MATSCIENCE REPORT—104

PROCEEDINGS OF THE SECOND CONFERENCE ON NUMBER THEORY

OOTACAMUND

3-7, August 1980

MATSCIENCE
THE INSTITUTE OF MATHEMATICAL SCIENCES
MADRAS-600 020, INDIA
June, 1981

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CONTENTS

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Participants of the Second Matscience Conference on Number Theory held at Ootacamund (August, 1980)

FOREWORD

This MATSCIENCE REPORT presents the Proceedings of the 'Second Conference on Number Theory' conducted by Matscience at the Hindustan Photo Films Club House, Ootacamund, during August 3-7, 1980. This conference is the 38th in the series of conferences conducted by Matscience on a wide range of topics in mathematical sciences.

The conference was inaugurated by Mr.P.R.S.Rao, the Managing Director, Hindustan Photo Films Manafacturing Company Limited, and Professor Alladi Ramakrishnan, Director, Matscience, released the Matscience Report on the Proceedings of the First Conference on Number Theory held at Mysore in August 1979.

Professor K.Ramachandra of the Tata Institute of Fundamental Research, Bombay and Dr.Krishnaswami Alladi of the Department of Mathematics, University of Michigan, Ann Arbor, U.S.A. were the principal speakers at the conference. Besides the active research workers in the field of number theory from various universities and research institutions, post-graduate teachers from colleges also participated. The organisers wish to thank all the participants for their enthusiastic cooperation in making the conference a success. The present report contains the papers in Number Theory contributed by the participants arranged according to the alphabetical order of the names of the authors except for the problems and some papers at the end which are of an interdisciplinary nature.

We are grateful to Mr.P.R.S.Rao, the Managing Director, H.P.F. Ootacamund and his colleagues for providing the conference hall, accommodation for the participants and all other necessary facilities and the kind hospitality shown to us. We wish to thank Mr.N.S.Sampath, Mr.R.Jayaraman and other supporting staff of our Institute for help rendered in organising the conference and bringing out this report.

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IRRATIONALITY ESTIMATES USING LEGENDRE POLYNOMIALS

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1. INTRODUCTION

It is generally very difficult to establish the irrationality of a given number. Therefore considerable interest remains in the subject even though its origins go back to Greek antiquity. Over the years several ingeneious methods have been developed, yet, one almost always ends up using the following irrationality criterion due to Dirichlet (see [14], p. 44):

A necessary and sufficient condition for a real number θ to be irrational is that there exist integer sequences $\{p_n\}$ and $\{q_n\}$ such that

$$0 \neq |q_n\theta - p_n| \to 0 \text{ as } n \to \infty. \tag{1.1}$$

To establish this criterion Dirichlet used his famous pigeon-hole principle. Thus the criterion is of an existential nature, and in general there is no hint to construct the sequences p_n and q_n.

For certain special numbers such as $\sqrt{2}$ and e the construction of these sequences is simple. For instance, by observing that $0 < \sqrt{2} - 1 < 1$ we see that the binomial theorem yields

$$0 < (\sqrt{2} - 1)^n = q_n\sqrt{2} - p_n \to 0, \text{ as } n \to \infty.$$

In the case of

$$e = \sum_{m=0}^{\infty} \frac{1}{m!} .$$

we set

$$q_n = n! \quad \text{and} \quad p_n = n! \sum_{m=0}^{n} \frac{1}{m!}, \tag{1.2}$$

and observe that

$$0 < q_n e - p_n = n! \sum_{m=n+1}^{\infty} \frac{1}{m!} < \frac{2}{n+1} \to 0, \text{ as } n \to \infty \tag{1.3}$$

The proof of the irrationality of e given in (1.3) makes use of the special divisibility properties of the factorial sequence (see 1.2)), and the fact that the series for e converges very rapidly. If either one of these features fail to hold the proof could become much more complicated or even break down. For instance, if we modify the series for e very slightly and consider

$$\sum_{m=0}^{\infty} \frac{1}{m!+1} ,$$

we get a number whose irrationality is yet to be confirmed - and perhaps will not be for some time ! Even for a simple series like

$$\sum_{m=1}^{\infty} \frac{1}{m^2}$$

the construction of the p_n and q_n is much more complicated, and does not even faintly resemble (1.2), because this series converges much more slowly.

For $s > 1$ let $\zeta(s)$ (Riemann zeta function) be defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} . \tag{1.4}$$

By a very special method involving the transcendence of π the numbers $\zeta(2n)$, $n = 1,2,3,\ldots$ were shown to be irrational (see [1]), but no one knows whether the numbers $\zeta(2n+1)$, $n = 1,2,\ldots$ are all irrational. In 1978, R. Apéry, a French mathematician surprised everyone by producing a truly remarkable proof of the irrationality of $\zeta(3)$ (see [16]), and further light on his result will be shed in Section 9.

The proof due to Apéry created renewed interest in irrationality and over the past three years gave rise to detailed investigation. Some of these investigations have revealed the dominant role played by 'Legendre polynomials'

$$P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} \left\{ x^n (1-x)^n \right\} \tag{1.5}$$

in such proofs, and also led to improvements of several earlier results. This article will review some of these results due to the author [1], the author in collaboration with Robinson [2], Beukers [6], [7], Bombieri [8], and Choodnovsky [9], [10], [11] and [12] in the context of several classical results.

Besides the proof the irrationality of a number θ, it is of considerable interest, for its own sake or for the purpose of applications, to derive an irrationality estimate for θ. More precisely, one would like to obtain lower bounds of the type

$$\left|\theta - \frac{p}{q}\right| \geq \phi(p,q)$$

for all rationals p/q, where ϕ depends on θ. It is in this sense that the Legendre polynomials $P_n(x)$ recently played an important role and improved substantially on irrationality estimates for certain numbers, provided by various classical approaches.

2. Irrationality type and measure.

We will be considering two kinds of irrationality estimates.

An irrational θ is of type at most τ, (or $\leq \tau$), if given $\varepsilon > 0$, there exists $q_0(\varepsilon)$, such that for all rationals p/q with $|q| \geq q_p(\varepsilon)$, we have

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{q^{\tau+\varepsilon}} \tag{2.1}$$

If no such τ exists, we say θ is of infinite type. We say that θ is of type τ, if it is of type $\leq \tau$, but is not of type $\leq \tau^*$ for any $\tau^* < \tau$. It is well known (see [14], p.42) that all irrationals are type ≥ 2. Therefore, if an irrational is of type ≤ 2, it is of type 2.

Sometimes we require an effective version of (2.1), for the purpose of applications. More precisely we want an inequality

$$\left|\theta - \frac{p}{q}\right| > \frac{c'}{q^{\mu}} \qquad (2.2)$$

valid for <u>all</u> rationals p/q, where μ in (2.2) is either constant (if θ is of finite type) or μ increasing finite, of q, tending to infinity with q, if θ is of infinite type. If μ is finite in (2.2) we say it is an irrationality measure for θ.

How does one arrive at inequalities like (2.1) and (2.2)? We state two lemmas below, the first as in [2], and second as in [1].

<u>Lemma 1:</u> Let K be either the rationals or an imaginary quadratic field. Let R be the ring of integers in K. Let θ be a complex number.

(a) Suppose there exists $Q > 1$ and $E > 1$, and $p_n, q_n \in R$ satisfying

$$|q_n| \leq Q^{n(1+o(1))}, \quad |q_n\theta - p_n| \leq E^{-n(1+o(1))} \qquad (2.3)$$

and

$$p_n q_{n+1} \neq q_n p_{n+1} \qquad (2.4)$$

Then $\theta \notin K$ and given $\varepsilon > 0$ there exist $b_0(\varepsilon)$ such that if $a, b \in R$ and $|b| > b_0(\varepsilon)$ then

$$\left|\theta - \frac{a}{b}\right| > \frac{1}{|b|^{\tau+\varepsilon}}, \quad \tau = \log(QE)/\log E. \qquad (2.5)$$

In particular θ is of type at most τ.

(b) Suppose there exist $Q > 1$, $E > 1$, $k_0 > 0$, $\ell_0 \geq 1/2$, and $p_n, q_n \in R$ satisfying (2.4) and

$$|q_n| < k_0 Q^n, \quad |q_n\theta - p_n| \leq \ell_0 E^{-n}. \tag{2.6}$$

Then for any $a, b \in R$ we have

$$\left|\theta - \frac{a}{b}\right| > \frac{c}{b^{\mu}},$$

where

$$\mu = (\log QE)/\log E \text{ and } c = \frac{1}{2K_0} Q^{\left(-2+\frac{\log 2b}{\log E}\right)}. \tag{2.7}$$

In particular, θ has irrationality measure μ.

We only prove part (b) of Lemma 1.

Let m be the least positive integer such that $|q_{m'}\theta - p_{m'}|^{-1} \geq 2b$ for all $m' > m$. From (2.6) we deduce that

$$m \leq \log(2|b|\ell_0) + 1. \tag{2.8}$$

Choose $n = m$ or $m+1$ so that $aq_n \neq bp_n$. Then

$$|q_n| \left|\theta - \frac{a}{b}\right| \geq |q_n| \left\{\left|\frac{p_n}{q_n} - \frac{a}{b}\right| - \left|\theta - \frac{p_n}{q_n}\right|\right\} \geq \frac{1}{|b|} - \frac{1}{2|b|} = \frac{1}{2|b|} \tag{2.9}$$

Thus (2.8) and (2.9) imply that

$$\left|\theta - \frac{a}{b}\right| \geq \frac{1}{2|b|q_n} > \frac{1}{2k_0 Q^{m+1}} = \frac{c}{b^{\mu}}$$

as claimed.

Unless the p_n, q_n are given in a certain convenient form it is usually cumbersome to check (2.4). Thus we state another lemma which does not require the verification of (2.4), provided the logarithms of the quantities $|q_n|, |p_n|$ and $|q_n\theta - p_n|$ can be estimated asymptotically.

Lemma 2. Let θ be a real number. Suppose p_n/q_n is a rational sequence satisfying

(i) $q_n \to \infty$

(ii) $q_{n+1} = q_n^{1+o(1)}$

(iii) For some $\lambda \in (0,1)$ we have

$$0 \neq \left| \theta - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{(1+\lambda)\,(1+o(1))}} .$$

Then θ is irrational of type at most $1 + \frac{1}{\lambda} = \tau$.

Essentially, conditions (i), (ii) and (iii) enable us to show that given p_n/q_n with q_n large, there exists p_m/q_m such that

$$q_m = q_n^{1+o(1)} \text{ and } p_m q_n \neq q_m p_n .$$

This is a substitute for (2.4), and Lemma 2 can be proved in a manner similar to Lemma 1. We omit the details and refer the reader to [1]. In other words, given the sequence p_n/q_n, one can extract a subsequence satisfying (2.4). The mere existence of this subsequence suffices, and we need not concern ourselves with the construction of this subsequence. Thus Lemma 2 is in certain cases more useful.

Inequalities like (2.1) and (2.2) have lots of applications. Liouville was the first to use them, when he deduced the existence of transcendental numbers by showing that all algebraic numbers are of finite type. More precisely, if α is algebraic, then it is of type $\leq \deg \alpha = \tau$. Thus

$$\ell = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$$

is transcendental since it is of infinite type. This fundamental observation of Liouville has undergone several improvements, some of which are considered in the next section.

3. The Thue Equation.

In 1909 A. Thue obtained the first significant improvement of Liouville's theorem, by showing that every algebraic number α of $\deg \alpha = n \geq 3$ is of type $< n$. More precisely, he showed that such an α is of type $\leq \frac{n}{2} + 1$. From this he deduced that if $F(x,y)$ is an irreducible binary form of degree $n \geq 3$, then for any integer m the Diophantine Equation (Thue's equation)

$$F(x,y) = m \tag{3.1}$$

can have only a finite number of solutions in integers x and y. For, we can rewrite (3.1) as

$$a(x-\alpha_1 y) \ldots (x-\alpha_n y) = m, \qquad (3.2)$$

where the α_i are algebraic. Thus if (3.1) had an infinite number of solutions in integers x and y, then at least one of the α_i will have infinitely many rational approximations x/y satisfying

$$\left|\alpha_i - \frac{x}{y}\right| \ll \frac{1}{y^n} \quad , \qquad (3.3)$$

and this violates Thue's theorem.

Over the decades this result of Thue underwent several improvements and Roth in 1955 finally established the deep result that all algebraic irrational numbers are of type 2. Thus, an irrational number of type > 2 is transcendental. The main difficulty with the application of the Roth and Thue theorems is that they are non-effective. If one wants for instance to effectively find all the solutions to (3.1), it is desirable to obtain effective versions of their inequalities. Therefore it is of considerable interest to derive irrationality measures μ (see 2.2) with $\mu < \deg \alpha$, by simple methods, even if the values μ are larger than the irrationality types provided by Thue and Roth. Baker [3], [4] in 1964 derived such measures for certain algebraic numbers, by the use of hypergeometric functions. In particular he showed that

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| \geq \frac{10^{-6}}{q^{2.955}} \quad ,$$

and thus deduced that the equation

$$x^3 - 2y^3 = m$$

has all its integer solutions satisfying

$$\max(|x|, |y|) < (3.10^5, m)^{23}.$$

Similar to Baker's result for $\sqrt[3]{2}$ and other algebraic numbers, we will derive in Section 7 using an even simpler method involving Legendre polynomials, irrationality measures $\mu < \deg \alpha$, for certain algebraic numbers α. As far as I know, this is the simplest way to obtain such non-trivial results for specific algebraic numbers.

4. The Classical Contin ued Fraction Method.

One of the first methods employed in proofs of irrationality was the continued fraction approach. To be more precise, this method enabled one to prove the irrationality of certain numbers θ which are of the form $f(p/q)$, $p, q, \in \mathbb{Z}$, where f is a function possessing a convergent continued fraction expansion. For instance if we consider the Ricatti differential equation

$$Y = Q_0 Y' + P_1 y'' \; (y = f(x)) \tag{4.1}$$

where $Q_0(x)$, $P_1(x) \in \mathbb{Z}[x]$, and iterate (4.1) without bothering about convergence, we get

$$\frac{y}{y'} = Q_0 + \cfrac{p_1}{Q_1 + p_2/(Q_2 + p_3/(Q_3 + \cdots} \tag{4.2}$$

where the Q_n and p_n are defined recursively by

$$Q_n = \frac{Q_{n-1} + p'_n}{1 - Q'_{n-1}} \; ; \; p_{n+1} = \frac{p_n}{1 - Q_n} \tag{4.3}$$

From (4.2) and (4.3) it follows that if the expansion is convergent at $x = p/q$, then (4.2) yields a sequence of rational approximations to $f(p/q) = \theta$. One would hope that these approximations are 'good enough' to establish the irrationality of θ.

There is however a major drawback. The nth-convergent p_n/q_n to (4.2) is such that p_n and q_n are rationals, and not necessarily integers. Thus if we clear these rationals of their denominators the quality of the approximation might diminish. In fact it might diminish to such an extent, that the new approximations thus generated may not satisfy Dirichlet's criterion (1.1), and thus the irrationality proof might break down. At any rate, even if the proof of irrationality survives, the irrationality estimate derived will not in general be good.

Historically, this method was used to establish the irrationality of certain classes of numbers, but the irrationality types and measures thus derived were often much weaker than what one would expect is the truth.

In 1761, Lambert confirmed the irrationality of π by this method, and later in 1765 established similar results for $\tan x$, e^x (and sonsequently $\log x$), for suitable rational x. The irrationality measures turned out to be poor.

The first good irrationality measure for π was due to Mahler who shows that $\mu \leq 42$. This result has been improved since then, and Apery's recent work shows that π is irrational of type $\leq 23.71 \ldots$.

The irrationality measures we discuss here by the Legendre Polynomial method, are very good, considering the simplicity of approach. In fact, in certain cases, the estimates are the best known to date. Without much further ado, we describe the procedure by considering first the exponential function, and the number π .

5. The Exponential Function and π .

We begin by deriving as in [1] a result for the exponential function.

Theorem 1. If s is rational and $\neq 0$, then e^s is irrational of type 2 .

Proof. First, consider

$$a_r = \int_0^1 e^{sx} x^r dx = \frac{(-1)^r r!}{s^{r+1}} \left(\left[1 - s + \frac{s^2}{2!} - \ldots + \frac{(-1)^r s^r}{r!}\right] e^s - 1 \right)$$

$$= (-1)^r u_r e^s - v_r, \quad r = 0,1,2,\ldots \; . \qquad (5.1)$$

Next, let $s = p/q$, where $p, q \in \mathbb{Z}$. Then (5.1) and (1.5) show that

$$p^{n+1} \int_0^1 p_n(x)\, e^{sx} dx = q_n e^s - p_n \, , \quad q_n, p_n \in \mathbb{Z} . \tag{5.2}$$

An n-fold integration by parts in (6.2) yields

$$\int_0^1 p_n(x)\, e^{sx} dx = \frac{p^n}{q^n} \cdot \frac{1}{n!} \int_0^1 e^{sx} x^n (1-x)^n dx . \tag{5.3}$$

If we combine (5.2) and (5.3) we arrive at

$$0 \neq \left| q_n e^s - p_n \right| \leq \frac{e^s p^{2n+1}}{(4q)^n n!} \to 0 \text{ as } n \to \infty \tag{5.4}$$

and that proves the irrationality of e^s.

To deduce the irrationality type, we note that (5.1) and (5.2) show that

$$p^n n! \ll q_n \ll 4^n p^n q^n n! . \tag{5.5}$$

On the other hand from (5.2) and (5.3), we get

$$q_n e^s - p_n = (n!)^{-1+o(1)}, \text{ as } n \to \infty , \tag{5.6}$$

because for a continuous function f, we have $\| f \|_n \to \| f \|_\infty$, as $n \to \infty$. So, (5.5) and (5.6) shows that the p_n, q_n satisfy the conditions of Lemma 2 with $\lambda = 1$, and therefore e^s is of type 2.

<u>Corollary 1.</u> If s is rational and $\neq 1$, then $\log s$ is irrational.

Next we prove

<u>Theorem 2.</u> π^2 is irrational.

<u>Proof.</u> (Beukers [7]):

For a positive integer n let

$$I_n = \int_0^1 \sin \pi t \; p_{2n}(t)\,dt. \tag{5.7}$$

An 2n-fold integration by parts of I_n yields

$$I_n = (-1)^n \frac{\pi^{2n}}{(2n)!} \int_0^1 \sin \pi t \cdot t^{2n}(1-t)^{2n} dt.$$

Therefore

$$0 \neq |I_n| \leq \left(\frac{\pi}{4}\right)^{2n} \cdot \frac{1}{(2n)!} \tag{5.8}$$

From (5.7) we see that I_n is a linear combination of terms like

$$\int_0^1 t^m \sin \pi t \, dt, \quad 0 \leq m \leq 2n . \tag{5.9}$$

A simple calculation of (5.9) shows that

$$I_n = \frac{1}{\pi} Q_n(\pi^{-2}) \tag{5.10}$$

where $Q_n(x) \in \mathbb{Z}[x]$ is of degree $\leq \frac{n}{2}$, Therefore if $\pi^2 = p/q$ is rational, then (5.10) shows that

$$\left| I_n \right| \geq 1/\pi \, p^n \qquad (5.11)$$

because $I_n \neq 0$. Estimates (5.8) and (5.11) are incompatible if n is large, and so we deduce that π^2 is irrational.

<u>Corollary 2.</u> π is irrational.

This proof unfortunately does not yield an irrationality type for π^2 or π, Note that while Theorem1 yields a good irrationality type for e^s, we do not derive any such estimate for the logarithm in Corollary 1, by simply viewing the logarithm as the inverse of the exponential function. It is thus desirable to have a method which directly constructs the p_n and q_n for the logarithm, and so enables us to obtain a good irrationality estimate. This is what we consider next.

6. <u>Irrationality Estimates for the Logarithm.</u>

For any complex number z let

$$\alpha(z) = \max\left(\left|\frac{(1 \pm \sqrt{1-z})^2}{z}\right|\right), \quad \beta(z) = \min\left(\left|\frac{(1 \pm \sqrt{1-z})^2}{z}\right|\right).$$

We then have the following result.

<u>Theorem 3.</u> Let K and R be as in Lemma 1. Let $r, s \in R$ satisfy

$$\text{(i)} \ \frac{r}{s} \notin [1, \infty) \ \text{and (ii)} \ \beta\left(\frac{r}{s}\right) \cdot |r| \cdot e < 1 .$$

Then $\log(1-\frac{r}{s}) \notin K$. Also, given $\varepsilon > 0$ there exist $b_0(\varepsilon)$ such that for all $a, b \in R$ with $|b| > b_0(\varepsilon)$ we have

$$\left| \log(1-\frac{r}{s}) - \frac{a}{b} \right| > \frac{1}{|b|^{\tau+\varepsilon}},$$

where

$$\tau = \tau_{r,s} = 1 + \frac{\log(\alpha(\frac{r}{s}).r) + 1}{\log(\alpha(\frac{r}{s}).r^{-1}) - 1}. \tag{6.2}$$

In particular $\log(1-\frac{r}{s})$ is irrational of type at most $\tau_{r,s}$.

To keep the exposition simple we only prove Theorem 3 for the case $r, s \in \mathbb{Z}^+$, $r < s$. For the case of general r and s we refer to [2] where this problem is considered in detail.

<u>Sketch of the Proof</u>. For any $n \in \mathbb{Z}^+$ and $z \notin [1,\infty)$ we have

$$\int_0^1 \frac{x^n dx}{1-zx} = \frac{-1}{z^{n+1}} \left[\log(1-z) + \sum_{k=1}^{n} \frac{z^k}{k} \right]. \tag{6.3}$$

Let

$$I_n(z) = \int_0^1 \frac{p_n(x)\,dx)}{1 - zx}. \tag{6.4}$$

From (6.3), (6.4) and (1.5) we see that

$$I_n(z) = -\frac{1}{z} p_n(\frac{1}{z}) \log(1-z) + Q_n(\frac{1}{z}) d_n^{-1}, \tag{6.5}$$

where $Q_n(x) \in \mathbb{Z}[x]$ is of degree n, $x \mid Q_n(x)$ and

$$d_n = \text{l.c.m.}\,[1,2,3,\ldots,n]\,. \tag{6.6}$$

An n-fold integration by parts in (6.4) shows that

$$I_n(z) = (-z)^n \int_0^1 \frac{x^n(1-x)^n dx}{(1-zx)^{n+1}}\,. \tag{6.8}$$

By letting $z = r/s$ in (6.5) we see from (6.8), (6.5) and (6.6) that

$$0 \neq q_n \log(1-\tfrac{r}{s}) - p_n = E_n,\quad q_n, p_n \in \mathbb{Z}, \tag{6.9}$$

where

$$q_n = -d_n r^n P_n(\tfrac{s}{r}),\; p_n = r^n Q_n(\tfrac{s}{r})\tfrac{r}{s},\; E_n = (-1)^n \frac{r^{n+1}}{s} I_n(\tfrac{r}{s})\, d_n\,. \tag{6.10}$$

From the Prime Number Theorem (see LeVeque [13], Vol.2 p.230-50) it follows that

$$d_n = e^{n(1+o(1))}. \tag{6.11}$$

Also, note that for $0 < \frac{r}{s} < 1$ we have

$$\sup_{0 \leq x \leq 1} \frac{x(1-x)}{1-\frac{r}{s}x} := \beta(\tfrac{r}{s})\,. \tag{6.12}$$

Therefore, by reasoning similar to (6.6) we get

$$E_n = \left\{e.r.\beta(\tfrac{r}{s})\right\}^{n(1+o(1))}. \tag{6.13}$$

On the other hand, it is well known that (see Szego [15], p.194)

$$P_n(\tfrac{s}{r}) = \alpha(\tfrac{r}{s})^{n(1+o(1))}. \tag{6.14}$$

Therefore

$$q_n = \left\{ e r \alpha\left(\frac{r}{s}\right) \right\}^{n(1+o(1))}. \tag{6.15}$$

Theorem 3 for $0 < \frac{r}{s} < 1$, $r, s \in \mathbb{Z}^+$, follows from Lemma 2, (6.9), (6.13) and (6.15).

While proving Theorem 3 for general r and s certain technical difficulties arise. First, it is not at all clear in (6.8) that $I_n \neq 0$ for infinitely many n. This fact can be established using the orthogonality of p_n. Next, the sup norm type estimate in (6.12) does not indicate the true size of E_n. A satisfactory asymptotic estimate for E_n can be derived from that of I_n, by establishing that the I_n satisfy a recurrence relation and using a theorem of Poincaré on recurrences. The proof of Theorem 3 can then by completed using Lemma 1, by establishing (2.4) by means of a lemma from Padé - Approximation Theory. For all details we refer to [2], for a complete treatment.

We mention in closing, that an effective version of Theorem 3 can be formulated (see Theorem 3 of [2]). In this case one has to calculate upper bounds for q_n and E_n explicitly. In particular one can show that for all positive rationals a/b,

$$\left|\log 2 - \frac{a}{b}\right| \geq \frac{(2000)^{-1}}{b^{4.871}}$$

and

$$\left|\log\left(1 + \frac{1}{m}\right) - \frac{a}{b}\right| > \frac{(22m)^{-3}}{b^{\mu_{1,m}}} \qquad (m \in \mathbb{Z}^+). \tag{6.6}$$

where

$$2 < \tau_{1,m} < \mu_{1,m} = 2 + o(1), \text{ as } m \to \infty .$$

Note finally that as $r, s \to \infty$ satisfying $\log r/\log s \to 0$, we have $\tau_{r,s} \to 2$. Irrationality estimates of the type provided by Theorem 3 and (6.16) are the best known yet for these logarithms.

7. Irrationality Estimates for k^{th}-roots.

The method described in Section 6 can be applied to

$$I_n = \int_0^1 \frac{P_n(x)\,dx}{(1-zx)^{\ell/k}}, \tag{7.1}$$

where ℓ, k are relatively prime integers and $z \notin [1,\infty)$. First, the substitution $u = 1 - zx$ shows that

$$\int_0^1 \frac{x^m\,dx}{(1-zx)^{\ell/k}} = \left(\frac{1}{z}\right)^{m+1} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \left\{ \frac{1-(1-z)^{j+1-\ell/k}}{j+1-\ell/k} \right\} \tag{7.2}$$

for all positive integers m. So, if we set

$$d_n(k, \ell) = \text{l.c.m.}\left[km + k - \ell,\ m = 1,2,\ldots,n\right],$$

we deduce from (7.1) and (7.2) that

$$d_n(k,\ell)\, z^{n+1} \int_0^1 \frac{P_n(x)\,dx}{(1-zx)^{\ell/k}} = (1-z)^{1-\frac{\ell}{k}} B_n(z) + A_n(z) \tag{7.3}$$

where $A_n(x)$ and $B_n(x) \in \mathbb{Z}[x]$ are of degree n.

We need now an estimate on the size of $d_n(k,\ell)$, similar to that on d_n provided by (6.11). The Prime Number Theorem for Arithmetic Progressions([13] , Vol.2, p. 256) implies that (for a Proof see Lemma 1 of [2])

$$d_n(k,\ell) = e^{f(k)n(1+o(1))}, \tag{7.4}$$

where

$$f(k) = \frac{k}{\phi(k)} \sum_{\substack{a=1\\(a,k)=1}}^{k} \frac{1}{a}; \tag{7.5}$$

and ϕ is Euler's function.

