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LEGENDRE POLYNOMIALS AND IRRATIONAL NUMBERS

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C O N T E N T S

§ 1. INTRODUCTION.

The purpose of this report is to present some of the most recent developments in the Theory of Irrationality together with many of the classical results in the subject. These recent developments were spurred by the spectacular proof by Roger Apery in June 1978, of the irrationality of

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \tag{1.1}$$

thereby solving one of the outstanding problems in the subject [16]. The result was spectacular not merely because it solved a longstanding problem, but also because of the manner in which the proof was presented. When Apery presented his proof in June 1978 at a conference in Marseilles, France in the form of a series of intriguing exercises, people did not know what to think of it. Subsequently at the Inernational Congress of Mathematicians at Helsinki in August 1978, a more coherent proof was presented. This was possible due to the efforts of Henri Cohen, Don Zagier, and Alf Van der Poorten, who in the intervening two months worked on Apery's ideas and filled several missing steps. We refer the reader to Van-der-Poorten's article which surveys closely the developments from Marseilles to Helsinki [16].

In December of 1978, Frits Beukers [7] by ingenious use of multiple integrals involving Legendre Polynomials gave proofs of the irrationality of $\zeta(2)$ and $\zeta(3)$ motivated by the ideas of Apery. At about the same time E. Bombieri [8], following upon one of the 'exercises' of Apery, gave proofs of the irrationality of the logarithm function at arguments $1+\frac{1}{m}, m\in\mathbb{Z}^+$ by the use of ordinary differential equations.

In January 1979, I investigated jointly with M.L. Robinson [1], [2], the one dimensional analogue of Beukers' method. Besides having connections with Bombieri's approach, one is led to the earlier work of Siegel [15] who had used similar methods to investigate the exponential function. The one-dimensional version yields explicit irrationality measures for logarithms and k^{th} roots of certain rationals. More especially one can prove inequalities of the type

$$\left|\log 2-\frac{p}{q}\right| \geq \frac{1}{q^{4.63+\epsilon}} \qquad \forall\, q \geq q_0(\epsilon),\ p,q\in\mathbb{Z} \qquad (1.2)$$

or

$$\left|\log 2-\frac{p}{q}\right| \geq \frac{10^{-10}}{q^{5.8}} \qquad \forall\, p,q\in\mathbb{Z} \qquad (1.3)$$

(Inequality (1.2) was realised by Bombieri also). Most recently we learnt that Beukers had also investigated the one-dimensional analogue of his original method and obtained similar results. Chudnovski has a general theory of proving irrationality of a large class of numbers. His theory is now in the process of development, and the only results in print are on this one-dimensional version [9].

We attempted to get irrationality measures for π by these methods. We were led quite surprisingly to the number $\pi/\sqrt{3}$ which seems to occupy a special place so far as these methods are concerned. One can show

$$\left|\frac{\pi}{\sqrt{3}} - \frac{p}{q}\right| \geq \frac{1}{q^{8.309986\ldots+\epsilon}} \quad \forall\, q \geq q_0(\epsilon). \tag{1.4}$$

Chudnovski also had realised (1.4), while Beukers independently had proved a slightly weaker form of it.

After the advent of transcendental number theory and its far reaching applications [6] , the theory of irrationality was ignored somewhat. Apery's proof created a resurgence of interest in the subject of irrationality. Till this some of the best results known to date were due to Baker [3] , [4] , [5] who used hypergeometric functions and improved on the earlier results of Siegel, Mahler and Shidlovski.

Here we shall start with the origins of the subject and rapidly survey the developments leading to ~~the present~~ results which we discuss ~~in detail~~. The summit of the discussion will as one would expect be the proof of irrationality of $\zeta(3)$ which we give towards the end. We will give Beukers' proof instead of Apery's since it is much easier to comprehend. For the discussion of the logarithm and k^{th} roots the approach shall be ours, since not unnaturally we are most acquainted with it ! The method of Beukers' or Chudnovskii in this regard, will be similar with minor differences.

Most recently Beukers and Chudnovski have extended their results to include the so-called di-logarithm and tri-logarithm functions. Chudnovski by very general methods claims results on the irrationality of a very wide class of numbers. I am afraid I cannot report on these here, since at the moment I am unfamiliar with their methods.

§2. Existence of Irrational Numbers.

Euclid was the first to demonstrate the existence of irrational numbers. We give below his celebrated proof of the irrationality of $\sqrt{2}$.

THEOREM A (Euclid) $\sqrt{2}$ is irrational.

Proof. Assume $\sqrt{2} = p/q$ with $p,q \in \mathbb{Z}$. We may take $p,q > 0$. Also we may assume $(p,q) = 1$. Then $2q^2 = p^2$. So p^2 is even which means p is even. Thus $2q^2 = 4p'^2$ giving $q^2 = 2p'^2$. So q^2 is even which means q is even. This says $(p,q) \geq 2$ contradicting our assumption $(p,q) = 1$. Thus $\sqrt{2}$ is irrational.

The above argument proves in general that the k^{th} root of a positive integer that is not a k^{th} power is irrational. This automatically implies the existence of infinitely many irrationals.

With the advent of set theory Cantor gave a proof of the existence of irrationals by countability arguments. His ideas were the following

DEFINITION 1. A set is called countable if it is either finite or is in one-to-one correspondence with the integers. Almost immediately from the definition one deduces

LEMMA A. A countable union of countable sets is countable

Proof. Let $S_1, S_2, S_3, \ldots,$ be countable sets. A process of counting may be prescribed by following the arrow in the diagram below:

$$\begin{array}{lcllll} S_1 & = & s_{11} \to s_{12} & s_{13} \to s_{14} & \cdots \\ S_2 & = & s_{21} \quad s_{22} & s_{23} \quad s_{24} & \cdots \\ S_3 & = & s_{31} \quad s_{32} & s_{33} \quad s_{34} & \cdots \\ \vdots & & & & \end{array} \tag{2.1}$$

LEMMA B. The rationals are countable.

Proof. The rationals in $[0,1)$ are countable by writing them in lexicographic ordering - namely

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \ldots \tag{2.2}$$

according to their denominators. Adding a fixed integer 'a' to each of these numbers gives a counting procedure for rationals in $[a, a+1)$. Lemma B now follows from Lemma A.

THEOREM B. The real numbers are uncountable.

Proof. Clearly it suffices to show that $[0,1)$ is uncountable, because subsets of countable sets are countable. Every $x \in (0,1)$ has a unique decimal expansion which is non-terminating. If x has a terminating expansion we can make it non-terminating by attaching an unending string of nines.

Assume $(0,1) = \{a_1, a_2, \ldots, a_n, \ldots\}$ is countable. Then

$$a_1 = .a_{11}\, a_{12}\, a_{13} \ldots \tag{2.3}$$

$$a_2 = .a_{21}\, a_{22}\, a_{23} \ldots$$

$$\vdots$$

where $0 \leq a_{ij} < 9$. For each i let b_i be chosen such that $b_i \neq a_{ii}$, $0 < b_i < 9$ and $b_i \in \mathbb{Z}$. Then

$$b = .b_1\, b_2\, b_3 \ldots$$

satisfies $b \in (0,1)$ but escapes the counting. So $(0,1)$ is uncountable and theorem B is proven.

COROLLARY. The set of irrational numbers is uncountable and hence non-empty.

<u>Proof.</u> Follows from Theorem B and Lemmas A and B.

The corollary shows that irrationals are so numerous that they can not be counted. Euclid's method is constructive whereas Cantor's is purely existential. It is desirable to have a criterion which will test the irrationality of a general real number x. Such a criterion will be provided in the next section.

3. <u>A Criterion for irrationality.</u>

The first practical criterion for testing the irrationality of a number was provided by Dirichlet.

THEOREM C (Dirichlet). Let x be a real number. A necessary and sufficient condition for x to be irrational is that there exists a sequence of pairs $(p_n, q_n) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \neq |q_n x - p_n| \to 0$ as $n \to \infty$.

Proof. Let x be rational. Say $x = \frac{a}{b}$, $b > 0$. Then: if $\frac{p}{q} \in \mathbb{Q}$ and $\frac{p}{q} \neq x$, we have

$$\left| x - \frac{p}{q} \right| = \left| \frac{a}{b} - \frac{p}{q} \right| \geqslant \frac{1}{bq} . \tag{3.1}$$

Consequently for all p, q with $qx - p \neq 0$ we have from (3.1)

$$| qx - p | \geqslant \frac{1}{b} . \tag{3.2}$$

Clearly (3.2) shows that if the sequence (p_n, q_n) exists then x should be irrational.

We now show that for irrational x, such a sequence does indeed exists. For this one makes use of the well known

Dirichlet Box Principle: If N+1 objects occupy N slots then at least one slot contains atleast 2 objects.

Pick $N \in \mathbb{Z}^+$ and consider the N+1 numbers $\{0, x\}$, $\{1, x\}, \cdots, \{Nx\}$ where $\{y\}$ represents the fractional part of y. That is $\{y\} = y - [y]$ where $[y]$ is the greatest integer $\leq y$. Clearly $0 \leq \{\ \} < 1$. So

$$0 \leq \{a, x\} < 1 , \quad a = 0, 1, 2, \cdots, N . \tag{3.3}$$

Also these N+1 numbers are pairwise distinct. For if

$$\{ax\} = \{bx\}$$

then

$$ax - [ax] = bx - [bx]$$

yielding

$$x = \frac{p}{q} \quad \text{with } p = [ax] - [bx] , \ q = a - b \neq 0 . \tag{3.4}$$

The irrationality of x shows that N+1 numbers in (3.3) are pairwise distinct. So for some pair a, b

$$0 < |\{ax\} - \{bx\}| < \frac{1}{N}$$

yielding

$$0 < |q_N x - p_N| < \frac{1}{N} \tag{3.5}$$

where $q_N = q$, $p_N = p$ in (3.4). Making $N \to \infty$ one gets the sequence claimed by the theorem.

Applications of the criterion.

Dirichlet's criterion can be used to prove the irrationality of certain numbers quite easily. For instance.

COROLLARY B $\sqrt{2}$ is irrational.

Proof. Clearly $1 < \sqrt{2} < 2$ so that $0 < \sqrt{2}-1 < 1$.

Then

$$0 \neq (\sqrt{2}-1)^n = q_n\sqrt{2} - p_n \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Corollary follows from (3.6).

COROLLARY C. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational.

Proof. Let $q_N = N!$. Then

$$q_N e = N! \sum_{m=0}^{N} \frac{1}{m!} + N! \sum_{m=N+1}^{\infty} \frac{1}{m!} = p_N + \epsilon_N, \ \epsilon_N > 0 \tag{3.7}$$

So

$$0 \neq q_N e - p_N = N! \sum_{m=N+1}^{\infty} \frac{1}{m!} = O\left(\frac{1}{N+1}\right) \to 0 \text{ as } N \to \infty. \tag{3.8}$$

Corollary C follows from (3.8).

Corollaries B and C represent situations when Dirichlet's criterion was easy to apply. But this is not true in general. Even minor variations in the terms of a series can produce great difficulties. For instance it is an unsolved problem of Erdős to determine the irrationality of the series

$$\sum_{n=2}^{\infty} \frac{1}{n!-1} . \tag{3.9}$$

In this case one does not know how to construct q_n and p_n.

Even in the case

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} , \quad k \in \mathbb{Z}^+ \tag{3.10}$$

it is not obvious how to construct q_N and p_N. For the natural choice is to set $q_N = \text{l.c.m}[1,2,\cdots,N]^K$ and $p_N = q_N \sum_{n=1}^{N} \frac{1}{n^k}$.

In this case $q_N \zeta(k) - p_N \not\to 0$ as $N \to \infty$, because the convergence of the series (3.10) is not rapid enough.

By entirely different methods, to be discussed in § 13, we will show $\zeta(k)$ is irrational if $k \in \mathbb{Z}^+, k \equiv 0 \pmod 2$. Such methods do not apply to $\zeta(k)$ when k is odd, and in particular to $\zeta(3)$. This was precisely why the irrationality of $\zeta(3)$ remained an unsolved problem for a long time. Indeed it is a measure of Apery's achievement that he cracked a problem that was unsuccessfully attempted by several outstanding mathematicians. Apery's method essentially amounts to a process of 'accelerating' the convergence of (3.10) when $k = 3$, without affecting its basic structure (see [16] for details).

The reader will have realised that Dirichlet's criterion is not too practical to apply to general situations. Apart from the age old ~~reductio ad absurdum~~ argument, it remains essentially the only criterion used to test irrationality.

§ 4. Transcendental Numbers.

We begin with a

DEFINITION. B. A complex number α is called algebraic if there exists a polynomial $0 \neq p(x) \in \mathbb{Z}[x]$ such that $p(\alpha) = 0$. Among all polynomials p for which $p(\alpha)=0$ there is one of smallest degree which is also irreducible. The degree of this polynomial is the degree of α, and the polynomial itself is called the defining polynomial of α. A complex number α not algebraic is called transcendental.

The existence of transcendental numbers was first observed by Liouville in his memoir (see [6]). He achieved this by means of the following observation:

THEOREM D. (Liouville). Let α be an algebraic number of degree n. Then there exists $C(\alpha) > 0$ such that $\forall p, q \in \mathbb{Z}$ with $\alpha \neq \frac{p}{q}$ we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C(\alpha)}{q^n} .$$

<u>Proof.</u> Let $p(x)$ be the defining polynomial of α. We ay assume $n = \deg p \geq 2$ because for n = 1 we know that the heorem is true. (see (3.1)). Since $p(\alpha)$ is irreducible we have or any $\frac{p}{q} \neq \alpha$, $p(\frac{p}{q}) \neq 0$. So

$$q^n p\left(\frac{p}{q}\right) \neq 0 \in \mathbb{Z} \tag{4.1}$$

implies

$$\left|q^n p\left(\frac{p}{q}\right)\right| \geq 1 . \tag{4.2}$$

But then by the mean value theorem

$$\left|q^n p\left(\frac{p}{q}\right)\right| = \left|q^n\left(p\left(\frac{p}{q}\right) - p(\alpha)\right)\right| = q^n |p'(\xi)| \, |\alpha - p/q| . \tag{4.3}$$

We may assumethat $\left|\alpha - \frac{p}{q}\right| < 1$ for otherwise the result is trivially true. But then

$$\sup_{|\xi - \alpha| \leq 1} |p'(\xi)| = O(1) . \tag{4.4}$$

Plainly (4.4), (4.3) and (4.2) imply Liouville's result if $C(\alpha)$ is suitably defined.

