frac{2\left[(2\alpha_1 + 3\alpha_2), \alpha_1\right]}{(\alpha_1, \alpha_1)} = 2 - 3 = -1 < 0.$$

There are no positive roots of higher levels. Hence the roots are,

$$\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + \alpha_2),$$
$$\pm(\alpha_1 + 3\alpha_2), \pm(2\alpha_1 + 3\alpha_2).$$

A simple induction on levels shows that any positive root β can be written as

$$\beta = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_k}$$

$\alpha_{i_j} \in \pi$ in a such a way that every p rticl~~e~~ sum

$$\alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_m} \quad m \leq k$$

is a root.

There exists unique isomorphism between two Lie algebras if their Cartan matrices are identical.

A simple system π is called <u>indecomposable</u> if it is impossible to partition π into non-vacuous non-overlapping sets π' and π'' such that $A_{ij} = 0$ for every $\alpha_i \in \pi'$, $\alpha_j \in \pi''$.

A Lie algebra is called simple if and only if the associated simple system π of roots is indecomposable.

<u>LEMMA</u>. Let $r(k)$

$$k = 1, \ldots\ldots, \ell$$

be the simple roots.

If the Weyl reflections corresponding to these simple roots we denote by S_k and if any weight W is such that

$$S_k W \leq W$$

then W is dominant. It is required to prove that

$$S_k W \leq W \text{ implies}$$

$$S_\alpha W \leq W \text{ for all } \alpha$$

$$S_k W = W - \frac{W.r(k)}{r(k).r(k)} r(k)$$

$$k = 1, \ldots \ell$$

$$S_k W \leq W$$

$$W.r(k) \geq 0$$

But $r(\alpha)$ (positive root) is expressible in terms of $r(k)$ with integer (non-negative) coefficients

$$W.r(\alpha) \geqslant 0$$

No reflection corresponding to any positive root can take W to anything higher $\therefore$ W is dominant.

APPENDIX 3.

Kostant's Formula

The simple proof we shall indicate is due to Cartier and to Steinberg (independently)*.

We first introduce the partition function $P(M)$ as the number of ways of writing M as a sum of ~~xmsstitk~~ positive roots, i.e., $P(M)$ is the number of solutions $(k_\alpha, k_\beta, \ldots, k_\rho)$ of $\sum_{\alpha>0} k_\alpha \alpha = M$ where the k_α are non-negative integers and $(\alpha, \beta, \ldots \rho)$ is the set of ~~negative~~ positive roots. From the definition, it follows that

$$P(0) = 1 \tag{1}$$

the only solution being $(0,0,\ldots,0)$ and since every positive root can be expressed in terms of simple roots with non-negative integer coefficients, we have

$$P(M) = 0 \tag{2}$$

unless $M = \sum m_i \lambda_i$ where the λ_i are the simple roots and m_i are non-negative integers. We consider the generating function

$$\sum_M P(M)\, e(M)$$

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*See N.Jacobson, Lie Algebras, Interscience Publishers 1962, Chapter VIII, page 260.

where

$$e(M) = x_1^{m_1} \cdots\cdots x_\ell^{m_\ell}$$

where $x_j = e(\lambda_j)$, it is clear that we have the identity

$$\sum_M P(M)\, e(M) = \sum P(m_1, \cdots m_\ell)\, x_1^{m_1} \cdots\cdots x_\ell^{m_\ell}$$

$$= \prod_{\alpha > 0} \left(1 + e(\alpha) + e(2\alpha) + \cdots\cdots\right)$$

$$= \prod_{\alpha > 0} \left(1 - e(\alpha)\right)^{-1}$$

(3)

Then it follows that

$$\left\{ \sum_M P(M)\, e(M) \right\} \prod_{\alpha > 0} \left(1 - e(\alpha)\right) = 1$$

Weyl's formula for character χ_Λ can be written as

$$\chi_\Lambda = \sum_{M \in D(\Lambda)} \gamma_M\, e(M) = \frac{\sum_{S \in W} \delta_S\, e\left[S(\Lambda + R_0)\right]}{\sum_{S \in W} \delta_S\, e(SR_0)} .$$

(4)

where S are the elements of the Weyl group and $R_o = \frac{1}{2} \sum_{\alpha > 0} \alpha$

γ_M is the inner multiplicity of the weight $M \in D(\Lambda)$. Hence

$$\sum_M \gamma_M \, e(-M) \cdot \sum_{S \in W} \delta_S \, e(-SR_o)$$

$$= \sum_{S \in W} \delta_S \, e\left[-S(\Lambda + R_o)\right] \tag{5}$$

Multiplying both sides by $e(R_o)$ we get

$$\sum_M \gamma_M \, e(-M) \sum_{S \in W} \delta_S \, e\,(R_o - SR_o)$$

$$= \sum_{S \in W} \delta_S \, e\left[R_o - S(\Lambda + R_o)\right] \tag{6}$$

It can however be proved that

$$\sum \delta_S \, e(SR_o) = e(R_o) \prod_{\alpha > 0} \left[1 - e(-\alpha)\right] \tag{7}$$

Multiplying both sides of Eq.(5) by $\sum P(M)\, e(M)$ we get

$$\sum \gamma_M\, e(-M) = \left(\sum \delta_S\, e\left[R_0 - S(\Lambda + R_0)\right]\right) \left(\sum P(M)\, e(M)\right)$$

$$= \sum_{S \in W} \delta_S\, P(M)\, e\left[M + R_0 - S(M + R_0)\right]$$

Comparing the coefficient of $e(-M)$ on both sides we get Kostant's formula for the inner multiplicity

$$\gamma_M^{\Lambda} = \sum_{S \in W} \delta_S\, P\left[S(\Lambda + R_0) - (M + R_0)\right]$$

APPENDIX 4.

EXTERNAL MULTIPLICITY

Expressions for external or outer multiplicity are obtained by the repeated use of the two character formulae, one which follows from the definition

$$\chi^{\Lambda}(m) = \sum_{m \in D(\Lambda)} \gamma_m \, e^{i(m,\phi)} \tag{1}$$

where the summation is over all the weights contained in the I.R. with highest weight Λ and the other is Weyl's formula

$$\chi^{\Lambda} = \frac{\sum \delta_S \exp i \left[S(\Lambda + R_o), \phi\right]}{\sum \delta_S \exp i \left[SR_o, \phi\right]} \tag{2}$$

where the summation is over all the elements of the Weyl group and R_o is half the sum of positive roots.

The product of two characters is given now given by*

$$\chi^{\Lambda} \chi^{\Lambda'} = \sum_{\Lambda''} \bar{\gamma}(\Lambda'') \chi^{\Lambda''} \tag{3}$$

*G. Racah, Group Theoretical Concepts and Methods in Elementary Particle Physics, (ed. F. Gursey), Gordan and Breach, N.Y. 1964.

where $\overline{\gamma}(\Lambda'')$ is the external multiplicity of the I.R. with highest weight Λ''.

Suppose we insert Eq.(1) for $\chi^{\Lambda'}$ and Eq.(2) for χ^{Λ} then Eq.(3) reads as

$$\frac{\sum_{S\in W} \delta_S \exp i\left[S(\Lambda+R_0),\phi\right]}{\sum_{S\in W} \delta_S \exp i\left[SR_0,\phi\right]} \sum_{m'\in D(\Lambda')} \gamma_{m'}\, e^{im'\phi}$$

$$= \frac{\sum_{m'} \gamma_{m'} \sum_{S\in W} \delta_S \exp i\left\{S(\Lambda+R_0)+m',\phi\right\}}{\sum_{S\in W} \delta_S \exp i\,(SR_0,\phi)}$$

$$= \sum_{\Lambda''} \overline{\gamma}(\Lambda'')\, \chi^{\Lambda''} \tag{4}$$

The l.h.s. of Eq.(4) looks like the character $\chi^{\Lambda+m'}$ provided $\Lambda+m'$ is a dominant weight i.e.

$$\overline{\gamma}(\Lambda'') = \gamma_{m'}$$

when

$$\Lambda + m' = \Lambda''$$

Eq.(4) is just Biedenharn's formula, and the condition for $\Lambda + m'$ to be dominant is that the I.R. $D(\Lambda)$ should dominate the I.R. $D(\Lambda')$. Then if $D(\Lambda)$ dominates over $D(\Lambda')$ then add to each weight of the I.R. $D(\Lambda')$ and the I.R. with highest weight $\Lambda + m'$, $m' \in D(\Lambda')$ just occurs $\gamma_{m'}$ times where $\gamma_{m'}$ is the inner multiplicity of the weight m' in the I.R. $D(\Lambda')$. The conditions for $D(\Lambda)$ to dominate $D(\Lambda')$ have been worked out for all classical groups*.

Suppose in Eq.(4), we use Weyl's formula for $\chi^{\Lambda''}$ also then we get

$$\sum_{m'} \gamma_{m'} \sum_{S \in W} \delta_S \exp i \left[S(\Lambda + R_0) + m', \phi \right]$$

$$= \sum_{\Lambda''} \bar{\gamma}(\Lambda'') \sum_{S'} \delta_{S'} \exp i \left[S'(\Lambda'' + R_0), \phi \right] \tag{5}$$

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*See for instance, A.J.Macfarlane, L.O'Raifeartaigh, and P.S.Rao, Jour. Math. Phys. 8, 536 (1967).

If we now insert Kostant's formula for $\gamma_{m'}$ in Eq.(5), we get

$$\sum_{m' \in D(\Lambda')} \sum_{s'' \in W} \delta_{s''} \; P \left[s\,(\Lambda' + R_0) - (m' + R_0) \right]$$

$$\sum_{s \in W} \delta_s \; \exp\, i \left[s(\Lambda + R_0) + m',\ \phi \right]$$

$$= \sum_{\Lambda''} \bar{\gamma}(\Lambda'') \sum_{s'} \delta_{s'} \exp\, i \left[s'(\Lambda'' + R_0), \phi \right]$$

which can now be rewritten as

$$\sum_{m' \in D(\Lambda')} \sum_{s, s'} \left(\delta_s \, \delta_{s''} \right) \; P \left[s \;\; (\Lambda' + R_0) - (m' + R_0) \right]$$

$$\exp\, i \left[s\,(\Lambda + R_0) + m',\ \phi \right]$$

$$= \sum_{\Lambda''} \bar{\gamma}(\Lambda'') \sum_{s'} \delta_{s'} \exp\, i \left[s'(\Lambda'' + R_0), \phi \right] \qquad (6)$$

Multiplly both sides by $\exp -i.(\Lambda''+R_0)$ and integrate over ϕ, then one gets using the orthogonality of the exponentials

$$\sum_{m' \in D(\Lambda')} \sum_{S,S'} \left(\delta_S \delta_{S'}\right) P \left[S(\Lambda'+R_0) - (m'+R_0)\right] \delta_{S(\Lambda+R_0)+m',\ \Lambda''+R_0}$$

$$= \sum_{\Lambda''} \bar{\gamma}(\Lambda'') \sum_{S'} \delta_{S'} \delta_{S'(\Lambda''+R_0),\ \Lambda''+R_0}$$

(7)

which after removing the summation over Λ'' on the r.h.s. reads as

$$\sum_{S,S''} \left(\delta_S \delta_{S''}\right) P \left[S(\Lambda'+R_0) + S(\Lambda+R_0) - (\Lambda''+2R_0)\right]$$

$$= \sum_{S'} \delta_{S'} \bar{\gamma} \left[S'(\Lambda''+R_0) - R_0\right] .$$

(8)

If Λ'' is dominant, $s'(\Lambda''+R_0)$ and $s'(\Lambda''+R_0)-R_0$ are dominant if and only if $s'=I$ and hence one gets, when Λ'' is dominant which it should be

$$\bar{\gamma}(\Lambda'') = \sum_{s,\,s''} \delta_s \, \delta_{s''} \, P\Big[s\,(\Lambda'+R_0) + s(\Lambda+R_0) - (\Lambda''+2R_0)\Big] \tag{9}$$

Eq.(9) is Steinberg's formula for outer multiplicity. This is a very useful formula.

If on the otherhand, one uses for all the characters in Eq.(3) Weyl's formula then we get

$$\sum_{s,s'\in W} (\delta_s \, \delta_{s'}) \exp i \left\{ s\,(\Lambda+R_0) + s'(\Lambda'+R_0), \phi \right\}$$

$$= \sum_{\Lambda''} \bar{\gamma}(\Lambda'') \sum_{s'',\,s\in W} (\delta_{s''} \, \delta_s) \exp i \left\{ s''(\Lambda''+R_0) + sR_0, \phi \right\} \tag{10}$$

Multiplying both sides by $\exp -i \left\{ (\bar{\Lambda} + 2R_0), \phi \right\}$

and integrating over ϕ , we get

$$\sum_{s, s' \in W} \delta_s \, \delta_{s'} \, \delta_{s(\Lambda + R_0) + s'(\Lambda' + R_0) \, , \, \bar{\Lambda} + 2R_0}$$

$$= \sum_{\Lambda''} \bar{\gamma}(\Lambda'') \sum_{s'', s \in W} (\delta_{s''} \, \delta_s) \; \delta_{s''(\Lambda'' + R_0) + sR_0 \, , \, \bar{\Lambda} + 2R_0} \, .$$

(11)

To remove the summation over Λ'' on the r.h.s. we set

$$s''(\Lambda'' + R_0) + sR_0 = \bar{\Lambda} + 2R_0$$

We have to solve for Λ'' in this equation. Multiplying throughout from the left by s'' , we get

$$(\Lambda'' + R_0) + s''s \, R_0 = s''(\bar{\Lambda} + 2R_0) \tag{12}$$

where we have used the fact that $(S'')^2 = I$. Then

$$\Lambda'' = S''\,(\bar{\Lambda} + 2R_0) - S''S R_0 - R_0$$

Eq.(11) becomes

$$\sum_{S'',S \in W} \bar{\gamma}\left[S''\,(\bar{\Lambda} + 2R_0) - S''S R_0 - R_0 \right] \delta_{S''}\,\delta_S$$

$$= \sum_{S,S' \in W} \delta_S\,\delta_{S'}\,\delta_{S(\Lambda + R_0) + S'(\Lambda' + R_0),\, \bar{\Lambda} + 2R_0}$$

(13)

The product $S''S$ is ggain another element of W, the argument of $\bar{\gamma}$ becomes

$$S''\,(\bar{\Lambda} + 2R_0) - S'''R_0 - R_0$$

$$= S''\,(\bar{\Lambda} + R_0) - R_0 \quad \text{since } S'' \text{ is summed}$$

and since $\bar{\Lambda}$ is dominant $S'' = I$ and thus Eq.(13) becomes

$$\bar{\tau}(\Lambda'') = \sum_{s \in W} \delta_s \, \delta_{(\Lambda + R_0) + s(\Lambda' + R_0),\, \Lambda'' + 2R_0} \tag{14}$$

When $\Lambda'' = \Lambda + \Lambda'$ then it follows immediately that

$$\bar{\tau}(\Lambda + \Lambda') = 1$$

which is of course a known result. Eq.(14) has been derived in a slightly differeent fashion by A.U. Klymyk, Soviet. Math. Dokl. Vol. 8, (1967), No.6 p.1531.

P A R T II.

SELF-CONSISTENT MODELS AND THE ORIGIN OF UNITARY SYMMETRY.

CHAPTER VI

ORIGIN OF UNITARY SYMMETRY AND CHARGE CONSERVATION IN STRONG INTERACTIONS

ABSTRACT

We discuss the relation of the existence of multiplets of strongly interacting particles and the possible unitary symmetry of their interactions. We present here a dynamical principle that concerns the one - particle propagators (two point functions) but yielding the existance of a (unitary) symmetry group for their trilinear interactions. We derive, as a by-product, electric charge (and hypercharge) conservation in the interaction of these particles.

ORIGIN OF UNITARY SYMMETRY AND CHARGE CONSERVATION IN STRONG INTERACTIONS*

Introduction:

One of the most remarkable features of elementary particles is their multiplet structure. The simplest such structure is the particle-antiparticle pairing with equal mass, spin, life-time etc. but with opposite electric charge, baryon number (and hypercharge) etc. We relate this regularity to the TOP invariance of the theory. In this sense, we may say that we understand the origin of particle-antiparticle symmetry.

Among the strongly interacting particles we find multiplets of particles having the same spin and, parity, but with slightly unequal masses. It is conventional to identify such a multiplet structure with the existence of an internal symmetry group, the multiplets constituting the various irreducible representations is now well established. This implies that the p-p, p-n and n-n forces are equal so that the strong interaction is invariant under rotations in the isotopic spin space. The slight difference between p-p and p-n forces is attributed to the weak electromagnetic interaction (relative to the strong

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*E.C.G.Sudarshan, L.O'Raifeartaigh and T.S.Santhanam, Phys.Rev. 136 B, 1092 (1964).

interaction strength) between the protons, although it is by no means true that this is the only possible mechanism of violation of charge independence.

It is now well established that there are regularities in the particle (resonance) spectrum which go beyond charge independence, in the sense that the multiplets can be further ~~spim~~ grouped together to constitute supermultiplets with the same spin, parity, baryon number and comparable masses which constitute irreducible representations of the special unitary group in three dimensions[1]. In this case, the departures from symmetry are not yet well understood. They are ascribed to a "small" part of the strong interaction themselves.

All along, the symmetry group was given to start with. Particles and resonances were accommodated in the various irreducible representations of the symmetry group. The missing components were looked for as particles or resonances in various strong interaction processes. The calculations have been carried out assuming the perturbations to be small and therefore neglected. But as to which multiplets occur, or as to the identification of particles with irreducible representations, the theory is silent. The Sakata model described the physical particles p, n and Λ to

1) See, for example, "The Eight-fold way" (Eds) M.Gell-Mann and Y.Ne'eman, (eds) Way", W.A.Benjamin, Inc., New York (1964).

belong to the "fundamental representation" of U(3). However, it did not yield the correct multiplicity structure to the other particles. The Gell-Mann-Ne'eman version of SU(3) started with the eight dimensional representation of SU(3) directly. There are at least two short-comings to this point of view; first, it does not tell which of the "smaller" representations actually occur; second, one has to coin reasons why certain representations do not make their presence. In the literature, such questions have been raised and to an extent explained[2].

There is a different line of development which makes such a connection more desirable[3]. In a dynamical scheme, when the particles or resonances appear in the direct channel of a two particle scattering process as a result of the exchange of these and other particles and resonances in the cross-channels, there are certain self-consistency conditions imposed on the number of particles and their coupling strengths and the multiplets that can be exchanged to give an attractive force are not arbitrary.

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(2) M.Gell-Mann, Physics **1**, 63 (1964)

(3) See for an exhaustive discussion E.C.G.Sudarshan, paper presented at the symposium 'Symmetry in Particle Physics', Chicago Meeting of the American Physical Society, November 1964. Syracuse University preprint 1206-SU-07 NYO-3399-07.

E.C.G.Sudarshan, Lectures on 'Origin of Symmetries', Matscience Report 33, (Madras).
Also see, E.C.G.Sudarshan, Symposia on Theoretical Physics, Edited by Alladi Ramakrishnan, Plenum Press, (New York) 1966.

R.E.Cutkosky, Brandeis Lectures (1965).

There is then the possibility of looking for the dynamical origin of symmetries, starting from the existence of (mass-spin-parity degenerate) multiplets of interacting particles and requiring self-consistency.

We might now ask ourselves the following question: Suppose we do not assume the existence of a symmetry group *a priori*, but we assert that not only are the masses and spins of the various members of a multiplet equal, but also the total squared transition matrix elements into members of other multiplets. Then the propagators of each of the particles belonging to multiplet are the same. Does this imply invariance of the interactions between the particles under a suitable continuous symmetry group ?

We shall see below that within a suitable dynamical framework, this question can be answered and the answer is "Yes". In view of the fundamental role played in this framework by the postulated equality of the propagators for members of a particle multiplet, we propose to elevate this postulate to the status of a dynamical principle, to be called the Smushkevich Principle. We can formulate it more precisely as follows: If the members of a boson multiplet have the associated fields $\phi^{\alpha}(x)$, Smushkevich Principle asserts

$$\langle 0 | T \left(\phi^{\alpha}(x) \, \phi^{\dagger \beta}(y) \right) | 0 \rangle = \delta^{\alpha\beta} \, \Delta_F^R (x-y) \tag{1}$$

Similarly, if the members of a fermion multiplet have the associated fields $\psi^{\alpha}(x)$, then,

$$\langle 0 | T\left(\psi^{\alpha}(x), \bar{\psi}^{\beta}(y) \right) | 0 \rangle = \delta^{\alpha\beta} S_F^R(x-y) \tag{2}$$

Because of the well-known relations connecting the spectral function of these two-point functions with the mass renormalization constant and with the physical mass, it follows that the masses and the self-masses of the various members of a multiplet are equal. A more useful and (possibly) equivalent statement of the Smushkevich Principle is the following:

"Topologically identical self-energy diagrams should give equal contributions to the propagators of components fields of a multiplet."

In discussing the conservation laws for strong interactions, we encounter two kinds of additive quantum numbers. An additive quantum number of the first kind has the same value for each member of an irreducible multiplet; each multiplet is associated with a fixed value for each of these quantum numbers. The most relevant example is the baryon number. On the other hand, an additive quantum number of the second kind, like electric charge or hypercharge, has different values for different members of a multiplet. We shall see below that we can derive the conservation law for additive quantum numbers of the second kind within our dynamical framework.