An n-fold integration by parts of I_n yields

$$I_n = \frac{1}{n!}\left[\prod_{j=1}^{n} (j + \frac{\ell}{k} - 1)\right](-z)^n \int_0^1 \frac{x^n(1-x)^n dx}{(1-zx)^{n+\ell/k}} \tag{7.6}$$

If $z = -r/s$ is rational, then the sup-norm argument used in the previous section shows that

$$|I_n| = \left(\left|\frac{r}{s}\right| \cdot \beta(r/s)\right)^{n(1+o(1))}. \tag{7.7}$$

On the other hand from (6.14), (7.2), (7.3) and (7.4) it follows that

$$\left|B_n(-\frac{r}{s})\right| \ll \left|\frac{r}{s}\right|^n P_n(-\frac{s}{r})\, d_n(k,\ell) \le \left\{\left|\frac{r}{s}\right| e^{f(k)} \cdot \alpha(-\frac{r}{s})\right\}^{n(1+o(1))} \tag{7.8}$$

So, the point is that for $z = -r/s$, one can clear (7.3) of denominators and obtain an expression

$$0 \neq q_n(1+\frac{r}{s})^{-\ell/k} - p_n = E_n\ ,\ p_n, q_n \in \mathbb{Z} \tag{7.9}$$

and use (7.8) and (7.7) to deduce that

$$|q_n| \le Q^{n(1+o(1))}, \quad |E_n| \le E^{-n(1+o(1))}.$$

where

$$Q = re^{f(k)}\alpha(-\frac{r}{s}) \quad \text{and} \quad E^{-t} = re^{f(k)}\beta(-\frac{r}{s}). \tag{7.10}$$

As in the general case of Theorem 1, the non-vanishing of $p_n q_{n+1} - q_n p_{n+1}$ follows from Padé Approximation Theory. Thus an irrationality type for $(1+\frac{r}{s})^{\ell/k}$ can be derived from (7.9) by the use of Lemma 1 and this is given below.

<u>Theorem 4.</u> Let ℓ, k, r, s be positive integers with $\ell < k$, $(\ell, k) = 1$ and $r < s$. Also let $|E| > 1$, where E is as in (7.10). Then $(1+\frac{r}{s})^{\ell/k}$ is irrational of type at most $\tau_{r,s,k} = \tau$ where

$$\tau = \frac{\log\{s(1+\sqrt{1+r/s})^2\} + f(k)}{\log\{(s/r^2)(1+\sqrt{1+r/s})^2\} - f(k)} + 1,$$

and $f(k)$ as in (7.5).

An effective version of Theorem 4 can be derived with a bit more care. One has for instance to use explicit upper bounds for $d_n(k, \ell)$ instead of the weaker estimate (7.4). For a detailed discussion on this effective version and on Theorem 4 itself, see [2].

As in Theorem 3, note here also that $\tau_{r,s,k} \to 2$ if $\log r/\log s \to 0$, as $r, s \to \infty$. Thus for such r and s one could have

$$3 \quad \tau_{r,s,k} \leq \deg\left\{(1+\frac{r}{s})^{\ell/k}\right\} = k,$$

if $k \geq 3$. This is, as we know, a useful fact in Diophantine Equations. For instance by taking $r = 1$, $s = 5831$, $\ell = 1$, and $k = 3$, we see from Theorem 4 that $\sqrt[3]{17}$ is irrational of type 2.5763... . The effective version of Theorem 4 derived in [2] shows that

$$\left| \sqrt[3]{17} - \frac{a}{b} \right| \geq \frac{10^{-20}}{|b|^{2.9}}$$

for all rationals a/b . Therefore, all the integer solutions of the equation

$$x^3 - 17y^3 = m \, , \quad m \in \mathbb{Z}$$

satisfy

$$\max (|x| , |y|) \leq 3(m.10^{20})^{10}.$$

The method underlying Theorem 4 differs significantly from the earlier approach of Baker [3], [4] who used hypergeometric functions, or the recent observations by Beukers and Choodnovsky [Lo] . I confess however, that Theorem 4 yields weaker irrationality types than these other more advanced methods. At any rate, to my knowledge, the method discussed in this section is the simplest one known so far to derive non-trivial irrationality types for special algebraic numbers.

9. Irrationality of $\zeta(2)$ and $\zeta(3)$

In June 1978, R. Apery produced remarkable proofs of the irrationality of

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ and } \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

His proof implicitly used Legendre Polynomials, but this fact was unnoticed until Beukers observed it. Beukers then rewrote the linear forms of Apéry for $\zeta(2)$ and $\zeta(3)$ in terms of Legendre Polynomials, and thus simplified the proof (see [6]).

The key observation is that for positive integers r and s

$$\int_0^1 \int_0^1 \frac{x^r y^s dxdy}{1-xy} \qquad (8.1)$$

is rational with denominator dividing d_r^2 if $r > s$, and is equal to

$$\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{r^2}$$

if $r = s$. Thus

$$d_n^2 \int_0^1 \int_0^1 \frac{p_n(x)\ (1-y)^n dxdy}{1-xy} = q_n \zeta(2) - p_n,\ p_n, q_n \in \mathbb{Z}. \qquad (8.2)$$

The non-vanishing of the expression in (8.2) and an upper bound for it can be easily derived by an n-fold integration by parts with respect to x which yields

$$0 \neq q_n \zeta(2) - p_n = d_n^2 \int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n dxdy}{(1-xy)^{n+1}} \tag{8.3}$$
$$\leq d_n^2\, \zeta(2) \left(\frac{\sqrt{5}-1}{2} \right)^{5n} .$$

Note that (6.11) implies that

$$d_n^2 \left(\frac{\sqrt{5}-1}{2} \right)^{5n} = \left\{ e\left(\frac{\sqrt{5}-1}{2}\right)^5 \right\}^{n(1+o(1))} \to 0,$$
$$\text{as } n \to \infty. \tag{8.4}$$

The irrationality of $\zeta(2)$ follows from (8.3) and(8.4).

The above method also yields an irrationality type $\leq$ 11.85 for $\zeta(2)$. The important thing is that this proof derives the irrationality of $\zeta(2)$ without appeal to the special properties of π.

The irrationality of $\zeta(3)$ is deduced from the above proof for $\zeta(2)$ by means of 'trick'. The basic observation is that

$$\frac{d}{d\sigma}\left(\sum_{n=1}^{\infty} \frac{1}{(n+\sigma)^s} \right)\Big|_{\sigma=0} = -s\, \zeta(s+1) .$$

Thus the integral representation for $\zeta(3)$ can be guessed from (8.1) by differentiating with respect to xy by taking r = s. In particular one starts with

$$I = -\int_0^1\int_0^1 \frac{\log xy . x^r y^s \, dxdy}{1-xy} = \int_0^1\int_0^1\int_0^1 \frac{x^r y^s \, dxdydz}{1-(1-xy)z}$$

and notes that I is rational with demominator d_r^3 if $r > s$, and

$$I = 2\left(\zeta(3) - 1 - \frac{1}{2^3} - \frac{1}{3^3} - \dots - \frac{1}{r^3}\right), \text{ if } r = s.$$

Thus

$$d_n^3 \int_0^1 \int_0^1 \int_0^1 \frac{p_n(x)\,p_n(y)\,dx\,dy\,dz}{1-(1-xy)z} = q_n\,\zeta(3) - p_n, p_n, q_n \in \mathbb{Z}. \tag{8.5}$$

The irrationality of $\zeta(3)$ follows from (8.5), using integration by parts n-times with respect to x and y. But here a clever substitution

$$w = \frac{1-z}{1-(1-xy)z}$$

is required. We do not want to elaborate on this, since we will simply be reproducing the exposition of Beukers [6]. The only unfortunate thing is that no one yet knows how to extend this beautiful proof to get the irrationality of $\zeta(s)$, for $s \in \mathbb{Z}^+$, $s \geq 4$.

Apéry's observation that $\zeta(2) = \pi^2/6$ has irrationality type ≤ 11.85 yields a uniform upper bound for the irrationality type of numbers $\pi/\sqrt{k}$, as the following observation shows:

$$\left|\frac{k^2}{k} - \frac{a^2}{b^2}\right| = \left|\frac{\pi}{\sqrt{k}} - \frac{a}{b}\right| \cdot \left|\frac{\pi}{\sqrt{k}} + \frac{a}{b}\right|.$$

Thus for all positive rationals k, $\pi/\sqrt{k}$ is irrational of type

$$\leq 2(11.85) = 23.70\,. \tag{8.6}$$

In particular this yields an irrationality type for π which is superior to Mahler's estimate.

9. Comparisons

Theorem 1 yields the best irrationality type for those Logarithms it treats. Choodnovsky [9] independently derived this result and later a whole lot more ! Bombieri [8] also proved Theorem 3, and other interesting results by the use of differential equations. We shall touch upon their ideas in Section 10 and Section 11.

Prèviously, the best known result for logarithms of special numbers were due to Baker [5], who used hypergeometric functions to derive the following: If $m \in \mathbb{Z}^+$, then for all $a, b \in \mathbb{Z}^+$

$$\left| \log(1 + \tfrac{1}{m}) - \tfrac{a}{b} \right| \geq \frac{c(m)}{|b|^{\beta_m}}, \tag{9.1}$$

where

$$\beta_1 = 12.5, \ \beta_2 = 7 \text{ and } \beta_m = 2 \frac{\log\left(\frac{4\sqrt{2}\,m^2}{m+1}\right)}{\log\left(\frac{\sqrt{2}\,m^3}{(m+1)^2}\right)}, \text{ for } m \geq 3,$$

and

$$c(1) = 10^{-10^5}, \quad c(m) = (\sqrt{2m})^{-10^4}, \ m \geq 2.$$

We note that $c(m) < (22m)^{-3}$ and

$$\tau_{1,m} \leq \mu_{1,m} \leq \beta_m. \tag{9.2}$$

Thus (6.16) is better than (9.1). All three quantities in (9.2) tend to 2 as $m \to \infty$.

Whereas the hypergeometric function method of Baker yields a weaker result for the logarithm, it scores over the Legendre Polynomial method in the most important situation - namely, in deriving irrationality types for algebraic numbers. In particular his results are superior to those provided by Theorem 4. But in view of the simplicity of the method underlying Theorem 4, it seems worthwhile to improve it without much loss of simplicity in treatment. Beukers and Choodnovsky have recently derived results like Theorem 3 by employing the more general Jacobi Polynomials

$$J_{m,n}(x) = \frac{x^{-m}}{n!} \left(\frac{d}{dx} \right)^n \left\{ x^{m+n}(1-x)^n \right\} .$$

Their method has more in common with Baker's than the method of Section 7.

For the specific irrational $\pi/\sqrt{3}$, Theorem 3 yields a better irrationality type, than that provided by Apery's method in (8.6). To see this take $r = \frac{1}{2} - \frac{\sqrt{3}i}{2}$, $s = 1$, and $K = \mathbb{Q}(\sqrt{-3})$ in Theorem 3. We then get

$$\tau = 8.309986 \ldots$$

and this is $\leq$ the irrationality type for $\pi/\sqrt{3}$ because

$$\log\left(1 - \frac{r}{s} \right) = \frac{\pi i}{3} .$$

Choodnovsky [9] who independently observed this, claimed in a brilliant lecture at Oberwolfach in May 1979, that even for a slight improvement of this particular result for $\pi/\sqrt{3}$, a major new idea will be required! Unfortunately, Theorem 1 does not yield an irrationality type for π, whereas the Apéry result (8.6) does.

10. <u>Differential Equations.</u>

Soon after Apery announced his result on the irrationality of $\zeta(3)$, Bombieri [8] viewed the problem as one involving differential equations. The motivation for this observation was a sketch of the proof of the irrationality of $\log 2$ in Exercise 8 of Vander Poorten's delightful article [15] on Apéry's proof. By developing this idea Bombieri not only established a form of Theorem 3, but also derived the results of Apéry on $\zeta(2)$ and $\zeta(3)$. In addition he could also derive Thue's theorem for certain k^{th} roots. To give a glimpse of the method we sketch Bombieri's version of VanderPoorten's Exercise 8.

Consider the differential operator

$$\mathcal{D}_0 = (x^2 - 2bx + 1)\frac{d}{dx} + (x-b)\left(\frac{d}{dx}\right)^0 .$$

If $b = 2a + 1$ is an odd integer notice that

$$y_0 = (x^2 - 2bx + 1)^{-1/2} \in \mathbb{Z}[[x]] \quad \text{and} \quad \mathcal{D}_0 y_0 = 0.$$

Next, define

$$y_1(x) = \int_0^x y_0(t)\,dt$$

and note that

$$\mathcal{D}_0 y_1 = 1.$$

The singularities for $\mathcal{D}_0$ are at α and β defined respectively by $2a \pm 2\sqrt{a(a+1)} + 1$. So $0 < \beta < 1 < \alpha$, and $\alpha\beta = 1$

But note that

$$y_1 - \omega\, y_0 \tag{10.1}$$

has no singularity at β , if $\omega = \frac{1}{2}\log(1+\frac{1}{a})$. An estimate on the size of the coefficients of (10.1) yields

$$q_n \log(1+\tfrac{1}{a}) - p_n = (e\beta)^{n(1+o(1))}, \tag{10.2}$$

where $q_n, p_n \in \mathbb{Z}$ and

$$|q_n| = (e\alpha)^{n(1+o(1))} \tag{10.3}$$

Theorem 3 for $r = 1$, $s = a$, follows from (10.2), (10.3) and Lemma 2.

It is interesting to note that y_0 is the generating function for the Legendre Polynomials $p_n(a)$.

Most recently Bombieri has pursued his method to the problem of obtaining transcendence measures, but that is beyond the scope of this article.

9. Recent Progress

The most significant progress in irrationality since Apéry's result, is due to Choodnovsky. By a variety of methods involving Legendre, Jacobi, Hermite and other orthogonal polynomials and in particular differential equations he has established irrationality estimates for a wide class of numbers (see [9], [10], [11] and [12]).

In particular he can estimate the irrationality type of $\log 3$, which is beyond the scope of Theorem 3. In addition he can derive a nice irrationality estimate for π directly, without involving π^2. Finally, he can discuss irrationality estimates at rational points x close to zero, for the dilogarithm and trilogarithm functions defined respectively by

$$\operatorname{dilog}(1+x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad \operatorname{trilog}(1+x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}.$$

Bombieri independently obtained similar results for these two functions.

In summary, the most powerful methods at present in irrationality involve differential equations and the clever use of orthogonal polynomials. The orthogonality occurs quite naturally in Padé - Approximations. At any rate, the Legendre Polynomial method we have discussed, besides being interesting and useful in its own right, has given us a glimpse of deeper methods and more interesting techniques that lie ahead. For an enlightening discussion on these more advanced methods, we refer to Choodnovsky's article [12], which treats these recent advances in the context of various classical approaches.

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Note the following corrections in the above text:-

a) In p.5 the second line after (2.2) should read as:-

constant .... increasing function .

b) In p.11, (4.2) - (4.3) should read as:-

$$\frac{y}{y'} = Q_0 + \left(\frac{P_1}{Q_1 + \frac{P_2}{Q_2 + P_3/Q_3 + \cdots}} \right) \tag{4.2}$$

where the Q_n and P_n are defined recursively by

$$Q_n = \frac{Q_{n-1} + P_n'}{1 - Q_{n-1}'} \; ; \; P_{n+1} = \frac{P_n}{1 - Q_n} . \tag{4.3}$$

c) In the following, the Legendre polynomial 'P_n' has been denoted by 'p_n' . Thd corresponding correct versions are:-

p.13, l.h.s. of (5.2): $p^{n+1}\int_0^1 P_n(x) e^{sx} dx$

p.13, l.h.s. of (5.3): $\int_0^1 P_n(x) e^{sx} dx$

p.14, r.h.s. of (5.7): $\int_0^1 \sin \pi t \, P_{2n}(t) dt$

p.16, r.h.s. of (6.4): $\int_0^1 \frac{P_n(x)\, dx}{1 - zx}$

p.16, r.h.s. of (6.5): $-\frac{1}{z} P_n\left(\frac{1}{z}\right) \log(1-z) + Q_n\left(\frac{1}{z}\right) d_n^{-1}$,

p.18, 4th line, 2nd paragraph: established...orthogonality of P_n. .....

p.23, l.h.s. of (8.2): $d_n^2 \int_0^1 \int_0^1 \frac{P_n(x)(1-y)^n \, dx dy}{1 - xy}$

p.25, l.h.s. of (8.5): $d_n^3 \int_0^1 \int_0^1 \int_0^1 \frac{P_n(x) P_n(y) \, dx dy dz}{1 - (1-xy)z}$

HARMONIC SUMS OF CERTAIN SUBSETS OF NONNEGATIVE INTEGERS

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ABSTRACT :

Let S be any subset of the set N of natural numbers. We denote the sum of the reciprocals of elements of S by H(S). In this paper, we describe a sufficient condition for H (S) to be finite. The condition has been described in terms of the basis representation of a natural number n in an arbitrarily fixed integer g greater than 1.

1. INTRODUCTION AND NOTATIONS :

Let S be any subset of the set N of all natural numbers. We define H (S), the harmonic sum of S, by $H(S) = \sum_{n \in S} \frac{1}{n}$ It may be noted that H(S), being a summation of positive terms, does not depend on any particular ordering of elements of S. Therefore, without any loss of generality, we assume that the elements of S are arranged in natural (increasing) order.

Let g be any fixed natural number > 1. It is well-known that every nonnegative integer n can be expressed uniquely in the form

(1.1) $$n = \sum_{i=0}^{\infty} e_i \; g^i$$

Where $o \leq e_1 < g$, and only finitely many e_i'S are nonzero This is called the basis representation of n in the base g.

Throughout this paper, a sequence shall always mean a strictly increasing infinite sequence of nonnegative integers. The complementing sequence $\{a'_n\}$ of the sequence $\{a_n\}$ is the strictly increasing sequence of nonnegative integers which are not elements of $\{a_n\}$. e.g the complementing sequence of the sequence of prime numbers (which is 2,3,5,7,11,...) is 0,1,4,6,8,9,10,... We make it clear that the complementing sequences can be finite or even empty. If for the sequence $\{a_n\}$ and an integer m, $m=a_r$ for some r, then we say that m is a member or an element of $\{a_n\}$ or m appears in $\{a_n\}$

Let $\{a_n\}$ be any sequence. We define $S\{a_n\}$ the set generated by $\{a_n\}$, by

$$S\{a_n\} = \{n/n \in N,\ P(n) \text{ is true}\}$$

where P (n) is : : :

''When n is expressed in the form (1.1), for each element i of $\{a_n\}$ the coefficient e_i of g^i is zero''.

In this paper, we prove the following-

Theorem: If $\{a_n\}$ is a sequence satisfying,

$$a_{n+1} - a_n \leq B \text{ for all } n,$$

then $H(S\{a_n\})$ is finite.

2. <u>A simple result</u> :

Theorem 2.1: Let S and T be any two infinite subsets of N. If $s_n \leq t_n$, where s_n is the n-th element of S and t_n that of T, then

$H(T) \leq H(S)$

Proof: $H(T) = \sum (t_n)^{-1} \leq \sum (s_n)^{-1} = H(S)$

3. A PARTICULAR CASE OF THEOREM 1.1

Theorem 3.1: Let d be any fixed natural number and if $a_n = nd$, then $H(S\{a_n\})$ is finite.

Proof: As easily observed, if $d=1$, then

$S\{a_n\} = \{1,2,3,\ldots\ldots, g^{-1}\}$

and $H(S\{a_n\})$ is obviously finite.

Let $d > 1$. and we write $D=d^2$ and S for $S\{a_n\}$ Define $R(n)=1/n$ for all $n \in N$ and $R(o)=0$. Now,

$$H(S) = \sum_{n\in S} R(n) = \sum_{\substack{n\in S\\ n<g^D}} R(n) + \sum_{\substack{n\in S\\ n\geq g^D}} R(n) = S_1+S_2, \text{ say}$$

Since S_1 is finite, it is sufficient to prove finiteness of S_2.

And

$$(3.1)\quad S_2 = \sum_{\substack{n\in S\\ n\geq g^D}} R(n) = \sum_{r=d}^{\infty} F(r),$$

where,

$$(3.2)\quad F(r) = \sum_{\substack{n\in S\\ g^{rd}\leq n<g^{(r+1)d}}} R(n)$$

Now, the number of elements of S satisfying $g^{rd} \leq n < g^{(r+1)d}$ does not exceed $g^{(r+1)d-r}$. For, every natural number $n < g^{(r+1)d}$ is of the form

$$\sum_{i=0}^{(r+1)d-1} e_i \; g^i \quad , \; 0 \leq e_i < g;$$

The number of e_i 's is, thus, $\leq (r+1)d$. Also, $e_i = 0$, if $i \in \{d, 2d, \ldots, rd\}$ and each of the remaining e_i's can have atmost g different values. Therefore, the number of terms on the right side of (3.2) is atmost $g^{(r+1)d-r}$ and since each term is $\leq g^{-rd}$, it follows that

(3.3) $F(r) \leq g^{d-r}$

On substituting (3.3) in (3.1), we obtain,

$$S_2 \leq \sum_{r=d}^{\infty} g^{d-r} < \frac{g}{g-1}$$

This completes the proof.

(Remark : It can be shown that $S_1 \leq d^2 \log g + \gamma + B_1 g$ where γ and B_1 are some constants independent of d and g)

4. MORE ABOUT COMPLEMENTING SEQUENCES :

Theorem 4.1: Let $\{a'_n\}$ be the complementing sequence of $\{a_n\}$, then a'_k exists only if

$a_n \geq n+k-1$ for some n.

Proof : We prove that if a_n n+k-1 , for all n, then a'_k does not exist. Let me be any arbitrarily large fixed natural number. Observe the first m-k terms of $\{a_n\}$;

$a_1, a_2, a_3, \ldots, a_{m-k}$.

Since $a_{m-k} \leq m-2$, it follows that out of first m-1 non-negative integers which do not appear in $\{a_n\}$ and are $\leq m-2$, do not exceed $(m-1)-(m-k)-k-1$. in number. Since m is arbitrary, it follows that a_k does not exist. This proves the theorem.

We now prove that the necessary condition described just now for the existence of a_k is sufficient also.

Theorem 4.2 : If $\{a'_n\}$ is the complementing sequence of $\{a_n\}$ and if $a_n \geq n+k-1$ for some n, then a'_k exists.

Proof: Let n_o be any integer satisfying $a_{n_o} \geq n_n+k-1$. On observing the first n_o elements of $\{a_n\}$

$$a_1, a_2, \ldots, a_{n_o} \quad (\geq n_o+k-1)$$

We see that out of first n_o+k nonnegative integers atmost n_o integers appear in $\{a_n\}$. Hence the existence of a_k' is guaranteed.

Theorem 4.3: Let $\{a'_n\}$ be the complementing sequence of $\{a_n\}$

If a_k exists, then

(4.1) $a'_k = n_o + k-2$,

Where, n_o is the smallest natural number satisfying

(4.2) $a_{n_o} \geq n_o+k-1$.

<u>Proof</u>: This is proved by induction on k. We note that $a_1 \geq o$ and $a_{n+m} \geq a_n+m$. Let k=1. If a'_1 exists, then let n_o

be the smallest natural number, which exists by theorem 4.2, satisfying the relation (4.2), with k=1, i.e.

(4.2') $a_{n_0} \geqslant n_0$.

If $n_0 = 1$, then $a_1 \geqslant 1$ and $a_1' = 0$, in which case (4.1) is true. If $n_0 > 1$, then for $1 \leq n < n_0$, $a_n < n$ and $a_{n_0} \geqslant$ no. Hence, $a_n = n-1$, if $1 \leq n < n_0$ and $a_{n_0} \geqslant n_0$. Therefore $a_1' = n_0 - 1$, which is nothing but (4.1). This completes the proof of the theorem in the case k=1.

Now, suppose that the Theorem is true for k-r. Then, if a_r' exists then

(4.3) $a_r' = n_0 + r - 2$,

where n_0 is the smallest natural number for which

(4.4) $a_{n_0} \geqslant n_0 + r - 1$.

Now, assume the existence of a_{r+1}' then a_r' must exist and (.3) is valid. Let t be the largest integer for which $a_t \leqslant a_r$. (If t does not exist, we then define t to be zero and in this case the set $\{a_1, a_2 \ldots \}$ in the following discussion to be regarded as the nullset.) Consider the set

$$\{a_1, a_2, \ldots\ldots, a_t\} \cup \{a_1', a_2', \ldots\ldots, a_r' (= n_0 + r - 2)\}$$

which consists of r+t distinct elements and they are, infact, $0, 1, 2, \ldots, n_0 + r - 2$. Hence we must have, $t + r = n_0 + r - 1$, i.e $t = n_0 - 1$. We now define $b_n = a_{t+n} - a_r' - 1$. It is easily verified that $\{b_n\}$ is a strictly increasing infinite sequence of non-negative integers. Let $\{b_n'\}$ be the complementing sequence of this sequence b_n. Does b_1' exist? Yes. For, if not, then

$b_n = n-1$ for all n. i.e. $a_{t+n} = a_r' + n$ for all n,

In view of (4.3), this reads as
$a_{(n_o-1)+n} = (n_o+r-2)+n$, since $t = n_o - 1$

Putting n=1,2,3,...., we find that all the integers $\geq n_o+r-1$ appear in $\{a_n\}$ and since $a_r' = n_o+r-2$, a'_{r+1} does not exist, which contradicts the hypothesis that a'_{r+1} exists. Thus b_1' exists. It is easily verified that $a'_{r+1} = b_1' + 1$. But, $b_j' = m_o - i$, where m_o is the smallest natural number satisfying,

(4.5) $$b_{m_o} \geq m_o$$

$$\text{i.e. } a_{t+m_o} \geq m_o + a_r' + 1$$

$$\text{i.e. } a_{n_o+m_o-1} \geq m_o + (n_o+r-2)+1$$
$$= (n_o+m_o-1)+(r+1)-1$$

or equivalently v_o is the smallest natural number satisfying

(4.6) $a_{v_o} \geq v_o + (r+1) - 1$ where $V_o = n_o + m_o - 1$

and
$$a'_{r+1} = b_1' + a_r' + 1$$
$$= (m_o-1) + (n_o+r-2)+1$$
$$= (n_o+m_o-1)+(r+1)-2$$
$$= V_o + (r+1) - 2$$

(4.7)

On comparing (4.6) and (4.7) with (4.1) and (4.2), we find that the theorem is true for k=r+1. This completes the proof.

5. NATURAL ORDERING OF S $\{a_n\}$:

Let $\{a_n\}$ be any sequence and s_n denote the n-th element of S $\{a_n\}$, when the elements are arranged in increasing order. We exhibit relations between S_n and the complementing sequence $\{a_n'\}$ of $\{a_n\}$

Theorem 5.1: Let $\{a_n'\}$ be the complementing sequence of $\{a_n\}$ and let n be any natural number. Further let

$$n = \sum_{i=0}^{m} e_i g^i , \; 0 \leq a_i < g, \; e_m \neq 0 .$$

Then s_n exists if and only if a'_{m+1} exists.

Proof: Every element of S $\{a_n\}$ is of the form

$$\sum_{i=0}^{k} e_i g^{a_i'} , \quad 0 \leq e_i < g, \; e_k \neq 0 .$$

Hence, if a'_{m+1} does not exist then the number of elements in S $\{a_n\} \leq g^m - 1$, while n is certainly $\geqslant g^m$. Thus s_n exists only if a'_{m+1} exists. Conversely if a'_{m+1} exists, then S $\{a_n\}$ contains atleast $g^{m+1}-1$ elements and since n $< g^{m+1}$, sn exists.

Theorem 5.2: Let $\{a_n'\}$ be the complementing sequence of $\{a_n\}$

Let $$n = \sum_{i=0}^{m} e_i g^i , \quad 0 \leq e_i < g, \quad e_m \neq 0 ;$$

be the representation of n in the base g. If n-th element

s_n of $S\{a_n\}$ exists, then is given by

(5.1)
$$s_n = \sum_{i=0}^{m} e_i g^{a'_{i+1}}$$

Proof: This is proved by induction on n. We note that, when n is represented in the base g with its leading co-efficient e_m non-zero, in view of the previous theorem a'_{m+1} exists.

If n=1, then m=o and a_1 exists and obviously $s_1=g^{a'_1}$, which satisfies (5.1) with n=1. Thus the theorem is true if n=1.

We now assume that the theorem is true if n=k and prove it for n=k+1. Let

(5.2) $$k=\sum_{i=0}^{m} e_i g^{i}, \quad 0 \leq e_i < g, \quad e_m \neq 0$$

and by the assumption, we have

(5.3)
$$s_k = \sum_{i=0}^{m} e_i g^{a'_{i+1}}$$

<u>Case (i)</u> In (5.3), $e_i=g-1$ for $o \leq i \leq m$. Then,

(5.4) $k+1=g^{m+1}$ and a'_{m+2} exists. (For, we are assuming that s_{k+1} exists)

Hence $g^{a'_{m+2}}$ is an element of $S\{a_n\}$

Let t be any natural number satisfying

$s_k < t < g^{a'_{m+2}}$.

i.e. $\sum_{i=0}^{m} (g-1)\, g^{a'_{i+1}} < t < g^{a'_{m+2}}$.