Indeed if α is a real number for which there exists a sequence of rationals p_n/q_n satisfying

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n^{\eta}} , \quad \eta \to \infty \tag{4.5}$$

then α is transcendental. It was this criterion which Liouville used to show

COROLLARY D. $\lambda = \sum_{n=0}^{\infty} \frac{1}{10^{n!}}$ is transcendental.

Proof. Set $q_n = 10^{n!}$ and $p_n = \left(\sum_{m=0}^{n} \frac{1}{10^{m!}} \right) . q_n$.

Then $\left| \lambda - \frac{p_n}{q_n} \right| = 0 \left(\frac{1}{q_n^n} \right)$. Hence the transcendence of λ.

In otherwords if a number is approximable extremely well then the number is transcendental. Unfortunately this is not at all a practical criterion because very few numbers can be approximated to the degree specified by (4.5) (see next section). Such numbers are called Liouville numbers. Ofcourse the countability argument in Theorem B if refined a bit further shows:

THEOREM E. (Cantor). The set of algebraic numbers is countable. The set of transcendental numbers is uncountable and hence non-empty.

Whereas Cantor's is purely existential, Liouville's test is impractical. The standard method of proving transcendence has been by the use of reduction-ed-absurdum. That is assume the given number is algebraic and reach a contradiction.

Such an approach has proved successful in establishing the transcendence of values of certain well behaved functions at rational or algebraic arguments. The most celebrated among such results is the theorem of Hermite-Lindemann which asserts

THEOREM F. If α is algebraic and $\neq 0$, e^{α} is transcendental

The theorem includes as a corollary the transcendence of e nd π. The transcendence of e is plain by setting $\alpha = 1$. To deduce he transcendence of π one simply notes $e^{i.\pi} = -1$, $i = \sqrt{-1}$, hence $i\pi$ is transcendental. Plainly π is also transcendental.

It follows from this that the inverse function $\log \alpha$ takes ranscendental values at all algebraic points $\alpha \neq 1$. Of course the ranscendence of a number trivially implies irrationality. Nevertheless none of these qualitative results say anything about explicit lower bounds for approximations of such numbers by rational sequences. One is then naturally led to the concept of 'irrationality measure' which we take up in the next section.

Explicit lower bounds for approximations to single or combinations of transcendental numbers has proved to be of great use in the Theory of Diophantine Equations. We refer to Baker's book for a full coverage of these ideas [6] .

§5. Irrationality Measure.

Liouville's result only says that algebraic numbers cannot be approximated to an exponent larger than their degree. The question as to whether certain or all algebraic numbers of degree n can be approximated to degree n was considered soon afterwards. Liouville's basic result underwent several improvements by Thue, Siegel, Dyson and Roth finally settled the issue.

THEOREM G. (Roth). Let α be an algebraic number and $\epsilon > 0$. Then there exists $C(\alpha, \epsilon) > 0$ such that for all p/q with $\alpha \neq p/q$

we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha,\epsilon)}{q^{2+\epsilon}} \tag{5.1}$$

This is equivalent to saying that given $\epsilon > 0$,

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^{2+\epsilon}} \tag{5.2}$$

has only finitely many solutions.

Roth's theorem is in principle a best possible result because it follows from Dirichlet's box principle that for any irrational number, algebraic or transcendental

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q^2} \tag{5.3}$$

admits infinitely many solutions. For this one simply has to observe in (3.4), (3.5) and (3.6) that $|q_N| \leq N$ whence

$$\left|\alpha - \frac{p_N}{q_N}\right| < \frac{1}{Nq_N} < \frac{1}{q_N^2} \quad . \tag{5.4}$$

Clearly as $N \to \infty$, (5.4) has to generate infinitely many solutions of (5.3). Indeed if ξ is a real number for which there is a fixed $\eta > 2$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\eta} \tag{5.5}$$

has infinitely many solutions then ξ is by Roth's theorem necessarily transcendental. Criterion (5.5) seems much easier to verify than (4.5) but is equally impractical. This is because a famous result of Khintchin says [10] .

THEOREM (Khintchin). Let $f(n)$ be function defined on $n \in Z^+$ with $f(n) > 0$. Suppose

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} < \infty .$$

Let S be the set of numbers ξ for which

$$\left| \xi - \frac{p_n}{q_n} \right| < \frac{1}{q_n f(q_n)}$$

has infinitely many solutions. Then S is a set of measure zero.

Khintchin's theorem says that the set of numbers violating Roth's criterion or satisfying (5.5) is a set of measure zero. On the other hand by Cantor's theorem, almost all numbers are transcendental. Thus to establish the transcendence of a given number ξ by the use of (5.5) is highly impractical.

Given a transcendental number, Khintchin's method in no way decides whether that number belongs to the set of measure zero or not. For certain special irrationals, (such as algebraic numbers), one knows exactly the degree of approximation, but in general the problem is open. It is convenient here to introduce the concept of irrationality measure.

DEFINITION. A number γ is a called an irrationality measure for a real number ξ, if given $\varepsilon > 0$

$$\left| \xi - \frac{p}{q} \right| \geqslant \frac{1}{q^{\gamma+\varepsilon}} \qquad \forall q \geqslant q_0(\varepsilon) \tag{5.6}$$

(the number ξ is necessarily irrational because if $\xi = \frac{a}{b}$, $b \geqslant 1$ then $p = Na$, $q = Nb$ with N sufficiently large violates (5.6)).

Instead of an inequality valid for sufficiently large q one may ask for explicit numbers $c > 0$, $\eta \geq \gamma$ such that

$$\left|\xi - \frac{p}{q}\right| > \frac{c(\xi)}{q^{\eta}} \qquad \forall\, p, q \, . \tag{5.7}$$

One should first note that for irrationals, both γ and η will have to be ≥ 2 in view of Dirichlet's criterion.

Generally so far as applications to Diophantine equations are concerned, the exponents η or γ yield information about the number of solutions whereas C says something about the size of the solutions. This feature is best discussed in Baker [6].

For a transcendental number or a number whose transcendence is yet undetermined it is useful to have an inequality like (5.6) or (5.7) because Khinchin's theorem is nonconstructive. Even for algebraic numbers it is desirable to have (5.7) with explicit values even if η is far removed from 2, because Roth's method at the moment does not permit an effective or explicit evaluation of $C(\alpha, \epsilon)$. Of course in view all foregoing remarks, if one does obtain an irrationality measure 2, it is a best possible result.

§ 6. Criterion for determining Irrationality Measure.

In this section we shall prove a criterion which will determine an irrationality measure for a number provided enough is known about a sequence of rational approximations. We shall actually discuss two criteria. Criterion 1 is classical. Criterion 2 is new in the sense that there are no explicit references. It is a minor but very useful variation of Criterion 1 as will be seen soon. We shall begin by proving

Criterion 1. Let θ be a real number and p_n/q_n a sequence of rationals satisfying

(i) $\dfrac{p_{n+1}}{q_{n+1}} \neq \dfrac{p_n}{q_n}$, $q_{n+1} > q_n$

(ii) $q_{n+1} = q_n^{1+o(1)}$

(iii) For some $\lambda \in (0,1)$

$$0 \neq \left|\theta - \frac{p_n}{q_n}\right| = O\left(\frac{1}{q_n^{1+\lambda}}\right) \tag{6.1}$$

Then $1+\frac{1}{\lambda}$ is an irrationality measure for θ.

Proof. It is clear for (6.1) and (3.1) that θ is irrational. Next let p/q be rational and defined δ by

$$\left|\theta - \frac{p}{q}\right| = \frac{1}{q^\delta}$$

We must show $\delta < 1+\frac{1}{\lambda} + \epsilon$ for large q.

Let us assume $\delta > 1+\frac{1}{\lambda}$. If q is sufficiently large, (i) shows we can find m such that

$$q_{m-1} \leq q^{\delta/(1+\lambda)} < q_m . \tag{6.2}$$

Let

$$l = \begin{cases} m & \text{if } \dfrac{p_m}{q_m} \neq \dfrac{p}{q} \\ m+1 & \text{otherwise}. \end{cases} \tag{6.3}$$

It follows from (6.3) and (i) that

$$\frac{p_l}{q_l} \neq \frac{p}{q} . \tag{6.4}$$

so (i), (ii), (6.1) and (6.2) show

$$q_\ell = q^{(\delta/_{1+\lambda})(1+o(1))} \quad ; \frac{1}{q_\ell^{1+\lambda}} < \frac{1}{q^\delta}. \tag{6.5}$$

Combining (6.4) and (6.5) yields

$$\begin{aligned} \frac{1}{q^{(1+(\delta/(1+\lambda)))(1+o(1))}} &= \frac{1}{q\, q_\ell} \leq \left|\frac{p}{q} - \frac{p_\ell}{q_\ell}\right| \leq \left|\theta - \frac{p}{q}\right| + \left|\theta - \frac{p_\ell}{q_\ell}\right| \\ &\leq \frac{1}{q^\delta} + O\left(\frac{1}{q_\ell^{1+\lambda}}\right) \\ &= O\left(\frac{1}{q^\delta}\right). \end{aligned} \tag{6.6}$$

Hence

$$\left(1+\frac{\delta}{1+\lambda}\right)(1+o(1)) > \delta$$

or equivalently

$$\delta < 1 + \frac{1}{\lambda} + \frac{\delta}{\lambda} o(1), \; q \to \infty .$$

So for large q , $\delta < 1 + \frac{1}{\lambda} + \epsilon$ and the proof is complete.

In some proofs checking (i) of criterion 1 presents difficulties. It is desirable to have a criterion which does not use (i) of Criterion 1. The next criterion circumvents this problem but requires a more elaborate proof.

Criterion 2. Let θ be a real number. Suppose p_n/q_n is a sequence of rationals satisfies

(i) $q_n \to \infty$

(ii) $q_{n+1} = q_n^{1+o(1)}$

ii) For some $\lambda \in (0,1)$

$$0 \neq \left|\theta - \frac{p_n}{q_n}\right| = \frac{1}{q_n^{(1+\lambda)(1+o(1))}} . \tag{6.7}$$

en $1+\frac{1}{\lambda}$ is an irrationality measure for θ.

<u>Proof.</u> First we will show that given p_n/q_n there exists p_m/q_m such that

$$p_m/q_m \neq p_n/q_n , \quad q_m = q_n^{1+o(1)} .$$

To see this let

$$\left|\theta - \frac{p_n}{q_n}\right| = \frac{1}{q_n^{1+\lambda+\epsilon_n}} . \tag{6.8}$$

By (iii) we know $\epsilon_r \to 0$. Set

$$\delta_n = \sup\{|\epsilon_n|, |\epsilon_{n+1}|, \dots\} \tag{6.9}$$

so that $\delta_n \to 0$ also.

Suppose $l > n$ and $\frac{p_l}{q_l} = \frac{p_n}{q_n}$. Then from (6.8)

$$q_n^{1+\lambda+\epsilon_n} = q_l^{1+\lambda+\epsilon_l}$$

or

$$q_l = q_n^{1+\frac{\epsilon_n - \epsilon_l}{1+\lambda+\epsilon_l}}$$

yielding for large n

$$q_n^{1-2\delta_n} \leq q_l \leq q_n^{1+2\delta_n} \tag{6.10}$$

as a consequence of (6.9).

Now let n be large and $m(n)$ the first integer greater than n such that

$$q_{m(n)} > q_n^{1+2\delta_n} . \tag{6.11}$$

The existence of $m(n)$ is insured by (i).

From (6.10) and (6.11) we know

$$\frac{p_{m(n)}}{q_{m(n)}} \neq \frac{p_n}{q_n} . \tag{6.12}$$

We claim that

$$q_{m(n)} = q_n^{1+o(1)} \tag{6.13}$$

If $m(n) = n + 1$, then (6.13) clearly follows from (ii). So assume $m(n) \geq n + 2$. Define

$$q_{n+1} = q_n^{1+\eta_n} , \quad n = 1,2,\ldots$$

From (ii) we know $\eta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Also

$$q_{m(n)} = q_n^{\prod_{j=n}^{m(n)-1} (1+\eta_j)} . \tag{6.14}$$

By our choice of $m(n)$ we have

$$\prod_{j=n}^{m(n)-1} (1 + \eta_j) > 1 + 2\,\delta_n$$

and we have

$$\prod_{j=n}^{m(n)-2} (1+\eta_j) \leq 1+2\delta_n .$$

Hence

$$1+2\delta_n < \prod_{j=n}^{m(n)-1} (1+\eta_j) \leq (1+2\delta_n)(1+\eta_{m(n)-1}) \tag{6.15}$$

and (6.13) plainly follows from (6.14) and (6.15).

We now imitate the proof of Criterion 1. Let p/q be rational and define δ by $\left|\theta - \frac{p}{q}\right| = \frac{1}{q^\delta}$. If $\delta \leq 1 + \frac{1}{\lambda}$ there is nothing to prove . So assume $\delta > 1 + \frac{1}{\lambda}$. For large q, one can find n such that

$$q^\delta = q_n^{(1+\lambda)(1+o(1))} \tag{6.16}$$

Define

$$K(n) = \begin{cases} n & \text{if } \frac{p_n}{q_n} \neq \frac{p}{q} \\ m(n) & \text{otherwise.} \end{cases} \tag{6.17}$$

From (6.12) and (6.17)

$$\frac{p_{K(n)}}{q_{K(n)}} \neq \frac{p}{q} \tag{6.18}$$

Also from (ii), (6.13), and (6.16)

$$q_{K(n)} = q^{\frac{\delta}{1+\lambda}(1+o(1))} \tag{6.19}$$

nally (6.7), (6.18) and (6.19) yield

$$\frac{1}{q^{(1+\frac{\delta}{1+\lambda})(1+o(1))}} \leq \left|\frac{p}{q} - \frac{p_{k(n)}}{q_{k(n)}}\right| \leq \left|\theta - \frac{p}{q}\right| + \left|\theta - \frac{p_{k(n)}}{q_{k(n)}}\right|$$

$$\leq \frac{1}{q^{\delta}} + \frac{1}{q_{k(n)}^{(1+\lambda)(1+o(1))}} = \frac{1}{q^{\delta(1+o(1))}} \tag{6.20}$$

whence

$$\delta \leq (1+\tfrac{1}{\lambda})(1+o(1)) \qquad q \to \infty.$$

proving Criterion 2.

Remarks. Although the p_n/q_n in Criterion 2 need not satisfy (i) of Criterion 1, the first criterion is not really weaker. Note that inequality (6.1) is changed to equality (6.7). What essentially (6.11), (6.12) and (6.13) convey is that if a sequence satisfies the conditions of Criterion 2 then one can extract a subsequence which satisfies the conditions of Criterion 1. But when applying Criterion 2 to a specific situation we need not bother ourselves with the selection of the subsequence. But if explicit estimates are required an accurate computation of $m(n)$ is essential. Moreover when criterion 2 is applied, (6.7) does show that the irrationality measure obtained is the best the method can give.

