Geometry of Linear Diophantine Equations

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Certificate

Certified that the work contained in the thesis entitled

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Abstract

The non-negative solutions of linear homogeneous Diophantine equations are studied using the geometric theory of convex polytopes. After a brief introduction to the theory of convex polytopes and its relation to solutions of linear homogeneous Diophantine equations, a theorem of Stanley, Bruggesser and Mani on the decomposition of the monoid of solutions is discussed in detail. An application of this theorem, due to Stanley, to prove a conjecture of Anand, Dumir and Gupta is explained.

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Notations and Conventions

I have followed the following notations and coneventions in this report.

Symbol	Meaning
$\mathbb{N}$	set of natural numbers $\{0, 1, 2, \dots\}$
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{Z}_{>0}$	$\{1, 2, 3, \dots\}$
$\mathbb{Z}_{\geq 0}$	$\{0, 1, 2, \dots\}$ same as $\mathbb{N}$
bd$A$	boundary of A
int$A$	interior of A
relint	relative interior of A
Conv$A$	convex hull of A
Aff$A$	affine hull of A
$[a, b]$	$\{x \mid x = \lambda a + (1-\lambda)b, 0 \leq \lambda \leq 1\}$
$[a, b)$	$\{x \mid x = \lambda a + (1-\lambda)b, 0 < \lambda \leq 1\}$
(a, b)	$\{x \mid x = \lambda a + (1-\lambda)b, 0 < \lambda < 1\}$
$L(x, y)$	line passing through x and y
$x_n \to x$	x_n converges to x
$\|x\|$	norm of x
$x \cdot y$	usual dot product of x and y
$X \cdot y$	$\{x \cdot y \mid x \in X\}$
$y + X$	$\{y + x \mid x \in X\}$
$\alpha + X$ for α scalar	$\{(\alpha + x_1, \dots, \alpha + x_n) \mid x = (x_1, \dots, x_n) \in X \subseteq \mathbb{R}^n\}$

$A \coprod B$	A disjoint union B
$\mathbb{N}A$	free monoid generated by A
$A - B$	elements of A not in B
$\emptyset$	the empty set
$f : A \longrightarrow B$	f is a map from A to B
$f(S)$ where $S \subseteq A$	$\{f(x) \mid x \in S\}$
$[a_{i,j}]_{m \times n}$ or simply $[a_{i,j}]$	an $m \times n$ matrix whose (i,j)th entry is $a_{i,j}$
$\delta_{i,j}$	$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
0	sometimes denotes the origin in $\mathbb{R}^n$
H^+ or H^-	A half space determined by the hyperplane H ($+$ or $-$ sign depends on the context)
$\Phi x = 0$	linear system of equations defined by the matrix Φ and the vector variable x
$\mathcal{C}_\Phi$	cone defined by the non-negative solutions of the linear system of equations $\Phi x = 0$
$S(P, w)$	facets of the polytope P visible from the point w
$U(P, w)$	facets of the polytope P not visible from the point w

Introduction

This thesis concerns the solutions of linear homogeneous Diophantine equations, namely, equations of the form

$$\Phi x = 0,$$

where Φ is an $m \times n$ matrix with integer entries, $x = (x_1, \ldots, x_n)^t$ is a column vector of variables, and we are only interested in solutions that are non-negative integers.

For example, one may wish to describe the set

$$E_\Phi = \{\beta \in \mathbb{N}^n | \Phi\beta = 0\}$$

of solutions in a compact form. One way of doing this is to study the formal power series:

$$E_\Phi(x) = \sum_{\beta \in E_\Phi} x^\beta,$$

where, for $\beta = (\beta_1 \ldots, \beta_n)$, x^β denotes the monomial $x_1^{\beta_1} \cdots x_n^{\beta_n}$. For example, for the equation

$$x_1 - x_2 = 0, \tag{1}$$

the formal power series is given by the rational function

$$E_\Phi(x) = \frac{1}{1 - x_1 x_2}.$$

This follows from the fact that the non-negative solutions to (1) form a free monoid, which is generated by $(1, 1)$. It turns out that $E_\Phi(x)$ is always a rational function. This can be deduced from the fact that every pointed convex cone can be triangulated (see [**Sta12**, Theorem 4.6.11]). An alternative approach uses Hilbert's basis theorem from commutative algebra (see [**Sta83**] Theorem 3.7).

In this thesis, in Theorem 2.16 we show that E_Φ can be expressed as a disjoint union of translates of free monoids. This not only gives the rationality of $E_\Phi(x)$, but also the non-negativity of the coefficients of the Ehrhart polynomial [**Sta80**], and lies at the heart of the theory of Ehrhart polynomials. This result also turns out to be the crucial ingredient needed to prove the conjectures of Anand, Dumir and Gupta (see Chapter 4) on integer stochastic matrices.

Most of the material that I have presented in this thesis is motivated and influenced by Stanley's work [**Sta83, Sta12, Sta82**]. Here I have emphasized the theory of Convex polytopes that appears in Chapter 1 and Chapter 2. Chapter 1 is an introduction to the general theory of convex polyhedra and its relation to the solutions of linear Diophantine equations. Chapter 2 is devoted to proving the shellability of the boundary complex of a convex polytope (due to Brugesser and Mani [**BM71**]) and the resulting decomposition of the monoid of non-negative integer solutions of a system of linear Diophantine equations (due to Stanley [**Sta80, Sta82**]). Chapter 3 is devoted to the reciprocity theorem, and Chapter 4 to the proof of the conjectures of Anand, Dumir and Gupta due to Stanley.

In the first two chapters I have tried to give the proofs in detail while in Chapter 3 and Chapter 4 it is not so. I hope that this idea of presentation will make the thesis smooth to read and enjoyable as well.

CHAPTER 1

Geometry of Solutions

1. Linear Homogeneous Diophantine Equations

Let Φ be an $m \times n$ matrix with integer entries. Let $x = (x_1, \ldots, x_n)^t$ be a column vector of n-variables. Consider the system

$$\Phi x = 0$$

of linear homogeneous equation with integral coefficients. Let

$$E_\Phi = \{\beta \in \mathbb{N}^n | \Phi\beta = 0\}$$

denote the set of non-negative integer solutions of the above system. Our main purpose is to understand E_Φ.

Note here that instead of non-negative integer solutions if we just ask for integral solutions then the problem is easy. The integer solutions of $\Phi x = 0$ form a subgroup of $\mathbb{Z}^n$ which is free. The rank of this free group is the nullity of Φ. To find all the integral solutions of $\Phi x = 0$ we just need to find a basis for this free group. But this is not the case for E_Φ. The set E_Φ is not a group, but rather, a monoid and further it can not be guaranteed that E_Φ is a free monoid.

Definition 1.1. *Given any subset $S \subset \mathbb{N}^n$ define $S(x) = \sum_{\beta \in S} x^\beta$.*

Here S is completely described by $S(x)$. In our case, understanding $E_\Phi(x)$ is equivalent to understanding E_Φ, which is what we are going to do in the following discussion.

The theory of E_Φ can be developed purely algebraically as well as geometrically. Here I have chosen the geometric way since it is more elegant and intuitive. The geometric theory of E_Φ proceeds by understanding the geometry of C_Φ, the convex hull of elements of E_Φ. It will turn out that C_Φ is a *pointed convex polyhedral cone.* To Make our argument systematic and clear we need to go through some basic results concerning convex polyhedra.

2. Basic Theory of Convex Polyhedra

We will work with $\mathbb{R}^n$ with its standard topology. $\mathbb{R}^n$ will be considered as a vector space over $\mathbb{R}$ with usual scalar multiplication and vector addition. The inner product is defined to be the dot product. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then $x \cdot y = \sum_{i=1}^{n} x_i y_i$.

Definition 1.2. *A vector $z \in \mathbb{R}^n$ is said to be an* affine combination *of k vectors $z_1, \ldots, z_k$ in $\mathbb{R}^n$ if there exist real numbers $\lambda_1, \ldots, \lambda_k$ such that $\lambda_1 + \ldots + \lambda_k = 1$ and*

$$z = \lambda_1 z_1 + \cdots + \lambda_k z_k.$$

If, in addition, the real numbers $\lambda_1, \ldots, \lambda_k$ can be chosen to be non-negative, then z is said to be a convex combination *of $z_1, \ldots, z_k$.*

Given a set $A \subset \mathbb{R}^n$, the set of all affine combinations formed from all finite subsets of A is called the affine hull *of A, and is denoted by* Aff$A$. *Similarly, the set of all convex combinations formed from all finite subsets of A is called the* convex hull *of A and is denoted by* Conv$A$. *If $A = \mathrm{Conv}A$, then A is called a* convex set.

Definition 1.3. *Given $x \neq y$ in $\mathbb{R}^n$, we denote by $L(x, y)$ the set of all affine combinations of x and y and call it the line through x and y. $L(x, y)$ is uniquely determined by the following property: if $x', y' \in L(x, y)$, with $x' \neq y'$, then $L(x, y) = L(x', y')$.*

If a subset H of $\mathbb{R}^n$ contains all lines $L(x, y)$ for all $x, y \in H$ such that $x \neq y$, then H is called a flat.

It follows from the above definition that for any subset $A \subset \mathbb{R}^n$, Aff$A$ is flat.

Theorem 1.4. *Every flat $H \in \mathbb{R}^n$ is the affine hull of a finite set of points in $\mathbb{R}^n$. Moreover, there exists $x \in \mathbb{R}^n$ and a linear subset V of $\mathbb{R}^n$ such that $H = x + V$.*

PROOF. Consider a strictly increasing sequence of flats in H:

$$x_1 \subsetneq \mathrm{Aff}\{x_1, x_2\} \subsetneq \mathrm{Aff}\{x_1, x_2, x_3\} \subsetneq \cdots$$

We claim that the above chain is of finite length. To prove this, define $B_{i-1} = \{x_1 - x_i, \ldots, x_{i-1} - x_i\}$. Suppose that the above sequence of flats has infinite length. Since we are in a finite dimensional vector space, there exists a least j such that B_j is a linearly dependent set. So there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}$, not all zero, such that

$$\alpha_1(x_1 - x_j) + \cdots + \alpha_{j-1}(x_{j-1} - x_j) = 0.$$

Note that if $\alpha_1 + \cdots + \alpha_{j-1} = 0$, then B_{j-1} will be linearly dependent (easy to check). This gives

$$\sum_{i=1}^{j-1} \frac{\alpha_i}{\sum_{l=1}^{j-1} \alpha_l} x_i = x_j.$$

Since x_j is an affine combination of $x_1, \ldots, x_{j-1}$, we get a contradiction to our assumption that the chain of flats has infinite length. This also shows the chain stops in $j-1$ steps. So $H = \text{Aff}\{x_1, x_2, \ldots, x_{j-1}\}$. But $\text{Aff}\{x_1, x_2, \ldots, x_{j-1}\} = x_j + \text{span} B_{j-1}$. □

If, for two vectors x and x', and two linear subspaces V and V', $x + V = x' + V'$, then it is easy to see that $V = V'$ and $x - x' \in V$.

Definition 1.5. *The dimension of a flat H is defined to be the dimension of V, where V is the linear subspace for which H can be written as $x + V$.*

Here is a result about convex sets stated without proof as the proof is easy and straightforward. A similar result also holds for affine sets or flats.

Lemma 1.6. *Intersection of convex sets is convex. Intersection of flats is a flat.*

Definition 1.7. *For $\alpha \in \mathbb{R}^n$ and $a \in \mathbb{R}$, define a hyperplane*

$$H_{\alpha,a} = \{x \in \mathbb{R}^n | \alpha.x = a\}.$$

We will often denote $H_{\alpha,a}$ by H_α or simply H when the context is clear.

Every hyperplane $H_{\alpha,a}$ determines two half-spaces $H^+_{\alpha,a} = \{x \in \mathbb{R}^n | \alpha.x \geq a\}$ and $H^-_{\alpha,a} = \{x \in \mathbb{R}^n | \alpha.x \leq a\}$.

Here $H_{\alpha,a}$ is flat, and is a linear subspace only if $a = 0$. $H^+_{\alpha,a}$ and $H^-_{\alpha,a}$ are convex sets having intersection $H_{\alpha,a}$.

Definition 1.8. *A convex polyhedron is an intersection of finitely many half-spaces (and hence is convex). The dimension of a convex polyhedron is defined to be the dimension of its affine hull.*

Definition 1.9. *Lat K be a convex polyhedron of dimension n in $\mathbb{R}^n$. A supporting hyperplane for a convex polyhedron K is a hyperplane H such that $K \subseteq H^+$ or $K \subseteq H^-$ and $K \cap H$ is nonempty.*

Now onwards, unless otherwise specified, we will always assume that our convex polyhedron K is of maximum dimension, i.e., if $K \subseteq \mathbb{R}^n$, then the dimension of K is n.