It is to be noted that for both boson and fermion multiplets, we may make an arbitrary unitary transformation of the particles belonging to a multiplet. This is tantamount to a redefinition of the "particles" constituting the multiplet. Under such a transformation, the additive quantum numbers of the first kind are unaltered; and the Smushkevich equations are unaltered, which is as it should be. Explicit use is made of this circumstance in the sequel.

In the following section, we illustrate the general method by considering the pion-nucleon system. Here, as well as in the general case, we shall assume a trilinear interaction involving two multiplets with n members each, and one multiplet with n^2-1 members. The pion-nucleon system corresponds to the choice $n = 2$, and we then deduce the invariance of the interaction under SU_2. In the following section we generalize this proof to deduce invariance under SU_n. The paper concludes with some comments on the primitive entries in the eight fold way realization of the SU_3 symmetry of strong interactions, and on the connection of the Smushkevich principle with the Smushkevich method in strong interaction physics[4].

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4) I.M.Smushkevich, Doklady Acad. Nauk SSSR, 103, 205 (1955). See R.E.Marshak and E.C.G.Sudarshan, Introduction to Elementary Particle Physics (Interscience Publishers, Inc., New York, 1961).

II. CHARGE INDEPENDENCE OF STRONG INTERACTIONS:-

In this section we wish to derive the charge independence (SU_2 invariance) of the pion-nucleon interaction (of the Yukawa type) from the Smushkevich principle <u>without assuming charge conservation.</u> We write the trilinear interaction in the form[5] (suppressing gamma matrices):

$$H_{int} = f^{\alpha}_{rs} \, N^{\dagger}_{r} \, N_{s} \, \pi^{\alpha} \, , \tag{3}$$

where summation over the repeated indices r, s, α is implied; r and s take on two values and α takes on three values. No generality is lost by taking the pion field to be Hermitian. Hermiticity of the Hamiltonian (3)

Fig.1. Pion diagrams.

then requires

$$\left(f^{\alpha}_{rs} \right)^{*} = f^{\alpha}_{sr} \tag{4}$$

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5) H.Yukawa, Proc.Phys.Math.Soc., Japan 17, 48 (1935).

We can now introduce a great deal of simplification in the formalism by considering the quantities $f^{\alpha}_{\gamma\delta}$ as a set of $n \times n$ matrices f^{α}. By Eq.(4), these matrices are Hermitian. Then, for the meson propagators computed in perturbation theory, the Smushkevich principle yields a series of relations of the form:

$$Sp\left(f^{\alpha} f^{\beta}\right) = A_1 \delta^{\alpha\beta} \tag{5a}$$

$$Sp\left(f^{\alpha} f^{\gamma} f^{\beta} f^{\gamma}\right) = A_2 \delta^{\alpha\beta} \tag{5b}$$

$$Sp\left(f^{\alpha} f^{\gamma} f^{\delta} f^{\beta} f^{\gamma} f^{\delta}\right) = A_3 \delta^{\alpha\beta} \tag{5c}$$

etc

As before, the summation over repeated indices is understood. These terms correspond to the propagator contributions from the diagrams indicated in Fig.1. Similarly, by considering the nucleon propagators, we obtain relations of the type:

$$f^{\alpha} f^{\alpha} = B_1 I \tag{6a}$$

$$f^{\alpha} f^{\beta} f^{\alpha} f^{\beta} = B_2 I \tag{6b}$$

$$f^{\alpha} f^{\beta} f^{\gamma} f^{\alpha} f^{\beta} f^{\gamma} = B_3 I \tag{6c}$$

etc

(where I is the $n\times n$ unit matrix), corresponding to the propagator contributions from the diagram shown in Fig.2.

Before embarking on the solution of these equations, we note that in any case the f^α will be undetermined up to the following two types of transformations:

(i) A unitary transformation

$$f^\alpha \longrightarrow f'^\alpha = U f^\alpha U^{-1}, \tag{7}$$

in the space of the N_r.

(ii) A real unitary (orthogonal) transformation

$$f^\alpha \longrightarrow f''^\alpha = V^{\alpha\beta} f^\beta \tag{8}$$

in the space of the π^α. In each case the corresponding linear transformations on the boson and fermion fields preserve Eqs. (1) and (2), as discussed in the introduction, as well as Eqs. (5) and (6).

Our aim will be to combine this freedom with the Smushkevich equations (5) and (6) to deduce that the f^α are proportional to the isotopic spin matrices τ^α.

We begin by using the transformation (8) to make

$$Sp(f^2) = Sp(f^3) = 0$$

and the transformation (7) to diagonalize the traceless

Fig. 2. Nucleon diagrams.

Hermitian matrix f^3 in the form

$$f^3 = g\,\tau^3 \tag{9}$$

At this point we make our first use of the Smushkevich equation (5) and the tracelessness of f^2 to obtain

$$f^2 = g_1 \tau^1 + g_2 \tau^2 , \qquad g_1^2 + g_2^2 = g^2$$

By suitable transformation of the kind (7), we can retain Eq.(9), but cast f^2 in the form

$$f^2 = g\,\tau^2 \tag{10}$$

A further use of (5a), together with (9) and (10), gives

$$f^1 = g_1' \tau^1 + g_0' I \; ; \quad g_1'^2 + g_0'^2 = g^2$$

Here the term containing the unit matrix I appears because f^1 is not necessarily traceless. We may now use Eq.(6) to deduce that

$$g_0 g_1' = 0$$

so that either g_0 or g_1' must vanish. Use of the Smushkevich equation (5b) eliminates the possibly that g_1' can vanish, so that we have

$$f^1 = \pm g \tau^1 \tag{11}$$

If we now consider the real orthogonal transformation

$$\pi^1 \longrightarrow \pm \pi^1, \quad \pi^2 \longrightarrow \pi^2, \quad \pi^3 \longrightarrow \pi^3$$

we finally obtain

$$f^\alpha = g \tau^\alpha$$

as required; i.e., the interaction Hamiltonian now assumes the familiar charge-independent form[6]:

$$H_{int} = g N^+ \tau^\alpha \phi^\alpha N \tag{12}$$

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6) H.Frohlich, W.Heitler and N.Kemmer, Proc.Roy.Soc.(London) 166, 154 (1938); N.Kemmer, Proc.Cambridge Phil.Soc. 34, 354 (1938).

We have thus established the charge independence of the pion-nucleon interaction[7]. It is important to note that we have not assumed charge conservation in this derivation. We may now deduce the conservation electric charge if it is defined as a linear sum of the 'third' component of isotopic spin and half the baryon number.

We might now ask whether the strange-particle interactions are also charge-independent. Clearly, the cascade hyperon-pion system behaves in just the same way. The nucleon-kaon-Σ-hyperon system behaves in essentially the same way, except that the triplet of Σ fields may not be taken Hermitian. But what about Σ-hyperoon-pion system for which all indices take on three values? It turns out that for this system the method fails, since a coupling scheme satisfying the Smushkevich equations (5) and (6) can be devised, which violates charge independence. For the nucleon-kaon-Σ-hyperon system, the preceding analysis does not apply directly, but it can be adapted to deduce charge independence (see Section III below).

We are then led to suggest that in a theory where charge independence is the highest symmetry of strong interactions, only

7) M.Grisaru has shown that it is possible to derive charge independence for the $NN\pi\pi$ coupling using the Smushkevich principle. We thank Professor Grisaru for communicating this result to us prior to publication.

the nucleon-pion, cascade hyperon-pion, nucleon-kaon-Σ-hyperon and cascade hyperon-kaon-Σ-hyperon trilinear couplings are fundamental, the other couplings being induced effects. It is interesting to note that the singlet Λ hyperon does not enter any of these reactions. Of course, if charge independence is a consequence of a larger symmetry group, these restrictions do not apply; they are replaced by other conditions.

In concluding this section, we point out that once charge independence is deduced, all the equations (5) and (6) are automatically satisfied.

III. UNITARY SYMMETRY:

Consider the derivation of SU_n invariance for a system consisting of two multiplets E and F containing n particles, each coupled trilinearly to a multiplet ϕ containing n^2-1 particles. We may write the effective interaction in the form

$$H_{int} = C^{\alpha}_{rs} E^{\dagger}_{r} F_s \phi^{\alpha} + \left(C^{\alpha}_{sr}\right)^{*} F^{\dagger}_{r} E_S \phi^{\dagger\alpha} \tag{13}$$

Once again, summation over repeated indices is implied, and we regard C^{α}_{rs} as elements of matrices C^{α}. Note, however, that the matrices C^{α} are in general not Hermitian, since E and F are distinct. If the interaction is invariant under SU_n, it could be cast in the form

$$H_{int} = g\, X^{\alpha}_{rs} \left\{ E_r^{+} F_s \phi^{\alpha} + F_r^{+} E_s \phi^{+\alpha} \right\}$$

where X^{α} are the (normalized) Hermitian generators of SU_n. Without loss of generality, we may normalize X^{α} by the relation

$$sp\,(X^{\alpha} X^{\beta}) = n\, \delta^{\alpha\beta} \tag{14}$$

In the case $E \equiv F$, the Smushkevich equations satisfied by the C^{α} may be written in the form

$$sp\,(C^{\alpha} C^{\beta}) = A_1 \delta^{\alpha\beta} = n\, G^2 \delta^{\alpha\beta} , \tag{15a}$$

$$sp\,(C^{\alpha} C^{\gamma} C^{\beta} C^{\gamma}) = A_2\, \delta^{\alpha\beta} \tag{15b}$$

$$sp\,(C^{\alpha} C^{\gamma} C^{\delta} C^{\beta} C^{\gamma} C^{\delta}) = A_3\, \delta^{\alpha\beta} \tag{15c}$$

etc

and

$$C^{\alpha} C^{\alpha} = B_1 I = (n^2-1) G^2 I , \tag{16a}$$

$$C^{\alpha} C^{\beta} C^{\alpha} C^{\beta} = B_2 I \tag{16b}$$

$$C^{\alpha} C^{\beta} C^{\gamma} C^{\alpha} C^{\beta} C^{\gamma} = B_3 I , \quad \text{etc} \tag{16c}$$

As in the pion-nucleon case, we have the possibility of making the transformations

$$C^{\alpha} \rightarrow C'^{\alpha} = U C^{\alpha} U^{-1} , \tag{17}$$

$$C^{\alpha} \rightarrow C''^{\alpha} = V^{\alpha\beta} C^{\beta} , \tag{18}$$

where U and V are unitary $(n \times n)$ and $(n^2-1) \times (n^2-1)$ dimensional matrices, respectively. Our aim is now to use Eqs. (15), (16), (17) and (18) to show that

$$C^{\alpha} = G X^{\alpha}$$

We can make a transformation of the type (18) to make the traces of all C^α vanish, except (possibly) that of C^1. The coupling matrices now take the form

$$C^\alpha = a^{\alpha\mu} X^\mu + \left(\frac{t}{n}\right) \delta^{\alpha 1} I , \tag{19}$$

where t is the trace of C^1.

Substituting (19) in (15a), and taking account of the tracelessness of X^μ, we get

$$n G^2 \delta^{\alpha\beta} = n a^{\alpha\mu} (a^{\beta\nu}) \delta^{\mu\nu} + \left(\frac{t^2}{n}\right) \delta^{\alpha\beta} \delta^{\alpha 1} ,$$

so that

$$a^{\alpha\mu}(a^{\beta\mu}) = \left[G^2 - \left(\frac{t}{n}\right)^2 \delta^{\alpha 1} \right] \delta^{\alpha\beta}$$

Let us define

$$b^{\alpha\mu} = \left\{ G^2 - \left(\frac{t}{n}\right)^2 \delta^{\alpha 1} \right\}^{-\frac{1}{2}} a^{\alpha\mu} , \tag{20}$$

so that

$$b^{\alpha\mu} b^{\beta\mu} = \delta^{\alpha\beta} , \tag{20}$$

and

$$C^{\alpha} = \left\{ G^2 - \left(\frac{t}{n}\right)^2 \delta^{\alpha 1} \right\}^{\frac{1}{2}} Y^{\alpha} + \frac{t}{n} \delta^{\alpha 1} I , \tag{21}$$

with

$$Y^{\alpha} = b^{\alpha\mu} X^{\mu} \tag{22}$$

The Y^{α} so defined satisfy Eqs. (15) and (16) by virtue of the properties of X^{α}. We have, in particular,

$$Sp\,(Y^{\alpha} Y^{\beta}) = n\,\delta^{\alpha\beta}$$

$$Sp\,(Y^{\alpha} Y^{\gamma} Y^{\beta} Y^{\gamma}) = n'\,\delta^{\alpha\beta}$$

$$Y^{\gamma} Y^{\gamma} = (n^2 - 1)\, I$$

From these equations we can show that

$$Sp\left([Y^{\alpha}, Y^{\gamma}]\,[Y^{\beta}, Y^{\gamma}] \right) = -k^2 \delta^{\alpha\beta}$$

where k^2 is a non-negative constant. Putting successively

$\alpha = \beta = 1$ and $\alpha = \beta = 2$, we get

$$\sum_{\gamma=3}^{n^2-1} sp\left([r^1, r^\gamma]\,[r^1, r^\gamma]\right) = \sum_{\gamma=3}^{n^2-1} sp\left([r^2, r^\gamma]\,[r^2, r^\gamma]\right) \tag{23}$$

On the other hand, from Eqs. (15) and (16), we can deduce

$$\sum_{\gamma=3}^{n^2-1} sp\left([c^1, c^\gamma]\,[c^1, c^\gamma]\right) = \sum_{\gamma=3}^{n^2-1} sp\left([c^2, c^\gamma]\,[c^2, c^\gamma]\right) .$$

Since C^1 enters only in the commutators on the left-hand side of this equation, the multiplet of the unit matrix in (21) does not contribute. We can thus rewrite the above equation in the form

$$G^2 \left\{ G^2 - \left(\frac{t}{n}\right)^2 \right\} \sum_{\gamma=3}^{n^2-1} Sp \left([Y^1, Y^\gamma][Y^1, Y^\gamma] \right)$$

$$= G^4 \sum_{\gamma=3}^{n^2-1} Sp \left([Y^2, Y^\gamma][Y^2, Y^\gamma] \right) . \tag{24}$$

Now, the traces occurring in Eqs. (23) and (24) are all negative definite. Hence, on comparing Eqs. (23) and (24), we deduce that

$$t = 0 .$$

Consequently, Eq.(21) becomes

$$C^\alpha = G\, Y^\alpha$$

If we now use the real unitary transformation

$$\phi^\alpha = b^{\alpha\mu} x^\mu ,$$

we may rewrite the interaction (13) in the form

$$H_{int} = G\, x^\alpha_{rs}\, E^+_r\, E_s\, x^\alpha + h.c ,$$

which is the required SU_n invariant form. We have thus deduced unitary symmetry for trilinear interactions from the Smushkevich principle.

IV. DISCUSSION:

It is gratifying to see that among symmetry groups of rank two, the dynamical framework considered above singles out SU_3. However, there are two circumstances that ought to be considered. First, none of the triplet representations of SU_3 has been discovered experimentally to date; secondly, the SU_3 symmetry is not exact, but is only approximate. The apparent nonexistence of the triplets may be accounted for by assuming that they are very heavy in mass. The eight-component multiplets may be taken to be the pseudoscalar meson octet comprising pions, kaons, antikaons, and the eta; or the corresponding vector-meson octet.

If we choose a theory with only one fundamental triplet (and ~~dkskk~~ its distinct antiparticle triplet), the quanta of these fields will have to have nonintegral values of baryon number and electric charge. On the other hand, we may choose two triplets, one with baryon number zero and one with baryon number one. Even in this case (unless new conservation laws are postulated), the electric charge would have fractional values. These entities would then obey an associated production rule, and could not decay into ordinary particles (with integral electric charges).

While such entities have been discussed recently in related contexts[8], in the present framework there is a

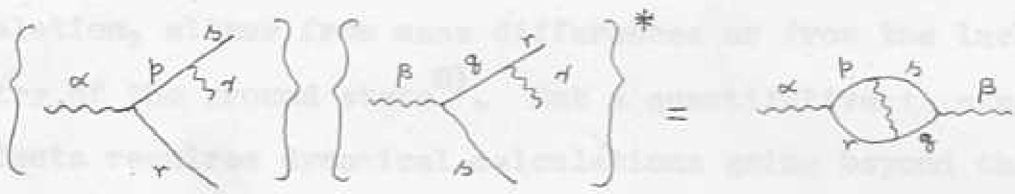

Fig.3. Illustrating the relation of Smushkevich's method and the Smushkevich principle.

primitive eight-fold multiplet which participates in the primitive trilinear interaction. This entails introducing a larger number of primitive entities than in the formulation in which the symmetry group is postulated; but on the other hand, the present work derives the symmetry from 'first' principles. Note that one of the triplets may be a baryon triplet, and the other one a meson triplet together with a baryon octet; for example, in the SU_2 case

8) M.Gell-Mann, Phys.Letters 8, 214 (1964); G.Zweig, CERN (unpublished).

we could consider the nucleon-kaon-Σ-hyperon coupling.

We must also take into account the breaking of the unitary symmetry. A clue to the pssible violation of the symmetry in the Smushkevich framework is provided by the structure of the one-particle propagator which is susceptible to spontaneous symmetry violation, either from mass differences or from the lack of symmetry of the ground state[9]. But a quantitativexx study of these effects requires dynamical calculations going beyond the algebraic techniques used here.

The Smushkevich principle used here is rather intimately related to the Smushkevich method in strong interaction physics[10].

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9) The question of broken symmetries is discussed by several authors in the Proceedings of the Seminar on Unified Theories of Elementary Particles, edited by D.Lurie and N.Mukunda (University of Rochester Pres, Rochester, 1963). See chap VII of this thesis

10) The extension of the Smushkevich method to invariance under arbitrary groups has been discussed in C.Dullemond, A.J. Macfarlane and E.C.G.Sudarshan, Phys. Rev.Letts. 10, 423 (1963);

A.J.Macfarlane, N.Mukunda and E.C.G.Sudarshan, Phys.Rev. 133, B 475, (1964); J.Math.Phys. 5, 576 (1964);

M.E.Mayer, Lectures on Strong and Electromagnetic Interactions (Brandeis University Press, Waltham, Massachusetts, 1963), Volume 1.

Consider, for example, the amplitude for the (virtual) process

$$\pi^{\alpha} \longrightarrow N_r + \bar{N}_s + \pi^{\beta}$$

Smushkevich equations for the π^{α} include the statement

$$\sum_{r\bar{s}\beta} M_{\alpha\to r\bar{s}\beta} M^{*}_{\alpha'\to r\bar{s}\beta} = \Gamma\,\delta_{\alpha\alpha'}$$

but in the framework of trilinear interactions (and use of perturbation theory) we have the additional result that

$$M_{\alpha\to r\bar{s}\beta} \sim M_{\alpha\to r\bar{t}}\, M^{*}_{\beta\to s\bar{t}}$$

so that

$$M_{\alpha\to r\bar{s}\beta} = \sum C^{\alpha}_{rt}\, C^{\dagger\beta}_{ts}$$

The Smushkevich equation for the production process now coincides with Eq.(15b), as illustrated in Fig.3. Similar comments apply to the other propagator diagrams we well.

We must also discuss the relation of the present work with a more limited application[11] in which charge (and hypercharge) conservation is imposed at the start. In this case the number of coupling constants are smaller, but so are the number of useful equations, since most of the Smushkevich equations become identities. The previous demonstrations of charge independence of pion-nucleon system required ~~an~~ the postulate of charge conservation. For the Σ-hyperon-pion system, for which, as mentioned above, the Smushkevich principle fails to yield SU_2 invariance if used alone, the Smushkevich principle is successful if we use charge conservation as well. But with sufficiently high multiplets, either method would fail; and the reason is simple. If charge conservation is imposed, for trilinear interactions, the number of coupling constants increases as the second

11) J.J.Sakurai, Phys.Rev. Letters 10, 446 (1963). In Sakurai's demonstration of charge independence of the pion-nucleon Yukawa interaction, he makes use of the conservation of electric charge explicitly. In his demonstration of SU_3 invariance, he imposes charge independence (and charge conjugation invariance) for the isotopic multiplets. But in such a framework, where the meson-octet components, are taken to be degenerate in masses, we cannot derive charge independence from 'first' principles using his method. In the present work on the interaction of two triplets and an octet, we do not impose charge independence, but derive it as a consequence of SU_2 invariance. See also the derivation of charge independence for pion-nucleon interaction by Frohlich, Heitler and Kemmer, Ref.6.

power of the multiplicity, but the number of useful Smushkevich equations increase linearly with the number of components of the multiplets. Without any such constraints, the number of coupling constants increases as the third power of the number of components, while the number of useful Smushkevich equations increase as the square. In view of this, it is curious to observe that usually only the lower-lying multiplets are in practice realized.