Then, it is easily seen that t can not be an element of $S\{a_n\}$. For, at least one i, the coefficient, say e_i, of g^i in the representation of t in the base g will be non-zero on one hand and this i will be some a_n on the other; which is not consistent with the definition of $S\{a_n\}$. Thus the element in $S\{a_n\}$ next to s_k is $g^{a_{m+2}}$ Therefore

$$s_{k+1} = g^{a'_{m+2}}, \text{ with } k+1 = g^{m+1} \text{ as required.}$$

<u>Case (ii)</u> In (5.3) at least one of the e_i 's is different from g-1. Here let r be the smallest for which $e_r \neq g-1$

Then
$$k = \sum_{i=0}^{r-1} (g-r)g^i + e_r g^r + \sum_{i=r+1}^{m} e_i g^i.$$

We, then, have by hypothesis
$$s_k = \sum_{i=0}^{r-1} (g-1) g^{a'_{i+1}} + e_r g^{a'_{r+1}} + \sum_{i=r+1}^{m} e_i g^{a'_{i+1}}$$

Now,

$$k+1 = (e_r+1) g^r + \sum_{i=r+1}^{m} e_i g^i \tag{5.5}$$

(Observe that $o \leq e_r+1 < g$ and therefore, (5.5) represents k+1 in the base g.)

We write, $$p = (e_r+1)g^{a'_{r+1}} + \sum_{i=r+1}^{m} e_i g^{a'_{i+1}}$$

Clearly, p is an element of $S\{a_n\}$. Also, it is easily shown that if $s_k < t < p$, then t can not be an element of $S\{a_n\}$ because of the same reasoning described in case (i). Hence,

$$s_{k+1} = p = (e_r+1)\, g^{a'_{r+1}} + \sum_{i=r+1}^{m} e_i g^{a'_{i+1}}$$

with $$k+1 = (e_r+1) g^r + \sum_{i=r+1}^{m} e_i g^i$$

Thus, in this case also, the required relation is true for n=k+1, if assumed it to be true for n=k. This completes the proof.

6. TOWARDS THE PROOF OF MAIN THEOREM:

Theorem 6.1: Let $\{a_n'\}$ and $\{b_n'\}$ be the complementing sequences of the sequences $\{a_n\}$ and $\{b_n\}$ respectively. Further, assume that $a_n \leq b_n$ for all n. Then, b_k' exists, whenever a_k' exists and $b_k' \leq a_k'$.

<u>Proof</u>: Suppose a_k exists. Then, by theorem 4.3,

(6.1) $a_k' = n_o+k-2$,

where n_o is the smallest natural number satisfying,

(6.2) $a_{n_o} \geq n_o+k-1$.

But, then, $b_{n_o} \geq a_{n_o} \geq n_o+k-1$ by (6.2) which shows, in view of theorem 4.2, that b_k' exists. Also,

(6.3) $b_k' = m_o+k-2$,

Where m_o is the smallest natural number for which

(6.4) $b_{m_o} \geq m_o+k-1$.

Therefore, $n_o \geqslant m_o$ and hence

$b'_k = m_o+k-2 \leqslant n_o+k-2 = a'_k$ by (6.1).

This proves the theorem.

Theorem 6.2: For the sequences $\{a_n\}$ and $\{b_n\}$ suppose that $a_n \leqslant b_n$ for all n. Then $H(S\{a_n\}) \leqslant H(S\{b_n\})$.

Proof: Let u_n and v_n denote the n-th elements of $S\{a_n\}$ and $S\{b_n\}$ respectively. If for a fixed natural number n, its representation in the base g is given by $n=\sum_{i=0}^{m} c_i g^i$, $0 \leq c_i \leq g-1$, $c_m \neq 0$, then by theorem 5.1. u_n exists if and only if a'_{m+1} exists. Suppose u_n exists- Then a'_{m+1}. exists. But then b'_{m+1} does exist by theorem 6.1. Now, with the help of the theorem 5.2. U_n and V_n are easily described as follows:

(6.5) $$u_n = \sum_{i=0}^{m} c_i g^{a'_{i+1}} \text{ and } v_n = \sum_{i=0}^{m} c_i g^{b'_{i+1}}$$

Since, $b'_i \leqslant a'_i$ for all i, it follows that $u_n \geqslant v_n$ and consequently by theorem 2.1, $H(S\{a_n\}) \leqslant H(S\{b_n\})$.

Proof of the main theorem: Let $\{a_n\}$ be any sequence for which $a_{n+1} - a_n \leqslant B_1$ for all n, where B_1 is any fixed positive integer. Let $b = \max\{a_1, B_1\}$. Consider b_n, where $b_n = nB$. Clearly $a_1 \leqslant B = b_1$. If $a_n \leqslant b_n$, then $a_{n+1} \leqslant a_n + B_1 \leqslant n \times B + B = (n+1)B = b_{n+1}$. Thus $a_n \leqslant b_n$ for all n. Hence $H(S\{a_n\}) \leqslant H(S\{b_n\})$. But the last expression here is finite by theorem 3.1 and the proof is completed.

A BRIEF SUMMARY OF SOME RESULTS IN THE ANALYTIC THEORY OF NUMBERS

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1. INTRODUCTION : I wish to speak about some results by me and some joint work with my colleague Mr. R. BALASUBRAMANIAN. The results are

1) A proof that $\frac{1}{T}\int_{T}^{2T}|\zeta(\frac{1}{2}+it)|^4 dt = O((\log T)^4)$, $T \geq 2$.

This result is not new. It is due to G.H. HARDY and J.E. LITTLEWOOD. But the method is new and simplifies the earlier (complicated) but very useful results on the hybrid fourth power moments of L-series due to H.L.MONTGOMERY. The new method turned out to be useful in density theorems. The method is very simple but sufficiently important. For a description of the method see (K. RAMACHANDRA, A simple proof of the mean fourth power estimate for $\zeta(\frac{1}{2}+it)$ and $L(\frac{1}{2}+it,\chi)$, Annali della Scoula Normale Superiore di Pisa, Classe di sci, Section IV, Vol.I (1974), 81 - 97).

2) Let k_1, k_2 be integers subject to $3 \leq k_1 \leq k_2$. Let χ_1 and χ_2 be two non-principal real characters mod k_1 and mod k_2 respectively such that for at least one integer $n (\geq 2)$, $\chi_1(n) \times \chi_2(n) = -1$. Put $L_1 = L(1,\chi_1)$ and $L_2 = L(1,\chi_2)$. Then $L_1 < \frac{A}{\log k_1}$ implies $L_2 > \frac{B}{\log k_2}\{(\log k_1)^{-2} k_2^{-cL_1}\}$,

where A, B, C are effective positive constants independent of k_1, k_2, χ_1 and χ_2. The proof of this new result uses only very simple facts from the theory of functions of a real variable. See my paper (K. RAMACHANDRA, one more proof of Siegel's theorem, HARDY - RAMANUJAN JOURNAL, Vol. 3 (1980))

3) Next I state a result due to myself and R. BALASUBRAMANIAN. (Ref. R. BALASUBRAMANIAN AND K. RAMACHANDRA, some problems of Analytic number theory III, HARDY - RAMANUJAN JOURNAL, Vol. 4 (1981), to appear). Let $F_0(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ be a series which converges at least for one complex number $s = \sigma + it$, (b_n being complex numbers) and hence in a half plane. Let $k \geq 2$ be an integer constant and A a constant $\geq 2k$. put $\alpha = \frac{1}{2} - \frac{1}{2k}$ and suppose that in ($\sigma \geq \frac{1}{2} - \frac{1}{A}$, $T \leq t \leq 2T$), $F_0(s)$ admits an analytic continuation and there M defined by $M = \max |F_0(s)|$ does not exceed Exp (T^B) Where B is a positive constant and $T \geq 10$. Put $F(s) = (F_0(s))^k$ $= \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ where $F_0(s)$ is absolutely convergent (note that if $F_0(s)$ is convergent at s it is absolutely convergent at s + 2). Let H(W) be an analytic on a curve L consisting of finitely many fixed straight lines all contained in ($\mathrm{Re}\, W \leq 1+\alpha$, $|\mathrm{Im}\, W| \leq B$) and we assume that $\frac{H(W)}{W}$ is analytic on L. Put $D(u) = \frac{1}{2\pi i} \int_L \frac{H(W)}{W} u^W dW$, Where $u > 0$. Next put $E(u) = \sum_{n \leq u} a_n - D(u)$ where $u > 0$.

Let $\alpha_1, \alpha_2, \alpha_3, \theta$ and C be positive constants subject to $\alpha \leq \frac{1}{2} - \frac{1}{A} < \alpha_1 < \alpha_2 < \alpha_3 < \frac{1}{2}$; $\theta < 1 - \frac{1}{k}$, and further assume that $\max\limits_{n \leq \frac{T}{100}} |b_n| > T^{-C}$ and $\max\limits_{w \text{ on } L} \left|\frac{H(w)}{w}\right| \leq C.$

Then there exists an effective constant $T_0 = T_0(k, A, B, C, \alpha_1, \alpha_2, \alpha_3)$ such that for all $T \geqslant T_0$ there holds

$$\max_{T \leq U \leq (M+T)^{T_0}} \int_U^{2U} \left|\frac{E(u)}{u^\alpha}\right|^2 \frac{du}{u}$$

$$> (T_0 \log T)^{-1} (V(\alpha))^k \left\{\frac{V(\alpha_1)}{V(\alpha_3)}\right\}^{\frac{\alpha_2 - \alpha}{\alpha_3 - \alpha_2}}$$

where $V(\sigma) = \frac{1}{T} \sum\limits_{n \leq T/100} |b_n|^2 \left(\frac{T}{n}\right)^{2\sigma}$ provided only that $\left(\sum\limits_{n \leq T} n^{-1} |b_n|^2\right)^k$ does not exceed T^θ times the right hand side of our main inequality.

<u>COROLLARY</u> $$\sum_{n=1}^{N} d(n) = N \log N + (2\gamma - 1)N + \Omega\left(N^{\frac{1}{4}} (\log N)^{-\frac{1}{2}}\right),$$

where $d(n)$ is defined by $\sum\limits_{n=1}^{\infty} \frac{d(n)}{n^s} = (\zeta(s))^2$, $(\operatorname{Re} s > 1)$ and γ is the Euler's constant.

The paper contains some other results as well. For Example $\Omega(N^{1/6})$ for the error term in the abelian group problem and $\Omega(N^{1/4}(\log N)^{-1/2})$ for the error term in the lattice point problem for a circle. There are $\Omega_\pm$ results also; but the exponents here fall short of optimal exponents.

In the case of Ω results the exponents are optimal but the log factors are not the best known. The theorem stated above is of interest because of its generality. For the history of the problem see the introduction to the joint paper cited above.

2. SOME REMARKS ON THE MEAN FOURTH POWER OF $\zeta(\frac{1}{2}+it)$

In a subsequent paper (J. London, Math. Soc., (1975))
I have shown that my method works even to prove

$$\frac{1}{T}\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^4 dt = \frac{1}{2\pi^2}(\log T)^4 + O((\log T)^3).$$

This is an old result of A.E. INGHAM and his proof was complicated. However these results are superseded by a result due to D.R. HEATH-BROWN which states that R H S can be replaced by $\frac{1}{2\pi^2}P(\log T)$ (where $P(x)$) is a monic polynomial of degree 4) plus an error which is $O(T^{(\varepsilon-\frac{1}{8})})$ for every fixed $\varepsilon > 0$. HEATH - BROWN'S result is very deep. My proof of the result quoted above goes through with very little modifications to prove a hybrid analogue for fourth power moments of L-series (For these and many other results see a paper by R. BALASUBRAMANIAN to appear). To give an idea of these results I give in a few lines a proof of

$$\frac{1}{T}\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^4 dt = O(T^{\varepsilon}) \qquad \text{---}(1)$$

valid for every fixed $\varepsilon > 0$. Put $s = \frac{1}{2}+it$, $T \leq t \leq 2T$, $0 < \varepsilon < \frac{1}{100}$; $X = T^{1+\varepsilon}$ and define $d(n)$ by

$\sum_{n=1}^{\infty} \frac{d(n)}{n^2} = (\zeta(z))^2$ where $\mathrm{Re}\, Z \geq 2$. Then, we have,

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} e^{-\frac{n}{x}} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta^2(s+w) x^w \Gamma(w)\, dw$$

$$= \zeta^2(s) + \frac{1}{2\pi i} \int_{|s+w-1|=\frac{1}{100}} \zeta^2(s+w) x^w \Gamma(w)\, dw + \frac{1}{2\pi i} \int_{\varepsilon-1-i\infty}^{\varepsilon-1+i\infty} \zeta^2(s+w) x^w \Gamma(w)\, dw$$

Hence (writing $\zeta^2(s+w) = \chi^2(s+w)\zeta^2(1-s-w)$,

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} e^{-\frac{n}{x}} - \frac{1}{2\pi i} \int_{|s+w-1|=\frac{1}{100}} \zeta^2(s+w) x^w \Gamma(w)\, dw - \frac{1}{2\pi i}(Q_1+Q_2),$$

where

$$Q_1 = \frac{1}{2\pi i} \int_{\varepsilon-1-i\infty}^{\varepsilon-1+i\infty} \chi^2(s+w) \Big(\sum_{n \geq T} \frac{d(n)}{n^{1-s-w}} \Big) x^w \Gamma(w)\, dw$$

and

$$Q_2 = \frac{1}{2\pi i} \int_{\varepsilon-1-i\infty}^{\varepsilon-1+i\infty} \chi^2(s+w) \Big(\sum_{n \leq T} \frac{d(n)}{n^{1-s-w}} \Big) x^w \Gamma(w)\, dw .$$

In Q_2 we move the line of integration to $\mathrm{Re}\, W = -\varepsilon$.
Using the asymptotics of the gamma functions, the functional equation gives

$$\chi^2(s+w) = O\Big((T + |\mathrm{Im}\, W|)^{\frac{1}{2} - \mathrm{Re}(s+w)} \Big)$$

where $\mathrm{Re}\, W$ is any fixed constant. Using this for $\mathrm{Re}\, W = \varepsilon - 1$, and $\mathrm{Re}\, W = -\varepsilon$ the required result (1) follows on using the trivial result

$$\frac{1}{T}\int_{T}^{2T}\Big|\sum_{Y\leq n\leq 2Y} a_n n^{it}\Big|^2 dt = O\Big(\sum_{Y\leq n\leq 2Y}|a_n|^2\left(\frac{Y}{T}\right)(YT)^{\delta}\Big)$$

where in one complex and δ is an arbitrary constant subject to $0 < \delta < 1$.

ON CERTAIN SUMS INVOLVING THE MAXIMAL K -FREE DIVISOR FUNCTION *

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§1. INTRODUCTION

Throughout this paper, m denotes a positive integer, p denotes a prime and x denotes a real number $\geqslant 1$. Let k denote a fixed integer $\geqslant 2$. As usual, m is called k-free if it is not divisible by the k -th power of any prime. Let $\gamma_k(m)$ denote the maximal k - free divisor of m. Let $\gamma(1) = 1$ and if $m > 1$, $\gamma(m)$ be the product of distinct prime factors of m. It is clear that $\gamma_2(m) = \gamma(m)$.

It is well-known that m is called squarefull if $p \mid m$ implies $p^2 \mid m$. Let $\ell(m)$ denote the characteristic function of the square-full integers; that is, $\ell(m) = 1$ or 0 according as m is square-full or not.

In this paper, we establish (See § 5) an asymptotic formula for $\sum_{m \leq x} \gamma_k(m)\ell(m)$ with an error term $E_k(x)$,

where $E_k(x) = O(x^{\frac{2k-1}{4}}\delta(x))$ or $O(x^{\frac{5}{4}}\delta(x))$ or $O(x^{\frac{4}{3}})$,

according as k is even or $k = 3$ or k is odd $\geqslant 5$ respectively,

where $\delta(x) = \exp\{-H \log^{3/5} 2x (\log\log 3x)^{-1/5}\}$,

H being a positive constant. On the assumption of the Riemann hypothesis, we prove that (See § 5)

$E_k(x) = O(x^{\frac{5k-3}{10}} W(x))$, whenever $k = 3$ or k is even,

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*Presented by : V. Sita Ramaiah

where $W(x) = \exp\{ H \log 2x (\log\log 3x)^{-1}\}$ In the case k=2, an asymptotic formula for the above sum has been established by R. Sita Rama Chandra Rao (4) with a weaker O - estimate of the error term Viz., $O(x^{3/4})$.

In §§ 2,3 and 4, we establish asymptotic formulae for the sums $\sum_{\substack{m \leq x \\ (m,n)=1}} (\gamma_k(m))^u$, $\sum_{\substack{m \leq x \\ (m,n)=1}} \gamma_k(m^2)$ and $\sum_{m \leq x} \gamma_k(m^2 t^3)$, with uniform order estimates of the error terms, where u is a fixed real number $\geqslant 1$, n denotes a fixed positive integer and t denotes a square-free integer. These formulae are required to establish the main results of this paper.

§ 2. An asymptotic formula for $\sum_{\substack{m \leq x \\ (m,n)=1}} (\gamma_k(m))^u$.

Throughout this section, u denotes a fixed real number $\geqslant 1$ and n denotes a fixed integer $\geqslant 1$. For $s > 0$, let

$$(2.1) \quad J_s(m) = m^s \sum_{d|m} \mu(d)\, d^{-s} = m^s \prod_{p|m} (1 - p^{-s}),$$

where μ is the Mobius function.

Clearly $J_1(m) = \varphi(m)$, the Euler tolient function. Also, it is clear that for $s > 1$,

$$(2.2) \quad \frac{m^s}{J_s(m)} = O(1).$$

(2.3) Lemma. We have

(2.4) $$J_s(m) = m^s \sum_{\substack{d\delta = m \\ (d,\delta)=1}} \frac{\mu^*(d)}{(\gamma(d))^s},$$

where $\mu^*(m) = (-1)^{\omega(m)}$, $\omega(m)$ being the number of distinct prime factors of m and (d,δ), as usual, the greatest common divisor of d and δ.

Proof. A proof of this lemma in case s=1 has been established by D. Suryanarayana and R. Sita Rama Chandra Rao (cf. (8), Lemma 2.12). The same proof works for any s.

Let q_k denote the characteristic function of the k - free integers. Then we have

(2.5) Lemma. For $0 \leq s < 1/k$,

(2.6) $$\sum_{\substack{m \leq x \\ (m,n)=1}} q_k(m) = \frac{x n^{k-1} \phi(n)}{\zeta(k) J_k(n)} + \Delta_k(x;n),$$

where

$$\Delta_k(x;n) = O\left(x^{\frac{1}{k}} \delta(x) \sigma^*_{-s}(n)\right),$$

uniformly,

(2.7) $$\sigma^*_{-s}(n) = \sum_{d|n} d^{-s} \mu^2(d),$$

$\zeta(s)$ being the Riemann Zeta function, and

(2.8) $$\delta(x) = \exp\left\{-H \log^{3/5} 2x \cdot (\log\log 3x)^{-\frac{1}{5}}\right\},$$

H being a positive constant.

(2.9) Lemma. If the Riemann hypothesis (R.H) is true, then the error term in the asymptotic formula (2.6) is given by

$$\Delta_k(x;n) = O\left(x^{\frac{2}{1+2k}} W(x)\,\theta(n)\right),$$

Where

(2.10) $\theta(n) = \sigma^*_{-0}(n) = 2^{\omega(n)}$,

and

(2.11) $W(x) = \exp\{H \log 2x \cdot (\log\log 3x)^{-1}\}$,

H being a positive constant.

(2.12) Remark. Formula (2.6) has been established by D. Suryanarayana and R. Sita Rama Chandra Rao (cf. (6), Theorem 3.1) with $\Delta_k(x;n) = O\left(x^{\frac{1}{k}}\delta(x)\sigma^*_{-s}(n)\varphi(n)n^{-1}\right)$, which, of course, implies Lemma 2.5. However, the term $\varphi(n)n^{-1}$ appearing in the order estimate of $\Delta_k(x;n)$ above is due to a lemma (cf. (6), Lemma 2.2) in the proof of which the authors (6) made use of the erroneous result viz.,

$\varphi(x,n) = O(x\varphi(n)n^{-1})$ for all $x \geq 1$ and $n \geq 1$,

Where $\varphi(x,n)$ is the number of positive integers $\leq x$ and prime to n. Thus from the methods of (6) what one gets actually is that $\Delta_k(x;n) = O\left(x^{\frac{1}{k}}\delta(x)\sigma^*_{-s}(n)\right)$.

Also, on the assumption of the Riemann hypothesis, the same authors (cf (6), Theorem 3.1, s = o) have established that $\Delta_k(x;n) = O\left(x^{2/(1+2k)}W(x)\theta(n)\varphi(n)n^{-1}\right)$.

The above mentioned remark applies here also, and we get only that $\Delta_k(x;n) = O(x^{\frac{2}{1+2k}} W(x)\theta(n))$.

(2.13) Remark. It is clear that for each $\varepsilon > 0$, $x^{\varepsilon}\delta(x)$ is increasing for large x. Using this it can be shown that

$$y^{\varepsilon}\delta(y) \leq a_1 x^{\varepsilon}\delta(x),$$

for all y, $1 \leq y \leq x$, where a_1 is a positive constant depending only on ε and the constant H in (2.8)

(2.14) Remark. It is clear that $W(x)$ is increasing for large x. Using this it can be shown that if $x \geq 1$, then

$$W(x) \leq a_2 W(y)$$

for all $y \geq x$, where a_2 is an absolute positive constant.

The following two lemmas (lemmas 2.15 and 2.17) are immediate consequences of lemmas 2.5, 2.9 and partial summation.

(2.15) Lemma. For $0 \leq \delta < 1/k$, we have

$$(2.16) \quad \sum_{\substack{m \leq x \\ (m,n)=1}} m^u q_k(m) = \frac{x^{u+1} n^{k-1}\phi(n)}{(u+1)\zeta(k) J_k(n)} + O\left(x^{u+1/k}\delta(x)\sigma^*_{-\delta}(n)\right),$$

where the O -constant depends only on k, s and u.

(2.17) Lemma. If the R.H is true, then the error term in the asymptotic formula (2.16) is given by

$$O\left(x^{u+\frac{2}{1+2k}} W(x)\theta(n)\right)$$

the O - constant being dependent only on k and u.

(2.18) Lemma. $\sum_{m=1}^{\infty} \frac{1}{m^a \gamma(m)}$ converges for every $a > 0$

Proof. Since the product $\prod_p \left\{1 + \sum_{j=1}^{\infty} \frac{1}{p^{ja}\gamma(p^j)}\right\}$

$= \prod_p \left\{1 + \frac{1}{p(p^a - 1)}\right\}$ converges for $a > 0$, the lemma follows from Theorem 41 of (2)

(2.19) Lemma. We have

$$(2.20) \quad \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu^*(m)\,\varphi(m)\,q_k(m)\,m^{k-2}}{(\gamma(m))^u J_k(m)} = \frac{\beta_{k,u}}{\beta_{k,u}(n)},$$

Where

$$(2.21) \quad \beta_{k,u} = \prod_p \left\{1 - \frac{(p^{k-1} - 1)}{p^u (p^k - 1)}\right\}$$

and

$$(2.22) \quad \beta_{k,u}(n) = \prod_{p \mid n} \left\{1 - \frac{(p^{k-1} - 1)}{p^u (p^k - 1)}\right\}.$$

Proof. Since

$$\frac{\mu^*(m)\,\varphi(m)\,q_k(m)\,m^{k-2}}{(\gamma(m))^u J_k(m)} = O\left(\frac{1}{m\,\gamma(m)}\right),$$

the series in (2.20) converges absolutely. Further the general term of the series is multiplicative in m. Now the lemma follows on expanding the series as an infinite product of Euler type (Cf. (3), Theorem 286)

(2.23) Lemma. For $0 < a < 1$,

$$\sum_{m>x} \frac{1}{m\gamma(m)} = O(x^{-1+a}).$$

Proof. From lemma 2.18 it follows that $\sum_{m \leq x} \frac{1}{\gamma(m)} = O(x^a)$;

lemma 2.23 follows from this and Partial summation.

(2.24) Lemma. For $0 < a < 1$, we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \frac{\mu^*(m)\varphi(m)q_k(m)\, m^{k-2}}{(\gamma(m))^u J_k(m)} = \frac{\beta_{k,u}}{\beta_{k,u}(n)} + O(x^{-1+a}).$$

Proof. Follows from lemmas 2.19 and 2.23

(2.25) Lemma. For $0 \leq s < 1/k$, we have

$$(2.26) \quad \sum_{\substack{m \leq x \\ (m,n)=1}} J_u(m)q_k(m) = \frac{x^{u+1} n^{k-1} \varphi(n)\beta_{k,u}}{(u+1)\zeta(k)J_k(n)\beta_{k,u}(n)} + O\left(x^{u+1/k}\,\delta(x)\,\sigma^*_{-s}(n)\right),$$

Where the O-constant depends only on k, s and u.

Proof. By lemmas 2.3, 2.15 and 2.19, we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} J_u(m)q_k(m) = \sum_{\substack{d \leq x \\ (d,n)=1}} \frac{d^u \mu^*(d)q_k(d)}{(\gamma(d))^u} \sum_{\substack{\delta \leq x/d \\ (\delta,dn)=1}} \delta^u q_k(\delta)$$

$$= \frac{x^{u+1} n^{k-1} \varphi(n)}{(u+1)\zeta(k) J_k(n)} \sum_{\substack{d \le x \\ (d,n)=1}} \frac{\mu^*(d)\varphi(d) q_k(d) d^{k-2}}{(\gamma(d))^{u} J_k(d)}$$

(2.27)

$$+ O\left(x^{u+\frac{1}{k}} \sigma^*_{-s}(n) \sum_{d \le x} \frac{\delta(x/d)\, \sigma^*_{-s}(d)}{d^{1/k} \gamma(d)} \right).$$

Since $\sigma^*_{-s}(d) = O(d^{\epsilon})$ for every $\epsilon > 0$, if $0 < \epsilon < 1/k$,

We have by Remark 2.13 and lemma 2.18,

$$(2.28) \sum_{d \le x} \frac{\delta(x/d)\, \sigma^*_{-s}(d)}{d^{1/k} \gamma(d)} = O\left(\delta(x) \sum_{d \le x} \frac{1}{d^{1/k - \epsilon} \gamma(d)} \right)$$

$$= O(\delta(x)).$$

Now the lemma follows from (2.28), (2.27) and lemma 2.24.

(2.29) <u>Lemma.</u> If the R.H. is true, then the error term in the asymptotic formula (2.26) is given by

$$O\left(x^{u+\frac{2}{1+2k}} W(x)\, \theta(n) \right),$$

the O-estimate being uniform in x and n.

<u>Proof.</u> Following the same procedure adopted in the proof of lemma 2.25, making use of lemma 2.17 and Remark 2.14 instead of lemma 2.15 and Remark 2.13, we get this lemma.

(2.30) <u>Remark.</u> Lemmas 2.25 and 2.29 have been established by V. Sita Ramaiah (5) in a complicated way. Formula (2.26) in case $u = 1$ has been established by D. Suryanarayana and P. Subrahmanyam (Cf. (9), Theorem 4.1) with an error term

$$O\left(\theta(n)\, x^{1+\frac{1}{k}}\, \delta(x)\right) + O\left(\theta^2(n)\, x\, \lambda(x/n)\right),$$

where $\lambda(x) = \log^{2/3} x\, (\log\log x)^{4/3}$. These two O - terms can be combined into a single O-term viz., $O(\theta^2(n)\, x^{1+\frac{1}{k}}\, \delta(x))$.