Definition 1.10. *A face of a convex polyhedron is its intersection with a supporting hyperplane. If the polyhedron has dimension n then its $n-1$ dimensional faces are called facets; the 0 dimensional faces are called vertices or extremal points. The set of all faces of a convex polyhedron K will be denoted by $\mathcal{F}(K)$.*

Note here that every face of a convex polyhedron is also a convex polyhedron. So the dimension of a face is well defined. Given a convex polyhedron K which is intersection of half-spaces, say $K = \cap_{i=1}^n H_i^+$, and a face $F = K \cap H$ of K, where H is a supporting hyperplane with $K \subseteq H^+$, F is expressed as an intersection of half-spaces by $F = \cap_{i=1}^n H_i^+ \cap H^-$.

Definition 1.11. *Let $K = \cap_{i=1}^n H_i^+$, then the family $\{H_i^+ | i \leq 1 \leq n\}$ is called irredundant if, for all $1 \leq i \leq n$, $\cap_{j \neq i} H_j^+ \neq K$.*

The following theorem gives a unique representation of a convex polyhedron in terms of half-spaces.

Theorem 1.12. *Let $K \subseteq \mathbb{R}^m$ be a convex polyhedron of dimension m. An expression $K = \cap_{i=1}^n H_i^+$, where $\{H_i^+ | i \leq 1 \leq n\}$ is an irredundant family, is unique. Also all the facets of K are given by $F_i = K \cap H_i$.*

Proof. If we can show that all the facets of K are given by $K \cap H_i$, then the uniqueness follows. First we will show that H_i is a supporting hyperplane for K. Let

$K_i = \cap_{j \neq i} H_j^+$. H_i^+ does not contain K_i because $K_i \neq K$. H_i^- does not contain K_i, for if it did, then $K \subseteq H_i^- \cap H_i^+ = H_i$, which contradicts the fact that K is of dimension m. So H_i intersects K_i in its interior. Therefore $H_i \cap K$ is of dimension $m-1$. Hence H_i is a supporting hyperplane for K intersecting it in a facet.

Conversely, given a facet F of K, consider the hyperplane $H = \text{Aff}F$. H is a supporting hyperplane, and we may assume that $K \subset K \cap H^+$. If H is not parallel to any H_i, then $H \cap H_1$ has dimension $n-2$. Since the boundary of K is contained in the union of the H_i's, H must intersect K in its interior, which contradicts the fact that H is a supporting hyperplane. But if H is parallel to H_i and supporting, then $H = H_i$. □

Corollary. Let $F_i := K \cap H_i$, then $\text{bd}K = \cup_{i=1}^n F_i$.

PROOF. Clearly $\cup_{i=1}^n F_i \subseteq \text{bd}K$. Let $x \in K$ such that $x \notin H_i$ for any i. Then there exists δ_i such that the ball $B(x, \delta_i) \subseteq H_i^+$. Let $\delta = \min\{\delta_i\}$. Then $B(x, \delta) \subseteq K$. So $x \notin \text{bd}K$. Hence $\text{bd}K \subseteq \cup_{i=1}^n F_i$. □

Corollary. Every face is contained in a facet.

PROOF. Since every face F is in the boundary of K, F is contained in $\cup_{i=1}^n F_i$, the union of all facets. Let H be the supporting hyperplane that determines F. Clearly $\text{Aff}F \subseteq H$. If F is not contained in any of the F_i's, then $\text{Aff}F$ intersects K in its interior, contradicting the assumption that H is supporting. So $F \subseteq F_i$ for some i. □

Till now we only know that the number of facets of a convex polyhedron is finite, but need not the number of faces. Also the faces are subsets of facets, but what kind of subset are they? We will soon see that the lower dimensional faces are also faces of the facets. So an induction argument on the dimension of faces shows that the total number of faces is also finite. To understand all this, we begin with the following lemma which says that faces are closed under intersection.

Lemma 1.13. *Any nonempty intersection of faces of a convex polyhedron is a face of the polyhedron.*

PROOF. Let $G_1, \ldots, G_r$ be r faces of the convex polyhedron K such that they have a nonempty intersection. Assume, without loss of generality, that each of them contains zero. Let H_{α_i} be the supporting hyperplane that determines G_i: $H_{\alpha_i} = \{x | x \cdot \alpha_i = 0\}$. Also assume that $K \subseteq H^+_{\alpha_i}$. Consider $H = \{x \mid x \cdot (\alpha_1 + \cdots + \alpha_i) = 0\}$. Then $K \subseteq H^+$ and $\cap_{i=1}^r G_i \subseteq K \cap H$. Now if $x \in K \cap H$, then $x \cdot (\alpha_1 + \cdots + \alpha_i) = 0$, so that $x \cdot \alpha_i = 0$ for all i. Therefore, $x \in G_i$. So we got $\cap_{i=1}^r G_i = K \cap H$ is a face of K with the supporting hyperplane H. □

Lemma 1.14. *Let G_1 and G_2 be two faces of K such that $G_1 \subsetneq G_2$. Then G_1 is a face of G_2.*

PROOF. Let H_1' be the supporting hyperplane for G_1, then $H_1 \cap \mathrm{Aff} G_2$ will be a supporting hyperplane for G_2 in $\mathrm{Aff} G_2$. So G_1 is a face of G_2. □

Corollary. Let G_1 and G_2 be two faces of the same dimension such that $G_1 \subseteq G_2$, then $G_1 = G_2$.

PROOF. If $G_1 \subsetneq G_2$ then G_1 is a face of G_2, hence their dimensions can not be equal. So we must have $G_1 = G_2$. □

Lemma 1.15. *A facet of a facet is intersection of two facets.*

PROOF. Let F_1 be a facet of K with facet plane H_1. In other words, $F_1 = H_1 \cap K = H_1 \cap_{i=1}^n H_i^+ = \cap_{i=2}^n H_i^+ \cap H_1$. The intersections $H_i^+ \cap H_1$ are half-spaces in H_1. So every facet of F_1 corresponds to the relative boundary of $(H_i^+ \cap H_1)$, which is $H_i \cap H_1$. So facets of F_1 are of the form $H_i \cap H_1 \cap K = (H_i \cap K) \cap (H_1 \cap K) = F_i \cap F_1$. □

Theorem 1.16. *Every face of K can be expressed as intersection of its facets.*

PROOF. Let F be a face of K. Then there exists a facet F_1 such that $F \subseteq F_1$. If $F = F_1$ then we are done. Otherwise F is a face of F_1 by Lemma 1.14, and so is contained in a facet of F_1 say F_1^1. If $F_1 = F_1^1$ then also we are done since every facet of a facet is intersection of two facets by Lemma 1.15. We will continue this process by constructing $F_1^1, F_1^2, F_1^3, \ldots$ and so on until $\mathrm{dim} F_1^j = \mathrm{dim} F$. But $F \subseteq F_1^j$, so by the corollary of Lemma 1.14, $F = F_1^j$.

Now we need to show that F_1^j is an intersection of facets. In fact, we ill show that it is an intersection of $j+1$ facets. We will do this by induction on the co-dimension of F_1^j. The base case is clear by Lemma 1.15, where the co-dimension is 2. Assume $F_1^{j-1} = F_1 \cap F_2 \cap \cdots \cap F_j$ by induction hypothesis. Here F_i are facets of K with facet planes H_i. Now $F_1^{j-1} = (K \cap H_1) \cap \cdots \cap (K \cap H_j) = (\cap_{i=1}^n H_i^+ \cap H_1) \cap \cdots \cap (\cap_{i=1}^n H_i^+ \cap H_j) = \cap_{i=j+1}^n (H_1 \cap \cdots \cap H_j \cap H_i^+)$. Here $H_1 \cap \cdots \cap H_j \cap H_i^+$ is half-space in $H_1 \cap \cdots \cap H_j$. We know that F_1^{j+1} is a facet of F_1^j, hence is of the form $H_1 \cap \cdots \cap H_j \cap H_i \cap K$ for some $i \geq j+1$. Therefore $F_1^{j+1} = (H_1 \cap K) \cap \cdots \cap (H_j \cap K) \cap (H_i \cap K) = F_1 \cap F_2 \cap \cdots \cap F_j \cap F_i$. This completes the proof. □

Corollary. A face of a face is a face.

Proof. Let F be a face of K and G be face of F. G is the intersection of facets of F. To show that G is a face of K we need to show it is an intersection of facets of K. We will be done by Theorem 2, if we can show that every facet of F is a face of K. From the proof of the Theorem 2, $F = F_1 \cap \cdots \cap F_j$ is an intersection of j facets represented by $F = \cap_{i=j+1}^n (H_1 \cap \cdots \cap H_j \cap H_i^+)$. So each facet of F is of the form $F_1 \cap \cdots \cap F_j \cap F_i$ for some $i \geq j+1$. Since every facet of F is intersection of facets of K, it is a face of K. □

Now we will look at a special type of convex polyhedron which is called a convex polytope. Formally,

Definition 1.17. *A bounded convex polyhedron is called a convex polytope.*

Here are two standard theorems that characterizes a convex polytope:

Theorem 1.18. *Let P be convex polytope and let* $\mathrm{Vert}P$ *be the set of all zero dimensional faces of P, then* $\mathrm{Vert}P$ *is nonempty and P is the convex hull of* $\mathrm{Vert}P$.

Proof. Since P is convex, $\mathrm{ConvVert}P \subseteq P$. So we need to show $P \subseteq \mathrm{ConvVert}P$. Suppose that $\mathrm{Vert}P = \{x_1, \ldots, x_l\}$, so that

$$\mathrm{ConvVert}P = \Big\{ \sum_{i=1}^{l} \lambda_i x_i \mid 0 \leq \lambda_i \leq 1, \sum_{i=1}^{l} \lambda_i = 1 \Big\}.$$

We will use induction on the dimension of P. If $\dim P = 1$, then P is a line segment and it is the convex hull of its end points. Now assume the result to be true for all polytopes up to dimension $m-1$. We will prove it for $\dim P = m$. If $y \in \mathrm{bd}P$, then y is in some face

F of P which is also a polytope. By induction hypothesis $y \in \text{ConvVert}F \subseteq \text{ConvVert}P$. So choose $y \in \text{int}P$. Consider a line passing through y. It intersects the boundary of the polytope at exactly two points (since P is bounded), say y_1 and y_2. We have $y_1, y_2 \in \text{ConvVert}P$ and $y = \lambda y_1 + (1-\lambda)y_2$ for some $0 < \lambda < 1$. Therefore $y \in \text{ConvVert}P$. □

The next theorem is the converse of the previous one. That means we want to show that if something is the convex hull of finitely many points then it is a convex polytope. Here let $P = \text{Conv}X$ where $X = \{x_1, x_2, \ldots, x_l\}$. Again we assume that P is of dimension m in $\mathbb{R}^m$. Before going into the theorem we need the following two lemmas.

Lemma 1.19. *If H is a supporting hyperplane for P, then there exists a subset Y of X such that $H \cap P = \text{Conv}Y$.*

Note: Here we extended the definition of a supporting hyperplane to convex sets by defining: *H is a supporting hyperplane for a convex set A if $H \cap A$ is nonempty and $A \subseteq H^+$.*

Proof. Let $y = \sum_{j=1}^{r} \lambda_i x_{i_j} \in H \cap P$ where $x_{i_j} \in X$ and $0 < \lambda_i \leq 1$. To prove the lemma it is enough to prove that each $x_{i_j} \in H \cap P$. Consider x_{i_1} and the line $L := L(x_{i_1}, \frac{\lambda_2}{\lambda_1 - 1}x_{i_2} + \cdots + \frac{\lambda_r}{\lambda_1 - 1}x_{i_r})$. Either $H \cap L = L$ or a singleton set. Clearly $y \in H \cap L$. Let $y' = \frac{\lambda_2}{\lambda_1 - 1}x_{i_2} + \cdots + \frac{\lambda_r}{\lambda_1 - 1}x_{i_r}$. Since x_{i_1} and y' are on the same side of H and y is on the line segment joining x_{i_1}, y', $L \subseteq H$. Hence $x_{i_1} \in H \cap P$. Similarly we can prove for each x_{i_j}. □

The next lemma is a very useful fact about closed convex sets and uses the *Hahn-Banach theorem.* Here just the statement of the Hahn-Banach theorem (only the case that is useful to us) is given.

Theorem (Hahn-Banach)**.** *Given a compact convex set K and a point x outside K there exists a linear functional f and a constant c such that $f(x) < c < f(K)$.*

Lemma 1.20. *If $A \subseteq \mathbb{R}^m$ is a compact convex set of dimension m and $x \in \text{bd}A$, then there exists a supporting hyperplane H of A containing x.*

Proof. Since $x \in \text{bd}A$ there is a sequence $x_n \to x$ and $\|x\| \leq 1$. By Hahn-Banach for each x_n there exists $y_n \in \mathbb{R}^m$ and $c_n \in \mathbb{R}$ such that $K \cdot y_n < c_n < x_n \cdot y_n$. Clearly $y_n \neq 0$.

