STUDIES IN WEIGHTED SPACES

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CONTENTS

INTRODUCTION

This thesis consists of two parts. The first five chapters comprising the first part are devoted to the study of one of the very interesting problems in the theory of approximation, namely the Bernstein problem of 'weighted approximation'. The remaining three chapters deal with multiplier transformations associated with 'weighted spaces' and constitute the second part.

Let $\gamma \geq 1$ be a continuous function defined on the set of real numbers, $\mathbb{R}$, satisfying the condition $\frac{|x|^n}{\gamma(x)} \to 0$ as $|x| \to \infty$ for all n. Let $\mathcal{C}_\gamma(\mathbb{R})$ denote the Banach space of continuous functions f, defined on $\mathbb{R}$, with the property that $\frac{f(x)}{\gamma(x)} \to 0$ as $|x| \to \infty$ and normed by $\|f\|_\gamma = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\gamma(x)}$.

In the problem formulated by Bernstein in 1924, it was examined whether it is possible to associate polynomials $P(x)$ with every such $f(x)$, fulfilling the criterion that $\sup_{-\infty < x < \infty} \frac{|f(x)-P(x)|}{\gamma(x)} < \varepsilon$, where ε is a preassigned positive number, i.e., whether the class of all polynomials, denoted by $\mathcal{P}$, is dense in the space $\mathcal{C}_\gamma(\mathbb{R})$ and γ is called the weight function.

In Chapter 1, we have stated the problem of Bernstein and its generalizations. The Banach spaces $\mathcal{C}_\gamma(X)$ and $L^p_\gamma(X)$ are defined, along with the linear subspaces $\mathcal{P}$, $E_{\gamma,0}(X)$,

$E_{\gamma,a}(X)$, $E^p_{\gamma,0}(X)$ and $E^p_{\gamma,a}(X)$. Chapter 2 contains a number of lemmas which we need in the proofs of the results given in Chapters 3, 4 and 5. In Chapter 3, we have used the method of Morgelyan to obtain various sets of necessary and sufficient conditions for the different linear subspaces to be dense in $\mathcal{C}_{\gamma}(X)$ and $L^p_{\gamma}(X)$. In Chapter 4, we discuss the Bernstein problem when the weight function γ is continuous and deduce the results analogous to those of Pollard. Chapter 5 deals with the case when $\mathcal{P}$ is not dense and the natural extension of the result of Hačatryan to $L^p_{\gamma}(\mathbb{R})$ is proved, thus providing a necessary and sufficient condition for $\mathcal{P}$ to be dense in $E^p_{\gamma,0}(\mathbb{R})$. Further, we have added a remark on best approximation.

The second part is concerned with the multiplier problem. This problem has been treated by various authors for different spaces (see for example De Leeuw [19] , Guy [22] , Hirschman [23-26] and Igari [27]). Hirschman [24] considered the space of complex-valued functions $f(n)$ defined on the additive group of integers, Z, with finite norm $\|f\|_p = \left(\sum_{-\infty}^{\infty} |f(n)|^p\right)^{1/p}$, where $1 \le p < \infty$ and observed that for $p = 2$, it is possible to find conditions that are sufficient to ensure that the corresponding multiplier transformation is a bounded one. He also analysed the space of complex-valued functions $f(\theta)$ defined on the set of real numbers modulo one, T, such that $\|f\|_p = \left(\int_Z |f(\theta)|^p \, d\theta\right)^{1/p}$ is finite and established the sufficient conditions for the associated multiplier transformation to be bounded. Devinâts and Hirschman [20] looked into the

Banach space $\ell^{2,\lambda}(Z)$ and studied the Banach algebra of those bounded linear transformations of $\ell^{2,\lambda}(Z)$ into itself which commute with convolution.

Introducing the requisite terminology in Chapter 6, we have set forth some sets of sufficient conditions on the multiplier function such that the corresponding multiplier transformation is a bounded transformation of $L^{2,\lambda}(T)$ into itself, with weight e^{λ}. and the results analogous to those of Hirschman [26] are given in Chapter 7. Chapter 8 deals with the problem for the space $\ell^{p,\lambda}(Z)$ and results similar to those given by Hirschman [24] for the case $\lambda = 0$ have been obtained.

Most of the results in the first part appeared in the Journal of Mathematical Analysis and Applications (GEETHA P.K., 'On Bernstein Approximation Problem', 25, No.2 (1969), pp.450-469) and the work presented in the second part is contained in a paper entitled 'On Multiplier Transformations' by UNNI K.R. and GEETHA P.K., MATSCIENCE preprint (revised version, July 1970).

Throughout, the numbers in square brackets indicate the papers and some standard books, which form the basic references. Each part has been endowed with an independent list of references and these are found at the end.

PART I

BERNSTEIN APPROXIMATION PROBLEM

CHAPTER 1

STATEMENT OF THE PROBLEM

Let $\mathbb{R}$ and $\mathbb{K}$ denote the set of real numbers and the set of complex numbers, respectively. Let δ be a real number such that $0 < \delta < \frac{1}{2}$. We set

$$\mathbb{R}_\delta = \bigcup_{n=-\infty}^{\infty} [n - \delta, n + \delta] .$$

Let X stand for either $\mathbb{R}$ or $\mathbb{R}_\delta$ and let Y be a function (in general, complex-valued) defined on X. Let $\mathcal{P}$ denote the class of all polynomials.

We consider the following Banach spaces of functions.

I. Suppose $Y \geq 1$ and $x^n/Y(x) \to 0$ as $|x| \to \infty$, $x \in X$, $n = 1, 2, \ldots$. We denote by

$\mathcal{C}_Y(X)$ = Banach space of continuous functions f on X such that $f(x)/Y(x) \to 0$ as $|x| \to \infty$ and normed by

$$\|f\|_Y = \sup_{x \in X} \frac{|f(x)|}{Y(x)}$$

$E_{Y,0}(X)$ = Class of all entire functions of exponential type zero, whose restrictions to X belong to $\mathcal{C}_Y(X)$.

$E_{Y,a}(X)$ = class of all entire functions of exponential type not greater than a, whose restrictions to X belong to $\mathcal{C}_Y(X)$.

II. Suppose $1 \le p < \infty$ and let Y satisfy

$$\int_X \left| \frac{x^n}{Y(x)} \right|^p dx < \infty \qquad n = 0, 1, 2, \dots .$$

We define

$L^p_Y(X)$ = Banach space of functions f such that

$$\int_X \left| \frac{f(x)}{Y(x)} \right|^p dx < \infty$$

and normed by

$$\|f\|_{Y,p} = \left(\int_X \left| \frac{f(x)}{Y(x)} \right|^p dx \right)^{1/p} .$$

$E^p_{Y,0}(X)$ = class of all entire functions of zero exponential type, whose restrictions to X belong to $L^p_Y(X)$.

$E^p_{Y,a}(X)$ = class of all entire functions of exponential type not exceeding a and which are Fourier transforms of measures supported by interval $(-a, a)$, and whose restrictions to X belong to $L^p_Y(X)$.

Here a is a nonnegative real number.

The Bernstein problem consists in asking for necessary and sufficient conditions in order that $\mathcal{P}$ is dense in $\mathcal{C}_Y(\mathbb{R})$. This problem was treated by various authors. A complete solution to this problem was given by Akhiezer and Bernstein [2], Mergelyan [15] and Pollard [17] . Koosis [11,12] obtained the conditions when $\mathbb{R}$ was replaced by X or Z, the set of integers. For $p = 2$, the analogous problem was treated by Levinson and McKean [14] and it was shown that Mergelyan's approach was applicable even to the case when $E^2_{\gamma,0}(\mathbb{R})$ is to be dense $L^2_\gamma(\mathbb{R})$. Akutowicz [3,4] further proved that the same sort of conditions hold also when $E^p_{\gamma,a}(\mathbb{R})$ is to be dense in $L^p_\gamma(\mathbb{R})$.

Let U denote any one of the linear subspaces $\mathcal{P}$, $E_{\gamma,0}(X)$, $E_{\gamma,a}(X)$ and let V denote any one of $\mathcal{P}$, $E^p_{\gamma,0}(X)$ $E^p_{\gamma,a}(X)$. We exploit the method of Mergelyan to investigate the conditions under which U (respectively V) is dense in $\mathcal{E}_\gamma(X)$ (respectively $L^p_\gamma(X)$). We shall also obtain results corresponding to Pollard's theorem. Hačatryan [10] obtained a necessary and sufficient condition for $\mathcal{P}$ to be dense in $E_{\gamma,0}(\mathbb{R})$. This result of Hačatryan is also extended in our case.

CHAPTER 2.

PRELIMINARY LEMMAS

We shall establish a few lemmas which we need in order to prove the main theorems later.

Let D denote any one of the linear subspaces $\mathcal{P}$, $E_{\gamma,0}(X)$, $E^{p}_{\gamma,0}(X)$, $E_{\gamma,a}(X)$, $E^{p}_{\gamma,a}(X)$. We will denote by $\|\cdot\|$, either $\|\cdot\|_{\gamma}$ or $\|\cdot\|_{\gamma,p}$, depending upon the space under consideration.

Let $\mathcal{M}_{\gamma}(D)$ denote the class of functions $f \in D$ such that $\|f\| \leq 1$. Let $\mathcal{N}_{\gamma}(D)$ denote the class of functions $g \in D$ which have no zeros in the upper half plane and for which $|g(x)| \geq 1$ and $\|g\| \leq 1 + \|1\|$. Set

$$M_{\gamma}(z,D) = \sup_{f \in \mathcal{M}_{\gamma}(D)} |f(z)|, \qquad z \in \mathbb{K}$$

$$\mathcal{A}(\gamma,D) = \sup_{f \in \mathcal{M}_{\gamma}(D)} \int_X \frac{\log |f(x)|}{1+x^2}\,dx$$

$$\mathcal{B}(\gamma,D) = \sup_{f \in \mathcal{N}_{\gamma}(D)} \int_X \frac{\log |f(x)|}{1+x^2}\,dx.$$

LEMMA 2.1. *Let* $z_0 \in \mathbb{K}$ *such that* $\operatorname{Im} z_0 \neq 0$. *If* $f \in D$, *then* $g \in D$, *where*

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

PROOF. This lemma has been proved by Akutowicz [4] for $D = E^{p}_{\gamma,a}(X)$. The proofs for the other cases are trivial.

LEMMA 2.2. *Let* $z_1, z_2 \in \mathbb{K}$ *such that* $\operatorname{Im} z_1 \neq 0$. *If* $f \in D$, *with* $f(z_1) = 0$, *then* $g \in D$, *where*

$$g(z) = \frac{z - z_2}{z - z_1} f(z). \tag{2.1}$$

PROOF. The proof of this lemma for $D = E^p_{Y,a}(X)$ is due to Akutowicz [3] . All other cases afford easy verification.

LEMMA 2.3. *When* $X = \mathbb{R}$ *or* $\mathbb{R}_\delta$, $\mathcal{A}(Y,D) < \infty$ *implies* $\mathcal{B}(Y,D) < \infty$.

PROOF. By definition

$$\mathcal{B}(Y,D) \leq \mathcal{A}((1 + \|1\|) Y,D) = \mathcal{A}(Y,D) + \log (1 + \|1\|).$$

The lemma is now obvious.

LEMMA 2.4. *When* $X = \mathbb{R}$ *or* $\mathbb{R}_\delta$, *if* $\mathcal{B}(Y,\mathcal{P}) < \infty$, *then* $\mathcal{A}(Y,\mathcal{P}) < \infty$.

PROOF. Let $P \in \mathcal{P}$. Then there exists $Q \in \mathcal{P}$ which has no zeros in the upper half plane such that

$$1 + |P(x)|^2 = |Q(x)|^2,$$

from which it follows that

$$1 \leq |Q(x)| \leq 1 + |P(x)| .$$

Now

$$\|Q\|_Y = \sup_{x \in X} \frac{|Q(x)|}{Y(x)}$$

and

$$\|Q\|_{Y,p} = \left(\int_X \left| \frac{Q(x)}{Y(x)} \right|^p dx \right)^{1/p} .$$

In either case, when $P \in \mathcal{M}_Y(\mathcal{P})$, we have

$$\|Q\| = 1 + \|1\| .$$

Therefore,

$$\mathcal{A}(Y, \mathcal{P}) \leq \mathcal{B}(Y, \mathcal{P})$$

and the conclusion of lemma follows.

LEMMA 2.5. *When* $X = \mathbb{R}$, *then* $\mathcal{B}(Y,D) < \infty$ *implies* $\mathcal{A}(Y,D) < \infty$, where $D = E_{Y,0}(X)$, $E^{p}_{Y,0}(X)$, $E_{Y,a}(X)$.

PROOF. Suppose $\mathcal{B}(Y,D) < \infty$. Let f be an entire function of exponential type τ, which belongs to D. Without loss of generality, we can assume that f is real. Then $1 + f^2(z)$ is an entire function of exponential type 2τ. By the given hypothesis, we have

$$\int_{\mathbb{R}} \frac{\log [1 + f^2(x)]}{1 + x^2}\, dx < \infty .$$

By a theorem of Akhieser [1], there exists an entire function g of exponential type τ, with no zeros in the upper half plane such that

$$|g(x)|^2 = 1 + f^2(x).$$

Clearly $g \in D$ and

$$\|g\| \leq 1 + \|f\|$$

so that if $f \in \mathcal{M}_Y(D)$, then $g \in \mathcal{N}_Y(D)$. Therefore $\mathcal{A}(Y,D) \leq \mathcal{B}(Y,D)$. Thus, $\mathcal{B}(Y,D) < \infty$ implies $\mathcal{A}(Y,D) < \infty$.

LEMMA 2.6. *The function* $M_Y(z,D)$ *has the following properties*:

(a) $\log M_Y(z,D)$ *is non-negative and subharmonic*

(b) $[M_\gamma(z,D)]^{-1}$ <u>is continuous in the half planes,</u>

$\operatorname{Im} z > 0$, $\operatorname{Im} z < 0$ <u>and</u> $M_\gamma(z,D) = M_\gamma(\bar{z},D)$.

(c) <u>If</u> $M_\gamma(z_0,D) = \infty$ <u>at some point</u> z_0 <u>with</u> $\operatorname{Im} z_0 \neq 0$, <u>then there exists a continuum</u> E, $z_0 \in E$, <u>disjoint from the real axis such that</u> $M_\gamma(z,D) = \infty$ <u>on</u> E.

PROOF. (a) First, we notice that the constant function $1 \in \mathcal{M}_\gamma(D)$. Since $1 \in D$, $M_\gamma(z,D) \geq 1$. The subharmonic property of $\log M_\gamma(z,D)$ follows from the fact that $\log|f(z)|$ is subharmonic for each $f \in D$.

(b) If $f \in \mathcal{M}_\gamma(D)$, then $\bar{\bar{f}} \in \mathcal{M}_\gamma(D)$ and it easily follows that

$$M_\gamma(z,D) = M_\gamma(\bar{z},D).$$

It is enough therefore to consider the upper half plane. Let $z_1, z_2 \in \mathbb{K}$ with $\operatorname{Im} z_j = b_j > 0$ for $j = 1,2$. If $f \in D$, then considering the function

$$g(z) = 1 + \frac{z - z_2}{z - z_1} \, \frac{f(z) - f(z_1)}{f(z_1)},$$

it follows that $g \in D$, by Lemma 2.2.