This O-estimate is clearly weaker than the one we obtain from (2.26) ($u = 1$). However, using their method (9), one can establish formula (2.26) ($u=1$) with error term $O(\theta(n)\, x^{1+1/k}\, \delta(x))$. But this O-estimate of the error term is also weaker than the one given in (2.26) ($u=1$), since $\sigma^{*}_{-s}(n) \leq \theta(n)$. Further, on the assumption of the Riemann hypothesis, D. Suryanarayana and P. Subrahmanyam (Cf. (9), Theorem 4.2) have established (2.26) ($u=1$) with an error term

$$O\left(\theta(n)\, x^{1+2/(1+2k)}\, W(x)\right) + O\left(\theta^2(n)\, x\, \lambda(x)\right).$$

These two O-terms can be again combined into a single O-term viz., $O\left(\theta^2(n)\, x^{1+2/(1+2k)}\, W(x)\right)$, which is weaker than the one we obtain from lemma 2.29 ($u=1$). However using their method (9), assuming the Riemann hypothesis, one can establish (2.26) ($u=1$) with an error term $O\left(\theta(n)\, x^{1+2/(1+2k)}\, W(x)\right)$.

This is same as the O-term of lemma 2.29 (u=1). Finally we may mention here that the methods of D. Suryanarayana and P. Subrahmanyam (9) adopted in establishing formula (2.26) (u=1) are complicated than ours.

(2.31) <u>Corollary</u> (Lemma 2.25, u=1, k=2). For $0 \leq s < 1/2$, we have

(2.32) $$\sum_{\substack{m \leq x \\ (m,n)=1}} \mu^2(m)\varphi(m) = \frac{x^2 n \varphi(n) d}{2\zeta(2) J_2(n)\beta(n)} + O\left(x^{\frac{3}{2}} \delta(x) \sigma^*_{-s}(n)\right)$$

Uniformly, where

(2.33) $$\alpha \equiv \prod_p \left\{1 - \frac{1}{p(p+1)}\right\},$$

and

(2.34) $$\beta(n) \equiv \prod_{p|n} \left\{1 - \frac{1}{p(p+1)}\right\}.$$

(2.35) <u>Corollary.</u> (Lemma 2.29, u=1, k=2). If the R.H. is true, then the error term in the asymptotic formula (2.32) is given by

$$O\left(x^{7/5} W(x)\theta(n)\right)$$

(2.36) <u>Remark.</u> Formulla (2. 32) (n=1) was originally established by S. Wigert (10) with an error term $O(x^{7/4})$, using analytic methods. This O-estimate was improved to $O(x^{3/2})$ by E. Cohen (Cf. (1), corollary 5. 1.2), using elementary methods.

(2.37) **Theorem.** For $0 \leq s < 1/k$, we have

(2.38) $$\sum_{\substack{m \leq x \\ (m,n)=1}} (\gamma_k(m))^u = \frac{x^{u+1}\zeta(u+1)\beta_{k,u}\varphi(n)J_{u+1}(n)n^{k-u-2}}{(u+1)\zeta(k)J_k(n)\beta_{k,u}(n)} + O(x^{u+1/k}\delta(x)\sigma^*_{-s}(n)),$$

where the O-estimate is uniform in x and n, $\beta_{k,u}$ and $\beta_{k,u}(n)$ are as given in (2.21) and (2.22)

Proof. Since $\sum_{d|m} J_u(d) = m^u$, we have

(2.39) $$(\gamma_k(m))^u = \sum_{d|\gamma_k(m)} J_u(d) = \sum_{d|m} J_u(d)\, q_k(d).$$

Now by (2.39), lemma 2.25 and Remark 2.13, we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} (\gamma_k(m))^u = \sum_{\substack{t \leq x \\ (t,n)=1}}' \sum_{\substack{d \leq x/t \\ (d,n)=1}} J_u(d)\, q_k(d)$$

$$= \frac{x^{u+1}\beta_{k,u}\varphi(n)n^{k-1}}{(u+1)\zeta(k)\beta_{k,u}(n)J_k(n)} \sum_{\substack{t \leq x \\ (t,n)=1}}' \frac{1}{t^{u+1}}$$

$$+ O\Big(x^{u+1/k}\delta(x)\sigma^*_{-s}(n)\sum_{t \leq x} 1/(t^{u+1/k-\varepsilon})\Big)$$

where $0 < \varepsilon < 1/k$. Since $u + \frac{1}{k} - \varepsilon > 1$ and

$$\sum_{t\le x,\,(t,n)=1} 1/t^{u+1} = \frac{\zeta(u+1)\,J_{u+1}(n)}{n^{u+1}} + O(x^{-u}),$$

We obtain Theorem 2.37, after using (2.2) and the fact

$$1/\beta_{k,u}(n) = O(1)\;.$$

(2.40) Theorem. If the R.H. is true, then the error term in the asymptotic formula (2.38) is given by

$$O(x^{u+2/(1+2k)}\,W(x)\,\theta(n))$$

Proof. Following the same procedure adopted in the proof of Theorem 2.37, making use of lemma 2.29 and Remark 2.14 instead of lemma 2.25 and Remark 2.13, we obtain this theorem.

(2.41) Remark. Taking $n = u = 1$ in Theorems 2.37 and 2.40, we obtain results of D. Suryanarayana and P. Subrahmanyam (cf. (9), Corollaries 4.3.1 and 4.4.1).

(2.42) Corollary (Theorem 2.37, u= 1, k =2). For $0 \le \delta < \frac{1}{2}$

(2.43) $$\sum_{\substack{m\le x\\(m,n)=1}} \gamma(m) = \frac{x^2\alpha\,\varphi(n)n^{-1}}{2\beta(n)} + O\left(x^{3/2}\delta(x)\sigma_{-\delta}^{*}(n)\right),$$

where α and $\beta(n)$ are given by (2.33) and (2.34).

(2.44) Corollary (Theorem 2.40, u=1, k=2). If the R.H. is true, then the error term in the asymptotic formula (2.43) is given by $O\left(x^{7/5}W(x)\theta(n)\right)$.

§ 3. An asymptotic formula for $\sum_{\substack{m\le x\\(m,n)=1}} \gamma_k(m^2)$.

First we have

(3.1) Theorem. Let k be odd. Then for $0 \leq s < 2/(k+1)$

$$(3.2)\quad \sum_{\substack{m\leq x\\(m,n)=1}} \gamma_k(m^2) = \frac{x^3\zeta(3)\,\beta_k\,\varphi(n)\,J_3(n)\,n^{\frac{k+1}{2}-4}}{3\zeta((k+1)/2)\,J_{\frac{k+1}{2}}(n)\,\beta_k(n)} + O\left(x^{2+2/(k+1)}\,\delta(x)\,\sigma^*_{-\delta}(n)\right),$$

Where

$$(3.3)\quad \beta_k \equiv \prod_p \left\{1-\frac{(p^{\frac{k-1}{2}}-1)}{p^2(p^{\frac{k+1}{2}}-1)}\right\}$$

and

$$(3.4)\quad \beta_k(n) \equiv \prod_{p\mid n}\left\{1-\frac{(p^{\frac{k-1}{2}}-1)}{p^2(p^{\frac{k+1}{2}}-1)}\right\}.$$

Proof. It is clear that for $j\geq 1$,

$$(3.5)\quad \gamma_k(p^j) = \begin{cases} p^j, & \text{if } j<k, \\ p^{k-1} & \text{if } j\geq k. \end{cases}$$

Further

$$(3.6)\quad \gamma_k(m^2) = \gamma^2_{\frac{k+1}{2}}(m).$$

For, it is clear that both sides of (3.6) are multiplicative in m. Hence it is enough if we verify (3.6) when $m = p^j$, a prime power. But this can be easily done using (3.5). Now, Theorem 3.1 follows from (3.6) and Theorem 2.37 with k replaced by $\frac{k+1}{2}$ and $u = 2$.

(3.7) Theorem. If the R.H. is true, then the error term in the asymptotic formula (3.2) is given by $O(x^{2+2/(k+2)}W(x)\theta(n))$.

Proof. Follows from (3.6) and Theorem 2.40 with k replaced by $\frac{k+1}{2}$ and u =2

Throughout the following we suppose that k is even and k ≥ 4. The following two lemmas (lemmas 3.8 and 3.10) are immediate consequences of corollaries 2.31, 2.35 and partial summation.

(3.8) Lemma. For $0 \leq s < 1/2$, we have

(3.9) $$\sum_{\substack{m \leq x \\ (m,n)=1}} m^{k-2} \mu^2(m)\varphi(m) = \frac{x^k \alpha n \varphi(n)}{k\zeta(2) J_2(n)\beta(n)} + O(x^{k-1/2}\delta(x)\sigma^*_{-s}(n)).$$

(3.10) Lemma. If the R.H is true, then the error term in the asymptotic formula in (3.9) is given by

$$O(x^{k-3/5} W(x)\theta(n)).$$

We write

(3.11) $$\gamma_k(m^2) = \sum_{d|m} f_k(d),$$

is that by the Mobius inversion formula (cf, (3), Theorem 267) we have

$$f_k(m) = \sum_{d|m} \mu(d)\gamma_k(m^2/d^2).$$

Hence by (3.5) we have for $j \geq 1$,

(3.12) $$f_k(p^j) = \gamma_k(p^{2j}) - \gamma_k(p^{2j-2})$$
$$= \begin{cases} J_2(p^j), & \text{if } j < k/2, \\ p^{k-2}\varphi(p), & \text{if } j = k/2, \\ 0, & \text{if } j > k/2. \end{cases}$$

(3.13) __Lemma.__ For $0 \leq s < 1/2$, we have

(3.14) $$\sum_{\substack{m \leq x \\ (m,n)=1}} f_k(m) = \frac{x^k \alpha A_k \varphi(n) n}{k \zeta(2) J_2(n) \beta(n) A_k(n)} + O\left(x^{k-1/2} \delta(x) \sigma^*_{-s}(n)\right),$$

where

(3.15) $$A_k \equiv \prod_p \left\{1 + \frac{(p^2-1)(p^{\frac{k(k-1)}{2}}-1)}{p^{\frac{k(k-1)}{2}}(p^2+p-1)}\right\},$$

and

(3.16) $$A_k(n) \equiv \prod_{p \mid n} \left\{1 + \frac{(p^2-1)(p^{\frac{k(k-1)}{2}}-1)}{p^{\frac{k(k-1)}{2}}(p^2+p-1)}\right\}.$$

__Proof.__ By (3.12) and lemma 3.8, we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} f_k(m) = \sum_{\substack{m_1, m_2 \leq x \\ (m_1,n)=(m_2,n)=1 \\ (m_1,m_2)=1}} J_2(m_1) q_{k/2}(m_1) m_2^{k-2} \varphi(m_2) \mu^2(m_2)$$

$$= \sum_{\substack{m_1 \leq x \\ (m_1, n) = 1}} J_2(m_1) q_{k/2}(m_1) \sum_{\substack{m_2 \leq x/m_1 \\ (m_2, n m_1) = 1}} m_2^{k-2} \varphi(m_2) \mu^2(m_2)$$

$$= \frac{x^k \alpha \varphi(n) n}{k \zeta(2) J_2(n) \beta(n)} \sum_{\substack{m_1 \leq x \\ (m_1, n) = 1}} \frac{q_{k/2}(m_1) \varphi(m_1)}{m_1^{k-1} \beta(m_1)}$$

(3.17)

$$+ O\left(x^{k-1/2} \sigma^{*}_{-8}(n) \sum_{m_1 \leq x} \frac{J(m_1) \delta(x/m_1) \sigma^{*}_{-8}(m_1)}{m_1^{k-1/2}}\right)$$

It is not difficult to show that

(3.18) $$\sum_{\substack{m_1 \leq x \\ (m_1, n) = 1}} \frac{q_{k/2}(m_1) \varphi(m_1)}{m_1^{k-1} \beta(m_1)} + \frac{A_k}{A_k(n)} + O\left(x^{3-k}\right).$$

Also, by Remark 2.13 and the fact $k \geq 4$,

we have for $0 < \varepsilon < 1/2$,

(3.19) $$\sum_{m_1 \leq x} \frac{J_2(m_1) \delta(x/m_1) \sigma^{*}_{-8}(m_1)}{m_1^{k-1/2}} \leq \sum_{m_1 \leq x} \frac{m_1^2 \delta(x/m_1) \sigma^{*}_{-8}(m_1)}{m_1^{7/2}}$$

$$= O\left(\delta(x) \sum_{m_1 \leq x} \frac{1}{m_1^{3/2 - \epsilon}}\right) = O(\delta(x)).$$

Now the lemma follows from (3.18), (3.19) and (3.17)

(3.20) Lemma. If the R.H. is true, then the error term in the asymptotic formula (3.14) is given by

$$O\left(x^{k-3/5} W(x) \theta(n)\right).$$

Proof. Following the same procedure adopted in the proof of lemma 3.13, making use of lemma 3.10 and Remark 2.14

instead of lemma 3.8 and Remark 2.13, we obtain lemma 3.20.

(3.21) Theorem. (With the notation of lemma 3.13). Let k be even and $k \geq 4$. Then for $0 \leq s < 1/2$, we have

(3.22)
$$\sum_{\substack{m \leq x \\ (m,n)=1}} \gamma_k(m^2) = \frac{x^k \alpha A_k \varphi(n) J_k(n) \zeta(k)}{k \zeta(2) J_2(n) \beta(n) A_k(n) n^{k-1}} + O(x^{k-1/2} \delta(x) \sigma_{-s}^{*}(n)),$$

the O-estimate being uniform in x and n.

Proof. By (3.11) and lemma 3.13, we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} \gamma_k(m^2) = \sum_{\substack{t \leq x \\ (t,n)=1}} \sum_{\substack{d \leq x/t \\ (d,n)=1}} f_k(d)$$

$$= \frac{x^k \alpha A_k \varphi(n) n}{k \zeta(2) J_2(n) \beta(n) A_k(n)} \sum_{\substack{t \leq x \\ (t,n)=1}} \frac{1}{t^k}$$

$$+ O\Big(x^{k-1/2} \sigma_{-s}^{*}(n) \sum_{t \leq x} \delta(x/t)/t^{k-1/2}\Big)$$

$$= \frac{x^k \alpha A_k \varphi(n) J_k(n) \zeta(k)}{k \zeta(2) J_2(n) \beta(n) A_k(n) n^{k-1}} + O(x) + O(x^{k-\frac{1}{2}} \delta(x) \sigma_{-s}^{*}(n)),$$

since $(A_k(n))^{-1} = O(1)$ and $\sum_{t \leq x} \delta(x/t)/t^{k-1/2} = O(\delta(x))$,

the later being a consequence of Remark 2.13.

Hence Theorem 3.21 follows.

(3.23) Theorem. Let k be even and $k \geqslant 4$. If the R.H. is true, then the error term in the asymptotic formula (3.22) is given by

$$O(x^{k-3/5} W(x)\,\theta(n)).$$

Proof. Following the same procedure adopted in the proof of Theorem 3.21, making use of lemma 3.20 and Remark 2.14 instead of lemma 3.13 and Remark 2.13, we obtain this theorem.

(3.24) Remark. For k =2, $\gamma_k(m^2) = \gamma(m^2) = \gamma(m)$.

Hence Corollary 2.42 gives a formula for $\sum\limits_{\substack{m \leq x \\ (m,n)=1}} \gamma_k(m^2)$

when k=2.

§ 4. An asymptotic formula for $\sum\limits_{m \leq x} \gamma_k(m^2 n^3)$.

Throughout this section, n denotes a square-free number. First we have.

(4.1) Lemma. Let g_k be the multiplicative function defined by

(4.2)
$$g_k(p^j) = \begin{cases} 0, \text{ if } p \nmid n, \\ \gamma_k(p^{2j+3})/\gamma_k(p^3), \text{ if } p \mid n, \end{cases}$$

for all $j \geqslant 1$. Then

(4.3)
$$\gamma_k(m^2 n^3) = \gamma_k(n^3) \sum_{\substack{dt=m \\ (d,n)=1}} \gamma_k(d^2)\, g_k(t).$$

Proof. It is clear that $\gamma_k(m^2n^3)/\gamma_k(n^3)$ is a multiplicative function of m. Hence for $s > 2$ we have

$$\sum_{m=1}^{\infty} \frac{\gamma_k(m^2n^3)}{m^s\gamma_k(n^3)} = \prod_p \left\{1 + \frac{\gamma_k(p^2n^3)}{p^s\gamma_k(n^3)} + \frac{\gamma_k(p^4n^3)}{p^{2s}\gamma_k(n^3)} + \cdots\right\}$$

$$= \prod_{p\nmid n}\left\{1 + \frac{\gamma_k(p^2)}{p^s} + \frac{\gamma_k(p^4)}{p^{2s}} + \cdots\right\}\prod_{p|n}\left\{1 + \frac{\gamma_k(p^5)}{p^s\gamma_k(p^3)} + \frac{\gamma_k(p^7)}{p^{2s}\gamma_k(p^3)} + \cdots\right\}$$

$$= \sum_{\substack{m=1\\(m,n)=1}}^{\infty} \frac{\gamma_k(m^2)}{m^s} \sum_{m=1}^{\infty} \frac{g_k(m)}{m^s}.$$

Hence (4.3) follows.

(4.4) Lemma. For $s > 2$,

$$\sum_{m=1}^{\infty} \frac{g_k(m)}{m^s} = O(F_s(n)),$$

where

(4.5) $$F_s(n) \equiv \sum_{d|n} \frac{2^{\omega(d)}}{J_{s-2}(d)} = \prod_{p|n}\left\{1 + \frac{2}{p^{s-2}-1}\right\}.$$

Proof. We have

$$\sum_{m=1}^{\infty} \frac{g_k(m)}{m^s} = \prod_{p|n}\left\{1 + \sum_{j=1}^{\infty} \frac{\gamma_k(p^{2j+3})}{p^{js}\gamma_k(p^3)}\right\}.$$

First suppose that k is odd. For k=3, $\gamma_k(p^3) = \gamma_k(p^{2j+3}) = p^2$, for all $j \geq 1$. Hence for k =3,

$$\sum_{m=1}^{\infty} \frac{g_k(m)}{m^s} = \prod_{p|n} \left\{1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right\} = \prod_{p|n}\left\{1+\frac{1}{p^s-1}\right\} \leq F_s(n).$$

For k=5, $\gamma_k(p^3) = p^3$ and $\gamma_k(p^{2j+3}) = p^4$ for all $j \geq 1$.

Hence for k=5,

$$\sum_{m=1}^{\infty} \frac{g_k(m)}{m^s} = \prod_{p|n}\left\{1+\frac{p}{p^s-1}\right\} \leq F_s(n).$$

Let $k \geq 7$ and $k = 2t+3$, so that $t \geq 2$. Then

$$\sum_{j=1}^{\infty} \frac{\gamma_k(p^{2j+3})}{p^{js}\gamma_k(p^3)} = \frac{1}{p^3}\left\{\sum_{j=1}^{t-1} \frac{p^{2j+3}}{p^{js}} + \sum_{j=t}^{\infty} \frac{p^{2t+2}}{p^{js}}\right\}$$

$$\leq \frac{1}{p^{s-2}-1} + \frac{p}{p^s-1} \leq \frac{2}{p^{s-2}-1}.$$

Hence

$$\sum_{m=1}^{\infty} \frac{g_k(m)}{m^s} \leq \prod_{p|n}\left\{1+\frac{2}{p^{s-2}-1}\right\} = F_s(n).$$

Suppose now that k is even. For k=2 or 4, we have

$$\sum_{m=1}^{\infty} \frac{g_k(m)}{m^s} = \prod_{p|n}\left\{1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right\} \leq F_s(n).$$

Let $k \geq 6$ and $b = \frac{k-4}{2}$ Clearly $2j+3 < k$, if and only if, $j \leq b$. Hence

$$\sum_{j=1}^{\infty} \frac{\gamma_k(p^{2j+3})}{\gamma_k(p^3)\,p^{js}} = \frac{1}{p^3}\left\{\sum_{j=1}^{b} \frac{p^{2j+3}}{p^{js}} + \sum_{j=b+1}^{\infty} \frac{p^{k-1}}{p^{js}}\right\}$$

$$\leq \frac{1}{p^{s-2}-1} + \frac{1}{p^s-1} \leq \frac{2}{p^{s-2}-1}.$$

Consequently,

$$\sum_{m=1}^{\infty} \frac{g_k(m)}{m^s} \leq F_s(n).$$

Hence lemma 4.4 follows.

(4.5)' Lemma. Let $s > 2$. Then for any θ, $2 < \theta < s$, we have

$$\sum_{t > x} g_k(t)/t^s = O(F_\theta(n)\, x^{\theta - s})$$

Proof. Follows from lemma 4.4 and partial Summation.

(4.6) Theorem. (With the notation of Theorem 3.1). Let k be odd and n be square-free. Then for $0 \leq s < 2/(k+1)$, we have

$$(4.7)\quad \sum_{m \leq x} \gamma_k(m^2 n^3) = \frac{x^2 \zeta(3)\, \beta_k \gamma_k(n^3) \varphi(n) J_3(n) n^{\frac{k+1}{2} - 4} G_k(n)}{3\zeta((k+1)/2)\, J_{\frac{k+1}{2}}(n)\, \beta_k(n)} + O\left(x^{2 + \frac{2}{k+1}} \delta(x)\, \sigma^*_{-s}(n) \gamma_k(n^3) F_\theta(n)\right),$$

Where $\theta = 2 + \frac{2}{k+1} - \varepsilon$, $0 < \varepsilon < \frac{2}{k+1}$

and

$$(4.8)\quad G_k(n) = \sum_{m=1}^{\infty} \frac{g_k(m)}{m^s},$$

$F_\theta(n)$ being given by (4.5) with s replaced by θ

Proof. By (4.3) and Theorem 3.1, we have

$$\sum_{m\le x}\gamma_k(m^2n^3) = \gamma_k(n^3)\sum_{t\le x} g_k(t)\sum_{\substack{d\le x/t\\(d,n)=1}}\gamma_k(d^2)$$

$$= \frac{\gamma_k(n^3)x^3\zeta(3)\beta_k J_3(n)\varphi(n)n^{\frac{k+1}{2}-4}}{3\zeta((k+1)/2)J_{\frac{k+1}{2}}(n)\beta_k(n)}\sum_{t\le x}\frac{g_k(t)}{t^3}$$

$$+ O\left(x^{2+\frac{2}{k+1}}\sigma^*_{-\delta}(n)\gamma_k(n^3)\sum_{t\le x}\frac{\delta(x/t)g_k(t)}{t^{2+2/(k+1)}}\right).$$

(4.9)

By lemma 4.5, we have

(4.10) $$\sum_{t\le x}\frac{g_k(t)}{t^3} = G_k(n) + O\left(F_\theta(n)\,x^{\theta-3}\right),$$

Where $G_k(n)$ is given by (4.8). Also, by Remark 2.13 and lemma 4.4,

(4.11) $$\sum_{t\le x}\frac{\delta(x/t)g_k(t)}{t^{2+2/(k+1)}} = O\left(\sum_{t\le x}\frac{\delta(x)g_k(t)}{t^{2+2/(k+1)-\epsilon}}\right)$$
$$= O\left(F_\theta(n)\delta(x)\right).$$

Substituting (4.10) and (4.11) into (4.9), Theorem 4.6 follows.

(4.12) <u>Theorem.</u> Let k be odd. If the R.H. is true, then the error term in the asymptotic formula (4.7) is given by

$$O\left(x^{2+\frac{2}{k+1}}W(x)\theta(n)\gamma_k(n^3)F_c(n)\right),$$

where $c = 2+\dfrac{2}{k+2}$

Proof. Following the same procedure adopted in the proof of Theorem 4.6, making use of Theorem 3.17 and Remark 2.14 instead of Theorem 3.1 and Remark 2.13.

(4.13) Theorem (With the notation of lemma 3.13). Let k be even, $k \geqslant 4$ and n be square-free. Then for $0 \leqq s < 1/2$, we have

$$(4.14)\quad \sum_{m \leq x} \gamma_k(m^2 n^3) = \frac{x^k \alpha\, \zeta(k) A_k \varphi(n) J_k(n) G_k(n)}{k\, \zeta(2) J_2(n) \beta(n) A_k(n) n^{k-3}}$$

$$+ O(x^{k-\frac{1}{2}} \delta(x) n^2 \sigma^*_{-8}(n) F_a(n)),$$

where $\quad a = k - \frac{1}{2} - \varepsilon, \quad 0 < \varepsilon < 5/2$

and

$$(4.15)\quad G_k'(n) = \sum_{m=1}^{\infty} \frac{g_k(m)}{m^k}.$$

Proof. By (4.3) and Theorem 3.21, we have

$$\sum_{m \leq x} \gamma_k(m^2 n^3) = \gamma_k(n^3) \sum_{t \leq x} g_k(t) \sum_{\substack{d \leq x/t \\ (d,n)=1}} \gamma_k(d^2)$$

$$(4.16)\quad = \frac{x^k \alpha A_k \zeta(k) \gamma_k(n^3) \varphi(n) J_k(n)}{k \zeta(2) J_2(n) \beta(n) A_k(n) n^{k-1}} \sum_{t \leq x} \frac{g_k(t)}{t^k}$$

$$+ O\left(x^{k-1/2} \gamma_k(n^3) \sigma^*_{-8}(n) \sum_{t \leq x} \frac{g_k(t)\, \delta(x/t)}{t^{k-1/2}}\right)$$

By lemma 4.5 (with $\theta = a$) we have

$$(4.17)\quad \sum_{t \leq x} \frac{g_k(t)}{t^k} = G_k'(n) + O\left(F_a(n) x^{a-k}\right),$$

Where $G'_k(n)$ is given by (4.15). It follows from Remark 2.13 and lemma 4.4 (with θ =a) the inner sum in the O-term of (4.16) is O ($\delta(x)F_a(n)$); Theorem 4.13 follows from this, (4.17), (4.16) and the fact that for square-free n and $k \geqslant 4$, $\gamma_k(n^3) = n^2$.

(4.18) Theorem. Let k be even and $k \geqslant 4$. If the R.H. is true, then the error term in the asymptotic formula (4.14) is given by

$$O\left(x^{k-3/5} W(x) n^2 \theta(n) F_b(n)\right),$$

where $b = k - 3/5$.

Proof. Following the same procedure adopted in the proof of Theorem 4.13, making use of theorem 3.23 and Remark 2.14 instead of Theorem 3.21 and Remark 2.13, we get this theorem.

(4.19) Theorem. If n is square-free and $0 \leq s < 1/2$, then

(4.20) $$\sum_{m\leq x} \gamma(mn) = \frac{\alpha x^2 \varphi(n) n^2}{2\beta(n) J_2(n)} + O\left(x^{3/2} \delta(x) n \sigma^*_{-s}(n)\right).$$

Proof. For Square-free n we have $\gamma(n) = n$ and $\gamma(m^2n^3) = \gamma(mn)$ for all me and n. Hence by (4.3) (k=2) we get

$$\gamma(mn) = n \sum_{\substack{dt = m \\ (d,n)=1}}' g_2(t)\gamma(d).$$

Hence by Corollary 2.42 and Remark 2.12 we have for $0 < \varepsilon < 1/2$,

$$\sum_{m\leq x} \gamma(mn) = n \sum_{t\leq x} g_2(t) \sum_{\substack{d\leq x/t \\ (d,n)=1}} \gamma(d)$$

$$(4.21)\qquad = \frac{\alpha x^2 \varphi(n)}{2\beta(n)} \sum_{t\leq x} \frac{g_2(t)}{t^2} + O\left(x^{3/2}\delta(x)\sigma^*_{-8}(n)\, n \sum_{t\leq x} \frac{g_2(t)}{t^{3/2-\varepsilon}}\right).$$

From (4.2) (k=2) it follows that

(4.22) $|g_2(t)| \leq 1$ for all t,

and

$$(4.23)\qquad \sum_{t=1}^{\infty} \frac{g_2(t)}{t^2} = \prod_{p|n} \left\{1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right\} = \frac{n^2}{J_2(n)}.$$

From (4.22) and (4.23) we have

$$(4.24)\qquad \sum_{t\leq x} \frac{g_2(t)}{t^2} = \frac{n^2}{J_2(n)} + O(x^{-1})$$

and from (4.22)

$$(4.25)\qquad \sum_{t\leq x} \frac{g_2(t)}{t^{3/2-\varepsilon}} \leq \sum_{t\leq x} \frac{1}{t^{3/2-\varepsilon}} = O(1),$$

since $0 < \varepsilon < 1/2$. Now Theorem 4.19 follows by substituting (4.25) and (4.24) into (4.21).