7. Legendre Polynomials.

In this section we shall discuss some properties of Legendre Polynomials that will be necessary to analyse the irrationality of the logarithm, exponentials and k^{th}-roots.

DEFINITION. Let $n \geq 0$ be an integer. $P_n(x)$ the nth Legendre Polynomials is defined by

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left[x^n(1-x)^n\right]. \tag{7.1}$$

One immediately has the following properties.

LEMMA 1. (a) $P_n(x) \in \mathbb{Z}[x]$ is of degree n.

(b) If $P_n(x) = \sum_{n=0}^{n} a_n(n)\, x^n$, then

$sgn\{a_n(n)\} = (-1)^n$.

(c) $P_n(x) = (-x)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(1-\frac{1}{x}\right)^k$

(d) $\left\{P_n(x)\right\}_{n=0}^{\infty}$ form an orthogonal base for the set of polynomials on $[0,1]$.

Proof. (a) follows from the observation that the product of n consecutive integers in a multiple of n !

(b) follows at once from (7.1).

To prove (c) we use Leibnitz's rule which gives

$$P_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{(n-k)!}x^{n-k}(-1)^{n-k}(1-x)^k\frac{n!}{k!}$$

$$= (-x)^n \sum_{k=0}^{n}\binom{n}{k}^2\left(1-\frac{1}{x}\right)^k.$$

o prove (d) we first show that if $p(x)$ is a polynomial of degree nd if $n > m$ then

$$\int_0^1 P_n(x)\, p(x)\, dx = 0 \tag{7.2}$$

o establish (7.2) integrate by parts $m+1$ times by integrating P_n nd differentiating p.

If follows from (7.2) that $\{P_n\}$ form an orthogonal system, for given P_n, P_m with $n \neq m$, assume that $n > m$. Apply (7.2) with $P_m = p$. Since each P_n is a polynomial of degree exactly n, the P_n form a base for the polynomials on $[0,1]$

LEMMA 2. Let $f(x)$ be a continuous function such that

$$C_n = \int_0^1 P_n(x)\, f(x)\, dx = 0 \quad \forall\, n > N$$

Then f is a polynomial of degree $\leq N$.

<u>Proof.</u> Consider the polynomial

$$h(x) = \sum_{n=0}^{N} \frac{C_n}{\int_0^1 P_n^2(t)\, dt} \cdot P_n(x) \tag{7.3}$$

Then $h(x)$ is a polynomial of degree $\leq N$.

By Lemma 1 (d) we have

$$\int_0^1 h(x)\, P_n(x)\, dx = \int_0^1 f(x)\, P_n(x)\, dx \qquad n = 0, 1, 2, \cdots \tag{7.4}$$

Next set $g(x) = h(x) - f(x)$. Then from (7.4)

$$\int_0^1 g(x)\, P_n(x)\, dx = 0, \quad n = 0, 1, 2, \cdots \tag{7.5}$$

Again by Lemma 1 (d)

$$\int_0^1 g(x)\, p(x)\, dx = 0 \tag{7.6}$$

for all polynomials $p(x)$. Since the polynomials are dense in the space of continuous functions, one deduces from (7.6) that

$$\int_0^1 g^2(x)\, dx = 0 \tag{7.7}$$

yielding $g(x) \equiv 0$, whence $f(x) = h(x)$. Lemma is proved.

COROLLARY 1. If f is a continuous function that is not a polynomial then $\int_0^1 f(x) P_n(x)\, dx$ is non-zero for infinitely many n.

Next we prove

LEMMA 3. If $x \notin [0,1]$ then $P_n(x) \neq 0$, $n = 0,1,2,\cdots$. That is all the zeros of $P_n(x)$ are in $[0,1]$.

Proof. The assertion is trivial for $P_0(x) = 1$. So let $n \geq 1$. From Lemma 1 (d). We have

$$\int_0^1 P_n(x)\, dx = 0 \tag{7.8}$$

which assures the existence of at least one point in (0,1) where $P_n(x)$ changes sign. Let $x_1, x_2, \cdots, x_\ell$ be all such points. Then the product $P_n(x)(x - x_1) \cdots (x - x_\ell)$ has constant sign on $[0,1]$ and is not identically zero. If $\ell < n$, then by (7.2)

$$\int_0^1 P_n(x)(x - x_1) \cdots (x - x_\ell)\, dx = 0. \tag{7.9}$$

since the integrand in (7.9) is non-zero and of constant sign, we must have $\ell \geq n$. Since P_n is of degree n, $\ell = n$. Lemma 3 is proved.

In the next section we shall deal with certain recurrence relations satisfied by P_n and certain integrals involving them.

8. Recurrence Relations and Poincare's Theorem.

In this section we shall prove that P_n and certain integrals involving them satisfy recurrence relations. The idea of doing this is to estimate the behavior of these quantities asymptotically. For this we will use a theorem due to Poincare which deals with the asymptotic behaviour of solutions of linear recurrences. We begin with

LEMMA 4. Let z be an arbitrary complex number. Then $P_n(z)$ satisfies the recurrence

$$n P_n(z) + (2n-1)(2z-1)P_{n-1}(z) + (n-1)P_{n-2}(z) = 0. \tag{8.1}$$

Proof. When $z = 0$, $P_n(z) = 1$, $n = 0, 1, 2, \cdots$, Lemma 4 is then trivially true because

$$n - (2n - 1) + n - 1 = 0.$$

When $z \neq 0$, we use Lemma 1 (c) which gives

$$P_n(z) = (-z)^n \sum_{k=0}^{n} \binom{n}{k}^2 \left(1 - \frac{1}{z}\right)^k \tag{8.2}$$

Set

$$B_n(\lambda) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k , \quad \lambda \in \mathbb{C} . \tag{8.3}$$

If we make the identification $\lambda = 1 - \frac{1}{z}$, then from (8.1) and (8.3) it is clear that (8.1) will be true if

$$n B_n(\lambda) - (2n-1)(1+\lambda) B_{n-1}(\lambda) + (n-1)(1-\lambda)^2 B_{n-2}(\lambda) = 0 . \tag{8.4}$$

Note that (8.4) will follow from (8.3) and the following identity:

$$n \binom{n}{k}^2 - (2n-1)\left\{ \binom{n-1}{k}^2 + \binom{n-1}{k-1}^2 \right\}$$
$$+ (n-1)\left\{ \binom{n-2}{k}^2 - 2\binom{n-2}{k-1}^2 + \binom{n-2}{k-2}^2 \right\} = 0 . \tag{8.5}$$

A proof of (8.5) can be given by means of straightforward but laborious computation. We omit the details Lemma 4 follows from (8.5).

LEMMA 5. Let $z \in \mathbb{C}$, $z \notin [0,1]$. Define

$$I_n(z) = \int_0^1 \frac{P_n(x)}{1 - \frac{1}{z}x}\, dx . \tag{8.6}$$

Then $I_n(z)$ satisfies the recurrence

$$n I_n(z) + (2n-1)(2z-1) I_{n-1}(z) + (n-2) I_{n-2}(z) = 0. \tag{8.7}$$

Proof. We compute (8.7) using Lemma 4. That is

$$n I_n(z) + (2n-1)(2z-1) I_{n-1}(z) + (n-1) I_{n-2}(z)$$

$$= \int_0^1 \frac{n P_n(x) + (2n-1) P_{n-1}(x) + (n-1) P_{n-2}(x)}{1 - \frac{1}{z} x} dx$$

$$\overset{(ii)}{=} \int_0^1 \frac{(2n-1)[2z-1-(2x-1)] P_{n-1}(x)}{1 - \frac{1}{z} x} dx$$

$$= (2n-1) 2z \int_0^1 P_{n-1}(x) dx = 0$$

proving (8.7).

Note that $P_n(z)$ and $I_n(z)$ satisfy the same recurrence.

$$n u_n + (2n-1)(2z-1) u_{n-1} + (n-1) u_{n-2} = 0 . \tag{8.8}$$

Whereas $P_n(z)$ satisfies (8.8) for all $z \in \mathbb{C}$, $I_n(z)$ satisfies (8.8) for all $z \in \mathbb{C}$, $z \notin [0,1]$.

LEMMA 6. (Poincare's Theorem). Let a, b, c be complex numbers and $\{u_n\}$ a sequence such that

$$a_n u_n + b_n u_{n-1} + c_n u_{n-2} = 0$$

where $a_n \to a$, $b_n \to b$, $c_n \to c$ as $n \to \infty$. Further assume that the associated characteristic polynomial

$$ax^2 + bx + c = 0 \tag{8.9}$$

has roots distinct in modulus. Then if $u_n \neq 0$ for infinitely many n, we have

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l \tag{8.10}$$

here l is a root of (8.9).

Proof. A clear exposition can be found in [13]. The arguments are elementary.

Suppose the u_n are as in Poincare's theorem. Then (8.10) shows that

$$u_n \neq 0 \ , \ \forall n \geq N \qquad (8.11)$$

So if $l \neq 0$ we have

$$\lim_{n \to \infty} \log \left| \frac{u_{n+1}}{u_n} \right| = \log |l|$$

Hence

$$\lim_{n \to \infty} \frac{\log \left| \frac{u_{N+1}}{u_N} \right| + \log \left| \frac{u_{N+2}}{u_{N+1}} \right| + \cdots + \log \left| \frac{u_n}{u_{n-1}} \right|}{n} = \log |l|$$

or

$$\lim_{n \to \infty} \frac{\log \left| \frac{u_n}{u_N} \right|}{n} = \log |l|$$

Thus

$$\lim_{n \to \infty} |u_n|^{1/n} = l$$

which is the same as saying

$$|u_n| = |l|^{n(1+o(1))} . \qquad (8.12)$$

We will now apply these ideas to $P_n(z)$ and $I_n(z)$. The idea is to get an estimate of the type (8.12) so that one may suitably apply this to Criterion 2 (see section 6).

LEMMA 7. The associated characteristic polynomial to the recurrence (8.8) is

$$x^2 + 2(2z-1)\ x + 1 = 0 \tag{8.13}$$

which has roots whose absolute values are

$$\alpha(z) = \max\{|2z-1 \pm 2\sqrt{z^2-z}|\} \tag{8.14}$$

$$\beta(z) = \min\{|2z-1 \pm 2\sqrt{z^2-z}|\}.$$

Further more $\alpha(z)\beta(z) = 1$. When $z \notin [0,1]$, $\alpha(z) \neq \beta(z)$ so that $0 < \beta(z) < 1 < \alpha(z)$. Therefore $|P_n(z)|$ and $|I_n(z)|$ behave either like $\alpha(z)^{n(1+o(1))}$ or $\beta(z)^{n(1+o(1))}$.

Proof. To see that (8.13) gives the characteristic polynomial divide (8.8) by n and let $n \to \infty$. Then

$$a_n \to 1,\ b_n \to 2(2z-1),\ c_n \to 1.$$

Next, (8.14) and the computation of $\alpha(z)\cdot\beta(z)$ follow plainly from (8.13). The asymptotic behavior of $|P_n(z)|$ and $|I_n(z)|$ follow from (8.12), Poincare's theorem, Lemma 2 and Lemma 3, which guarantee that $|P_n(z)|$ and $|I_n(z)|$ are non-zero for infinitely many n. Lemma 7 is proven.

We are now in a position to obtain measures of irrationality for various irrational numbers.

§ 9. Irrationality measures for the exponential function.

We will prove that for all rational s, $s \neq 0$, e^s has irrationality measure 2. The method is related to Siegel's well known proof of the irrationality of e^s. The proof that we present here is more in line with the methods of the following sections and differs from Siegel's approach in some respects.

THEOREM 1. For all rational s, $s \neq 0$, e^s is irrational and has irrationality measure 2.

Proof. We begin by considering the integral

$$a_n = \int_0^1 x^n e^{sx}\,dx \;, \quad n = 0, 1, 2, \ldots \tag{9.1}$$

The integral can be explicitly evaluated by partial integration. We have

$$a_n = \left[\frac{1}{s} - \frac{n}{s^2} + \frac{n(n-1)}{s^3} \cdots + (-1)^n \frac{n!}{s^{n+1}}\right] e^s + (-1)^{n+1}\frac{n!}{s^{n+1}} \tag{9.2}$$

$$= (-1)^n \frac{n!}{s^{n+1}} \left\{ \left[1 - s + \frac{s^2}{2!} - \frac{s^3}{3!} \cdots + (-1)^n \frac{s^n}{n!}\right] e^s - 1 \right\}.$$

If we write

$$a_n = (-1)^n u_n e^s - v_n \tag{9.3}$$

then from (9.2)

$$s^{-(n+1)} n! \ll u_n \ll n!\, s^{-(n+1)} \tag{9.4}$$

for sufficiently large n.

Let $s = \frac{p}{q} > 0$. Consider the integral

$$\int_0^1 P_n(x)\, e^{xs}\,dx = e^s \sum_{r=0}^{n} a_r(n)(-1)^r u_r - \sum_{r=0}^{n} a_r(n)\, v_r \,. \tag{9.5}$$

It is clear from (9.2) that

$$p^{n+1} \int_0^1 P_n(x)\, e^{sx}\,dx = q_n e^s - p_n \tag{9.6}$$

where $q_n, p_n \in \mathbb{Z}$.

We will use (9.5) to estimate q_n asymptotically. First note

$$q_n = p^{n+1} \sum_{r=0}^{n} a_r(n)(-1)^r u_r \qquad (9.7)$$

Using Lemma 1(a) we have

$$a_r(n)(-1)^r > 0 \qquad (9.8)$$

Also Lemma 1 c with $z = -1$ yields

$$1 \leq a_r(n)(-1)^r \leq \sum_{k=0}^{n} \binom{n}{k}^2 \leq \left[\sum_{k=0}^{n} \binom{n}{k}\right]^2 \leq 4^n . \qquad (9.9)$$

So by (9.8), (9.9), (9.7) and (9.4)

$$q_n << p^{n+1} 4^n n! \, q^{n+1} \qquad (9.10)$$

and

$$q_n >> p^{n+1} n! \qquad (9.11)$$

So (9.10) and (9.11) together yield

$$q_n = (n!)^{1+o(1)} . \qquad (9.12)$$

Finally integrating (9.6) by parts n times yields

$$p^{n+1} \int_0^1 P_n(x) e^{sx} dx = \frac{p^{2n+1}}{q^n} \frac{1}{n!} \int_0^1 e^{sx} x^n (1-x)^n dx . \qquad (9.13)$$

Note that

$$0 \leq x(1-x) \leq \sup_{0 \leq x \leq 1} x(1-x) = \frac{1}{2} . \qquad (9.14)$$

Since for a Lebesgue integrable function on 0,1] one has $\|f\|_n \to \|f\|_\infty$ we have from (9.14)

$$\int_0^1 e^{sx} x^n (1-x)^n \, dx = \left(\tfrac{1}{2}\right)^{n(1+o(1))} \tag{9.15}$$

Combing (9.13) and (9.15) yields

$$0 \neq p^{n+1} \int_0^1 P_n(x) e^{sx} \, dx = \left(\frac{1}{n!}\right)^{1+o(1)} \tag{9.16}$$

Comparing (9.12), (9.16) and (9.6) shows that p_n/q_n is a sequence of approximations to e^s satisfying the conditions of Criterion 2 with $\lambda = 1$. Thus $1 + \frac{1}{\lambda} = 2$ is a measure of irrationality for e^s if s is a positive rational.