Some other comments are in order. With strong interactions one may be skeptical about the relevance of using algebraic relations deduced by considering peturbation diagrams. However, it is to be noted that we do not use the perturbation-theoretic estimates for the actual amplitudes, but only their dependence on the 'internal' labels. What is even more to the point is that similar equations are obtained as self-consistency relations in the strong coupling limit. We may think of the Smushkevich equations as reflecting the self-consistency of the trilinear vertex and the orthogonality and completeness of 'wave functions' of members of a multiplet considered as bound states of members of the other two multiplets. In the same spirit, we may also think of the trilinear interactions between the three multiplets with n, n and n^2-1 members as itself being caused by the direct coupling of four multiplets with n members each, which leads to n^2-1 bound states. Perhaps these considerations are of relevance to the theory of strongly interacting particles.

C H A P T E R VII

BROKEN SYMMETRY AND THE SMUSHKEVICH PRINCIPLE.

ABSTRACT

An attempt is made to incorporate broken SU(2) symmetry of isotopic spin in a dynamical scheme based on the Smushkevich Principle. It is found that a unique solution to the problem is possible only if conservation of electric charge is assumed. The possibility of extending this method to higher symmeties is discussed.

CHAPTER VII.

BROKEN SYMMETRY AND THE SMUSHKEVICH PRINCIPLE*

1. INTRODUCTION:

In the recent past, there have been many attempts[1-10] to derive the unitary symmetry of strongly interacting particles from purely dynamical considerations. One such and perhaps the most successful, attempt is to deduce the internal symmetry group from the ~~so~~called Smushkevich principle[7]. This principle is the dynamical requirement that the 'clothed' propagators of a given set of particles are equal and can be most conveniently expressed as a set of equations (SOS equations), by taking recourse to perturbation theory. This method has been used to derive the SU(n)

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* P.Narayanasamy and T.S.Santhanam, Nuovo Cimento ~~(in press)~~. 59A, 20, 1969

1. R.E.Cutkosky, Phys. Rev. 131,1888 (1963).
2. R.E.Cutkosky, Ann. Phys. (NY) 23, 415 (1963).
3. H.M.Chan, P.C.Decelles and J.E.Paton, Phys.Rev.Lett.11,521 (1963).
4. R.H.Capps, Phys.Rev.Letters 10, 312 (1963).
5. E.C.G.Sudarshan, Phys. Letters 9, 286 (1964).
6. J.J.Sakurai, Phys. Rev. Letters 10, 446 (1963).
7. E.C.G.Sudarshan, L.O'Raifeartaigh and T.S.Santhanam,Phys. Rev. 136, B 1092(1964) (We refer to this paper as SOS). See Chapter VI.
8. E.C.G.Sudarshan, 'Symmetry in Particle Physics', Proceedings of the Chicago Meeting of the American Physical Society (Nov. 1964).
9. E.C.G.Sudarshan, 'Theory of Approximate Symmetries', Seminar on High-Energy Physics and Elementary Particles, ICTP, Trieste, 1965, Proceedings (IAEA, Vienna) - A complete list of references for earlier work can be found in this paper.
10. A.K.Bose and J.Patera, Phys. Rev. Letters 14, 729 (1965).

invariance of a trilinear interaction between sets of particles of multiplicity n, n and $n^2 - 1$. The remarkable feature of this derivation is that the conservation of electric charge is not assumed but is rather a consequence of the SOS equations. More recently, there have been several attempts to extend the method to the case of trilinear interaction among vector mesons[11] and to the trilinear interaction among three sets of particles of arbitrary multiplicities[9].

An advantage of this method consists in the fact that the internal symmetry group comes equipped with the representation of the symmetry group.

However, it is well known that a realistic description of strong interactions is not possible in terms of an exact symmetry, as all symmetries are badly broken. It is thus worthwhile to investigate the structure of a broken symmetry in the framework of such dynamical schemes. The SOS equations are the most suited for this purpose. In fact, Sudarshan has discussed the possibility of introducing the symmetry violation in this manner[8].

It is the purpose of this paper to investigate the solution of the SOS equations in the presence of a symmetry violation. The lack of symmetry can be realized through the presence of an additional term in the SOS equation for the propagator which transforms as an irreducible representation of the group.

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11) O'Fleischmann R.Musto, L.O'Raifeartaigh and P.S.Rao, ~~Syracuse University Preprint~~. Phys. Rev. 158, 1588 (1967)

In section 2 we investigate the SOS equations for the simple case of broken SU(2) of iso-spin, when the nucleon propagator is a linear combination of an invariant plus a small term transforming like the third component of iso-spin. This method of introducing symmetry violation is analogous to that employed in calculations[12] involving spontaneous symmetry breakdown in field theory. By incorporating the symmetry ~~violating~~ violation in this manner, we get three sets of solutions, satisfying all the SOS equations (at least up to the sixth order); these are analyzed in Section 3. We are unable to obtain a unique solution without using charge conservation. If charge conservation is assumed, a unique solution follows which is used to obtain sum rules for the coupling constants. In the last section we discuss the possibility of extending this method to higher symmetries.

2. SOLUTION OF THE SOS EQUATIONS.

We consider the most general trilinear interaction among the nucleons and pions, assuming only the baryon conservation:

$$H_{int} = g^{\alpha}_{rs} \, N^{+}_{r} \, N_{s} \, \phi^{\alpha} , \qquad r,s = 1,2 ; \quad \alpha = 1,2,3 \tag{1}$$

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12) R. Arnowitt and S. Deser, Phys. Rev. 138, B 712 (1965).

The meson field ϕ can be chosen to be real and the g's can be treated as matrices in the (r,s) space. The hermiticity of the Hamiltonian implies

$$g_\alpha^{+} = g_\alpha , \tag{2}$$

implying that g's are hermitian matrices.

Suppose the symmetry breaking manifests through the following extended Smushkevich principle

$$\langle 0 | T(\psi_r(x)\,\bar{\psi}_s(y)) | 0 \rangle = S_F(x-y)\left[k_1\,\delta_{rs} + k_2\,(\tau_3)_{rs}\right],$$

and we also assume that the breaking in meson self energy is only in the fourth order. The SOS equations take the form

$$Sp\; g^\alpha g^\beta = k_1\,\delta^{\alpha\beta}, \tag{3}$$

$$Sp\; g^\alpha g^\gamma g^\beta g^\gamma = k_2\,\delta^{\alpha\beta} + k_3\,\delta_{\alpha 1}\,\delta_{\beta 1}, \tag{4}$$

$$Sp\left(g^\alpha g^\gamma g^\delta g^\beta g^\gamma g^\delta\right) = k_4\,\delta^{\alpha\beta} + k_5\,\delta_{\alpha 1}\,\delta_{\beta 1}, \tag{5}$$

for the meson self energies, and

$$g^\alpha g^\alpha = m_1 I + m_2 \tau_3 , \tag{6}$$

$$g^{\alpha} g^{\beta} g^{\alpha} g^{\beta} = m_3 I + m_4 \tau_3 , \tag{7}$$

for the nucleon self energies. (We adopt summation convention for repeated indices.) It is known that when the symmetry-breaking terms in the above equations (k_3, k_5, m_2 and m_4) are absent, the interaction is SU(2) invariant. We assume here that, in the second order, the symmetry-breaking term is present only in the nucleon self energy. m_2 is arbitrary (positive or negative but non-zero) parameter signifying the extent of symmetry violation. However, the meson self energy may develop symmetry-violating terms in higher orders as indicated in Eqs. (4) and (6). We could have in fact postulated the SOS equations straightaway. The presence of these terms are indeed dictated by the self consistency of the SOS equations, as will be seen later.

Since g's are hermitian matrices, they may be expanded in terms of a complete set of 2 x 2 matrices. We can choose this set to be ($\tau_1, \tau_2, \tau_3, I$) where the τ's are the usual Pauli matrices. So, we have

$$g^1 = a_i \tau_i + a_4 I , \tag{8}$$

$$g^2 = b_i \tau_i + b_4 I , \tag{9}$$

$$g^3 = c_i \tau_i + c_4 I , \quad (i = 1, 2, 3) \tag{10}$$

where a, b, c are real.

We notice that Eqs. (3)-(7) are left invariant under rotations about the third axis in the isospin space. This property can be exploited to set a_2 equal to zero. (We could have chosen any one of a_1, b_1, c_1, a_2, b_2, c_2 to be zero without affecting the generality of the arguments.) Using Eqs. (8), (9) and (10), the meson self energy Eq.(3) yields

$$a_i b_i + a_4 b_4 = b_i c_i + b_4 c_4 = c_i a_i + c_4 a_4 = 0 \tag{11}$$

$$a_i a_i + a_4^2 = b_i b_i + b_4^2 = c_i c_i + c_4^2 = \frac{1}{2} k_1 , \tag{12}$$

and the nucleon self energy Eq. (6) gives

$$a_1 a_4 + b_1 b_4 + c_1 c_4 = 0 , \tag{13}$$

$$b_2 b_4 + c_2 c_4 = 0 , \tag{14}$$

$$2 \ (a_3 a_4 + b_3 b_4 + c_3 c_4) = m_2 , \tag{15}$$

and

$$3 k_1 = 2 m_1 \tag{16}$$

From the fourth order meson equation, Eq. (4), we obtain

$$a_i b_j \left[\varepsilon_{1ij}\, c_1 c_4 + \varepsilon_{2ij}\, c_2 c_4 + \varepsilon_{3ij}\, (c_3 c_4 - m_2) \right] = 0 ,$$

$$c_i a_j \left[\varepsilon_{1ij}\, b_1 b_4 + \varepsilon_{2ij}\, b_2 b_4 + \varepsilon_{3ij}\, (b_3 b_4 - m_2) \right] = 0 , \tag{17}$$

$$b_i c_j \left[\varepsilon_{1ij}\, a_1 a_4 + \varepsilon_{3ij}\, (a_3 a_4 - m_2) \right] = 0 ,$$

and

$$m_2 \left(c_3 a_4 + c_4 a_3 \right) + c_4 a_4 \left(\tfrac{1}{2} k_1 - 4 b_4^2 \right) = 0 ,$$

$$m_2 \left(a_3 b_4 + a_4 b_3 \right) + a_4 b_4 \left(\tfrac{1}{2} k_1 - 4 c_4^2 \right) = 0 ,$$

$$m_2 \left(b_3 c_4 + b_4 c_3 \right) + b_4 c_4 \left(\tfrac{1}{2} k_1 - 4 a_4^2 \right) = 0 . \tag{18}$$

In deriving Eqs. (17) and (18), the explicit use of Eq.(6) has been made, thus manifesting the effect of symmetry violation in higher order meson self energy. The fourth order nucleon self energy, Eq.(7) implies

$$\begin{aligned} \tfrac{3}{2} k_1 + 4 k_1 \left(a_4^2 + b_4^2 + c_4^2 \right) &- 4 \left(a_4^4 + b_4^4 + c_4^4 \right) \\ &- 6 \left(a_4^2 b_4^2 + b_4^2 c_4^2 + c_4^2 a_4^2 \right) \\ &- 2 \left[\left(\varepsilon_{ijk}\, a_i b_j \right)^2 + \left(\varepsilon_{ijk}\, b_i c_j \right)^2 \right. \\ &\qquad \left. + \left(\varepsilon_{ijk}\, c_i a_j \right)^2 \right] = m_3 , \end{aligned} \tag{19}$$

$$m_4 = i m_2 . \tag{20}$$

We also need the following equations:

$$\begin{aligned} \det(abc)\, c_4 a_4^2 &= 0 , \\ \det(abc)\, a_4 b_4^2 &= 0 \\ \det(abc)\, b_4 c_4^2 &= 0 , \end{aligned} \tag{21}$$

which follow from the sixth order meson equation, Eq. (5). Here $\det(abc) = \varepsilon^{ijk} a_i b_j c_k$ .with Now Eq.(17) with $a_2 = 0$ gives

$$m_2 a_1 b_2 = 0 \tag{22}$$

which means that either a_1 or b_2 is zero. If $b_2 = 0$, then the second equation in Eq.(17) yields

$$m_2 b_1 c_2 = 0 \tag{23}$$

and similarly if $c_2 = 0$, then the last equation in Eq.(17) implies

$$m_2 c_1 a_2 = 0 . \tag{24}$$

If Eqs. (22)-(24) are analyzed carefully, making extensive use Eqs. (11)-(21), we can deduce the following choices for the structure of g^α :

$$
\begin{aligned}
g^1 &= a_3\tau_3 + a_4 I ,\\
g^2 &= b_1\tau_1 ,\\
g^3 &= c_2\tau_2 .
\end{aligned}
\tag{I}
$$

$$
\begin{aligned}
g^1 &= a_3\tau_3 + a_4 I ,\\
g^2 &= b_1\tau_1 \pm c_1\tau_2 ,\\
g^3 &= c_1\tau_1 \mp b_1\tau_2 ,
\end{aligned}
\tag{II}
$$

and

$$
\begin{aligned}
g^1 &= a_1\tau_1 + a_3\tau_3 + a_4 I ,\\
g^2 &= b_1\tau_1 + b_3\tau_3 + b_4 I ,\\
g^3 &= c_1\tau_1 + c_3\tau_3 + c_4 I
\end{aligned}
\tag{III}
$$

It should be emphasized that (III) is not really a solution of the SOS equations since there are a number of supplementary conditions to be satisfied by the constants a , b , c . This will be discussed in the next section.

3. DISCUSSION OF THE SOLUTIONS:

We first consider solution (I). Now, Eqs. (12) and (15) imply

$$a_3^2 + a_4^2 = b_1^2 = c_2^2 = \tfrac{1}{2} k_1 ,$$
$$2 a_3 a_4 = m_2 \tag{25}$$

which enables us to express a_3, a_4, b_1 and c_2 in terms of the two constants k_1 and m_2. The pion-nucleon interaction thus has the form

$$H_{int} = (a_3 + a_4) \bar{p} p \pi^0 + (a_4 - a_3) \bar{n} n \pi^0$$
$$+ b_1 (\bar{n} p \pi^+ + \bar{p} n \pi^-) \tag{26}$$

The physical coupling constants may then be identified as

$$g^2_{pn} = g^2_{np} = \tfrac{1}{2} k_1 ,$$
$$g^2_{pp} = \tfrac{1}{2} k_1 + m_2 ,$$
$$g^2_{nn} = \tfrac{1}{2} k_1 - m_2 . \tag{27}$$

This immediately leads to a sum rule

$$g^2_{pn} = g^2_{np} = \tfrac{1}{2} (g^2_{pp} + g^2_{nn}) \tag{28}$$

The above relation is weaker than that based on charge independence and is consistent with charge symmetry. It should be mentioned here that the solution is consistent with the meson trace equation only when a symmetry-violating term is present, as in Eq. (5). We thus have the requirement of self consistency of the solution and the SOS equations. Such self consistency may be understood in terms of the situation mentioned earlier, namely, that the meson self energy develops symmetry violation in the higher orders as a result of the symmetry breaking in the lowest order nucleon self energy. This is also true in the case of solutions (II) and (III).

As for solution (II), we have

$$a_3^2 + a_4^2 = b_1^2 + c_1^2 = \tfrac{1}{2} k_1$$

$$2\, a_3 a_4 = m_2 \tag{29}$$

For this case, it is clear from the SOS equations that it is not possible to express g^{α} in terms of the constants k_1 and m_2 alone. However, if one imposes the conservation of electric charge, it can be shown immediately that this choice will no longer satisfy the equations.

We now turn our attention to the solution (III) which involves, besides Eqs. (13)-(15), the following supplementary conditions:

$$
\begin{aligned}
&a_1 b_1 + a_3 b_3 + a_4 b_4 = 0\,, \\
&b_1 c_1 + b_3 c_3 + b_4 c_4 = 0\,, \\
&c_1 a_1 + c_3 a_3 + c_4 a_4 = 0\,, \\
&a_1^2 + a_3^2 + a_4^2 = b_1^2 + b_3^2 + b_4^2 = c_1^2 + c_3^2 + c_4^2 \\
&\qquad = \tfrac{1}{2} k_1
\end{aligned}
\tag{20}
$$

Perhaps, if one could go beyond the sixth order, one may be able to eliminate this case as an independent solution of the SOS equations, if not altogether ruled out. However, if one assumes charge conservation (as in the case of solution (II)), so this solution can be eliminated.

Thus, if one assumes charge conservation, one is led to a unique solution of the SOS equations, viz., solution (I). This solution is not entirely unexpected. It is true that one is able to deduce the symmetry group as a unique solution of the SOS equations without having to assume charge conservation in some special cases; for example, the Yukawa interaction of nucleons and mesons. However, for the case of $\Sigma\Sigma\pi$ system, even when the symmetry is not broken, one does not have a unique solution without assuming conservation of electric charge[7),9)]. In fact, for the trilinear interaction between vector mesons, it is known that additional requirements (e.g. complete antisymmetry of g in all the indices[11)]) are needed in order to obtain a unique solution.

4. REMARKS:

The method used here to include symmetry breaking can be extended to higher symmetry groups by letting the quark propagator transform like a linear combination of an invariant and a specific diagonal operator belonging to the adjoint representation. This will cause a mass difference between the quarks. For instance, in SU(3) the quark propagator can be assumed to transform like $aI + b\lambda_8$. In the corresponding exact symmetry situation one has the SU(3) invariance of the trilinear interaction involving quark, antiquark and mesons. Such symmetry breaking will then cause a mass splitting between the strange and non-strange quarks. One can also derive sum rules for coupling constants[13].

---

13) This method of obtaining the sum rules will then be analogous to the phenomenological method where the broken symmetry manifests itself in he the form of a few tadpole diagrams, e.g., S.Coleman and S.L.Glashow, Phys. Rev. 134 B 671 (1964).

APPENDIX 5.

We summarize below some interesting relationships between the n-j symbols of SU(2)*. The problem we were faced in this chapter may be immediately realized as the inverse problem, the solution of which may not always be straighforward and unique.

We use the abbreviation

$$[x] = 2x+1 .$$

We start with the usual orthogonal properties of the $6j$ symbols

$$\sum_x [x] \begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix} \begin{Bmatrix} c & d & x \\ a & b & q \end{Bmatrix} = \frac{\delta(p,q)}{[p]} , \tag{1}$$

$$\sum_x [x] (-1)^{p+q+x} \begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix} \begin{Bmatrix} c & d & x \\ b & a & q \end{Bmatrix} = \begin{Bmatrix} c & a & q \\ d & b & q \end{Bmatrix} \tag{2}$$

The coupling diagrams are given by Figs. A_1 and A_2.

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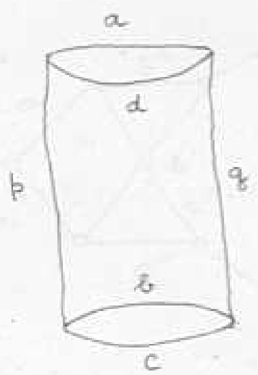
*See for instance, B.R.Judd. 'Operator Techniques in Atomic Spectroscopy', McGraw-Hill Book Company Inc., (New York) 1963, page 62.



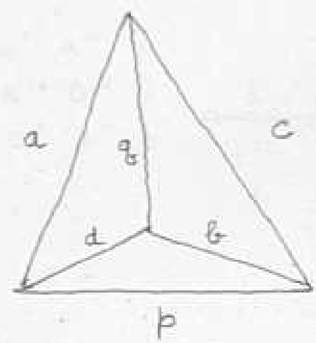
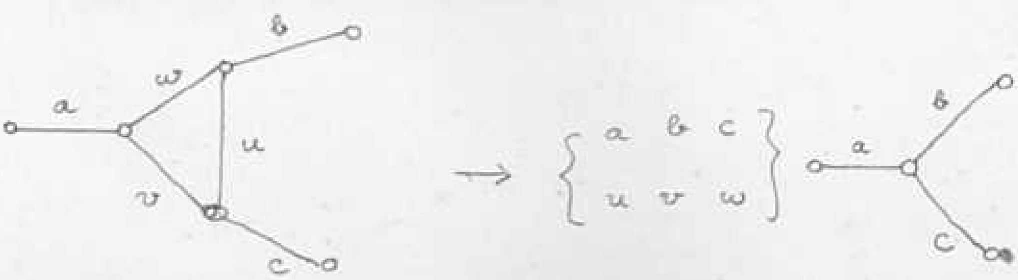


Fig. 1 Fig. 2

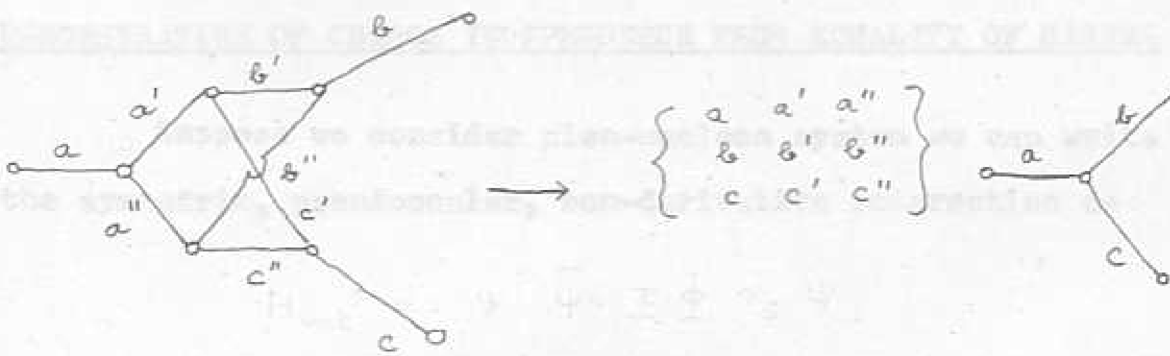
Appropriate reduction techniques of nj-symbols, we summarise through the following diagpams

(1) Double links are removed by the substitution

(2) Triangles are eliminated as follows



(3)



and so on . For more details, see R.E.Cutkosky, Brandeis Lectures (1965).