We can assume $\|y_n\|$ is bounded, because for each $0 < \lambda \leq 1$, $K \cdot \lambda y_n < \lambda c_n < x_n \cdot \lambda y_n$, and if $\|y_n\| > 1$, then for some $\lambda < 1/\|y_n\|$ we replace λy_n instead of y_n. So y_n has a convergent subsequence. Without loss of generality we assume $y_n \to y$. So we have $K \cdot y \leq x \cdot y$. Now our required hyperplane is $H = \{z \in \mathbb{R}^n | z.y = x.y\}$. □

Theorem 1.21. *P, the convex hull of the finite set X, is a polytope.*

Proof. We only need to show that P is a convex polyhedron, since the boundedness of P is clear. Let $y \in \mathbb{R}^n$ such that $y \notin P$. S be the set of all affine combinations of at most $m-1$ points of X. Intersection of P with a supporting hyperplane will be called a face. By Lemma 1.19, S contain all faces of P of dimension at most $m-2$. Let M denote the union of cones spanned by S with vertex y. By Bair's category since $\dim P = m$, $\text{int}P - M$ is nonempty. For $x \in \text{int}P - M$, consider the ray $R = \{\lambda x + (1-\lambda)y | \lambda \geq 0\}$ with beginning point y. Let $\lambda_0 = \inf\{\lambda \geq 0 | \lambda x + (1-\lambda)y \in P\}$. Since P is compact and $y \notin P$, λ_0 exists and nonzero. So $x_0 := \lambda_0 x + (1-\lambda_0)y \in \text{bd}P$. So x_0 is in some face of P, by Lemma 1.20. Since $x \notin M$, x_0 does not belongs to any element of C. So F must have dimension $m-1$ and $H = \text{Aff}F$ is a supporting hyperplane of P that contain y and P in two opposite sides. There fore $P = \cap_{H \in \Lambda} H^+$, where Λ is the collection of all supporting hyperplanes H of P such that $H \cap P$ is $m-1$ dimensional and $P \subseteq H^+$. Note that H is determined uniquely by $H \cap P$, and so by Lemma 1.19, Λ is finite. □

Here Theorem 1.18 and Theorem 1.21 suggest an equivalent definition of polytope, namely, the convex hull of finitely many points. While working with them we will choose the more convenient one.

Now we will introduce another special type of convex polyhedron, namely the convex polyhedral cone.

Definition 1.22. *A convex polyhedral cone $\mathcal{C}$ is intersection of half-spaces where all the hyperplanes of the half-spaces pass through a common point, say o.*

Usually o is assumed to be the origin $(0, 0, \ldots, 0)$ of $\mathbb{R}^m$. So every facet plane of $\mathcal{C}$ is linear. From now on, we will assume that o is the origin, unless otherwise specified.

Intuitively we understand that a cone has a unique vertex called the apex and in the above case we guess that the apex must be o. But here $\mathcal{C}$ may not have a vertex at all.

But for some cases our intuition is true. To explain it more clearly note that if $\mathcal{C}$ has a vertex then it must be the intersection of m facet planes of $\mathcal{C}$ (as usual we assume that $\mathcal{C}$ has dimension m and is in $\mathbb{R}^m$). Suppose the m facet planes of C whose intersection is a vertex of $\mathbb{R}^m$, are represented by a system of m linearly independent equations as:

$$a_{1,1}x_1 + \cdots + a_{1,m}x_m = 0$$
$$\vdots$$
$$a_{m,1}x_1 + \cdots + a_{m,m}x_m = 0$$

The vector of variables $(x_1, \ldots, x_m)$ represents a point in $\mathbb{R}^m$. So if $\mathcal{C}$ has a vertex then it must be 0. But to say when exactly it has a vertex, we need to introduce the following definition; the Lemma following it answers our question.

Definition 1.23. *$\mathcal{C}$ is called pointed if it does not contain a line, equivalently, if $v \in \mathcal{C}$ then $-v \notin \mathcal{C}$.*

Lemma 1.24. *$\mathcal{C}$ has a vertex if and only if it is pointed.*

Proof. Suppose that $\mathcal{C}$ is pointed. Let $\mathcal{C}$ have n facets with facet planes given by:

$$a_{1,1}x_1 + \cdots + a_{1,m}x_m = 0$$
$$\vdots$$
$$a_{n,1}x_1 + \cdots + a_{n,m}x_m = 0$$

Since v and $-v$ ($v \neq 0$) both can not be the solution of the above system the solution space must be zero dimensional so 0 is vertex of $\mathcal{C}$. Now given that 0 is a vertex of $\mathcal{C}$, we must have a supporting hyperplane H for $\mathcal{C}$ which will intersect it at 0. For any $v \neq 0$ in $\mathcal{C}$, v and $-v$ are in opposite sides of H, so $-v \notin \mathcal{C}$. This shows $\mathcal{C}$ is pointed. □

The only vertex of $\mathcal{C}$ (when it is pointed) is 0. Next come the one dimensional faces, which are called extremal rays. The role of extremal rays for a pointed polyhedral cone $\mathcal{C}$ is same as the role of vertices for a convex polytope. In fact a pointed convex polyhedral cone $\mathcal{C}$ is some way equivalent to a convex polytope P. To understand this correspondence we need the following definition:

Definition 1.25. *If H is a (possibly affine) hyperplane that intersects $\mathcal{C}$ in such a way that every ray coming out of 0 intersects H at a unique point, we call $\mathcal{C} \cap H$ is a nondegenerate cross section of $\mathcal{C}$.*

Theorem 1.26. *Let $\mathcal{C}$ be a pointed convex polyhedral cone. Then there exists a hyperplane H such that $P := \mathcal{C} \cap H$ is a nondegenerate cross section of $\mathcal{C}$. Further*

(i) *P is a convex polytope.*

(ii) *$\psi : \mathcal{F}(\mathcal{C}) - \{0\} \longrightarrow \mathcal{F}(P)$ defined by $\psi(F) = F \cap H$ is a bijection and* $\mathrm{dim}F = \mathrm{dim}(F \cap H) - 1$.

Proof. The half-spaces that define $\mathcal{C}$ be given by the following set of inequalities.

$$a_1 \cdot x \geq 0$$
$$a_2 \cdot x \geq 0$$
$$\vdots$$
$$a_n \cdot x \geq 0,$$

where $a_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ and $x = (x_1, \ldots, x_n)$. Define the hyperplanes

$$H^0 = \{x \mid (a_1 + \cdots + a_n) \cdot x = 0\}$$

and

$$H^1 = 1 + H^0.$$

It is easy to check that H^0 is a supporting hyperplane for $\mathcal{C}$, $\mathcal{C} \cap H^0 = \{0\}$ and $\mathcal{C} \subseteq (H^0)^+$. Now we claim that $\mathcal{C} \cap H^1$ is a nondegenerate cross section. For $x \in \mathcal{C}$ and $x \neq 0$, the ray through x originating from 0 is given by $R = \{\lambda x | \lambda \geq 0\}$. If $\lambda x \in H^1$ then $(\sum_{i=1}^n a_i).x = 1$. Note here that $(\sum_{i=1}^n a_i).x > 1$, since $x \in (H^0)^+ - H^0$. Therefore the unique value for λ for x is $\frac{1}{(\sum_{i=1}^n a_i).x}$. This proves that $\mathcal{C} \cap H^1$ is nondegenerate.

Proof of (i). P is clearly a convex polyhedron. To show that it is a polytope we only need to show that P is bounded. If P is not bounded, then it contains a ray $R(x_0, \alpha) = \{x_0 + t\alpha | t \geq 0, x_0 \in P, \alpha \in H^0\}$. Every ray from 0 that intersects $R(x_0, \alpha)$ is of the form $R_t = \{\lambda(x_o + t\alpha) | \lambda \geq)\}$. Let $A = \cup_{t \geq 0} R_t \subseteq \mathcal{C}$. Note that any element of the form $\lambda x_0 + s\alpha \in A$ for all $\lambda, s \geq 0$. If we fix s, since $s = \lambda t$, we have $\lambda = s/t$. When $t \to \infty$, $\lambda \to 0$. So $s\alpha \in \overline{A} \subseteq \mathcal{C}$. Choose $s = 1$. This gives contradiction to the fact that $\mathcal{C} \cap H^0 = \{o\}$. So P must be bounded and hence is a polytope.

Proof of (ii). To say that ψ is well defined we must show that for any face F of $\mathcal{C}$, $F \cap H^1$ is a face of P. Since every ray passing through an interior point of $\mathcal{C}$ lies in $\{0\} \cup \text{int}\mathcal{C}$ and intersects H^1 at a nonzero point, $H^1 \cap \text{int}\mathcal{C} \neq \emptyset$,. Therefore $\dim P = \dim H^1 = m-1$. Let H be a supporting hyperplane for $\mathcal{C}$ such that $H \cap \mathcal{C} = F$, then $H \cap H^1$ is a supporting hyperplane in H^1 for P that intersects P at $F \cap H^1$. So $F \cap H^1$ is a face of P.

To show ψ is one to one, let F and G be two faces of $\mathcal{C}$ such that $H^1 \cap F = H^1 \cap G$. Consider $\{\lambda x \mid x \in H^1 \cap F, \lambda \geq 0\}$, which is equal to the set $\{\lambda x | x \in H^1 \cap G, \lambda \geq 0\}$ since H^1 intersects $\mathcal{C}$ at nondegenerate cross section. So $F = G$.

To show ψ is onto, consider a face F' of P with supporting hyperplane H' in H^1. Then $\text{Aff}(0, H')$ will be a supporting hyperplane of $\mathcal{C}$ that intersects $\mathcal{C}$ at a face F such that $F \cap \mathcal{C} = F'$. So $\psi(F) = F'$.

Now the only thing left is to show $\dim\psi(F) = \dim F - 1$. $\text{Aff} F$ is intersection of $\dim F$ number of facet planes that are not parallel to H^0 and so to H^1. Therefore $\dim\psi(F) = \dim\text{Aff} F \cap H^1 = \dim\text{Aff} F - 1 = \dim F - 1$. □

Corollary. If $\mathcal{C}$ is convex pointed polyhedral cone then it is the convex hull of its extremal rays.

Proof. Every extremal ray of $\mathcal{C}$ corresponds to a vertex of P. Now given $x \in \mathcal{C}$, $x \neq 0$, there exists $\lambda > 0$ such that $\lambda x \in P$. Since λx is in the convex hull of vertices of P (by Theorem 1.18), x is convex hull of extremal rays of $\mathcal{C}$. □

Here we conclude this section and will go back to our original problem in the next section.

3. The Polyhedral Cone of a System of Linear Equations

Our interest is in the non-negative integral solutions of the system of linear equations $\Phi x = 0$. Instead of non-negative integral solutions, if we ask for all non-negative real-valued solutions, the set of solutions is a convex pointed polyhedral cone in $\mathbb{R}^n$ and the integral solutions will form a lattice in this cone. To justify this let $\Phi = [a_{i,j}]_{m\times n}$. The non-negative solutions will be given by the intersection of the following half-spaces:

$$(*)\qquad \begin{aligned} a_{1,1}x_1 + \cdots + a_{1,n}x_n &\geq 0 \\ a_{1,1}x_1 + \cdots + a_{1,n}x_n &\leq 0 \\ &\vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n &\geq 0 \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n &\leq 0 \\ x_1 &\geq 0 \\ &\vdots \\ x_n &\geq 0 \end{aligned}$$

Note that all the hyperplanes for these half-spaces pass through origin. So the set of non-negative real-valued solutions, say $\mathcal{C}_\Phi$, is a convex polyhedral cone. To see that it is pointed, consider the hyperplane $H = \{(y_1, \ldots, y_n) | \sum_{i=1}^n y_i = 0\}$. Clearly H is a supporting hyperplane for $\mathcal{C}_\Phi$, intersecting it only at the origin $\{o\}$. So $\mathcal{C}_\Phi$ is pointed. We will always assume $\mathcal{C}_\Phi$ is nonzero.

The special thing about $\mathcal{C}_\Phi$ in comparison to other pointed cones is that all its faces are uniquely determined by its support:

Definition 1.27. *Given $x \in \mathbb{R}^n$ such that $x = (x_1, \ldots, x_n)$, define its support to be* $\mathrm{Supp}(x) = \{i | x_i \neq 0\}$. *For a subset A of $\mathbb{R}^n$,* $\mathrm{Supp}(A) = \cup_{x\in A}\mathrm{Supp}(x)$.

Theorem 1.28. *Let B_n be all the subsets of the set $\{0, 1, \ldots, n\}$. Define a map f : $\mathcal{F}(\mathcal{C}_\Phi) \longrightarrow B_n$ by $f(F) = \mathrm{Supp}(F)$, then f is one to one.*

Proof. We will prove it by induction on n. For the base case consider $n = 2$. In this case if $\mathcal{C}_\Phi \neq 0$ then $\mathcal{C}_\Phi$ is a ray that starts from the origin. Clearly, $\mathrm{Supp}(0) = \emptyset$ and $\mathrm{Supp}(\mathcal{C}_\Phi) \neq \emptyset$. Assume that the induction hypothesis is true up to $n - 1$. Now let $\mathcal{C}_\Phi$ be a cone in $\mathbb{R}^n$. Let the coordinate planes be $G_i = \{x | x_i = 0\}$. If $\dim\mathcal{C}_\Phi = 1$ the theorem is trivial. So assume $\dim\mathcal{C}_\Phi \geq 2$.