(4.26) <u>Theorem.</u> If the R.H. is true, then the error term in the asymptotic formula (4.20) is given by

$$O\left(x^{7/5} W(x)\, n\theta(n)\right).$$

<u>Proof.</u> Following the same procedure adopted in the proof of Theorem 4.19, making use of Corollary 2.44 and Remark 2.14, instead of Corollary 2.42 and Remark 2.13, we obtain this theorem.

§ 5. An asymptotic formula for $\sum_{m \le x} \gamma_k(m)\,\ell(m)$

Let $\ell(m)$ denote the characteristic function of the square-full integers. Then we have :

(5.1) Lemma (cf. (7), p. 198). $\ell(m) = \sum_{d^2 n^3 = m} \mu^2(n)$

(5.2) Theorem. We have

(5.3)
$$\sum_{m \le x} \gamma(m)\ell(m) = \frac{x}{2} + O(x^{3/4}\delta(x)).$$

Proof. By lemma 5.1 and Theorem 4.19,

$$\sum_{m \le x} \gamma(m)\ell(m) = \sum_{n \le x^{1/3}} \mu^2(n) \sum_{d \le x^{1/2}/n^{3/2}} \gamma(dn)$$

$$= \frac{\alpha x}{2} \sum_{n \le x^{1/3}} \frac{\mu^2(n)\varphi(n)n^{-1}}{J_2(n)\beta(n)} + O\Big(x^{3/4} \sum_{n \le x^{1/3}} \frac{\delta(x^{1/2}/n^{3/2})\,\sigma^*_{-\delta}(n)}{n^{5/4}}\Big)$$

(5.4)
$$= \frac{\alpha x}{2} \sum_{n=1}^{\infty} \frac{\mu^2(n)\varphi(n)n^{-1}}{J_2(n)\beta(n)} + O(x^{2/3}) + O(x^{3/4}\delta(x)).$$

By (2.34) and (2.33), we have

(5.5)
$$\sum_{n=1}^{\infty} \frac{\mu^2(n)\varphi(n)n^{-1}}{J_2(n)\beta(n)} = \prod_p \Big\{1 + \frac{(p-1)p^{-1}}{(p^2-1)\big(1-\frac{1}{p(p+1)}\big)}\Big\}$$

$$= \prod_p \Big\{1 + \frac{1}{p^2+p-1}\Big\} = \prod_p \Big\{1 - \frac{1}{p(p+1)}\Big\}^{-1} = \frac{1}{\alpha}$$

Substituting (5.5) into (5.4), Theorem 5.2 follows.

(5.6) **Theorem.** If the R.H. is true, then the error term in the asymptotic formula (5.3) is given by

$$O\left(x^{7/10} W(x)\right)$$

Proof. Following the same procedure adopted in the proof of Theorem 5.21, making use of Theorem 4.26 instead of Theorem 4.19, we obtain this theorem.

(5.7) **Theorem.** Let k be even and $k \geqslant 4$. Then

$$(5.8)\quad \sum_{m \leq x} \gamma_k(m)\,\ell(m) = \frac{x^{k/2}\zeta(k)\,T_k}{k\,\zeta(2)} + O\left(x^{\frac{2k-1}{4}}\,\delta(x)\right),$$

where

$$(5.9)\quad T_k = \begin{cases} \prod_p \left\{ \dfrac{2p^6 + p^5 - 2p^4 + 1}{p^2(p+1)} \right\}, \text{ if } k = 4 \\[2ex] \prod_p \left\{ \dfrac{p^{\frac{5k-8}{2}}(p^2+p+1)\,A_k(p) + (p^k - 1)\,G_k'(p)}{p^{(5k-6)/2}(p+1)} \right\}, \end{cases}$$

$$(5.10)\quad A_k(p) = 1 + \left\{ \frac{(p^2-1)(p^{k(k-1)/2} - 1)}{p^{k(k-1)/2}(p^2+p+1)} \right\} \left[\text{if } k \geqslant 6\right.$$

and

$$(5.11)\quad G_k'(p) = 1 + p^{-(k-2)(k-4)/2}\left\{ \frac{p^{(k-2)(k-4)/2} - 1}{p^{k-2} - 1} + \frac{1}{p^k - 1} \right\}.$$

Proof. By lemma 5.1 and Theorem 4.13 we have

$$\sum_{m\le x}\gamma_k(m)\,l(m)=\sum_{n\le x^{\frac13}}\mu^2(n)\sum_{m\le x^{1/2}/n^{3/2}}\gamma_k(m^2n^3)$$

(5.12) $$=\frac{x^{k/2}\zeta(k)\,\alpha A_k}{k\,\zeta(2)}\sum_{n\le x^{\frac13}}\frac{\mu^2(n)\varphi(n)J_k(n)G'_k(n)}{n^{\frac{5k-6}{2}}J_2(n)\beta(n)}$$

$$+\,O\Big(x^{\frac{2k-1}{4}}\sum_{n\le x^{\frac13}}\frac{\delta(x^{\frac12}/n^{\frac32})\,\sigma^*_{-s}(n)F_a(n)n^2}{n^{3(2k-1)/4}}\Big)$$

By lemma 4.4 and (4.15),

$$G'_k(n)=O(F_k(n))=O(n^{\varepsilon}),$$

for every $\varepsilon>0$, so that

(5.13) $$\sum_{n>x^{\frac13}}\frac{\mu^2(n)\varphi(n)J_k(n)G'_k(n)}{n^{\frac{5k-6}{2}}J_2(n)\beta(n)}=O\Big(\sum_{n>x^{\frac13}}n^{-\left(\frac{3k-4}{2}-\varepsilon\right)}\Big)$$

$$=O(x^{1-k/2+\varepsilon}),$$

for every $\varepsilon>0$. Further it is not difficult to show that

(5.14) $$\alpha A_k\sum_{n=1}^{\infty}\frac{\mu^2(n)\varphi(n)J_k(n)G'_k(n)}{n^{\frac{5k-6}{2}}J_2(n)\beta(n)}=T_k,$$

Where T_k is given by (5.9)

Also, by Remark 2.13 and the fact that $\sigma^*_{-s}(n)=O(n^{\varepsilon})$ and $F_a(n)=O(n^{\varepsilon})$, for every $\varepsilon>0$, it follows that

the inner sum in the O-term of (5.12) is $O(\delta(x))$;
Theorem 5.7 follows from this, (5.13), (5.14) and (5.12).

(5.15) Theorem. Let k be even and ≥ 4. If the R.H. is true, then the error term in the asymptotic formula (5.9) is given by

$$O\left(x^{\frac{5k-3}{10}} W(x)\right).$$

Proof. Following the same procedure adopted in the proof of Theorem 5.8, making use of Theorem 4.18 and Remark 2.14 instead of Theorem 4.13 and Remark 2.13, we obtain this theorem.

(5.16) Theorem. Let k be odd. Then.

$$(5.17)\quad \sum_{m\leq x} \gamma_k(m)\ell(m) = \frac{x^{3/2}\zeta(3)\beta_k T_k'}{3\,\zeta((k+1)/2)} + \Delta_k(x),$$

where

$$(5.18)\quad \Delta_k(x) = \begin{cases} O(x^{5/4}\delta(x)), & \text{if } k=3, \\ O(x^{4/3}), & \text{if } k\geq 5, \end{cases}$$

$$(5.19)\quad T_k' = \begin{cases} \prod_p \{1+p^{\frac{1}{2}}(p^3+p^2-1)^{-1}\}, & \text{if } k=3, \\ \prod_p \left\{1+\dfrac{(p^3+p-1)}{p^{1/2}(p^4+p^3+p^2-p-1)}\right\}, & \text{if } k=5, \\ \prod_p \left\{1+\dfrac{p^{(k-10)/2}(p-1)(p^3-1)G_k(p)}{(p^{(k+1)/2}-1)\beta_k(p)}\right\}, & \text{if } k\geq 7 \end{cases}$$

where for k $\geqslant$ 7,

(5.20) $$G_k(p) = 1 + \frac{(p^{\frac{k-5}{2}} - 1)}{p^{\frac{k-5}{2}}(p-1)} + \frac{1}{p^{\frac{k-7}{2}}(p^3-1)},$$

(5.21) $$\beta_k(p) = 1 - \frac{(p^{\frac{k-1}{2}} - 1)}{p^2(p^{\frac{k+1}{2}} - 1)},$$

and β_k is as given in (3.3)

<u>Proof.</u> By lemma 5.1 and Theorem 4.6 we have

$$\sum_{m \leq x} \gamma_k(m)\ell(m) = \sum_{n \leq x^{\frac{1}{3}}} \mu^2(n) \sum_{m \leq x^{\frac{1}{2}}/n^{3/2}} \gamma_k(m^2 n^3)$$

(5.22)
$$= \frac{x^{3/2}\zeta(3)\beta_k}{2\zeta((k+1)/2)} \sum_{n \leq x^{\frac{1}{3}}} \frac{\mu^2(n)\gamma_k(n^3)\varphi(n)J_3(n)n^{\frac{k+1}{2}-4}G_k(n)}{n^{9/2}J_{\frac{k+1}{2}}(n)\beta_k(n)}$$

$$+ O\Big(x^{1+\frac{1}{k+1}} \sum_{n \leq x^{\frac{1}{3}}} \mu^2(n) n^{-3(1+1/(\frac{2}{k+1}))} \delta(x^{\frac{1}{2}}/n^{3/2})\sigma^{*}_{-s}(n)\gamma_k(n^3)$$

Let

(5.23) $$B_k(n) = \frac{\mu^2(n)\gamma_k(n^3)\varphi(n)J_3(n)n^{\frac{k+1}{2}-4}G_k(n)}{n^{9/2}J_{\frac{k+1}{2}}\beta_k(n)}.$$

If n is square-free, then $\gamma_k(n^3) = n^2$ or n^3 according as k = 3 or k $\geqslant$ 5. Hence we have

(5.24) $$B_k(n) = \begin{cases} O(G_k(n)n^{-5/2}), & \text{if } k = 3, \\ O(G_k(n)n^{-3/2}), & \text{if } k \geqslant 5. \end{cases}$$

For any b $>$ 2, using a standard argument, it follows from (4.5) that

(5.25) $$\sum_{n \leq x} F_b(n) = O(x).$$

Further by (4.8), lemma 4.4 and (5.25),

$$\sum_{n \leq x} G_k(n) = O(x),$$

so that by partial summation, for any $s > 1$, we have

$$\sum_{n>x} \frac{G_k(n)}{n^s} = O(x^{1-s}).$$

Hence by (5.24),

(5.26) $$\sum_{n > x^{\frac{1}{3}}} B_k(n) = \begin{cases} O(x^{-1/2}), \text{ if } k=3. \\ O(x^{-1/6}), \text{ if } k \geq 5. \end{cases}$$

Also, it is not difficult to show that

(5.27) $$\sum_{n=1}^{\infty} B_k(n) = T_k',$$

where T_k' is given by (5.19)

Let $C_k(x)$ denote the sum in the O-term in (5.22).

For k=3, using Remark 2.13 we can show that

$$C_k(x) = O(\delta(x)),$$

so that by (5.23), (5.27), (5.26) and (5.22), we have (for k=3),

$$\sum_{m \leq x} \gamma_k(m) \ell(m) = \frac{x^{3/2} \zeta(3) \beta_k T_k'}{3\zeta((k+1)/2)} + O(x) + O(x^{5/4}\delta(x)).$$

which implies Theorem 5.16 in case k=3. For $k \geq 5$ using $\delta(x) < 1$, we have

$$C_k(x) \leq \sum_{n \leq x^{\frac{1}{3}}} n^{-3/k+1} \sigma^*_{-s}(n) F_\theta(n) \mu^2(n)$$

(5.28) $$= L(x), \text{ say.}$$

We can suppose in the following that $s > 0$. Now by (2.7) and (5.25) (with $b = \theta$), we get.

$$\sum_{n \leq x} \sigma^*_{-s}(n) F_\theta(n) \mu^2(n) \leqslant \sum_{d\delta \leq x} d^{-s} F_\theta(d) F_\theta(\delta)$$

$$= \sum_{d \leq x} d^{-s} F_\theta(d) \sum_{\delta \leq x/d} F_\theta(\delta)$$

$$= O\Big(x \sum_{d \leq x} d^{-(s+1)} F_\theta(d)\Big) = O(x),$$

since $s > 0$ and $F_\theta(d) = O(d^\varepsilon)$ for every $\varepsilon > 0$ implies

$$\sum_{d \leqslant x} d^{-(s+1)} F_\theta(d) = O(1).$$

It now follows from partial summation and (5.28) that

$$L(x) = O\left(x^{\frac{1}{3} - \frac{1}{k+1}}\right)$$

so that

(5.29) $$c_k(x) = O\left(x^{\frac{1}{3} - \frac{1}{k+1}}\right).$$

Hence for $k \geqslant 5$, it follows from (5.29), (5.23), (5.27) (5.26) and (5.22) that

(5.30) $$\sum_{m \leq x} \gamma_k(m) \lambda(m) = \frac{x^{3/2} \zeta(3) \beta_k T_k'}{2\zeta((k+1)/2)} + O(x^{\frac{4}{3}}) + O(x^{\frac{4}{3}}),$$

which proves Theorem 5.16 when $k \geqslant 5$.

(5.31) <u>Theorem.</u> If the R.H. is true, then the error term in the asymptotic formula (5.18) (k=3) is given by

$$\Delta_3(x) = O(x^{6/5} W(x)).$$

<u>Proof</u>. Following the same procedure adopted in the proof of Theorem 5.16 ($k=3$), making use of Theorem 4.12 instead of Theorem 4.6, we obtain Theorem 5.31.

(5.32) <u>Remark</u>. Eq. (5.30) shows that our method does not give any improvement in the order estimate of $\Delta_k(x)$ for $k \geq 5$ even on the assumption of the Riemann hypothesis.

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ON ABSTRACT MOBIUS INVERSION

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ABSTRACT :

This paper deals with some extensions of the principle of Mobius Inversion considered by G.C. Rota who introduced the theory of incidence algebras in 1964. Three theorems have been proved with a view to obtaining inversion formulae for arithmetic functions of two variables. The following identities have been obtained as illustrations :

(1) If $\phi(r)$ denotes Euler's phi-function

$$\sum_{d|(n,r)} \frac{\phi(n/d)\,\phi(r/d)}{d\,\phi(nr/d^2)} = 1.$$

(2) If $\tau(r)$ represents the number of divisors of r , then

$$\sigma_2(g)\, nr/g^2 = \sum_{t|n} \sum_{d|r} \tau((t,d))\phi(n/t)\,\phi(r/d) \;;\; g=(n,r)$$

where $\sigma_2(r)$ denotes the sum of the squares of the divisors of r.

(3) If C(n,r) denotes Ramanujan's Sum and $\mu(r)$ is the Mobius function,

$$\sum_{d|(n,r)} C(n/d,\, r/d)\, d\,\mu(d) = \mu(r).$$

1. INTRODUCTION :

In 1964, G.C. Rota (1) introduced the theory of Mobius functions in the context of incidence algebras which he used as a powerful tool for the study of combinatorial theory. The subsequent papers of D.A. Smith (2), (3), (4), (5), give a detailed account of the major results obtained in the area of generalized arithmetic function algebras. In (6), Robert Spira has also given a lucid exposition of the principle of abstract Mobius inversion, using the description of Mobius function due to G.C. Rota (1). The purpose of this paper is to extend the idea of abstract Mobius inversion to other inversion formulae which have applications to arithmetic functions of two variables. We formulate our results in a setting similar to that used by Spira (6)

2. PRELIMINARIES :

Let P be a non-empty set which is partially ordered by the relation $\leq$ on P. Suppose that for $x, y \in P$, the segment

$$[x,y] = \{ z \mid x \leq z \text{ and } z \leq y \}$$

is finite. Under this assumption, the partially ordered set is said to be locally finite. P is also assumed to have a minimal element denoted by 0. That is, $0 \leq x$ for all $x \in P$. Further, we need P to be a semi-lattice. That is, there exists for all $x, y \in P$, a greatest lower bound g (called the meet of x and y and denoted by $x \wedge y$) such that $g \leq x, y$ and $z \leq x, y$ imply that $z \leq g$. By $g \leq x, y$, we mean $g \leq x$

and $g \leqslant y$.

Let K be an arbitrary field. We say that a function $f : P \times P \to K$ is an incidence function if $f(x,y) = 0$, except possibly when $x \leq y$. The composition of two incidence functions f and g is defined by

$$(1.1) \qquad (f.g)(x,y) = \begin{cases} \sum\limits_{x \leq z \leq y} f(x,z)\, g(z,y) & , \text{ if } x \leq y \\ 0 , & \text{otherwise.} \end{cases}$$

Clearly, $(f.g)$ is an incidence function. It is easy to verify that the incidence functions form a ring with identity under the operations of addition and composition (1.1). The identity element under composition is

$$(1.2) \qquad \delta(x,y) = \begin{cases} 1 & , \text{ if } x = y \\ 0 & , \text{ if } x \neq y. \end{cases}$$

In fact, the set $\mathcal{A}$ of incidence functions forms a K-algebra with pointwise addition, scalar multiplication and composition. $\mathcal{A}$ is called an incidence algebra over P and K. Defining

$$(1.3) \qquad e(x,y) = \begin{cases} 1 & , \text{ if } x \leqslant y \\ 0 & , \text{ otherwise} \end{cases}$$

$$(1.4) \qquad \mu(x,y) = \begin{cases} 1 & , \text{ if } x = y \\ -\sum\limits_{x \leq z \leq y} \mu(x,z) & \\ 0 & , \text{ otherwise} \end{cases}$$

we note that

(1.5) $\mu . e = e. \mu = \delta$

If we consider the point-wise product $fg(x,y) = f(x,y)\, g(x,y)$, A is a commutative K-algebra with identity $e(x,y)$ (1.3).

Now, the theorem on abstract Mobius inversion may be stated as follows :

For $f, g \in A$ such that $g(0,y) = \sum_{0 \leq x \leq y} f(0,x)$

then $f(0,y) = \sum_{0 \leq x \leq y} g(0,x)\, \mu(x,y)$

In the case of Z^+, the set of positive integers, taking partial order as 'divides' we obtain the classical Mobius inversion formula as a special case of the above result.

Next, we denote the set of all finite segments of P by S. Let F be a mapping from $S \times S \to K$. $F([u,x],[v,y])$ is defined to be zero unless $u \leq x$ and $v \leq y$. In the same manner, we may define a mapping P_f : $S \times S \to K$ by letting

$$P_f([u,x],[v,y]) = \begin{cases} f(u,x) & \text{whenever } [u,x] = [v,y] \\ 0 & \text{otherwise.} \end{cases}$$

We call P_f , the principal function determined by the incidence function f.

2. INVERSION FORMULAE:

Let $\mathcal{F}$ be the set of incidence function on $S \times S$. For $F, G \in \mathcal{F}$ we define the product (F.G) as

$$(2.1) \qquad (F.G)([u,x],[v,y]) = \sum_{\substack{u \leq w \leq x \\ v \leq z \leq y}} F([u,w],[v,z])\,G([w,x],[z,y])$$

where, as indicated, the summation is over all w in $[u,x]$ and z in $[v,y]$.

It is easy to observe that the above composition is associative. We need the following elementary functions which are required in the illustration of the composition (2.1) and which lead to the proposed inversion formulae.

$$(2.2) \qquad \Delta([u,x],[v,y]) = \delta(u,x)\ \delta(v,y)$$

$$(2.3) \qquad E([u,x],[v,y]) = e(u,x)\ e(v,y)$$

$$(2.4) \qquad M([u,x],[v,y]) = \mu(u,x)\ \mu(v,y)$$

where δ, e and μ are as defined in (1.2), (1.3) and (1.4) respectively.

Lemma : 1. $(E.M)([u,x],[v,y]) = \Delta([u,x],[v,y])$

For, from (2.3) and (2.4), we have

$$(E.M)([u,x],[v,y]) = \sum_{u \leq w \leq x}\sum_{v \leq z \leq y} e(u,w)\,e(v,z)\ \mu(w,x)\ \mu(z,y)$$

$$= \sum_{u \leq w \leq x}\sum_{v \leq z \leq y} \mu(w,x)\ \mu(z,y)$$

$$= \sum_{u \leq w \leq x} \mu(w,x) \sum_{v \leq z \leq y} \mu(z,y)$$

$$= \delta(u,x)\ \delta(v,y) \quad \text{, by (1.4)}$$

$$= \Delta([u,x],[v,y])$$

This proves the lemma. It may also be noted that (M.E) = Δ

Lemma : 2

$$(P_\delta \cdot F)([u,x],[v,y]) = \begin{cases} F([u,x],[u,y]) & \text{if } u = v \\ 0, & \text{otherwise} \end{cases}$$

Proof:

$$(P_\delta \cdot F)([u,x],[v,y]) = \sum_{\substack{u \le w \le x \\ v \le z \le y}} P_\delta([u,w],[v,z])F([w,x],[z,y])$$

$$= \begin{cases} \sum\limits_{\substack{u \le w \le x \\ u \le w \le y}} \delta(u,w)\, F([w,x],[w,y]) & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} = \begin{cases} F([u,x],[u,y]) & \text{if } u = v \\ 0, & \text{otherwise.} \end{cases}$$

Hence the lemma.

Lemma : 3

$$(P_e \cdot F)([u,x],[v,y]) = \begin{cases} \sum\limits_{u \le w \le x \wedge y} F([w,x],[w,y]), & \text{if } u = v \\ 0, & \text{otherwise.} \end{cases}$$

Proof follows on lines similar to that of Lemma 2.

Lemma : 4

$$(P_\mu \cdot P_e)([u,x],[v,y]) = \begin{cases} P_\delta([u,x],[u,y]), & \text{if } u = v \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Using the definition of P_μ, we have

$$(P_\mu \cdot P_e)([u,x],[v,y]) = \begin{cases} \sum\limits_{\substack{u \leq w \leq x \\ u \leq w \leq y}} \mu(u,w) P_e([w,x],[w,y]) & \text{if } u = v \\ 0, \text{ otherwise} \end{cases}$$

$$= \begin{cases} \sum\limits_{u \leq w \leq x \wedge y} \mu(u,w) P_e([w,x],[w,y]) & \text{if } u = v \\ 0, \text{ otherwise} \end{cases}$$

$= ($ 0, if $x \neq y$, by the definition of P_e

But,

$$(P_\mu \cdot P_e)([u,x],[v,x]) = \begin{cases} \sum\limits_{u \leq w \leq x} \mu(u,w), & \text{if } u = v \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \delta(u,x), & \text{if } u = v \\ 0, & \text{otherwise} \end{cases}$$

So,

$$(P_\mu \cdot P_e)([u,x],[v,y]) = \begin{cases} P_\delta([u,x],[u,y]) & \text{if } u = v \\ 0, \text{ otherwise} \end{cases}$$

This completes the proof of the lemma.

Now, we come to the main results embodied in the following theorems :

THEOREM: 1. If $G([u,x],[v,y]) = \sum\limits_{\substack{u \leq w \leq x \\ v \leq z \leq y}} F([u,w],[v,z])$

then,

$$F([u,x],[v,y]) = \sum_{\substack{u \leq w \leq x \\ v \leq z \leq y}} G([u,w],[v,z])\mu(w,x)\mu(z,y)$$

Proof: We note that in terms of the composition (2.1),

$$G([u,x],[v,y]) = (F.E)([u,x],[v,y])$$

and

$$\sum_{\substack{u \leq w \leq x \\ v \leq z \leq y}} G([u,w],[v,z])\ \mu(w,x)\ \mu(z,y) = (G.M)([u,x],[v,y]).$$

Simplification of (G.M) leads to

$$\begin{aligned}(G.M)([u,x],[v,y]) &= ((F.E).M)([u,x],[v,y]) \\ &= (F.(E.M))([u,x],[v,y]) \\ &= (F.\Delta)([u,x],[v,y]), \text{ by lemma 1} \\ &= F([u,x],[v,y])\end{aligned}$$

Hence the theorem.

THEOREM: 2. If $G([u,x],[v,y]) = \sum_{u \leq z \leq x \wedge y} F([z,x],[z,y])$

then $F([u,x],[u,y]) = \sum_{u \leq z \leq x \wedge y} \mu(u,z)\ G([z,x],[z,y])$

Proof: By lemma 3, we have

$$\sum_{u \leq w \leq x \wedge y} F([w,x],[w,y]) = (P_e.F)([u,x],[u,y])$$

But, $\sum_{u \leq z \leq x \wedge y} \mu(u,z)\ G([z,x],[z,y]) = (P_{\mu} \cdot G)([u,x],[u,y])$

$$= (P_{\mu} \cdot (P_e \cdot F))([u,x],[u,y])$$

$$= ((P_{\mu} \cdot P_e) \cdot F)([u,x],[u,y])$$

$$= (P_{\delta} \cdot F)([u,x],[u,y]), \text{ by lemma 4}$$

$$= F([u,x],[u,y]), \text{ by lemma 2}$$

This proves the theorem.

Next, we define $(e.e)(x,y)$ as $\tau(x,y) = |[x,y]|$ (which may be considered as the number of chains of length 2 contained in $[x,y]$). Therefore,

$$\sum_{u \leq z \leq x \wedge y} E([z,x],[z,y]) = \tau(u, x \wedge y)$$

and so, $(P_e \cdot E)([u,x],[u,y]) = \tau(u, x \wedge y)$.

THEOREM: 3. If $G([u,x],[u,y]) = \sum_{\substack{u \leq w \leq x \\ u \leq z \leq y}} F([w,x],[z,y])$

then $$\sum_{u \leq z \leq x \wedge y} G([z,x],[z,y]) = \sum_{\substack{u \leq w \leq x \\ u \leq z \leq y}} \tau(u, w \wedge z)\ F([w,x],[z,y])$$

Proof : From the structure of G , we have

$$G([u,x],[u,y]) = (E.F)([u,x],[u,y])$$

Now,

$$\sum_{u \leq z \leq x \wedge y} G([z,x],[z,y]) = (P_e \cdot G)([u,x],[u,y])$$

$$= (P_e \cdot (E.F))([u,x],[u,y])$$

$$= ((P_e \cdot E).F)([u,x],[u,y])$$

$$= \tau(u, x \wedge y).F([u,x], [u,y])$$

$$= \sum_{\substack{u \leq w \leq x \\ u \leq z \leq y}} \tau(u, w \wedge z) F([w,x], [z, y])$$

This completes the proof of the theorem.

3. ARITHMETIC FUNCTIONS OF TWO VARIABLES:

The following inversion formulae (3.1) and (3.2) follow as special cases of Theorems 1 and 2 respectively.

(3.1) If $f(n,r)$ and $g(n,r)$ are two arithmetic functions which are such that

$$g(n,r) = \sum_{t \mid n} \sum_{d \mid r} f(t,d)$$

then, $$f(n,r) = \sum_{t \mid n} \sum_{d \mid r} g(t,d)\ \mu(n/t)\ \mu(r/d)$$

For, considering Z^+, the set of positive integers with partial order 'divides', let

$F([1,n], [1,r]) = f(n,r)$ and $G([1,n], [1,r]) = g(n,r)$.

Then, using Theorem 1 we obtain

$$G([1,n], [1,r]) = \sum_{1 \leq t \leq n \quad 1 \leq d \leq r} F([1,t], [1,d])$$

and this implies and is implied by

$$F([1,n], [1,r]) = \sum_{\substack{1 \leq t \leq n \\ 1 \leq d \leq r}} G([1,t], [1,d])\ \mu(t,n)\ \mu(d,r)$$

As $\mu(t,n) = \mu(n/t)$ [6] and $\mu(d,r) = \mu(r/d)$, (e.1) follows.