APPENDIX 6

DEMONSTRATION OF CHARGE INDEPENDENCE FROM EQUALITY OF MASSES

Suppose we consider pion-nucleon system we can write the symmetric, pseudoscalar, non-derivative interaction as

$$H_{int} = g\,\overline{\psi}\,\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}\,\gamma_5\,\psi, \tag{1}$$

where ψ denotes the nucleon isospinor and ϕ the pseudoscalar fields. If the total isospin is conserved, we have

$$\underset{\sim}{T} = \frac{1}{2}\int \overline{\psi}(x)\,\gamma_0\,\underset{\sim}{\tau}\,\psi(x)\,d^3x + \int \partial^0\,\phi^{\dagger}(x)\,\underset{\sim}{t}\,\phi(x)\,d^3x . \tag{2}$$

The first term on the r.h.s denotes the isospin current due to the nucleons and the second term denotes that due to pions. Suppose ψ and ϕ undergo the following gauge transformations

$$\begin{aligned}\psi \rightarrow \psi' &= U\psi U^{-1},\\ \phi \rightarrow \phi' &= U\phi U^{-1},\\ U &= \exp i\left(\underset{\sim}{T}\cdot\underset{\sim}{\theta}\right).\end{aligned} \tag{3}$$

Here U is an operator in the larger Hilbert space containing both ψ and ϕ. We have

$$\psi_r \to \psi_r' = \sum_s \left[\exp \tfrac{i}{2}(\underset{\sim}{\tau}\cdot\underset{\sim}{\theta})\right]_{rs} \psi_s ,$$

$$\phi_\alpha \longrightarrow \phi_\alpha' = \sum_\beta \left[\exp \tfrac{i}{2}(\underset{\sim}{t}\cdot\underset{\sim}{\theta})\right]_{\alpha\beta} \phi_\beta \qquad (4)$$

If the interaction is invariant under this transformation, the equation of motion must remain unchanged. Hence, the mass is unaltered under these transformations. Consider the propagator

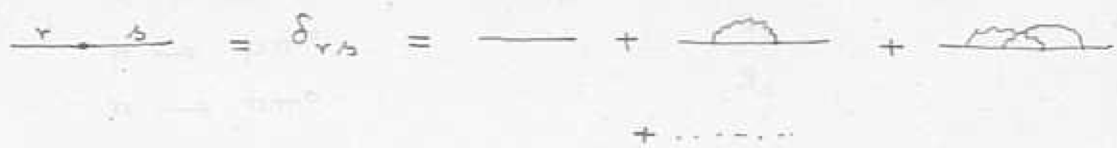
$$S_{rs}' = \langle (\bar{\psi}_r(x),\ \psi_s(y))_+ \rangle_0$$

$$= \delta_{rs}\ S'(x-y) \qquad (5)$$

If the vacuum is invariant under these transformations

$$S_{rs}' = \langle 0|\, U^{-1} U (\bar{\psi}_r(x)\, \psi_s(y))\, U^{-1} U\, |0\rangle \qquad (6)$$

In perturbation theory (6) may be developed as



(7)

The first one on the r.h.s. corresponds to the bare propagator and is the same for all particles of equal mass. If we consider one particle diagrams, each of the diagrams contribute the same. Now, let us ask the converse problem. If diagrams contribute the same to the masses (and therefore self masses), does this imply the invariance of the interaction under isospin transformations, implying thereby the origin of charge independence ?

Sakurai[*] has made a simple and explicit derivation of the isospin symmetry by assuming trilinear interactions between pions and nucleons and by equating the contributions of various self energy diagrams. Since the first bare term is the same in all cases, let us look at the second terms. The various π-N cases are

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[*] J.J.Sakurai, Phys. Rev. Lett. **10**, 446 (1963). It should be restressed that in our derivation of charge independence from Shmushkevich principle, charge conservation is <u>not assumed; but is derived</u>.

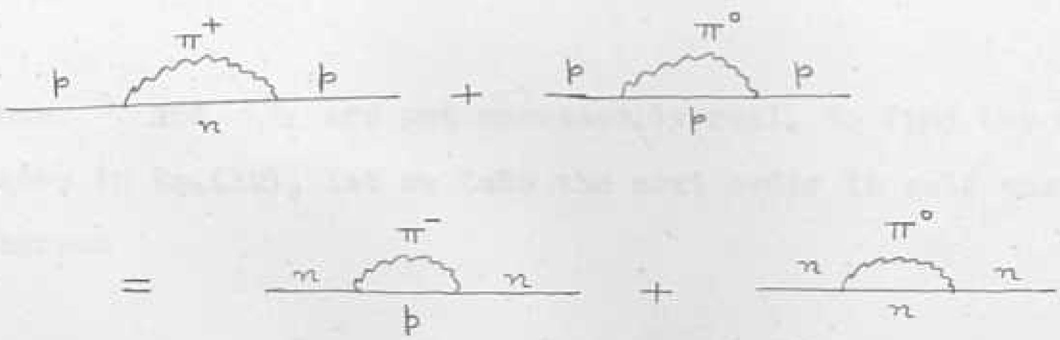
	coupling constants
$p \to n\pi^+$	g_1
$n \to p\pi^-$	g_2
$p \to p\pi^\circ$	g_3
$n \to n\pi^\circ$	g_4
$\pi^+ \to p\bar{n}$	g_1
$\pi^- \to n\bar{p}$	g_2
$\pi^\circ \to p\bar{p}$	g_3
$\pi^\circ \to n\bar{n}$	g_4

(8)

The g's denote the various coupling constants for the various reactioons. From charge independence, one knows that

$$g_1 = g_2 = \sqrt{2}\, g_3 = -\sqrt{2}\, g_4 \tag{9}$$

Suppose we now explicitly evaluate the various self energy contributions.and equate those of p and n we have

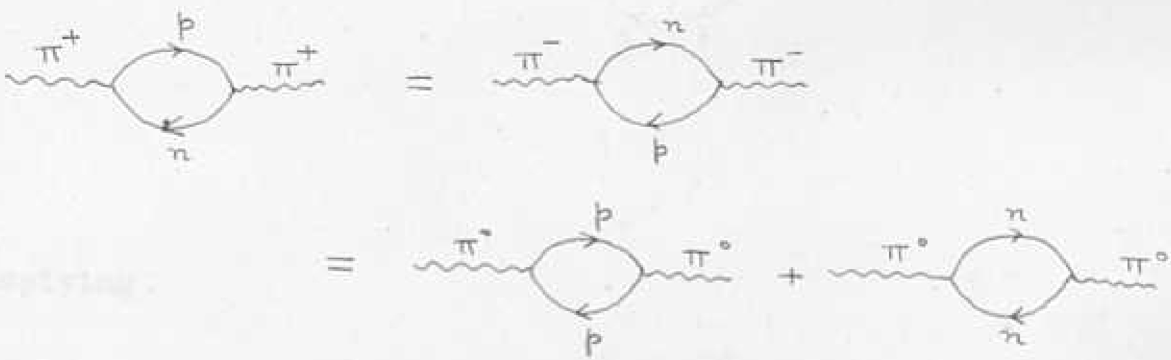


implying the relation

$$|g_1|^2 + |g_3|^2 = |g_2|^2 + |g_4|^2 \tag{10}$$

g_3 and g_4 should be real

Similarly if we compute pion self energies



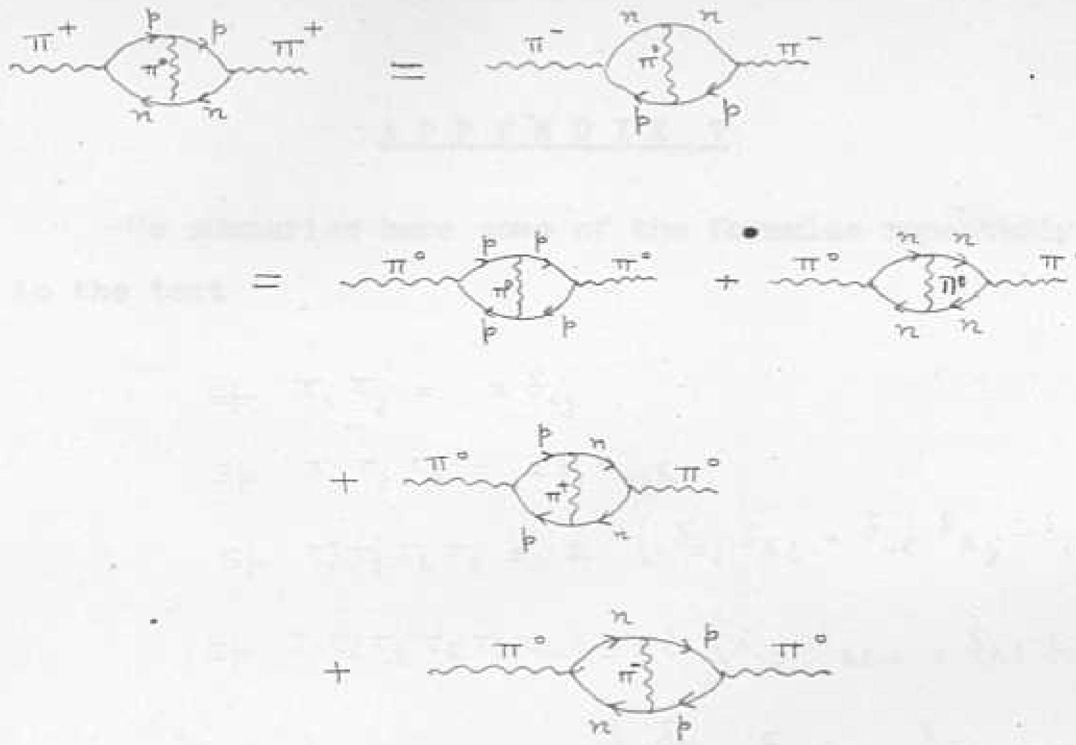
implying

$$|g_1|^2 = |g_2|^2 = |g_3|^2 + |g_4|^2 \tag{11}$$

Solving Eqs. (10) and (11), we get

$$|g_1|^2 = |g_2|^2 = 2\,|g_3|^2 = 2\,|g_4|^2 \tag{12}$$

Since g_1 and g_2 are not necessarily real, to find the relative phases in Eq.(12), let us take the next order in self energy diagrams



implying

$$g_1^2 g_3 g_4 = g_3^4 + g_4^4 + g_3 g_4 (g_1^2 + g_2^2)$$

or

$$g_3^4 + g_4^4 + g_3 g_4 g_2^2 = 0 \tag{13}$$

It then trivially follows that

$$g_3 g_4 \leq 0 \tag{14}$$

Eq.(14) along with Eq.(12) yields Eq.(9). Thus, Sakurai has shown that the implications of charge independence can be derived from just the equality of masses !

APPENDIX 7

We summarize here some of the formulae repeatedly used in the text

$$\mathrm{Sp}\ \tau_i\tau_j = 2\delta_{ij},$$

$$\mathrm{Sp}\ \tau_i\tau_j\tau_k = 2i\,\varepsilon_{ijk},$$

$$\mathrm{Sp}\ \tau_i\tau_j\tau_k\tau_l = 2\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - \delta_{ik}\delta_{jl}\right),$$

$$\mathrm{Sp}\ (\tau_i\tau_j\tau_k\tau_l\tau_m) = 2i\left(\delta_{ij}\,\varepsilon_{klm} + \delta_{kl}\,\varepsilon_{ijm} + \delta_{lm}\,\varepsilon_{ijk} - \delta_{km}\,\varepsilon_{ijl}\right),$$

$$\begin{aligned}\mathrm{Sp}\ (\tau_i\tau_j\tau_k\tau_l\tau_m\tau_n) = 2\big(&\delta_{ij}\delta_{kl}\delta_{mn} - \delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} \\ &- \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jm}\delta_{kl} \\ &- \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{il}\delta_{kj}\delta_{mn} \\ &- \varepsilon_{ijn}\varepsilon_{klm} + \varepsilon_{ijm}\varepsilon_{kln}\big),\end{aligned}$$

$$[g^1, g^2] = a_4\,\underset{\sim}{b}\cdot\underset{\sim}{\tau} + b_4\,\underset{\sim}{a}\cdot\underset{\sim}{\tau} + 2i\,\det(ab\tau),$$

$$[g^2, g^3] = b_4\,\underset{\sim}{c}\cdot\underset{\sim}{\tau} + c_4\,\underset{\sim}{b}\cdot\underset{\sim}{\tau} + 2i\,\det(bc\tau),$$

$$[g^3, g^1] = c_4\,\underset{\sim}{a}\cdot\underset{\sim}{\tau} + a_4\,\underset{\sim}{c}\cdot\underset{\sim}{\tau} + 2i\,\det(ca\tau),$$

$$\{g^1, g^2\}_+ = 2a_4\,\underset{\sim}{b}\cdot\underset{\sim}{\tau} + 2b_4\,\underset{\sim}{a}\cdot\underset{\sim}{\tau},$$

$$\{g^2, g^3\}_+ = 2b_4\,\underset{\sim}{c}\cdot\underset{\sim}{\tau} + 2c_4\,\underset{\sim}{b}\cdot\underset{\sim}{\tau},$$

$$\{g^3, g^1\}_+ = 2c_4\,\underset{\sim}{a}\cdot\underset{\sim}{\tau} + 2a_4\,\underset{\sim}{c}\cdot\underset{\sim}{\tau}$$

APPENDX 8.

PROOF OF SOME IDENTITIES SATISFIED BY THE GENERATOR OF SU_n

Let X^α be the (n^2-1) hermitian generators of SU_n. Then any traceless $n\times n$ matrix can be expanded in terms of X^α with complex coefficients. We choose the generators so that

$$sp\,(x^\alpha x^\beta) = n\,\delta^{\alpha\beta}$$

If for any two matrices A,B we define the scalar product by

$$(A, B) = sp\,(A^+ B)$$

the traceless $n\times n$ matrices constitute a unitary vector space of n^2-1 dimensions. If u is any $n\times n$ unitary matrix and

$$x'^\alpha = u^+ x^\alpha u = V^{\alpha\beta} x^\beta ,$$

then

$$(x'^\alpha, x'^\beta) = (x^\alpha, x^\beta)$$

so that V is a real unitary $(n^2-1)\times(n^2-1)$ matrix. The matrices u furnish the fundamental representation of SU_n while the matrices V furnish the adjoint representation of SU_n.

Now consider for any u

$$u^{+} x^{\alpha} x^{\alpha} u = V^{\alpha\mu} V^{\alpha\nu} x^{\mu} x^{\nu} = x^{\mu} x^{\mu}$$

so that $x^{\alpha} x^{\alpha}$ commutes with every unitary matrix. Hence $x^{\alpha} x^{\alpha}$ must be a multiple of the identity

$$x^{\alpha} x^{\alpha} = k_1 I$$

Similarly

$$u^{+} x^{\alpha} x^{\beta} x^{\alpha} x^{\beta} u = V^{\alpha\mu} V^{\beta\nu} V^{\alpha\rho} V^{\beta\sigma} x^{\mu} x^{\nu} x^{\rho} x^{\sigma}$$

$$= x^{\mu} x^{\nu} x^{\mu} x^{\nu},$$

so that

$$x^{\alpha} x^{\beta} x^{\alpha} x^{\beta} = k_2 I$$

In a similar fashion

$$x^{\alpha} x^{\beta} x^{\gamma} x^{\alpha} x^{\beta} x^{\gamma} = k_3 I,$$

and so on.

By the choice of generators we already have assured the validty of the relation

$$sp(x^{\alpha} x^{\beta}) = n\, \delta^{\alpha\beta}.$$

Putting $\alpha = \beta$ and summing we get

$$sp\,(x^\alpha x^\alpha) = n\,(n^2-1) .$$

But since

$$x^\alpha x^\alpha = k_1 I ,$$

this implies that

$$k_1 = n^2 - 1 , \quad x^\alpha x^\alpha = (n^2-1)\, I .$$

We can also consider

$$t^{\alpha\beta} = sp\,(x^\alpha x^\gamma x^\beta x^\gamma) .$$

Then (remembering that V is real unitary),

$$t^{\alpha\beta} = sp\,(u^+ x^\alpha x^\gamma x^\beta x^\gamma u) = V^{\alpha\mu} V^{\beta\nu} t^{\mu\nu}$$
$$= (V t V^{-1})^{\alpha\beta} ,$$

so that the $(n^2-1) \times (n^2-1)$ matrices V and t commute. But since V furnishes the adjoint representation of SU_n this implies that $t^{\alpha\beta}$ must be a multiplet of the unit matrix $\delta^{\alpha\beta}$. We may then write

$$sp\,(x^\alpha x^\gamma x^\beta x^\gamma) = k_2' \, \delta^{\alpha\beta}$$

Similarly we can show that

$$\mathrm{Sp}\left(X^{\alpha} X^{\gamma} X^{\delta} X^{\beta} X^{\gamma} X^{\delta} \right) = k_3' \, \delta^{\alpha\beta},$$

etc. The constants $k_2', k_3', \ldots$ are related to $k_2, k_3, \ldots$ Since if we put $\alpha = \beta$ and since we can obtain

$$\mathrm{Sp}\left(X^{\alpha} X^{\gamma} X^{\alpha} X^{\gamma} \right) = (n^2-1)\, k_2' = n k_2 ,$$

$$\mathrm{Sp}\left(X^{\alpha} X^{\gamma} X^{\delta} X^{\alpha} X^{\gamma} X^{\delta} \right) = (n^2-1)\, k_3' = n k_3 ,$$

etc.

APPENDIX 9.

PROOF OF THE IDENTITIES (15) AND (16) OF CHAPTER VI.

Let X^α be the generators of SU_n and u be any $n \times n$ unitary matrix. Then

$$u^+ X^\alpha u = V^{\alpha\beta} X^\beta$$

where $V^{\alpha\beta}$ is a real unitary matrix. Consider the matrix

$$M = X^\alpha X^\alpha.$$

Then

$$u^+ M u = V^{\alpha\beta} V^{\alpha\gamma} X^\beta X^\gamma = X^\beta X^\gamma \delta_{\beta\gamma} = X^\beta X^\beta = M.$$

But the only matrices which commute with every matrix is a multiplet of the unit matrix I. Hence

$$X^\alpha X^\alpha = k_1 I.$$

Similarly

$$u^+ X^\alpha X^\beta X^\alpha X^\beta u = V^{\alpha\mu} V^{\beta\rho} V^{\alpha\tau} V^{\beta\sigma} X^\mu X^\rho X^\tau X^\sigma$$
$$= X^\mu X^\rho X^\mu X^\rho.$$

APPENDIX 10

Solutions of the Smushkevich problem (3 x 3 x 3):

We give below the counter example of a set of three (3 x 3) matrices different from the set of matrices obeying SU(2) algebra which satisfy all the Smushkevich equations. Thus the solution of the Smushkevich problem (3 x 3 x 3) is not unique. However, if we insist of charge conservation, the solution corresponding to SU(2) is uniquely picked up.

The set of matrices

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & f & 0 \\ f & 0 & g \\ 0 & g & 0 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 0 & ig & 0 \\ -ig & 0 & if \\ 0 & -if & 0 \end{pmatrix}$$

with $$f^2 + g^2 = 1,$$

satisfy all the SOS equations

$$c^\alpha c^\alpha = m I,$$
$$\mathrm{Sp}\, c^\alpha c^\beta = n\, \delta^{\alpha\beta},$$
$$\mathrm{Sp}\, c^\alpha c^\gamma c^\beta c^\gamma = n'\, \delta^{\alpha\beta},$$
$$\mathrm{Sp}\, c^\alpha c^\gamma c^\delta c^\beta c^\gamma c^\delta = n''\, \delta^{\alpha\beta},$$

and so on, where $C^{\alpha} = I_{\alpha}$. Although, it looks that there are infinite sets of solutions, it has been shown* that the solutions fall into one of the three classes characterized by three different algebraic properties

$$\alpha : \quad [c_\alpha , c_\beta] = 0 ,$$

$$\beta : \quad [c_\alpha , c_\beta] = \frac{-i}{\sqrt{2}} \varepsilon_{\alpha\beta\gamma} c_\gamma ,$$

$$\gamma : \quad \{c_\alpha , c_\beta\} = \frac{1}{\sqrt{2}} c_\gamma \quad (\text{cyclic})$$

Standard Solutions for the algebra (3,3,3)

	c_1	c_2	c_3
α :	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
β :	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & -i \\ -i & i & 0 \end{pmatrix}$
γ :	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

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*H. Leutwyler and E.C.G. Sudarshan, ~~Syracuse preprint, to be published in~~ Phys. Rev. 156, 1637 (1967).