If s is negative one simply notes that e^{-s} and e^s have the same irrationality measure. Thus theorem 11 is proved.

COROLLARY 2. The function log s is irrational for all rational s, whenever $s \neq 1$.

§10. Irrationality measures for logarithms of certain rationals.

Proving or measuring the irrationality of the exponential function at non-zero rational arguments, shows that the inverse function log s takes irrational values at rational points other than 1. This procedure unfortunately does not yield satisfactory irrationality measures for the logarithm. We demonstrate here a method of proving directly the irrationality of the logarithm. The interest of the method lies in the fact that this yields good irrationality measures. We need some preliminary results.

LEMMA 8. Let n be a positive integer and d_n = l.c.m. $[1, 2, \ldots, n]$. Then $d_n = e^{n(1+o(1))}$.

<u>Proof.</u> $\log d_n = \sum\limits_{p^\alpha \le n,\; p^{\alpha+1} > n} \log p^\alpha = \sum\limits_{p^\beta \le n} \log p = \psi(n)$

where ψ is the Tchebychev function. By the Prime Number Theorem, $\psi(x) \sim x$. Lemma 7 clearly follows from this estimate of $\log d_n$.

THEOREM 2. Let p, q be positive integers satisfying

$$(1-\sqrt{1+p/q})^2 \cdot q \cdot e < 1 \qquad (10.1)$$

Then $\log(1 + p/q)$ is irrational and has irrationality measure

$$\mu_{p,q} = \frac{\log\{q(1+\sqrt{1+p/q})^2\}+1}{\log\{q/p^2(1+\sqrt{1+p/q})^2\}-1} + 1 . \qquad (10.2)$$

We list some immediate corollaries.

COROLLARY 3. $\log(1 + 1/m)$ is irrational for all $m \in \mathbb{Z}^+$ and $\mu^*_{1,m} \longrightarrow 2$.

COROLLARY 4. $\log 2$ has irrationality measure $4.622\cdots$

COROLLARY 5. If p, q satisfy (10.1) and $\log p/\log q \longrightarrow 0$ as $q \longrightarrow \infty$ then $\mu_{p,q} \longrightarrow 2$.

<u>Proof of Theorem.</u> We begin by investigating $I_n(z)$. Suppose $z \in \mathbb{C}$; $z \notin [0, 1]$. By long division one has

$$\int_0^1 \frac{x^n}{1-\frac{1}{z}x}\,dx = \int_0^1 \left\{ \left(\sum_{n=1}^{n} \frac{(-1)^{n-1} x^{n-n}}{(-\frac{1}{z})^n} \right) + \frac{z^n}{1-\frac{x}{z}} \right\} dx$$

$$= -z^{n+1} \log\left(1-\frac{1}{z}\right) - \sum_{n=1}^{n} \frac{z^n}{n-n+1} . \qquad (10.3)$$

Thus

$$I_n(z) = \int_0^1 \frac{P_n(x)}{1-\frac{1}{z}x}\,dx = \left\{-z P_n(z)\, d_n \log(1-\tfrac{1}{z}) - Q_n(z)\right\} d_n^{-1} \quad (10.4)$$

where d_n is defined in Lemma 7 and $Q_n(x) \in \mathbb{Z}[x]$ is of degree n. Also integration by parts n-times of $I_n(z)$ gives

$$I_n(z) = \left(-\frac{1}{z}\right)^n \int_0^1 \frac{x^n(1-x)^n}{(1-\frac{1}{z}x)^{n+1}}\,dx \quad . \tag{10.5}$$

Now set $z = -q/p$. Note that $z \notin [0,1]$. It follows from (10.5) that

$$I_n(-q/p) \neq 0 \quad \forall n \, . \tag{10.6}$$

Also since

$$\sup_{0\le x\le 1} \frac{x(1-x)}{(1+p/q\, x)} = \left(1-\sqrt{1+p/q}\right)^2 q^2/p^2 \tag{10.7}$$

we know from (10.5) and (10.7) that

$$0 \le I_n(-q/p) \le \left\{(1-\sqrt{1+p/q})^2\, q/p\right\}^n . \tag{10.8}$$

Now define

$$p_n = p^{n+1} Q_n(-q/p), \quad q_n = q \cdot d_n \cdot p^n P_n(-q/p) \, . \tag{10.9}$$

Then from (10.4) and (10.6) once has

$$0 \neq q_n \log(1+p/q) - p_n = d_n p^{n+1} I_n(-q/p), \quad p_n, q_n \in \mathbb{Z} \, . \tag{10.10}$$

The irrationality of $\log(1+p/q)$ follows from (10.10) and Dirichlet's theorem because (10.9), and Lemma 7 imply

$$0 \le d_n p^{n+1} I_n(-q/p) \le \left(1-\sqrt{1+p/q}\right)^{2n} q^n d_n = o(1) . \tag{10.11}$$

In order to obtain irrationality measures we will show that p_n / q_n satisfies the conditions of Criterion 2. First it is clear from (10.7), (10.5) and the fact that $\|f\|_n \to \|f\|_\infty$.

$$I_n(-q/p) = \beta(-q/p)^{n(1+o(1))} . \tag{10.12}$$

We claim that

$$q_n = \{e \cdot p \cdot \alpha(-q/p)\}^{n(1+o(1))} . \tag{10.13}$$

By Lemma 7,

$$P_n(-q/p) = \ell^{n(1+o(1))} \tag{10.14}$$

with $\ell = \beta(-q/p)$ or $\alpha(-q/p)$. If $\ell = \beta(-q/p)$ then (10.1) implies that $|q_n| < 1$ for all large n and hence zero. Therefore by (10.10) and (10.11) $p_n \to 0$ and hence $I_n = 0$ for all large n. This contradicts (10.6) and hence established (10.13).

It is clear now from (10.13), (10.12) and (10.10) that p_n/q_n satisfies all the conditions of Criterion 2 with

$$\lambda = -\log\{e \cdot p \cdot \beta(-q/p)\} / \log\{e \cdot p \cdot \alpha(-q/p)\} . \tag{10.15}$$

Theorem follows from (10.15) and Criterion 2.

Corollary clearly follows from Theorem 2 if one observes that for all $m \in \mathbb{Z}^+$

$$\left\{1-\sqrt{1+\frac{1}{m}}\right\}^2 \cdot m \cdot e < 1. \qquad (10.16)$$

To realise (10.16) we note

$$\left\{1-\sqrt{1+\frac{1}{m}}\right\}^2 m = \frac{m}{\left\{1+\sqrt{1+\frac{1}{m}}\right\}^2 m^2} = \frac{1}{\left\{1+\sqrt{1+\frac{1}{m}}\right\}^2 m} < \frac{1}{4} \qquad (10.17)$$

Corollaries 2 and 3 follow from (10.2) by simple computation.

11. The irrational number $\pi/\sqrt{3}$.

It is desirable to have measures of irrationality for a much wider class of numbers $\log(1 - \frac{1}{n})$ than the ones considered in the previous section. Our original attempt to investigate this was to get good irrationality measure for π. In the course of these investigations we were naturally led to the number $\pi/\sqrt{3}$ which occupies a special place due to the properties of the ring of algebraic integers in $Q(\sqrt{-3})$. We shall discuss the irrationality of π itself, later

THEOREM 3. $\frac{\pi}{\sqrt{3}}$ is irrational and has irrationality measure

$$\frac{2\log\left|1+e^{\frac{\pi i}{6}}\right|+1}{2\log\left|1+e^{\frac{\pi i}{6}}\right|-1} + 1 = 8.309986\ldots$$

Proof. In (10.4) take $z = e^{\pi i/3}$. It follows from Lemma 7 that there exists N such that

$$I_n(e^{\pi i/3}) \neq 0 \quad \forall n \geq N. \tag{11.1}$$

Also since

$$\sup_{0 \leq x \leq 1} \frac{x(1-x)}{|1-e^{-\pi i/3}x|} = \frac{1}{2\sqrt{3}} \tag{11.2}$$

(11.2) and (10.5) show that

$$|I_n(e^{\pi i/3})| < \frac{2}{\sqrt{3}}\left(\frac{1}{2\sqrt{3}}\right)^n. \tag{11.3}$$

Next since $\log(1 - e^{-\pi i/3}) = \pi i/3$, we deduce from (10.4) that

$$I_n(e^{\pi i/3}) = \left\{ e^{-\pi i/3} P_n(e^{\pi i/3})\, d_n \frac{\pi i}{\sqrt{3}} - Q_n(e^{\pi i/3}) \right\} d_n^{-1}. \tag{11.4}$$

Define

$$\left.\begin{aligned} p_n &= Q_n(e^{\pi i/3}) \\ q_n &= e^{-\pi i/3} P_n(e^{\pi i/3})\, d_n \end{aligned}\right\} \tag{11.5}$$

Note that p_n, q_n are algebraic integers in $Q(\sqrt{-3})$. Using (11.5) and (11.1), (11.4) becomes

$$0 \neq q_n \frac{\pi i}{3} - p_n = d_n I_n(e^{\pi i/3}),\ n \geq N. \tag{11.6}$$

Since the ring of integers in $Q(\sqrt{-3})$ is a discrete lattice, the irrationality of $\pi/\sqrt{3}$ follows from (11.6) and (11.1) because

$$|I_n(e^{\pi i/3})|\, d_n \leq \frac{2}{\sqrt{3}}\left(\frac{1}{2\sqrt{3}}\right)^n d_n = o(1),\ n \to \infty \tag{11.7}$$

whence $\pi i/3 \notin Q(\sqrt{-3})$. Comparing imaginary parts yields the irrationality of $\pi/\sqrt{3}$.

To obtain the desired irrationality measure we will find a sequence $\{f_n/g_n\}$ satisfying the conditions of Criterion 2.

Note that Lemma 7 and (11.?) imply

$$|I_n(e^{\pi i/3})| = \{\beta(e^{\pi i/3})\}^{n(1+o(1))} \quad (11.8)$$

We claim that

$$|q_n| = \{e\alpha(e^{\pi i/3})\}^{n(1+o(1))} \quad (11.9)$$

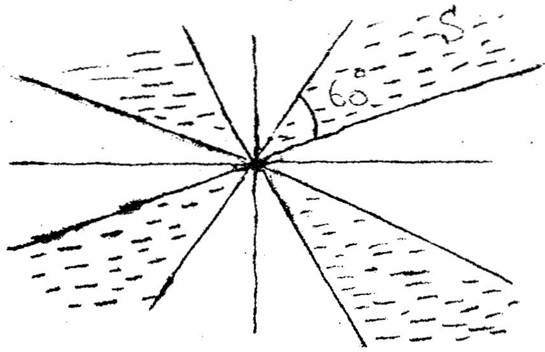
By lemma

$$|P_n(e^{\pi i/3})| = \ell^{n(1+o(1))} \quad (11.10)$$

where $\ell = \alpha(e^{\pi i/3})$, or $\beta(e^{\pi i/3})$. If $\ell = \beta(e^{\pi i/3})$ then $|q_n| < 1$ for large n and hence zero. Therefore by (11.6) $p_n = 0$ for large n, contradicting (11.1). This establishes $\ell = \alpha(e^{\pi i/3})$ and (11.9) follows from (11.10) and (11.5).

We now seek a sequence of rational approximations to $\pi/\sqrt{3}$ which satisfy the conditions of Criterion 2. Separating real and imaginary parts in (11.6) gives rise to two rational approximations to $\pi/\sqrt{3}$ but condition (ii) and (iii) in Criterion 2 might not hold. For this reason we introduce a rotation argument.

Let S be the symmetrical shaded region below



In the complex plane multiplication by $e^{\pi i/3}$ is rotation by 60°.

Let

$$(q_n', p_n', \delta_n') = \begin{cases} (q_n, p_n, 1) & \text{if } q_n \in S \\ (e^{i\pi/3} q_n, e^{i\pi/3} p_n, e^{i\pi/3}) & \text{otherwise}. \end{cases} \tag{11.11}$$

Then $q_n' \in S$, so

$$\begin{aligned} A|q_n'| &\le |\operatorname{Re} q_n'| \le |q_n'| \\ A|q_n'| &\le |\operatorname{Im} q_n'| \le |q_n'| \end{aligned} \qquad A = \sin\frac{\pi}{8} \tag{11.12}$$

Consider the equation

$$q_n' \frac{\pi i}{3} - p_n' = \delta_n' I_n(e^{\pi i/3}). \tag{11.13}$$

Let

$$2q_n' = a_n + i b_n\sqrt{3}, \quad 2p_n' = r_n + i s_n\sqrt{3} \tag{11.14}$$

where $a_n, b_n, r_n, s_n \in \mathbb{Z}$.

Separating real and imaginary parts in (11.13) gives

$$\left|b_n \frac{\pi}{\sqrt{3}} + r_n\right| = \left|2 \operatorname{Re}(\delta_n' I_n(e^{\pi i/3}))\right|$$

$$\left|a_n \frac{\pi}{\sqrt{3}} - 3 s_n\right| = \left|2\sqrt{3} \operatorname{Im}(\delta_n' I_n(e^{\pi i/3}))\right|.$$

Let

$$(g_n, f_n) = \begin{cases} (|b_n|, |\gamma_n|) & \text{if } |Re(\delta_n' I_n(e^{\pi i/3}))| > \frac{1}{2}|\delta_n' I_n(e^{\pi i/3})| \\ (|a_n|, |3\delta_n|) & \text{otherwise} \end{cases} \qquad (11.15)$$

Then the above arguments show

$$g_n = \{\epsilon\,\alpha(e^{\pi i/3})\}^{n(1+o(1))} \qquad (11.16)$$

So naturally $g_n \longrightarrow \infty$; $g_{n+1} = g_n^{\,1+o(1)}$.