(3.2) If $f(n,r)$ and $g(n,r)$ are such that $g(n,r) = \sum_{d \mid (n,r)} f(n/d, r/d)$

then, $$f(n,r) = \sum_{d \mid (n,r)} \mu(d)\, g(n/d, r/d)$$

and conversely.

To prove (3.2), we observe that for Z^+ under the partial order 'divides' $n \wedge r = (n,r)$ whenever $n, r \in Z^+$, where (n,r) denotes the g.c.d of n and r. Clearly, (3.2) is a particular case of Theorem 2.

(3.3) If $f(n,r)$ and $g(n,r)$ are such that

$$g(n,r) = \sum_{t \mid n} \sum_{d \mid r} f(n/t, r/d)$$

then

$$\sum_{d \mid (n,r)} g(n/d, r/d) = \sum_{t \mid n} \sum_{d \mid r} ((t,d))\, f(n/t, r/d)$$

where $\tau(r)$ denotes the number of divisors of r.

(3.3) may be deduced from Theorem 3. The derivation is similar to that of (3.1) and (3.2). The details of proof are omitted.

4. APPLICATIONS TO IDENTITIES:

As applications of the inversion formulae proved in § 3, we give below a few identities connected with Euler's totient $\phi(r)$ and Ramanujan's Sum $C(n,r)$.

$$(4.1)\qquad \sum_{d \mid (n,r)} \frac{\phi(n/d)\,\phi(r/d)}{d\,\phi(nr/d^2)} = 1$$

Proof: It is well known that $\phi(nr) = \dfrac{\phi(n)\,\phi(r)\,g}{\phi(g)}$; $g = (n,r)$

That is, $$\frac{\phi(g)}{g} = \frac{\phi(n)\,\phi(r)}{\phi(nr)}$$

Or, $$\sum_{d \mid g} \frac{r}{d}\,\mu(d) = \frac{r\,\phi(n)\,\phi(r)}{\phi(nr)}$$

Appealing to (3.2) we obtain

$$\sum_{d \mid g} \frac{(r/d)\,\phi(n/d)\,\phi(r/d)}{\phi(nr/d^2)} = r$$

which reduces to (4.1) on cancellation of r.

$$(4.2)\qquad \sum_{t \mid n}\sum_{d \mid r} \tau((t,d))\,\phi(n/t)\,\phi(r/d) = \sigma_2(g)\,nr/g^2 \,;\quad g = (n,r).$$

where $\sigma_2(r)$ denotes the sum of the squares of the divisors of r.

Proof: As $\sum_{d \mid r} \phi(d) = r$, we have

$$\sum_{t \mid n}\sum_{d \mid r} \phi(n/t)\,\phi(r/d) = nr$$

If $\sigma_2(r)$ denotes the sum of the squares of the divisors of r , it is easy to see that

$$\sum_{d \mid g} nr/d^2 = \sigma_2(g)\,nr/g^2 \;;\; g = (n,r)$$

(In (3.3) , take $f(n,r) = \phi(n)\phi(r)$ and $g(n,r) = nr$.

Then, it follows from (3.3) that

$$\sigma_2(g)\ \ nr/g^2 = \sum_{t \mid n} \sum_{d \mid r} \tau((t,d))\ \phi(n/t)\ \phi(r/d).$$

Remark : It may be pointed out that (4.2) is proved in [7, Section V) using the properties of functions of [illegible] greatest common divisor.

Next, let C(n,r) denote Ramanujan's sum defined by

$$C(n,r) = \sum_{\substack{h (\mathrm{mod}\ r) \\ (h,r) = 1}} \exp\left(2\pi i h n/r \right)$$

where the summation is over a reduced residue system mod r.

It is known that

$$C(n,r) = \sum_{d \mid (n,r)} \mu\,(r/d)\ d.$$

(4.3) $$\sum_{d \mid g} C(n/d, r/d)\ d\ \mu(d) = \mu\,(r)\ ;\quad g = (n,r).$$

Proof : We note that $$\frac{C(n,r)}{r} = \sum_{d \mid g} \frac{\mu\,(r/d)}{r/d}$$

Therefore, from (3.2) we get

$$\sum_{d \mid g} \frac{C(n/d, r/d)}{r/d}\ \mu\,(d) = \frac{\mu\,(r)}{r}$$

from which (4.3) follows immediately.

CONCLUDING REMARKS :

The results obtained in this paper were the outcome of an attempt to study Mobius Inversion in the context of arithmetic functions of two variables.

There are many identities involving $C(n,r)$, $\phi(r)$ and other related arithmetical functions. Some of them (see for example (4.1), (4.2) and (4.3)) can be brought under some sort of an inversion principle. However, the following identities for $\phi(r)$ are not quite revealing from the point of view of inversion:

(4.4) $$\frac{\phi(nr)}{nr} = \sum_{t \mid n} \sum_{d \mid r} (t,d) \quad \mu(t) \quad \mu(d)/ td$$

Vaidyanathaswamy [7] obtains the above identity from the identical equation for totients.

An identity due to S.S.Pillai is the following

(4.5) $$\phi(nr) = \sum_{d \mid (n,r)} \phi(n/d) \ \phi(r/d) \ d$$

whenever n and r do not have a common unitary divisor greater than 1.

It is shown in [7] that (4.5) is a special case of a restricted Busche-Ramanujan Identity.

(4.6) $$\sum_{d \mid (n,r)} (nr/d^2) \ d \ \mu(d) = \phi(n/u) \ \phi(r/u) \ \phi_2(u)$$

Where u is the greatest common square-free unitary divisor of

(n,r) and $$\phi_2(r) = r \prod_{p \mid r} (1 - 2/p)$$

We observe that (4.6) can be deduced from the multiplicative property of $\phi(nr)$ considered as a function of two variables n, r.

Yet another identity satisfied by $\phi(r)$ is

$$(4.7) \qquad \phi(n) \quad \phi(r) = \sum_{d \mid (n,r)} \phi(nr/d) \quad \mu(d)$$

(4.7) is due to Venkataraman [8]. It does not seem to be the result of a Mobius Inversion, as it is obtained as a property of symmetric multiplicative functions [8].

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MAXIMAL K-th POWER AND THE MAXIMAL K-th POWER UNITARY DIVISOR OF AN INTEGER *

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1. INTRODUCTION:

Let k be a fixed integer $\geqslant 2$. A positive integer n is called a k-th power, if in its prime power factorization each exponent is a multiple of k. By a k-th power, unitary divisor of n we mean a divisor d of n such that d is a k-th power and also d unitarily divides n. Let $\Gamma_k(n)$ denote the maximal k-th power divisor of n and $\Gamma_k^*(n)$ denote the maximal k-th power unitary divisor of n. Further a positive integer n is called unitarily k-free, if n is not divisible unitarily by the k-th power of any integer > 1. The concept of unitarily k-free integers was introduced by E. Cohen(cf (3) §I) Let Q_k^* denote the set of unitarily k-free integers and $q_k^*(n)$ denote the characteristic function of the set Q_k^*. Let $\Gamma_{u,k}^*(n)$ denote the maximal unitarily k-free, unitary divisor of n. It is easy to observe that $\Gamma_{u,2}^*(n) \equiv \Gamma_u^*(n)$ the maximal exponentially odd, unitary divisor of n. (Integers in whose prime power factorizations each exponent is odd, are called exponentially odd integers).

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*Presented by : P.Subrahmanyam.

In this paper, we establish asymptotic formulae for the summatory functions of (1) $\Gamma_k(n)$ (2) $\Gamma_k^*(n)$ (3) $\frac{1}{\Gamma_k(n)}$ and (4) $\frac{1}{\Gamma_k^*(n)}$. As consequences of the asymptotic formulae for the functions (3) and (4) we obtain asymptotic formulae for the summatory functions of $T_k(n)$ and $\Gamma_{u,k}^*(n)$ respectively, where $T_k(n)$ denotes the divisor of n conjugate to the greatest k-th power divisor of n.

In § 2. We prepare the necessary background and establish the main results of this paper in § 3. We discuss the consequences of the Riemann hypothesis in § 4.

§ 2. Preliminaries.

In this section we introduce some notation, state some lemm's already established and then prove some lemmas which are needed in our present discussion. Let $\mu(n)$ and $\varphi(n)$ denote respectively the Mobius function and the Euler lotient function. Let $\mu^*(n)$ denote the unitary analogue of the Mobius function defined by $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of $n > 1$. Let $\theta(n)$ denote the number of square-free divisors of n. Let the constant e_k be defined, by

(2.1)
$$e_k = \zeta(k)\prod_p\left(1-\frac{2}{p^k}+\frac{1}{p^{k+1}}\right)$$

Where $\zeta(s)$ is the Riemann Zeta function defined by $\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}$ for $s > 1$. Let $E_k(n)$ be the function defined by $E_k(1)=1$ and

(2.2) $$E_k(n) = n\prod_{p|n}\left(\frac{p(p^k-1)}{p^{k+1}-2p+1}\right)$$

Remark 2.1 Clearly $\theta(n) \leq \tau(n)$ the number of divisors of n.

In our present discussion we need the following elementary estimates.

(2.3) $$\sum_{m\leq x}\frac{1}{m} = \log x + \gamma + O\left(\frac{1}{x}\right) \quad \text{(cf.[1] Theorem 3.2(a))}$$

(2.4) $$\sum_{m>x}\frac{1}{m^s} = O(x^{1-s}) \text{ if } s>1 \quad \text{(cf.[1] Theorem 3.2(c))}$$

(2.5) $$\sum_{m\leq x}\tau(m) = O(x\log x) \quad \text{(cf.[5], Theorem 320)}$$

(2.6) $$\sum_{m\leq x}\frac{\tau(m)}{m} = O(\log^2 x) \quad \text{(cf.[1], p.70, Problem 3)}$$

Lemma 2.1 (cf. [2], lemma (2'3) ii) For $k \geq 0$

(2.7) $$\sum_{m\leq x,\ (m,n)=1} m^k = \frac{x^{k+1}}{k+1}\,\frac{\varphi(n)}{n} + O(\theta(n)x^k)$$

Lemma 2.2 (cf. [5], Theorem 303)

(2.8) $$\sum_{m=1}^{\infty} q_k(m)/m^s = \zeta(s)/\zeta(ks)$$

Where $q_k(n)$ is the characteristic function of the k-free integers

(Integers in whose cannonical representation each exponent is $< k$)

Lema 2.3 (cf. [3], (3.7) and (3.1) as $r \to \infty$)

(2.9) $$q_k^*(n) = \sum_{\substack{d^k\delta = n\\ (d,\delta)=1}}\mu^*(d)$$

Lemma 2.4 (cf [11], Lemma 2·6, $s = k+1$)

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu^*(m)\varphi(m)}{m^{k+1}} = \zeta(k)\prod_{p}\left(1-\frac{2}{p^k}+\frac{1}{p^{k+1}}\right)\prod_{p|n}\left\{\frac{p(p^k-1)}{p^{k+1}-2p+1}\right. \tag{2.10}$$

Lemma 2.5 (cf (11), Lemma 2.4, s = k+1)

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{\mu^*(m)\varphi(m)}{m^{k+1}} = O\left(\frac{\theta(n)\,\delta(x)}{x^{k-1}}\right) \tag{2.11}$$

Where $\delta(x)$ is given by

$$\delta(x) = \begin{cases} \exp\{-A\log^{3/5}x\,(\log\log x)^{-1/5}\}; & x \geq \\ 1; & 0<x<3. \end{cases} \tag{2.12}$$

A being a positive absolute constant.

Lemma 2.6 (cf. (11), Lemma 2.13, s = k+1) If the Riemann hypothesis is true, then for $x \geqslant 3$,

$$\sum_{\substack{m>x \\ (m,n)=1}} \frac{\mu^*(m)\varphi(m)}{m^{k+1}} = O\left(\frac{\theta(n)\,\omega(x)\log x}{x^{k-\frac{1}{2}}}\right). \tag{2.13}$$

Uniformly where $\omega(x)$ is given by

$$\omega(x) = \begin{cases} \exp\{A\log x\,(\log\log x)^{-1}\} & \text{for } x\geq 3 \\ 1 & \text{for } 0<x<3 \end{cases} \tag{2.14}$$

A being a positive constant.

Lemma 2.7 (cf. (10), lemma 2.8) For $x \geqslant 3$

(2.15) $$M_n^*(x) \equiv \sum_{\substack{m \le x \\ (m,n)=1}} \mu^*(m) = O(\theta(n)\, x\, \delta(x))$$

Uniformly where $\delta(x)$ is given by (2.12)

Lemma 2.8 (cf.(10) Lemma 2.16). If the Riemann hypothesis is true, then for $x \geq 3$,

(2.16) $$M_n^*(x) \equiv \sum_{\substack{m \le x \\ (m,n)=1}} \mu^*(m) = O\left(\theta(n)\, x^{\frac{1}{2}}\, \omega(x) \log x\right)$$

Uniformly where $\omega(x)$ is given by (2.14)

Lemma 2.9 (cf.(12), satz. 1 P.192). For $x \geq 3$

(2.17) $$\sum_{m \le x} q_k(m) = \frac{x}{\zeta(k)} + O(x^{1/k}\, \delta_k(x))$$

Uniformly where $\delta_k(x)$ is given by

(2.18) $$\delta_k(x) = \begin{cases} \exp\left\{-\frac{A}{2} k^{-8/5} \log^{3/5} x \,(\log\log x)^{-\frac{1}{5}}\right\} & \text{for } x \geq 3 \\ 1 & \text{for } 0 < x < 3 \end{cases}$$

Lemma 2.10. (cf (9), Theorem 3.2, n=1) If the Riemann hypothesis is true then for $x \geq 3$

(2.19) $$\sum_{m \le x} q_k(m) = \frac{x}{\zeta(k)} + O\left(x^{\frac{2}{2k+1}}\, \omega(x)\right)$$

Uniformly where $\omega(x)$ is given by (2.14)

Lemma 2.11 (cf (7), lemma 2.4). If $f(n)$ and $g(n)$ are multiplicative then

(2.20) $$h(n) = \sum_{d^k \delta = n} f(d)\, g(\delta), \quad k \geq 1$$

is also multiplicative.

Lemma 2.12 (cf (8), Theorem 2.4) If f(n) and g(n) are multiplicative then

(2.21) $$h'(n) = \sum_{\substack{d^k\delta = n \\ (d,\delta)=1}} f(d)\, g(\delta), \quad k \geq 1$$

is also multiplicative

Lemma 2.13 For $x \geq 3$

(2.22) $$\sum_{\substack{m \leq x \\ (m,n)=1}} q_k^*(m) = e_k \frac{\varphi(n)\, E_k(n)}{n^2} + O\left(\theta(n)\, x^{\frac{1}{k}}\, \delta_k(x)\right\}$$

Uniformly, where e_k, $E_k(n)$ and $\delta_k(x)$ are respectively given by (2.1), (2.2) and (2.18)

Proof. By lemma 2.3 we have

$$\sum_{\substack{m \leq x \\ (m,n)=1}} q_k^*(m) = \sum_{\substack{m \leq x \\ (m,n)=1}} \sum_{\substack{d^k\delta = m \\ (d,\delta)=1}} \mu^*(d) = \sum_{\substack{d^k\delta \leq x \\ (d\delta,n)=1 \\ (d,\delta)=1}} \mu^*(d)$$

Let $z = x^{\frac{1}{k}}$ and $0 < \rho = \rho(z) < 1$ where $\rho(z)$ will be suitably chosen later. If $d^k\delta \leq x$ then both $d > \rho z$ and $\delta > \rho^{-k}$ cannot simultaneously hold good and so

$$\sum_{\substack{m \leq x \\ (m,n)=1}} q_k^*(m) = \sum_{\substack{d^k\delta \leq x \\ d \leq \rho z \\ (d\delta,n)=(d,\delta)=1}} \mu^*(d) + \sum_{\substack{d^k\delta \leq x \\ \delta \leq \rho^{-k} \\ (d\delta,n)=(d,\delta)=1}} \mu^*(d) - \sum_{\substack{d \leq \rho z \\ \delta \leq \rho^{-k} \\ (d\delta,n)= \\ (d,\delta)=1}} \mu^*(d)$$

(2.23) $$= S_1 + S_2 - S_3, \quad \text{say}$$

By Lemma 2.1 (k=o) Remark 2.1 and (2.5) we have

$$S_1 = \sum_{\substack{d \leq \rho z \\ (d,n)=1}} \mu^*(d) \sum_{\substack{\delta \leq x/d^k \\ (dn,\delta)=1}} 1 = \sum_{\substack{d \leq \rho z \\ (d,n)=1}} \mu^*(d) \left\{ \frac{x}{d^k} \frac{\varphi(dn)}{dn} + O(\theta(dn)) \right\}$$

$$= \frac{x\varphi(n)}{n} \sum_{\substack{m \le \rho z \\ (m,n)=1}} \frac{\mu^*(m)\varphi(m)}{m^{k+1}} + O\left(\theta(n) \sum_{m \le \rho z} \tau(m)\right)$$

$$= \frac{x\varphi(n)}{n} \sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\mu^*(m)\varphi(m)}{m^{k+1}} - \frac{x\varphi(n)}{n} \sum_{\substack{m > \rho z \\ (m,n)=1}} \frac{\mu^*(m)\varphi(m)}{m^{k+1}}$$

$$+ O(\theta(n)\rho z \log \rho z)$$

Hence by lemmas 2.4, 2.5 and (2.1), (2.2) we have

(2.24) $$S_1 = \frac{e_k E_k(n)\varphi(n)}{n^2} x + O(\theta(n)\rho^{1-k} z\, \delta(\rho z)) + O(\theta(n)\rho z \log \rho z)$$

We have by lemma 2.7

$$S_2 = \sum_{\substack{\delta \le \rho^{-k} \\ (\delta,n)=1}} \sum_{\substack{d \le \sqrt[k]{x/\delta} \\ (d,\delta n)=1}} \mu^*(d) = \sum_{\substack{\delta \le \rho^{-k} \\ (\delta,n)=1}} M^*_{\delta n}\left(\sqrt[k]{\frac{x}{\delta}}\right) = O\left(\sum_{\substack{m \le \rho^{-k} \\ (m,n)=1}} \theta(mn) \sqrt[k]{\frac{x}{m}}\, \delta\left(\sqrt[k]{\frac{x}{m}}\right)\right)$$

Since $\delta(x)$ is monotonic decreasing and $\sqrt[k]{\frac{x}{m}} \ge \rho z$ we have $\delta\left(\sqrt[k]{\frac{x}{m}}\right) \le \delta(\rho z)$. Also, we have by Remark 2.1, (2.5) and by partial summation

(2.25) $$\sum_{m \le \rho^{-k}} \frac{\theta(m)}{m^{\frac{1}{k}}} = O\left(\rho^{1-k} \log \frac{1}{\rho}\right)$$

Hence

(2.26) $$S_2 = O\left(\theta(n)\rho^{1-k} z\, \delta(\rho z) \log \frac{1}{\rho}\right)$$

Also, by lemma 2.7 and Remark 2.1 and (2.5) we have

$$S_3 = \sum_{\substack{\delta \le \rho^{-k} \\ (\delta,n)=1}} \sum_{\substack{d \le \rho z \\ (d,\delta n)=1}} \mu^*(d) = \sum_{\substack{\delta \le \rho^{-k} \\ (\delta,n)=1}} M^*_{\delta n}(\rho z) = O\left(\sum_{\substack{m \le \rho^{-k} \\ (m,n)=1}} \theta(mn)\rho z\, \delta(\rho z)\right)$$

$$= O\Big(\theta(n)\rho z\,\delta(\rho z)\sum_{m\le\rho^{-k}}\theta(m)\Big) = O\Big(\theta(n)\rho z\,\delta(\rho z)\sum_{m\le\rho^{-k}}\tau(m)\Big)$$

(2.27) $$S_3 = O\left(\theta(n)\rho^{1-k} z\,\delta(\rho z)\log\frac{1}{\rho}\right)$$

Hence, by (2.23), (2.24), (2.26) and (2.27) we have

(2.28) $$\sum_{\substack{m\le x\\(m,n)=1}} q_k^*(m) = \frac{e_k E_k(n)\varphi(n)}{n^2} + O\left(\theta(n)\rho^{1-k} z\,\delta(\rho z)\log\frac{1}{\rho}\right) + O(\theta(n)\rho z\log\rho z)$$

Now choosing $\rho = \rho(X) = \left\{\delta\left(x^{\frac{1}{2k}}\right)^{-\frac{1}{2}}\right\}^{\frac{1}{k}}$ and writing $f(x) = \log^{3/5}\left(x^{\frac{1}{2k}}\right)\left(\log\log\left(x^{\frac{1}{2k}}\right)\right)^{-\frac{1}{5}}$ as in (cf.(6), Theorem 4.1, (4.12), (4.13) and following the same theorem referred here we get the first and second O - terms of (2.28) are each of

$$O\left(\theta(n)\,x^{\frac{1}{k}}\,\delta_k(x)\right).$$

This completes the proof of lemma 2.13.

<u>Lemma 2.14</u> If the Riemann hypothesis is true, then for $x \geq 3$

(2.29) $$\sum_{\substack{m\le x\\(m,n)=1}} q_k^*(m) = \frac{e_k E_k(n)\varphi(n)}{n^2}x + O\left(\theta(n)\,x^{\frac{2}{2k+1}}\,\omega(x)\right)$$

Uniformly where $\omega(x)$ is given by (2.14).

<u>Proof</u>. Following the same procedure adopted in proving lemma 2.13 and making use of lemmas 2.6 and 2.8 instead of lemmas 2.5 and 2.7 we get the following instead of (2.28)

(2.30) $$\sum_{\substack{m\le x\\(m,n)=1}} q_k^*(m) = \frac{e_k Q(n) E_k(n)\,x}{n^2} + O\left(\theta(n)\rho^{\frac{1}{2}-k} z^{\frac{1}{2}}\omega(\rho z)\log(\rho z)\right) + O(\theta(n)\rho z\log(\rho z))$$

Now choosing $\rho = z^{-\frac{1}{2k+1}}$ We see that $0 < \rho < 1$ and $\rho^{\frac{1}{2}-k} z^{\frac{1}{2}} = \rho z = x^{2/2k+1}$ since $\omega(x)$ is monotonic increasing, we have $\omega(\rho z) \leq \omega(z)$ and also $\log \rho z \leq \log z$. Hence the first and second O-terms in (2.30) are each of $O(\theta(n) x^{\frac{2}{2k+1}} \omega(x^{\frac{1}{k}}) \log x) = O(\theta(n) x^{\frac{2}{2k+1}} \omega(x))$.

Hence lemma 2.14 follows.

<u>Remark 2.2.</u> The Case n=1 of lemmas 2.13 and 2.14 have been proved by D.Suryanarayana and R. Sita Ramachandra Rao (cf (10), Theorem 3.1 and Theorem 3.2)

Now we establish the following identities.

<u>Lemma 2.15</u>

$$(2.31)\qquad \Gamma_k(n) = \sum_{d^k\delta = n} d^k q_k(\delta)$$

$$(2.32)\qquad \frac{1}{\Gamma_k(n)} = \sum_{d^k\delta = n} q_k(\delta)/d^k$$

$$(2.33)\qquad \Gamma_k^*(n) = \sum_{\substack{d^k\delta = n\\(d,\delta)=1}} d^k q_k^*(\delta)$$

$$(2.34)\qquad \frac{1}{\Gamma_k^*(n)} = \sum_{\substack{d^k\delta = n\\(d,\delta)=1}} q_k^*(\delta)/d^k$$

Where $q_k(n)$ and $q_k^*(n)$ denote respectively the characteristic functions of the set of k-free and unitarily k-free integers.

<u>Proof.</u> Since $q_k(n)$ and $q_k^*(n)$ are multiplicative, it follows by lemmas 2.11 and 2.12 that the right side sums of (2.31) - (2.34) are multiplicative, also $\Gamma_k(n)$, $\frac{1}{\Gamma_k(n)}$, $\Gamma_k^*(n)$ and $1/\Gamma_k^*(n)$ are multiplicative. It therefore suffices to verify (2.31) - (2.34)

at n = p^{α} a prime power.

For proving (2.31) we note that any $\alpha \geqslant$ k can be uniquely written as $\alpha = tk + r$, $0 \le r < k$ so that

$$\sum_{d^k\delta = p^\alpha} d^k q_k(\delta) = \begin{cases} 1^k q_k(p^\alpha) & \text{for } \alpha < k \\ p^{tk} q_k(p^r) & \text{for } \alpha \geqslant k \end{cases} = \begin{cases} 1 & \text{if } \alpha < k \\ p^{tk} & \text{if } \alpha \geqslant k \end{cases}$$

$$= \Gamma_k(p^\alpha)$$

and t the greatest multiple of k in α.

Hence (2.31) follows. (2.32) can be proved in a similar way. For proving (2.33)

$$\sum_{\substack{d^k\delta = p^\alpha \\ (d,\delta)=1}} d^k q_k^*(\delta) = \begin{cases} 1^k q_k^*(p^\alpha) & \text{if } k \nmid \alpha \\ p^\alpha q_k^*(1) & \text{if } k \mid \alpha \end{cases} = \begin{cases} 1 & \text{if } k \nmid \alpha \\ p^\alpha & \text{if } k \mid \alpha \end{cases}$$

$$= \Gamma_k^*(p^\alpha)$$

Thus (2.33) follows. In a similar way (2.34) can be proved.

§ 3. First we establish.

<u>Theorem 3.1</u> For $x \geqslant 2$,

(3.1) $$\sum_{m \le x} \Gamma_k(m) = \frac{\zeta\left(\frac{k+1}{k}\right)}{(k+1)\zeta(k+1)} x^{1+\frac{1}{k}} + O(x \log x)$$

<u>Proof</u>. By (2.31), lemmas 2.1 and 2.2 and (2.3) We have

$$\sum_{m \le x} \Gamma_k(m) = \sum_{m \le x} d^k q_k(\delta) = \sum_{\delta \le x} q_k(\delta) \sum_{d \le \left(\frac{x}{\delta}\right)^{\frac{1}{k}}} d^k$$

$$= \sum_{\delta \le x} q_k(\delta) \left\{ \frac{1}{k+1}\left(\left(\frac{x}{\delta}\right)^{\frac{1}{k}}\right)^{k+1} + O\left(\left(\left(\frac{x}{\delta}\right)^{\frac{1}{k}}\right)^k\right)\right\}$$

$$= \frac{x^{1+\frac{1}{k}}}{1+k} \sum_{m \le x} \frac{q_k(m)}{m^{1+\frac{1}{k}}} + O\left(x \sum_{m \le x} \frac{1}{m}\right)$$

$$= \frac{x^{1+\frac{1}{k}}}{k+1}\left\{\sum_{m=1}^{\infty}\frac{q_k(m)}{m^{1+\frac{1}{k}}} - \sum_{m>x}\frac{q_k(m)}{m^{1+\frac{1}{k}}}\right\} + O(x\log x)$$

$$= \frac{x^{1+\frac{1}{k}}}{k+1}\left\{\frac{\zeta(\frac{k+1}{k})}{\zeta(k+1)} + O\left(\frac{1}{x^{\frac{1}{k}}}\right)\right\} + O(x\log x)$$

$$= \frac{\zeta((k+1)/k)}{(k+1)\zeta(k+1)}\, x^{1+\frac{1}{k}} + O(\log x)$$

This completes the proof of theorem 3.1

Corollary 3.1 For $x \geqslant 2$

(3.2) $$\sum_{m\leq x}\Gamma_2(m) = \frac{\zeta(3/2)}{3\zeta(3)}\, x^{3/2} + O(x\log x)$$

Where $\Gamma_2(m)$ is the maximal square divisor of m

Theorem 3.2. For $x \geqslant 3$,

(3.3) $$\sum_{m\leqslant x}\frac{1}{\Gamma_k(m)} = \frac{\zeta(2k)}{\zeta(k)}\, x + O\left(x^{\frac{1}{k}}\,\delta_k(x)\right)$$

Where $\delta_k(x)$ is given by (2.18)

Proof. By (2.32) and lemma 2.9, we have

$$\sum_{m\leq x}\frac{1}{\Gamma_k(m)} = \sum_{m\leq x}\sum_{d^k\delta=m}\frac{q_k(\delta)}{d^k} = \sum_{d\leq x^{\frac{1}{k}}}\frac{1}{d^k}\sum_{\delta\leq\frac{x}{d^k}} q_k(\delta)$$

$$= \sum_{d\leq x^{\frac{1}{k}}}\frac{1}{d^k}\left\{\frac{x}{\zeta(k)d^k} + O\left(\left(\frac{x}{d^k}\right)^{\frac{1}{k}}\delta_k\left(\frac{x}{d^k}\right)\right)\right\}$$

(3.4) $$= \frac{x}{\zeta(k)}\sum_{m\leq x^{\frac{1}{k}}}\frac{1}{m^{2k}} + O\left(\sum_{m\leq x^{\frac{1}{k}}}\frac{1}{m^k}\left(\frac{x}{m^k}\right)^{\frac{1}{k}}\delta_k\left(\frac{x}{m^k}\right)\right)$$

Since $x^{\frac{1}{k}}\,\delta_x(x)$ is monotonic increasing for sufficiently large x_2 x

We have

$$\sum_{m\leq x^{\frac{1}{k}}} \frac{1}{m^k}\left(\frac{x}{m^k}\right)^{\frac{1}{k}} \delta_k\left(\frac{x}{m^k}\right) = O\left(x^{\frac{1}{k}}\delta_k(x)\sum_{m\leq x^{\frac{1}{k}}}\frac{1}{m^k}\right)$$

$$= O\left(x^{\frac{1}{k}}\delta_k(x)\right).$$

Further by the definition of $\zeta(s)$ given in (2.1) and by (2.4) we have

$$\sum_{m\leq x^{\frac{1}{k}}} \frac{1}{m^{2k}} = \sum_{m=1}^{\infty} \frac{1}{m^{2k}} - \sum_{m>x^{\frac{1}{k}}} \frac{1}{m^{2k}}$$

$$= \zeta(2k) + O\left(\left(x^{\frac{1}{k}}\right)^{1-2k}\right) = \zeta(2k) + O\left(x^{\frac{1}{k}-2}\right)$$

Thus from (3.4) and the above discussion, we have

$$\sum_{m\leq x} \frac{1}{\Gamma_k(n)} = \frac{\zeta(2k)}{\zeta(k)}x + O\left(x^{\frac{1}{k}-1}\right) + O\left(x^{\frac{1}{k}}\delta_k(x)\right)$$

$$= \frac{\zeta(2k)}{\zeta(k)}x + O\left(x^{\frac{1}{k}}\delta_k(x)\right)$$

Hence theorem 3.2 follows.

