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BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER DIFFERENTIAL EQUATIONS

By

RAVI P. AGARWAL

MATSCIENCE

THE INSTITUTE OF MATHEMATICAL SCIENCES

MADRAS-600 020 (INDIA)

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Preface

This report deals with boundary value problems for ordinary differential equations of higher order. Most of the results discussed here are new which we have obtained recently and are in the process of publication in scientific Journals. The systematic arrangement of these results was made when I visited Mathematics and Statistics Department, Marathwada University, Aurangabad for six weeks (February--March 1979) under U.G.C. Visiting Professorship Programme and delivered about fifteen lectures. I am thankful to Professor B.G. Pachpatte and Professor D.Y.Kasture for several discussions during this period. Some results were presented at the Mathematics Department, Indian Institute of Technology, Madras, when I was invited to give a seminar talk. Of course these results I have discussed several times in several different forms here in MATSCIENCE. It is really a pleasure to thank Professor Alladi Ramakrishnan, Director, Matscience, for his constant encouragement and help throughout the preparation of this report.

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R.P. Agarwal
MATSCIENCE, The Institute of Mathematical Sciences,
Madras-600 020. (India)

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1. Introduction:

Boundary value problems(BVPs) for ordinary linear or nonlinear differential equations occur in many branches of applied mathematics, theoretical physics, and engineering, the most significant among them being the boundary layer theory, the study of stellar interiors and control and optimization theory.

For a given differential equation of order $n (\geq 2)$ or a system of differential equations, when the conditions are prescribed at two points, we say two point problem. If the conditions are prescribed at more than two points we say multipoint problem. For a given dynamical system with n degrees of freedom, there may be available exactly n states observed at n different times. A mathematical description of such a system results in an n-point BVP. The discretization of certain BVPs for partial differential equations over irregular domains with the method of lines also forms MPBVP.

Example 1.1. In the problem of the motion of a particle of mass m under the action of a given force $\vec{F}(t, \vec{r}, \vec{r}')$ it is frequently necessary to find the law of motion if at the initial time $t = t_0$ the particle was located in a position characterized by the radius vector $\vec{r}_0$ and at time t_1 it has to reach point $\vec{r} = \vec{r}_1$.

The problem reduces to integrating the differential equation of motion

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}(t, \vec{r}, \vec{r}\,')$$

with the boundary conditions $\vec{r}(t_0) = \vec{r}_0, \vec{r}(t_1) = \vec{r}_1$.

Example 1.2: A natural and most prolific source of nonlinear differential equations with two point boundary conditions is the calculus of variations. Consider the problem of finding the extrema of the functional

$$\int_a^b F(t, x(t), x'(t))\, dt$$

under the conditions $x(a) = \alpha$, $x(b) = \beta$. Suitable hypothesis on F lead to a second order BVP (Euler equation $F_x - \frac{d}{dt} F_{x'} = 0$ with $x(a) = \alpha$, $x(b) = \beta$).

Example 1.3 [1], [2] The transverse displacement of an elastically imbedded rail to a distributed transverse load is described by the linear fourth order equation

$$(EI(\xi)\, u'')'' + k(\xi)\, u = q(\xi)$$

where $EI(\xi)$ is the flexural rigidity, $k(\xi)$ the elastic resistance of the supporting material and $q(\xi)$ the load density. For a freely supported rail the boundary conditions are

$u''(-L) = u'''(-L) = u''(L) = u'''(L) = 0$ and

correspond to vanishing moments and shear forces at the rail ends. Here we shall assume that the rail is hinged in a complicated manner at its endpoints and that the moments and shear forces have to be determined at the rail ends from the measured displacements at four different points along the rail.

When EJ, g and K are given by

$$EJ(\xi) = EJ_0\left(2-(\xi/L)^2\right), \quad g(\xi) = g_0\left(2-(\xi/L)^2\right)$$

the equation for the elastic rail can be converted into the nondimensional form

$$\left[(2-t^2)v''\right]'' + kv = 2-t^2$$

where $t = \xi/L$, $V = \left(EJ_0/g_0L^4\right)u$, $k = \left(L^4/EJ_0\right)K$

$= 40$ (we assume).

By defining z through the relation

$$z = (2-t^2)v''$$

the above equation can further be reduced to

$$z'' + 40v - 2 + t^2 = 0, \quad (2-t^2)v'' - z = 0.$$

We shall assume that the following (dimensionless) displacements were observed

$$t_1 = 0.2, \quad v(t_1) = \alpha_1 = 0.0448156$$

$$t_2 = 0.4, \quad v(t_2) = \alpha_2 = 0.0433224$$

$$t_3 = 0.6, \quad v(t_3) = \alpha_3 = 0.0410152$$

$$t_4 = 0.8, \quad v(t_4) = \alpha_4 = 0.0381534.$$

If we let $x = (y \;\; y' \;\; z \;\; z')^T$ then the above system becomes

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & f(t) & 0 \\ 0 & 0 & 0 & 1 \\ -40 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2-t^2 \end{bmatrix} \tag{1.1}$$

$$(f(t) = 1/(2-t^2))$$

and the boundary conditions can also be written as

$$x_1(t_i) = \alpha_i \qquad (i = 1, 2, 3, 4) \tag{1.2}$$

where t_i and α_i are defined earlier.