Claim 1. Every face has to be contained in some G_i.

If $x \in \mathcal{C}_\Phi$ that is not in any G_i then there exists an open ball around x, say B, which does not intersect any G_i. Let the solution space of $\Phi x = 0$ be V, then $B \cap \mathcal{C}_\Phi = B \cap V$, which is a open ball in V and hence in $\mathcal{C}_\Phi$. So $x \in \text{int}\mathcal{C}_\Phi$. This says that every element of the boundary of $\mathcal{C}_\Phi$ is contained in some G_i, again since G_i are flat and faces of $\mathcal{C}_\Phi$ have at most dimension n every face of $\mathcal{C}_\Phi$ is contained in some G_i.

Claim 2. If F_1, F_2 are two distinct faces of $\mathcal{C}_\Phi$ then $\text{Supp}(F_1) \neq \text{Supp}(F_2)$.

If F_1 and F_2 do not lie in same G_i then clearly we are done. Otherwise F_1 and F_2 are faces of $\mathcal{C}_\Phi \cap G_i$ and we are done by induction. This completes the proof of Claim 2 and hence proof of the theorem. □

To understand the non-negative integral solutions of E_Φ we need to deal with some kind of generating set for E_Φ. One way is as follows.

Definition 1.29. *$\beta \in E_\Phi$ is called a completely fundamental solution if, for all positive integers n and $\alpha, \alpha' \in E_\Phi$ such that $n\beta = \alpha + \alpha'$, we have $\alpha = i\beta$ and $\alpha' = (n-i)\beta$ where i and $n-i$ are positive integers. We will denote the set of all completely fundamental solutions by $CF(E_\Phi)$ or just CF when the context is clear.*

Theorem 1.30. *$\beta \in CF$ if and only if β satisfies the following two properties:*

(i) *β is in some extremal ray say R_β of C_Φ.*

(ii) *Every $\beta' \in R_\beta \cap E_\Phi$ satisfies $\beta - \beta' \in E_\Phi$.*

Proof. Let $\beta \in CF$. If $\text{Supp}\beta$ is not minimal then there exists $\alpha \in E_\Phi$ such that $\text{Supp}\alpha \subset \text{Supp}\beta$. Now choose a large enough n such that $n\beta - \alpha \in E_\Phi$. This contradicts the fact that $\beta \in CF$. So $\text{Supp}\beta$ is minimal, hence by Theorem 1.26 β is contained in an extremal ray R_β. This completes the proof of (i). For (ii), we know that $R_\beta \cap E_\Phi$ is a singly generated monoid. Because it is a completely fundamental element, β must be the generator of this monoid which implies (ii).

For the converse, suppose β is an element of E_Φ that satisfies conditions (i) and (ii). Suppose that for some positive integer n, $n\beta = \alpha + \alpha', \alpha, \alpha' \in E_\Phi$. Since β is in an extremal ray R_β, $\alpha, \alpha' \in R_\beta$. By (ii) since β is the generator of the monoid $R_\beta \cap E_\Phi$, $\alpha = i\beta$ and $\alpha' = i'\beta$ where $i, i' \in \mathbb{Z}_{>0}$ and $i + i' = n$. This shows $\beta \in CF$. □

Corollary. $CF(E_\Phi)$ is finite and unique.

Elements of CF are the generators of the extremal rays of $\mathcal{C}_\Phi$. So every element of E_Φ can be expressed as a positive rational linear combination of elements of CF. But this expression may not be unique. For example, consider $\Phi = [1 \quad -1 \quad 1 \quad -1]$, then

$$CF = \{(1,1,0,0),(1,0,0,1),(0,0,1,1),(0,1,1,0)\}.$$

Now, $(1,1,1,1) = (1,1,0,0)+(0,0,1,1) = (1,0,0,1)+(0,1,1,0)$. But if elements of CF are linearly independent, then we can guarantee the uniqueness.

Definition 1.31. *A pointed convex polyhedral cone is called simplicial if non-zero vectors chosen from its distinct extremal rays are linearly independent.*

We will call a hyperplane rational if the equation that defines it has rational coefficients. Similarly a convex polyhedron is said to be rational if all the intersecting half-spaces are defined by rational hyperplanes.

Definition 1.32. *A simplicial monoid is defined to be the set of lattice points of a rational simplicial cone.*

For a simplicial monoid S, $CF(S)$ is a linearly independent set. Let $CF(S) = \{\alpha_1,\dots,\alpha_t\}$. So CF is t dimensional. Here $\alpha_1,\dots,\alpha_t$ are called quasi-generators of S. Define

$$D_S = \{\gamma \in S | \gamma = a_1\alpha_1 + \cdots + a_t\alpha_t, 0 \leq a_i < 1\}.$$

Lemma 1.33. *S be a simplicial monoid with quasi-generators $\alpha_1,\dots,\alpha_t$. Then every element $\gamma \in S$ can be written uniquely in the form $\gamma = \beta + a_1\alpha_1 + \cdots + a_t\alpha_t$ where $\beta \in D_S$ and a_i are non-negative integers.*

Proof. Suppose that γ is expressed in terms of the quasi-generators as $\gamma = b_1\alpha_1 + \cdots + b_t\alpha_t$, where b_i's are positive rationals. If $b_i' = b_i - [b_i]$, then $\gamma = (b_1'\alpha_1 + \cdots + b_t'\alpha_t) + ([b_1]\alpha_1 + \cdots + [b_t]\alpha_t)$. We choose $\beta = b_1'\alpha_1 + \cdots + b_t'\alpha_t$ and $a_i = [b_i]$. Now to prove the uniqueness of the expression let $\gamma = \beta + a_1\alpha_1 + \cdots + a_t\alpha_t = \beta' + a_1'\alpha_1 + \cdots + a_t'\alpha_t$ such that $\beta, \beta' \in D_S$ and $a_i \in \mathbb{Z}_{\geq 0}$. So $\beta - \beta' = (a_1 - a_1')\alpha_1 + \cdots + (a_t - a_t')\alpha_t$. Since $\beta, \beta' \in D_S$, $a_i - a_i' \in (-1,1)$ and is integer as a_i, a_i' are integers. So $a_i - a_i = 0$, hence $\beta = \beta'$. Also

$a_1\alpha_1 + \cdots + a_t\alpha_t = a'_1\alpha_1 + \cdots + a'_t\alpha_t$. By the linear independence of α_i, $a_i = a'_i$. This completes the proof of uniqueness of the expression of γ. $\square$

In the beginning of the chapter we have remarked that any subset $S \subseteq \mathbb{N}$ is completely described by its generating function $S(x)$. The above lemma helps us to write $S(x)$ for a simplicial monoid S in a nice form.

Corollary. We have

$$S(x) = \frac{\sum_{\beta \in D_S} x^\beta}{\prod_{i=1}^t (1 - x^{\alpha_i})}.$$

PROOF. Indeed,

$$\begin{aligned} S(x) &= \sum_{\beta \in D_S} \sum_{a_i \in \mathbb{N}} x^{\beta + a_1\alpha_1 + \cdots + a_t\alpha_t} \\ &= \sum_{\beta \in D_S} x^\beta \Big(\sum_{a_i \in \mathbb{N}} \prod_{i=1}^t (x^{\alpha_i})^{a_i} \Big) \\ &= \frac{\sum_{\beta \in D_S} x^\beta}{\prod_{i=1}^t (1 - x^{\alpha_i})} \end{aligned}$$

as claimed. $\square$

This shows that the generating function of a simplicial monoid is a rational function, where the numerator is a polynomial with positive coefficients. Can we use this to express the generating function function of E_Φ in such a nice form? We will now see that this is done by decomposing $\mathcal{C}_\Phi$ into simplicial cones.

Definition 1.34. *A triangulation of a pointed convex polyhedral cone $\mathcal{C}$ is a collection $\Gamma = \{\sigma_1, \ldots, \sigma_r\}$ of simplicial cones such that the following three properties are satisfied:*

(i) $\cup_{i=1}^t \sigma_i = \mathcal{C}$

(ii) *If $\sigma \in \Gamma$ then every nonzero face of σ also in Γ*

(iii) *$\sigma_i \cup \sigma_j$ is a common face of σ_i and σ_j.*

Here is an important theorem about triangulation of a pointed polyhedral cone.

Theorem 1.35. *A pointed convex polyhedral cone $\mathcal{C}$ possess a triangulation Γ whose extremal rays are the extremal rays of $\mathcal{C}$.*

We are skipping the proof of this theorem in this chapter and will come back to it in the next chapter after introducing the concept of pulling of vertices of a convex polytope. Although a proof without using pulling of vertices can be given (by induction on the dimension of P using Theorem 2.10, see also [**BR07**, Appendix B]), we prefer to use pulling , since it gives a *shellable* triangulation (Corollary to Theorem 2.10). Our next theorem is just a corollary of the above one, but separately given as a theorem to emphasize its importance.

Theorem 1.36. *For a system of linear diophantine equations* $\Phi x = 0$*,*

$$E_\Phi(x) = \frac{p(x)}{\prod_{\beta \in CF}(1 - x^\beta)}$$

Here $p(x)$ *is a polynomial in* $x_1, \dots, x_n$*.*

PROOF. Let Γ be a triangulation of $\mathcal{C}_\Phi$ such that the extremal rays of Γ are the extremal rays of $\mathcal{C}_\Phi$. Let $\sigma_1, \dots, \sigma_t$ be the maximal elements of Γ (with respect to inclusion). Let S_i be the lattice points of σ_i, which is a simplicial monoid. Since $C_\Phi = \cup_{i=1}^t \sigma_i$, $E_\Phi = \cup_{i=1}^t S_i$. Fix the notation $[t] = \{1, \dots, t\}$ for each positive integer t. For any subset A of $[t]$ define $S_A = \cap_{i \in A} S_i$. If S_A is non-zero then it is a simplicial monoid. By corollary to the Lemma 1.33, $S_A(x) = \frac{p_A(x)}{\prod_{\beta \in CF(S_A)}(1 - x^\beta)}$. Now we will use the principle of inclusion exclusion to write a formula for $E_\Phi(x)$:

$$E_\Phi(x) = \sum_{A \subseteq [t]} (-1)^{|A|-1} \frac{p_A(x)}{\prod_{\beta \in CF(S_A)}(1 - x^\beta)}.$$

By our triangulation $CF_\Phi = \cup_{A \subseteq [t]} CF(S_A)$. So $E_\Phi(x) = \frac{p(x)}{\prod_{\beta \in CF}(1 - x^\beta)}$. (Note here that $p(x)$ may not have positive coefficients.) □

From the above theorem we know that $E_\Phi(x)$ is a rational function where the denominator is given with respect to completely fundamental elements of $E_\Phi(x)$. But *to know the numerator exactly* is a difficult task. The next two chapters of this article are developed in this direction.

Notes

In this chapter, Section 2 serves as an introduction to the theory of convex polytopes and convex polyhedra. I have referred to Grunbaum's book [**Grü03**] for this section. The proof of Theorem 1.12 and a part of the proof of Theorem 1.21 are taken from this book. Materials of Section 3, Theorems 1.30, 1.35, 1.36 and Lemma 1.33, are taken from [**Sta12**] (see 4.6). For a brief introduction to generating functions see [**Sta78**].

CHAPTER 2

The Stanley-Bruggesser-Mani Decomposition

In this section we will go through two important techniques regarding a polytope P. The first one is called *pulling of vertices* which is used to triangulate faces of P without introducing new vertices; this is something similar to Theorem 1.35 in case of convex polyhedral cone. In fact we will prove Theorem 1.35 using pulling. The second one is of great importance as it proves two well known conjectures, namely the *Upper bound Conjecture* proved by Macmullen [**McM70**] and the *ADG Conjecture* proved by Stanley [**Sta82**]. It is called the *shelling of facets of a polytope.* Finally, in the third section we will see the Stanley-Bruggesser-Mani decomposition of the monoid E_Φ, which uses both shelling and pulling and is a beautiful as well as useful theorem about E_Φ.

1. Pulling the Vertices of a Polytope

We will begin with a n-dimensional polytope in $\mathbb{R}^n$.

Definition 2.1. *Let H be a hyperplane in $\mathbb{R}^n$ such that P is contained in one of the half-spaces determined by H. Say $P \subseteq H^+$. For some $w \in \mathbb{R}^n - P$, we say w is beneath H if $w \in \text{int}H^+$ and w is beyond H if $w \in \text{int}H^-$. If $w \in \mathbb{R}^n - P$, and for all facet planes H of P, $w \notin H$, we say w is admissible. Further for some facet F of P, we say that an admissible point w is beyond or beneath F if and only if it is so for the facet plane $\text{Aff}F$.*

The following theorem explains the faces of the new polytope that are obtained by adding a new vertex to a given polytope.