Corollary 3.2 For $x \geq 3$,

$$\sum_{m\leq x} \frac{1}{\Gamma_2(m)} = \frac{\zeta(4)}{\zeta(2)}x + O\left(x^{\frac{1}{2}}\delta_2(x)\right) \tag{3.5}$$

Remark 3.1 We have $T_k(n) \times \Gamma_k(n) = n$, Where $T_k(n)$ is the divisor of n conjugate to the greatest k-th power divisor of n. so that $T_k(n) = n/\Gamma_k(n)$

Now we have the following.

Theorem 3.3. For $x \geq 3$.

$$\sum_{m\leq x} T_k(m) = \frac{x^2}{2}\frac{\zeta(2k)}{\zeta(k)} + O\left(x^{1+\frac{1}{k}}\delta_k(x)\right). \tag{3.6}$$

Proof. This follows by theorem 3.2 and Abels identity (cf. (1), Theorem 4.2)

Corollary 3.3 For $x \geqslant 3$

$$\sum_{m\leq x} T_2(m) = \frac{x^2}{2}\frac{\zeta(4)}{\zeta(2)} + O(x^{3/2}\delta_2(x)) \tag{3.7}$$

Remark 3.2 From (3.7) it follows that the average order of T_2 (m) is $\frac{\zeta(4)}{2\zeta(2)}x = \frac{\pi^2 x}{30}$ a result that has been established by E-Cohen (cf (4), Corollary 2.23).

Theorem 3.4 For $x \geqslant 2$

$$\sum_{m\leq x}\Gamma_k^*(m) = \frac{x^{1+\frac{1}{k}}}{(k+1)}\sum_{m=1}^{\infty}\frac{q_k^*(m)\varphi(m)}{m^{2+\frac{1}{k}}} + O(x\log^2 x) \tag{3.8}$$

Proof: By (2.33), lemma 2.1, Remark 2.1, (2.4) and (2.6) we have

$$\sum_{m\leq x}\Gamma_k^*(m) = \sum_{m\leq x}\sum_{\substack{d^k\delta = m\\(d,\delta)=1}} d^k q_k^*(\delta) = \sum_{\delta\leq x} q_k^*(\delta)\sum_{\substack{d\leq\left(\frac{x}{\delta}\right)^{\frac{1}{k}}\\(d,\delta)=1}} d^k$$

$$= \sum_{m\leq x} q_k^*(m)\left\{\frac{1}{k+1}\left(\frac{x}{m}\right)^{1+\frac{1}{k}}\frac{\varphi(m)}{m} + O\left(\theta(m)\left(\left(\frac{x}{m}\right)^{\frac{1}{k}}\right)^k\right)\right\}$$

$$= \frac{x^{1+\frac{1}{k}}}{k+1}\sum_{m\leq x}\frac{q_k^*(m)\varphi(m)}{m^{2+\frac{1}{k}}} + O\left(x\sum_{m\leq x}\frac{\theta(m)}{m}\right)$$

$$= \frac{x^{1+\frac{1}{k}}}{(k+1)}\sum_{m=1}^{\infty}\frac{q_k^*(m)\varphi(m)}{m^{2+\frac{1}{k}}} + O\left(x^{1+\frac{1}{k}}\sum_{m>x}\frac{1}{m^{1+\frac{1}{k}}}\right) + O\left(x\sum_{m\leq x}\frac{\tau(m)}{m}\right)$$

$$= \frac{x^{1+\frac{1}{k}}}{(k+1)}\sum_{m=1}^{\infty}\frac{q_k^*(m)\varphi(m)}{m^{2+\frac{1}{k}}} + O(x) + O(x\log^2 x)$$

Hence the proof of theorem 3.4 is complete.

Corollary 3.4. For $x \geqslant 3$,

$$\sum_{m\leq x}\Gamma_2^*(m) = \frac{x^{3/2}}{3}\sum_{m=1}^{\infty}\frac{q_2^*(m)\varphi(m)}{m^{5/2}} + O(x\log^2 x) \tag{3.9}$$

Where $\Gamma_2^*(m)$ denotes the maximal square, unitary divisor of m.

Theorem 3.5. For $x \geq 3$

$$(3.10) \quad \sum_{m\leq x} \frac{1}{\Gamma_k^*(m)} = x e_k \sum_{n=1}^{\infty} \frac{\varphi(n)\,E_k(n)}{n^{2(k+1)}} + O\left(x^{\frac{1}{k}}\delta_k(x)\right)$$

Where e_k and $E_k(n)$ are respectively given by (2.1) and (2.2)

Proof. By (2.34) and lemma 2.13 we have

$$\sum_{m\leq x}\frac{1}{\Gamma_k^*(m)} = \sum_{m\leq x}\sum_{\substack{d^k\delta=m\\(d,\delta)=1}} \frac{q_k^*(\delta)}{d^k} = \sum_{\substack{d^k\delta\leq x\\(d,\delta)=1}} \frac{q_k^*(\delta)}{d^k}$$

$$(3.11) \quad = \sum_{d\leq x^{\frac{1}{k}}} \frac{1}{d^k} \sum_{\substack{\delta\leq x/d^k\\(\delta,d)=1}} q_k^*(\delta) = \sum_{n\leq x^{\frac{1}{k}}} \frac{1}{n^k}\left\{\frac{e_k\varphi(n)\,E_k(n)}{n^2}\,\frac{x}{n^k} + O\left(\theta(n)\left(\frac{x}{n^k}\right)^{\frac{1}{k}}\right.\right.$$

$$\left.\left.\times\,\delta_k\left(\frac{x}{n^k}\right)\right)\right\} = e_k\cdot x\sum_{n\leq x^{\frac{1}{k}}}\frac{\varphi(n)\,E_k(n)}{n^{2(k+1)}} + O\left(\sum_{n\leq x^{\frac{1}{k}}}\frac{\theta(n)}{n^k}\left(\frac{x}{n^k}\right)^{\frac{1}{k}}\delta_k\left(\frac{x}{n^k}\right)\right)$$

We have $\frac{1}{E_k(n)} = O\left(\frac{1}{n}\right)$ since $n\prod_{p|n}\left(\frac{p(p^k-1)}{p^{k+1}-2p+1}\right) > n\prod_{p|n}\left(1-\frac{1}{p^2}\right)$

$> n\prod_p\left(1-\frac{1}{p^2}\right) = \frac{n}{\zeta(2)}$ All since $x^{\frac{1}{k}}\delta_k(x)$ is monotanic increasing for sufficiently large x we have

$$\sum_{m\leq x^{\frac{1}{k}}}\frac{\theta(m)}{m^k}\left(\frac{x}{m^k}\right)^{\frac{1}{k}}\delta_k\left(\frac{x}{m^k}\right) = O\left(x^{\frac{1}{k}}\delta_k(x)\sum_{m\leq x^{\frac{1}{k}}}\frac{\theta(m)}{m^k}\right)$$

$$= O\left(x^{\frac{1}{k}}\delta_k(x)\sum_{m\leq x}\frac{1}{m^{k-\epsilon}}\right) = O\left(x^{\frac{1}{k}}\delta_k(x)\right)$$

follows 1, Remark 2.1 and $\tau(m) = O(m^{\epsilon})$ for every $\epsilon > 0$.

Thus from (3.11) and the above discussion we have

$$\sum_{m\leq x}\frac{1}{\Gamma_k^*(m)} = x e_k\left\{\sum_{n=1}^{\infty}\frac{\varphi(n)\,E_k(n)}{n^{2(k+1)}} + O\left(\sum_{n\geq x^{\frac{1}{k}}}\frac{1}{n^{2k}}\right)\right\} + O\left(x^{\frac{1}{k}}\delta\right.$$

$$= x e_k \sum_{n=1}^{\infty} \frac{\varphi(n) E_k(n)}{n^{2(k+1)}} + O\left(x^{\frac{1}{k}-1}\right) + O\left(x^{\frac{1}{k}} \delta_k(x)\right)$$

$$= x e_k \sum_{n=1}^{\infty} \frac{\varphi(n) E_k(n)}{n^{2(k+1)}} + O\left(x^{\frac{1}{k}} \delta_k(x)\right)$$

This completes the proof of theorem 3.5

<u>Corollary 3.5</u> For $x \geq 3$

(3.12) $$\sum_{m \leq x} \frac{1}{\Gamma_2^*(m)} = x e_2 \sum_{n=1}^{\infty} \frac{\varphi(n) E_2(n)}{n^6} + O\left(x^{\frac{1}{2}} \delta_2(x)\right)$$

<u>Remark 3.3</u> We note that each positive integer n can be written uniquely as $n = \Gamma_{u,k}^*(n) \Gamma_k^*(n)$ where $\Gamma_{u,k}^*(n)$ denotes the maximal unitarily k-free, unitary divisor of n.

As a consequence of Remark 3.3, Theorem 3.5 and Abels identity (cf (1), Theorem 4.2) we have the following.

<u>Theorem 3.6</u> For $x \geq 3$

(3.13) $$\sum_{m \leq x} \Gamma_{u,k}^*(m) = \frac{x^2}{2} e_k \sum_{n=1}^{\infty} \frac{\varphi(n) E_k(n)}{n^{2(k+1)}} + O\left(x^{1+\frac{1}{k}} \delta_k(x)\right)$$

<u>Corollary 3.6</u> For $x \geq 3$

(3.14) $$\sum_{m \leq x} \Gamma_u^*(n) = \frac{x^2}{2} e_2 \sum_{n=1}^{\infty} \frac{\varphi(n) E_2(n)}{n^6} + O\left(x^{3/2} \delta_2(x)\right)$$

Where $\Gamma_u^*(n)$ denote the maximal exponentially odd, unitary divisor of m.

<u>Theorem 3.7.</u> For $x \geq 2$,

(3.15) $$\sum_{m \leq x} \frac{1}{\Gamma_{u,k}^*(m)} = x^{\frac{1}{k}} \sum_{m=1}^{\infty} \frac{q_k^*(m) \varphi(m)}{m^{2+\frac{1}{k}}} + O\left(\log^3 x\right)$$

Proof. This follows by Remark 3.3. Abels identity (cf (1), Theorem 4.2 and theorem 3.4)

§ 4. Here we discuss the consequences of the Riemann hypothesis.

Theorem 4.1. If the Riemann hypothesis is true, then for $x \geqslant 3$ the O - term in (3.3) can be replaced by $O(x^{2/2k+1}\,\omega(x))$ where $\omega(x)$ is given by (2.14)

Proof. Following the same-procedure adopted in the proof of Theorem 3.2 and making use of lemma 2.10 instead of lemma 2.9 and noting that $\omega(x)$ is monotonic increasing we get theorem 4.1.

Theorem 4.2. If the Riemann hypothesis is true, then for $x \geqslant 3$, the O-term in (3.6) can be replaced by $O(x^{1+\frac{2}{2k+1}}\,\omega(x))$

Proof. Following the same procedure adopted in the proof of theorem 3.3 and making use of theorem 4.1 instead of theorem 3.2 and noting that $\omega(x)$ is monotonic increasing we get theorem 4.2.

Theorem 4.3 of the Riemann hypothesis is true, then for $x \geqslant 3$ the O-term in (3.10) can be replaced by $O(x^{2/2k+1}\,\omega(x))$

Proof. Following the same procedure adopted in the proof of theorem 3.5 and making use of lemma 2.14 instead of lemma 2.13 and noting that $\omega(x)$ is monotonic increasing we get theorem 4.3.

Theorem 4.4 If the Riemann hypothesis is true, then for $x \geq 3$, the O-term in (3.13) can be replaced by $O\left(x^{1+\frac{2}{2k+1}}\,\omega(x)\right)$

Proof. Following the same procedure adopted in the proof of theorem 3.6 and making use of theorem 4.3 instead of theorem 3.5, we get theorem 4.4.

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REPRESENTING INTEGERS AS SUMS OF TWO RELATIVELY PRIME COMPOSITE INTEGERS

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It has been known that every sufficiently large integer is a sum of two integers each of which is a product of at most three primes. In this note we prove a much more modest but also a slightly different result. Just and Schaumberger (1) have proved that every even integer greater than 38 is a sum of two odd composite integers. Here, we prove the following

Theorem : Every sufficiently large integer is the sum of two relatively prime composite integers.

Proof: Clearly

$$n = a + b,\ (a, b) = 1 \Rightarrow (n, a) = 1 \text{ and } a \leq n;$$

Conversely

$$(a, n) = 1,\ a \leq n \Rightarrow n = a + (n - a) \text{ with } (a, n - a) = 1.$$

Thus

$$n = a + b,\ (a, b) = 1 \text{ iff } (n, a) = 1 \text{ and } a \leq n.$$

Let $a_1, a_2, \ldots, a_k$ be the $k = \phi(n)$ integers not exceeding n and relatively prime to n $(n \geq 3)$. Then

$$\left.\begin{array}{l} a_1 + (n - a_1) \\ a_2 + (n - a_2) \\ \ldots \quad \ldots \quad . \\ a_k + (n - a_k) \end{array}\right\} \qquad (I)$$

are all the representations of n as a sum of two relatively prime integers. Of course in (I), each representation occurs twice since $a_i + (n - a_i)$ also occurs as $a_j + (n - a_j)$ where $a_j = n - a_i$. So the number of distinct representations is $1/2\ \phi(n)$.

The representations $a_i + (n - a_i)$ and $a_j + (n - a_j)$ will be called duplicates of each other if $a_i + a_j = n$.

Now we want a_i and $n - a_i$ not only to be relatively prime but also composite. If $n = a + b$, a, b composite and $(a, b) = 1$, we shall call this a relevent representation.

First, note that the set $\{a_1, a_2, \ldots, a_k\}$ contains all the primes upto n except those that divide n. So this set contains $\pi(n) - W(n)$ primes, where $\pi(n)$ is the number of primes not exceeding n and $W(n)$ denotes the number of prime divisors of n. In addition to these $\pi(n) - W(n)$ noncomposite numbers, the set contains one more noncomposite number, viz. 1. Therefore, out of the k representations (I), $\pi(n) - W(n) + 1$ are certainly not relevent. The duplicate of an irrelevent representation is itself irrelevent. So there are at most $2\{\pi(n) - W(n) + 1\}$ representations in (I) which are not relevent. Hence the number of relevent representations in (I) is at least.

$$\phi(n) - 2\{\pi(n) - W(n) + 1\}.$$

Eliminating duplications, we see that the number of distinct relevent representations of n is at least $\frac{1}{2}\phi(n) - \{\pi(n) - W(n) + 1\}$.

We have therefore proved that if $f(n)$ is the number of ways in which n can be represented as a sum of two relatively prime composite integers, then for $n \geqslant 3$,

$$f(n) \geqslant \left\{\tfrac{1}{2}\phi(n) - \pi(n)\right\} + \left\{W(n) - 1\right\}.$$

Now (i) $W(n) - 1 \geqslant 0$,

(ii) $$\liminf_{n \to \infty} \frac{\phi(n)}{n/\log\log n} = e^{-C},$$

where C is Euler's constant (2) and by Prime Number Theorem

(iii) $$\lim_{n \to \infty} \frac{\pi(n)}{n/\log n} = 1.$$

Thus as $n \to \infty$, $\phi(n)$ has a larger order than $\pi(n)$ consequently as $n \to \infty$, we find that $f(n) \to \infty$. This not only proves the theorem as stated but actually proves the following:

Given a positive integer t, every sufficiently large integer can be expressed as a sum of two relatively prime composite integers in at least t different ways.

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PARTITIONS WITH CONGRUENCE CONDITIONS AND COLOR RESTRICTIONS

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Abstract:

In this paper the author has obtained a convergent series for $p(n;\mathcal{A},m,\mathcal{S})$, the number of partitions of a positive integer n into the parts congruent to $\pm a_j$ (mod m), with any part congruent to $\pm a_j$ (mod m) appearing in atmost δ_j colors and allowing the repetitions of the parts with the same color, where

$$a_j \in \mathcal{A} : \{a_1, a_2, ---, a_r\}, 1 \leq a_j < \frac{m}{2}, \delta_j \in \mathcal{S} : \{\delta_1, \delta_2, ---, \delta_r\}.$$

using the Hardy-Ramanujan-Rademacher method.

§1. Introduction: In this paper we obtain a convergent series and asymptotic formulae for $p(n;\mathcal{A},m,\mathcal{S})$ the number of partitions of a positive integer n into the parts congruent to $\pm a_j$ (mod m), with any part congruent to $\pm a_j$ (mod m), appearing in atmost δ_j colors and allowing repetitions of the parts with the same color where $a_j \in \mathcal{A} : \{a_1, a_2, ---, a_r\}$

$$1 \leq a_j < \frac{m}{2}, \quad \delta_j \in \mathcal{S} : \{\delta_1, \delta_2, ---, \delta_r\}.$$

To obtain the convergent series for the partition function $p(n;\mathcal{A},m,\mathcal{S})$ we follow the Farey circle dissection method of Rademacher [4]. For this we need the behaviour of their generating functions in the neighbourhood of a rational point on a circle of radius $<$ 1 and concentric to the unit circle.

§2: Transformation formula for the generating function of $p(n;a,m,\delta)$

The generating functions of $p(n;a,m,\delta)$ is given by

$$F(x;a,m,\delta)=F(x)=\prod_{a_j\in a}\left[\prod_{\nu=0}^{\infty}(1-x^{m\nu+a_j})^{-\delta_j}\right]\times\left[\prod_{\nu=1}^{\infty}(1-x^{m\nu-a_j})^{-\delta_j}\right]$$

$$=\sum p(n;a,m,\delta)x^n=\sum_{n=0}^{\infty}p(n)x^n$$

with $p(0)=1$.

To study the behaviour of F(x) in the neighbourhood of $\frac{h}{k}$ for any two integers h,k such that $(h,k)=1$, $0\leq h<k$, we take $x=\left\{\exp\left(\frac{2\pi i h}{k}-\frac{2\pi z}{k}\right)\right\}$, $Re\, z>0$ and study the the transformation $x\Rightarrow X$ where $X=\left\{\exp\left(\frac{2\pi i h'\gamma}{k}-\frac{2\pi}{Kz}\right)\right\}$

h' being a fixed solutions of hh' $\equiv$ -1(mod k),

$$K=\{m,k\},\ (\text{l.c.m. of } m,k),\ d=(m,k)$$

with $k=dk'$, $m=dm'$ and α, γ being a pair of fixed integers, satisfying $\alpha k'+\gamma m'=1$.

We set $m\nu \pm a_j = gK + \mu_j$, $0 < \mu_j < K$.

Using Mellin's formula, properties of the Hurwitz Zeta function and following the method of Subrahmanya Sastri [5], we obtain the transformation formula as

$$F(x) = 2^{-\tau^0} \chi(d,h,k) \prod_{a_j \in \mathcal{a}^0} \left(\operatorname{Cosec}^{\delta_j} \frac{\pi \ell_j^0}{m'} \right) \times$$

$$\times \exp\left\{ \frac{\pi}{6mk} \left(\frac{B}{z} - Az \right) \right\} H(x;\, \mathcal{a}, \rho, d, k) \quad (2.1)$$

where $\ell_j^0 \equiv \ell \left[\frac{a_j}{d} \right] (\mathrm{mod}\ m')$, $0 < \ell_j^0 < \frac{m'}{2}$,

$h a_j \equiv \pm b_j\ (\mathrm{mod}\ d)$ whichever yields $0 \leqslant b_j \leqslant \left[\frac{d}{2} \right]$

$\chi(d,h,k)$ is exponential of a generalised Dedekind sum viz

$\chi(d,h,k) = \exp\{\pi i\, T(d,h,k)\}$ where

$$T(d,h,k) = \sum_{a_j \in \mathcal{a}} \delta_j \sum_{\mu_j} \left(\left(\frac{h\mu_j}{k} \right)\right) \left(\left(\frac{\mu_j}{K} \right)\right), \quad 0 < \mu_j \equiv \pm a_j (\mathrm{mod}\, m) < K,$$

$$H(x;\mathcal{a},\rho,d,k) = \prod_{a_j \in \mathcal{a}} \left\{ \prod_{\nu=0}^{\infty} \left(1 - \rho_j x^{d\nu + b_j}\right)^{-\delta_j} \times \right.$$

$$\left. \times \prod_{\nu=1}^{\infty} \left(1 - \rho_j x^{d\nu - b_j}\right)^{-\delta_j} \right\} \quad (2.2)$$

with $\rho_j = e^{\pm 2\pi i \ell a_j/m}$ according as $\mu_j \equiv \pm a_j (\mathrm{mod}\ m)$,

$\mathcal{a}^0 = \{a_j \mid a_j \in \mathcal{a},\, a_j \equiv 0\ (\mathrm{mod}\ d)\}$,

$A = \sum_j \delta_j A_j = \sum_j \delta_j (m^2 - 6ma_j + 6a_j^2)$

$B = \sum_j \delta_j B_j = \sum_j \delta_j (d^2 - 6db_j + 6b_j^2)$

§3: Convergent Series for $p(n; a, \varphi, m)$: :

In this section we deal with the main problem of determining p(n) in the form of convergent series using the transformation equation (2.1) obtained in §2. For this purpose, we express p(n) as contour integrals on the circle $|x| = \exp\{-2\pi N^{-2}\}$ using Cauchy's integral formula and obtain the integrals as convergent series using the Farey circle dissection method of Rademacher [4].

For this we shall first find the estimate of an exponential sum of the roots of unity involving $\chi(d, h, k)$ taken over the integers belonging to a reduced system of residues (mod k) as the trivial estimate O(k) will not suffice out purpose. We adopt essentially the methodology of Lehner [3], Hagis (Jr.) [1] and Subrahmanya Sastri [5] and obtain the following result.

If $A \neq 12mn$ the exponential sum

$$S(n, \nu, \eta) = \sum_{\substack{h \,(\mathrm{mod}\, k) \\ (h,k)=1 \\ h \equiv \eta \,(\mathrm{mod}\, d)}} \chi(d, h, k) \exp\left\{\frac{-2\pi i}{k}(hn - h'\gamma\nu)\right\} \tag{3.1}$$

(where $hh' \equiv -1 \pmod k$), is subject to the estimate $O(n^{\frac{1}{3}} k^{\frac{2}{3}+\epsilon})$ we shall now find a convergent series for p(n). Applying Cauchy's integral formula and Farey circle dissection method, following the method of Subrahmanya Sastri [5], we obtain the following theorem.

THEOREM 1: If $n \geqslant 0$, $n \neq A/12m$, then for the number p(n) of partitions of n into any positive summands congruent to $\pm a_j$ (mod m) with any part congruent to $\pm a_j$ (mod m) appearing in almost δ_j colors and allowing the repetitions of the parts with the same color, where

$$a_j \in \mathfrak{a}\{a_1, a_2, \ldots, a_r\},\ 1 \leq a_j < \frac{m}{2},\ \delta_j \in \mathcal{S}:\{\delta_1, \delta_2, \ldots, \delta_r\},$$

we have the convergent series representation

$$p(n) = 2\pi \sum_{\substack{k=1 \\ (k,m)=d}}^{\infty} \sum_{\substack{0<\eta<d \\ (\eta,d)=1}} \left[2^{-\tau^{\circ}} \prod_{a_j \in \mathfrak{a}^{\circ}} \operatorname{Cosec}^{\delta_j^{\circ}} \frac{\pi \delta_j^{\circ}}{m'} \right] \times$$

$$\times \sum_{\nu < \frac{B}{12d}} c(\nu, \eta)\, \mathfrak{S}(n, \nu, \eta)\, L(n, \nu, \eta)$$

where $\mathfrak{S}(n,\nu,\eta)$ is given by (3.1)

$$L(n,\nu,\eta) = \frac{(B-12\nu d)^{\frac{1}{2}}}{k(12mn-A)^{\frac{1}{2}}} I_1\left\{\frac{\pi}{3mk}(12mn-A)^{\frac{1}{2}}(B-12d\nu)^{\frac{1}{2}}\right\} \quad \text{if } n > \frac{A}{12m},$$

$$\frac{(B-12\nu d)^{\frac{1}{2}}}{k(A-12mn)^{\frac{1}{2}}} J_1\left\{\frac{\pi}{3mk}(A-12mn)^{\frac{1}{2}}(B-12d\nu)^{\frac{1}{2}}\right\} \quad \text{if } n < \frac{A}{12m},\ A > 0,$$

$I_1(z)$, $J_1(z)$ being the Bessel functions of the first order whose expressions are given by

$$I_1(z) = -i\, J_1(z) = \sum_{\lambda=0}^{\infty} \left\{ \left(\frac{z}{2}\right)^{2\lambda+1} \frac{1}{\lfloor\lambda \lfloor\lambda+1} \right\}$$

and $C(\nu, \eta)$ is the coefficient of x^{ν} in $H(x; a, \mathcal{S}\, d, k)$ in (2.2).

Here we exclude $n = \frac{A}{12m}$, for in this case, the exponential sums in (3.1) do not admit of a better general estimate than $O(k)$ which will not be sufficient for the purpose. For this particular value of n, p(n) can be obtained directly by finding the coefficient of $x^{A/12m}$ in (2.1).

§4. <u>Special Case</u>

We consider the case $a : \{a_j \mid 1 \leq a_j < \frac{m}{2},\ (a_j, m) = 1\}$ and $\mathcal{S} = \{s_j \mid s_j = s, \forall j\}$. This gives us the formula for the number of partitions of a positive integer n into parts relatively prime to m with each part appearing in at most clolors and allowing the repetitions of the parts with the same color and we obtain the following theorem from Theorem 1.