Example 1.4 [3]. The line method for partial differential equations lies midway between analytical and grid methods. The basis of the method is substitution of finite differences for derivatives with respect to one independent variable, and retention of the derivatives with respect to the remaining variables. This approach replaces a given differential equation by a system of differential equations with a smaller number of independent variables.

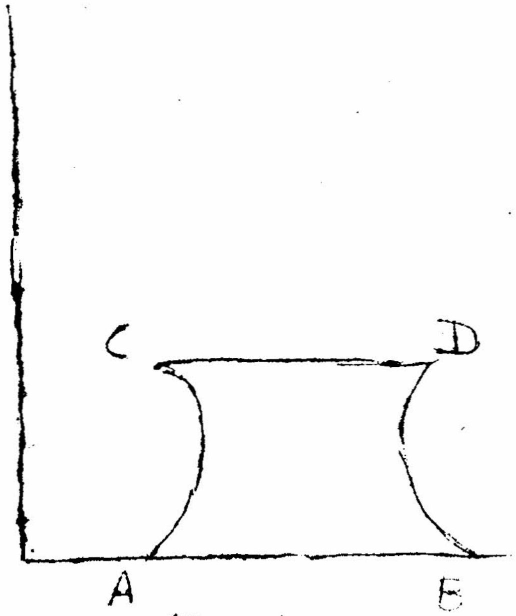
Assume that it is required to integrate the equation

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2}$$

$$+ D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y)$$

in the region Ω (Fig.1).
For definiteness, we assume that the equation is elliptic. Boundary conditions on the boundary S(ACDB) are given.

We draw lines parallel to the x axis, assuming that the distance between two adjacent lines is constant and equal to h. Assume that the region Ω is intersected by the lines $y = y_0 + kh = y_k$ $(k = 0, 1, \cdots, n)$. We set $y = y_k$ in the equation and substitude difference ratios for the derivatives with respect to y. For examples, we can set



(Fig.1)

$$\left.\frac{\partial u}{\partial y}\right|_{y=y_k} = \frac{1}{h}\left[u_{k+1}(x) - u_k(x)\right]$$

where $u_k(x) = u(x, y_k)$. Similarly,

$$\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{y=y_k} = \frac{1}{h}\left[u'_{k+1}(x) - u'_k(x)\right]$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=y_k} = \frac{1}{h^2}\left[u_{k+1}(x) - 2u_k(x) + u_{k-1}(x)\right].$$

We substitute this into the equation, in which we have already set $y = y_k$, and thus obtain a system of n ordinary linear equations in n+2 unknown functions

$$u_0(x),\ u_1(x),\ \dots,\ u_n(x),\ u_{n+1}(x)$$

In the region Ω, the missing two equations can be obtained from the boundary conditions on the line segments AB and CD of the boundary. The boundary conditions for the unknown functions $u_k(x)$ can easily be obtained from the boundary conditions for the function $u(x, y)$ on AC and BD.

Thus we obtain a system of second order equations with multi-point boundary conditions.

Example 1.5. [4] . We consider similarity solution of the unsteady flow of gas through a semi-infinite porous medium, initially filled with gas at a uniform pressure P_0. At time $t = 0$ the pressure at the outflow face is suddenly reduced from P_0 to P_1 and thereafter maintained at this lower pressure. In terms of a dimension free quantity ω, defined by

$$\omega(z) = \alpha^{-1}\left(1 - \frac{P^2(z)}{P_0^2}\right), \quad \alpha = 1 - \frac{P_1^2}{P_0^2}$$

the problem takes the form

$$\omega''(z) + \frac{2z}{\sqrt{1 - \alpha\,\omega(z)}}\,\omega'(z) = 0 \qquad (1.3)$$

$$\omega(0) = 1, \quad \omega(+\infty) = 0 \qquad (1.4)$$

(see the original paper [4] for the details of the physical problem and the reduction of the equation $\nabla^2(P^2) = A^2 \frac{\partial P}{\partial t}$ to an ordinary differential equation). This is an example of infinite interval.

Example 1.6. In the case of BVP's a small change in the boundary conditions can lead to significant changes in the behaviour of the solution.

Let us consider the initial value problem (IVP)

$$x''' + x = 0, \quad x(0) = c_1, \quad x'(0) = c_2, \quad x''(0) = c_3.$$

It has the unique solution

$$x(t) = \frac{1}{3}(c_1 - c_2 + c_3)\, e^{-t} + \frac{1}{3}(2c_1 + c_2 - c_3) \times$$
$$e^{t/2} \cos \frac{\sqrt{3}}{2} t + \frac{1}{\sqrt{3}}(c_2 + c_3)\, e^{t/2} \sin \frac{\sqrt{3}}{2} t$$

for any set of values c_1, c_2, c_3.