Theorem 2.2. *Let P be a n-dimensional polytope in $\mathbb{R}^n$, and let $w \in \mathbb{R}^n - P$. Then all the faces of the polytope $P' := \text{Conv}(\{w\} \cup P)$ other than the vertex w are characterized as follows:*

(i) *A face F of P is a face of P' if and only if there is a facet F'' of P which contains F, and w is beneath F'' with respect to P.*

(ii) *If F is a face of P, then $F' = \text{Conv}(\{w\} \cup F)$ is a face of P' if and only if at least one of the following two conditions holds*

(a) $w \in \text{Aff}F$

(b) *w is beyond at least one facet of P containing F and beneath another.*

Proof. P' is a polytope, and since $w \notin P$, w is a vertex of P'. First we will show that the faces of P' are either of type (i) or (ii). Let's begin the proof of this fact with the following claim:

Claim. Let H be a supporting hyperplane for P' (we exclude the case when H intersects P' only at w). Then $H \cap P$ is nonempty.

If H does not contain w, consider $\lambda x + (1-\lambda)w \in H$ for some $0 < \lambda \leq 1$ and $x \in P$. If $\lambda < 1$ then x and w lie in two different open half spaces determined by H which gives a contradiction to the fact that H is supporting for P'. So $\lambda = 1$ and $x \in H$. Now if H contains w and some other element, say $\lambda x + (1-\lambda)w$, where $\lambda > 0$ and $x \in P$ then the line through x and w contained in H. So $x \in H$. This shows in any case $H \cap P \neq \emptyset$.

The above claim shows that H is a supporting hyperplane for P and so $H \cap P$ is a face of P. If $w \notin H$ then $H \cap P'$ is a face of type (i), and if $w \in H$, $H \cap P'$ is a face of type (ii).

Proof of (i). Let F be a face contained in a facet F'' of P and w be beneath F'' with respect to P. Then $\text{Aff}F''$ is a supporting hyperplane of P', intersecting it at F''. So F'' is a face of P' too. Since F is a face of F'', it is a face of P'. To prove the other implication, let F be a common face of P and P'. If all the facets of P' containing F contain w, then their intersection which is F also contains w. This is a contradiction to $w \notin F$. So we must have a facet F'' of P' that contains F and $w \notin F''$. F'' is a face of P, since it is face of type (i) of P'. Clearly w is beneath F'' with respect to P.

Proof of (ii). Suppose, for a face F of P, $\text{Conv}(\{w\} \cup F)$ is a face of P'. Choose $x \in \text{relint}F$ and $y \in \text{int}P$ such that $A = \text{Aff}\{x, y, w\}$ is a two dimensional face (this is possible if we are assuming $n \geq 2$; for $n = 1$ the statement of the theorem is trivial). Let $P_1 = P \cap A$. P_1 is a polygon and $F_1 = F \cap A$ is a face of it. $P_1' = P' \cap A = \text{Conv}(\{w\} \cup P_1)$. If $F_1' = \text{Conv}(F_1 \cup \{w\})$ is a face of P_1' then either F_1 is a vertex or $\text{Aff}F_1$ contains w, in which case F_1 is an edge. If $w \in \text{Aff}F_1$, then clearly $w \in \text{Aff}F$. For the other case, when

F_1 is a vertex, it is intersection of two edges of P_1. It's easy to check that w is beneath one of these edges and beyond the other. The same is true for the facets containing these edges. Now we will prove, if (a) or (b) of (ii) is satisfied, then $\mathrm{Conv}(\{w\} \cup F)$ is a face of P'. The case of (a) is trivial. For (b), assume that $0 \in F$. Let F_1 and F_2 be two facets of P such that $F \subseteq F_1 \cap F_0$, w is beyond F_1 and beneath F_2. Let $H_1 = \{x | x.a_1 = 0\}$ be $\mathrm{Aff} F_1$, $H_2 = \{x | x.a_2 = 0\}$ be $\mathrm{Aff} F_2$ and $P \subseteq H_1^+ \cap H_2^+$. Lat $H_0 = \{x | x.a_0 = 0\}$ be a supporting hyperplane for P that intersects it at F. By our assumption $w \cdot a_1 < 0$ and $w \cdot a_2 > 0$. If $w \in H_0$, then H_0 will be a supporting hyperplane for P' intersecting it at $\mathrm{Conv}(\{w\} \cup F)$ and so we are done. Otherwise $w \cdot a_0 > 0$ or $w \cdot a_0 < 0$. We will only treat the case $w \cdot a_0 > 0$, as other case is similar. Let $c = \frac{w \cdot a_0 + w \cdot a_2}{-w \cdot a_1} > 0$, and $b := a_0 + a_2 + c a_1$. Define $H = \{x | x.b = 0\}$. It is easy to check that H is a supporting hyperplane that intersects P' at $\mathrm{Conv}(\{w\} \cup F)$. □

Definition 2.3. *Consider* $\mathrm{bd}P$. *Let w be an admissible point. The set of all visible points of* $\mathrm{bd}P$ *from w is defined as* $S(P, w) = \cup_{F \in \Lambda_v} F$ *where*

$$\Lambda_v = \{F | F \text{ is a facet of } P \text{ and } w \text{ is beyond } F\}.$$

Similarly the set of invisible points on $\mathrm{bd}P$ *is defined as* $U(P, w) = \cup_{F \in \Lambda_u} F$ *where*

$$\Lambda_u = \{F | F \text{ is a facet of } P \text{ and } w \text{ is beneath } F\}.$$

Lemma 2.4. *Given a polytope P with a vertex v, there exists an admissible w such that $S(P, w)$ is the union of the facets that contain v and $U(P, w)$ is the union of the facets that do not contain v.*

Proof. Let $F_1, \dots, F_r$ be all the facets that contain v. Let $H_i = \mathrm{Aff} F_i$ and $P \subseteq \cap_{i=1}^r H_i^+$. Since $\cap_{i=1}^r H_i^+$ has nonempty interior so does $\cap_{i=1}^r H_i^-$. Choose a point $w \in \mathrm{int} \cap_{i=1}^r H_i^-$, then w is beyond $F_1, \dots, F_r$ with respect to P and $[w, v) \subseteq \mathrm{int} \cap_{i=1}^r H_i^-$. Consider F, a facet of P that does not contain v. Then $\mathrm{Aff} F$ either does not intersect $[w, v)$ or intersects it exactly at one point. So we can choose w such that $\mathrm{Aff} F$ does not intersect $[w, v)$ for any facet F of P. Since v and w are on the same side of $\mathrm{Aff} F$ for any facet F not containing v, w is beneath F. □

From the above lemma there exists a w such that it is beyond all the facets that contain v and beneath all the facets that do not contain v; also $[w, v)$ does not intersect any facet plane of P.

Definition 2.5. *Let v be a vertex of a polytope P and w be an admissible point such that $[w, v)$ does not intersect any facet plane and v is an interior point of $P' = \mathrm{Conv}(\{w\} \cup P)$, then we say P' is obtained from P by pulling the vertex v to w.*

Lemma 2.6. *We can pull any vertex of a polytope to some admissible point.*

Proof. The w we described in Lemma 2.4 is a point to which v can be pulled. Here we need to verify that v is an interior point of P'. If not, then v is in some facet of P'. From Theorem 2.2, every facet of P' is a face of P not containing v or of the form $\mathrm{Conv}(\{w\} \cup F)$. So v is in a facet of P' which is of the form $\mathrm{Conv}(\{w\} \cup F)$. Since F is a face of P and w is admissible, $w \notin \mathrm{Aff} F$. Again by Theorem 2.2, w is beyond a facet of P containing F. So $\mathrm{Conv}(\{w\} \cup F) \cap P = F$ and since $v \notin F$ we have $v \notin \mathrm{Conv}(\{w\} \cup F)$. We saw in any case v can not be contained in a facet of P', hence v must be an interior point of P'. □

Pulling of vertices brings changes to the structure of the faces of a polytope. We will see that the number of faces of a polytope increases with pulling. But after a finite number of steps the number of faces remains same, which is a kind of 'saturation point' while pulling. At that stage the faces of the polytope become *simplices*. We discuss this briefly in our next theorem.

Definition 2.7. *A cell complex Λ is defined to be a finite set of polytopes in $\mathbb{R}^n$ such that, given $\sigma_1, \sigma_2 \in \Lambda$, we have $\sigma_1 \cap \sigma_2 \in \Lambda$ and is a common face of σ_1 and σ_2. Each element of Λ is called a cell. $A = \cup_{\sigma \in \Lambda} \sigma$ is called a geometric realization of Λ. Conversely, Λ is called a cell complex structure on A.*

We will also use the term 'cell complex' to refer to the geometric realization of a cell complex.

Definition 2.8. *A polytope is called a simplex if it is convex hull of $k + 1$ vertices and is of dimension k. A triangulation of a subset A of $\mathbb{R}^n$ is a cell complex structure on*

A *whose every element is a simplex. A simplicial cell complex is a cell complex (or the geometric realization of a cell complex) whose every element is a simplex.*

Note that if a polytope A is a simplex then the convex hull of any proper subset of vertices of A gives a face of A.

The following theorem gives a way to triangulate the boundary of a polytope by pulling its vertices.

Theorem 2.9. *By successively pulling the vertices of a polytope P finitely many times we can obtain a polytope P' such that* $\text{bd}P'$ *is a simplicial cell complex. Further let* $\Gamma' = \{F | F \in \mathcal{F}(P')\}$ *and define* $f : \text{Vert}P' \to \text{Vert}P$ *by setting* $f(v') = v$ *if* v' *is pulled from* v *successively. Construct* $\Gamma = \{F | F = \text{Conv}(f(\text{Vert}F')), F' \in \Gamma'\}$. *Then* Γ *is a triangulation of* $\text{bd}P$.

Proof. If $\dim P = 1$ the above theorem is trivial. If $\dim P = 2$ then P is a polygon. Since all the faces are one dimensional, $\text{bd}P$ is simplicial. So assume $\dim P \geq 3$. Let v be a vertex of P and let P_1 be obtained from P by pulling v to v'.

Claim. If all the i dimensional faces of P are simplices then all the $i+1$ dimensional faces of P_1 containing v' are simplices.

By Theorem 2.2, the faces of P_1 that contain v' are of the form $\text{Conv}(\{v'\} \cup F)$, where F is a face of P. If $\dim\text{Conv}(\{v'\} \cup F) = i+1$, then $\dim F = i$. Since all i dimensional faces of P are simplices, we have that F is a simplex and so $\text{Conv}(\{v'\} \cup F)$ is a simplex. This proves our claim.

By the above claim if we successively pull all the vertices of the polytope P at least once and obtain a new polytope P_2, then all the two dimensional faces of P_2 are simplices. Continuing this way, we obtain a polytope P' all of whose faces are simplices. Note that the number of vertices of P is equal to the number of vertices of P'.

Now to show Γ is a triangulation of $\text{bd}P$ we need to show that the following two properties hold:

(1) $\text{bd}P = \cup_{G \in \Gamma} G$

(2) If G_1, $G_2 \in \Gamma$ then $G_1 \cap G_2 \in \Gamma$ and is a common face of G_1 and G_2.

We will prove it by using induction on the number of pullings. For the base case, suppose we obtained P_1 from P by pulling a vertex v to v_1. Let $\Gamma_1 = \{\mathrm{Conv}(f_1(\mathrm{Vert}F))|F \in \mathcal{F}(P_1)\}$, where $f_1 : \mathrm{Vert}P_1 \longrightarrow \mathrm{Vert}P$ is defined by $f_1(v_1) = v$ and f_1 is identity on all other vertices. We will show condition (1) and (2) for Γ_1.

For (1), note that all the faces which do not contain v are faces of P_1. So we consider $x \in \mathrm{bd}P$ that is contained in a facet F_1 which contains v. The line $L(x, v)$ intersects the boundary of F_1 at a unique point other than v, say u. We claim that u is in a facet of F_1 that does not contain v. We will prove this by induction on the dimension of F_1. Let u be in a facet F_2 of F_1 that contains v (otherwise the proof of our claim is clear). The line $L(x, v)$ is contained in $\mathrm{Aff}F_2$ and intersects the boundary of F_2 at u. By induction hypothesis there exists a facet of F_2 not containing v and containing u. But this facet of F_2 is the intersection of a facet of F_1 not containing v with F_2. So we have proved that there is a facet F of F_1 that does not contain v but contains u. F is the intersection of two facets, one of which contains v and other does not contain v. So $\mathrm{Conv}(\{v\} \cup F) \in \Gamma_1$. But $x \in \mathrm{Conv}(\{v\} \cup F)$ since $x \in [v, u]$. This completes the proof of the fact that $\mathrm{bd}P = \cup_{G\in\Gamma_1} G$.