Finally one deduces from (11.15) and (11.16)

$$\left|\frac{\pi}{\sqrt{3}} - \frac{f_n}{g_n}\right| = \frac{1}{g_n^{(1+\lambda)(1+o(1))}} \qquad (11.17)$$

where

$$\lambda = \frac{\log|\alpha(e^{\pi i/3})\,\epsilon^{-1}|}{\log|\alpha(e^{\pi i/3})\,\epsilon|} . \qquad (11.18)$$

Hence f_n/g_n satisfies the conditions of Criterion 2 and theorem 3 follows from (11.17) and (11.18).

<u>Remarks.</u> Unlike the results of the previous section, the use of the sup norm alone to estimate the growth of $I_n(e^{\pi i/3})$ would give a result weaker that Theorem 3. Of course the sup norm has its use, for it enabled us to determine using Lemma 7 the actual growth rate of $I_n(e^{\pi i/3})$. (see (11.8) and (11.3)).

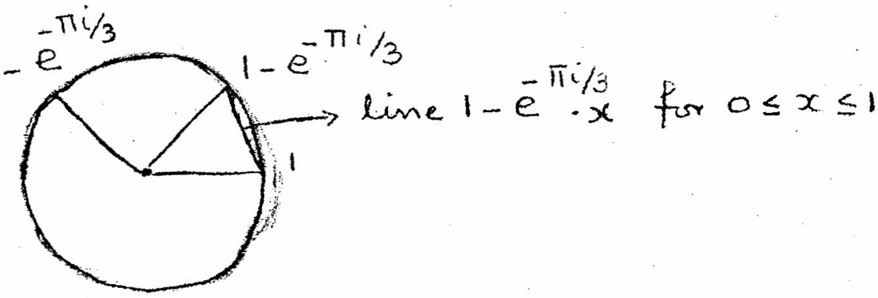
From (11.8) we see that the actual growth rate of $I_n(e^{\pi i/3})$ is

$$|I_n(e^{\pi i/3})| = (2-\sqrt{3})^{n(1+o(1))} \tag{11.19}$$

whereas the sup norm gives

$$|I_n(e^{\pi i/3})| \leq \left(\frac{1}{2\sqrt{3}}\right)^{n(1+o(1))} \tag{11.20}$$

(Perhaps an appeal to the following figure will illustrate the behaviour of the sup norm.



The suppremum of $x(1-x)/(1-e^{-\pi i/3}x)$ is easily seen from the figure to attain its value at $x = 1/2$, the mid point).

It is worth noting that (11.19) is of a genuinely smaller order of magnitude that (11.20) because

$$(2-\sqrt{3}) = \frac{1}{2+\sqrt{3}} < \frac{1}{2\sqrt{3}}\,. \tag{11.21}$$

In fact (11.21) shows that if one tries to prove the non-vanishing of $I_n(e^{\pi i/3})$ for large n, by estimating $I_n(e^{\pi i/3})$ in the neighbourhood of $x = 1/2$, one would run into trouble. Fortunately the orthogonality of $P_n(x)$ enabled one to prove Lemma 7 and consequently use Poincare's theorem.

If one does use the sup norm, then Criterion 1 must be applied. In this case one will have to check condition (ii) of Criterion 1. This may be done by means of a trick, which will be described in § 16.

<u>An alternative argument:</u> We will present below an alternate argument which will avoid the rotation idea used in the proof ofthe theorem. We will consider a variation of (11.4) namely

$$e^{\pi i/3} I_n(e^{\pi i/3})\, d_n = d_n P_n(e^{\pi i/3}) \frac{\pi i}{3} - Q_n(e^{\pi i/3}) \tag{11.22}$$

From the defining equation of $P_n(x)$ it follows that

$$P_n(x) = (-1)^n P_n(1-x) \tag{11.23}$$

Setting $x = e^{\pi i/3}$ yields

$$P_n(e^{\pi i/3}) = (-1)^n P_n(1-e^{\pi i/3}) = (-1)^n P_n(e^{-\pi i/3}) = \overline{(-1)^n P_n(e^{\pi i/3})} \tag{11.24}$$

where $\overline{}$ denotes conjugation. Therefore $P_n(e^{\pi i/3})$ is either purely real or purely imaginary.

Let us assume without loss of generality that $P_n(e^{\pi i/3})$ is imaginary. Then

$$d_n P_n(e^{\pi i/3}) \frac{\pi i}{3} \tag{11.25}$$

is purely real. From (11.3) or (11.8) we have

$$0 \le |\mathrm{Im}(e^{\pi i/3} d_n I_n(e^{\pi i/3}))| = o(1) \tag{11.26}$$

Since the integers in $Q(\sqrt{-3})$ is a discrete lattice, (11.25), (11.26) and (11.22) imply

$$\mathrm{Im}\,(Q_n(e^{\pi i/3})) = 0 \tag{11.27}$$

because $|\mathrm{Im}\,(Q_n(e^{\pi i/3}))| = o(1)$. Finally (11.27) implies that

$$\mathrm{Im}(e^{\pi i/3}\, d_n I_n(e^{\pi i/3})) = 0 . \tag{11.28}$$

Similar arguments work if we assume $P_n(e^{\pi i/3})$ is real. So if (11.12) replaces (11.4), unlike the earlier case, there is no need to have two approximations to $\pi/\sqrt{3}$, since the components of (11.12) are all either purely real or purely imaginary. Thus Criterion 2 can be applied without any trouble, yielding theorem 3.

Whereas this alternate argument appears simpler than the rotation idea, the rotation argument has its own merits. One should note that the alternate argument worked because of the special properties of $e^{\pi i/3}$ (see 11.24), whereas the rotation idea is applicable to more general situations.

12. Irrationality measure of k^{th} roots of certain rationals.

In this section we will measure the irrationality of the k^{th} roots of certain rationals by the use of Legendre Polynomials. The method described here deviates in some respects from the ones of the earlier sections. We begin with

LEMMA 9. Let k, ℓ be positive integers with $1 \le \ell < k$, $(\ell, k) = 1$. Denote by

$$d_n(k, \ell) = \mathrm{l.c.m.}\{km - \ell \mid m = 1, 2, \ldots, n\}. \tag{12.1}$$

Then

$$d_n(k,l) = e^{f(k)n(1+o(1))} \tag{12.2}$$

where

$$f(k) = \frac{k}{\varphi(k)} \sum_{\substack{a=1 \\ (a,k)=1}}^{k} \frac{1}{a} \tag{12.3}$$

and $\varphi(k)$ is the Euler function.

<u>Proof.</u> We begin with

$$\log d_n(k,l) = \sum_{p^\alpha \| (\text{some } km-l),\ p=\text{prime}} \alpha \log p \tag{12.4}$$

where $p^\alpha \| (\text{some } km-l)$ means that α is the largest exponent to which p divides any one of the members $km-l$, $m = 1,2,\ldots,n$. We divide (12.4) into two parts-namely

$$\Sigma_1 = \sum_{p \le \sqrt{kn-l}} \alpha \log p \quad , \qquad \sum_{p \geqslant \sqrt{kn-l}} \alpha \log p = \Sigma_2 \ . \tag{12.5}$$

Clearly

$$0 \le \Sigma_1 \le O\Big(\log n \sum_{p \le \sqrt{kn-l}} \log p\Big) = O(\sqrt{n} \log n) \tag{12.6}$$

by the use of the Prime Number Theorem. In Σ_2 each α is either 0 or 1. Moreover α is zero if $p \geqslant Kn - l$. Thus

$$\sum_2 = \sum_{\substack{\sqrt{kn-\ell} \le p \le kn-\ell \\ p \mid km-\ell \text{ for some } m,\ 1 \le m \le n}} \log p \qquad (12.7)$$

Pick a residue class $j \pmod k$, with $(j,k) = 1$, and let $p \equiv j \pmod k$. Then $p \mid km - \ell$ implies

$$km - \ell = \lambda p \quad , \qquad \lambda \geq 1 \qquad (12.8)$$

or equivalently

$$\lambda p \equiv -\ell \pmod k \qquad (12.9)$$

giving

$$\lambda \equiv -j^{-1} \ell \pmod k \qquad (12.10)$$

subject to the condition

$$0 \le \lambda p \le kn - \ell \quad , \quad \lambda \geq 0 \qquad (12.11)$$

Let a_j be the least positive residue class mod k such that $a_j \equiv -j^{-1} \ell \pmod k$. Clearly (12.11) implies $\lambda \geq a_j$. So from (12.11)

$$p \le \frac{kn-\ell}{\lambda} \le \frac{kn-\ell}{a_j} \qquad (12.12)$$

Therefore

$$\sum_{\substack{\sqrt{kn-\ell} \le p \le kn-\ell \\ p \equiv j \pmod k,\ p \mid \text{some } km-\ell,\ m=1,2,\cdots n}}' \log p = \sum_{\sqrt{kn-\ell} \le p \le \frac{kn-\ell}{a_j}}' \log p = \frac{kn}{\varphi(k) a_j} (1+o(1)) \qquad (12.13)$$

by the use of the Prime Number Theorem for Arithmetic Progressions. Therefore from (12.13) we deduce

$$\sum_2 = \frac{kn}{\varphi(k)} \Big(\sum_{\substack{j=1 \\ (j,k)=1}}^{k} \frac{1}{a_j} \Big) (1+o(1)) \tag{12.14}$$

From the definition of a_j it is clear that if j runs through a complete set of reduced residues mod k, so does a_j. Therefore

$$\Sigma_2 = f(k).n.(1+o(1)) \tag{12.15}$$

Lemma follows from (12.15) (12.6) and (12.5).

THEOREM 4. Let ℓ, k, p, q be positive integers with $1 \leq \ell < k$, $(\ell, k) = 1$ and $1 \leq p \leq q, (p,q) = 1$. Suppose k, p, q satisfy

$$\left\{1 - \sqrt{1+p/q}\right\}^2 \cdot q \cdot e^{f(k)} < 1 \tag{12.16}$$

where $f(k)$ is given by (12.2). Then $(1 + \frac{p}{q})^{\ell/k}$ is irrational with irrationality measure

$$\mu'_{p,q,k} = \frac{\log\left\{q\left(1+\sqrt{1+p/q}\right)^2\right\} + f(k)}{\log\left\{\frac{q}{p^2}\left(1+\sqrt{1+p/q}\right)^2\right\} - f(k)} + 1 \tag{12.17}$$

We record a few immediate corollaries.

COROLLARY 6. $(1 + \frac{1}{m})^{\ell/k}$ is irrational for all $m \in \mathbb{Z}^+$, $m \geq m_0(k)$ and $\mu_{1,m,k} \longrightarrow 2$ as $m \longrightarrow \infty$.

COROLLARY 7. If p, q satisfy (12.16) and $\log p/\log q \to 0$ as $q \to \infty$ then $\mu_{p,q,k} \to 2$ for fixed k.

Proof of Theorem 4. Define

$$J_n = \int_0^1 \frac{P_n(x)}{(1+\frac{p}{q}x)^{\ell/k}}\,dx \tag{12.18}$$

Integrating (12.18) by parts n-times

$$J_n = \frac{\prod_{j=1}^{n}\left(j+\frac{\ell}{k}-1\right)}{n!}\left(\frac{p}{q}\right)^n \int_0^1 \frac{x^n(1-x)^n}{(1+\frac{p}{q}x)^{(\ell/k)+n}}\,dx \neq 0 \tag{12.19}$$

A short calculation shows that

$$\sup_{0\le x\le 1} \frac{x(1-x)}{(1+p/q\,x)} = \left\{1-\sqrt{1+p/q}\right\}^2 q^2/p^2 . \tag{12.20}$$

From the definition of $a_r(n)$ in § 7 we have from (12.18),

$$J_n = \sum_{r=0}^{n} a_r(n) \int_0^1 \frac{x^r}{(1+\frac{p}{q}x)^{\ell/k}}\,dx \tag{12.21}$$

The substitution $u = 1 + p/q\, x$ leads to

$$J_n = \sum_{r=0}^{n} a_r(n)\left(\frac{q}{p}\right)^{r+1}\int_1^{1+\frac{p}{q}} \frac{(u-1)^r}{u^{\ell/k}}\,du$$

$$= -\sum_{r=0}^{n} a_r(n)\left(-\frac{q}{p}\right)^{r+1}\left(\sum_{j=0}^{r}\binom{r}{j}\frac{(1+\frac{p}{q})^{j+1-\ell/k}-1}{j+1-\frac{\ell}{k}}\cdot(-1)^j\right) \tag{12.2?}$$

Next let

$$q_n = -p^{n+1} d_n(k,\ell) \sum_{r=0}^{n} a_r(n) \left(-\frac{q}{p}\right)^{r+1} \sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{\left(1+\frac{p}{q}\right)^{j+1}}{j+1-\frac{\ell}{k}}$$

(12.23)

$$p_n = -p^{n+1} d_n(k,\ell) \sum_{r=0}^{n} a_r(n) \left(-\frac{q}{p}\right)^{r+1} \sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{1}{j+1-\frac{\ell}{k}} .$$

Then it follows from (12.23) and (12.1) that $p_n, q_n \in \mathbb{Z}$. So from (12.22) and (12.23)

$$0 \neq -p^{n+1} d_n(k,\ell) J_n = q_n \left(1+\frac{p}{q}\right)^{-\ell/k} - p_n . \tag{12.24}$$

The irrationality of $(1 + p/q)^{-\ell/k}$ follows from (12.24) because Lemma 9 and (12.20) imply

$$p^{n+1} d_n(k,\ell) J_n = o(1) . \tag{12.25}$$

In order to obtain the irrationality measure we will show that the sequence $\{p_n/q_n\}$ satisfies the conditions of Criterion 2.