On the real axis, we have

$$\frac{g(x)}{\gamma(x)} \leq \frac{|x - z_2|}{|x - z_1|} \, \frac{|f(x)|}{|f(z_1)|} \, \frac{1}{\gamma(x)} + \frac{|z_1 - z_2|}{|x - z_1|} \, \frac{1}{\gamma(x)}$$

$$\leq \left(1 + \frac{|z_1 - z_2|}{b_1}\right) \frac{|f(x)|}{|f(z_1)|} \, \frac{1}{\gamma(x)} + \frac{|z_1 - z_2|}{b_1} \, \frac{1}{\gamma(x)}.$$

Hence,

$$\|g\| \leq \left(1 + \frac{|z_1 - z_2|}{b_1}\right) \frac{\|f\|}{|f(z_1)|} + \frac{|z_1 - z_2|}{b_1} \, \|1\|$$

$$\leq \left(1 + \frac{|z_1 - z_2|}{b_1}\right) \frac{1}{|f(z_1)|} + \frac{|z_1 - z_2|}{b_1} \, \|1\|, \quad \text{if } f \in \mathcal{M}_\gamma(D),$$

$$= A, \text{ say}.$$

Thus, $\frac{g}{A} \in \mathcal{M}_\gamma(D)$ and $M_\gamma(z_2, D) \geq \frac{|g(z_2)|}{A} \geq \frac{1}{A}$. Therefore,

$$\frac{1}{M_\gamma(z_2,D)} \leq A = \left(1 + \frac{|z_1 - z_2|}{b_1}\right) \frac{1}{|f(z_1)|} + \frac{|z_1 - z_2|}{b_1} \|1\|$$

which implies that

$$\frac{1}{M_\gamma(z_2,D)} \leq \left(1 + \frac{|z_1 - z_2|}{b_1}\right) \frac{1}{M_\gamma(z_1,D)} + \frac{|z_1 - z_2|}{b_1} \|1\|$$

which in turn gives,

$$\frac{1}{M_\gamma(z_2,D)} \leq \frac{1}{M_\gamma(z_1,D)} + \frac{|z_1 - z_2|}{b_1} (1 + \|1\|)$$

since $\frac{1}{M_\gamma(z,D)} \leq 1$.

Interchaning z_1 and z_2, we obtain a similar inequality which together with the preceding one yields

$$\left|\frac{1}{M_\gamma(z_2,D)} - \frac{1}{M_\gamma(z_1,D)}\right| \leq \frac{|z_2 - z_1| (1 + \|1\|)}{\min (b_1, b_2)} .$$

The continuity of $\frac{1}{M_\gamma(z,D)}$ is now immediate.

(c) The method of proof is that of Mergelyan [15] .

We set

$$\gamma_1(x) = \gamma(x) (1 + x^2)^{1/2}$$

$$\omega(x) = \gamma(x) (1 + x^2)^{-1/2}.$$

For each $z_0 \in \mathbb{K}$, with $\operatorname{Im} z_0 \neq 0$,

$$I_{z_0}(x) = (x - z_0)^{-1} \qquad x \in X.$$

It is clear that $I_{z_0} \in \mathcal{C}_\gamma(X)$ and $L^p_\gamma(X)$.

LEMMA 2.7. *If $M_\gamma(z_0, D) = \infty$ for some point z_0, with $\operatorname{Im} z_0 \neq 0$, then for every $\varepsilon > 0$, there exists $f \in D$ such that*

$$\|I_{z_0} - f\|_\omega < \varepsilon , \qquad (2.4)$$

where $D = \mathcal{P}, E_{\gamma,0}(X), E_{\gamma,a}(X)$.

PROOF. Let $N > 0$. By hypothesis, there exists $f_1 \in \mathcal{M}_\gamma(D)$ such that $|f_1(z_0)| > N$. Let

$$f(z) = \frac{f_1(z) - f_1(z_0)}{(z_0 - z)\, f_1(z_0)} .$$

Then $f \in D$ by Lemma 2.1.

Further, on the real axis,

$$\|I_{z_0} - f\|_\omega = \sup_{x \in X} \left\{ \frac{|I_{z_0}(x) - f(x)|}{\omega(x)} \right\} = \sup_{x \in X} \left\{ \frac{|I_{z_0}(x) - f(x)|}{\gamma(x)(1+x^2)^{-1/2}} \right\}$$

$$= \sup_{x \in X} \left\{ \frac{\left| \frac{1}{x - z_0} - \frac{f_1(x) - f_1(z_0)}{(z_0 - x) f_1(z_0)} \right|}{\gamma(x)\,(1+x^2)^{-1/2}} \right\}$$

$$= \sup_{x \in X} \left\{ \frac{1}{|x - z_0|} \cdot \frac{|f_1(x)|}{\gamma(x)\,(1+x^2)^{-1/2}\,|f_1(z_0)|} \right\}$$

Now, $f_1 \in \mathcal{M}_\gamma(D)$ and therefore $\|f_1\|_\gamma = \sup_{x \in X} \frac{|f_1(x)|}{\gamma(x)} \leq 1$. Hence

$$\|I_{z_0} - f\|_\omega \leq \sup_{x \in X} \left(\frac{\sqrt{1+x^2}}{|x - z_0|} \right) \cdot \frac{1}{N} < \frac{C(z_0)}{N} .$$

Since $M_\gamma(z_0, D) = \infty$, we have, as $N \to \infty$,

$$\|I_{z_0} - f\|_\omega < \varepsilon .$$

LEMMA 2.8. *If* $\operatorname{Im} z_0 \neq 0$ *and if for every* $\varepsilon > 0$ *there exists* f *satisfying* (2.4), *then* $M_\gamma(z_0, D) = \infty$, *where* $D = \mathcal{P}, E_{\gamma,0}(X), E_{\gamma,a}(X)$.

PROOF. Let $\varepsilon > 0$ be given and set

$$K(z_0) = \sup_{x \in X} \left(\frac{|x - z_0|}{\sqrt{1+x^2}} \right).$$

Let

$$f_1(z) = \frac{1 - (z - z_0)\, f(z)}{\varepsilon\, K(z_0)}.$$

Then $f_1 \in D$. Further,

$$\| f_1 \|_\gamma = \sup_{x \in X} \frac{\left| \frac{1 - (x - z_0) f(x)}{\varepsilon K(z_0)} \right|}{\gamma(x)} = \sup_{x \in X} \frac{\left| \frac{1 - (x - z_0) f(x)}{\varepsilon K(z_0)} \right|}{\omega(x)\sqrt{1+x^2}}$$

$$= \sup_{x \in X} \frac{\left| \frac{1}{x - z_0} - f(x) \right|}{\omega(x)} \cdot \frac{1}{\varepsilon K(z_0)} \cdot \frac{|x - z_0|}{\sqrt{1+x^2}}$$

$$\leq 1 .$$

Therefore $f_1 \in m_\gamma(D)$ and

$$f_1(z_0) = \frac{1}{\varepsilon\, K(z_0)}$$

which shows that

$$M_\gamma(z_0, D) = \infty.$$

LEMMA 2.9. <u>If</u> $M_\gamma(z_0, D) = \infty$ <u>for some point</u> z_0, <u>with</u> $\operatorname{Im} z_0 \neq 0$, <u>then for every</u> $\varepsilon > 0$, <u>there exists</u> $f \in D$ <u>such that</u>

$$\| I z_0 - f \|_{\omega, p} < \varepsilon \qquad (2.6)$$

where $D = \mathcal{P}$, $E^p_{\gamma,0}(X)$, $E^p_{\gamma,a}(X)$..

PROOF. Let $N > 0$. By hypothesis, there exists $f_1 \in m_\gamma(D)$ such that $|f_1(z_0)| > N$. With the same choice of $f(z)$ as in Lemma 2.7, we get

$$\leq \left(\max_{x \in X} \frac{\sqrt{1+x^2}}{|x-z_0|}\right)^p \int_{-\infty}^{\infty} \left|\frac{f_1(x)}{\gamma(x)}\right|^p \frac{1}{|f_1(z_0)|^p}\, dx .$$

Since $f_1 \in \mathcal{M}_\gamma(D)$, $\|f_1\|_{\gamma,p} \leq 1$. Hence

$$\|I_{z_0} - f\|^p_{\omega,p} \leq \left(\frac{C(z_0)}{N}\right)^p .$$

As N is arbitrary, (2.5) follows.

LEMMA 2.10. *If* $\operatorname{Im} z_0 \neq 0$ *and if for every* $\varepsilon > 0$ *there exists* f *satisfying* (2.5), *then* $M_\gamma(z_0, D) = \infty$, *where* $D = \mathcal{P}$, $E^p_{\gamma,0}(X)$, $E^p_{\gamma,a}(X)$.

PROOF. The proof is similar to that of Lemma 2.8.

$$\|f_1\|^p_{\gamma,p} = \int_X \left|\frac{f_1(x)}{\gamma(x)}\right|^p dx = \int_X \left|\frac{1-(x-z_0)f(x)}{\varepsilon K(z_0)\gamma(x)}\right|^p dx$$

$$= \int_X \left|\frac{\frac{1}{x-z_0} - f(x)}{\omega(x)}\right|^p \left|\frac{x-z_0}{\sqrt{1+x^2}}\right|^p \left(\frac{1}{\varepsilon K(z_0)}\right)^p dx$$

$$\leq \left(\max_{x\in X} \frac{|x-z_0|}{\sqrt{1+x^2}}\right)^p \frac{1}{\varepsilon^p (K(z_0))^p} \int_X \left|\frac{I_{z_0}(x) - f(x)}{\omega(x)}\right|^p dx \leq 1,$$

which implies that $f_1 \in \mathcal{M}_\gamma(D)$. The rest of the proof is the same as that of Lemma 2.8.

LEMMA 2.11. *Let* $z_1, z_2, \ldots$ *be any infinite sequence of complex numbers tending to a finite complex number* z_0, *with* $\operatorname{Im} z_0 \neq 0$. Put

$$S = \left\{ \frac{1}{x-z_k}, \frac{1}{x-\bar{z}_k};\ k = 1, 2, \cdots \right\}.$$

Then the finite linear combinations of the elements of S *are dense in the space* $\mathcal{C}_\gamma(\mathbb{R})$.

For a proof see Fuchs [9, pp.45-46.].

CHAPTER 3

SOLUTION OF THE PROBLEM: MERGELYAN'S APPROACH

We shall now present a complete solution to the Bernstein problem, using the technique employed by Mergelyan.

THEOREM 3.1. Let $X = \mathbb{R}$ or $\mathbb{R}_\delta$. Then any one of the following conditions is necessary and sufficient for $\mathcal{P}$ to be dense in $\mathcal{C}_\gamma(X)$.

(a) $M_{\gamma_1}(z, \mathcal{P}) = \infty$, $\quad$ Im $z \neq 0$

(b) $\mathcal{A}(\gamma_1, \mathcal{P}) = \infty$,

(c) $$\int_X \frac{\log M_{\gamma_1}(x, \mathcal{P})}{1 + x^2}\, dx = \infty .$$

PROOF. The case $X = \mathbb{R}$ has been proved by Mergelyan [15]. We consider the case $X = \mathbb{R}_\delta$.

(a) As the class $\mathcal{K}$ of all continuous functions with compact support ; is dense in $\mathcal{C}_\gamma(X)$ and as the polynomials can be uniformly approximated by linear combinations of such functions, the proof is completed by using Lemma 2.7 and Lemma 2.8.

Let $\mathcal{P}$ be dense in $\mathcal{C}_\gamma(X)$. Since $\frac{1}{x - z_0}$ belongs to $\mathcal{C}_\gamma(X)$ for every z_0, Im $z_0 \neq 0$, Lemma 2.8 gives

$$M_{\gamma_1}(z_0, \mathcal{P}) = \infty, \qquad \text{Im } z_0 \neq 0.$$

Now suppose that for every z_0, with Im $z_0 \neq 0$,

$$M_{\gamma_1}(z_0, \mathcal{P}) = \infty.$$

Then $\frac{1}{x-z}, \frac{1}{x-\bar{z}}$ $(z \in E)$ can be uniformly approximated by polynomials according to Lemma 2.7, and the system of linear combinations of the functions $\frac{1}{x-z}, \frac{1}{x-\bar{z}}$ $(z \in E)$ is dense in $\mathscr{C}_\gamma(X)$ by Lemma 2.11. Thus $\mathcal{P}$ is dense in $\mathscr{C}_\gamma(X)$.

(b) If P be a real polynomial, there exists a constant $C(\delta)$ which depends only on δ such that

$$\int_{-\infty}^{\infty} \frac{\log[1+P^2(x)]}{1+x^2}\,dx \leq C(\delta) \int_{\mathbb{R}_\delta} \frac{\log[1+P^2(x)]}{1+x^2}\,dx.$$

This has been proved by Koosis [11, p.231].

Suppose $\mathcal{A}(\gamma_1,\mathcal{P}) < \infty$. Let $P \in \mathcal{M}_{\gamma_1}(\mathcal{P})$. It is enough to consider real polynomials P. Let

$$|Q(x)|^2 = 1 + |P(x)|^2.$$

Then, $\|P\| \leq 1$ implies

$$\left\|\tfrac{1}{2} Q\right\| \leq 1$$

and

$$\int_{\mathbb{R}_\delta} \frac{\log \frac{1}{2}|Q(x)|}{1+x^2}\,dx \leq \mathcal{A}(\gamma_1,\mathcal{P}).$$

This in turn implies that

$$\int_{\mathbb{R}_\delta} \frac{\log[1+P^2(x)]}{1+x^2}\,dx \leq 2\,\mathcal{A}(\gamma_1,\mathcal{P}) + \pi \log 4.$$

Then

$$\log|Q(i)| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|Q(x)|}{1+x^2}\,dx$$

$$\leq \frac{C(\delta)}{\pi}\left(\mathcal{A}(\gamma_1,\mathcal{P}) + \pi\log 2\right).$$

Hence $M_{\gamma_1}(i, \mathcal{P}) < \infty$ and $\mathcal{P}$ is not dense in $\mathscr{C}_\gamma(\mathbb{R}_\delta)$.

Conversely,

$$\int_{\mathbb{R}_\delta} \frac{\log|Q(x)|}{1+x^2}\,dx \le \int_{-\infty}^{\infty} \frac{\log|Q(x)|}{1+x^2}\,dx$$

$$= \pi \log|Q(i)| ,$$

so that if $\mathcal{A}(\gamma_1, \mathcal{P}) = \infty$, then $M_{\gamma_1}(i, \mathcal{P}) = \infty$ also. Hence $\mathcal{P}$ is dense in $\mathscr{C}_\gamma(\mathbb{R}_\delta)$.

(c) This is proved by the arguments of Mergelyan.

If $P \in \mathcal{M}_{\gamma_1}(\mathcal{P})$, then $|P(x)| \le M_{\gamma_1}(x, \mathcal{P})$ so that

$$\mathcal{A}(\gamma_1, \mathcal{P}) \le \int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(x, \mathcal{P})}{1+x^2}\,dx$$

$$\le C(\delta) \int_{\mathbb{R}_\delta} \frac{\log M_{\gamma_1}(x, \mathcal{P})}{1+x^2}\,dx.$$

This establishes the necessity of the condition.

We shall now prove that if $\mathcal{P}$ is not dense in $\mathscr{C}_\gamma(\mathbb{R}_\delta)$, then

$$\int_{\mathbb{R}_\delta} \frac{\log M_{\gamma_1}(x, \mathcal{P})}{1+x^2}\,dx < \infty .$$

As $\mathcal{P}$ is not dense in $\mathscr{C}_\gamma(\mathbb{R}_\delta)$, there exists $g \in L^q$ such that

$$\int_{-\infty}^{\infty} \frac{x^n}{\gamma(x)}\, g(x)\,dx = 0 \tag{3.1}$$

and

$$\int_{-\infty}^{\infty} |g(x)|^q \, dx < \infty. \tag{3.2}$$

Let

$$F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} \, \frac{g(x)}{Y(x)} \, dx. \tag{3.3}$$

The function F is analytic in the half planes $\operatorname{Im} z > 0$, $\operatorname{Im} z < 0$ and is not identically zero. From (3.3) we obtain

$$F(z)\, P(z) = \int_{-\infty}^{\infty} \frac{P(x)}{x-z} \, \frac{g(x)}{Y(x)} \, dx$$

for any polynomial P. If $P \in \mathcal{M}_{Y_1}(\mathcal{P})$, we have

$$|F(z)P(z)| \leq \int_{-\infty}^{\infty} \left| \frac{P(t)}{\gamma(t)\sqrt{1+t^2}} \right| \frac{\sqrt{1+t^2}}{|t-z|} \cdot |g(t)| \, dt$$

$$\leq \left(\sup_{-\infty < t < \infty} \left| \frac{t-i}{t-z} \right| \right) \|P\|_{\gamma_1, p} \, \|g\|_q$$

$$\leq \left(\sup_{-\infty < t < \infty} \left| \frac{t-i}{t-z} \right| \right) \|g\|_q$$

$$\leq (1 + |z|)\, M \qquad \text{for } |\operatorname{Im} z| \geq 1$$

where M is a constant.