To show condition (2) holds for Γ_1, we will characterize the intersection of two elements σ_1 and σ_2 of Γ_1 in the following three cases:

(a) σ_1 and σ_2 are faces of P not containing v.
(b) σ_1 is a face of P not containing v but $\sigma_2 = \mathrm{Conv}(\{v\} \cup F_2)$
(c) $\sigma_1 = \mathrm{Conv}(\{v\} \cup F_1)$ and $\sigma_2 = \mathrm{Conv}(\{v\} \cup F_2)$

Case (a) clearly satisfies (2). For Case (b) first we will show that if $\sigma_1 \cap F_2 = \emptyset$ then $\sigma_1 \cap \sigma_2 = \emptyset$. If not, let $x \in \sigma_1 \cap \sigma_2 \neq \emptyset$. Let H be a supporting hyperplane for P determining σ_1 and $P \subseteq H^+$. Let the line $L(x, v)$ intersect F_2 at u. Then $x \in [v, u]$. But $[v, u] \subseteq \mathrm{int}(H^+)$ since $\sigma_1 \cap F_2 = \emptyset$. This contradicts the assumption that $x \in \sigma_1$. Now suppose $\sigma_1 \cap F_2 \neq \emptyset$. Any $x \in \sigma_1 \cap \sigma_2$ is of the form $x = \lambda v + (1 - \lambda)y$ for some $y \in F_2$ and $0 < \lambda \leq 1$. We will show that $x \in \sigma_1 \cap F_2$. If $\lambda = 1$ then we are done, so assume $\lambda \neq 1$. If $y \in \sigma_1 \cap F_2$ then $v \in \sigma_1$ since $x \in \sigma_1$, which is not possible. If $y \notin \sigma_1$ then the supporting hyperplane for P determining σ_1 has to separate y and v giving a contradiction. It follows that λ must be 1 and so $x \in \sigma_1 \cap F_2$. We conclude that

$\sigma_1 \cap \sigma_2 = \sigma_1 \cap F_2$, which is a common face of σ_1 and σ_2, and is in Γ_1. For Case (c), since $\text{Conv}(\{v\} \cup (F_1 \cap F_2)$ is a common face of F_1 and F_2 and is contained in Γ_1, we only need to show $\sigma_1 \cap \sigma_2 = \text{Conv}(\{v\} \cup (F_1 \cap F_2)$. Clearly $\text{Conv}(\{v\} \cup (F_1 \cap F_2) \subseteq \sigma_1 \cap \sigma_2$. For the other inclusion let $x \in \sigma_1 \cap \sigma_2$. Then x can be written in the following two forms: $x = \lambda v + (1 - \lambda)y_1$ and $x = \lambda' v + (1 - \lambda')y_2$, where $y_1 \in F_1$, $y_2 \in F_2$ and $0 < \lambda, \lambda' \leq 1$. Then the line through x, v intersects F_1 at y_1 and F_2 at y_2. But this line intersects bd$P$ at a unique point other than v. So $y_1 = y_2$, $\lambda = \lambda'$ and $x \in \text{Conv}(\{v\} \cup (F_1 \cap F_2))$. This completes the proof of property (2) for Γ_1.

Suppose that, after successively pulling the vertices of P $m+1$ times, we obtained P'. Suppose that P' was obtained from P_m by pulling the vertex v_m to w. Let w correspond to the vertex v of P by f. Let Γ_m be the cell complex structure induced by P_m on bd$P$. By induction hypothesis Γ_m satisfies (1) and (2). We have to prove that Γ satisfies (1) and (2). To prove (1), we will only consider those elements in bd$P$ that lie in some facet of P containing v. Let x be a point on such a facet. Since Γ_m satisfies (1), $x \in G \in \Gamma_m$, where G corresponds to G', a face of P_m containing v_m. Let x correspond to x' in G'. Every line through x' and v_m intersects the boundary of G' in some facet G'' of G' such that $v_m \notin G''$. Then $\text{Conv}(\{w\} \cup G'')$ is a face of P' and so $\text{Conv}(\{v\} \cup G_1) \in \Gamma$ (where G'' correspond to G_1 in P). Since $x \in \text{Conv}(\{v\} \cup G_1)$ this completes the proof of (1). For the case (2), if $\sigma_1, \sigma_2 \in \Gamma$, we only need to consider the cases when at least one of σ_1, σ_2 is not in Γ_m. Proof for these cases are similar to that of case (b) and (c) of Γ_1. □

Theorem 2.10. *P be a polytope and v be a vertex of P. Let Λ' be a triangulation of the set of facets of* bd$P$ *that do not contain v. Define $A = \{\text{Conv}(v, F) | F \in \Lambda'\}$ and let Λ be the set consisting of all the elements of A along with their faces. Then Λ is a triangulation of P.*

Proof. Let $x \in P$ and $x \neq v$. The line through x and v intersects a facet of P not containing v and so an element of Λ', say it F. Then $x \in \text{Conv}(\{v\} \cup F)$. This shows that $P = \cup_{\sigma \in \Lambda} \sigma$. Now to show the intersection property, let $\sigma_1 = \text{Conv}(\{v\} \cup F_1)$ and $\sigma_2 = \text{Conv}(\{v\} \cup F_2)$, where F_1 and F_2 are in Λ'. If $x \in \sigma_1 \cap \sigma_2$, then the line through x and v intersects bd$P$ at a unique point other than v. So the line intersects $F_1 \cap F_2$

and hence $x \in \mathrm{Conv}(\{v\} \cup F_1 \cap F_2)$ which is a common face of σ_1 and σ_2 and is in Λ. Verifying the intersection property for the other cases is trivial. □

Corollary. Let Γ be as in Theorem 2.9. Let v be the vertex that was pulled last. Let $\Lambda' = \{F \in \Gamma | v \notin F\}$. Then Λ constructed in Theorem 2.10 is a triangulation of P. Also $\mathrm{bd}\Lambda' := \{F | F \in \Lambda', F \subseteq \mathrm{bd}P\}$ is the same as Γ.

The proof of the above corollary is clear from the construction of Γ in Theorem 2.9 and by the construction of Λ in Theorem 2.10.

2. Shelling and the Bruggesser-Mani Theorem

Definition 2.11. *Let P be a polytope. A line G is said to be in general position with respect to P if*

(1) *it is not parallel to any of the facet planes of P*
(2) *it intersects the facet planes of P at distinct points.*

Lemma 2.12. *Given an admissible point w with respect to the polytope P, there exists a line in general position passing through w that intersects* $\mathrm{int}P$.

Proof. Let $F_1, \ldots, F_r$ be the facets of P. The set of all lines that pass through w and are parallel to F_i forms a hyperplane H_i through w. Let $H_{i,j}$ be the hyperplane passing through $\mathrm{Aff}F_i \cap \mathrm{Aff}F_j$ and w. Since P is n dimensional, $\mathrm{int}P \nsubseteq \cup_{i=1}^{r} H_i \cup_{i,j} H_{i,j}$. So there exists a line in general position that passes through w and $\mathrm{int}P$. □

Definition 2.13. *Let G be a line in general position as described in Lemma 2.12. Then G intersects* $\mathrm{bd}P$ *at two distinct points p_1 and p_2. Suppose that $(p_1, w]$ does not intersect P and $(p_2, w]$ intersects P. Let the connected components of $G - \mathrm{int}P$ be G_1 and G_2 where w, $p_1 \in G_1$ and $p_2 \in G_2$. We will give a linear ordering on $G - \mathrm{int}P$ as follows; given $x, y \in G - \mathrm{int}P$ we will say $y \geq x$ if*

(1) *$x \in G_1$ and $y \in G_2$ or if,*
(2) *$x, y \in G_1$ and $x \in [p_1, y]$ or if,*
(3) *$x, y \in G_2$ and $y \in [x, p_2]$.*

Lemma 2.14. *Let $t \in G_1$. For a facet F of P, let $g(F)$ be the point of intersection of G with* Aff$F$*. Let $t \neq g(F)$ for any F. Then*

$$S(P,t) = \bigcup_{\substack{F \text{ is a facet of } P \\ g(F)<t}} F$$

and

$$U(P,t) = \bigcup_{\substack{F \text{ is a facet of } P \\ g(F)>t}} F.$$

PROOF. $F \subseteq S(P,t)$ if and only if t and P are on different sides of Aff$F$. Since $g(F) < t$ and $t \in G_1$, clearly t and p_1 are on different sides of Aff$F$ and therefore t and P are on different sides of Aff$F$. Conversely, given $t > p_1$ and $g(F) < t$, Aff$F$ intersects the line joining t and p_1 and so t is beyond Aff$F$. This completes the proof of the expression for $S(P,t)$. The expression for $U(P,t)$ follows since it contains all the facets that are not contained in $S(P,t)$. □

We now come to the theorem for shellability, which is due to Bruggesser and Mani [**BM71**, Section 4, Prop. 2]:

Theorem 2.15 (Bruggesser and Mani)**.** *Let $P \subseteq \mathbb{R}^n$ be an n-dimensional polytope and $w \in \mathbb{R}^n$ be an admissible point with respect to P. Then we can arrange facets of $S(P,w)$ (Similarly $U(P,w)$) as $F_1, \ldots, F_r$ such that for all $i \geq 2$, $F_i \cap (\cup_{j=1}^{i-1} F_j)$ is a union of facets of F_i. This technique of arranging facets is called the shelling of facets of $S(P,w)$ (or $U(P,w)$).*

PROOF. We will apply induction on the number of facets. If there is only one facet the result is trivial. Assume by induction hypothesis that the result is true for any $S(P,w)$ (or $U(P,w)$) having fewer than r facets. Consider a line G in general position that intersects P in its interior and passes through w. By Lemma 2.12 such a line exists. Suppose the line from w intersects P at p_1 and leaves it at p_2. We will arrange the facets of P as $\{F_i\}_{i=1}^k$ such that according to the linear order described in Definition 2.13; $i \leq j$ if and only if $g(F_i) \leq g(F_j)$. Therefore $p_1 = g(F_1) < g(F_2) < \cdots < g(F_k) = p_2$.

Case 1. $S(P,w)$; where $S(P,w)$ has r facets.

Let i_0 be the largest integer such that $g(F_{i_0}) < w$. By Lemma 2.14, $i_0 = r$ and $S(P,w) = \cup_{i=1}^{r} F_i$. We have assumed $r > 1$. Let t be a point on G such that $g(F_{r-1}) < t < g(F_r)$. Again by Lemma 2.14 $S(P,t) = \cup_{i=1}^{r-1} F_i$. By induction hypothesis, the given ordering of the facets is a shelling of $S(P,t)$. To show $S(P,w)$ has a shelling we need to show $S(P,t) \cap F_r$ is union of facets of F_r. We claim $S(P,t) \cap F_r = S(F_r, g(F_r))$. Note that $g(F_r)$ is admissible with respect to F_r. In $\mathrm{Aff} F_r$, $g(F_r)$ is beyond $F_i \cap F_r$ if $i < r$ and beneath $F_i \cap F_r$ if $i > r$. So $S(P,t) \cap F_r$ is union of facets of F_r.

Case 2. $U(P,w)$; where $U(P,w)$ has r facets and for all facets F in $U(P,w)$, $g(F) \in G_2$.

In this case we can find w' sufficiently faraway from p_2 in G_2 such that $U(P,w) = S(P,w')$. Now by Case 1, the result follows.

Case 3. $U(P,w)$ where $U(P,w)$ has at least one facet in G_1, and $U(P,w)$ is a union of r facets.

Let j_0 be the smallest number such that $F_{j_0} \in U(P,w)$. Then $j_0 = k - r + 1$. Choose $t \in G_1$ such that $g(F_{j_0}) < t < g(F_{j_0+1})$. By Lemma 2.14, $U(P,t) = \cup_{i=j_0+1}^{k} F_i$, and so $U(P,w) = U(P,t) \cup F_{j_0}$. By the induction hypothesis, the given order on facets is a shelling for $U(P,t)$. We have to show that $U(P,t) \cap F_{j_0}$ is a union of facets of F_{j_0}, which is equivalent to showing that $U(P,t) \cap F_{j_0} = U(F_{j_0}, g(F_{j_0}))$. In $\mathrm{Aff} F_{j_0}$, $g(F_{j_0})$ is beyond $F_i \cap F_{j_0}$ if $i < j_0$ and beneath $F_i \cap F_{j_0}$ if $i > j_0$. So $U(P,t) \cap F_{j_0} = U(F_{j_0}, g(F_{j_0}))$, that is, $U(P,t) \cap F_{j_0}$ is union of facets of F_{j_0}. This completes the proof of Case 3. □

3. The Stanley-Bruggesser-Mani Decomposition

We will return to the problem of understanding non-negative integral solutions of the linear system Φ. Here the same notation as there in Chapter 1 will be used. As before the cone $\mathcal{C}_\Phi$ will have dimension n and will be contained in $\mathbb{R}^n$.