Recall that for a bounded measurable function g on $[0,1]$ $\|g\|_n \to \|g\|_\infty$. Therefore from (12.19) and (12.20) we deduce

$$J_n = \left\{ \left(1-\sqrt{1+\frac{p}{q}}\right)^2 \cdot \frac{q}{p} \right\}^{n(1+o(1))} . \tag{12.26}$$

Next we claim that there exists constants c_1, c_2 such that for all large r (r, c, c_2 depending only on p, q, ℓ, k)

$$0 < \frac{c_2}{r} \leq \sum_{j=0}^{r} \binom{r}{j} (-1)^j \frac{\left(1+\frac{p}{q}\right)^{j+1}}{j+1-\frac{\ell}{k}} \leq c_1 . \tag{12.27}$$

To realise (12.27) one first notes that the sum there is

$$\left(1+\frac{p}{q}\right)^{\frac{\ell}{k}} \int_0^{1+\frac{p}{q}} u^{-\ell/k} (1-u)^r \, du . \tag{12.28}$$

The last inequality in (12.27) clearly follows from (12.28) because

$$\int_0^{1+\frac{p}{q}} u^{-\ell/k} (1-u)^r \, du \leq \int_0^2 u^{-\ell/k} du = O(1) . \tag{12.29}$$

To prove the first inequality, first assume $0 < p/q < 1$. Then

$$\int_0^{1+p/q} = \int_0^{1-p/q} + \int_{1-p/q}^{1} + \int_1^{1+p/q} \geq \int_0^{1-p/q} . \tag{12.30}$$

since $u^{-\ell/k}$ is a strictly decreasing function of u. Next for large r

$$\int_0^{1-p/q} u^{-\ell/k} (1-u)^r du \geq \left(1-\frac{1}{r}\right)^r \int_0^{1/r} u^{-\ell/k} du \geq \left(1-\frac{1}{r}\right)^r \frac{1}{r} \int_0^1 u^{-\ell/k} du \tag{12.3}$$
$$\geq \frac{c_2}{r}$$

So (12.27) follows from (12.30) and (12.31) when $p/q < 1$, When $p/q = 1$ one has

$$\int_0^2 u^{-\ell/k}(1-u)^n\,du = \int_0^1 + \int_1^2 \gg \int_0^1 (1-u)^n\,du \gg \frac{1}{n} \tag{12.32}$$

With (12.27) established we use Lemma 1b to deduce

$$-a_n(n)\left(-\frac{q}{p}\right)^{n+1} > 0 \tag{12.33}$$

Therefore from (12.33, (12.27) and (12.23)

$$\frac{c_4}{n} p^n d_n(k,\ell)\, P_n\left(-\frac{q}{p}\right) q \le q_n \le c_3\, p^n d_n(k,\ell)\, P_n\left(-\frac{q}{p}\right) q. \tag{12.34}$$

We claim that

$$q_n = \left\{ p\, e^{f(k)} \alpha\left(-\frac{q}{p}\right)\right\}^{n(1+o(1))} \tag{12.35}$$

By Lemma 7,

$$P_n\left(-\frac{q}{p}\right) = \ell^{\,n(1+o(1))} \tag{12.36}$$

where $\ell = \alpha(-\frac{q}{p})$ or $\beta(-\frac{q}{p})$. If $\ell = \beta(-\frac{q}{p})$ then (12.36) implies that $|q_n| < 1$ for large n and hence zero. So by (12.25) $p_n = o(1)$ for large n contradicting (12.19). So $\ell = \alpha(-\frac{q}{p})$ and (12.35) now follows from (12.36). Thus $q_n \longrightarrow \infty$ and $q_{n+1} = q_n^{1+o(1)}$. Finally (12.35), (8.14) (12.26), (12.24) and a short calculation shows

$$\left| \left(1+\frac{p}{q}\right)^{-\ell/k} - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{(1+\lambda)(1+o(1))}} \tag{12.37}$$

where

$$\lambda = \frac{\log\left\{ \frac{q}{p^2}\left(1+\sqrt{1+p/q}\right)^2 \right\} - f(k)}{\log\left\{ q\left(1+\sqrt{1+\frac{p}{q}}\right)^2 \right\} + f(k)} \tag{12.38}$$

Thus all the conditions of Criterion 2 are satisfied and hence $(1 + p/q)^{-\ell/k}$ has irrationality measure $\mu_{p,q,k}$. The same is then true for $(1 + \frac{p}{q})^{\ell/k}$. Theorem 4 is proved.

Remarks. To get irrationality measures for k^{th} roots Beukers uses the more general Jacobi Polynomials given by

$$P_{a,b}(x) = \frac{1}{x^a}\cdot\frac{1}{(b-a)!}\left(\frac{d}{dx}\right)^{b-a}\left\{x^b(1-x)^{b-a}\right\}$$

where a, b are specially chosen. The rational approximations as well as that measures that are obtained are different. Both methods however yield measures of irrationality for $(1 + \frac{p}{q})^{\ell/k}$ that converge to 2 as $\log p/\log q \longrightarrow 0$. The use of $P_n(x)$ seemed to us more direct since it was similar to the methods of sections 11, 10 and 9.

§ 13. The Zeta-Function at even integer arguments.

It is the purpose of this section to show that the Riemann-Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad R_e\, s > 1 \tag{13.1}$$

takes irrational values at the positive even integers. We shall first obtain a closed evaluation of $\zeta(s)$ at values $s = 2k$, $k \in \mathbb{Z}^+$. This is achieved by means of the following well known result from Fourier Analysis.

LEMMA 10. Let $g(x)$ be a continuous function on $(-\pi, \pi)$. Define the Fourier coefficients of g by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx \;;\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx \tag{13.2}$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g^2(x)\,dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{13.3}$$

The proof of Lemma 10 is by the use of the orthogonality of the trigonometric polynomials cos nx, sin nx.

We can now prove

THEOREM 5. There exists a rational p_k/q_k for each $k \in \mathbb{Z}^+$ such that

$$\zeta(2k) = \frac{p_k}{q_k}\pi^{2k}$$

Proof. For a positive integer k, let $g(x) = x^k$, $-\pi < x < \pi$ and

$$a_n(k) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^k \cos nx\, dx, \quad b_n(k) = \int_{-\pi}^{\pi} x^k \sin nx\, dx \tag{13.4}$$

$$a_0(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^k\, dx .$$

Setting $k = 1$ gives

$$a_n(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos nx \;\; x^k\, dx = 0, \quad n = 0, 1, 2, \dots \tag{13.5}$$

because cosine(t) is an even function. One evaluates

$$b_n(1) = \frac{-x\cos nx}{\pi n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n}\, dx = \frac{2(-1)^{n+1}}{n} . \tag{13.6}$$

So using (13.3) one has from (13.5) (13.6)

$$4\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3}$$

yielding

$$\zeta(2) = \frac{\pi^2}{6} . \tag{13.7}$$

This proves Theorem 5 when $k = 1$.

When k = 2 one has

$$b_n(2) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0, \quad n = 1, 2, \ldots \tag{13.8}$$

since sin is an odd function. On the other hand

$$a_0(2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{\pi^2}{3} \tag{13.9}$$

whilst

$$a_n(2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx = \frac{x^2 \sin nx}{\pi n}\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin nx . 2x}{n} dx$$

$$= (-1)^n \frac{4}{n^2} \tag{13.10}$$

So from (13.8), (13.9), (13.10) and (13.3) we have

$$16 \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{\pi^4 . 2}{9} = \frac{2}{5}\pi^4 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\, dx$$

yielding

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{13.11}$$

So theorem 5 is proved for k = 2.

To establish the theorem ~~for general~~ we will prove by induction on k that if k is even

$$(k \text{ even}) \quad a_n(k) = \frac{r_k}{s_k}\,\frac{1}{n^k}, \quad b_n(k) = 0 \qquad n = 1, 2, \cdots \tag{13.12}$$

on the other hand if k is odd

$$(k \text{ odd}) \quad a_n(k) = 0, \qquad b_n(k) = \frac{r_k}{s_k}\,\frac{1}{n^k}, \quad n = 1, 2, \cdots \tag{13.13}$$

where r_k, s_k are integers depending only on k.

Observe first that (13.12) and (13.13) are established, for $k = 1, 2$, (see 13.5, 13.6, 13.8, 13.10).

Let k be odd and $\geqslant 3$. Then

$$a_n(k) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^k \cos nx \, dx = 0 \tag{13.14}$$

since Cos is an even function. On the other hand

$$b_n(k) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^k \sin nx \, dx = -\frac{x^k \cos nx}{\pi \cdot n}\Bigg|_{-\pi}^{\pi} + \frac{k}{\pi}\int_{-\pi}^{\pi} \frac{x^{k-1}\cos nx}{n}\, dx$$

$$= \frac{k}{n}\, a_n(k-1) = \frac{k}{n}\,\frac{r_{k-1}}{s_{k-1}}\,\frac{1}{n^{k-1}} = \frac{r_k}{s_k}\,\frac{1}{n^k}. \tag{13.15}$$

Next if k is even

$$b_n(k) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^k \sin nx \, dx = 0 \tag{13.16}$$

since sin is an odd function, whilst

$$a_n(k) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^k \cos nx \, dx = \frac{x^k \sin nx}{\pi \cdot n}\Big|_{-\pi}^{\pi} - \frac{k}{\pi}\int_{-\pi}^{\pi} \frac{x^{k-1}\sin nx}{n}\, dx$$

$$= \frac{-k}{n} b_n(k-1) = -\frac{k}{n}\frac{\lambda_{k-1}}{S_{k-1}}\frac{1}{n^{k-1}}$$

$$= \frac{\lambda_k}{S_k}\frac{1}{n^k} \tag{13.17}$$

Thus (13.14) then (13.17) establish (13.12), (13.13) by double induction.

Now observe that

$$a_0(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^k dx = \frac{2\pi^k}{(k+1)} \quad \text{if } k \text{ is even}$$

$$= 0 \quad \text{if } k \text{ is odd} \tag{1318}$$

So by (13.8) and (13.12), (13.13)

$$a_0^2(k) + \frac{\lambda_k^2}{S_k^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{2k} dx$$

$$= \frac{2\pi^{2k}}{2k+1} \tag{13.19}$$

yielding

$$\zeta(2k) = \frac{p_k}{q_k}\, \pi^{2k}$$

Theorem 5 is proved.

THEOREM 6. For $k = 1,2,3,\ldots,$ $\zeta(2k)$ is irrational.

Proof. Theorem 6 is an immediate consequence of the fact that π is a transcendental number (see § 4 and [6]). The transcendence of π implies that all its powers are transcendental and hence irrational.

Theorem 6 has an interesting interpretation in the theory of primes.

COROLLARY 8. The number of prime numbers is infinite.

Proof. For every even integer $2k$, $k \in \mathbb{Z}$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \prod_{p} \left(1 - \frac{1}{p^{2k}}\right)^{-1} \tag{13.20}$$

where the product goes over all primes. Identity (13.20) attributed to Euler is an immediate consequence of ~~uniqueness of~~ unique factorization. If the number of primes were finite the right hand side would be rational whereas the left side is irrational. This contradiction establishes the corollary.

Remarks. No simple closed evaluations of the type described by theorem 5 have been found for $\zeta(2k+1)$ for any integer $k > 0$. This was precisely the reason why the problem of deciding the irrationality of the zeta function at the positive odd integers remains unsolved today. Several mathematicians (including Ramanujan) have tried to get useful closed evaluations of $\zeta(2k+1)$ but in vain. It was generally believed that all the numbers $\zeta(2k+1)$ were equally difficult to deal with until Apery's proof showed that $\zeta(3)$ was a simpler number than the values of the Zeta function at larger odd integers.

14. Irrationality of $\zeta(2)$.

We shall now present Beukers's proof of the irrationality of $\zeta(2)$. His ideas are motivated by the identities of Apery. Baukers obtains a proof of the irrationality of $\zeta(2)$ which does not involve a closed evaluation of the sum. We begin with

LEMMA 11. Let r and s be non-negative integers. If $r > s$ then

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy}\, dx\, dy \tag{14.1}$$

is a rational number with denominator that is a divisor of d_r^2. If $r = s$ then

$$\int_0^1 \int_0^1 \frac{x^r y^r}{1-xy}\, dx\, dy = \zeta(2) - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots - \frac{1}{r^2}. \tag{14.2}$$

<u>Proof.</u> Expand $(1-xy)^{-1}$ into a power series and perform double integration term by term. One then evaluates the integral

in (14.1) as

$$\sum_{k=0}^{\infty} \frac{1}{(k+r+1)(k+s+1)} = \sum_{k=0}^{\infty} \frac{1}{(r-s)} \left[\frac{1}{k+s+1} - \frac{1}{k+r+1} \right]$$

$$= \frac{1}{r-s} \left\{ \frac{1}{s+1} + \cdots + \frac{1}{r} \right\} \tag{14.3}$$

when $r \neq s$. The first assertion of Lemma 11 follows from (14.3).

When $r = s$, it is easily seen that the first sum in (14.3) is precisely the quantity in (14.2). Lemma 11 is proven.

THEOREM 7. $\zeta(2)$ is irrational.

<u>Proof.</u> For a positive integer n denote by

$$B_n = \int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} \, dx\, dy \tag{14.4}$$

It is clear that

$$B_n = \{ p_n + q_n \zeta(2) \} d_n^{-2}, \quad p_n, q_n \in \mathbb{Z}. \tag{14.5}$$

An n-fold integration by parts with respect to x transforms B_n into

$$B_n = (-1)^n \int_0^1 \int_0^1 \frac{y^n (1-y)^n x^n (1-x)^n}{(1-xy)^{n+1}} \, dx\, dy \tag{14.6}$$

To realise (14.6) from (14.4) one integrates $P_n(x)$ and differentiates $(1-xy)^{-1}$, and uses (7.1).

Now one computes

$$\sup_{\substack{0\le x\le 1\\ 0\le y\le 1}} \frac{x(1-x)y(1-y)}{1-xy} = \left(\frac{\sqrt{5}-1}{2}\right)^5 \tag{14.7}$$

Note that (14.6) implies $B_n \neq 0$. So from (14.5), (14.6) and (14.7) one has

$$\begin{aligned} 0 \neq |p_n + q_n \zeta(2)| &\le d_n^2 \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \int_0^1\int_0^1 \frac{dx\,dy}{1-xy} \\ &= d_n^2 \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2) = o(1) \end{aligned} \tag{14.8}$$

because by Lemma 8

$$d_n^2 = e^{2n(1+o(1))} \tag{14.9}$$

and

$$0 < e^2\left(\frac{\sqrt{5}-1}{2}\right)^5 < 1 \tag{14.10}$$

The irrationality of $\zeta(2)$ follows from (14.8) and Dirichlet's criterion.

<u>Remarks.</u> One can also obtain a measure of irrationality for $\zeta(2)$. Since we are dealing with positive functions in (14.6) and (14.7) it is clear that

$$B_n = \left(\frac{\sqrt{5}-1}{2}\right)^{5n(1+o(1))} \tag{14.11}$$

If one evaluates the p_n, q_n in (14.5), one finds these are identical to the sequences generated by Apery. These sequences may be shown to satisfy a recurrence relation from which one can deduce the asymptotic behavior. (see [16] for details). It turns out that

$$q_n = \left\{ e^2 \left(\frac{\sqrt{5}+1}{2}\right)^5 \right\}^{n(1+o(1))} = \left[\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \right] d_n^2 \tag{14.12}$$

So p_n/q_n is a sequence of approximations to $\zeta(2)$ satisfying the conditions of Criterion 2 with

$$\left| \zeta(2) - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{(1+\lambda)(1+o(1))}} \tag{14.13}$$

where

$$\lambda = \frac{5 \log\left(\frac{\sqrt{5}+1}{2}\right) + 2}{5 \log\left(\frac{\sqrt{5}+1}{2}\right) - 2} + 1 \tag{14.14}$$

It follows from Criterion 2 that $\zeta(2)$ has irrationality measure

$$1 + \frac{1}{\lambda} = 11.851 \ldots \tag{14.15}$$

Since $\zeta(2) = \pi^2/6$, the following is immediate from (14.15).