P A R T III

APPLICATION OF SYMMETRY PRINCIPLES TO PARTICLE RE-ACTIONS

CHAPTER VIII

REPRESENTATION MIXING EFFECTS IN SU(3) SYMMETRY.*

ABSTRACT

The effect of mixing the irreducible representations of SU(3) group as a mechanism of breaking the symmetry is studied in various interactions. It is found that many results obtained earlier by using the symmetry-breaking in the conventional way are reproduced while in addition this offers a simpler way of understanding the symmetry-breaking effects.

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INTRODUCTION.

It is well known in nuclear physics that the symmetry breaking many times manifests in mixing various rotation levels of different symmetry properties. (For example, there is a finite D-state admixture to the dominant S-state deuteron ground state wave function). We study in this paper the consequences of a model in which the baryons and the isobars belong to an admixture of irreducible representations (IRs) [8] and [10] of SU(3) and assume only charge independence of meson baryon interaction. The motivation for this comes from a simple observation

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*Alladi Ramakrishnan, T.S.Santhanam and A.Sundaram, (Preprint).

that an isotriplet with Y (Hypercharge) = 0 and an isodoublet with Y = -1 occur in both the lowest lying IRs [8] and [10] of SU(3). The recently discovered Roper resonance (1400 Mev) has all the quantum numbers of the necleon and the problem therefore is now to accommodate it in the SU(3) scheme. This representation mixing model can easily accommodate it. Of course, the problem will be now ~~found~~ to find the other particles.

Using this model, we study the baryon-meson couplings. We find relations for instance, for the strong decays of isobars which have been earlier obtained in the broken SU(3) model where the breaking was introduced as a perturbation in the interaction, and these relations are consistent with experiments[1]. We also study the effect of such a mixing in the baryon-baryon-meson couplings. The consequences of this model for mass relations and magnetic moments for baryons have been discussed.

THE MODEL

We assume that the baryons and the isobars belong to the reducible representation

$$|B\rangle = \left\{ \alpha\, [8] + \beta\, [10] \right\} ,$$

$$\alpha^2 + \beta^2 = 1 \tag{1}$$

1) See for a recent clear analysis, M.Goldberg, J.Leitner, R.Musto and L.O'Raifeartaigh, Nuovo Cimento, 45, 169 (1966).

of SU(3). It is immediately apparent that for nucleons and Λ, $\beta = 0$ since they have no counterparts in the representation $[10]$ and for Ω^-, $\alpha = 0$ since it has no counterpart in $[8]$. The parameter β/α measures the amount of mixing.

STRONG DECAYS

We first study the strong decays of isobars in this model. If we denote the matrix element as follows,

$$\langle \alpha [8] + \beta [10] , 8 \mid \alpha' [8] + \beta' [10] \rangle$$

$$= \lambda_1 \langle 8, 8 \mid 8_1 \rangle + \lambda_2 \langle 8,8 \mid 8_2 \rangle \qquad (2)$$

$$+ \lambda_3 \langle 10, 8 \mid 10 \rangle + \lambda_4 \langle 8,8 \mid 10 \rangle$$

then the contributions to the various relevent decays are given in the tables[2] 1 and 2.

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2) The relevant C.G.Coefficients have been taken from J.J.de Swart, Revs. Mod. Phys. 35, 916 (1963) and P.McNamee and F.Chilton, Revs. Mod. Phys. 36, 1005, (1964).

TABLE 1.

Decay	λ_1	λ_2	λ_3	λ_4
$Y_1^* \longrightarrow \Sigma\pi$	0	$\sqrt{2/3}$	$\sqrt{1/3}$	$\sqrt{1/6}$
$Y_1^* \longrightarrow \Lambda\pi$	$\frac{1}{\sqrt{5}}$	0	0	-1/2
$\Xi^* \longrightarrow \Xi\pi$	$-\frac{3}{2\sqrt{5}}$	1/2	$1/2\sqrt{2}$	1/2
$N^* \longrightarrow N\pi$	0	0	0	$\frac{-1}{\sqrt{2}}$

Table 2.

Decay	λ_1	λ_2	λ_3	λ_4
$N^* \longrightarrow \Sigma K$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$Y_1^* \longrightarrow N\bar{K}$	$-\sqrt{\frac{2}{10}}$	$\frac{1}{\sqrt{6}}$	0	$-\frac{1}{\sqrt{6}}$
$Y_1^* \longrightarrow \Sigma\eta$	$\frac{1}{\sqrt{5}}$	0	0	$\frac{1}{2}$
$Y_1^* \longrightarrow \Xi K$	$\sqrt{\frac{2}{10}}$	$-\frac{1}{\sqrt{6}}$	$-\sqrt{\frac{1}{3}}$	$\frac{1}{\sqrt{6}}$
$\Xi^* \longrightarrow \Sigma\bar{K}$	$\frac{3}{2\sqrt{5}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\Xi^* \longrightarrow \Lambda\bar{K}$	$-\frac{1}{2\sqrt{5}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\Xi^* \longrightarrow \Xi\eta$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2}$
$\Omega^- \longrightarrow \Xi\bar{K}$	0	0	$\frac{1}{\sqrt{2}}$	1

Eliminating the parameters, among the observed strong decays, we get one interesting sum rule

$$2\,G(N^{*} \longrightarrow N\pi) - 2G(\Xi^{*} \to \Xi\,\pi) = 3G(Y_1^{*} \to \Lambda\,\pi) - \frac{3}{2}\,G(Y_1^{*} \longrightarrow \Sigma\pi). \tag{3}$$

This sum rule has been earlier obtained by various people[3] in broken SU(3), where the symmetry breaking was assumed in a completely different way. This sum rule is well satisfied experimentally[1]. There are many more relations that are predicted among the other strong decay modes for which sufficient experimental data is not yet available

$$G(\Omega \to \Xi\bar{K}) = -\sqrt{6}\;G(Y_1^{*}\to N\bar{K}) + 2G(\Xi^{*}\to\Xi\,\pi) + \sqrt{2}\,G(N^{*}\to N\pi) \tag{4}$$

$$2G(\Xi^{*}\to\Sigma\bar{K}) = -\sqrt{6}\;G(Y_1^{*}\to N\bar{K}) + \sqrt{6}\;G(Y_1^{*}\to\Sigma\pi) + \sqrt{2}\;G(N^{*}\to N\pi) \tag{5}$$

$$2G(\Xi^{*}\to\Lambda\bar{K}) = \sqrt{6}\;G(Y_1^{*}\to N\bar{K}) - \sqrt{2}\;G(N^{*}\to N\pi) + 2\;G(Y_1^{*}\to\Lambda\,\pi) \tag{6}$$

$$2G(N^{*}\to\Sigma K) = \sqrt{6}\;G(Y_1^{*}\to\Xi K) - \sqrt{2}\;G(N^{*}\to N\pi) + 2G(\Xi^{*}\to\Xi\,\pi) + 6G(Y_1^{*}\to\Lambda\pi) \tag{7}$$

$$2G(Y_1^{*}\to\Sigma\eta) = \frac{2\sqrt{6}}{3}\;G(Y_1^{*}\to\Xi K) - G(Y_1^{*}\to N\bar{K}) + \frac{4}{3}\left\{2G(\Xi^{*}\to\Xi\,\pi) - \sqrt{2}\;G(N^{*}\to N\pi) + 6G(Y_1^{*}\to\Lambda\,\pi).\right. \tag{8}$$

$$2G(\Xi^* \to \Xi\,\eta) = \frac{2}{3}\sqrt{6}\left\{G(Y_1^* \to \Xi K) - G(Y_1^* \to N\bar{K})\right\}$$

$$+ \frac{1}{3}\left\{2G(\Xi^* \to \Xi\pi) + \sqrt{2}\; G(N^* \to N\pi)\right\} \qquad (9)$$

These relations also have been predicted in the broken SU(3) model[3]. An analysis of $\bar{K}N$ data may provide information to check many of these sum rules.

BARYON-BARYON-MESON COUPLINGS.

The baryon-baryon meson couplings are all expressible in terms of five parameters, after eliminating which we get the following sum rules

$$\sqrt{2}\; g_{\Lambda NK} = g_{NN\eta} + g_{NN\pi}\,, \qquad (10)$$

$$g_{NN\pi} - g_{NN\eta} + 2\, g_{\Lambda\Lambda\eta} = 0\,. \qquad (11)$$

$$\sqrt{\frac{3}{2}}\; g_{\Sigma NK} = 2\, g_{NN\eta} - g_{NN\pi} + g_{\Sigma\Lambda\pi}\,. \qquad (12)$$

$$2\, g_{\Xi\Xi\pi} = \sqrt{\frac{3}{2}}\; g_{\Sigma\Sigma\pi} + g_{\Sigma\Sigma\eta}$$

$$+ 2(g_{NN\eta} - g_{NN\pi})\,, \qquad (13)$$

$$g_{\Sigma K\Xi} = \sqrt{\frac{3}{2}}\; g_{\Sigma\Sigma\pi} - g_{\Sigma\Sigma\eta} - \frac{1}{2}\, g_{NN\eta}$$

$$+ \frac{5}{2}\, g_{NN\pi} - 2\, g_{\Sigma\Lambda\pi}\,, \qquad (14)$$

$$g_{\Xi\Xi\eta} = \frac{1}{2}\sqrt{\frac{3}{2}}\; g_{\Sigma\Sigma\pi} + \frac{3}{2}\, g_{\Sigma\Sigma\eta} + g_{NN\eta} - g_{NN\pi} \qquad (15)$$

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3) C.Dullemond, A.J.Macfarlane and E.C.G.Sudarshan, Phys. Rev.Lett. 10, 423 (1963); V.Singh and V.Guptha, Phys.Rev. 135B, 1442 (1964). C.Bechi, E.Eberle and M.Morpurgo, Phys.Rev. 136B, 808 (1964). M.Konuma and K.Tomozawa, Phys.Lett, 10, 347 (1964).

$$2\, g_{\Xi\Lambda K} = 3\, g_{NN\eta} - g_{NN\pi} + 2\, g_{\Sigma\Lambda\pi} \tag{16}$$

The present knowledge of the coupling constants does not permit a check of these sum rules. However it should be remarked that using forward dispersion relation, Lusignoli et al[4] have recently estimated $N\Lambda K$ and $N\Sigma K$ and found substantial deviation from exact SU(3) predictions.

MASS RELATIONS AND MAGNETIC MOMENTS

The masses of all baryons are expressible in terms of six parameters and therefore no useful prediction is obtained. However, in the case of electromagnetic interactions, (assuming that the electromagnetic current transforms like the T^1_1 component of the octet of SU(3)) the following relatives are obtained among the magnetic moments of baryons

$$\mu_\Lambda = \frac{1}{2}\mu_n \quad , \tag{17}$$

$$\mu_{\Sigma^-} = \mu_{\Xi^-} \quad , \tag{18}$$

$$\mu_{\Sigma^+} + \mu_{\Sigma^-} = 2\mu_{\Sigma^0} \quad , \tag{19}$$

$$\mu_{\Xi^0} = 4\mu_\Lambda + 2\mu_{\Sigma^0} \tag{20}$$

4) M.Lusignoli et al., Phys. Lett. 21, 210 (1966).

The relations (18)-(20) are those predicted by exact SU(3) of which it is well known that the relation (19) follows from just charge independence.

The same set of relations are also obtained for e.m. mass differences.

CONCLUSIONS

The model so far discussed* is essentially different from the models[3)] which introduce the symmetry breaking effects through a linear combination of operators. Such types of breaking the symmetry have the following undesirable features[5)]. The assumption that the mass operator transforms like the IR [8] in the case of SU(3) yields the Gell-Mann-Okubo formula which works well for both the baryons and the mesons. On the otherhand in SU(6), the simplest transformation property of the mass operator as the IR [35] or even a simple linear combination of certain representations, is certainly inadequate, since for the mesons one has to assume some different linear combination of representations. One may argue that similar uncertainty is there in the parameters characterizing the mixing of the representation. However, it is hoped that a critical analysis of various experimental informations may be used

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5) H.R.Rubinstein, Phys. Letts. 22, 210 (1965).

*Recently a similar model has been tried to accommodate the Roper resonance by F.Halzan and M.Konuma, preprint RIEP-69, March, 1968, Kyoto, Japan. I thank Dr.C.Shaw for useful discussion on this point.

to fix these parameters approximately. The method is not altogether strange since we are already familiar with the $\omega - \phi$ mixing[6] and is quite similar to the 'configuration mixing' in nuclear spectroscopy.

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6) J.J.Sakurai, Phys. Rev. Lett. **9**, 472 (1962),
S. Okubo, Physics Letters, **8**, 163 (1963).

CHAPTER IX

P-WAVE NON-LEPTONIC DECAYS OF HYPERONS IN SU(6) AND REPRESENTATION MIXING*

ABSTRACT.

Assuming that the final baryons in the parity conserving non-leptonic hyperon decays belong to the completely antisymmetric representation [20] of SU(6), it is shown that the p-wave amplitude $B(\Sigma^- \to n + \pi^-) = 0$.

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It is well known that the SU(6) theory[1] has been able to explain[2] the parity violating s-wave non-leptonic decays of hyperons and certain relations have been obtained consistent with experiments, one of them being $S(\Sigma^+_+) = 0$. On the other hand, the relations obtained for the parity conserving p-wave decays are not all consistent with experiments. This may be due to the inadequacy of the theory to accommodate the orbital angular momentum. In this note we show that if one assumes that the final baryons in the parity-conserving decays belong to the completely anti-symmetric representation [20], one gets, in addition to the predictions of the $\Delta I = \frac{1}{2}$ rule (which, of course, is built in the theory through octet dominance), the important relation

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*T.S.Santhanam, Physics Letters, 21,234 (1966).

1) F.Gursey and L.A.Radicati, Phys.Rev.Letters 13, (1964) 173, A.Pais, Phys.Rev.Letters 13 (1964) ~~222~~,175, F.Gursey, A.Pais and L.A.Radicati, Phys.Rev.Letters 13 (1964) 299, B.Sakita, Phys.Rev. 136 (1964) B 1756.

2) S.P.Rosen and S.Pakvasa, Phys.Rev.Letters 13 (1964) 773, K.Kawarabayashi, Phys.Rev.Lett. 14 (1965) 86, M.Suzuki, Phys.Lett.14(1965) 64, G.Altarelli, F.Buccella and R.Gatto, Phys.Letts.14 (1966) 70, P.Babu, Phys.Rev.Letters 14 (1965) 166.

$B(\Sigma_-^-) = 0.$

It is assumed in the following that the Hamiltonian for the $p.c$ decayss transforms as a spurion with orbital angular momentum $\ell = 1$. The initial state of the baryon is assumed to belong to pure $[56]$ representation with $\ell = 0$ since it is decaying at rest. On the other hand, the final baryon is assumed to transform as a mixture of both $[56]$ with $\ell = 0$ and $[20]$ with $\ell = 1$ representations. Then, the non-leptonic parity-conserving decay can be described through the interaction (assumpting that the final baryon belongs to the mixed representation $\alpha\,[20] + \beta\,[56]$)

$$\begin{aligned} H_{pc} = {} & a\ \overline{\psi}_{[\alpha\beta(3,m)]}\ \psi^{\{\alpha\beta'(2,m)\}}\ M^{\beta}_{\beta'} \\ & + b\ \overline{\psi}_{\{\alpha\beta(3,m)\}}\ \psi^{\{\alpha\beta\gamma\}}\ M^{(2,m)}_{\gamma} \\ & + c\ \overline{\psi}_{\{\alpha\beta(3,m)\}}\ \psi^{\{\alpha\beta'(2,m)\}}\ M^{\beta}_{\beta'} \\ & + h.c \end{aligned} \tag{1}$$

$$\alpha = (A, i), \quad \beta = (B, j), \quad \text{and} \quad \gamma = (C, k),$$

$$A, B, C = 1, 2, 3, \qquad i, j, k = 1, 2$$

The expansions for the [56] representation $\psi^{\{\alpha\beta\gamma\}}$ and the [20] representation $\psi^{[\alpha\beta\gamma]}$ in terms of their SU(3) x SU(2) contents are given by

$$\psi^{\{\alpha\beta\gamma\}} = \chi^{ijk} d^{ABC} + \frac{\sqrt{2}}{6} \Big[\left(2\,\varepsilon^{ij} \chi^{k} + \varepsilon^{jk} \chi^{i} \right) \varepsilon^{ABD} b^{C}_{D} + \left(\varepsilon^{ij} \chi^{k} + 2\,\varepsilon^{jk} \chi^{i} \right) \varepsilon^{BCD} b^{A}_{D} \Big], \tag{2}$$

$$\psi^{[\alpha\beta\gamma]} = \frac{1}{6}\sqrt{6}\; \chi^{ijk} \varepsilon^{ABC} + \frac{1}{6}\sqrt{6} \left[\varepsilon^{jk} \chi^{i} \varepsilon^{ABD} b^{C}_{D} - \varepsilon^{ij} \chi^{k} \varepsilon^{BCD} b^{A}_{D} \right]$$

Here χ^{i} and χ^{ijk} stand for spin $\frac{1}{2}$ and $\frac{3}{2}$ wave functions respectively, b^{A}_{B} is the baryon octet tensor and d^{ABC} is the decouplet tensor. For the parity-conserving decays

$$M^{\beta}_{\beta'} \sim i\,\sigma^{j}_{j'}\, P^{B}_{B'} . \tag{3}$$

$P^B_{B'}$, is the usual octet of pseudoscalar mesons.

Using the interaction form (1), after somewhat lengthy, but straightforward, calculation, we get

$$
\begin{aligned}
B(\Sigma^+_0) &= \tfrac{1}{2}\sqrt{2}\,(-3a+10b+c),\\
B(\Xi^0_0) &= \tfrac{1}{3}\sqrt{3}\,(2a-9b+3c),\\
B(\Lambda^0_0) &= -\tfrac{1}{2}\sqrt{3}\,(a-2b+c),\\
B(\Xi^-_-) &= -\tfrac{1}{3}\sqrt{6}\,(2a-9b+3c),\\
B(\Lambda^0_-) &= \tfrac{1}{2}\sqrt{6}\,(a-2b+c),\\
B(\Sigma^-_-) &= -10b,\\
B(\Sigma^+_+) &= (-3a+c),
\end{aligned}
\tag{4}
$$

where the notation is standard.

Now, if one assumes that the final baryon belongs to a pure $[20]$ representation (b = c = 0), (the corresponding Euler wave function of the three quark system has $\ell = 1$), then one obtains in addition to the predictions of $\Delta I = \frac{1}{2}$ rule the following relations without assuming anything else:

$$B(\Sigma^-_-) = 0, \tag{5}$$

$$B(\Sigma^+_0) = -\sqrt{3}\,B(\Lambda^0_-),$$

$$\downarrow$$

$$B \tag{6}$$

$$B(\Xi^-_-) = -\frac{4}{3}\, B(\Lambda^0_-) \tag{7}$$

Apart from sign, the second relation is well satisfied ~~experimentall~~ experimentally[3]. However, the third relation is not.

If one has all the three terms, one gets (of course in addition to the predictions of $\Delta I = \frac{1}{2}$ rule), the following sum rule:

$$11\, B(\Lambda^0_-) + \sqrt{3}\, B(\Sigma^+_0) + 6\, B(\Xi^-_-) = \frac{6}{5}\sqrt{6}\, B(\Sigma^-_-), \tag{8}$$

which is not consistent with experiments. This is due to the inconsistency of Eq.(7) with experiments.

The absence of [56] in the final state can be qualitatively argued as follows. If one associates an Euler wave function with $\ell = 1$, with the completely antisymmetric [20] representation, this could be thought of as being responsible for inducing the $\ell = 1$ spurion behaviour to the p-c Hamiltonian. ~~can be assoc ciated~~ The completely symmetric [56] representation can be associated with $\ell = 0$ so that it cannot contribute to the p-c decays.

One point has to be emphasized, that the [20] representation is not the one with $J = \frac{3}{2}^-$ but rather with $S = \frac{1}{2}^+$, $\ell = 1$ where ℓ is the intrinsic orbital angular momentum of the three quark wave function.

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3) The experimental information has been taken from R.H.Dalitz, Lecture Notes given at the International School of Physics, 'Enrico Fermi' on Weak Interactions organized by the Italian Physical Society at Varenna in June 1964. We take $B(\Sigma^+_0) = 3.6 \pm 0.35$.

CHAPTER X.

REPRESENTATION MIXING IN STATIC SU(6) AND THE RELATION BETWEEN G_A AND $(D/F)_{AX}$ *

ABSTRACT

Relations between G_A and $(D/F)_{AX}$ are obtained in the frame work of static non-relativistic SU(6) theory using the mixed representations $[56]$ and $[20]$ (with $\ell = 0$ and $\ell = 1$ for the baryons).

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Recently GATTO et. al[1] have obtained a relation between $(D/F)_{AX}$ and G_A using the algebra of chiral $U(3) \times U(3)$ currents of Gell-Mann[2] which is in excellent agreement with experiments. More recently, Harari[3] has come out with the idea of representation mixing and has been able to reproduce many interesting results in a much simpler way. In this paper, we show that in the non-relativistic static SU(6) theory[4], one can deduce similar relations if one believes in representation mixing.

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*T.S.Santhanam, I.C.T.P., preprint. I.C/66/33 (unpublished).