Theorem 2.16 (Stanley, Bruggesser and Mani). *There exist free submonoids $E_1, \dots, E_t$ of E_Φ of rank n and $\delta_1, \dots, \delta_t \in E_\Phi$ such that E_Φ is disjoint union of $\delta_i + E_i$, i.e.,*

$$E_\Phi = \coprod_{i=1}^{t} (\delta_i + E_i).$$

Proof. Let $H^1 = \{(x_1, \dots, x_n) | \sum_{i=1}^{n} x_i = 1, x_i \in \mathbb{R}^n\}$ and $P := H^1 \cap \mathcal{C}_\Phi$ be a nondegenerate cross section of $\mathcal{C}_\Phi$. Then P is an $n-1$ dimensional polytope. By pulling the vertices of P successively we triangulate bd$P$ as described in Theorem 2.9. Let P' be the polytope obtained from P by pulling its vertices and whose boundary complex corresponds to the triangulation of bd$P$ by f (see Theorem 2.9). Let v be the vertex of P which corresponds to the last vertex pulled to obtain P'. Assume that v corresponds to w of P'. There exists a point w' such that $U(P', w')$ is the union of all the facets of P' not containing w. The facets of P' not containing w will correspond to the $n-1$ dimensional elements, say $F_1, \ldots, F_r$, in the triangulation of bd$P$ that do not contain v. Let $G_i := \text{Conv}(\{v\} \cup F_i)$. By Corollary to the Theorem 2.10 the G_i's, along with their faces, give a triangulation of P. As P is a nondegenerate cross section of $\mathcal{C}_\Phi$, by Theorem 1.26, this triangulation of P gives a triangulation of $\mathcal{C}_\Phi$. Let $C(G_i)$ be the cone of G_i, which is the union of all the rays having source 0 and intersecting G_i. The $C(G_i)$'s are the n dimensional elements in the triangulation of $\mathcal{C}_\Phi$. Let Q_i be the monoid of lattice points in $C(G_i)$. Since $C(G_i)$'s are simplicial cones, Q_i are simplicial monoids and so $CF(Q_i)$'s are linearly independent sets. Recall that $D_{Q_i} = \{\sum_{\beta \in CF(Q_i)} a_\beta \beta \mid 0 \leq a_\beta < 1\}$. By Lemma 1.33, $Q_i = \coprod_{\gamma \in D_{Q_j}} (\gamma + \mathbb{N}CF(Q_i))$, where $\mathbb{N}CF(Q_i)$ is the free monoid of $CF(Q_i)$. Since $U(P', w')$ is shellable, so are $F_1, \ldots, F_r$. Assume that our indexing of F_i matches with the ordering for shelling, i.e., $(\cup_{i=1}^{j-1} F_i) \cap F_j$ is a union of facets of F_j. Then $(\cup_{i=1}^{j-1} G_i) \cap G_j$ is union of facets of G_j and $(\cup_{i=1}^{j-1} C(G_i)) \cap C(G_j)$ is union of facets of $C(G_j)$. We will prove the theorem with the following claims.

Claim 1. There is a unique face G_j' of G_j which is minimal with respect to being not contained in $\cup_{i=1}^{j-1} G_i$.

The set $\cup_{i=1}^{j-1} G_i$ is a union of facets of G_j. Enumerate the facets of G_j as $F_1', F_2', \ldots$. For each k, let x_k denote the unique vertex of G_j that is not contained in F_k'. Let X be the set of all such x_k's. We will show that $G_j' := \text{Conv}X$ has the required property. Note $F_j \nsubseteq \cup_{i=1}^{j-1} G_i$, so X does not contain all the vertices of G_j and hence G_j' is a face of G_j. If $G_j' \subseteq \cup_{i=1}^{j-1} G_i$ then $G_j' \subseteq F_k'$ for some facet F_k' of G_j which is contained in $\cup_{i=1}^{j-1} G_i$. Therefore $x_k \notin G_j'$, which contradicts the definition of G_j'. Hence $G_j' \nsubseteq \cup_{i=1}^{j-1} G_i$. Any face of G_j' is contained in $\cup_{i=1}^{j-1} G_i$ because it misses some of the $x_k \in X$, and is therefore contained in F_k', showing that G_j' is minimal with respect to not being contained in $\cup_{i=1}^{j-1} G_i$.

To show the uniqueness, let F be a face of G_j not contained in $\cup_{i=1}^{j-1} G_i$. Then F is not contained in F'_k for any k. So each $x_k \in X$ is in F, which gives $G'_j \subseteq F$. By minimality of G'_j, $F = G'_j$. This proves the uniqueness of G'_j, completing the proof of Claim 1.

Let T_j be the set of all completely fundamental elements of $CF(Q_j)$ that are in extremal rays of G'_j. Given $\gamma \in D_{Q_j}$, we define $\hat{\gamma} = \gamma + \sum \beta$ where $\beta \in T_j$ and γ linearly depends on $CF(Q_j) - \{\beta\}$. To complete the proof of the theorem we need to show the following claim.

Claim 2. The monoid E_Φ has a decomposition into a disjoint union of translates of free monoids:

$$E_\Phi = \coprod_{j=1}^{r} \coprod_{\gamma \in D_{Q_j}} (\hat{\gamma} + \mathbb{N}CF(Q_j))$$

To prove the above claim, we will show that given $\eta \in E_\Phi$, we have $\eta \in \hat{\gamma} + \mathbb{N}CF(Q_j)$ if and only if j is the minimum index such that $\eta \in Q_j$.

Let j be minimum such that $\eta \in Q_j$. Then $\eta = \gamma + \sum_{\beta_i \in CF(Q_j)} a_i \beta_i$ where $a_i \in \mathbb{N}$ and $\gamma \in D_{Q_j}$. For some $\beta_i \in T_j$, if η is linearly dependent on $CF(Q_j) - \{\beta_i\}$ then there exists a facet of $C(G_j)$ containing η and contained in some $C(G_i)$ with $i < j$. This contradicts the minimality of j. So η depends on every element of T_j in Q_j. Therefore if γ depends on $CF(Q_j) - \{\beta_i\}$ for some $\beta_i \in T_j$ then $a_i \geq 1$. So $\eta - \hat{\gamma} \in \mathbb{N}CF(Q_j)$.

If $\eta \in \hat{\gamma}' + \mathbb{N}CF(Q_l)$ for some $l > j$ and some $\gamma' \in D_{Q_l}$, then η has an expression $\hat{\gamma}' + \sum_{\beta'_i \in CF(Q_l)} a'_i \beta'_i$. Here $\hat{\gamma}'$ has to linearly depend on all elements of T_l in Q_l. Since $\eta \in C(G_l) \cap C(G_j)$, there exists $\beta' \in T_l$ such that η is linearly independent of β' in Q_l, which contradicts to the fact that $\hat{\gamma}'$ linearly depends on all elements of T_l in Q_l. So such an l should not exist, and so j is unique such that $\eta \in \hat{\gamma} + \mathbb{N}CF(Q_j)$ for some $\gamma \in D_{Q_j}$. This completes the proof of Claim 2 and hence the proof of the theorem. □

Notes

The main ideas in this chapter are 'pulling the vertices of a convex polytope' and 'shelling the boundary complex of a convex polytope'. For the results about techniques of pulling I referred to [MS71]. Theorem 2.2 is from this book. Theorem 2.15 is proved by Bruggesser and Mani in [BM71]. Even though our way of defining visible and invisible facets differs from that of [BM71], the idea of the proof of Theorem 2.15 remains the same. Here I only discussed the shelling of visible and invisible facets, but the shelling of all the facets of a polytope is an easy consequence [**BM71**, Section 2, corollary of Proposition 2]. The last section is from [**Sta82**]. For the proof of Theorem 2.16, I have followed [**Sta82**, Section 5].

CHAPTER 3

The Reciprocity Theorem

The contents of this chapter do not quite depend on the results of the previous chapter. The only thing used in this chapter from the previous chapter is the existence of triangulations of $\mathcal{C}_\Phi$. Here we will mainly focus on the results about poset structure of the triangulation. But before going into that we need to know some results about finite posets mainly the *Mobius inversion formula.* These results, being incidental to the main emphasis of this thesis, are stated without proof.

Definition 3.1. *A finite set $\mathcal{P}$ with a binary relation $\leq$ is called a poset (finite poset) if for all $x, y, z \in \mathcal{P}$*

(i) $x \leq x$

(ii) *if $x \leq y$ and $y \leq x$ then $x = y$*

(iii) *if $x \leq y$ and $y \leq z$ then $x \leq z$*

Definition 3.2. *The Mobius function μ on $\mathcal{P} \times \mathcal{P}$ is defined as follows:*

for all $x \in \mathcal{P}$ we have $\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, y)$

The significance of Mobius function appears in the Mobius inversion formula:

Theorem 3.3 (Mobius Inversion Formula)**.** *Let $f, g : \mathcal{P} \longrightarrow \mathbb{Q}$, then $g(x) = \sum_{y \leq x} f(y)$ if and only if $g(x) = \sum_{y \leq x} g(y)\mu(y, x)$ for all $x \in \mathcal{P}$*

A proof of the Mobius inversion formula can be found in [**Sta12**, 3.7.1].

To use Mobius inversion we should know the Mobius function for the corresponding poset. There is not always a nice expression for the Mobius function of a poset. Our interest, fortunately, is in the following nice class of posets:

Let Γ be any triangulation of the convex pointed polyhedral cone $\mathcal{C}_\Phi$. $\hat{\Gamma}$ be the poset of elements of Γ ordered by inclusion along with a highest element $\hat{1}$, i.e., for any $\sigma \in \Gamma$ we have $\sigma < \hat{1}$. Let $\dim \mathcal{C}_\Phi = n$. The formula for the Mobius function on $\hat{\Gamma}$ is given by;

Theorem 3.4. *For* $\sigma, \tau \in \Gamma$

$$\mu(\sigma,\tau) = \begin{cases} (-1)^{\dim\tau - \dim\sigma} & \text{if } \sigma \leq \tau < \hat{1} \\ (-1)^{n-\dim\sigma+1} & \text{if } \sigma \nsubseteq \mathrm{bd}\mathcal{C}_\Phi \text{ and } \tau = \hat{1} \\ 0 & \text{if } \sigma \subseteq \mathrm{bd}\mathcal{C}_\Phi \text{ and } \tau = \hat{1} \end{cases}$$

The proof of the above theorem needs some results on simplicial complexes. See [**Sta12**, 3.8.9, 4.6.2] for details.

Definition 3.5. *For a simplex* $\sigma \in \Gamma$ *define* E_σ *to be the set* $\sigma \cap \mathbb{N}^n$ *of lattice points in* σ *and* $\overline{E_\sigma}$ *to be the set* $\{x \in E_\sigma \mid x \notin E_\tau \text{ for all } \tau < \sigma\}$ *of lattice points in the relative interior of* σ*. Define* $E_{\hat{1}} = E_\Phi$, $\overline{E_{\hat{1}}} = \emptyset$, $\overline{E_\Phi} = E_\Phi - \mathrm{bd}\mathcal{C}_\Phi$ *and* $\overline{\Gamma} = \{\sigma \in \Gamma | \sigma \notin \mathrm{bd}\mathcal{C}_\Phi\}$.

Lemma 3.6. *We have*

$$\overline{E_\Phi}(x) = \sum_{\sigma \in \overline{\Gamma}} \overline{E_\sigma}(x)$$

and

$$E_\Phi(x) = -\sum_{\sigma \in \Gamma} \mu(\sigma, \hat{1}) E_\sigma(x) \ .$$

Proof. $\overline{E_\Phi}(x) = \sum_{\sigma \in \overline{\Gamma}} \overline{E_\sigma}(x)$ is clear from the definition. For the formula for E_Φ note $E_{\hat{1}}(x) = E_\Phi(x)$ and $\overline{E_{\hat{1}}}(x) = 0$. Clearly $E_\sigma(x) = \sum_{\tau \leq \sigma} \overline{E_\tau}(x)$ for all $\sigma, \tau \in \hat{\Gamma}$. In particular

$$E_\Phi(x) = E_{\hat{1}}(x) = \sum_{\tau \in \hat{\Gamma}} \overline{E_\tau}(x) = \sum_{\tau \in \Gamma} \overline{E_\tau}(x).$$

Applying Mobius inversion formula

$$0 = \overline{E_{\hat{1}}} = \sum_{\sigma \in \hat{\Gamma}} E_\sigma(x)\mu(\sigma, \hat{1}) = E_\Phi(x)\mu(\hat{1}, \hat{1}) + \sum_{\sigma \in \Gamma} E_\sigma(x)$$

so

$$E_\Phi(x) = -\sum_{\sigma \in \Gamma} \mu(\sigma, \hat{1}) E_\sigma(x).$$

□

Any point in the interior of a simplicial cone depends on all its extremal rays: For simplicial monoids, we have:

Lemma 3.7. *If $\{\eta_1, \ldots, \eta_t\}$ is the set of quasi-generators of E_σ then*

$$\overline{E_\sigma} = \{\sum_{i=1}^{t} a_i\eta_i \in E_\sigma | a_i > 0\}$$

Lemma 3.8. *For a simplicial cone σ we have*

$$\overline{E_\sigma}(x) = (-1)^t E_\sigma(1/x)$$

PROOF. We have

$$\begin{aligned} E_\sigma(1/x) &= \Big(\sum_{\beta\in D_{E_\sigma}} x^{-\beta}\Big)\prod_{i=1}^{t}(1-x^{-\eta_i})^{-1} \\ &= (-1)^t\Big(\sum_{\beta\in D_{E_\sigma}} x^{\hat{\eta}-\beta}\Big)\prod_{i=1}^{t}(x^{\eta_i}-1)^{-1}, \qquad (2) \end{aligned}$$

where $\hat{\eta} = \sum_{i=1}^{t}\eta_i$. But $D_{E_\sigma} = \{\sum_{i=1}^{t} a_i\eta_i | 0 \le a_i < 1\}$. So $\hat{\eta} - D_{E_\sigma} = \{\sum_{i=1}^{t} a_i\eta_i | 0 < a_i \le 1\}$. Thus

$$(\hat{\eta} - D_{E_\sigma}) \oplus \mathbb{N}CF(\sigma) = \Big\{\sum_{i=1}^{t} a_i\eta_i \in E_\sigma | a_i > 0\Big\} = \overline{E_\sigma}.$$