Thus,

$$|P(z)| \leq M \, \frac{1 + |z|}{|F(z)|} \qquad \text{for } |\operatorname{Im} z| \geq 1 .$$

Now, F is analytic and bounded for $\operatorname{Im} z \geq 1$. By Carleman's theorem,

$$\int_{-\infty}^{\infty} \frac{\log |F(x+i)|}{1+x^2}\, dx > -\infty .$$

Put

$$M \cdot \frac{1 + |x+i|}{|F(x+i)|} = m(x).$$

Then

$$\int_{-\infty}^{\infty} \frac{\log m(x)}{1+x^2}\, dx = L < \infty$$

and

$$\log |P(x+i)| < \log m(x), \qquad P \in m_{\gamma_1}(\mathcal{P}).$$

Since $\log |P(z)|$ is subharmonic, we have the estimate.

$$\log |P(z)| \leq \frac{1-y}{\pi} \int_{-\infty}^{\infty} \frac{\log m(t)}{(t-x)^2 + (1-y)^2}\, dt, \qquad \operatorname{Im} z < 1,$$

so that

$$\log |P(x)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log m(t)}{1 + (x-t)^2}\, dt = \phi(x), \text{ say.}$$

Thus, we obtain

$$\log M_{\gamma_1}(x, \mathcal{P}) \leq \phi(x).$$

We shall now show that the integral

$$\int_{-N}^{N} \frac{\phi(x)}{1+x^2}\, dx$$

is bounded as $N \to \infty$.

$$\int_{-N}^{N} \frac{\phi(x)}{1+x^2}\,dx = \int_{-\infty}^{\infty}\left[\int_{-N}^{N}\frac{dx}{(1+x^2)\,(1+(x-t)^2)}\right]\log m(t)\,dt$$

$$< C\int_{-\infty}^{\infty}\frac{\log m(t)}{1+t^2}\,dt = CL < \infty .$$

This, together with the inequality of Koosis, proves the result.

THEOREM 3.2. <u>Any one of the following conditions is necessary and sufficient for</u> $E_{\gamma,0}(X)$ <u>to be dense in</u> $\mathscr{C}_\gamma(X)$, <u>where</u> $X = \mathbb{R}$ <u>or</u> $\mathbb{R}_S$.

(a) $M_{\gamma_1}(z, E_{\gamma_1,0}(X)) = \infty$, $\qquad \operatorname{Im} z \neq 0$,

(b) $$\sup_{f \in \mathcal{M}_{\gamma_1}(E_{\gamma_1,0}(X))} \int_X \frac{\log^+|f(x)|}{1+x^2}\,dx = \infty ,$$

(c) $$\int_X \frac{\log M_{\gamma_1}(x, E_{\gamma_1,0}(X))}{1+x^2}\,dx = \infty .$$

PROOF. As in the case of Theorem 3.1, the proof of (a) is deduced from the corresponding Lemmas 2.7 and 2.8 of Chapter 2.

We shall prove the necessity of (b) and (c). It is enough to show that if

$$\sup_{f \in \mathcal{M}_{\gamma_1}(E_{\gamma_1,0}(X))} \int_X \frac{\log^+|f(x)|}{1+x^2}\,dx = K < \infty , \tag{3.4}$$

then $E_{\gamma,0}(X)$ is not dense in $\mathscr{C}_\gamma(X)$. It is enough to assume (3.4) even for the smaller set consisting of those functions which are real on the real axis, for any other can be written as the sum of two, one real and one purely imaginary on the real axis. If f is an entire function of exponential type zero, we can split it into an even part f_1 and an odd part f_2. First, we assert that there exists a constant $C(\delta)$, which depends only on δ, such that

$$\int_{-\infty}^{\infty} \frac{\log\left[1+f^2(x)\right]}{1+x^2}\,dx \le C(\delta)\int_X \frac{\log\left[1+f^2(x)\right]}{1+x^2}\,dx \qquad (3.5)$$

where (3.5) is obvious with $C(\delta) = 1$, when $X = \mathbb{R}$. We shall prove (3.5) only for even functions. The proof for odd functions is similar. The Hadamard factorization gives

$$f(z) = z^{2m} \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right) ,$$

where $0 < \lambda_1 \le \lambda_2 \le \ldots$. As in [14] we can ignore the zero of f at the origin, by considering the function

$$f_\lambda(z) = \lambda^{2m}\left(1 - \frac{z^2}{\lambda^2}\right)^m \frac{f(z)}{z^{2m}} .$$

f_λ is an even entire function of exponential type zero and $\left|\frac{f_\lambda(z)}{f(z)}\right| \to 1$ as $|z| \to \infty$, so that $f_\lambda \in E_{\gamma,0}(X)$ and $\|f_\lambda - f\| \to 0$ as $\lambda \to 0$.

Following Levinson and McKean [14], we consider the function

$$g(z) = \prod_{|\lambda_n| < d}\left(1 - \frac{z^2}{\lambda_n^2}\right)\prod_{n > d\Lambda}\left(1 - \frac{z^2\lambda^2}{n^2}\right) .$$

depending upon a small positive number λ and a large integral number d. It was asserted there that given $\lambda > 0$, $\varepsilon > 0$ and $A < \infty$, there exists $d_1 = d_1(\lambda, \varepsilon, A)$ and a universal constand B such that for $d \geq d_1$,

(i) $|f(x) - g(x)| < \varepsilon$ $\qquad |x| < A$

(ii) $|g(x)| < B|f(x)|$ $\qquad A \leq |x| < \frac{d}{2}$

(iii) $|g(x)| < B$ $\qquad |x| \geq \frac{d}{2}$

(iv) $g \in L^2(\mathbb{R}^1)$.

Since the entire function $g(z)$ differs from $\sin \pi\lambda z$ by a rational factor, it is of exponential type $\pi\lambda$ and bounded on the real axis. Now the function

$$G(z) = 1 + g^2(z)$$

is of exponential type $2\pi\lambda$, real and bounded on the real axis and satisfies the inequality $G(x) \geq 1$. There then exists an entire function $h(z)$ of exponential type $\pi\lambda$, having no zeros in the upper half plane and satisfying

$$|h(x)|^2 = 1 + g^2(x), \qquad -\infty < x < \infty .$$

Now, applying the technique of Koosis [11] to h we obtain

$$\pi\lambda + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\delta}\int_{-\delta/2}^{\delta/2}\frac{dt}{(x-t)^2+1}\right)\log|h(x)|\,dx \leq$$

$$\leq \frac{K_{\delta/2}}{\delta(1-\delta^2)} \times \int_{\mathbb{R}_\delta}\frac{\log|h(x)|}{1+x^2}\,dx + \pi\lambda\left\{1 - \frac{\log\sin\pi\delta/2}{\pi}\right\}$$

from which we deduce that

$$\int_{-\infty}^{\infty}\frac{\log|h(x)|}{1+x^2}\,dx \leq C(\delta)\left\{\int_{\mathbb{R}_\delta}\frac{\log|h(x)|}{1+x^2}\,dx - \pi\lambda\log\sin\pi\delta/2\right\}$$

(see Koosis [11], p.234) with $C(\delta)$, a constant, depending only on δ. In terms of g, this gives

$$\int_{-\infty}^{\infty} \frac{\log[1+g^2(x)]}{1+x^2}\,dx \le C(\delta)\left\{\int_{\mathbb{R}_\delta} \frac{\log[1+g^2(x)]}{1+x^2}\,dx - 2\pi\lambda \log \sin \pi\delta/2\right\} \qquad (3.6)$$

The application of (i), (ii) and (iii) yields

$$\sup_x \frac{|f(x)-g(x)|}{\gamma(x)} \le \sup_{|x|<A} \frac{|f(x)-g(x)|}{\gamma(x)} + \sup_{A\le|x|<d/2} \frac{|f(x)-g(x)|}{\gamma(x)} + \sup_{|x|\ge d/2} \frac{|f(x)-g(x)|}{\gamma(x)}$$

$$\le \varepsilon \cdot \sup_{|x|<A} \frac{1}{\gamma(x)} + \sup_{A\le|x|<d/2} (B+1)\frac{|f(x)|}{\gamma(x)} + \sup_{|x|\ge d/2} \frac{B+|f(x)|}{\gamma(x)}$$

$\longrightarrow 0$ as $d \to \infty$, $A \longrightarrow \infty$ and $\varepsilon \to 0$ in that order so that

$$\sup \frac{|g(x)|}{\gamma(x)} \longrightarrow \sup \frac{|f(x)|}{\gamma(x)}$$

and

$$g(x) \longrightarrow f(x)$$

for all x under these conditions.

Then, since (3.4) is satisfied, letting $d \to \infty$, $A \longrightarrow \infty$ and $\varepsilon \to 0$ and applying Lebesgue dominated convergence theorem, (3.6) gives

$$\int_{-\infty}^{\infty} \frac{\log[1+f^2(x)]}{1+x^2}\,dx \le C(\delta)\left\{\int_{\mathbb{R}_\delta} \frac{\log[1+f^2(x)]}{1+x^2}\,dx - 2\pi\lambda \log \sin \pi\delta/2\right\}.$$

Since the left hand side is independent of λ, we let $\lambda \to 0$ to obtain (3.6). Then, under the hypothesis (3.4), as in the proof of Theorem 3.1, we get

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2}\, dx \le 2\, C(\delta)\, (\, K + \pi \log 2)$$

provided $\|f\|_{Y_1} \le 1$. Now it follows that

$$\log|f(z)| \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \log^+ |f(t)|dt \,, \qquad z = x+iy\,,\ y>0 \qquad (3.7)$$

from Nevanlinna's theorem, letting $R \to \infty$ and using

$$\lim_{R\to\infty} \sup R^{-1} \log|f(Re^{i\theta})| \le 0$$

so that

$$\log|f(i)| \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+|f(t)|}{1+t^2}\, dt \le 2\, \frac{C(\delta)}{\pi}(K+\pi\log 2)$$

which implies that

$$M_{Y_1}(i,\, E_{Y_1,0}) < \infty$$

and $E_{Y,0}(X)$ is not dense in $\mathscr{C}_Y(X)$.

This establishes the necessity of conditions (b) and (c).

To complete the proof, it is sufficient if we show that when $E_{Y,0}(X)$ is not dense in $\mathscr{C}_Y(X)$,

$$\int_X \frac{\log^+|f(t)|}{1+t^2}\, dt \le \int_X \frac{\log M_{Y_1}(t,\, E_{Y_1,0}(X))}{1+t^2}\, dt \qquad (3.8)$$

for each $f \in \mathcal{M}_{Y_1}(E_{Y_1,0}(X))$.

Since $E_{\gamma,0}(X)$ is not dense in $\mathscr{C}_\gamma(X)$, there exists a function σ of bounded variation over X such that

$$\int_X \frac{f(t)}{\gamma(t)}\, d\sigma(t) = 0 \qquad \text{for all } f \in E_{\gamma,0}(X)$$

and

$$\int_X |d\sigma(t)| = 1.$$

Suppose $f \in E_{\gamma_1,0}(X)$. Define

$$g(t) = \frac{f(t) - f(z)}{t - z} .$$

Then, clearly $g \in E_{\gamma,0}(X)$.

Extending the above functional to the whole of $\mathscr{C}_\gamma(X)$, we obtain

$$\int_X \frac{f(t)}{(t-z)} \frac{1}{\gamma(t)}\, d\sigma(t) = f(z) \int_X \frac{1}{(t-z)} \frac{1}{\gamma(t)}\, d\sigma(t)$$

for $f \in E_{\gamma_1,0}(X)$, $\operatorname{Im} z \neq 0$.

Setting

$$F(z) = \int_X \frac{1}{(t-z)} \frac{1}{\gamma(t)}\, d\sigma(t)$$

we find that

$$\begin{aligned}
|F(z) f(z)| &= \left| \int_X \frac{f(t)}{t-z} \frac{1}{\gamma(t)}\, d\sigma(t) \right| \\
&\leq \int_X \left| \frac{f(t)}{\gamma(t)\sqrt{1+t^2}} \right| \frac{\sqrt{1+t^2}}{|t-z|}\, |d\sigma(t)| \\
&\leq \|f\|_{\gamma_1} \cdot M\sqrt{1+t^2} \\
&\leq M\sqrt{1+t^2},
\end{aligned}$$

M being a constant.

Thus

$$|f(z)| \le M \frac{\sqrt{1+t^2}}{|F(z)|}.$$

Hence

$$|f(x+i)| \le M \frac{\sqrt{1+t^2}}{|F(x+i)|}$$

so that

$$M_{\gamma_1}(x+i) \le M \frac{\sqrt{1+t^2}}{|F(x+i)|}$$

and

$$\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(x+i)}{1+x^2}\,dx < \infty .$$

Then

$$\log|f(x)| \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t+i)}{1+(x-t)^2}\,dt, \qquad f \in E_{\gamma_1,0}(X).$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t)}{1+t^2}\,dt \le \left(\int_{-\infty}^{\infty} \log M_{\gamma_1}(t+i)\,dt\right)\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)(1+(x-t)^2)}\right)$$

$$= 2\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t+i)}{t^2+4}\,dt$$

$$< \infty .$$

This completes the proof of the theorem.

THEOREM 3.3. Any one of the following conditions is necessary and sufficient for $\mathcal{P}$ to be dense in $L^p_\gamma(\mathbb{R})$.

(a) $M_{\gamma_1}(z, \mathcal{P}) = \infty,$ $\qquad \operatorname{Im} z \neq 0$

(b) $\mathcal{A}(\gamma_1, \mathcal{P}) = \infty,$

(c) $$\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, \mathcal{P})}{1 + t^2}\, dt = \infty.$$

PROOF. Since $\mathcal{K}$, the class of continuous functions with compact support, is dense in $L^p_\gamma(\mathbb{R})$ and since every function in $\mathcal{K}$ can be approximated by finite linear combinations of functions of the form $\frac{1}{x-z}$, $\operatorname{Im} z \neq 0$, in $L^p_\gamma(\mathbb{R})$, (a) is immediate.

(b) follows by verbatim proof of Theorem 3.1 (b).

Since

$$\mathcal{A}(\gamma_1, \mathcal{P}) \leq \int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, \mathcal{P})}{1 + t^2}\, dt,$$

necessity of (c) follows.

We shall now prove that if $\mathcal{P}$ is not dense in $L^p_\gamma(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, \mathcal{P})}{1 + t^2}\, dt < \infty.$$

There exists $g \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\int_{-\infty}^{\infty} \frac{t^n}{\gamma(t)}\, g(t)\, dt = 0, \qquad n = 0, 1, 2, \ldots$$

and

$$\|g\|_q = 1 = \left(\int_{-\infty}^{\infty} |g(t)|^q\, dt \right)^{1/q}.$$

If we set

$$F(z) = \int_{-\infty}^{\infty} \frac{1}{t-z} \cdot \frac{1}{\gamma(t)} g(t)\,dt$$

then F is analytic in the half planes $\operatorname{Im} z > 0$, $\operatorname{Im} z < 0$ and it is not identically zero.

For each polynomial P, we have

$$P(z)F(z) = \int_{-\infty}^{\infty} \frac{P(t)}{\gamma(t)} \frac{1}{t-z} g(t)\,dt .$$

Let $P \in \mathcal{M}_{\gamma_1}(\mathcal{P})$. Then $\|P\|_{\gamma_1,p} \leq 1$ and

$$|F(z)P(z)| = \left| \int_{-\infty}^{\infty} \frac{P(t)}{\gamma(t)} \cdot \frac{\sqrt{1+t^2}}{t-z} \cdot \frac{g(t)}{\sqrt{1+t^2}}\,dt \right|$$

$$\leq \int_{-\infty}^{\infty} \left| \frac{P(t)}{\gamma_1(t)} \right| \left| \frac{\sqrt{1+t^2}}{t-z} \right| |g(t)|\,dt$$

$$\leq \max_{t\in\mathbb{R}} \left| \frac{t-i}{t-z} \right| \|P\|_{\gamma_1,p} \|g\|_q$$

$$\leq \max_{t\in\mathbb{R}} \left| \frac{t-i}{t-z} \right|$$

by the application of Holder's inequality and the facts that $\|P\|_{\gamma_1,p} \leq 1$, $\|g\|_q = 1$.