1. R.Gatto, L.Maiani and G.Preparata, Phys.Rev.Letts.16, 377 (1966).
2. M.Gell-Mann, Physics 1, 63 (1964).
3. H.Harari (preprint). After the completion of this work, we became aware of the preprint by N.Cabibbo and H.Ruegg (CERN preprint) where they have got similar conclusions in the framework of SU(3)x SU(3) chiral algebra.
4. F.Gursey and L.A.Radicati, Phys.Rev.Letts. 13, 173 (1964), A.Pais, Phys.Rev.Letts. 13, 175 (1964), F.Gursey, A.Pais and L.A.Radicati, Phys. Rev. Letts. 13, 299 (1964), B.Sakita, Phys.Rev. 136, B 1756 (1964).

Let us assume that the baryons belong to the representation $[56']$ where

$$[56'] = \alpha\,[56] + \beta\,[20]$$

$$\text{with } \alpha^2 + \beta^2 = 1 \tag{1}$$

where $[56]$ and $[20]$ are the completely symmetric and antisymmetric representations respectively of SU(6). We know that in the non-relativistic limit,

$$\bar{\psi}\,\gamma_\mu \gamma_5\,\psi \rightarrow \phi^+ \vec{\sigma}\,\phi, \text{ for space components}$$

$$\rightarrow\ 0 \qquad \text{for the time component}$$

$$\text{and} \quad \bar{\psi}\,\gamma_\mu\,\psi \rightarrow \phi^+ \phi \qquad \text{for the time component}$$

$$\rightarrow\ 0 \qquad \text{for the space components.}$$

$$\mu = 0, 1, 2, 3$$

Here ψ and ϕ stand for four and two component spinors respectively. Evaluating the matrix element

$$M = \langle \alpha[56] + \beta[20] \,|\, -G_A \vec{\sigma} + G_V \,|\, \alpha[56] + \beta[20] \rangle, \tag{2}$$

using the usual SU(3) x SU(2) expansions for the $[56]$ and $[20]$ representations, assuming that $(-G_A\vec{\sigma} + G_V)$ transform like the $[35]$ representation, one finds

$$M = -G_A \left\{ Tr(\bar{B}BP)\left(\frac{8}{3}\alpha^2 - 1\right) + Tr(\bar{B}PB)\left(\frac{4}{3}\alpha^2 - 1\right)\right\} + G_V \left\{ \frac{1}{3} Tr(\bar{B}BP - \bar{B}PB)\right\} \tag{3}$$

where B stands for the octet of baryons. As a consequence, we find

$$-G_A = \frac{D + F}{3F - D} \tag{4}$$

Suppose now that the $[20]$ representation has $\ell = 1$ so that in the expansion $(\overline{20}, 20)$ the roles of $\vec{\sigma}$ and 1 are interchanged (as is usually done for the p-wave pseudoscalar meson Yukawa coupling in static SU(6) theory). In this case we get

$$M = -G_A \left\{ Tr(\bar{B}BP)\left(2\alpha^2 - \frac{1}{3}\right) + \frac{1}{3} Tr(\bar{B}PB)\right\} + G_V \left\{ Tr(\bar{B}BP)\left(1 - \frac{2}{3}\alpha^2\right) + Tr(\bar{B}PB)\left(1 - \frac{4}{3}\alpha^2\right)\right\}, \tag{5}$$

so that one finds immediately

$$-G_A = \frac{D + F}{3(D - F)} \tag{6}$$

Relation (6) has been obtained by GATTO et. al[1] by saturating chiral $U(3) \times U(3)$ algebra with $[56]$ representation $(\ell = 0)$ and $[20]$ representation $\ell = 1$. Of course, the discussion of orbital angular momentum ~~is~~ in non-relativistic SU(6) does not carry much sense. Subsequently several people have repeated the same calculation in $SU(6)_W$ model and they also get Eqs. (4) and (6)..

CHAPTER XI

RADIATIVE DECAYS OF MESONS IN HIGHER SYMMETRY MODELS.*

ABSTRACT.

The prediction of the groups $SU(3) \times SU(3)$ collinear and $SU(6)_W$ on the radiative decays of mesons in presented.

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The study of the radiative decays of mesons is the most interesting case for comparison with experiments, since many decays are energetically possible, and slowly experimental information is getting available. It is well known that for a real photon only magnetic transitions are allowed. Hence, there is only one form factor $G_M(q^2)$. In addition, $SU(6)_W$ relates $P^{(1)}$ to $P^{(8)}$ where $P^{(1)}$ and $P^{(8)}$ are the singlet and octet of pseudoscalar mesons. Without further knowledge of $G_M(q^2)$, we assume it to vary slowly with q^2 using, however, a phase space factor calculated with the physical masses. For the decays $V \to P + \gamma$ or $P \to V + \gamma$ we start with the interaction

$$\varepsilon^{\alpha\beta\gamma\delta} \, \partial_\alpha A_\beta \, \partial_\gamma V_\delta \, P ,$$

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* H. Ruegg, W. Rühl and T. S. Santhanam, Helv. Phys. Acta, 40, 9 (1967).

where V denotes the vector meson, A the photon field and P the pseudoscalar meson.

In momentum space, with the momentum $\vec{q}$ of the photon along say the third axis, one typical term will be

$$G_M(q^2) \; q_3 \; M_V \cdot \varepsilon^{3102} \cdot A_1 V_2 P .$$

Assuming the symmetry relations for $G_M(q^2)$, one gets a phase space factor proportional to q^3.

Considering the decay of a vector monet into a pseudoscalar nonet and a photon, SU(3) alone (with C invariance) describes these decays in terms of three coupling constants[1).]

$$g_{88}, \quad g_{81}, \quad g_{18} .$$

The predictions of $S[U(3)\times U(3)]$ collinear and $SU(6)_W$ are

$$g_{18} = \sqrt{2}\; g_{88} \qquad S\;[U(3)\times U(3)]_{coll} ,$$

$$g_{18} = \sqrt{2}\; g_{88} = g_{81} \qquad SU(6)_W$$

In the following table, the predictions of SU(3), $S\;[U(3)\times U(3)]$, and $SU(6)_W$ are given.

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1) M.Gourdin, Unitary Symmetry, North-Holland Publishing Company, Amsterdam.
See Also S.Okubo, Lectures on Unitary Symmetry, University of Rochester report 1963.

Remarks. Column 6: The values in $\lambda = -0.643$ and $\sin\alpha = \pm 0.183$ were used. These have been determined using the quadratic mass formula. This choice minimises $\varphi \rightarrow \pi^0\gamma$.

Column 7: To give a rough idea for the order of magnitude, the input $\Gamma(\omega\pi^0\gamma) = 1.2$ MeV was used, although this is only an upper limit with large errors (10%). The five first decay widths are then calculated using $U(3) \otimes U(3)$, the others using $SU(6)_W$, and multiplying with q^3.

Column 8: The values for Γ are taken from Rosenfeld et al., UCRL - 8030 (Rev.1.10.1965). The widths of $X_0 = \eta'$ is unknown, but the values of the Table can be used to give a lower limit to it. With

$$\frac{\Gamma(X_0 \rightarrow \rho\gamma)}{\Gamma(X_0 \rightarrow all)} < \frac{1}{4} ,$$

one gets

$$\Gamma_{X_0} > 0.24 \text{ MeV}$$

or

$$\Gamma_{X_0} > 0.40 \text{ MeV}$$

according to the two possible solutions. This is a prediction of $SU(6)_W$ only.

As is apparent from the Table, three decays are particularly well suited for comparison with experiment, because q^3 has nearly the same value.

From collinear $S\left[U(3) \times U(3)\right]$ one gets:

$$\frac{\Gamma(\rho\pi\gamma)}{\Gamma(\omega\pi\gamma)} = 0.104$$

In addition, $SU(6)_W$ predicts:

$$\frac{\Gamma(\varphi\eta\gamma)}{\Gamma(\omega\pi\gamma)} = 0.203 \quad \text{or} \quad 0.33 ,$$

where the two solutions refer to the two possible signs of sin α. From $S\left[U(3) \otimes U(3)\right]$ one gets the relation for amplitudes, corrected for phase space

$$0.284 \; |\omega\eta\gamma| = 0.093 \; |\varphi\eta\gamma| + 0.06 \; |\omega\pi\gamma|$$

Only upper limits are known for the experimental quantities involved.

Finally we remark that Becchi and Morpurgo[2] calculate the absolute rate of $\omega \longrightarrow \pi\gamma$, using a quark model and a quark magnetic moment deduced from the proton magnetic moment. They find

$$\Gamma(\omega\pi\gamma) = 1.17 \text{ MeV},$$

in agreement with experiment. They also get the other results of the Table, although the physical assumptions of their quark model are different from $SU(6)_W$.

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2) C.Becchi and G.Morpurgo, Phys.Rev. 140 B, 687 (1965)

TABLE

Predictions of SU(3), S [U(3) X U(3)] and $SU(6)_W$ for the decays $M_1 \rightarrow M_2\gamma$

Transition	g_{88}	g_{18}	g_{81}	$q^3 \times 10^{-6}$	g^2	Γ_{if} (MeV)	Γ_{if}/Γ
$\rho^+\pi^-\gamma$	1			50.66	1	0.12	10^{-3}
$\rho^0\pi^0\gamma$	1			50.90	1	0.12	10^{-3}
$K^{*+}K^-\gamma$	1			29.37	1	0.07	1.5×10^{-3}
$K^{*0}\bar{K}^0\gamma$	-2			28.73	4	0.28	6.0×10^{-3}
					From S [U(3) ⊗ U(3)] $\sqrt{2}\, g_{88} = g_{18}$		
$\varphi\pi^0\gamma$	$\sqrt{3}\cos\lambda$	$\sqrt{3}\sin\lambda$		125.75	0.06	0.02	1.5×10^{-2}
$\omega\pi^0\gamma$	$-\sqrt{3}\cos\lambda$	$\sqrt{3}\cos\lambda$		54.88	8.94	1.2 = Input	10^{-1}
					From $SU(6)_W$ $2\, g_{88} = g_{18} = g_{81}$		
$\rho^0\eta\gamma$	$\sqrt{3}\cos\alpha$		$\sqrt{3}\sin\alpha$	6.40	4.63 1.57	0.073 0.025	6×10^{-4} 2×10^{-4}
$\varphi\eta\gamma$	$-\cos\lambda\cos\alpha$	$\sin\lambda\cos\alpha$	$\cos\lambda\sin\alpha$	47.44	2.10 3.40	0.24 0.40	7.3×10^{-2} 12.1×10^{-2}
$\omega\eta\gamma$	$\sin\lambda\cos\alpha$	$\cos\lambda\cos\alpha$	$-\sin\lambda\sin\alpha$	7.92	0.36 0.07	0.007 0.002	6×10^{-4} 1.7×10^{-4}
$X^0\rho^0\gamma$	$-\sqrt{3}\sin\alpha$		$\sqrt{3}\cos\alpha$	5.33	4.38 7.43	0.057 0.096	
$\varphi X^0\gamma$	$\cos\lambda\sin\alpha$	$-\sin\lambda\sin\alpha$	$\cos\lambda\cos\alpha$	0.20	1.88 0.57	0.0009 0.0003	3×10^{-4} 10^{-4}
$X^0\omega\gamma$	$-\sin\lambda\sin\alpha$	$-\cos\lambda\sin\alpha$	$-\sin\lambda\cos\alpha$	4.10	0.66 0.95	0.007 0.009	

C H A P T E R XII

CURRENT ALGEBRA FROM EIGHT DIMENSIONAL FIELDS*

ABSTRACT

The algebras formed by the integrated currents constructed out of unrenormalized Heisenberg fields of strongly interacting particles are discussed.

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INTRODUCTION.

Following the suggestion of Gell-Mann[1] that the algebra generated by current operators can provide a useful tool in understanding the symmetries of strongly interacting particles, there have been a large number of investigations[2] in this field. In all these investigations, one starts with a set of current densities constructed from the fundamental (quark) fields of a symmetry group, whose space integrals close among themselves under equal time commutation (ETC), thereby forming an algebra. The assersion is that once the algebra is formed, we can forget the way by which we obtained them. In fact, it is claimed that we could have straightaway postulated this algebra as a model. In what follows, we attempt to see that if we start with eight fields instead of the three quark fields where do we end ?

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*P.Narayanasamy, T.Pradhan and T.S.Santhanam, ICTP preprint 1966 (unpublished).

1. M.Gell-Mann, Phys.Rev.**125**, 1067 (1962), Physics **1**, 63 (1964).
2. S.Fubini and G.Furlan, Physics **1**, 228 (1965), S.L.Adler, Phys.Rev.Letts. **14**, 1051 (1965), W.I.Weisberger, Phys.Rev.Letts. **14**, 1047 (1965), B.W.Lee, Phys.Rev.Letts. **14**, 673 (1965) and others.

2. ALGEBRA OF VECTOR CURRENTS.

We first consider the set of eight vector currents

$$V_\mu^i(t) = \sum_{r,s=1}^{8} \int d^3x \, \overline{\psi}_r(\vec{x},t) \, K_{rs}^i \, \gamma_\mu \, \psi_s(\vec{x},t),$$

$$i = 1, \ldots\ldots, 8 \tag{1}$$

constructed out of the eight known baryon fields*

$\psi_1 = \Sigma^+$, $\psi_2 = \Sigma^-$, $\psi_3 = \Sigma^0$, $\psi_4 = p$, $\psi_5 = n$, $\psi_6 = \Xi^0$, $\psi_7 = \Xi^-$

and $\psi_8 = \Lambda$, such that they have the (I, Y) quantum numbers of the mesons $\pi^+, \pi^-, \pi^0, K^+, K^0, \bar{K}^0, K^-$ and η respectively. The K_{rs}^i are arbitrary numbers. By using the equal time anticommutation rule**

$$\left\{ \psi_r^\dagger(\vec{x},t), \psi_s(\vec{x}',t) \right\} = \delta_{rs} \, \delta(\vec{x}-\vec{x}'), \tag{2}$$

---

*For our purpose these baryon fields along with the pseudoscalar and scalar fields to be discussed later in this paper can be taken as fundamental fields.

** Actually we need only the weaker relation, for our purposes

$$\left[\psi_r^\dagger \psi_s, \psi_t^\dagger \psi_u\right] = \delta_{st} \, \psi_r^\dagger \psi_u - \delta_{ru} \, \psi_t^\dagger \psi_s .$$

for the Heisenberg field operators of the baryons, it is easy to show that the time components of V^i_μ close among themselves, without any condition on baryon masses, under ETC* provided the matrices K^i obey the commutation relations

$$[K^i, K^j] = -K^i_{jk} K^k , \tag{3}$$

where the non-zero K^i_{jk} are given in Table 1 and happen to be identical to the matrix elements of the canonical form[4)] of the F-matrices used in the SU(3) symmetry of Gell-Mann and Ne'eman[5)]. Our currents, therefore, form an algebra isomorphic to SU(3). These currents, however, are not quite general, none of them contains terms bilinear in Σ and Λ. One cannot therefore use these currents to describe electromagnetic processes such as $\Sigma^0 \rightarrow \Lambda^0 + \gamma$.

With eight baryons it is not possible to incorporate such terms in the currents without unduly enlarging the algebra**. On the other hand, the situation is different if one starts with nine baryons.

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* We neglect the derivatives of delta functions, which, strictly speaking, should occur in these commutation relations. For details see reference (3).

** One can have, for example, an algebra isomorphic to U(8) with 64 currents constructed out of the eight baryon fields. These currents will certainly have terms bilinear in Σ and Λ.

3. J.Schwinger, Phys.Rev.Letters 3, 296 (1959), T.Pradhan, Nucl.Phys. 9, 124 (1958).
4. P.Tarjanne, Ann. Acad.Sci., Fennicae, Series A VI, Physica 105 (1962).

The time components of the vector currents

$$V^{i}_{\mu\pm}{}^{\xi} = \sum_{r,s=1}^{9} \int d^3x \, \psi_r^{\dagger}(\vec{x},t)\, M^{i(\pm)}_{rs} \gamma_0 \gamma_\mu \psi_s(\vec{x},t)$$

$$i = 1, 2, \ldots, 8 \tag{4}$$

constructed out of the Heisenberg fields of these nine baryons close among themselves under ETC provided

$$M^{i(\pm)}_{rs} = \frac{1}{2}\left(K^{i}_{rs} \pm L^{i}_{rs} \right)$$

$$r, s = 1, 2, \ldots, 9 \tag{5}$$

where the matrices L^i obey the commutation relations

$$[L^i, L^j] = -K^i_{jk} K^k,$$

with $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (6)

~~with~~
~~The non-zero~~ with

$$K^i_{j9} = K^i_{9j} = 0$$

The non-zero matrix elements of L^i are given in Table 2. It will be seen that terms bilinear in Σ and Λ appear in these currents The algebra is isomorphic to SU(3) x SU(3) (non-chiral)*. The baryon

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*A quark model based on this group has been considered by J. Schwinger For details, see ref.(6). It has also been discussed by A. Salam and J.C. Ward in the context of double gauge groups. For details see ref.(7).

6) J. Schwinger, Phys. Rev. Letts. 12, 237 (1964).
7) A. Salam and J.C. Ward, Phys. Rev. 136, B 763 (1964).

The baryon $Y_0^*(1405)$ can be taken as the ninth baryon.

It will be noticed that the algebra is determined by the number of baryons used. In general, with n baryons, currents have to be constructed using $(n \times n)$ matrices. Since the complete set of n^2 of these $(n \times n)$ matrices form the U(n) algebra, the currents will also form U(n) algebra. Since such an algebra is very large, one has to look for the smallest sub-algebra of $(n \times n)$ matrices which can account for all physical processes. The number of baryons needed to start with cannot be decided from any principle. One has to take into consideration the simplicity of approach with no loss of generality that gives results consistent with observed data. Our entire approach to current algebra from this point of view ~~wi~~ is phenomenological. Quark current algebra is also phenomenological to the same extent because one can have various models with varying numbers of quarks. One uses only those that are compatible with observed facts.

So far we have considered currents constructed out of baryon fields. The mesons also contribute to these currents. In fact, their contribution would be necessary if we want some of these currents to be conserved, or if we want to account for their weak decays and electromagnetic interactions. With nine baryons and 9 observed pseudoscalar mesons (X^0 being the ninth one) we can have SU(3) x SU(3) non-chiral algebra whose elements are

$$V^{i(\pm)}(t) = \sum_{r,s=1}^{9} \int d^3x \left(\psi_r^{+} M_{rs}^{i(\pm)} \psi_s + i\, \phi_r^{+} M_{rs}^{i(\pm)} \partial_0 \phi_s \right).$$

(7)

3. ALGEBRA OF VECTOR AND AXIAL VECTOR CURRENTS.

The time components of the axial vector currents of the baryon (we shall hereafter consider the case of nine baryons only) fields

$$A_k^{i}(t) = \sum_{r,s=1}^{9} \int d^3x\ \psi_r^{+} \gamma_5 K_{rs}^{i} \psi_s ,$$

$$i = 1, \cdots, 8$$

(8)

do not close among themselves, However, along with the time components of the vector currents

$$V^{i}(t) = \sum_{r,s=1}^{9} \int d^3x\ \psi_r^{+} K_{rs}^{i} \psi_s ,$$

(9)

they form the chiral SU(3) x SU(3) algebra. This algebra suffers from the same drawback as that of the vector SU(3) algebra, i.e., the currents do not contain terms bilinear in Σ and Λ and cannot therefore account for weak decays of the type $\Sigma^- \to \Lambda e \bar{\nu}$.

For applications to weak interactions one needs the bigger algebra $SU(3) \times SU(3) \times SU(3) \times SU(3)$* formed by the currents

$$A^{i(\pm)}(t) = \sum_{r,s=1}^{9} \int d^3x \; \psi_r^\dagger(\vec{x},t)\gamma_5 \left(K^i_{rs} \pm L^i_{rs}\right) \psi_s(\vec{x},t),$$

(10a)

$$V^{i(\pm)}(t) = \sum_{r,s=1}^{9} \int d^3x \; \psi_r^\dagger(\vec{x},t) \left(K^i_{rs} \pm L^i_{rs}\right) \psi_s(\vec{x},t).$$

(10b)

As for meson contributions to these currents, we encounter some difficulties. It is not possible to construct axial vector currents that are bilinear in pseudoscalar fields. Although one can construct such currents from trilinear products of pseudoscalr fields, the commutation relations become very complex. On the other hand, if we introduce a nonet of scalar mesons, (the so-called σ mesons[9]),

*The corresponding symmetry group has been discussed by Y.Nambu and P.Freund (ref.(8)).

8) P.G.O.Freund and Y.Nambu, Phys.Rev.Letters 25, 714 (1964).