Comparing with (2) gives the identity of the lemma. □

The following reciprocity theorem is a generalization of the above result to E_Φ:

Theorem 3.9 (Reciprocity Theorem)**.** *For the given system of linear equations Φ we have*

$$E_\Phi(1/x) = (-1)^n \overline{E_\Phi}(x)$$

PROOF. We have

$$\begin{aligned} E_\Phi(x) &= -\sum_{\sigma\in\Gamma}\mu(\sigma,\hat{1})E_\sigma(x) && \text{by Lemma 3.6} \\ &= -\sum_{\sigma\in\hat{\Gamma}}(-1)^{n+1-\dim\sigma}E_\sigma(x) && \text{by Theorem 3.4} \\ &= (-1)^n\sum_{\sigma\in\hat{\Gamma}}(-1)^{\dim\sigma}E_\sigma(x), \end{aligned}$$

which implies

$$\begin{aligned} E_\Phi(1/x) &= (-1)^n \sum_{\sigma\in\hat{\Gamma}} (-1)^{\dim\sigma} E_\sigma(1/x) \\ &= (-1)^n \sum_{\sigma\in\overline{\Gamma}} \overline{E_\sigma}(x) && \text{by Lemma 3.8} \\ &= (-1)^n \overline{E_\Phi}(x), \end{aligned}$$

which proves the theorem. □

Corollary. Given $\gamma \in \mathbb{Z}^n$ the following two conditions are equivalent

(i) $E_\Phi(1/x) = (-1)^n x^\gamma E_\Phi(x)$

(ii) $\overline{E_\Phi} = \gamma + E_\Phi$

Proof. By the reciprocity theorem, (i) is equivalent to

$$\overline{E_\Phi}(x) = x^\gamma E_\Phi(x),$$

which is clearly equivalent to (ii). □

Notes

For this chapter I mainly referred to [**Sta12**, Chapter 2, 3]. Rota's article [**Rot64**] is a good exposition of Mobius functions.

CHAPTER 4

The ADG Conjecture

This chapter is about an application of all the results from the previous chapters. I have chosen the *Anand-Dumir-Gupta Conjecture* (ADG conjecture) because it was the main motivation behind the theory of E_Φ. Before stating this conjecture I would like to begin with the following combinatorial problem:

> *Suppose n distinct things, each replicated r times, are distributed among n persons equally. In how many ways can we do this?*

If $r = 1$ then it is same as giving n distinct things to n persons, which can be done in $n!$ ways. If $n = 1$ the answer is 1, for $n = 2$ the problem is equivalent to find the number of ways we can partition r in 2 different parts which is $r + 1$. For further discussion let us fix the notation $H_n(r)$ for this count. MacMahon [**Mac04**] showed that

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}.$$

A nice general formula for $H_n(2)$ is given by Anand, Dumir and Gupta [**ADG66**] as

$$\sum_{n\geq 0} \frac{H_n(2)x^n}{(n!)^2} = \frac{e^{x/2}}{\sqrt{1-x}}.$$

But to describe $H_n(r)$ in complete generality seems to be a difficult problem. Anand, Dumir and Gupta in their paper [**ADG66**] gave some conjectures about $H_n(r)$ which help us to calculate $H_n(r)$ for certain values and gives some descriptions about a general formula for $H_n(r)$. I will not state this conjecture as it appears in [**ADG66**], rather I will give an equivalent formulation that appears in [**Sta83**, 1.1]:

Conjecture (Anand, Dumir and Gupta)**.** *For every positive integer n*

$$\sum_{r\geq 0} H_n(r)\lambda^r = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^{(n-1)^2+1}},$$

with $h_0 + h_1 + \cdots + h_d \neq 0$*,* $d = n^2 - 3n + 2$ *and* $h_i = h_{d-i}$ *for* $i = 0, 1, \ldots, d$*. In addition to this, Stanley conjectured that* h_i *is a non-negative integer and* $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$*.*

We will soon see the proof of the conjecture except the last part i.e. $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$, which still stands as an open problem [**Sta83**, 1.1].

Before going to the proof of ADG, let us take another look at our distribution problem. If $a_{i,j}$ denotes the number of things of type j that the ith person gets then the matrix $[a_{i,j}]$ determines the distribution uniquely. So the calculation of $H_n(r)$ is equivalent to counting the number of matrices of the form $[a_{i,j}]_{n\times n}$ where $a_{i,j} \in \mathbb{N}$ and $\sum_{i=1}^n a_{i,j} = \sum_{j=1}^n a_{i,j} = r$. If we consider the system of linear equations

$$\sum_{i=1}^n x_{i,j} = \sum_{k=1}^n x_{k,l}$$

for $i, j, k, l \in \{1, \ldots, n\}$. The non-negative integral solutions of this system of equations give the set of all matrices in our required form. Let the above system be Φ and $\mathcal{C}_\Phi$ be the cone of non-negative real valued solutions of Φ. Then $\mathcal{C}_\Phi$ is a cone in $\mathbb{N}^{n^2}$ of dimension $(n-1)^2+1$. To investigate E_Φ further, we need to know what its completely fundamental elements are. The following lemma tells us how to find $CF(E_\Phi)$.

Lemma 4.1. *Every element of* E_Φ *can be written as sum of permutation matrices. Recall that a permutation matrix is of the form* $[\delta_{i,\sigma(i)}]$*, where* $\sigma \in S_n$*, the group of permutations of* n *elements.*

The proof of the above lemma uses Hall's Marriage Condition [**Hal86**, Theorem 5.1.1]:

Theorem (Hall's Marriage Condition)**.** *Let* $G_1, \ldots, G_n$ *be* n *sets such that for all* $k \leq n$ *union of any* k *sets has at least* k *elements then we can choose distinct representatives* $g_i \in G_i$ *for each* i*.*

Proof of Lemma 4.1. Let $A = (a_{i,j})_{n\times n}$ be a matrix in E_Φ with

$$\sum_{i=1}^n a_{i,j} = \sum_{j=1}^n a_{i,j} = r\ .$$

Let $G_i = \{j | a_{i,j} \neq 0\}$. We claim that G_i's satisfy Hall's Marriage Condition. Consider any k of the G_i's say $G_{i_1}, \ldots, G_{i_k}$. Let

$$y = \sum_{l=1}^{k} \sum_{j=1}^{n} a_{i_l,j} = rk$$

If $\cup_{l=1}^{k} G_{i_l} = \{j_1, \ldots, j_s\}$ then

$$y = \sum_{l=1}^{k} \sum_{t=1}^{s} a_{i_l,j_t} \leq \sum_{i=1}^{n} \sum_{t=1}^{s} a_{i,j_t} = rs$$

So $rk \leq rs$ that implies $k \leq s$. Since G_i satisfy Hall's condition, we can choose distinct representatives from each G_i say it j_i. Clearly $a_{i,j_i} \geq 1$. Let P be the permutation matrix $[\delta_{i,j_i}]$. Then $A - P \in E_\Phi$. Say $A - P = [a'_{i,j}]$. Then

$$\sum_{i=1}^{n} a'_{i,j} = \sum_{j=1}^{n} a'_{i,j} = r - 1 \ .$$

Now by induction on r we can write A as sum of permutation matrices. □

A matrix in E_Φ is called an integer stochastic matrix. An integer stochastic matrix $[a_{i,j}]$ said to have line sum r if

$$\sum_{i=1}^{n} a_{i,j} = \sum_{j=1}^{n} a_{i,j} = r \ .$$

Let P be a permutation matrix. If

$$nP = n_1 A_1 + n_2 A_2$$

for some $A_1, A_2 \in E_\Phi$ and $n_1, n_2 \in \mathbb{N}$ then the (i,j)th entry of P is non-zero if and only if (i,j)th entry of at least one of the A_i is non-zero. But in P exactly one entry of each row and each column is non-zero so for A_i either it is zero or exactly the same entry of it is nonzero. So A_i are multiples of P and $n = n_1 + n_2$. This shows that the set of all completely fundamental elements of E_Φ are the set of all permutation matrices.

By Theorem 2.16

$$E_\Phi = \coprod_{i=1}^{t} (\delta_i + E_i)$$

where $E_1, \ldots, E_t \subseteq E_\Phi$ are free monoids of rank $(n-1)^2+1$ and $\delta_1, \ldots, \delta_t \in E_\Phi$. In fact, from the proof of Theorem 2.16, $CF(E_i) \subseteq CF(E_\Phi)$, and

$$E_\Phi(x) = \sum_{i=1}^{t} \frac{x^{\delta_i}}{\prod_{\eta \in CF(E_i)}(1-x^\eta)} \ .$$

Specializing the variables $(x_{1,1}, \ldots, x_{n,n})$ as

$$x_{i,j} = \begin{cases} \lambda & \text{if } i = 1, \\ 1 & \text{otherwise,} \end{cases}$$

$E_\Phi(x)$ becomes

$$E_\Phi(\lambda) = \sum_{i=1}^{t} \frac{\lambda^{a_i}}{(1-\lambda)^{(1-n)^2+1}},$$

where a_i is the line sum of δ_i. Note for any $A \in E_\Phi$ having line sum a, x^A becomes λ^a after substitution. So

$$E_\Phi(\lambda) = \sum_{r \geq 0} H_n(r)\lambda^r = \frac{p(\lambda)}{(1-\lambda)^{(n-1)^2+1}}$$

Here

$$p(\lambda) = \sum_{i=1}^{t} \lambda^{a_i} = h_0 + h_1\lambda + \cdots + h_d\lambda^d \quad \text{(say)}$$

is a polynomial with non-negative integral coefficients, i.e,

$$h_i \geq 0$$

and clearly

$$h_0 + h_1 + \cdots + h_d \neq 0 \ .$$

Note that $H_n(0) = 1$ implies $h_0 = 1$. Now to find the degree of $p(\lambda)$ recall the Corollary of the Reciprocity Theorem 3.9. It is easy to see that

$$\overline{E_\Phi} = [1]_{n\times n} + E_\Phi.$$

Hence

$$E_\Phi(1/x) = (-1)^{(n-1)^2+1} x^{[1]} E_\Phi(x),$$

which implies

$$E_\Phi(1/\lambda) = (-1)^{(n-1)^2+1}\lambda^n E_\Phi(\lambda).$$

After simplification we get

$$(h_0\lambda^d + h_1\lambda^{d-1} + \cdots + h_d)\lambda^{(n-1)^2+1-d} = \lambda^n(h_0 + \cdots + h_d\lambda^d).$$

Equating the powers of λ on both the sides, we get

$$(n-1)^2 + 1 = n + d \quad \Rightarrow d = n^2 - 3n + d.$$

Also note that

$$h_0\lambda^d + h_1\lambda^{d-1} + \cdots + h_d = h_0 + h_1\lambda \cdots + h_d\lambda^d,$$

which says

$$h_i = h_{d-i}.$$

This completes the part of the ADG conjecture that we wanted to prove. $\square$

Notes

The statement of the ADG conjecture given here is an extension due to Stanley [**Sta83**, 1.1] of the original formulation of Anand, Dumir and Gupta [**ADG66**]. A proof of the original conjecture without using the Stanley-Bruggesser-Mani decomposition appears in [**Sta12**, Section 4.6]. One may refer to [**Sta83**, Chapter 1] for an algebraic proof of the ADG Conjecture.

We saw that

$$\sum_{r\geq 0} H_n(r)\lambda^r = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^{(n-1)^2+1}}$$

from which it follows that

$$H_n(r) = \binom{r+(n-1)^2}{(n-1)^2}h_0 + \binom{r-1+(n-1)^2}{(n-1)^2}h_1 + \cdots + \binom{r-d+(n-1)^2}{(n-1)^2}h_d.$$

This says we know a formula for $H_n(r)$ if we know all the h_i's. One way to find the values of the h_i's is by interpolation. We already have $h_i = h_{d-i}$, $h_0 = 1$ and $h_1 = n! - n$. So to interpolate $p(\lambda)$ we need to know the values of $p(\lambda)$ at another $[d/2] - 2$ points. It turns out that the volume of the polytope having extremal points at permutation matrices gives

$p(1)$ [**BR07**, Lemma 3.19]. But it is an open problem to find an *easy* way to compute $p(1)$. Now what about $p(\lambda)$, is it easier to compute $p(\lambda)$ than $p(1)$? Computing $p(\lambda)$ is also an open problem. For further discussion of this see [**DG95**] and [**DS98**].

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