Therefore

$$|P(z)| \leq M \cdot \frac{1+|z|}{|F(z)|} ,$$

where M is a constant. The rest of the proof is the same as that of Theorem 3.1.

The following theorems can be proved analogously (see Akutowicz [3,4]).

THEOREM 3.4. <u>Any one of the following conditions is necessary and sufficient for</u> $E^p_{\gamma,0}(\mathbb{R})$ <u>to be dense in</u> $L^p_\gamma(\mathbb{R})$.

(a) $M_{\gamma_1}(z, E^p_{\gamma_1,0}(\mathbb{R})) = \infty$, $\quad$ Im $z \neq 0$

(b) $\sup_{f \in \mathcal{M}_{\gamma_1}(E^p_{\gamma_1,0}(\mathbb{R}))} \int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{1+t^2}\, dt = \infty$

(c) $$\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, E^p_{\gamma_1,0}(\mathbb{R}))}{1+t^2}\, dt = \infty.$$

PROOF. (a) is trivial since Lemmas 2.6, 2.7 and 2.8 hold in this case also.

Now suppose that $E^p_{\gamma,0}(\mathbb{R})$ is dense in $L^p_\gamma(\mathbb{R})$. Let $f \in E^p_{\gamma_1,0}(\mathbb{R})$. Then

$$\log|f(z)| \le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} \log^+ |f(t)|\, dt, \qquad z = x+iy,\ y>0.$$

Applying (a), as in the case of Theorem 3.1, we get the necessity of the conditions (b) and (c).

To establish the sufficiency of these conditions, it is enough to prove that when $E^p_{\gamma_1,0}(\mathbb{R})$ is not dense in $L^p_\gamma(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{1+t^2}\, dt \le \int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, E^p_{\gamma_1,0}(\mathbb{R}))}{1+t^2}\, dt < \infty$$

for $f \in \mathcal{M}_{\gamma_1}(E^p_{\gamma_1,0}(\mathbb{R}))$. We also notice that

$$\log M_{\gamma_1}(z, E^p_{\gamma_1,0}(\mathbb{R})) \le \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, E^p_{\gamma_1,0}(\mathbb{R}))}{1+t^2}\, dt,$$

where $z = x + iy$, $y > 0$.

As in Theorem 3.3, there exists a function $g \in L^q$ such that

$$\int_{-\infty}^{\infty} \frac{f(t)}{\Upsilon(t)}\, g(t)\, dt = 0 \quad \text{for all } f \in E^p_{\Upsilon,0}(\mathbb{R})$$

and

$$\int_{-\infty}^{\infty} |g(t)|^q\, dt = 1,$$

with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in E^p_{\Upsilon_1,0}(\mathbb{R})$, $\phi(t) = \frac{f(t) - f(z)}{t - z}$ is the sum of our members of $E^p_{\Upsilon,0}(\mathbb{R})$, so that

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-z} \frac{g(t)}{\Upsilon(t)}\, dt = \left(\int_{-\infty}^{\infty} \frac{1}{t-z} \frac{g(t)}{\Upsilon(t)}\, dt \right) f(z).$$

Setting

$$F(z) = \int_{-\infty}^{\infty} \frac{1}{t-z} \frac{g(t)}{\Upsilon(t)}\, dt$$

we find, as in Theorem 3.3,

$$|f(x+i)| \le \frac{\text{const}}{|F(x+i)|} \sqrt{1+x^2}$$

so that

$$M_{\Upsilon_1}(x+i) \le \frac{\text{const}}{|F(x+i)|} \sqrt{1+x^2}$$

and

$$\int_{-\infty}^{\infty} \frac{\log M_{\Upsilon_1}(t, E^p_{\Upsilon_1,0}(\mathbb{R}))}{1+t^2}\, dt < \infty .$$

The proof is now completed as in Theorem 3.2.

THEOREM 3.5. <u>Any one of the following conditions is necessary and sufficient for</u> $E^p_{\gamma,a}(\mathbb{R})$ <u>to be dense in</u> $L^p_\gamma(\mathbb{R})$.

(a) $M_{\gamma_1}(z, E^p_{\gamma_1,a}(\mathbb{R})) = \infty$, $\operatorname{Im} z \neq 0$

(b) $\mathcal{A}(\gamma_1, E^p_{\gamma_1,a}(\mathbb{R})) = \infty$

(c) $$\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, E^p_{\gamma_1,a}(\mathbb{R}))}{1+t^2}\, dt = \infty.$$

PROOF. The necessity is established using earlier arguments. (a) is trivial and can be proved as before, using Lemmas 2.6, 2.8 and 2.10.

Suppose $E^p_{\gamma,a}(\mathbb{R})$ is dense in $L^p_\gamma(\mathbb{R})$. Then, by (a)

$$M_{\gamma_1}(i, E^p_{\gamma,a}(\mathbb{R})) = \infty.$$

If $\phi \in \mathcal{m}_{\gamma_1}(E^p_{\gamma_1,a}(\mathbb{R}))$, then

$$\log|\phi(x+iy)| \le ay + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|\phi(t)|}{y^2+(x-t)^2}\, dt, \quad y > 0$$

so that

$$\log|\phi(i)| \le a + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\phi(t)|}{1+t^2}\, dt$$

$$\le a + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log M_{\gamma_1}(t, E^p_{\gamma_1,a}(\mathbb{R}))}{1+t^2}\, dt$$

and the necessity of (b) and (c) is immediate.

To prove the sufficiency we use the following lemma [3].

LEMMA. Suppose $E^p_{\gamma,a}(\mathbb{R})$ is not dense in $L^p_\gamma(\mathbb{R})$. Then for each $L \in (E^p_{\gamma,a})^\perp$, there exists a function B_L holomorphic in the upper half plane such that

$$f(z)\, B_L(z) = L(I_z\, f), \qquad f \in E^p_{\gamma,a}(\mathbb{R}).$$

We complete the proof as follows. Suppose $E^p_{\gamma,a}(\mathbb{R})$ is not dense in $L^p_\gamma(\mathbb{R})$. Then for each $f \in \mathcal{M}_{\gamma_1}(E^p_{\gamma_1,a}(\mathbb{R}))$, we have

$$\begin{aligned} |f(z)\, B_L(z)| &= \left| \int_{-\infty}^{\infty} \frac{f(t)}{t-z} \frac{g(t)}{\gamma(t)}\, dt \right| \\ &\leq \sup_{-\infty < t < \infty} \left| \frac{t-i}{t-z} \right| \\ &\leq A < \infty . \end{aligned}$$

Setting

$$m(t) = \frac{A}{B_L(t+i)}, \qquad -\infty < t < \infty,$$

using a theorem of Paley and Wiener, we have

$$\int \frac{\log|B_L(t+i)|}{1+t^2}\, dt > -\infty$$

which implies

$$\int_{-\infty}^{\infty} \frac{\log m(t)}{1+t^2}\, dt < \infty.$$

Now $\log|f(z)|$ is subharmonic and $\log|f(x+iy)| \leq \log m(x)$ for real x.

Therefore

$$\log|f(x+iy)| \leq a_0 - ay + \frac{1-y}{\pi} \int_{-\infty}^{\infty} \frac{\log m(t)}{(t-x)^2 + (1-y)^2}\, dt$$

in the half plane $y < 1$, a_0 being a certain constant. In particular, for $y = 0$,

$$\log |f(x)| \leq a_0 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log m(t)}{(t-x)^2+1}\, dt \equiv K(x).$$

As in Theorem 3.1

$$\int_{-\infty}^{\infty} \frac{\log |f(x)|}{1+x^2}\, dx \leq \int_{-\infty}^{\infty} \frac{\log M_{Y_1}(t, E^p_{Y_1,a}(R))}{1+t^2}\, dt$$

$$\leq \int_{-\infty}^{\infty} \frac{K(t)}{1+t^2}\, dt$$

$$< \infty ,$$

thus completing the proof.

CHAPTER 4

SOLUTION OF THE PROBLEM WHEN γ IS CONTINUOUS

So far, we have imposed no continuity condition on γ. It was therefore necessary to introduce the function γ_1 and the conditions were obtained in terms of γ_1. We shall now see that when γ is a continuous function, the same conditions, with γ_1 replaced by γ, will provide the solution to our problem.

In this chapter, we shall investigate the necessary and sufficient conditions for D to be dense in $\mathcal{C}_\gamma(\mathbb{R})$, $\mathcal{P}$ to be dense in $\mathcal{C}_\gamma(\mathbb{R})$ and $\mathcal{C}_\gamma(\mathbb{R}_S)$ and for $\mathcal{P}$ to be dense in $L^p_\gamma(\mathbb{R})$, assuming that γ is continuous. We shall also treat the non-dense case and ask for the closure of the set of polynomials in $\mathcal{C}_\gamma(\mathbb{R})$ and in $L^p_\gamma(\mathbb{R})$.

THEOREM 4.1. <u>Let $\gamma \geq 1$ be continuous. Then D is dense in $\mathcal{C}_\gamma(\mathbb{R})$ if and only if</u>

$$\mathcal{A}(\gamma, D) = \infty \tag{4.1}$$

PROOF. Since

$$\gamma(x) \leq \gamma_1(x)$$

it follows that

$$\mathcal{A}(\gamma, D) \leq \mathcal{A}(\gamma_1, D),$$

so that the sufficiency of (4.1) is clear.

We shall now prove the necessity. Suppose that

$$\mathcal{A}(\gamma, D) < \infty.$$

We shall then show that the closure of D in $\mathcal{C}_\gamma(\mathbb{R})$ consists of at most entire functions of exponential type. It is enough to show that the set $\mathcal{M}_\gamma(D)$ constitutes a normal family. It suffices to show this even for the smaller set consisting of these elements which are real on the real axis, for any other can be written as the sum of two elements, one real and one purely imaginary on the real axis.

Let $f \in D$ such that $\|f\| \leq 1$. Set

$$|g(x)|^2 = 1 + f^2(x),$$

where g has no zeros in the upper half plane and $g \in D$. In fact, this is trivial when $D = \mathcal{P}$ and in the case $D = E_{Y,0}$ or $E_{Y,a}$ it follows that

$$\int_{-\infty}^{\infty} \frac{\log |f(x)|}{1 + x^2}\, dx \leq A(Y,D) < \infty$$

by (4.2)

By P̊isson formula, we have, since $\log |g(z)|$ is subharmonic,

$$\log |g(z)| \leq a.|y| + \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log|g(t)|}{(x-t)^2+y^2}\, dt, \quad z = x+iy, \; y \neq 0,$$

where $a = 0$ when $D = \mathcal{P}$ or $E_{Y,0}$.

Setting

$$\phi(t) = \frac{\log|g(t)|}{1+t^2} \geq 0$$

we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \phi(t)\, dt \leq A(Y,D) + \log 2$$

$$= L .$$

The rest of the proof is similar to that of Akhiezer

[2, pp. 111-112] . Thus we get

$$|f(z)| \le e^{6L+a}\, e^{(5L+a)|z|}\,, \qquad z \in K\,.$$

To prove the above inequality, we find

$$\log|g(z)| \le a|y| + \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\phi(t)}{(x-t)^2+y^2}\,dt + \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{t^2\phi(t)}{(x-t)^2+y^2}\,dt$$

$$\le a|y| + \frac{L}{|y|} + \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{t^2\phi(t)}{(x-t)^2+y^2}\,dt\,.$$

Integration by parts yields

$$\frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{t^2\phi(t)}{(x-t)^2+y^2}\,dt = \frac{|y|}{\pi}\left\{\int_{-\infty}^{\infty}\phi(t)\,dt - \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\tau}\phi(\tau)\,d\tau\right]\frac{d}{dt}\left(\frac{t^2}{(x-t)^2+y^2}\right)dt\right\}$$

$$\le L|y| + L|y|\int_{-\infty}^{\infty}\left|\frac{d}{dt}\left(\frac{t^2}{(x-t)^2+y^2}\right)\right|dt\,.$$

Now

$$\int_{-\infty}^{\infty}\left|\frac{d}{dt}\left(\frac{t^2}{(x-t)^2+y^2}\right)\right|dt = \frac{2\,(x^2+y^2)}{y^2}\,.$$

Hence

$$\log|g(z)| \le a|y| + L\left\{\frac{1}{|y|} + |y| + \frac{2(x^2+y^2)}{|y|}\right\}$$

$$= a|y| + L\cdot\frac{1+2x^2+3y^2}{|y|} \qquad (4.3)$$

We thus find that

$$\log|g(x \pm i)| \le a + L(4 + 2x^2).$$

Now the function

$$F(z) = e^{-2Lz^2}\, g(z)$$

is regular in the strip $|\operatorname{Im} z| \le 1$ and bounded on the sides of this strip, that is

$$|F(x \pm i)| \leq e^{4L+a}, \qquad -\infty < x < \infty .$$

Moreover it converges uniformly to 0 for $|x| \to \infty$. Therefore, by the maximum principle it follows that in the whole strip $|\text{Im } z| \leq 1$,

$$|F(z)| \leq e^{4L+a}$$

so that we get

$$|g(z)| \leq e^{4L+a} e^{2Lx^2} .$$

In particular,

$$|g(z)| \leq e^{6L+a}$$

in the square $|x| \leq 1, |y| \leq 1$.

From (4.3), it follows that for $|x| \geq 1$,

$$|g(x \pm ix)| \leq e^{a|x|} e^{L} e^{5L|x|} .$$

Therefore, on the bisectors $y = \pm x$, $-\infty < x < \infty$,

$$|g(z)| \leq e^{6L+a} e^{(5L+a)|x|} .$$

By the Phragmen-Lindelof principle, for every $z \in \mathbb{K}$ we have

$$|g(z)| \leq e^{6L+a} e^{(5L+a)|z|} .$$

We are now in a position to complete the proof of our theorem.

Assume that

$$A(\gamma, D) < \infty .$$

Suppose that it is possible to approximate a function $g \in \mathcal{E}_\gamma(\mathbb{R})$ with any desired degree of accuracy by means of elements of D, that is, we assume that there exists a sequence of functions $\{f_n\}$

in D for which

$$\lim_{n \to \infty} \| f_n - g \| = 0 .$$

Since

$$\| f_n \| \le \| g \| + \| f_n - g \|$$

there exists a constant C such that

$$| f_n(x) | \le C \gamma(x), \qquad -\infty < x < \infty .$$

Then, by what has already been proved, letting $A = e^{6L+a}$, $B = 5L + a$, we have

$$| f_n(z) | \le C A e^{B|z|}, \qquad n = 1, 2, \dots ,$$

and thus $\{f_n\}$ forms a normal family.

Therefore, it is possible to select a subsequence from $\{f_n\}$ which converges to an entire function G of exponential type less than or equal to B uniformly in any finite part of the plane.

At the same time,

$$G(x) = f(x)$$

for all n, whenever γ is finite. By the continuity of γ, it follows that g is the restriction to the real axis of an entire function G of exponential type not exceeding B and hence cannot be an arbitrary function of the class $\mathcal{E}_\gamma(\mathbb{R})$. Thus, D is not dense in $\mathcal{E}_\gamma(\mathbb{R})$. This completes the proof.