COROLLARY 9. For all $k \in \mathbb{Z}^+$, the numbers π^2/k have irrationality measure $11.851 \ldots$.

§15. Irrationality of $\zeta(3)$ - a celebrated problem.

In this section we shall prove the irrationality of $\zeta(3)$. The proof we shall give is due to Beukers. As remarked earlier, Beukers' proof is essentially that of Apery's but much more easy to understand. In fact the approximating sequence p_n/q_n of rationals that Beukers obtains is identical to Apery's. The proof makes very clever use of triple integrals involving Legendre Polynomials. The proof is an extension of the proof of the irrationality of $\zeta(2)$ presented in the earlier section. We begin with

LEMMA 12. Let r and s be non-negative integers. If $r > s$ then the integral

$$-\int_0^1\int_0^1 \frac{\log xy \,.\, x^r y^s}{1-xy}\,dx\,dy \tag{15.1}$$

is a rational number whose denominator divides d_r^3. If $r = s$

$$-\int_0^1\int_0^1 \frac{\log xy \; x^r y^r}{1-xy}\,dx\,dy = 2\left[\zeta(3)-\frac{1}{1^3}-\frac{1}{2^3}-\cdots-\frac{1}{r^3}\right] \tag{15.2}$$

Proof. Let $\sigma > 0$. Consider the integral

$$\int_0^1\int_0^1 \frac{x^{r+\sigma}\,y^{s+\sigma}}{1-xy}\,dx\,dy \tag{15.3}$$

.

Expanding $(1 - xy)^{-1}$ power series and integrating term by term transforms the integral to

$$\sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} . \tag{15.4}$$

One can rewrite this sum as

$$\sum_{k=0}^{\infty} \frac{1}{r-s}\left[\frac{1}{k+s+\sigma+1} - \frac{1}{k+\sigma+r+1}\right] = \frac{1}{r-s}\left\{\frac{1}{s+\sigma+1} + \cdots + \frac{1}{r+\sigma}\right\}. \tag{15.5}$$

when $r = s$.

If we differentiate with respect to σ and put $\sigma = 0$, the integral changes to

$$\int_0^1\int_0^1 \frac{\log xy}{1-xy}\, x^r y^s \, dx\, dy \tag{15.6}$$

while (15.5) changes to

$$\frac{1}{r-s}\left\{\frac{1}{(s+1)^2} + \cdots + \frac{1}{r^2}\right\}. \tag{15.7}$$

The first part of Lemma 12 follows from (15.6) and (15.7).

When $r = s$ it follows from (15.4) by differentiation with respect to σ and setting $\sigma = 0$ that

$$-\int_0^1\int_0^1 \frac{\log xy \; x^r y^r}{1-xy}\, dx\, dy = \sum_{k=0}^{\infty} \frac{-2}{(k+r+1)^3} \tag{15.8}$$

and (15.2) follows from (15.8). Lemma 12 is proved.

We are now in a position to prove

THEOREM 8. $\zeta(3)$ is irrational.

Proof. We begin by considering the integral

$$C_n = -\int_0^1\int_0^1 \frac{\log xy \, . P_n(x) P_n(y) \, dx\, dy}{1-xy} \tag{15.9}$$

It is clear from Lemma 1(a) that

$$C_n = (p_n + q_n \zeta(3)) \, d_n^{-3}, \quad p_n, q_n \in \mathbb{Z} \tag{15.10}$$

We now notice

$$-\frac{\log xy}{1-xy} = \int_0^1 \frac{dz}{1-(1-xy)z} \tag{15.11}$$

Combining (15.9), (15.11) yields

$$C_n = \int_0^1\int_0^1\int_0^1 \frac{dx\, dy\, dz \, P_n(x) P_n(y)}{1-(1-xy)z} \tag{15.12}$$

If we integrate C_n by parts with respect to x n-times, by integrating $P_n(x)$ and differentiating $\{1-(1-xy)z\}^{-1}$ one obtains

$$C_n = \int_0^1\int_0^1\int_0^1 \frac{(xyz)^n (1-x)^n P_n(y)}{\{1-(1-xy)z\}^{n+1}} \, dx\, dy\, dz \tag{15.13}$$

by the use of (7.1).

Next we use the substitution

$$w = \frac{(1-z)}{1-(1-xy)z} \quad \text{or} \quad z = \frac{1-w}{1-(1-xy)w} \tag{15.14}$$

yielding

$$dz = \frac{-xy\,dw}{\left\{1-(1-xy)w\right\}^2}\ , \quad 1-(1-xy)z = \frac{xy}{1-(1-xy)w} \tag{15.15}$$

These substitutions convert C_n to

$$C_n = (-1)^n \int_0^1\int_0^1\int_0^1 \frac{(1-x)^n(1-w)^n P_n(y)}{1-(1-xy)w}\,dx\,dy\,dw\ . \tag{15.16}$$

Finally an n-fold integration by parts with respect to y of (15.16), integration of $P_n(y)$ and differentiation of $\left\{1-(1-xy)w\right\}^{-1}$, yields as a consequence of (7.1),

$$C_n = \int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{\left\{1-(1-xy)w\right\}^{n+1}}\,dx\,dy\,dw\ . \tag{10.17}$$

Next we notice that by symmetry $x(1-x)\ (y)(1-y)\ w(1-w)\left\{1-(1-xy)w\right\}^{-1}$ can attain its maximum only when $x = y$. One can then show after a certain computation that

$$\sup_{\substack{0\le x\le 1\\ 0\le y\le 1\\ 0\le w\le 1}} \frac{x(1-x)y(1-y)w(1-w)}{\left\{1-(1-xy)w\right\}} = (\sqrt{2}-1)^4. \tag{15.18}$$

It is clear from (15.18) and (15.17) that

$$0 \leq c_n \leq (\sqrt{2}-1)^{4n} \int_0^1 \int_0^1 \int_0^1 \frac{dx\,dy\,dz}{1-(1-xy)z} \leq (\sqrt{2}-1)^{4n}\, 2\, \zeta(3) \tag{15.19}$$

Finally (15.10) and (15.19) yields

$$0 \leq |p_n + q_n \zeta(3)| = d_n^3 c_n \leq d_n^3 (\sqrt{2}-1)^{4n} \cdot 2\, \zeta(3) \tag{15.20}$$

The irrationality of $\zeta(3)$ follows from (15.20) because by Lemma 8

$$d_n^3 = e^{3n(1+o(1))} \tag{15.21}$$

whilst

$$0 < e^3 (\sqrt{2}-1)^4 < 1 \tag{15.22}$$

yielding in (15.20)

$$0 \neq | p_n + q_n \zeta(3) | = o(1), \quad n \to \infty \tag{15.23}$$

The proof of Theorem 8 is complete.

<u>Irrationality measure for $\zeta(3)$.</u> The above method yields an irrationality measure for $\zeta(3)$. The discussion shall proceed along the lines of § 14.

Since we are dealing with a positive function in (15.17) it follows from (15.18) that

$$c_n = (\sqrt{2}-1)^{4n(1+o(1))} \tag{15.24}$$

To calculate q_n one uses Lemma 12, (15.9) (15.10) (7.1), and Lemma 1 (b). One can compute

$$q_n = d_n^3 \sum_{r=0}^{n} a_r^2(n) = \left[\sum_{r=0}^{n} \binom{n}{r}^2 \binom{n+r}{r}^2 \right] d_n^3 \tag{15.25}$$

The asymptotic sige of q_n can be estimating by calculating

$$f(n) = \max_{0 \le r \le n} \binom{n}{r}^2 \binom{n+r}{r}^2. \tag{15.26}$$

One can set $n = \lambda r$, with $\lambda > 1$, and the maximum in (15.26) will be attained at a value near λn where

$$\lambda = \sqrt{2} \tag{15.27}$$

So (15.27) will enable us to estimate $f(n)$ asymptotically. Once this is done, one simply notes

$$f(n) \le \sum_{r=0}^{n} \binom{n}{r}^2 \binom{n+r}{r}^2 \le n\, f(n) \tag{15.28}$$

yielding

$$\sum_{r=0}^{n} \binom{n}{r}^2 \binom{n+r}{r}^2 = f(n)^{1+o(1)}. \tag{15.29}$$

The upshot of the whole procedure is that after a certain computation one can establish

$$q_n = \left\{ e^3 (\sqrt{2}+1)^4 \right\}^{n(1+o(1))} \tag{15.30}$$

Once this is done it is easy to verify that

$$0 \neq \left| \zeta(3) - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{(1+\lambda)(1+o(1))}} \tag{15.31}$$

where

$$\lambda = \frac{4 \log\{\sqrt{2}+1\} - 3}{4 \log\{\sqrt{2}+1\} + 3}, \quad 0 < \lambda < 1 . \tag{15.32}$$

Thus p_n/q_n satisfies all the conditions of Criterion 2 and $\zeta(3)$ has irrationality measure

$$1 + \frac{1}{\lambda} = 13.4178 \ldots$$

The sequence (p_n, q_n) which comes out of Beuker's integral is identical to the sequence (p_n, q_n) generated by Apery. One can show after a lot of manipulations (see [16]) that q_n satisfies a two term recurrence with coefficients tending to a constant. One can then use Poincare's theorem which yields (15.30). Apery used the classical Criterion 1 instead of Criterion 2 which involved verification of condition (i) of Criterion 1. This was achieved by means of a trick which will be described in the next section.

For the moment we collect the above arguments into

THEOREM 9. $\zeta(3)$ has irrationality measure 13.4178... .

Remarks. It is desirable to have a proof of the irrationality of $\zeta(k)$, $k = 2,3,\ldots$, by the above method. At the moment no one seems to know how to do this. The difficulty seems to be the following. If one represents $\zeta(k)$ by means of a k-dimensional integral then these are no 'obvious' substitions like (15.14) which yield an estimate like (15.23) when $k \geqslant 4$. The number 3 or the dimension 3 is small enough for such tricks to take place. Thus from Apery's point of view, $\zeta(3)$ is a simpler number compared to the values of the Zeta function at larger integers. This is different from the ideas of § 13. (see remarks at the end of § 13).

§ 16. Explicit Lower Bounds.

Let θ be irrational. Another problem (closely related to finding an irrationality measure) is to exhibit constants c, k (depending only on θ) such that

$$\left|\theta - \frac{a}{b}\right| \geqslant \frac{c}{b^k} \quad \forall\, a, b \in \mathbb{Q} \tag{16.1}$$

We shall use estimates from Theorem 2 to show that

$$\left|\log 2 - \frac{a}{b}\right| \geqslant \frac{10^{-10}}{b^k} \geqslant \frac{10^{-10}}{b^{5.8}} \tag{16.2}$$

with

$$k = 1 + \frac{\log 24}{\log\left\{(\sqrt{2}+1)^2/3\right\}} < 5.8 \tag{16.3}$$

To see this let $p = q = 1$ in Theorem 2. The corresponding q_n in (10.9) becomes

$$q_n = d_n\, P_n(-1) \tag{16.4}$$

From Lemma (c) one deduces

$$P_n(-1) = \sum_{k=0}^{n} \binom{n}{k}^2 2^k \tag{16.5}$$

Rosser and Schoenfeld [14] have shown

$$d_n < \xi^n , \qquad \xi = 2.826 \tag{16.6}$$

By (16.4), (16.5), (16.6)

$$2^n \leq q_n \leq 8^n \xi^n \tag{16.7}$$

because

$$\sum_{k=0}^{n} \binom{n}{k}^2 2^k \leq 2^n \left(\sum_{k=0}^{n} \binom{n}{k} \right)^2 = 8^n . \tag{16.8}$$

Also q_n is a monotone increasing sequence tending to infinity. Note also that (10.5) and (10.8) yield

$$0 \leq I_n(-1) \leq (\sqrt{2}-1)^{2n} . \tag{16.9}$$

We claim that the sequence p_n/q_n has the property $p_n/q_n \neq p_{n-1}/q_{n-1}$. Th see this let $A_n = P_n(-1)$ and $B_n = Q_n(-1)/d_n$ By Lemmas 4 and 5 and (10.4) wel know

$$\begin{aligned} n A_n - 3(2n-1) A_{n-1} + (n-1) A_{n-2} &= 0 \\ n B_n - 3(2n-1) B_{n-1} + (n-1) B_{n-2} &= 0 \end{aligned} \tag{16.10}$$

Eliminating A_{n-1}, B_{n-1} from (16,10) gives

$$n \left[A_n B_{n-1} - B_n A_{n-1} \right] = (n-1) \left[A_{n-1} B_{n-2} - B_{n-1} A_{n-2} \right] . \tag{16.11}$$

Since $A_1 B_0 \neq A_0 B_1$, iteration of (16.11) shows that $A_n/B_n \neq A_{n-1}/B_{n-1}$. This shows that the same is then true of p_n/q_n as claimed.

Now let a/b be any rational and $\alpha = \log\left((\sqrt{2}+1)^2/\xi\right)/\log 8\xi$. Let m be the first integer such that

$$q_m^{\alpha} \geq 2b \tag{16.12}$$

Let

$$q_n = \begin{cases} q_m & \text{if } \frac{a}{b} \neq \frac{p_m}{q_m} \\ q_{m+1} & \text{otherwise}. \end{cases} \tag{16.13}$$

Using Lemma 4 with $z = -1$, it easily follows that

$$q_k < 6k\, q_{k-1} \tag{16.14}$$

So (16.12), (16.13), (16.14) all imply

$$(2b)^{1/\alpha} > q_{m-1} \geq \frac{q_n}{36n(n-1)}. \tag{16.15}$$

Note that (16.9), (16.7) and (16.4) show

$$\left|\log 2 - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^{1+\alpha}} \tag{16.16}$$

Now observe

$$\frac{1}{bq_n} \leq \left|\frac{a}{b} - \frac{p_n}{q_n}\right| \leq \left|\log 2 - \frac{a}{b}\right| + \left|\log 2 - \frac{p_n}{q_n}\right|. \tag{16.17}$$

So (16.16) and (16.17) yield

$$\left|\log 2 - \frac{a}{b}\right| \geq \frac{1}{bq_n} - \frac{1}{q_n^{1+\alpha}}. \tag{16.18}$$

It follows from (16.12) and (16.18) that

$$\left|\log 2 - \frac{a}{b}\right| \geq \frac{1}{q_n^{1+\alpha}} = \frac{q_n^{k\alpha-1-\alpha}}{q_n^{k\alpha}}. \tag{16.19}$$

Thus (16.15) gives

$$\left|\log 2 - \frac{a}{b}\right| \geqslant \frac{q_n^{k\alpha-1-\alpha}}{\{36n(n-1)(2b)^{1/\alpha}\}^{k\alpha}} = \left[\frac{q_n^{k\alpha-1-\alpha}}{2^k\{36n(n-1)\}^{k\alpha}}\right]\frac{1}{b^k} . \quad (16.20)$$

Inequality (16.12) follows from (16.20) because from (16.7)

$$\frac{q_n^{k\alpha-1-\alpha}}{2^k\{36n(n-1)\}^{k\alpha}} \geqslant \frac{2^{n(k\alpha-1-\alpha)}}{2^k\{36n(n-1)\}^{k\alpha}} \geqslant 10^{-10} .$$

Arguments similar to the above with $z = -m$ in (10.4) yield

THEOREM 10. Let $m \in \mathbb{Z}^+$. Then

$$\left|\log\left(1+\frac{1}{m}\right) - \frac{a}{b}\right| \geqslant \frac{(10m)^{-10}}{b^{k_m}}$$

for all rationals a, b where

$$k_m = \frac{\log(4(m+1)\cdot 3)}{\log\{(m(1+\sqrt{1+\frac{1}{m}})^2)/3\}} + 1 . \quad (16.21)$$

We omit the details in the proof of theorem 10.