9) M.Gell-Mann and M.Levy, Nuovo Cimento 16, 705 (1960).

there is absolutely no problem. The currents of the SU(3) x SU(3) x SU(3) x SU(3) algebra along with the meson contributions are given by

$$A^{i(\pm)}(t) = \sum_{r,s=1}^{9} \int d^3x \left(\psi_r^{+} \gamma_5 M_{rs}^{i(\pm)} \psi_s + i \sigma_r^{+} M_{rs}^{i(\pm)} \partial_0 \phi_s + i \phi_r^{+} M_{rs}^{i(\pm)} \partial_0 \sigma_s \right) = -i \int d^3x \; a^{i(\pm)}(\vec{x},t) , \qquad (11)$$

$$V^{i(\pm)}(t) = \sum_{r,s=1}^{9} \int d^3x \left(\psi_r^{+} M_{rs}^{i(\pm)} \psi_s + i \phi_r^{+} M_{rs}^{i(\pm)} \partial_0 \phi_s + i \sigma_r^{+} M_{rs}^{i(\pm)} \partial_0 \sigma_s \right), \qquad (12)$$

where

$$M^{i(\pm)} = K^{i} \pm L^{i} .$$

Besides this $[SU(3)]^4$ algebra, one can have the following algebras with vector and axial vector currents

1. SU(3) x SU(3) with currents A_L^i and V_K^i ,
2. SU(3) x SU(3) x SU(3) with currents V_K^i, V_L^i and $A^{i(+)}$
3. SU(3) x SU(3) x SU(3) with currents V_K^i, V_L^i and $A^{j(-)}$

All these have been discussed by Nambu and Freund[8] in the context of symmetry groups.

TABLE 1.

$K_{23}^{(1)} = -K_{31}^{(1)} = 2$ $K_{54}^{(1)} = K_{76}^{(1)} = \sqrt{2}$	$K_{32}^{(2)} = 2$ $K_{45}^{(2)} = \sqrt{2}$
$K_{84}^{(4)} = -K_{78}^{(4)} = -\sqrt{3}$ $K_{34}^{(4)} = -K_{73}^{(4)} = -1$ $K_{25}^{(4)} = -K_{62}^{(4)} = -\sqrt{2}$	$K_{48}^{(7)} = -K_{87}^{(7)} = -\sqrt{3}$ $K_{43}^{(7)} = -K_{37}^{(7)} = -1$ $K_{52}^{(7)} = -K_{26}^{(7)} = -\sqrt{2}$
$K_{85}^{(5)} = -K_{68}^{(5)} = -\sqrt{3}$ $K_{35}^{(5)} = -K_{63}^{(5)} = 1$ $K_{14}^{(5)} = -K_{72}^{(5)} = -\sqrt{2}$	$K_{58}^{(6)} = -K_{86}^{(6)} = -\sqrt{3}$ $K_{53}^{(6)} = -K_{36}^{(6)} = 1$ $K_{41}^{(6)} = -K_{27}^{(6)} = -1$
$K_{44}^{(3)} = -K_{55}^{(3)} = K_{66}^{(3)} = K_{77}^{(3)} = 1$ $K_{11}^{(3)} = -K_{22}^{(3)} = 2$	$K_{44}^{(8)} = K_{55}^{(8)} = \sqrt{3}$ $K_{66}^{(8)} = K_{77}^{(8)} = -\sqrt{3}$

<u>Table 2</u>

$L^{(1)}_{54} = L^{(1)}_{76} = \sqrt{2}$	$L^{(2)}_{45} = -L^{(2)}_{76} = \sqrt{2}$
$L^{(1)}_{28} = L^{(1)}_{81} = \sqrt{\frac{4}{3}}$	$L^{(2)}_{82} = L^{(2)}_{18} = \sqrt{\frac{4}{3}}$
$L^{(1)}_{29} = L^{(1)}_{91} = \sqrt{\frac{8}{3}}$	$L^{(2)}_{92} = L^{(2)}_{81} = \sqrt{\frac{8}{3}}$
$L^{(4)}_{84} = L^{(4)}_{78} = -\sqrt{\frac{1}{3}}$	$L^{(7)}_{48} = L^{(7)}_{87} = -\sqrt{\frac{1}{3}}$
$L^{(4)}_{34} = L^{(4)}_{73} = 1$	$L^{(7)}_{43} = L^{(7)}_{37} = 1$
$L^{(4)}_{26} = L^{(4)}_{61} = \sqrt{2}$	$L^{(7)}_{43} = L^{(7)}_{16} = \sqrt{2}$
$L^{(4)}_{91} = L^{(4)}_{79} = -\sqrt{\frac{8}{3}}$	$L^{(7)}_{19} = L^{(7)}_{97} = -\sqrt{\frac{8}{3}}$
$L^{(5)}_{85} = L^{(5)}_{68} = -\sqrt{\frac{1}{3}}$	$L^{(6)}_{58} = L^{(6)}_{86} = -\sqrt{\frac{1}{3}}$
$L^{(5)}_{63} = L^{(5)}_{35} = -1$	$L^{(6)}_{36} = L^{(6)}_{53} = -1$
$L^{(5)}_{14} = L^{(5)}_{72} = \sqrt{2}$	$L^{(6)}_{41} = L^{(6)}_{27} = \sqrt{2}$
$L^{(5)}_{92} = L^{(5)}_{69} = -\sqrt{\frac{8}{3}}$	$L^{(6)}_{29} = L^{(6)}_{96} = -\sqrt{\frac{8}{3}}$

Table 2 (contd.)

$L^{(3)}_{44} = -L^{(3)}_{55} = 1$	$L^{(8)}_{44} = L^{(8)}_{55} = -\sqrt{\frac{1}{3}}$
$L^{(3)}_{66} = -L^{(3)}_{77} = -1$	$L^{(8)}_{66} = L^{(8)}_{77} = -\sqrt{\frac{1}{3}}$
$L^{(3)}_{38} = -L^{(3)}_{83} = \sqrt{\frac{4}{3}}$	$L^{(8)}_{11} = L^{(8)}_{22} = \sqrt{\frac{4}{3}}$
$L^{(3)}_{39} = L^{(3)}_{93} = \sqrt{\frac{8}{3}}$	$L^{(8)}_{88} = -\sqrt{\frac{4}{3}}$
	$L^{(8)}_{89} = L^{(8)}_{98} = -\sqrt{\frac{16}{3}}$

CHAPTER XIII

INFINITE MOMENTUM LIMIT AND THE ALGEBRA OF CURRENTS*

ABSTRACT

The importance of infinite momentum limit to obtain covariant results using the algebra of currents is studied. It is shown that unless one goes beyond the single particle approximation, it is very difficult to envisage an exponential behaviour of the electromagnetic form factors.

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Recently, there have been some attempts[1),2)] to find fomal solutions of the commutator algebra of the fourier transform of the current densities using the $p_z \to \infty$ technique. In particular, Barnes and Kazes[2)] have considered the algebra of vector current densities (actually their Fourier Transform) with SU(2) as the internal symmetry algebra. By approximating the matrix elements of the commutator with single particle diagonal matrix elements and using the $p_z \longrightarrow \infty$ limit, they obtain the isovector ~~s~~ form factors at finite momenta. The purpose of this paper ~~Letter~~ is to analyse their results in an arbitrary Lorentz frame. The conclusions are that the Cabibbo-Radicati sum rule[3)] is

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*T.S.Santhanam A.Sundaram and K.Venkatesan (Preprint) 1967.

1) R.Dashen and M.Gell-Mann, Phys.Rev.Lett. 17, 340 (1966).
2) K.J.Barnes and E.Kazes, Phys.Rev.Lett., 17, 978 (1966).
3) N.Cabibbo and L.A.Radicati, Phys.Lett., 19, 697 (1966).
4) C.G.Bollini and J.J.Giambiagi, Nuovo Cim. 21, 107 (1961).
I.Saavedra, Nucl. Phys. 74, 677 (1965).

independent of the Lorentz frame one uses and one always gets only a harmonic q^2-dependence for the form factors, no matter what momentum limit one employs so long as we limit ourselves to <u>finite</u> number of diagonal single particle matrix elements. The only way ~~perha~~ perhaps to get a more realistic exponential q^2-dependence is to go beyond the single particle matrix elements or to take infinite superposition of single particle matrix elements.

Following Barnes and Kazes, we consider the one dimensional commutator algebra of vector currents

$$\left[V^+(q'), V^-(q)\right] = 2V^3\,(q+q'), \tag{1}$$

$$\left[V^3(q'), V^\pm(q)\right] = V^\pm\,(q+q'), \tag{2}$$

where

$$V_i(q) = \int e^{iqx}\, V_{i_0}(x)\, dx \tag{3}$$

The V_{i_0} are the vector current densities (fourth component) and i refers to the ~~i~~isospin index. Differentiating Eq. (2) with respect to q' and taking the limit $q' \to 0$, we get the Heisenberg type equation

$$\left[V^{3\prime}(0), V^\pm(q)\right] = \pm V^{\pm\prime}(q). \tag{4}$$

with the well known iterated solution

$$V^{\pm}(q) = e^{\pm q\, V^{3\prime}(0)}\; V^{\pm}(0)\; e^{\mp q\, V^{3\prime}(0)} . \tag{5}$$

This is the relation one gets using the algebra.

Consider the following matrix element of $V_i\,(x=0)$ between the nulceon states which form the approximate representation of the algebra

$$\begin{aligned} M_i &= \langle p+q \,|\, V_i(x=0) \,|\, p\rangle \\ &= \bar{v}\,(p+q) \left\{ F_1^{v}(q^2)\,\gamma_0 + \frac{i\,F_2^{v}}{2m}(q^2)\,\sigma_{0x}\,q_x \right\} \tau_i\, v(p) \end{aligned} \tag{6}$$

By using the Foldy-Wouthuysen transformation one can write Eq.(6) as

$$M_i = U^{+}(0)\; L\,(p+q) \left\{ F_1^{v}(q^2)\,\gamma_0 + \frac{F_2^{v}(q^2)}{2m}\,\gamma_x\,q_x \right\} L^{-1}(p)\,\tau_i\, v(0), \tag{7}$$

where

$$L\,(p+q) = \exp\; \tfrac{1}{2}\,\theta\,\gamma_x ,$$

and

$$L^{-1}(p) = \exp\; -\tfrac{1}{2}\,\theta'\,\gamma_x ,$$

with

$$\theta = \operatorname{arc\,tg} \frac{|p+q|}{m} ,$$

and $$\theta' = \text{arc tg} \frac{|p|}{m} .$$

The θ is related to the angle appearing in the Lorentz transformation

$$\mathcal{L}(p) = e^{\frac{1}{2}\omega \frac{\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}} ,$$

$$\omega = \text{arc tgh} \frac{|\vec{p}|}{E(\vec{p})} ,$$

as

$$\frac{1}{2}\theta = \text{arc tg} \left(\text{tgh} \frac{1}{2}\omega \right)$$

(see Ref. (4) for instance).. Hence $V_i(q)$ may be represented by[5)]

$$V^{\pm,3}(q) = \left\{ F_1^v(q^2) + \frac{F_2^v(q^2)}{2m} \gamma_\alpha q_\alpha \right\} e^{\frac{1}{2}(\theta - \theta')\gamma_\alpha} \frac{\tau^{\pm,3}}{2} , \tag{8}$$

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4) C.G.Bollini and J.J.Giambiagi, Nuovo Cimento 21, 107 (1961) I.Saavedra, Nucl. Phys., 74, 677 (1965.

5) It is clear that V_i is a function of ~~only~~ q only when $p = o$ or $p \to \infty$. However if we look at the first few derivatives, it becomes immediately obvious that V's are functions of ~~only~~ q^2 only when we go to the frame $p \to \infty$.

so that

$$V^{\pm}(0) = F_1^{v}(0)\,\frac{\tau^{\pm}}{2}\,,$$

$$V^{3'}(0) = \frac{F_2^{v}(0)}{2m}\,\frac{\tau^{3}}{2}\,. \tag{9}$$

Substituting Eq.(9) in Eq.(5) one easily finds[6)]

$$F_1^{v}(q^2) = F_1^{v}(0)\cos\left\{\frac{\sqrt{-q^2}}{2m}F_2(0) - \frac{1}{2}(\theta-\theta')\right\},$$

$$F_2^{v}(q^2) = F_1^{v}(0)\,\frac{2m}{\sqrt{-q^2}}\,\sin\left\{\frac{\sqrt{-q^2}}{2m}F_2(0) - \frac{1}{2}(\theta-\theta')\right\}, \tag{10}$$

---

6) The presence of p dependent term $\theta - \theta'$ in the form factors is essentially due to the single particle approximation we use. For example, the matrix element $\langle p | V_i V_j | p' \rangle$ consists of two Feynman diagrams which we have approximated by a single Feynman diagram. When we take the infinite momentum limit, the cross diagram does not contribute so that the p-dependent term $\theta - \theta'$ naturally drops out from the form factors. We thank Professor Zaccariasen for discussions on this point.

which as can be verified goes to the Barnes-Kazes form for $p \to \infty$. From (10) one can get

$$F_1^{v^2}(q^2) - \frac{q^2}{4m^2} F_2^{v^2}(q^2) = 1 \tag{11}$$

Actually Eq.(11) is independent of any momentum limit. From Eq.(11) by reexpressing in terms of the Sachs form factors and differentiating with respect to q^2 one can get the Cabibbo-Radiacati sum rule without the continuum term.

One can easily see that for small values of p and q the zero of $F_1^{v}(q^2)$ is pushed upto $q^2 = 30.7\ f^{-2}$ from around $15\ f^{-2}$. However, the zero of $F_2^{v}(q^2)$ is brought down. A detailed comparison is not worthwhile, since in any case one can get only sine or cosine forms or perhaps Bessel functions. To get an exponential form one has to think for an infinite superposition which can be done perhaps if we include many particle intermediate states, a neat way of doing this is yet unclear.

CHAPTER XIV

STUECKELBERG FIELDS AND CURRENT ALGEBRA SUM RULES*

ABSTRACT

Stueckelberg formulation of vector meson fields is applied formally to explain in a natural way the $A_1 - \pi$ mixing and several consequences of this model are studied.

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One of the most important sum rules obtained from current algebra is the so-called Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (K.S.R.F.) relation[1)]

$$F_\rho^2 = 2 m_\rho^2 F_\pi^2 , \tag{1}$$

where F_ρ and F_π are the coupling constants for the decays $\rho \to vac$ and $\pi \to vac$ respectively. Doubts have been raised on the methods used in deriving this relation[2)]. This sum rule along with Weinberg's first sum rule[3)] in fact predicts the famous relation $m_{A_1} = \sqrt{2}\, m_\rho$. In this ~~letter~~ paper, we give a plausible derivation of the equation (1) using Stueckelberg fields[4)].

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* T.S.Santhanam, Nuovo Cimento, 57A, 440(1968).

1) K.Kawarabayashi and M.Suzuki, Phys.Rev.Lett. 16, 255 (1966).
2) Riazuddin and Fayyazuddin, Phys.Rev. 147, 1071 (1966).
2) S.Okubo, Lecture notes on Asymptotic Symmetry and Algebras of Currents, Ravalpindi report.
3) S.Weinberg, Phys. Rev. Lett. 18, 507 (1967).
4) see next page.

We show that not only we need symmetry breaking term in Weinberg's[4] second sum rule, but also in the first sum rule if we have to account for the observed ratiom $\frac{F_K}{F_\pi} = 1.28$ (or 1.26)[5].

We define Stueckelberg vector field U_μ as

$$U_\mu = A_\mu + \frac{1}{x} \partial_\mu B, \tag{2}$$

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4) For a detailed discussion on Stueckelberg fields see S.Kamefuchi, Matscience Report 14. 'The Stueckelberg formalism of vector meson fields' (1963). Recently this fact has also been realized by T.W. Chen and R.E.Pugh, Phys. Rev. Lett. 20, 880 (1968). In using Stueckelberg fields, of course, we have ignored several formal diffeculties that one encounters. Firstly, the A field has a non vanishing divergence and so one has to some how interpret its fourth component. The second difficulty is the fact that both A and B fields obey the same equation of motion and so, it will be a crude approxmation to take $m_{A_1} = m_\pi$. I thank Professors S.Okubo, S.Kamefuchi and E.C.G.Sudarshan for pointing out to me the various problems that arise in using Stueckelberg fields. Nevertheless, this model is attractive in its own right. We believe that this is the best place to study PCAC as the elimination of the B-field has ~~much~~ to do with the conservation of current.

5) N.Brene, M.Ross and A.Sirlin (to be published). The non-renormalization of F_K/F_π when there is no symmetry breaking in the first Weinberg sum rule has been earlier noted in a different context by
C.S.Lai, Phys. Rev. Lett., 20, 509 (1968).
R.J.Oakes , Phys. Rev. Lett., 20, 513 (1968).

where A_μ is the vector field and B is a scalar field. $\varkappa$ is the mass of the vector field. The free field Lagrangian in the Stueckelberg formalism is given by

$$\mathcal{L} = -\frac{1}{2}\left[(\partial_\mu A_\nu)(\partial_\mu A_\nu) + \varkappa^2 A_\nu A_\nu\right] - \frac{1}{2}\left[(\partial_\mu B)(\partial_\mu B) + \varkappa^2 B^2\right], \tag{3}$$

from which follow the free field commutation relations

$$\left[A_\mu(x), A_\nu(x')\right] = i\,\delta_{\mu\nu}\,\Delta\,(x-x'), \tag{4a}$$

and

$$\left[B(x), B(x')\right] = i\,\Delta\,(x-x'). \tag{4b}$$

We shall not discuss here the problem of elimination of B-field and non-conservation of currents. The interaction responsible for leptonic decays of vector (or axial vector) mesons in this formalsim is given by

$$\mathcal{L}_{lep} \sim g\, J^{\mu}_{lep}\, v_\mu, \tag{5}$$

so that we get the following interesting relations

$$F_{A_1} = m_{A_1} F_\pi ,$$

$$F_{K_A} = m_{K_A} F_K ,$$

$$F_\rho = m_\rho F_\sigma ,$$

$$F_{K^*} = m_{K^*} F_k . \tag{6}$$

We have denoted here by σ and k , the non-strange and strange scalar excitations. These, in conjunction with Weinberg's first sum-rule

$$F_\sigma^2 + \frac{F_\rho^2}{m_\rho^2} = F_\pi^2 + \frac{F_{A_1}^2}{m_{A_1}^2} ,$$

$$= F_k^2 + \frac{F_{K^*}^2}{m_{K^*}^2} = F_K^2 + \frac{F_{K_A}^2}{m_{K_A}^2} , \tag{7}$$

yield the following relations

$$F_\rho^2 = (2 F_\pi^2 - F_\sigma^2) m_\rho^2 ,$$

$$F_{K^*}^2 = (2 F_K^2 - F_k^2) m_{K^*}^2 ,$$

$$F_\sigma^2 = F_k^2 = F_\pi^2 = F_K^2 . \tag{8}$$

The first may be recognized to be the K.S.R.F. relation when we ignore the scalar excitation σ . However, this is very difficult in view of the last equality in Eq.(8). This equality can be avoided if we do not have the last two relations in Eq.(6) which is very hard to be understood in view of Eq.(5). Even in this case we always get $\frac{F_K}{F_\pi} = 1$. From these considerations it is clear that unless we introduce symmetry breaking term either in Weinberg's first sum rule or K.S.R.F. type relations we will always end up with $F_K/_{F_\pi} = 1$, even if we introduce a breaking term in Weinberg's second sum rule.

PART IV

CLIFFORD ALGEBRA AND ITS GENERALIZATIONS

CHAPTER XV

ON THE REPRESENTATIONS OF GENERALIZED CLIFFORD ALGEBRA*

Prefatory note: This chapter which has been introduced for the sake of completeness comprises the work done by the group at Matscience of which the author was one of the collaboratoss. It is in pursuance of a programme of establishing the hitherto unobserved connections between the unitary groups and the clifford algebra inititated by Ramakrishnan.

ABSTRACT

This chapter deals with two particular aspects of the programme mentioned above in finding the irreducible representations of the generalized clifford algebra and to establish the connection between the generalized clifford algebra and the self-representation of the unitary groups.

***

Hitherto we have been dealing with the work on unitary groups and their irreducible representations. It was noticed by Ramakrishnan that the non-diagonal Gell-Mann matrices λ have the property $\lambda^3 = \lambda$, while in the case of diagonal matrices, those representing I_z, U_z and V_z also obey the same relation. Since in a different context Ramakrishnan was concerned with matrices which

* Alladi Ramakrishnan, T.S.Santhanam and P.S.Chandrasekharan, to be published in J.Math. and Phys. Sciences, I.I.T., Madras.

satisfy the relation that L^2 is a multiple of unit matrix or more generally A^m = (constant) I, a programme was initiated to investigate the study of possible connections and it turned out that such a connection exists and we here study two particular aspects of the work in which the author collaborated with Ramakrishnan.