THEOREM 4.2. <u>Let</u> $\gamma \ge 1$ <u>be continuous. Then</u> $\mathcal{P}$ is <u>dense in</u> $\mathcal{E}_\gamma(\mathbb{R}_3)$ <u>if and only if</u>

$$\mathcal{A}(\gamma, \mathcal{P}) = \sup_{P \in \mathcal{M}_\gamma(\mathcal{P})} \int_{\mathbb{R}_\delta} \frac{\log |P(x)|}{1+x^2}\, dx = \infty . \tag{4.4}$$

PROOF. We need to prove only the necessity of (4.4). Assume

$$\sup_{P \in \mathcal{M}_\gamma(\mathcal{P})} \int_{\mathbb{R}_\delta} \frac{\log |P(x)|}{1+x^2}\, dx < \infty .$$

Then, for all polynomials P which are real on the real axis and with $\|P\| \le 1$, there exists a constant L_δ which depends only on δ such that

$$\int_{-\infty}^{\infty} \frac{\log |P(x)|}{1+x^2}\, dx \le L_\delta (K + \pi \log 2) .$$

Using the formula

$$\log |P(z)| \le \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |P(t)|}{1+t^2}\, dt ,$$

the argument of Theorem 3.6 will show that the set of polynomials P satisfying

$$\|P\| \le 1$$

forms a normal family. The closure of this set can only consist of analytic functions. Thus $\mathcal{P}$ is not dense in $\mathcal{C}_\gamma(\mathbb{R}_\delta)$.

THEOREM 4.3. <u>Suppose</u> $\gamma \ge 1$ <u>is continuous. If</u> $\mathcal{P}$ <u>is dense in</u> $\mathcal{C}_\gamma(\mathbb{R})$, <u>then</u>

$$\int_{-\infty}^{\infty} \frac{\log \gamma(x)}{1+x^2}\, dx = \infty .$$

PROOF. From the inequality

$$\frac{p}{A+B}\int_{-A}^{B}\log\left|\frac{f(x)}{\gamma(x)}\right|dx \le \log\left\{\frac{1}{A+B}\int_{-A}^{B}\left|\frac{f(x)}{\gamma(x)}\right|^{p}dx\right\}$$

where $f \in \mathcal{M}_{\gamma}(D)$ and A and B are arbitrary constants, it follows that

$$\int_{-A}^{B}\log\left|\frac{f(x)}{\gamma(x)}\right|dx \le 0$$

or

$$\int_{-A}^{B}\log|f(x)|\,dx \le \int_{-A}^{B}\log\gamma(x)\,dx .$$

Now,

$$\int_{-n}^{n+1}\frac{\log|f(x)|}{1+x^2}dx \le \frac{1}{1+n^2}\int_{-n}^{n+1}\log|f(x)|\,dx$$

$$\le \frac{1}{1+n^2}\int_{-n}^{n+1}\log\gamma(x)\,dx$$

$$\le \frac{1+(n+1)^2}{1+n^2}\int_{-n}^{n+1}\frac{\log\gamma(x)}{1+x^2}dx$$

$$\le 2\int_{-n}^{n+1}\frac{\log\gamma(x)}{1+x^2}dx .$$

Hence, as $n \to \infty$,

$$\int_{-\infty}^{\infty}\frac{\log|f(x)|}{1+x^2}dx \le 2\int_{-\infty}^{\infty}\frac{\log\gamma(x)}{1+x^2}dx .$$

Thus, $f \in \mathcal{M}_{\gamma}(D)$ implies that

$$\int_{-\infty}^{\infty}\frac{\log|f(x)|}{1+x^2}dx \le 2\int_{-\infty}^{\infty}\frac{\log\gamma(x)}{1+x^2}dx . \qquad (4.5)$$

But, by Theorem 4.1, when D is dense in $\mathcal{C}_{\gamma}(\mathbb{R})$,

$$\mathcal{A}(\gamma, D) = \infty$$

and thus, using (4.5), the proof is completed.

THEOREM 4.4. (Analogue of Pollard's theorem). Let $\gamma \geq 1$ be continuous. A necessary and sufficient condition that $\mathcal{P}$ is dense in $\mathcal{C}_\gamma(\mathbb{R}_\delta)$ is that the following hold:

(a) $$\int_{\mathbb{R}_\delta} \frac{\log \gamma(x)}{1+x^2}\,dx = \infty ,$$

(b) there exists a sequence of polynomials $\{p_n\}$ such that for each real x,

$$\lim_{n \to \infty} |p_n(x)| = \gamma(x)$$

and

$$|p_n(x)| \leq M\gamma(x)$$

where M is a constant.

PROOF. NECESSITY.

The necessity of (a) has been proved by Theorem 4.3. To prove that of (b), choose a continuous function F such that $\frac{F(x)}{\gamma(x)}$ is equal to one on $\mathbb{R}_\delta \cap [-n,n]$, zero outside $(-n-\delta, n+\delta)$ linear on $[-n-\delta,-n]$ and $[n,n+\delta]$.

Since $\mathcal{P}$ is dense in $\mathcal{C}_\gamma(\mathbb{R}_\delta)$, there exists a polynomial p_n such that

$$\| F - p_n \|_\gamma \leq \frac{1}{2^n}$$

which implies that

$$\left|\frac{p_n(x)}{\gamma(x)}\right| \leq \|p_n\|_\gamma \leq \|F\|_\gamma + \|F - p_n\|_\gamma \leq M_1 + \frac{1}{2^n} \leq M_1 + \frac{1}{2} = M$$

where $\|F\|_\gamma = M_1$ and M is the resultant constant, so that

$$|p_n(x)| \leq M\,\gamma(x), \qquad \text{as } n \to \infty$$

and

$$\left|1 - \frac{p_n(x)}{\gamma(x)}\right| \leq \frac{1}{2^n}$$

which gives

$$\lim_{n \to \infty} |p_n(x)| = \gamma(x).$$

The proof is thus completed is in Akhiezer [2, p.118].

SUFFICIENCY. Let us suppose that the conditions (a) and (b) hold. It is enough to prove that there exists a sequence of polynomials $\{P_n\}$ such that

$$\|P_n\|_\gamma \leq C \tag{4.6}$$

where C is a constant and

$$\sup_n \int_{\mathbb{R}_\delta} \frac{\log |P_n(x)|}{1+x^2} dx = \infty . \tag{4.7}$$

This would imply that $A(\gamma, \mathcal{P}) = \infty$ and the density will be an immediate consequence.

Let us consider the polynomial

$$1 + |p_n(x)|^2 .$$

It can be represented in the form

$$1 + |p_n(x)|^2 = |P_n(x)|^2$$

where the polynomials P_n and p_n are of the same degree.

Further

$$1 \leq |P_n(x)| = \sqrt{1+|P_n(x)|^2} \leq \sqrt{1+M^2(\gamma(x))^2} \leq \left(\sqrt{1+M^2}\right)\gamma(x)$$

so that for the polynomials P_n, (4.6) is satisfied with $C = \sqrt{1+M^2}$. It remains to prove (4.7).

Suppose

$$\sup_n \int_{\mathbb{R}_\delta} \frac{\log |P_n(x)|}{1+x^2} dx = A < \infty . \tag{4.8}$$

Since the condition (b) is satisfied,

$$\lim_{n\to\infty} |P_n(x)| = \sqrt{1 + (\gamma(x))^2}$$

and since

$$|P_n(x)| \geq 1,$$

by Fatou's lemma and by virtue of (4.8) we find that the integral

$$\frac{1}{2}\int_{\mathbb{R}_\delta} \frac{\log\left[1 + (\gamma(x))^2\right]}{1+x^2}\,dx$$

exists and does not exceed A. This implies that

$$\int_{\mathbb{R}_\delta} \frac{\log \gamma(x)}{1+x^2}\,dx$$

does not exceed A, contradicting (a). This complies the proof.

THEOREM 4.5. Let γ be continuous. Then $\mathcal{P}$ is dense in $L^p_\gamma(\mathbb{R})$ if and only if the following conditions hold:

(a)
$$\int_{-\infty}^{\infty} \frac{\log \gamma(x)}{1+x^2}\,dx = \infty ,$$

(b) there exists a sequence of polynomials $\{P_n\}$ such that

$$\|P_n\| \leq M, \qquad n = 1, 2, \ldots,$$

and

$$\lim_{n\to\infty} |P_n(x)| = \frac{\gamma(x)}{\sqrt{1+x^2}}$$

where M is a constant.

PROOF. (NECESSITY). We have to prove only the necessity of (b), since that of (a) has already been established. As $\mathcal{K}$, the class of continuous functions with compact support, is dense in $L^p_\gamma(\mathbb{R})$, we choose $F \in L^p_\gamma(\mathbb{R})$ such that

$$F(x) = \begin{cases} \dfrac{\gamma(x)}{\sqrt{1+x^2}}, & x \in [-n,n] \\ 0, & x \notin [-n,n]. \end{cases}$$

Then

$$\|F\|_{\gamma,p} \le \left(\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{p/2}}\, dx \right)^{\frac{1}{p}} = \alpha .$$

Since $\mathcal{P}$ is dense in $L^p_\gamma(\mathbb{R})$, there exists p_n such that

$$\|F - p_n\|_{\gamma,p} \le \frac{1}{2^n}.$$

It then follows that

$$\|p_n\|_{\gamma,p} \le \|F\|_{\gamma,p} + \|F - p_n\|_{\gamma,p} \le \alpha + \frac{1}{2^n} \le \alpha + \frac{1}{2}.$$

Further,

$$\int_{-n}^{n} \left| \frac{1}{\sqrt{1+x^2}} - \frac{p_n(x)}{\gamma(x)} \right|^p dx \le \frac{1}{2^{np}}$$

from which we obtain

$$\lim_{n\to\infty} \left| \frac{1}{\sqrt{1+x^2}} - \frac{p_n(x)}{\gamma(x)} \right| = 0 \quad a.e.$$

By continuity, we have

$$\lim_{n\to\infty} p_n(x) = \frac{\gamma(x)}{\sqrt{1+x^2}} .$$

(SUFFICIENCY). To prove the sufficiency, it is enough to prove that if the conditions (a) and (b) are satisfied, then there exists a sequence of polynomials $\{P_n\}$ such that

$$\|P_n\|_{\gamma_1,p} \le C$$

where C is a constant and

$$\sup_n \int_{-\infty}^{\infty} \frac{\log |P_n(x)|}{1+x^2}\,dx = \infty .$$

Set

$$|P_n(x)|^2 = 1 + |p_n(x)|^2 .$$

Then we have

$$\|P_n\|_{\gamma_1,p} \leq \|1\|_{\gamma_1,p} + \|p_n\|_{\gamma_1,p} \leq \|1\|_{\gamma_1,p} + \|p_n\|_{\gamma,p}$$

$$\leq \|1\|_{\gamma_1,p} + M = C .$$

Now suppose that

$$\int_{-\infty}^{\infty} \frac{\log |P_n(x)|}{1+x^2}\,dx < A < \infty , \qquad n = 1, 2, \ldots$$

Since the conditions of the theorem are satisfied, we have

$$\lim_{n\to\infty} |P_n(x)| = \lim_{n\to\infty} \sqrt{1 + |p_n(x)|^2} = \sqrt{1 + \frac{(\gamma(x))^2}{1+x^2}} .$$

Also, as $|P_n(x)| \geq 1$, by Fatou's lemma we have

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log\left[1 + \frac{(\gamma(x))^2}{1+x^2}\right]}{1+x^2}\,dx$$

exists and does not exceed A. This ~~sum~~ implies that

$$\int_{-\infty}^{\infty} \frac{\log \gamma(x)}{1+x^2}\,dx - \int_{-\infty}^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx$$

exists and does not exceed A, which in turn gives

$$\int_{-\infty}^{\infty} \frac{\log \gamma(x)}{1+x^2}\,dx$$

does not exceed A. This contradicts condition (a) and establishes the sufficiency. This completes the proof of the theorem.

CHAPTER 5.

THE NON-DENSE CASE AND A REMARK ON BEST APPROXIMATION

We shall now consider the non-dense case and find the closure of the class $\mathcal{P}$ in $L^p_\gamma(\mathbb{R})$.

It was observed by Mergelyan [15] that for any $\gamma(x) \geq 1$, there are two possibilities: Either

$$M_\gamma(z, \mathcal{P}) = \infty, \qquad \operatorname{Im} z \neq 0,$$

or there is a function $\varepsilon(r) \to 0$ as $r \to \infty$ ($|z| = r$) such that

$$M_\gamma(z, \mathcal{P}) < A e^{r\,\varepsilon(r)}$$

where A is a constant independent of r. From this, it follows that either $\mathcal{P}$ is dense in $\mathcal{E}_\gamma(\mathbb{R})$ or the closure of $\mathcal{P}$ is contained in $E_{\gamma,0}(\mathbb{R})$.

Analogously, we can prove that

THEOREM 5.1. Either $\mathcal{P}$ is dense in $L^p_\gamma(\mathbb{R})$ or the closure of $\mathcal{P}$ in $L^p_\gamma(\mathbb{R})$ is contained in $E^p_{\gamma,0}(\mathbb{R})$.

PROOF. The proof is the same as that of Mergelyan [15] .

Hačatryan has proved that the following theorem is true.

THEOREM. Suppose that $\mathcal{P}$ is not dense in $\mathcal{E}_\gamma(\mathbb{R})$. Then $\mathcal{P}$ is dense in $E_{\gamma,0}(\mathbb{R})$ if and only if

$$M_{\gamma_1}(z, \mathcal{P}) = M_{\gamma_1}(z, E_{\gamma_1,0}(\mathbb{R})), \qquad \operatorname{Im} z \neq 0.$$

A natural extension to $L^p_\gamma(\mathbb{R})$ is therefore immediate.

THEOREM 8.2. *Suppose* $\mathcal{P}$ *is not dense in* $L^p_\gamma(\mathbb{R})$. *Then* $\mathcal{P}$ *is dense in* $E^p_{\gamma,0}(\mathbb{R})$ *if and only if*

$$M_{\gamma_1}(z, \mathcal{P}) = M_{\gamma_1}(z, E^p_{\gamma_1,0}(\mathbb{R})), \quad \text{Im } z \neq 0 \tag{8.1}$$

PROOF. Suppose (8.1) holds. Let $L \in \mathcal{P}^\perp$. Then there exists $\phi \in L^q$ such that

$$L(t^n) = 0 = \int_{-\infty}^{\infty} \frac{t^n}{\gamma(t)} \phi(t)\, dt, \qquad n = 0,1,2,\ldots,$$

with

$$\|L\| = \|\phi\|_q = 1$$

Therefore, for $P \in \mathcal{P}$,

$$L\left[\frac{P(t) - P(z)}{t-z}\right] = 0 . \tag{8.2}$$

We extend L to $L^p_\gamma(\mathbb{R})$. Then (8.2) implies

$$L\left[\frac{P(t)}{t-z}\right] = P(z)\cdot L\left[\frac{1}{t-z}\right] . \tag{8.3}$$

The function

$$F(z) = L\left[\frac{1}{t-z}\right]$$

is holomorphic for $\text{Im } z \neq 0$. Further

$$\left| L\left[\frac{P(t)}{t-z}\right]\right| \leq \|\phi\|_q \, \|P\|_{\gamma_1,p} \left\|\frac{t-i}{t-z}\right\|_\infty = o(1), \quad \text{as } |\text{Im} z| \to \infty,$$

for $P \in \mathcal{M}_{\gamma_1}(\mathcal{P})$.

Therefore

$$|F(z)| \leq o(1)\left(\sup_{P \in \mathcal{M}_{\gamma_1}(\mathcal{P})} |P(z)|\right)^{-1} = \frac{o(1)}{M_{\gamma_1}(z, \mathcal{P})} . \tag{8.4}$$

Let $f \in E^p_{\gamma_1,0}(\mathbb{R})$. Then

$$\frac{f(t) - f(z)}{t-z} \in E^p_{\gamma,0}(\mathbb{R})$$

for every z.