Similar computations can be carried out for $\pi/\sqrt{3}$. In this case one can use (11.3) which gives an explicit upper bound. The rational approximations p_n/q_n to $\pi/\sqrt{3}$ can be shown to satisfy condition (1) of Criterion 1. This is done by using ideas similar to (16.10) and (16.18) with $z = e^{\pi i/3}$ replacing $z = -1$. Beukers has carried out the computations and the result he obtains is

$$\left|\frac{\pi}{\sqrt{3}} - \frac{p}{q}\right| \geqslant \frac{1}{q^{21}} . \quad (16.22)$$

Even for the numbers $\zeta(2)$ and $\zeta(3)$ the method described in § 14 and § 15 will yield explicit lower bounds. Upper bounds for the integrals follow from (15.19) and (14.7). To show that p_n/q_n satisfy condition (i) of Criterion 1 one uses the following idea :

For $\zeta(3)$, the A_n, B_n satisfy

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n-1)^3 u_{n-2} = 0. \quad (16.23)$$

Eliminating the central term in (16.23), as in (16.10) yields an identity similar to (16.11). It follows from this by iteration that $p_n/q_n \neq p_n/q_{n-1}$.

For $\zeta(2)$ the A_n, B_n satisfy

$$n^2 u_n - (11n^2 - 11n + 3)u_{n-1} - (n-1)^2 u_{n-2} = 0. \quad (16.23)$$

Again the process described above shows $p_n/q_n \neq p_{n-1}/q_{n-1}$. Identities (16.22) and (16.23) are due to Apery.

Once these are established one can obtain by the method described for log 2, explicit lower bounds for approximations to $\zeta(2)$ and $\zeta(3)$. As far as we know no one seems to have carried out these computations fully.

§ 17. Comparisons.

The estimates we have obtained for the logarithm improve earlier results. Baker [5] combined ideas of Siegel and Mahler and proved the following :

$$\left| \log\left(1+\frac{1}{m}\right) - \frac{a}{b} \right| > \frac{c(m)}{q^{\beta_m}} \quad (17.1)$$

where

$$\beta_1 = 12\cdot 5\ , \ \beta_2 = 7$$

$$\beta_m = 2\ \frac{\log\{4\sqrt{2}\ m^2/(m+1)\}}{\log\{\sqrt{2}\ m^3/(m+1)^2\}}$$

and

$$C(1) = 10^{-10^5}\ ,\ C(m) = (\sqrt{2}\ m)^{-10^4}\ ,\ m \geq 2\ .$$

It should be noted that Theorem 10 is stronger than (17.1) because

$$C(m) < (10m)^{-10} \text{ and } 2 \leq K_m \leq \beta_m\ ,\ m \in \mathbb{Z}^+\ .$$

Of course if $\mu_{1,m}$ is as in Theorem 2 we also have

$$2 \leq \mu_{1,m} < K_m\ ,\ m \in \mathbb{Z}^+\ .$$

In fact

$$\mu_{1,m} = 2 + \frac{2}{\log m} + o\left(\frac{1}{\log^2 m}\right)$$

$$K_m = 2 + \frac{2\log 3}{\log m} + o\left(\frac{1}{\log^2 m}\right)$$

$$\beta_n = 2 + \frac{2\log 4}{\log m} + o\left(\frac{1}{\log^2 m}\right).$$

Bombieri [8] recently used differential equations to obtain Theorem 2 along with other results.

The results of Theorem 4 are certainly not the best known. The Thue-Siegel-Roth theorem (see § 5) states that any algebraic irrational number has irrationality measure 2. However for the special algebraic numbers of Theorem 4 our method is much simpler.

Baker [3], [4] obtained effective irrationality measures which are superior to theorem 4 in certain cases. For instance with p = 3, q = 125 in Theorem 4 Baker shows

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > \frac{c}{q^{2.96}}, \qquad c > 0 \tag{17.2}$$

This shows that if N is a prescribed integer then the Diophantine equation

$$x^3 - 2y^3 = N \tag{17.3}$$

has only finitely many solutions. Fur suppose that (17.3) has infinitely many solutions. Then writing

$$x^3 - 2y^3 = (x - \sqrt[3]{2}\,y)(x^2 + \sqrt[3]{2}\,xy + \sqrt[3]{4}\,y^2) \tag{17.4}$$

one deduces that (17.3) and (17.4) imply

$$\left| \sqrt[3]{2} - \frac{x}{y} \right| = o\left(\frac{1}{y^3}\right)$$

has infinitely many solutions, contradicting (17.2). The value of c will yield a bound on the size of the solutions. For details see [ ].

We know from § 14 (Corollary) that Apery showed that π^2/k $k \in \mathbb{Z}^+$ has irrationality measure 11.851. That is if $k \in \mathbb{Z}^+$ and $\varepsilon > 0$ then

$$\left| \frac{\pi^2}{k} - \frac{p}{q} \right| > \frac{1}{q^{11.851+\varepsilon}} \qquad \forall\, q \geq q_0(\varepsilon, k)$$

Consequently for all $k \in \mathbb{Z}^+$

$$\left|\frac{\pi}{\sqrt{k}} + \frac{p}{q}\right| \cdot \left|\frac{\pi}{\sqrt{k}} - \frac{p}{q}\right| = \left|\frac{\pi^2}{k} - \frac{p^2}{q^2}\right| \geq \frac{1}{q^{23.702+\epsilon}} \qquad q \geq q_1(\epsilon, k)$$

whence

$$\left|\frac{\pi}{\sqrt{k}} - \frac{p}{q}\right| \geq \frac{1}{q^{23.702+\epsilon}} \quad \forall q \geq q_2(\epsilon, k) \tag{17.5}$$

Theorem 3 is superior to (17.5) when $k = 3$.

Mignotte [12] utilised ideas of Mahler to show that π has irrationality measure 20.81 (which is superior to (17.5) when $k = 1$). A substantial improvement of Mignotte's results using our methods would be most desirable. The simplicity of our approach induces certain limitations on such attempts which will be discussed in the next section.

18. Merits and Limitations of our method.

The function $x^n(1-x)^n$ appears in the proof of the result that e^s is irrational whenever $s \neq 0$ is rational. One deduces from this that $\log s$ is irrational so long as s is rational and $s \neq 1$. As noted in § 10 this procedure does not yield satisfactory measures of irrationality for the logarithm. By replacing $x^n(1-x)^n$ by $P_n(x)$ one is able to prove simultaneously the irrationality of e^s for all $s \in Q$, $s \neq 0$ and $\log s$ for certain $s \neq 0$. The advantage of this is that (see § 10) one gets good irrationality measures for the logarithm. It would be nice to extend these results to a larger class $\log(1 - \frac{1}{z})$ then the numbers z considered here.

In Theorem 2 the only restriction is inequality (10.1) This is necessary in the method because in (10.11) the condition o(1) is necessary to deduce irrationality. Inequality (10.1) is also consistent with irrationality measure $\mu_{p,q}$ because (10.1) is equivalent to

$$\log\left\{\frac{q}{p^2}\left(1+\sqrt{1+\frac{p}{q}}\right)^2\right\}-1>0 \tag{18.1}$$

which shows the denominator in (10.2) is positive. As a consequence of (18.1), $\mu_{p,q} > 2$. On the other hand if we permit the denominator in (10.2) to be negative then $\mu_{p,q} < 1$ contradicting (5.3).

While Theorem 2 apparently admits the logarithms of certain rationals greater than one, some rationals less than one are also allowed because

$$\log\left(1+\frac{p}{q}\right) = -\log\left(1-\frac{p}{q+p}\right). \tag{18.2}$$

Similarly in Theorem 4 inequality (12.16) assures that (12.25) holds (which is necessary to deduce irrationality and be consistent with the irrationality measure $\mu_{p,q,k}$). A rational greater than one, $1+\frac{p}{q}$, was chosen to compensate for the change of sign of $a_r(n)$ in (12.33). Note that Theorem 4 also applies to certain rationals less than one because

$$\left(1+\frac{p}{q}\right)^{\ell/k} = \left(1-\frac{p}{q+p}\right)^{-\ell/k}. \tag{18.3}$$

If $z \in Q(\sqrt{-d})$, $d \in Z^+$ is an algebraic integer then our method shows.

that $\log(1-\frac{1}{z}) \notin Q(\sqrt{-d})$. The proof of this which uses the discreteness of the integer lattice is similar to the proof of Theorem 3. But this does not yield information about both the real and imaginary parts of $\log(1-\frac{d}{z})$ in general. When $d = 1$ our method yields

$$\left| \log\left(1-\frac{1}{z}\right) - \frac{p_n}{q_n} \right| = \frac{o(1)}{|q_n|} \tag{18.4}$$

where $p_n, q_n \in Q(i)$ are algebraic integers. But then separating real and imaginary parts involves the expression

$$\frac{p_n}{q_n} = \frac{a_n}{b_n} + i\frac{c_n}{s_n} \tag{18.5}$$

One loses a lot in doing this because in general $|s_n| \gg |q_n^2|$.

Thus the quantity $\frac{o(1)}{|q_n|}$ in (18.4) is not small enough to ensure irrationality because generally it is of size $1/|q_n|^{(1+\lambda)(1+o(1))}$, $0 < \lambda < 1$. If there were some way of getting around this difficulty one would have an irrationality measure for π, because if $z = i$, $\log(1-\frac{1}{z}) = \frac{\log 2}{2} + \frac{i\pi}{4}$. At the moment this does not seem likely.

On the other hand one might write $\frac{i\pi}{2} = \log i$ with $z = -(i-1)^{-1}$. In this situation one does not have to separate real and imaginary parts. Unfortunately $-(i-1)^{-1}$ is not a Gaussian integer. To compensate for this multiplication by $(i-1)^n$ is necessary to get a linear form in $\log(1-\frac{1}{z})$ and 1 with integer coefficients. But then the $o(1)$ in (18.4) no longer holds destroying the proof of irrationality.

An interesting case does occur when $z = e^{\pi i/3}$ (yielding $1 - \frac{1}{z} = z = e^{\pi i/3}$ so that $\log(1 - \frac{1}{z})$ is purely imaginary). This is precisely the case discussed by Theorem 3 yielding the result for $\pi/\sqrt{3}$.

What about the method of Beukers discussed in § 14 and § 15? Can one use the two and three dimensional integrals to prove the irrationality of a whole class of numbers which include $\zeta(2)$, $\zeta(3)$ as special cases? This is discussed in the next section.

§ 19. Generalizations.

The Riemann Zeta Function and the logarithm are particular cases of a general function - the k-logarithm function. Let $k, z \in \mathbb{C}$. Define

$$L_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \tag{19.1}$$

whenever the series is convergent. One notes that if $|z|<1$ then (19.1) is well defined for all $k \in \mathbb{C}$. When $|z| = 1$, $z \neq 1$ is defined by (19.1) only for $\operatorname{Re} k > 0$. When $|z| > 1$ the series diverges no matter what k is.

It should be noted that

$$L_k(1) = \zeta(k), \text{ (Riemann Zeta function }, k>1) \tag{19.2}$$

and

$$L_1(z) = -\log(1-z) \tag{19.3}$$

Special interest attaches to $L_2(z)$ and $L_3(z)$ the Di-logarithm and tri-logarithm functions, because these arise as natural generalizations of $\zeta(2)$ and $\zeta(3)$ in much the same way as $-\log(1-z)$ generalizes log 2.

By considering the Jacobi Polynomials,

$$J_{a,b}(x) = \frac{1}{x^a(b-a)!}\left(\frac{d}{dx}\right)^{b-a}\left\{x^b(1-x)^{b-a}\right\}$$

which is a generalisation of the Legendre Polynomials, Beukers has by the use of the two and three dimensional integrals of the type described in § 14 and § 15 proved the irrationality of the Dilog and Trilog functions at certain rational values z near zero. Of course Chudnovski's general theory of irrationality which involves the construction of Pade-approximations to functions satisfying linear differential equations includes a discussion of the Dilogarithm and Trilogarithm functions. The function $L_k(z)$ for $k \in \mathbb{Z}^+$, $k \geq 4$ seems at the moment difficult to tackle. I am afraid I cannot elaborate on the methods of Beukers or Chudnovski since I am unfamiliar with these recent techniques.

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ERRATA

Page	Line	Instead of	Read
1	14	Inernational Con ress	International Congress
6	9	Corollary	Corollary A
15	1	be function	be a function
18	17	satisfies	satisfying
23	18	Le nitz's	Leibnitz's
25	19	(.2)	(7.2)
26	10	beh vior	behaviour
28	4	$\equiv$	$=$
32	last	$\frac{1}{2}$	$\frac{1}{4}$
33	3	$\frac{1}{2}$	$\frac{1}{4}$
33	4	Combing	Combining
50	7	ineequality	inequality
53	3	achiev d	achieved
59	9	Baukers	Beukers
64	7	σ = 0	$\sigma = 0$
66	10	(10.17)	(15.17)
68	7	λn	n/λ
69	8	Beuker'	Beukers'
70	4	$\zeta(3)$	$\zeta(3)$
74	12	(16.23)	(16.24)
74	14	(16.22)	(16.24)
76	7	Fur	For
76	13	[ ]	[4]
77	21	logar thm	logarithm
81	6	$J_{a,b}(x)$	$P_{a,b}(x)$
83	12	Intelligenar	Intelligencer