It has been shown by Ramakrishnan[1)] that there are $(2\nu+1)$ anticommuting matrices of dimension $2^\nu \times 2^\nu$ satisfying the two clifford ~~ak~~ conditions

$$\text{I} \qquad \mathcal{L}_i \mathcal{L}_j = -\mathcal{L}_j \mathcal{L}_i, \quad i,j = 1,\ldots,2\nu+1,$$

$$\text{II} \qquad \mathcal{L}_i^2 = I \tag{1}$$

If we form p.fold products ($p = 0,1,\ldots 2\nu$) of the $\mathcal{L}$'s, we obtain an aggregate of $2^{2\nu}$ matrices constituting the elements of the clifford algebra C_n ($n = 2\nu$). Out of these 2^n elements, only $(2\nu + 1)$ base elements $\mathcal{L}_i$ satisfy both the clifford conditions I and II while the other elements obey only the second clifford condition of (1). In conformity with the mathematical literature we denote the 2^n elements of C_n by

1) Alladi Ramakrishnan, ~~T.S.Santhanam and P.S.Chandrasekharan~~ J. Math. Anal. Appl. Vol. 20, p. 9-16 (1967).

$$e_1^{\rho_1} e_2^{\rho_2} \ldots\ldots e_n^{\rho_n},$$

$$\rho = 0 \text{ or } 1 \tag{2}$$

It has been pointed out[2)] that there are three methods of generating the $(2\nu + 1)$ base elements which can represented as matrices of dimension $2^{\nu} \times 2^{\nu}$ of C_n, the first being traced to the primary derivation of the α-matrices by Dirac[3),4),5)], the second one due to Ramakrishnan[1)] and the third is due to Rasevskii[6)]. While the first two methods which have been shown to be equivalent[2)], generate the 2ν independent base matrices of C_n of dimension $2^{\nu} \times 2^{\nu}$ from the $2\nu - 2$ (= n-1) independent base matrices of C_{n-1} of dimension $2^{\nu-1} \times 2^{\nu-1}$, in the third method of Rasevskii[6)] the n-independent base elements of C_n are generated as a mapping on vectors constructed out of the complete set of 2^{ν} elements of C_{ν}.

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2) Alladi Ramakrishnan, T.S.Santhanam and P.S.Chandrasekhran, 'L-matrices and the fundamental theorem in spinor theory', to be published in the Symposia in Theoretical Physics and Mathematics, Vol.10, Plenum Press, New York.

3) P.A.M.Dirac, 'Quantum Theory of the Electron', Prog.Roy.Soc., London, (A) 117, 610-624 (1928).

4) R.Brauer and H.Weyl, Spinors in n dimensions, Amer.J.Math., 57, 425-449 (1935).

5) H.Boerner, 'Representations of Groups', North-Holland Publishing Co., Amsterdam, 1963. Chapter VIII, page 268.

6) P.K.Rasevskii, American Mathematical Society Translations, series 2, Vol.6, (1957) 1.

Recently Yamazaki[7] has formulated a new algebra (we call it generalized clifford algebra (C_n^m g,c,a) of m^n elements generated from a set of n basic elements $e_1, \ldots, e_n$ obeying

$$e_i e_j = \omega e_j e_i , \qquad i \neq j = 1, \ldots, n , \quad (i<j)$$
$$e_i^m = I , \tag{3}$$

where ω is ~~the~~ a m^{th} primitive root of unity. The m^n elements of the algebra C_n^m with the multiplication given by Eq.(3) can be shown to form a vector space and are obtained as

$$e_1^{k_1} e_2^{k_2} \cdots e_n^{k_n} , \quad 0 \leq k_1, \ldots, k_n \leq m-1 . \tag{4}$$

Recently A.O.Morris[8] has obtained explicit representations of C_n^m using the method of Brauer and Weyl. In this chapter we attempt to find the irreducible representations of C_n^m using the method of Rasevskii. The stimulus for study of this method as well as the g.c.a. is a direct outcome of the series of papers published by A.Ramakrishnan.

We can write an arbitrary element A of C_n^m as

$$A = a_0 1 + a_{1^{i_1} 2^{i_2} \ldots n^{i_n}} e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n} ,$$
$$0 \leq i_1, \ldots, i_n \leq m-1 . \tag{5}$$

7) K.Yamazaki, J.Fac.Sci.University of Tokyo, Set I, 10 (1964) 147-195, See page 191.
8) A.O.Morris, Quar.J.Math. Oxford (2), 18, 1967, 7-12.

where we have used the summation convention of repeated indices.

We now dévide the m^n elements into m sets as

$$A = A_0 + A_1 + \cdots + A_{m-1} , \tag{6}$$

where A_i contains terms of degree $i \bmod m$, each having m^{n-1} elements.

We now construct after Raseveskii, the representation of the 2n base elements of G.C.A. C^m_{2n} obeying

$$e'_i e'_j = \omega e'_j e'_i , \quad i < j, \ i, j = 1, \ldots 2n ,$$
$$e'^m_i = I .$$

The first n elements of C^m_{2n} are obtained as the mapping

$$A \xrightarrow{\hat{E}_i} A e_i \tag{7}$$

The other n elements are obtained as

$$A \xrightarrow{\hat{E}_{n+i}} \zeta e_i \left[\omega^{m-1} A_0 + \omega^{m-2} A_1 + \cdots + 1 A_{m-1} \right] , \tag{8}$$

where

$$\zeta = 1 \quad \text{for } m \text{ odd}$$
$$= \omega^{\frac{1}{2}} \quad \text{for } m \text{ even.}$$

The affinors $\hat{E}_1, \ldots, \hat{E}_{2n}$ can be shown to furnish a representation of the generalized Clifford Algebra C_{2n}^{m}.

Case 1. For the first n elements, $\hat{E}_1, \ldots, \hat{E}_n$ the proof is obvious since

$$A \xrightarrow{\hat{E}_i \hat{E}_j} A e_i e_j \quad ; \tag{9}$$

But

$$A \xrightarrow{\hat{E}_j \hat{E}_i} A e_j e_i = \omega^{m-1} A e_i e_j$$

$$\therefore \quad \hat{E}_i \hat{E}_j = \omega \hat{E}_j \hat{E}_i \quad , \tag{10}$$

and

$$A \xrightarrow[m \text{ times}]{\hat{E}_i \cdots \hat{E}_i} A \underbrace{e_i \cdots e_i}_{m\text{-times}} = A$$

$$\therefore \quad \left(\hat{E}_i\right)^m = I \tag{11}$$

Therefore by eq.(9), (10), (11) it follows that $\hat{E}_i$, $i = 1, \ldots n$, obey the algebra C_{2n}^{m}.

Case 2. To prove that $\hat{E}_{n+i}$ also obey the algebra C_{2n}^{m} we proceed as follows:

$$A \xrightarrow{\hat{E}_{n+i}} \zeta e_i \left[\omega^{m-1} A_0 + \cdots + 1\, A_{m-1}\right]. \tag{12}$$

Notice that the degree of the terms has been increased by one.

Hence

$$A \xrightarrow{\hat{E}_{n+j}\,\hat{E}_{n+i}} \zeta\, e_j e_i \left[\omega^{2m-3} A_0 + \omega^{2m-5} A_1 + \cdots + \omega^{m-1} A_{m-1} \right] \tag{13}$$

IIly

$$A \xrightarrow{\hat{E}_{n+i}\,\hat{E}_{n+j}} \zeta\, e_i e_j \left[\omega^{2m-3} A_0 + \cdots + \omega^{m-1} A_{m-1} \right] \tag{14}$$

Thus,

$$\hat{E}_{n+i}\,\hat{E}_{n+j} = \omega\, \hat{E}_{n+j}\,\hat{E}_{n+i}.$$

It is then not hard to prove that

$$\left(\hat{E}_{n+i} \right)^m = I$$

<u>Case 3</u>: We shall prove now, that $\hat{E}_i$ and $\hat{E}_{n+j}$ obey the algebra

$$A \xrightarrow{\hat{E}_i\,\hat{E}_{n+j}} \cancel{e_i}\ \cancel{e_j}\ \zeta\, e_j \left[\omega^{m-1} A_0 + \cdots + 1\, A_{m-1} \right] e_i\,, \tag{15}$$

and

$$A \xrightarrow{\hat{E}_{n+j}\,\hat{E}_i} \zeta e_j \left[\omega^{m-2} A_0 + \omega^{m-3} A_1 + \cdots\cdots + \omega^{m-1} A_{m-1}\right] e_i . \tag{16}$$

From (15) and (16) it is clear that

$$\hat{E}_i \hat{E}_{n+j} = \omega \, \hat{E}_{n+j} \hat{E}_i . \tag{17}$$

Thus Eq.(7) and (8) yield a representation of the 2n basic elements of C_{2n}^{m}.

We shall demonstrate the above procedure for the case m = 3, n = 2, i.e. to obtain the representation of C_4^3 from C_2^3.

We start with two basic elements e_1, e_2 of C_2^3 obeying the G.C.A. The complete set of 9 elements of C_2^3 is given by

$$1,\ e_1,\ e_2,\ e_1^2,\ e_1e_2,\ e_2^2,\ e_1^2e_2,\ e_1e_2^2,\ e_1^2e_2^2$$

An arbitrary element of C_2^3 is therefore written as

$$\begin{aligned} A = \ & a_0 1 + a_1 e_1 + a_2 e_2 + a_{11} e_1^2 + a_{12} e_1 e_2 \\ & + a_{22} e_2^2 + a_{1^2 2} e_1^2 e_2 + a_{12^2} e_1 e_2^2 \\ & + a_{e_1^2 e_2^2} e_1^2 e_2^2 . \end{aligned} \tag{18}$$

$a_{1^2 2^2}$

Now

$$A = A_0 + A_1 + A_2 ,$$

where

$$A_0 = a_0 + a_{1^2 2} e_1^2 e_2 + a_{12^2} e_1 e_2^2 ,$$

$$A_1 = a_1 e_1 + a_2 e_2 + a_{1^2 2^2} e_1^2 e_2^2 ,$$

$$A_2 = a_{11} e_1^2 + a_{12} e_1 e_2 + a_{22} e_2^2 . \tag{19}$$

The mapping $A \xrightarrow{\hat{E}_1} Ae_1$ **yields the matrix:**

$$\hat{E}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \end{bmatrix} , \tag{20}$$

The mapping $A \xrightarrow{\hat{E}_2} Ae_2$ **yields the matrix:**

$$\hat{E}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{21}$$

The mapping $\quad A \xrightarrow{\hat{E}_3} \zeta\, e_1 \left[\omega^2 A_0 + \omega A_1 + 1.A_2 \right]$

gives the matrix (here $\zeta = 1$ since $m = 3$)

$$\hat{E}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 & 0 \end{bmatrix} \tag{22}$$

The mapping $\quad A \xrightarrow{\hat{E}_4} \zeta\, e_2 \left(\omega^2 A_0 + \omega A_1 + 1.A_0 \right)$ gives the last matrix

$$\hat{E}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{23}$$

It is importante to note that in contrast to the other method, the method due to Rasevskii does not require the explicit form of the matrices of C_n^m to construct the representation of C_{2n}^m.

If P, Q are the base elements of the generalized clifford algebra C_2^3, the explicit representations of which have been obtained by Morris as follows

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{bmatrix}$$

then the Gell-Mann matrices can be expressed in terms of the complete set of elements of C_2^3, P, Q, PQ, P^2, Q^2, P^2Q, PQ^2, P^2Q^2. This is a non-trivial statement since the connection has become possible since the elements of the generalised clifford algebra are automatically traceless. Thus, just as the generators of orthogonal group can be expressed in terms of elements obeying clifford algebra in particular representations (called the spinor representations), the self-representation of the special unitary group can be built from elements obeying generalised clifford algebra.

CHAPTER XVI

CLIFFORD ALGEBRA AND MASSLESS PARTICLES*

ABSTRACT

It is shown that an equation of motion of a spin half particle involving a complete set of mutually anticommuting (2n+1) matrices furnishing the representation of Clifford Algebra C^{n}_{2n} of order 2^{n} can be reduced to the ordinary Dirac form involving only four anticommuting matrices when the particle is massive. In the case of massless particles it reduces to an equation involving five anticommuting matrices one occurring in a singular idempotent combination with the unit matrix. It is further shown that the above two forms are connected by a singular idempotent operator.

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1. INTRODUCTION

The freedom one has in linearising the Klein-Gordan equation has been used to describe massless spin-half particles (neutrino) in two different ways. The first, well-known, of course, is to describe it through a wave equation of the Dirac type in which the mass parameter is set equal to zero so that the equation describing it is

$$\gamma_{\mu} \partial^{\mu} \psi = 0 \qquad (1)$$

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* T.S.Santhanam and P.S.Chandrasekaran, to be published in Progr. Theor. Phys. (Japan), Vol.41. 264, (1969).

The equivalence of this equation to the familiar two component from is well known[1]. An alternate inequivalent way, however, is to use singular idempotent matrices without explicitly putting the mass parameter equal to zero. This has been known in the literature long back[2]. Recently, attention has been drawn to this fact by Z.Tokuoka[3] and N.D. Sen Gupta[4]. In this case one uses all the five mutually anticommuting matrices in four dimensions and the equation of motion can be written as

$$\left(\gamma_\mu \partial^\mu + m_1 (1 \pm \gamma_5)\right) \psi = 0 \tag{2}$$

It is very clear that equations (1) and (2) are inequivalent. The parameter m_1 (not connected with the mass, as it does not make its appearance in the K.G. equation) has been interpreted as the degree of chirality.

In this paper, we show the following: Even in the case of a linear equation involving the complete set of (2n+1) mutually anticommuting matrices forming the base elements of the Clifford Algebra C_{2n}^{2} ~~and~~ of 2^{2n} dimension, it is shown that this equation can be reduced to an equation of the Dirac type involving only four anticommuting matrices, when the spin half particle~~s~~ is massive. On the

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1) T.D.Lee and C.N.Yang, Phys. Rev. 106, (1957), 1671.
L.Landau, Nucl. Phys. 3, (1957), 127.
A.Salam, Nuovo Cimento, 5, (1957), 307.
K.M.Case, Phys. Rev. 107, (1957), 307.
2) Harish-Chandra, Proc.Roy.Soc. A186, (1946), 502, H.J.Bhabha, Rev. Mod.Phys. 21, (1949), 451.
3) Z.Takuoka, Prog.Theor.Phys. 37, (1967), 603.
4) N.D.Sengupta, Nucl. Phys. B4, (1968), 147.

otherhand, in the case of massless particles, the equation of motion will involve five anticommuting matrices, one of them in a singular idempotent combination with the unit matrix. We show that in this case also, we can reduce the equation to the original Dirac form without the mass term.

2. THE CASE OF FOUR DIMENSIONS

The equation of motion which uses all the five mutually anticommuting matrices in four dimensions can be written as

$$\left(\gamma_\mu \partial^\mu + m_1 \gamma_5 + m_2 \right) \psi = 0 , \tag{2a}$$

where γ_μ and γ_5 are hermitian. It follows that the square of the Hamiltonian

$$H^2 = |\vec{p}|^2 + (m_2^2 - m_1^2) \tag{3}$$

If we multiply by an operator[4)]

$$O = \frac{1}{\sqrt{m_2^2 - m_1^2}} (m_2 - m_1 \gamma_5) , \tag{4}$$

we get

$$(\gamma'_\mu \partial_\mu + m') \psi = 0 , \tag{5}$$

where

$$\gamma_\mu' = O\gamma_\mu , \tag{6}$$

and

$$m' = \left(m_2^2 - m_1^2\right)^{\frac{1}{2}}$$

Now, the γ_μ''s are not hermitian. However by a transformation

$$\gamma_\mu' = e^{-\gamma_5 \varphi} \gamma_\mu e^{\gamma_5 \varphi} , \tag{7}$$

with

$$\tanh \varphi = \frac{m_1}{m_2} ,$$

the equation (5) reduces to

$$\left(\gamma_\mu \partial^\mu + m'\right) \psi' = 0 \tag{8}$$

where

$$\psi' = e^{\gamma_5 \varphi} \psi \tag{9}$$

What has been now demonstrated is the fact that although the complete set of five anticommuting matrices was used in writing down the equation of motion, by a suitable transformation, the equation can be brought to the standard Dirac form. The demonstration is true when $m_1^2 \neq m_2^2$. The transformation (4) and (7) do not exist when $m_1^2 = m_2^2$ in which case the equation of motion descrbing the massless spin-half particle splits into two as

$$\left(\gamma_\mu \partial^\mu + m_1 (1 \pm \gamma_5)\right) \psi_\pm = 0 \,, \tag{10}$$

and the two equations are connected by a charge conjugation operation. It is interesting to note that $(1 \pm \gamma_5)$ is singular and idempotent. Arguments have been advanced that m_1 can be interpreted as the degree of Chirality[4].

3. EQUATION IN 2^n DIMENSIONS.

We show below that in the case of massive spin half particles, even if we describe it by a wave function in 2^n dimensions through an equation containing (2n+1) parameters, it could ~~anti~~ be still brought to the standard Dirac form involving only four anticommuting matrices by a suitable transformation. In the case of massless particles, the equation can be reduced to the form of equation (10) involving five anticommuting matrices. In this, case, the equation involves a singular idempotent matrix.

It is knwon that there are (2n+1) mutually anticommuting matrices of dimension $2^n \times 2^n$ which form the complete set of base elements satisfying the Clifford algebra C_{2n} of dimension 2^{2n}. These matrices are easily constructed using the elegant method developed by A.Ramakrishnan[5]. The most general wave equation of a spinor particle which makes use of all these (2n+1) matrices can be written as

$$\left(\Gamma_\mu \partial^\mu + m_1 \Gamma_4 + m_2 \Gamma_5 + \dots + m_{2n-3} \Gamma_{2n} + m_{2n-2}\right)\psi = 0 , \tag{11}$$

where $$\Gamma_\mu = \Gamma_{0,1,2,3} ,$$

and $$\{ \Gamma_i , \Gamma_j \} = 2\,\delta_{ij} , \quad i,j = 1, \dots\dots 2n+1 ,$$

with $$\Gamma_\mu \partial^\mu = \Gamma_0 \partial^0 - \Gamma_1 \partial^1 - \Gamma_2 \partial^2 - \Gamma_3 \partial^3 .$$

In this case we have

$$H^2 = |\vec{p}|^2 + m^2_{2n-2} - \left(m_1^2 + \dots\dots + m^2_{2n-3} \right) \tag{13}$$

If we operate on equation (11) by

$$O' = \frac{m_{2n-2} - \left(m_1 \Gamma_4 + \dots + m_{2n-3} \Gamma_{2n} \right)}{\left\{ m^2_{2n-2} - \sum_{i=1}^{2n-3} m_i^2 \right\}^{\frac{1}{2}}} , \tag{13a}$$

5) A.Ramakrishnan, Jour. Math. Anal. and Appl. Vol.20 (1967), 9.

we get

$$\left(\Gamma_\mu' \partial^\mu + m' \right) \psi = 0 ,$$

(14)

where

$$m' = \left\{ m_{2n-2}^2 - \sum_{i=1}^{2n-3} m_i^2 \right\}^{\frac{1}{2}}$$

(15)

and

$$\Gamma_\mu' = O \Gamma_\mu$$

(15)

This can be ~~x~~brought to the normal Hermitian form by a transformation,

$$\Gamma_\mu' = e^{-\chi \varphi} \Gamma_\mu e^{\chi \varphi} ,$$

where

$$\chi = \frac{m_1 \Gamma_4 + \cdots + m_{2n-3} \Gamma_{2n}}{\left\{ \sum_{i=1}^{2n-3} m_i^2 \right\}^{\frac{1}{2}}} ,$$

and

$$\tanh \varphi = \left\{ \sum_{i=1}^{2n-3} m_i^2 \right\}^{\frac{1}{2}} \Big/ m_{2n-2} .$$

such that

$$\left(\Gamma_\mu \partial^\mu + m' \right) \psi' = 0 , \qquad (16)$$

with
$$\psi' = e^{x\varphi}\psi$$

We have shown that a spin half particle can be described by a 2^n dimensional wave function obeying the standard Dirac equation (16) if it is massive. Again, the whole procedure fails if

$$m_{2n-2}^2 = \sum_{i=1}^{2n-3} m_i^2$$

In this case let us define a

$$\Gamma = \frac{1}{m_{2n-2}} \left(m_1 \Gamma_4 + \cdots + m_{2n-3} \Gamma_{2n} \right) \tag{17}$$

so that
$$\Gamma^2 = I ,$$

and
$$\{\Gamma, \Gamma_\mu\} = 0$$

Now, equation (11) takes the form

$$\left(\Gamma_\mu \partial^\mu + m_{2n-2} (1 \pm \Gamma) \right) \psi_\pm = 0 , \tag{18}$$

which is in the same form as equation (16). $(1 \pm \Gamma)$ are again singular and idempotent. The parameters $m_1, m_2, \cdots, m_{2n-3}$ appear only in the representation of Γ matrices.

Thus we have shown[6)] that even in the general case of Clifford algebra C_{2n} in 2^{2n} dimension, a massive spin half particle can be described through the Dirac type equation with only four anti-commuting matrices and a massless spin half particle can be described through an equation involving five anticommuting matrices one of them in a singular idempotent combination with the unit matrix.

4. DISCUSSION.

It is worth mentioning here that even in the case of massless particles, we can still describe it through an equation involving only four anticommuting matrices and in this case one gets a wave function which is an eigenstate of the Chirality operator, For, if we multiply equation (10), by the singular operator $(1 \pm \gamma_5)$ we get

$$\gamma_\mu \partial^\mu \psi'_\pm = 0 \tag{19}$$

Equation (19) involves only four anticommuting matrices and this is just the ~~equx~~ Dirac equation with zero mass. It is, interesting to note that equation (19) is γ_5 invariant and $\psi'_\pm$ are eigenstates of γ_5. This is also true in the case of equation (18).

6) In fact just one matrix whose square is unity is sufficient in the case of massive spin 1/2 particles although relativistic invariance requires four anticommuting matrices. We thank Professor E.C.G.Sudarshan for a pertinent question in this connection.