Then, we can show that

$$\Phi(z) = L\left[\frac{f(t)-f(z)}{t-z}\right]$$

$$= L\left[\frac{f(t)}{t-z}\right] - f(z)F(z) \tag{5.5}$$

is an entire function of exponential type zero. It follows that

$$L\left[\frac{f(t)}{t-iy}\right] = o(1)$$

Therefore, from (5.4) and (5.5) we get

$$|\Phi(iy)| \le o(1) + |f(iy)|\,|F(iy)|$$

$$\le o(1) + \frac{o(1)}{M_{\gamma_1}(iy,\mathcal{P})}\cdot M_{\gamma_1}(iy, E^p_{\gamma_1,0}(\mathbb{R})) \tag{5.6}$$

Consequently, since by hypothesis, (5.1) holds, using (5.6) we have

$$|\Phi(iy)| = o(1) \qquad \text{as } |y| \to \infty,$$

which gives

$$\Phi(z) \equiv 0.$$

Thus

$$L\left[\frac{f(t)-f(z)}{t-z}\right] = 0,$$

which implies that

$$L\left[\frac{f(t)}{t-z_0}\right] = 0,$$

where z_0 is a zero of $f(z)$.

Let $f \in E^p_{\gamma,0}(\mathbb{R})$. Consider the function

$$g(t) = \|f\|^{-1}(t-i)\,f(t).$$

Then $g(i) = 0$, $g \in \mathcal{M}_{\gamma_1}(E^p_{\gamma_1,0}(\mathbb{R}))$ and so

$$0 = L\left[\frac{g(t)}{t-i}\right] = L\left[\|f\|^{-1} f(t)\right] = \|f\|^{-1} L\,[f(t)]\,.$$

Thus, the functional L vanishes for all $f \in E^p_{\gamma,0}(\mathbb{R})$ and the closure of $\mathcal{P}$ is therefore $E^p_{\gamma,0}(\mathbb{R})$.

REMARK 5.3. Suppose γ has the representation

$$\gamma(x) = \gamma(0)\exp\left(\int_0^x \frac{\omega(t)}{t}\,dt\right). \tag{5.7}$$

Assume

$$\int_{-\infty}^{\infty} \frac{\log \gamma(x)}{1+x^2}\,dx = \infty.$$

Then

$$\int_{-\infty}^{\infty} \frac{\log \omega(x)}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{\log \gamma(x)}{1+x^2}\,dx - \int_{-\infty}^{\infty} \frac{\log\sqrt{1+x^2}}{1+x^2}\,dx$$

implies

$$\int_{-\infty}^{\infty} \frac{\log \omega(x)}{1+x^2}\,dx = \infty.$$

Then

$$\int_{-\infty}^{\infty} \left|\frac{f(x)-P(x)}{\gamma(x)}\right|^p dx = \int_{-\infty}^{\infty} \left|\frac{f(x)-P(x)}{\omega(x)}\right|^p \frac{1}{(1+x^2)^{p/2}}\,dx,$$

so that if $\mathcal{P}$ is dense in $\mathcal{C}_\omega$, then $\mathcal{P}$ is dense in L^p_γ also. Thus, the condition (5.7) implies that the polynomials are dense in L^p_γ, when $1 < p < \infty$.

Setting $p(x) = \log \gamma(x)$, $q(x)$, the function inverse to $p(x)$, θ any number satisfying $0 < \theta < 1$, $K = \theta^{2n+2}/(1-\theta^2)$ and $\delta = \log [19h(0)] / eh(1)$, it is easy to prove from the theorem of Mergelyan [16] that easy

PROPOSITION 8.4. *There exists an absolute constant* $C > 0$ *such that the inequality*

$$E_n\left(\gamma, \frac{1}{x-a}\right) < C \cdot \frac{\gamma(0)}{\gamma^2(1)} \, \frac{1-K}{|\operatorname{Im} a|} \exp\left\{ \frac{|\operatorname{Im} a|}{\pi(1+|a|^2)} \times \int_0^{e^{-1}\theta q(n-\delta)} \frac{\log \gamma(x)}{1+x^2}\, dx \right\}$$

holds for all n *satisfying the conditions* $K < 1$ *and* $n > \delta$, *where* $E_n[\gamma, 1/(x-a)]$ *denotes* $\inf \|I_a - P\|_{\gamma,p}$ *for all polynomials of degree not exceeding* n *and* a *is a non real complex number.*

PART II

ON MULTIPLIER TRANSFORMATIONS

CHAPTER 6

MULTIPLIERS ON WEIGHTED SPACES

Let λ be a real number such that $-\frac{1}{2} < \lambda < \frac{1}{2}$. Let T denote the set of real numbers modulo one and Z the additive group of integers. For $1 \leq p < \infty$, we denote by $\ell^{p,\lambda}(Z)$, the vector space of complex-valued functions f defined on Z such that

$$N_{p,\lambda}[f] = \left(\sum_{n \in Z} |f(n)|^p (|n| + 1)^{p\lambda} \right)^{1/p}$$

is finite, while $L^{p,\lambda}(T)$ denotes the space of those complex-valued functions f defined on T for which

$$\|f\|_{p,\lambda} = \left(\int_T |f(\theta)\theta^{\lambda}|^p \, d\theta \right)^{1/p}$$

is finite.

If $f \in \ell^{2,0}(Z)$, its Fourier transform

$$f^{\wedge}(\theta) = \sum_{n \in Z} f(n)\, e^{2\pi i n \theta}, \qquad \theta \in T,$$

exists as a limit in the mean, of order 2, of the partial sums of the series on the right, and the inversion formula

$$f(n) = \int_T f^{\wedge}(\theta)\, e^{-2\pi i n \theta}\, d\theta$$

is valid. Let $h^{\wedge}$ be a bounded measurable function defined on T. We set

$$Hf(n) = \int_T f^{\wedge}(\theta)\, h^{\wedge}(\theta)\, e^{-2\pi i n \theta}\, d\theta$$

for $n \in Z$, $f \in \ell^{2,0}(Z)$. Such a transformation H, determined by $h^\wedge$, is called a multiplier transformation. If

$$N_{p,\lambda}[H] = \text{l.u.b.} \left\{ N_{p,\lambda}[Hf]/N_{p,\lambda}[f] \text{ , } f \in \ell^{2,0}(Z) \cap \ell^{p,\lambda}(Z), \ f \not\equiv 0 \right\}$$

is finite, then H has a unique extension as a bounded linear transformation of $\ell^{p,\lambda}(Z)$ into itself, with norm $N_{p,\lambda}[H]$, since $\ell^{2,0}(Z) \cap \ell^{p,\lambda}(Z)$ is dense in $\ell^{p,\lambda}(Z)$.

Similarly for $f \in L^{2,0}(T)$, we set

$$f^\wedge(n) = \int_T f(\theta)\, e^{-2\pi i n\theta}\, d\theta.$$

Let $h^\wedge$ be a bounded function defined on Z. Then the multiplier transformation H, associated with $h^\wedge$, is defined by

$$Hf(\theta) = \sum_{n \in Z} h^\wedge(n)\, f^\wedge(n)\, e^{2\pi i n\theta} .$$

If

$$\|H\|_{p,\lambda} = \text{l.u.b} \left\{ \|Hf\|_{p,\lambda} / \|f\|_{p,\lambda}, \ f \in L^{2,0}(T) \cap L^{p,\lambda}(T), \ f \not\equiv 0 \right\}$$

is finite, then H has a unique extension as a bounded linear transformation of $L^{p,\lambda}(T)$ into itself.

An important problem in this connection is to find sufficient conditions on the multiplier function $h^\wedge$, which will guarantee that the multiplier transformation H associated with $h^\wedge$ is a bounded transformation. In [24], Hirschman has investigated this problem when $\lambda = 0$ and obtained some conditions different from the most familiar result that if $h^\wedge$ is of bounded variation on T, then H is bounded for $1 < p < \infty$. In [26], he considered the problem for $\ell^{2,\lambda}(Z)$ and obtained the following result in terms of bounded β-variation of a function, which terminology we shall explain later in chapter 7.

THEOREM A. <u>Let</u> $h^\wedge$ <u>be defined on</u> T <u>and let</u> H <u>be the corresponding multiplier transformation.</u> <u>If</u> $V_\beta[h^\wedge]$ <u>is finite,</u> $(\beta > 2)$ <u>then</u>

$$N_{2,\lambda}[H] < \infty \qquad \text{if } |\lambda| < \frac{1}{\beta},$$

<u>where</u> $V_\beta[h^\wedge]$ <u>denotes the β-variation of</u> $h^\wedge$.

We extend the results of Hirschman to $\ell^{p,\lambda}(Z)$ in chapter 8. In chapter 7, the result analogous to Theorem A is proved for $L^{2,\lambda}(T)$.

CHAPTER 7

MULTIPLIERS ON $L^{2,\lambda}(T)$

In this chapter, we discuss multiplier transformations defined on $L^{2,\lambda}(T)$. Let $h^\wedge$ be a bounded function defined on Z and let H be the corresponding multiplier transformation defined on $L^{2,\lambda}(T)$. If $I(H)$ is the set of all indices λ for which $\|H\|_{2,\lambda}$ is finite, then it is easy to verify that

(a) if $\lambda_1, \lambda_2 \in I(H)$ and if $\gamma = (1-\eta)\lambda_1 + \eta\lambda_2$, $0<\eta<1$, then $\gamma \in I(H)$ and $\|H\|_{2,\gamma} \le \|H\|_{2,\lambda_1}^{1-\eta} \cdot \|H\|_{2,\lambda_2}^{\eta}$.

(b) if $\lambda \in I(H)$, then $-\lambda \in I(H)$ and $\|H\|_{2,\lambda} = \|H\|_{2,-\lambda}$.

The first of these results is a consequence of the Riesz-Thorin convexity theorem [29] , while the second follows from the fact that the conjugate space of $L^{2,\lambda}(T)$ is $L^{2,-\lambda}(T)$.

We shall give two lemmas, which will be needed later.

LEMMA 7.1. *If* $f(\theta) \sim \sum_{n\in Z} f^\wedge(n)\, e^{2\pi i n\theta}$, *then for* $0 \le \lambda < \frac{1}{2}$,

(a) $\sum_{n\in Z} |f^\wedge(m+n)|^2 (|n|+1)^{-2\lambda} \le A'(\lambda)\,(N_{2,\lambda}[f]\ )^2$,

(b) $\sum_{n\in Z} |f^\wedge(m+n)|^2 (|n|+1)^{2\lambda} \ge A''(\lambda)\,(N_{2,-\lambda}[f])^2$

for all $m \in Z$, *where* $A'(\lambda)$ *and* $A''(\lambda)$ *are positive constants depending only on* λ.

PROOF. This can be easily deduced from Hirschman [23,p.61] .

LEMMA 7.2. *If* $f \in L^{2,\lambda}(T)$ *and if* $a_n = \int_T f(\theta)\, e^{-2\pi i n\theta}\, d\theta$, *then for* $0 < \lambda < \frac{1}{2}$,

$$A' \int_T |f(\theta)\,\theta^{\lambda}|^2 d\theta \le \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} |a_{n+m} - a_m|^2 n^{-1-2\lambda} \le A'' \int_T |f(\theta)\,\theta^{\lambda}|^2 d\theta$$

where A' *and* A'' *are positive constants, depending only on* λ.

PROOF. See Hirschman [23, p.52].

Let $\mathcal{M}_\lambda$ denote the set of all bounded multiplier transformations defined on $L^{2,\lambda}(T)$.

THEOREM 7.3. *Suppose* $0 < \lambda < \frac{1}{2}$ *and* $H \in \mathcal{M}_\lambda$. *Then there exists a constant* $A(\lambda)$, *depending only on* λ, *such that for any* $f \in L^{2,\lambda}(T)$,

$$\sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f\hat{}(m)|^2 |h\hat{}(m+n) - h\hat{}(m)|^2 \le A(\lambda)\, \|H\|_{2,\lambda}^2 \|f\|_{2,\lambda}^2 .$$

PROOF. It is easy to verify that

$$|h\hat{}(m+n)| \le \|H\|_{2,\lambda} .$$

Now using this and the relation

$$f\hat{}(m)\,[\,h\hat{}(m+n) - h\hat{}(m)] =$$
$$= [f\hat{}(m+n)\, h\hat{}(m+n) - f\hat{}(m)\, h\hat{}(m)] + [f\hat{}(m) - f\hat{}(m+n)]\, h\hat{}(m+n)$$

we obtain

$$|f\hat{}(m)|^2\, |h\hat{}(m+n) - h\hat{}(m)|^2 \le 2\,|f\hat{}(m+n)\, h\hat{}(m+n) - f\hat{}(m)\, h\hat{}(m)|^2$$
$$+ 2\,\|H\|_{2,\lambda}^2\, |f\hat{}(m+n) - f\hat{}(m)|^2 .$$

Multipling by $n^{-1-2\lambda}$ and summing over m and n, we get

$$\sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m)|^2 |h^\wedge(m+n) - h^\wedge(m)|^2 \leq$$

$$\leq 2 \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m+n) h^\wedge(m+n) - f^\wedge(m) h^\wedge(m)|^2 +$$

$$+ 2 \|H\|^2_{2,\lambda} \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m+n) - f^\wedge(m)|^2$$

$$\leq 2 A''(\lambda) \int_T |H f(\theta)\, \theta^\lambda|^2 d\theta + 2 A''(\lambda) \|H\|^2_{2,\lambda} \int_T |f(\theta)\, \theta^\lambda|^2 d\theta$$

by virtue of Lemma 7.2. The result is now obvious.

THEOREM 7.4. Let $0 < \lambda < \frac{1}{2}$. There exists a constant $A(\lambda)$ depending only on λ such that if $h^\wedge$ is defined on Z satisfying

$$|h^\wedge(m)| \leq C, \qquad m \in Z$$

and

$$\sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m)|^2 |h^\wedge(m+n) - h^\wedge(m)|^2 \leq C^2 \|f\|^2_{2,\lambda},$$

for every $f \in L^{2,\lambda}(T)$, then $H \in \mathcal{M}_\lambda$ and $\|H\|_{2,\lambda} \leq C.\, A(\lambda)$.

PROOF. We have

$$f^\wedge(m+n) h^\wedge(m+n) - f^\wedge(m) h^\wedge(m) = f^\wedge(m)[h^\wedge(m+n) - h^\wedge(m)] + [f^\wedge(m+n) - f^\wedge(m)] h^\wedge(m+n)$$

so that

$$|f^\wedge(m+n) h^\wedge(m+n) - f^\wedge(m) h^\wedge(m)|^2 \leq 2 |f^\wedge(m)|^2 |h^\wedge(m+n) - h^\wedge(m)|^2 + 2C^2 |f^\wedge(m+n) - f^\wedge(m)|^2.$$

Multiplying by $n^{-1-2\lambda}$ and summing over m and n, we get the desired result using Lemma 7.2, since

$$\sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m+n) h^\wedge(m+n) - f^\wedge(m) h^\wedge(m)|^2 \le$$

$$\le 2 \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m)|^2 |h^\wedge(m+n) - h^\wedge(m)|^2 + 2C^2 \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m+n) - f^\wedge(m)|^2$$

$$\le 2C^2 \|f\|_{2,\lambda}^2 + 2C^2 A''(\lambda) \int_T |f(\theta)\theta^\lambda|^2 d\theta$$

$$\le 2(A''(\lambda)+1) C^2 \|f\|_{2,\lambda}^2$$

and thus

$$\|Hf\|_{2,\lambda}^2 \le 2C^2 (1 + A''(\lambda)) \|f\|_{2,\lambda}^2$$

which yields

$$\|H\|_{2,\lambda} \le C \cdot A(\lambda)$$

where $A(\lambda) = \sqrt{2(1+A''(\lambda))}$.

Notice that Theorems 7.3 and 7.4 correspond to the result of Devinatz and Hirschman [20, Lemmas 3d, 3e] .

Before we come to the main result, we need the following definition.

DEFINITION 7.5. <u>If</u> $g^\wedge$ <u>is a function defined on</u> Z,

$$V_\beta[g^\wedge] = \text{l.u.b.} \left\{ \sum_{k=0}^{N-1} |g^\wedge(n_{k+1}) - g^\wedge(n_k)|^\beta \right\}^{1/\beta},$$

<u>the least upper bound being taken over all sets of integers</u> $n_0 < n_1 < \cdots < n_N$ <u>is called the β-variation of</u> $g^\wedge$.