<u>Theorem 2</u>: For $n > 0$, $n \neq \dfrac{s\,\varphi(m)\, M \mu(M)}{24m}$ the convergent series for $p^*(n, s)$ the number of partitions of n into parts relatively prime to m, each part appearing in

atmost 's' colors and allowing repetitions of the parts with the same color is given by

$$p^*(n,s) = 2\pi T(m) \sum_{\substack{d \mid m \\ \mu(D)=1}} \sum_{k=d}^{\infty} \sum_{\nu < \frac{s\varphi(m)D}{24d}} c(\nu)\, S(n,\nu)\, L(n,\nu)$$

where

$$S(n,\nu) = \sum_{\substack{0<\eta<d \\ (\eta,d)=1}} S(n,\nu,\eta), \quad c(\nu) = c(\nu,\eta),$$

$$L(n,\nu) = L(n,\nu,\eta) \text{ for all } \eta$$

and

M = product of all distinct primes dividing m

D = product of all distinct primes dividing d

If we put s = 1 in Theorem 2, we obtain Subrahmanya Sastri's result [5] . If we put, in particular, that m is square free and if s = 1, then D = d and we obtain Iseki's result [2] . If we put m = p, an odd prime p > 3, s = 1, we obtain Hagis result [1] .

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ON THE USE OF MATRIX METHODS IN CERTAIN NUMBER THEORETIC PROBLEMS

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ABSTRACT:

Matrix theory is used to study a spectral decomposition of arithmetic functions introduced recently by Krishnaswami Alladi. Also there is included a brief sketch of the recent work of the author and Santhanam on the use of the concept of generalized matrix inverse to solve elegantly certain set of linear diaphontine equations occurring in the study of the structure of the Lie group SU(3)

1. INTRODUCTION

In two earlier papers[1,2] the author has used a matrix approach to understand the Dirichlet products and inverses of arithmetic functions and hence to derive certain number theoretic identities. Particularly the Mobius function $\mu(n)$ has been represented in the interesting form

$$\mu(n) = \sum_{k=1}^{\Omega(n)} (-1)^k \left\{ \sum_{t=1}^{k} (-1)^{k-t} \binom{k}{t} \left[\prod_{i=1}^{r} \binom{\alpha_i + t - 1}{\alpha_i} \right] \right\} ; \quad \forall\, n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1, \tag{1.1}$$

resulting from the identification that the matrix M with elements

$$M_{k\ell} = \begin{cases} \mu\left(\frac{k}{\ell}\right) & \text{if } \ell \mid k, \\ 0 & \text{other wise,} \end{cases} \tag{1.2}$$

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*Section 3 of the paper presents the joint work of the author and T.S.Santhanam, mentioned in the article by T.S.Santhanam in this proceedings, pp 139-142 is the inverse of the matrix E with elements

$$E_{k\ell} = \begin{cases} 1 & \text{if } \ell | k, \\ 0 & \text{other wise.} \end{cases} \tag{1.3}$$

It is clear that if matrices are associated with arithmetic functions in the above fashion then the Dinchlet product can be understood simply as a matrix product.

Recently Krishnaswami Alladi[3] has shown that if we define

$$\sum_{d|n} \mu(d) f(d) = f^*(n) \tag{1.4}$$

then for a square-free function

$$f_+ = \frac{1}{2}(f + f^*)$$

and (1.5)

$$f_- = \frac{1}{2}(f - f^*)$$

(1.6)

are such that $f_+^* = f_+ , \; f_-^* = -f_-$

and $f^{**} = f.$ (1.7)

In section 2 below we shall understand this result as a special case of a spectral decomposition of any arithmetic function in terms of matrix theory.

Section 3 gives a brief report of a recent attempt of the author and Santhanam to study certain sets of linear diaphontine equations occuring in the problem of internal multiplicity structure of weights for the SU(3)-group[4,5] using the concept of generalized matrix inverse.

2. A SPECTRAL DECOMPOSITION OF ARITHMETIC FUNCTIONS:

Let us rewrite the product relation in (1.4) as

$$\begin{pmatrix} \mu(1) & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \mu(1) & \mu(2) & 0 & \cdot & \cdot & \cdot & \cdot \\ \mu(1) & \mu(2) & \mu(3) & \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & & & \end{pmatrix} \begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ \vdots \end{pmatrix} = \begin{pmatrix} f^*(1) \\ f^*(2) \\ f^*(3) \\ \vdots \end{pmatrix}. \tag{2.1}$$

Since $\mu(n)$ takes only three values (0, +1, -1) the eigen value equation

$$\sum_{d|n} \mu(d) g(d) = \lambda g(n) \tag{2.2}$$

admits only three eigen values, namely

$$\lambda = 0, \pm 1. \tag{2.3}$$

If we now solve the eigen value equations (2.2) for the particular values of λ given by (2.3) it is found easily that any arithmetic function f(n) is such that (f_0, f_+, f_-) defined by

$$\begin{aligned} f_0(n) &= f(n) - f(\gamma(n)) \\ f_+(n) &= \tfrac{1}{2}\{f(\gamma(n)) + f^*(\gamma(n))\} \\ f_-(n) &= \tfrac{1}{2}\{f(\gamma(n)) - f^*(\gamma(n))\} \end{aligned}$$

or

$$\begin{pmatrix} f_+(n) \\ f_-(n) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} f(\gamma(n)) \\ f^*(\gamma(n)) \end{pmatrix} \tag{2.4}$$

where

$$\gamma(n) = \prod_{i=1}^{r} p_i \ ; \ \text{for } n = \prod_{i=1}^{r} p_i^{\alpha_i}. \tag{2.5}$$

obey

$$f_0^*(n) = 0, \quad f_+^*(n) = f_+(n), \quad f_-^*(n) = -f_-(n). \tag{2.6}$$

Then we have

$$f(n) = f_0(n) + f_+(n) + f_-(n), \quad f^{**}(n) = f(\gamma(n)). \tag{2.7}$$

Thus for any arithmetic function there is a spectral decomposition having the properties given by (2.1) and (2.4) - (2.7). The result of Krishnaswami Alladi given in section-1 for square-free functions with $f_0(n) = 0$ is seen to be a special case of the above general result for any arithmetic function.

Krishnaswami Alladi[-3] has also defined a generalized Mobius function

$$\mu_z(n) = \begin{cases} 1 & \text{for } n = 1 \\ z^{\omega(n)} & \text{for square-free } n > 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

associated with the inversion formula

$$\sum_{d|n} f(d) = F(n) \Leftrightarrow f(n) = \sum_{d_1|n} F\left(\frac{n}{d_1}\right)\mu_z(d_1) \sum_{d_2|d_1} \mu_z(d_2) \sum_{d_3|d_2} \mu_z(d_3) \cdots$$

$$\cdots \sum_{d_{n-1}|d_{n-2}} \mu_z(d_{n-1}) \tag{2.9}$$

so that $z = -1$ corresponds to the well-known Mobius inversion formula. It is quite clear ~~that~~ analoguous to this developement one can easily generalize the above-discussed spectral decomposition of arithmetic functions also by replacing $\{-1 = 2\sqrt{1}\}$ by $\{z_n = \exp\left(\frac{i2\pi}{n}\right)\}$. Now since the equation $g^*(n) = \sum_{d|n} \mu_z(d) g(d) = \lambda g(n)$ has (n+1) eigen values $\{0, z_n^k ; k = 0, 1, 2, \cdots n-1\}$ the analogue of (2.4) in this case should involve the n-dimensional Sylvester or finite Fourier transform matrix

3. ON A SET OF DIAPHONTINE EQUATIONS:

In the study of the multiplicity structure of the weights of SU(3) - group using Kostant's formula Santhanam[5] (See also 4 for earlier literature) has found that the set of linear diaphontine equations

$$\begin{pmatrix} k_1 = a_1 + a_3 \\ k_2 = a_2 + a_3 \end{pmatrix} \text{ or } \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad (3.1)$$

with known integer values for (k_1, k_2) are to be solved for the unknown integer values of (a_1, a_2, a_3). Recently the author and Santhanam have reexamined this problem as follows Let us denote

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = C. \quad (3.2)$$

Then by the well-known theorem [6,7] on the general solution of the equation of the type (3.1) it follows that we can write

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} C_g^{-1} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \left(C_g^{-1} C \right) \right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (3.3)$$

where (x_1, x_2, x_3) are arbitrary and C_g^{-1} is the generalized inverse of the matrix C. In this case we have

$$C_g^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix},$$

$$C_g^{-1}C = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \tag{3.4}$$

if we compute C_g^{-1} using the standard procedure due to Greville (Cf. 7 for the details). Now using (3.4), (3.3) can be explicitly written as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2k_1 - k_2 + x_1 + x_2 - x_3 \\ -k_1 + 2k_2 + x_1 + x_2 - x_3 \\ k_1 + k_2 - x_1 - x_2 + x_3 \end{pmatrix}. \tag{3.5}$$

Let us now denote

$$X = x_1 + x_2 - x_3 ,$$
$$Y = k_1 + k_2 - X . \tag{3.6}$$

Then (3.5) becomes

$$a_1 = k_1 - \frac{Y}{3} ,$$

$$a_2 = k_2 - \frac{Y}{3} ,$$

$$a_3 = \frac{Y}{3} . \tag{3.7}$$

Since (x_1, x_2, x_3) or X can be chosen arbitrarily (3.6) and (3.7) indicate clearly that the integer solutions for ((a_1, a_2, a_3) are obtained choosing

$$Y = 0,3,6, \ldots\ldots, 3 \min (k_1, k_2). \tag{3.8}$$

Thus leading to the result that total number of solutions $= 1 + \min (k_1, k_2)$ (3.9)

for the set of diaphontine equations (3.1). The same result has been derived earlier in a different fashion, a review of which has been given by Santhanam[8] in these proceedings also. For higher order Lie groups also the sets of linear diaphontine equations to be solved in the above problem are of the same structure as (3.1) though involving higher dimensional matrices in place of the simple C-matrix obtained for SU(3). We hope that the above elegant method based on the use of generalized matrix inverse can be adopted fruitfully in the other cases also. We plan to discuss this in future elsewhere.

ACKNOWLEDGEMENTS:

I wish to thank Professor Alladi Ramakrishnan for kind and constant encouragement. I am grateful to Dr. Krishnaswami Alladi for very useful discussions. My thanks are also due to Professor T.S. Santhanam for discussions and fruitful collaboration.

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DIOPHANTINE EQUATIONS AND PARTITION FUNCTIONS

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Many years ago, when I was working on the problem of 'multiplicity' in classical groups, I came across the problem of solving, more precisely to find the degeneracy of the solutions, some linear diophantine equations. I solved the problem by using the method of generating functions.

The multiplicity M of a weight m which belongs to the irreducible representation of a group G with the highest weight Λ is given by Kostaut's formula[1)]

$$M^{\Lambda}(m) = \sum_{S \in W} \delta_S P[m + R_0 - S(\Lambda + R_0)] \quad (1)$$

The sum is over the elements S of the discrete Weyl group W, $\delta_S = \pm 1$ depending on whether the action of S permutes even or odd. P is the partion function defined by

$P(A)$ = number of ways

A can be written as

$$A = \sum_{\alpha_\mu \in \Sigma^+} a_\mu \alpha_\mu, \qquad (2)$$
$$\alpha_\mu = \text{integers} \geq 0$$

The α's belong to the space Σ^+ of the positive roots of G and

$$R_0 = \frac{1}{2} \sum_{\mu \in \Sigma^+} \alpha_\mu \tag{3}$$

Since A can be written in terms of simple roots β as

$$A = \sum_{i=1}^{l} k_i \beta_i \tag{4}$$

$$\beta_i \in \Pi$$

$$k_i = \text{integers} \geq 0$$

The β_i's belong to the space Π of the simple roots, l is the rank of the group. Since α's can be expressed in terms of β's with non-negative coefficients, we see that the value of $\underline{P}(A)$ is equal to the degeneracy of the solutions of the Diophantine equations.

$$k_i = C_{i\mu} a_\mu \tag{5}$$

$$i = 1, \ldots, l$$

$$\mu = 1, \ldots, N$$

$$N = \text{number of positive roots of } G.$$

The problem is that given a set of non-negative integers k_i, to find the number of non-negative a_μ's which will satisfy the above equation.

In the simple case of $A_2 \sim su(3)$

$$C_{i\mu} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and Gruber and I[2] found the solution as

$$P = 1 + \min(k_1, k_2) \tag{6}$$

In the more complicated case of G_2, the matrix is

$$C_{i\mu} = \begin{pmatrix} 1 & 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Radhakrishnan and I[3] solved to find the solution in complete generality. Belinfante[4] has succeeded in writing a neat computer programme for the same.

Let us now discuss the general case. We define the generating function[5] for the group G as

$$f(x_1, \ldots, x_\ell) = \prod_{\mu=1}^{N} \frac{1}{1 - x_1^{c_{1\mu}} \cdots x_\ell^{c_{\ell\mu}}}$$

$$|x| \leq 1 \tag{7}$$

It can be easily seen that P is nothing but the coefficient of $x_1^{k_1} \cdots x_\ell^{k_\ell}$ in the Taylor expansion of f. The merit of this method is that one can set recursion relations for P and the problem can be solved by reducing the calculation to that of calculating P for low rank groups. For instance, for the case of unitary groups. As, one finds[5] that

$$P_{A_\ell}(k_1,\dots,k_\ell) = \sum_{i_{\ell-1}=i_{\ell-2}}^{\min(k_{\ell-1},k_\ell)} \sum_{i_{\ell-2}=i_{\ell-3}}^{k_{\ell-2}} \cdots \sum_{i_1=0}^{k_1} P_{A_{\ell-1}}(k_1-i_1,\dots,k_{\ell-1}-i_{\ell-1}) \tag{8}$$

In ref.5), I have built such recursions for all classical groups Recently, Jagannathan and I[7] solved the simple Su(3) problem using the method of generalized inverses.

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BROUWER'S FIXED POINT THEOREM

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It is very well known that all familiar proofs of the Brouwer's fixed point theorem use either combinatorial arguments, homology theory, differential forms or methods from geometric topology. Compare (1), (2), (3). The aim of this lecture is to present a proof of Brouwer's fixed point theorem, which is completely non-combinatorial and also to illustrate the power of the differentiable approach to topological questions. It is pertinent to mention that John Milner (4) has recently presented a very elementary proof of this classical theorem which is strangely very simple.

Let us recall that the Brouwer's fixed point theorem says that every continuous self map of the closed unit ball in R^n has a fixed point.

<u>Theorem</u>: Let $V_1^n = \{ x = (x_1, x_2, \ldots x_n) \in R^n / \; \| x \| \leq 1 \}$ be the unit ball in R^n. Suppose $f : V_1^n \to V_1^n$ is a continuous map, then there exists a point $x \in V_1^n$ such that $f(x) = x$.

<u>Proof</u> : <u>Step I</u> :

We first claim that it suffices to prove Brouwer's Theorem for differentiable maps. Suppose $f : V_1^n \to V_1^n$ is continuous. Let $\varepsilon > 0$. Let $\pi_\varepsilon : V_1^n \to V_{1-\varepsilon}^n$ be the retraction of V_1^n onto its subset $V_{1-\varepsilon}^n$

in formulas, we write

$$\pi_\varepsilon(x) = \begin{cases} x & \text{if } \|x\| < 1-\varepsilon \\ \dfrac{x}{\|x\|}(1-\varepsilon) & \text{if } \|x\| \geq 1-\varepsilon \end{cases}$$

It is clear that $d(f, \pi_\varepsilon f) \leq \varepsilon$ where d is the supremum metric. Consider the maps $\pi_{\frac{1}{m}} f : V_1^n \to V_{1-\frac{1}{m}}^n$, $m = 2, 3, \ldots$.
By Weierstrass approximation Theorem, we can find C^∞ maps ϕ_m such that $d(\pi_{\frac{1}{m}} f, \phi_m) \leq \frac{1}{m}$.

In particular this means that $\phi_m(V_1^n) \subseteq V_1^n$.

Assuming Brouwer's theorem for smooth maps, there is an

$$x^m \in V_1^n \text{ such that } \phi_m(x^m) = x^m, \; m = 2, 3, \ldots$$

Since V_1^n is compact (we may pass through a subsequence if necessary) we may assume $x^m \to x_0 \in V_1^n$. Using continuity of f and the triangle inequality, we note that x_0 is a fixed point for f.

<u>Step II</u> **:** To prove Brouwer's theorem for smooth maps, it is enough to prove that there is no smooth map $\phi : V_1^n \to S_1^n$ such that ϕ restricted to S_1^n is the identity.
($S_1^n = \{x \in \mathbb{R}^n , \|x\| = 1$). It is a folk-lore result that this last statement is actually equivalent to Brouwer's theorem and we do not step here to prove it. So what remains

then is to prove the no-retraction theorem for smooth maps. Suppose there were a C^{∞} map $\emptyset : V_1^n \to S_1^n$ whose restriction to S_1^n is the identity map. To say that $\emptyset$ is smooth on V_1^n means that there is an extension of $\emptyset$ to an open neighbourhood $N(V_1^n)$ of V_1^n which is C^{∞}, now $\emptyset : N(V_1^n) \to S_1^n$ is a smooth map from an n-dimensional manifold onto an (n-1) - dimensional submanifold. Let $p \in S_1^n$ be a regular value for $\emptyset$. (Thanks to Sard. Such a p exists!) Note that $\emptyset^{-1}(p)$ is a one dimensional submanifold of $N(V_1^n)$, since Codim of $\emptyset^{-1}(p)$ in $N(V_1^n)$ equals Codim of $\{p\}$ in S_1^n, which is n-1. Let K be the connected component of $\emptyset^{-1}(p)$ containing p. Remembering the result that every connected one-dimensional manifold is diffeomorphic either to an open interval or a circle, (5), we will have to consider two possibilities for K.

Step III: (a) K is diffeomorphic to an open interval. Now K is a closed subset of $N(V_1^n)$ and so $K \cap V_1^n$ is closed in V_1^n. Parametrize K by $\{\theta(s) : -\infty < s < +\infty\}$ and let $\theta(0)=p$. First of all, K must pierce S_1^n at p, i.e. it can not be that $K \subseteq V_1^n$ or $K \subseteq N(V_1^n) / V_1^n$. For otherwise K would be tangent to S_1^n at p which would contradict the regularity of $\emptyset$ at p. As $\theta(0) = p$, $\theta(-\varepsilon)$ must lie either inside or outside V_1^n for small $\varepsilon > 0$. Let us assume that $\theta(-\varepsilon)$ lies in the interior of V_1^n. We then claim that $\theta(s) \in \text{Int } V_1^n$ for all $s < 0$. Otherwise $q = \theta(s_o) \in S_1^n$ for some $s_o < 0$. Now $q \neq p$ for otherwise K would not be diffeomorphic to an open interval. But $\emptyset$ restricted to S_1^n is the identity and so $q = \emptyset(q) = \emptyset(\theta(s_0)) = p$. Hence s_0 cannot exist. Similarly $\theta(s) \in N(V_1^n) / V_1^n$ for all $s > 0$. Consider the set of points in V_1^n of the form $\theta(s)$, $s < 0$. Let $\{s_n\}$ be a sequence of real numbers tending monotonely to $-\infty$ and consider the sequence of points $\{\theta(s_n)\}$. Passing to a subsequence if necessary, we may assume $\lim_n \theta(s_n) = \alpha \in V_1^n$ exists. Now $\emptyset\,[\theta(s_n)] = p$ for all n and by continuity of $\emptyset$, $\emptyset(\alpha) = p$. Thus $\alpha \in \emptyset^{-1}(p) \cap \overline{K} \cap V_1^n = K \cap V_1^n$ since $K \cap V_1^n$ is closed in V_1^n. But this clearly contradicts the fact that K is diffeomorphic to an open interval. Hence case (a) is completed.

(b) K is diffeomorphic to a circle - as in case (a), it cannot be that K lies completely inside V_1^n or completely inside $N(V_1^n) / V_1^n$. But since K is essentially a circle, K must pierce S_1^n, in two points, say at p, q, $q \neq p$. But, as in case (a), $q = \emptyset(q) = \emptyset(p) = p$ a contradiction. This proves case (b) and completely proves no-retraction theorem.

<u>APPLICATIONS</u> : As a surprisingly concrete application of Brouwer, we can prove the theorem of Frobenius.

<u>Theorem</u> : If the entries in an nxn real matrix K are all non negative, then K has a real non negative eigenvalue.

<u>An idea of the Proof</u>: May assume that K is nonsingular; otherwise 0 is an eigenvalue. Let K also denote the

associated linear operator on R^n, and consider the map

$f : x \to Kx / |Kx|$ restricted to $S_1^{n-1} \to S_1^{n-1}$.

Note that this maps the 'first quadrant'.

$F = \{ (x_1, x_2, \ldots x_n) \in S^{n-1} : \text{all } x_i \geq 0 \}$ into itself.

We satisfy ourselves that F is homeomorphic to V_1^{n-1} and then invoke Brouwer.

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PROBLEMS

Problem 1 (by Krishnaswami Alladi, Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA)

Let $P(n)$ denote the largest prime factor of an integer $n > 1$ and put $P(1) = 1$. Let p denote a prime number. If f is a bounded arithmetic function satisfying

$$\lim_{x\to\infty} \frac{\log x}{x} \sum_{p\le x} f(p) = c, \tag{1}$$

then

$$\lim_{x\to\infty} \frac{1}{x} \sum_{1\le n\le x} f(P(n)) = c. \tag{2}$$

The converse of this statement is not true. In fact one can easily construct a bounded function f satisfying (2) such that the limit in (1) does not even exist.

Denote by $\eta_f(x) = \left| \frac{1}{x} \sum_{1\le n\le x} f(P(n)) - c \right|$. Show that if $\eta_f(x) \to 0$ sufficiently rapidly, as $x \to \infty$, then (1) holds with the same value of c. We conjecture that

$$\eta_f(x) = O((\log x)^{-2})$$

suffices.

PROBLEM 2. (by K.Ramachandra, School of Mathematics, Tata Institute of Fundamental Research, Bombay)

Prove that

$$\sum_{d_1|n} \sum_{d_2|n} \mu(d_1)\mu(d_2)/\{d_1, d_2\} = \frac{\phi(n)}{n} .$$

SOLUTION.

Let $$f(n_1, n_2) = \sum_{d_1|n_1} \sum_{d_2|n_2} \frac{\mu(d_1)\mu(d_2)}{\{d_1, d_2\}} .$$

Note $\mu(n_1)\mu(n_2)$ and $\{n_1, n_2\}$ are multiplicative arithmatic functions of n_1 and n_2.

$$f(n_1, n_2) = \prod_p \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \frac{\mu(p^i)\mu(p^j)}{\{p^i, p^j\}}$$

where $n_1 = \prod p^{\alpha}$, $n_2 = \prod p^{\beta}$; $\alpha, \beta \geq 0$

Putting $n_1 = n_2$ so that $\alpha = \beta$, we get

$$f(n, n) = \prod \left(1 - \frac{1}{p}\right) = \frac{\phi(n)}{n} .$$

GENERALIZATION. (by N.Balasubramanian, Joint Cipher Bureau, Ministry of Defence, Government of India, New Delhi).

This is a particular case of quite a general class of results which can be obtained with the same case, using the technique of multiplicativity in several arithmetic variables (cf. N. Balasubramanian, Proceedings of the First Conference on Number Theory, Matscience Report 101, (1980) 47-62)

Let

$$f(n_1, n_2) = \sum_{d_1|n_1} \sum_{d_2|n_2} \mu(d_1)\mu(d_2)/\{d_1, d_2\}$$

Note that $\{\,,\,\}$ the l.c.m. is multiplicative strictly. Hence

$$f(n_1, n_2) = \prod_p f(p^\alpha, p^\beta)$$

where

$$n_1 = \prod p^\alpha \, ; \; n_2 = \prod p^\beta \, , \quad \alpha, \beta \geqslant 0$$

So

$$f(n_1, n_2) = \prod_p \cdot \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \frac{\mu(p^i, p^j)}{p^{\max(i,j)}} = \frac{\phi(\{n_1, n_2\})}{\{n_1, n_2\}}.$$

PROBLEM 3. (by N.Balasubramanian, Joint Cipher Bureau, Ministry of Defence, Government of India, New Delhi.)

In the theory of arithmetic functions with multiplicativity in more than one variable on which Vaidyanathaswamy, Kesava Menon and others have worked extensively the convolution

$$\sum_1^\infty \cdots \sum_1^\infty \frac{f(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}} \times \sum_1^\infty \cdots \sum_1^\infty \frac{g(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}$$

$$= \sum_1^\infty \cdots \sum_1^\infty \frac{h(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}$$

where $$h(n_1,\ldots,n_r) = \sum_{d_1|n_1}\cdots\sum_{d_r|n_r} f(d_1,\ldots,d_r)\,g\left(\frac{n_1}{d_1},\ldots,\frac{n_r}{d_r}\right)$$

plays a fundamental role in obtain ing multi-dimensional extension of Mobius inverseion and other techniques. In this connection I have defined certain simple arithmetic function such as

$\pi(n_1,\ldots,n_r) = 1$ or 0 according as the n's are coprime or not

$\delta(n_1,\ldots,n_r) = 1$ or 0 according as the n's are all equal or not.

These functions, along with the g.c.d. and l.c.m., I have shown, to be multiplicative strictly and they could be wielded to obtain the results of all the earlier work (mentioned in the beginning) in a unified manner. The solution of the above problem by Ramachandra is a case in point. The problem now proposed by me is to extend this technique to the convolution over unitary divisors of a natural number through defining a suitable function like the ones mentioned above.

PROBLEM 4. (by R. Sivaramakrishnan, Department of Mathematics, University of Calicut, Calicut,).

D is an integral domain in which primes are units defined by using the divisibility property. It is known that every non-unit in D is a finite product of distinct primes. What are the necessary and sufficient conditions under which D will become a unique factorization domain ? The solution to this problem would lead to a proof that the ring of number - theoretic functions is a unique factorization domain, without appealing to the methods already given by Cashwell and Everett (Pac. J. Math. 1959)).

PROBLEM 5. (by A.M.Vaidya, and V.S.Joshi, Department of Mathematics, and Statistics, South Gujarat University, Surat)

Let n_1 be any positive integer written in the decimal scale. Add the sum of the digits of n_1 to n_1. Suppose we get n_2. Do the same to n_2 to get n_3 and so on. We get what is called the digit-addition series of n_1. Prove that

(i) if $(n, 3) = 1$, the digit-addition series of n will merge with that of 1 after a finite number of steps.

(ii) if $3 | n$ and $9 \nmid n$, the digit-addition series of n will merge with that of 3.

(iii) if $9 | n$ then the digit-addition of n will merge with that of 9.

(e.g. note that digit-addition series of 86 is 86, 100, 101, 103, ... and that of 77 is 77, 91, 101, 103, ... so that we say that the two series merge at 101).

PROBLEM 6. (by A.M.Vaidya and V.S.Joshi, Department of Mathematics and Statistics, South Gujarat University, Surat).

If sum of the digits of n in the decimal scale is k, and $k | n$, we say that n is a Harshad snumber or specifically a Harshad number for k. Let $f(k)$ be the least Harshad number for k. E.G. $f(1) = 1$, $f(11) = 209$, $f(12) = 48$. Find estimates of $f(k)$. Also determine the asymptotic density of Harshad numbers and the behaviour of the harmonic series of the Harshad numbers.

PROBLEM 7. (by R.Jagannathan, Matscience, The Institute of Mathematical Sciences, Madras, India).

If J is a positive integer and $n, r, s = 0, \pm 1, \pm 2 \cdots$ then evaluate the sum

$$\frac{2}{(2J+1)^2} \sum_{\substack{r=-J \\ (\neq n)}}^{J} \frac{1}{(r-n)} \left\{ \sum_{s=-J}^{J} s \exp\left[\frac{i 2\pi s (r-n)}{2J+1} \right] \right\}^2$$

for any given value of J and n with $|n| \leq J$. This sum occurs in a problem in physics.

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ERRATA

1) In p.51 the authors' names and addresses should be as :-

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3) Cf. p.150. For Problem 2, K.Ramachandra indicated a line of proof based on first principles. Both the solution and the generalization given here in p.150 are by N.Balasubramanian.