We shall first obtain a result analogous to the lemma of Hirschman [26] .

THEOREM 7.6. <u>Suppose that</u> $0 < \lambda < \frac{1}{2}$. <u>Let $h^\wedge$ be of bounded 1-variation on</u> Z. <u>Then, if</u> H <u>is the corresponding multiplier transformation, we have</u>

$$\|H\|_{2,\lambda}^2 \leq B(\lambda)\left\{\|h^\wedge\|_\infty^2 + \|h^\wedge\|_\infty V_1[h^\wedge]\right\}$$

<u>where</u> $B(\lambda)$ <u>is a finite constant depending only on</u> λ <u>and</u>

$$\|h^\wedge\|_\infty = \sup_{n\in Z} |h^\wedge(n)|.$$

PROOF. By virtue of Theorem 7.4, we need to estimate only the quantity

$$M = \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m)|^2 |h^\wedge(m+n) - h^\wedge(m)|^2.$$

Now

$$M \leq 2\|h^\wedge\|_\infty \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m)|^2 \sum_{k=1}^{n} |h^\wedge(m+k) - h^\wedge(m+k-1)|$$

$$= 2\|h^\wedge\|_\infty \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{k=1}^{n} \sum_{m=-\infty}^{\infty} |f^\wedge(m-k)|^2 |h^\wedge(m) - h^\wedge(m-1)|$$

$$= 2\|h^\wedge\|_\infty \sum_{m=-\infty}^{\infty} |h^\wedge(m) - h^\wedge(m-1)| \sum_{k=1}^{\infty} |f^\wedge(m-k)|^2 \sum_{n=k}^{\infty} n^{-1-2\lambda}$$

$$\leq \frac{1}{\lambda}\|h^\wedge\|_\infty \sum_{m=-\infty}^{\infty} |h^\wedge(m) - h^\wedge(m-1)| \sum_{k=1}^{\infty} |f^\wedge(m-k)|^2 k^{-2\lambda}$$

$$\leq C(\lambda)\|h^\wedge\|_\infty V_1[h^\wedge]\, \|f\|_{2,\lambda}^2,$$

using Lemma 7.1. As in the proof of Theorem 7.4, we have

$$|f^\wedge(m+n)h^\wedge(m+n) - f^\wedge(m)h^\wedge(m)|^2 \leq 2|f^\wedge(m)|^2 |h^\wedge(m+n) - h^\wedge(m)|^2 + 2|h^\wedge(m+n)|^2 |f^\wedge(m+n) - f^\wedge(m)|^2$$

and therefore

$$\sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m+n)\, h^\wedge(m+n) - f^\wedge(m)\, h^\wedge(m)|^2 \leq$$

$$\leq 2 \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |f^\wedge(m)|^2 |h^\wedge(m+n) - h^\wedge(m)|^2 + 2 \sum_{n=1}^{\infty} n^{-1-2\lambda} \sum_{m=-\infty}^{\infty} |h^\wedge(m+n)|^2 |f^\wedge(m+n) - f^\wedge(m)|^2$$

$$\leq 2C(\lambda) \|h^\wedge\|_\infty V_1[h^\wedge] \|f\|^2_{2,\lambda} + 2\|h^\wedge\|^2_\infty A''(\lambda) \|f\|^2_{2,\lambda}$$

by virtue of Lemma 7.2 .

Thus

$$\|Hf\|^2_{2,\lambda} \leq B(\lambda) \left\{ \|h^\wedge\|^2_\infty + V_1[h^\wedge] \|h^\wedge\|_\infty \right\} \|f\|^2_{2,\lambda}$$

and this gives

$$\|H\|^2_{2,\lambda} \leq B(\lambda) \left\{ \|h^\wedge\|^2_\infty + V_1[h^\wedge] \|h^\wedge\|_\infty \right\},$$

where $B(\lambda) = 2 \max \{C(\lambda), A(\lambda)\}$.

LEMMA 7.7. Let $h^\wedge$ be a real valued function defined on Z. For each $\beta > 1$, there exists a constant $C(\beta)$ depending only on β such that for each $h^\wedge$, for which $V_\beta[h^\wedge]$ is finite and for $\varepsilon > 0$ there exists $h^\wedge_\varepsilon$ with the properties:

(a) $\|h^\wedge - h^\wedge_\varepsilon\| < \varepsilon$

(b) $V_1[h^\wedge_\varepsilon] \leq C(\beta)\, V_\beta[h^\wedge]\, \varepsilon^{1-\beta}$

where $\|\cdot\|_\infty$ is defined as in Theorem 7.6.

PROOF. This lemma corresponds to Lemma 3 of Hirschman [26] and is proved by the arguments used in [24] .

We now come to the main result in this section and it is the analogue of Theorem A stated in chapter 6.

THEOREM 7.8. <u>Let</u> $h^\wedge$ <u>be defined on</u> Z <u>and let</u> H <u>be the corresponding multiplier transformation on</u> $L^{2,\lambda}(T)$. <u>If</u> $V_\beta[h^\wedge]$ <u>is finite, where</u> $\beta > 2$, <u>then</u>

$$\|H\|_{2,\lambda} < \infty \qquad \text{if} \quad |\lambda| < \frac{1}{\beta}.$$

PROOF. First we obtain a sequence of functions $\{g_m^\wedge\}$ such that

$$h^\wedge = \lim_{m \to \infty} g^\wedge_m$$

pointwise on Z. This construction is given by Hirschman [24] and by Edwards [21, Volume 2, p.270]. We shall not give the details here. Assuming without loss of generality that $h^\wedge(0) = 0$, a real-valued function h^* defined on the entire real line is obtained by interpolating linearly between successive values of $h^\wedge(n)$ so that $h^*(x)/_{x=n} = h^\wedge(n)$. Then, for each non-negative integer m, a function $g^\wedge_m$ is constructed satisfying

$$V_1[g_m^\wedge] \leq 2^{(\beta-1)m} V_\beta[h^\wedge]^\beta \tag{7.1}$$

and

$$\|h^\wedge - g^\wedge_m\|_\infty \leq 2^{-m}. \tag{7.2}$$

Moreover

$$V_\beta[g_m^\wedge] \leq V_\beta[h^\wedge].$$

The proof of our theorem is completed following the arguments of Hirschman [26]. Define a sequence of functions $\{h_m^\wedge\}_{m=1}^\infty$ on Z as follows:

$$h_1^\wedge(n) = g_1^\wedge(n)$$

$$h_m^\wedge(n) = g_m^\wedge(n) - g_{m-1}^\wedge(n).$$

Then

$$\hat{h}(n) = \sum_{m=1}^{\infty} h^{\wedge}_m(n)$$

and

$$V_1[h^{\wedge}_m] \leq C \cdot 2^{(\beta-1)m} V_\beta [h^{\wedge}]^\beta ,$$

$$\|h^{\wedge}_m\|_\infty \leq C \cdot 2^{-m} .$$

If H_m is the multiplier transformation associated with $h^{\wedge}_m$, then

$$\|H\|_{2,\lambda} \leq \sum_{m=1}^{\infty} \|H_m\|_{2,\lambda} .$$

Choose α, $\lambda < \alpha < \frac{1}{2}$. By Theorem 7.6,

$$\|H_m\|_{2,\alpha} = O\left[(2^{-m})^2 + 2^{-m} \cdot 2^{m(\beta-1)}\right]^{1/2} = O\left(2^{m(\frac{\beta}{2}-1)}\right) .$$

On the other hand, by Parseval's equality

$$\|H_m\|_{2,0} = \|h^{\wedge}_m\|_\infty = O(2^{-m}) .$$

Putting $\lambda = (1-\theta)\, 0 + \theta\alpha$, $0 < \theta < 1$, we obtain by virtue of the Riesz-Thorin convexity theorem the relation

$$\|H_m\|_{2,\lambda} = O\left(2^{m(-1+(\beta\lambda/2\alpha))}\right) .$$

The series $\sum_{m=1}^{\infty} \|H_m\|_{2,\lambda}$ is conveggent if $\beta\lambda < 2\alpha$ or $\lambda < \frac{2\alpha}{\beta}$.

Since α is arbitrary subject to the condition $\lambda < \alpha < \frac{1}{2}$, it is always possible to choose α so that $\lambda < \frac{2\alpha}{\beta}$, if $0 < \lambda < \frac{1}{\beta}$. Thus we have proved the theorem for $0 < \lambda < \frac{1}{\beta}$. The case when $\lambda = 0$ being trivial, the theorem follows by the duality argument given at the beginning of this chapter.

CHAPTER 8

MULTIPLIERS ON $\ell^{p,\lambda}(Z)$

We shall consider the problem for $\ell^{p,\lambda}(Z)$ and obtain some results analogous to those obtained by Hirschman [24], for the case $\lambda = 0$.

Let $f \in \ell^{2,0}(Z)$. If

$$h(k) = \int_T h^\wedge(\theta)\, e^{-2\pi i k\theta}\, d\theta, \qquad k \in Z,$$

Then

$$Hf(n) = \int_T f^\wedge(\theta)\, h^\wedge(\theta)\, e^{-2\pi i n\theta}\, d\theta$$

$$= \sum_{k\in Z} f(n-k)\, h(k).$$

The series on the right converges absolutely for each n, by Parseval's relation, since $k \in \ell^{2,0}(Z)$ and $f \in \ell^{2,0}(Z)$ by our assumption.

Let $f \in \ell^{p,\lambda}(Z)$ and $g \in \ell^{q,-\lambda}(Z)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Define

$$\{f,g\} = \sum_{n\in Z} f(n)\, g(-n).$$

Then

$$|\{f,g\}| \leq \Big|\sum_{n\in Z} f(n)\, g(-n)\Big| \leq N_{p,\lambda}[f]\, N_{q,-\lambda}[g]$$

and every bounded linear functional L on $\ell^{p,\lambda}(Z)$ is of the form $\{f,g\}$ with $\|L\| = N_{q,-\lambda}[g]$.

Let H be a multiplier transformation defined on $\ell^{p,\lambda}(Z)$. We claim that H is also a multiplier on $\ell^{q,-\lambda}(Z)$, associated with the same function $h^\wedge$ and $N_{p,\lambda}[H] = N_{q,-\lambda}[H]$.

Suppose $g \in \ell^{2,0}(Z) \cap \ell^{p,\lambda}(Z)$. Then

$$\{Hf, g\} = \sum_{n\in Z} Hf(n)\, g(-n) = \sum_{n\in Z} \Big\{ \sum_{j\in Z} f(n-j)h(j) \Big\} g(-n)$$

$$= \sum_{n\in Z} g(-n) \sum_{j\in Z} f(n-j)h(j) = \sum_{n\in Z} g(-n) \sum_{m\in Z} f(m)h(n-m)$$

$$= \sum_{m\in Z} f(m) \sum_{n\in Z} g(-n)h(n-m) = \sum_{m\in Z} f(m) \Big\{ \sum_{j\in Z} g(-m-j)h(j) \Big\}$$

$$= \sum_{m\in Z} f(m)\, Hg(-m) = \{f, Hg\}.$$

If $N_{p,\lambda}[g] \le 1$, we have

$$N_{q,-\lambda}[H] = \underset{N_{p,\lambda}[g]\le 1}{\mathrm{l.u.b.}} \; |\{f, Hg\}|$$

$$\le \underset{N_{p,\lambda}[g]\le 1}{\mathrm{l.u.b.}} \; N_{q,-\lambda}[f]\, N_{p,\lambda}[Hg]$$

$$\le N_{q,-\lambda}[f]\, N_{p,\lambda}[H]$$

This implies that

$$N_{q,-\lambda}[H] \le N_{p,\lambda}[H].$$

Similarly interchanging the roles of p and q, the reverse inequality can be established. Thus we have

$$N_{p,\lambda}[H] = N_{q,-\lambda}[H].$$

THEOREM 8.1. <u>If</u>

(a) $|h^\wedge(\theta)| \le A$, $\qquad \theta \in T$,

(b) $|h^\wedge(\theta) - h^\wedge(\theta+t)| \le A|t|^\alpha$, $\quad \frac{1}{2} < \alpha \le 1$,

<u>then</u> H <u>is a bounded linear transformation of</u> $\ell^{p,\lambda}(Z)$ <u>into itself,</u> <u>where</u> $1 < p < \infty$, $|\lambda| < \alpha - \frac{1}{2}$.

PROOF. Let

$$\hat{s}_k(\theta) = \sum_{|n| \le 2^k} h(n) e^{2\pi i n\theta}$$

be the partial sum of order 2^k, of the Fourier series for $h^\wedge$. Given $\varepsilon > 0$, it is easily seen that

$$\| \hat{s}_k - h^\wedge \|_\infty \le A\,C(\alpha,\varepsilon)\, 2^{-k(\alpha-\varepsilon)}$$

(see Hirschman [24, p.223]) so that if

$$\hat{h}_k = \hat{s}_k - \hat{s}_{k-1}$$

then

$$\| \hat{h}_k \|_\infty \le A\,C(\alpha,\varepsilon)\, 2^{-k(\alpha-\varepsilon)} \tag{8.1}$$

where $\|\cdot\|_\infty$ is defined on T. Let H_k be the multiplier transformation associated with $\hat{h}_k$. Then

$$H_k f(n) = \int_T f^\wedge(\theta)\, \hat{h}_k(\theta)\, e^{-2\pi i n\theta}\, d\theta = \sum_{j \in Z_k} f(n-j) h(j)$$

where $Z_k = \{ n \in Z,\ 2^{k-1} < |n| \le 2^k \}$. It is also easy to verify that

$$N_{r,\lambda}[H_k] \le \Big(\sum_{j \in Z_k} |h(j)|^r (1+|j|)^{r\lambda} \Big)^{\frac{1}{r}}, \quad r = 1,2. \tag{8.2}$$

Using the relation

$$\sum_{j \in Z_k} |h(j)| \le A\,C(\alpha,\varepsilon)\, 2^{k(\frac{1}{2}-\alpha+\varepsilon)},$$

(Zygmund [29]) and by virtue of (8.2) it follows that

$$N_{1,\lambda}[H_k] \le A\,C(\alpha,\varepsilon)\, 2^{k(\frac{1}{2}-\alpha+|\lambda|+\varepsilon)} \tag{8.3}$$

From (8·2) we have

$$N_{2,\lambda}[H_k] \le \Big\{ \sum_{j \in Z_k} |h(j)|^2 (1+|j|)^{2\lambda} \Big\}^{1/2}$$

$$\le \sum_{j \in Z_k} |h(j)| (1+|j|)^\lambda .$$

Thus

$$N_{2,\lambda}[H_k] \leq A\,C(\alpha,\varepsilon)\,2^{k(\frac{1}{2}-\alpha+|\lambda|+\varepsilon)} . \tag{8.4}$$

Suppose $1 < p \leq 2$. Putting $\frac{1}{p} = \frac{1-\omega}{1} + \frac{\omega}{2}$, $0 < \omega < 1$, we obtain from (8.3) and (8.4), by virtue of the Riesz-Thorin convexity theorem

$$N_{p,\lambda}[H_k] \leq A\,C(\alpha,\varepsilon)\,2^{k(\frac{1}{2}-\alpha+|\lambda|+\varepsilon)} .$$

If $|\lambda| < \alpha - \frac{1}{2}$, we can choose ε so small that

$$\sum_{k=0}^{\infty} N_{p,\lambda}[H_k] < \infty .$$

Further, since

$$h^{\wedge}(\theta) = \sum_{k=0}^{\infty} h^{\wedge}_k(\theta),$$

the convergence being uniform in θ, it is easy to show that

$$Hf(n) = \sum_{k=0}^{\infty} H_k\, f(n)$$

and

$$N_{p,\lambda}[H] \leq \sum_{k=0}^{\infty} N_{p,\lambda}[H_k] < \infty.$$

The theorem is therefore true for $1 < p \leq 2$. The regular conjugacy argument gives the result for $2 \leq p < \infty$.

Now we state two results of Devinatz and Hirschman [20] as lemmas.

LEMMA 8.2. *If* $0 < \lambda < \frac{1}{2}$, *then there exist positive constants* $A_1(\lambda)$ *and* $A_2(\lambda)$ *depending only on* λ *such that*

$$\left(N_{2,\lambda}[f]\right)^2 - |f(0)|^2 \le A_1(\lambda) \int_0^1\int_0^1 \left\{ |f^\wedge(\theta) - f^\wedge(\phi)|^2 (\sin\pi|\theta-\phi|)^{-1-2\lambda} \right\} d\theta\, d\phi$$

and

$$\left(N_{2,\lambda}[f]\right)^2 - |f(0)|^2 \ge A_2(\lambda) \int_0^1\int_0^1 \left\{ |f^\wedge(\theta) - f^\wedge(\phi)|^2 (\sin\pi|\theta-\phi|)^{-1-2\lambda} \right\} d\theta\, d\phi .$$

LEMMA 8.3. *Let* $0 < \lambda < \frac{1}{2}$. *There exists a constant* $A''(\lambda)$ *such that if* $h^\wedge$ *is a measurable function defined on* T *satisfying*

$$\|h^\wedge\|_\infty \le C$$

with $h(0) = 0$, *and if*

$$\int_T |f^\wedge(\theta)|^2 d\theta \int_T |h^\wedge(\theta) - h^\wedge(\phi)|^2 (\sin\pi|\theta-\phi|)^{-1-2\lambda} d\phi \le C^2 \left(N_{2,\lambda}[f]\right)^2$$

for every $f \in \ell^{2,\lambda}(Z)$, *then*

$$N_{2,\lambda}[H] \le C \cdot A''(\lambda).$$

We now prove

THEOREM 8.4. *Suppose* $h^\wedge$ *satisfies the conditions*

(a) $|h^\wedge(\theta)| \le A$, $\theta \in T$,

(b) $|h^\wedge(\theta) - h^\wedge(\theta+t)| \le B|t|^\alpha$, $0 < \alpha \le 1$.

Then there exists a constant C *such that*

$$\int_T |f^\wedge(\theta)|^2 \int_T |h^\wedge(\theta) - h^\wedge(\phi)|^2 (\sin\pi|\theta-\phi|)^{-1-2\lambda} d\theta\, d\phi \le CAB\left(N_{2,\lambda}[f]\right)^2,$$

where $0 < \lambda < \frac{\alpha}{2}$.

PROOF. We consider the quantity

$$M = \int_T |f^\wedge(\theta)|^2 \int_T |h^\wedge(\theta) - h^\wedge(\phi)|^2 (\sin \pi|\theta-\phi|)^{-1-2\lambda} \, d\theta \, d\phi$$

$$\leq 2 \|h^\wedge\|_\infty \int_T |f^\wedge(\theta)|^2 \int_T |h^\wedge(\theta) - h^\wedge(\phi)| (\sin \pi|\theta-\phi|)^{-1-2\lambda} \, d\theta \, d\phi \, .$$

Since $h^\wedge$ satisfies (b), we have

$$\int_T |h^\wedge(\theta) - h^\wedge(\phi)| (\sin \pi|\theta-\phi|)^{-1-2\lambda} \, d\phi \leq I_1 + I_2 \, ,$$

where

$$I_1 \leq B \int_\theta^{\theta+\frac{1}{2}} |\theta-\phi|^\alpha (\sin \pi|\theta-\phi|)^{-1-2\lambda} \, d\phi$$

and

$$I_2 \leq B \int_{\theta-\frac{1}{2}}^{\theta} |\theta-\phi|^\alpha (\sin \pi|\theta-\phi|)^{-1-2\lambda} \, d\phi \, .$$

Considering I_1 and making the substitution $u = \phi - \theta$,

$$I_1 \leq B \int_0^{1/2} u^\alpha (\sin \pi u)^{-1-2\lambda} \, du \, ,$$

which, by virtue of the inequality

$$\sin \pi u \geq cu, \qquad 0 \leq u \leq \tfrac{1}{2}$$

gives

$$I_1 \leq B \int_0^{1/2} u^\alpha \cdot u^{-1-2\lambda} \, du = B \int_0^{1/2} u^{\alpha-1-2\lambda} \, du < \infty \, ,$$

if $\alpha - 2\lambda > 0$. Similarly, we find that if $\alpha - 2\lambda > 0$,

$$I_2 < \infty .$$

Hence, there exists a constant C which depends on λ and α such that

$$\int_T |h^\wedge(\theta) - h^\wedge(\phi)| (\sin \pi|\theta-\phi|)^{-1-2\lambda} \, d\phi \leq CB \, .$$

Thus,

$$M \le CB \|h^\wedge\|_\infty \int_T |f^\wedge(\theta)|^2 d\theta$$

$$\le CBA \int_0^1 |f^\wedge(\theta)|^2 \theta^{-2\lambda} d\theta \quad , \qquad \text{by (a)},$$

when $\lambda > 0$. Now applying Lemma 7.1, we obtain

$$M \le C\,A\,B \cdot A'(\lambda)\,(N_{2,\lambda}[f])^2 < C\,A\,B\,(N_{2,\lambda}[f])^2,$$

where $A'(\lambda)$ can be included in C.

THEOREM 8.5. *Suppose* $h^\wedge$ *satisfies the conditions* (a) *and* (b) *of Theorem 8.4. Then if* $0 < \lambda < \frac{\alpha}{2}$, *there exists a constant* C *which depends on* α *and* λ, *such that if* H *is the associated multiplier transformation with* $h(0) = 0$, *then*

$$\left(N_{2,\lambda}[H]\right)^2 \le CAB.$$

PROOF. As the conditions (a) and (b) of Theorem 8.4 are satisfied by the function $h^\wedge$, we have

$$\int_T |f^\wedge(\theta)|^2 \int_T |h^\wedge(\theta) - h^\wedge(\phi)|^2 \left(\sin\pi|\theta-\phi|\right)^{-1-2\lambda} d\theta\, d\phi \le CAB\left(N_{2,\lambda}[f]\right)^2$$

where $0 < \lambda < \frac{\alpha}{2}$. Applying Lemmas 8.2 and 8.3 and making use of the fact that $h(0) = 0$, we have the desired result.

THEOREM 8.6. *Suppose* $h^\wedge$ *satisfies the conditions*

(a) $|h^\wedge(\theta)| \le A$, $\qquad \theta \in T$,

(b) $|h^\wedge(\theta) - h^\wedge(\theta+t)| \le A.|t|^\alpha$, $\quad \frac{1}{2} \le \alpha < 1$.

Then H *is a bounded linear transformation of* $\ell^{p,\lambda}(Z)$ *into itself, where* $\frac{\alpha}{2} > |\lambda| > \alpha - \frac{1}{2}$ *and*

$$\frac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|} < p < \frac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|-2\alpha}.$$

PROOF. Suppose $s^{\wedge}_k$ is defined as in the proof of Theorem 8.1 and let H_k be the associated multiplier transformation given there. Then, since

$$\|s^{\wedge}_k\|_\infty \le A\,C(\alpha,\varepsilon)\,2^{-k(\alpha-\varepsilon)}$$

and, as can be easily verified,

$$|h^{\wedge}_k(\theta) - h^{\wedge}_k(\theta+t)| \le A\,C(\alpha,\varepsilon)\,2^{\varepsilon k}\,|t|^{\alpha},$$

we have, by virtue of Theorem 8.5,

$$(N_{2,\lambda}[H_k])^2 \le A\cdot 2^{-k(\alpha-\varepsilon)}$$

which implies that

$$N_{2,\lambda}[H_k] \le A\,2^{-\frac{k}{2}(\alpha-\varepsilon)} \tag{8.5}$$

Now suppose that $\dfrac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|} < p \le 2$. Then, if $\dfrac{1}{p} = \dfrac{1-\omega}{1} + \dfrac{\omega}{2}$, we have $\omega > \dfrac{1-2\alpha+2|\lambda|}{1-\alpha+2|\lambda|}$. By the Riesz-Thorin convexity theorem (this is possible since $0<\omega<1$ under the condition that $|\lambda| > \alpha - \frac{1}{2}$) we obtain from (8.3) and (8.5)

$$N_{p,\lambda}[H_k] \le A\,2^{k[(\frac{1}{2}-\alpha+|\lambda|+\varepsilon)(1-\omega) - \omega(\alpha-\varepsilon)/2]} \tag{8.6}$$

Under the above condition on ω, it is possible to choose ε small enough such that the quantity in the exponent of (8.6) is negative. With such a choice of $\varepsilon > 0$, it follows that

$$N_{p,\lambda}[H] \le \sum_{k=0}^{\infty} N_{p,\lambda}[H_k] < \infty$$

if $\dfrac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|} < p \le 2$. The result for $2 \le p < \dfrac{2(1-\alpha+2|\lambda|)}{1+2|\lambda|-2\alpha}$ follows by the conjugacy argument. This comples the proof.

In Theorems 8.1 and 8.6 we have assumed that $\alpha > \frac{1}{2}$. We have not asserted that these theorems are the best possible. There are multiplier transformations for some p and λ even if $\alpha < \frac{1}{2}$ as can be seen from the following result.

THEOREM 8.7. <u>If</u> $h^{\wedge}$ <u>satisfies the conditions</u> (a) <u>and</u> (b) <u>of Theorem 8.4, then</u> H <u>is a bounded linear transformation of</u> $\ell^{p,\lambda}(Z)$ <u>into itself if</u> $\frac{2}{1+2(\alpha-\lambda)} < p < 2$ <u>and</u> λ <u>is a non-negative number such that</u> $\alpha > \lambda > \alpha - \frac{1}{2}$.

PROOF. With the same notation as in Theorem 8.1, we have

$$N_{2,0}[H_k] \le AC(\alpha,\varepsilon)2^{-k(\alpha-\varepsilon)}. \tag{8.7}$$

Let $\gamma = \frac{2-p}{p}$. Then $\frac{1}{p} = \frac{1-\gamma}{2} + \frac{\gamma}{1}$. Assume that $\lambda = (1-\gamma)0 + \gamma\eta$.

Applying the Riesz-Thorin convexity theorem to (8.7) and to

$$N_{1,\eta}[H_k] \le AC(\alpha,\varepsilon)2^{k(\frac{1}{2}-\alpha+\eta+\varepsilon)} \tag{8.8}$$

we have

$$N_{p,\lambda}[H_k] \le AC(\alpha,\varepsilon)2^{k[(\frac{1}{2}-\alpha+\eta+\varepsilon)\gamma + (\alpha-\varepsilon)(-1)(1-\gamma)]}. \tag{8.9}$$

The exponent on the right hand side of (8.9) is negative if $\gamma < \frac{2\alpha}{1+\eta}$, or equivalently, $1 < \frac{2\alpha}{\gamma+2\lambda}$, since γ is positive. This in turn gives $\gamma < 2(\alpha-\lambda)$ which implies $\frac{2}{1+2(\alpha-\lambda)} < p$. Thus, for non-negative γ satisfying the above condition, the theorem is true. Hence H is a bounded linear transformation of $\ell^{p,\lambda}(Z)$

into itself $\frac{2}{1+2(\alpha-\lambda)} < p < 2$, where λ is non-negative and satisfies the condition $\alpha > \lambda > \alpha - \frac{1}{2}$.

It is observed that if $\alpha < \frac{1}{2}$, then $\lambda > \alpha - \frac{1}{2}$ is satisfied by any non-negative λ. In particular, when $\lambda = 0$, the range for p reduces to $\frac{2}{1+2\alpha} < p < 2$ and this is exactly the result given by Hirschman [24, Theorem 2a].

DEFINITION 8.8. <u>A function</u> g <u>defined on an interval</u> $I = a \leq x \leq b$ <u>is said to be of bounded</u> β<u>-variation</u> $(1 \leq \beta < \infty)$ <u>if</u>

$$V_\beta[g] = \text{l.u.b.}\left(\sum_{k=0}^{n} |g(x_{k+1}) - g(x_k)|^\beta\right)^{1/\beta}$$

<u>is finite, where the least upper bound is taken over all finite sets</u> $a \leq x_0 < x_1 \cdots < x_n \leq b$.

THEOREM 8.9. <u>Suppose</u> $h^\wedge$ <u>satisfies the conditions</u>

(a) $|h^\wedge(\theta)| \leq A$, $\qquad \theta \in T$,

(b) $V_\beta[h^\wedge] = V_\beta < \infty$, $\qquad \beta > 2$,

(c) $|h^\wedge(\theta+t) - h^\wedge(\theta)| \leq L|t|^\delta$, $\qquad \frac{1}{2} < \delta \leq 1$.

<u>Then</u> H <u>is a bounded linear transformation of</u> $\ell^{p,\lambda}(z)$ <u>into itself, where</u> $\frac{2\delta-1}{\beta} < |\lambda| < \delta - \frac{1}{2}$ <u>and</u> $1 < p < \infty$.

PROOF. It is possible to construct a sequence of functions $\{g^\wedge_k(\theta)\}$ satisfying the following conditions:

(A) $V_1[g^\wedge_k] \leq 2^{(\beta-1)k} V_\beta[h^\wedge]^\beta$,

(B) $|g^\wedge_k(\theta) - g^\wedge_k(\phi)| \leq L|\theta-\phi|^\delta$,

(C) $\|g^\wedge_k\|_\infty \leq A$,

(D) $\|g^\wedge_{k+1} - g^\wedge_k\|_\infty \leq C.\, 2^{-k}$.

See Hirschman [24, Theorems 2e, 2f].

Setting

$$h^\wedge_0(\theta) = g^\wedge_0(\theta),$$

$$h^\wedge_k(\theta) = g^\wedge_k(\theta) - g^\wedge_{k-1}(\theta), \quad k = 1,2,\ldots,$$

we obtain

$$h^\wedge(\theta) = \sum_{k=0}^{\infty} h^\wedge_k(\theta)$$

pointwise on T. Then

$$N_{p,\lambda}[H] \leq \sum_{k=0}^{\infty} N_{p,\lambda}[H_k].$$

Further, it follows that $h^\wedge_k$ satisfies the conditions

$$|h^\wedge_k(\theta)| < C.\, 2^{-k}$$

$$V_1[h^\wedge_k] \leq C.2^{(\beta-1)k}$$

and

$$|h^\wedge_k(\theta) - h^\wedge_k(\phi)| \leq C.2^{-k}|\theta-\phi|^\delta.$$

Now, $h^\wedge_k$ satisfies the hypotheses of Theorem 8.1 with $A = C.2^{-k}$, and therefore

$$N_{1,\lambda}[H_k] \leq C.2^{-k} \sum_{m=0}^{\infty} 2^m \left(\tfrac{1}{2} - \delta + |\lambda| + \varepsilon\right)$$

If $|\lambda| < \delta - \frac{1}{2}$, we can choose ε so small that

$$N_{1,\lambda}[H_k] \leq C.2^{-k}. \tag{8.10}$$

Now suppose that $\frac{2\delta-1}{\beta} < \lambda < \delta - \frac{1}{2}$. Choose γ such that $\lambda < \gamma < \delta - \frac{1}{2}$. Then by what is proved in Hirschman [6, p. 855], we obtain

$$N_{2,\lambda}[H_k] \leq C.2^{k(-1 + \frac{\beta\lambda}{2\gamma})} \tag{8.11}$$

Setting $\frac{1}{p} = \frac{1-\omega}{1} + \frac{\omega}{2}$, $0<\omega<1$, we obtain by Riesz-Thorin convexity theorem that

$$N_{p,\lambda}[H_k] \leq C.\,2^{k(-1 + \frac{\beta\lambda}{2\gamma})\omega - k(1-\omega)} \tag{8.12}$$

The exponent on the right hand side is negative if $\frac{\beta\lambda}{2\gamma}\omega < 1$ or $\omega < \frac{2\gamma}{\beta\lambda} < \frac{2(\delta - \frac{1}{2})}{2\delta-1} = 1$. Thus $N_{p,\lambda}[H] < \infty$ if $\frac{2\delta-1}{\beta} < \lambda < \delta - \frac{1}{2}$.

The rest of the arguments can be easily completed.

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