However, the BVP

$$x''' + x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x(b_1) = \epsilon \quad (\epsilon \neq 0 \text{ and real})$$

where b_1 is the first positive root of the equation

$$2 \sin\left(\frac{\sqrt{3}}{2} b - \frac{\pi}{6}\right) + e^{-3/2\, b} = 0$$

has no solution; the problem

$$x''' + x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x(b) = \epsilon, \quad 0 < b < b_1$$

has the unique solution

$$x(t) = \frac{\epsilon\, e^{\frac{1}{2}(t-b)}\left[e^{-\frac{3}{2}t} + 2\sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)\right]}{\left[2\sin\left(\frac{\sqrt{3}}{2}b - \frac{\pi}{6}\right) + e^{-\frac{3}{2}b}\right]},$$

while the problem

$$x''' + x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x(b_1) = 0$$

has an infinite number of solutions

$$x(t) = k\left[e^{-t} + 2e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)\right]$$

where k may have any value.

Example 1.7. [1] The BVP

$$x'' = \frac{3}{2}x^2, \quad x(0) = 4, \quad x(1) = 1 \tag{1.5}$$

possesses two solutions, one of which, namely $x(t) = 4/(1+t)^2$ is expressible in elementary terms while the other involves elliptic functions. This clearly stays in the interval $[1,4]$ whereas the latter decreases from 4 to below -10 and then increases to 1.

It is easy to see that BVP (1.5) is equivalent to the following problem

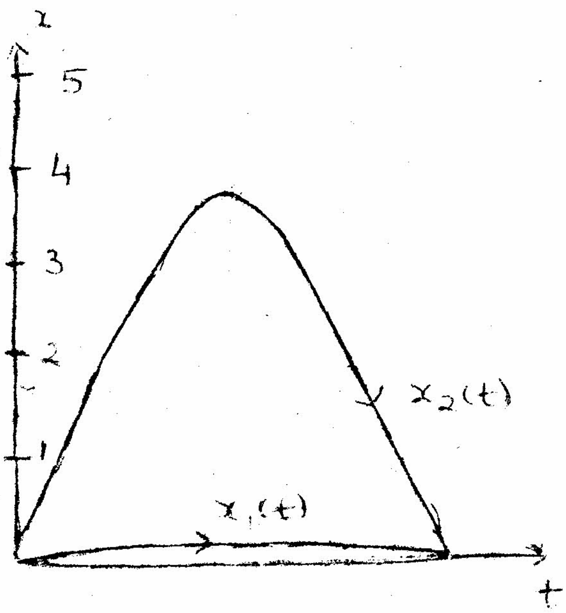
$$\begin{aligned} w''(t) &= \frac{3}{2}\left(w(t) + 4 - 3t\right)^2 \\ w(0) &= w(1) = 0 \end{aligned} \tag{1.6}$$

where the first solution is now $w(t) = \dfrac{4}{(1+t)^2} - (4-3t)$.

<u>Example 1.8</u> [5] The BVP

$$x'' + \lambda e^{x} = 0, \quad x(0) = 0 = x(1) \tag{1.7}$$

for $\lambda = 1$ has exactly two solutions (see Fig.2). This problem arises in applications involving the diffusion of heat generated by positive temperature-dependent sources. For instance, it arises in the analysis of Joule losses in electrically conducting solids with λ representing the square of the constant current and e^{x} the temperature-dependent resistance, or in frictional heating with representing the square of the constant shear stress and the temperature-dependent fludity.



(Fig.2)

<u>Example 1.9</u>. [5] The problem for the stationary temperature distribution in a bar whose ends $t = \pm 1$ are kept at the temperature $x = 1$ and which transfers heat to the environment at the temperature $x = 0$ proportional to $x + \frac{1}{4} x^2$ is

$$x'' - \left(x(t) + \frac{1}{4} x^2(t) \right) = 0$$

$$x(-1) = 1, \quad x(1) = 1 . \tag{1.8}$$

where $f(t, x, x')$ is continuous and satisfies a uniform Lipschitz condition

$$|f(t,x,x') - f(t,\bar{x},\bar{x}')| \leq L_0 |x - \bar{x}| + L_1 |x' - \bar{x}'| \qquad (2.3)$$

on $[a_1, a_2] \times R^2$, has a long history, going back to Picard [6] 1893 (it appears in the literature that before him the main attack was to construct the solution of only those problems for which the existence and uniqueness was ensured). He showed that if $(a_2 - a_1)$ is sufficiently small, then the sequence $\{x_n(t)\}$ of functions generated on using iteration procedure

$$x''_{n+1}(t) + f(t, x_n(t), x'_n(t)) = 0$$

$$x_{n+1}(a_1) = A, \quad x_{n+1}(a_2) = B$$

$$n = 0, 1, \ldots$$

with $x_0(t)$ known, converges to the solution of the BVP (2.1), (2.2). In this way, he obtained existence and uniqueness over all intervals $[a_1, a_2]$ of length less than h, where

$$\frac{1}{2} L_0 h^2 + L_1 h < 1.$$

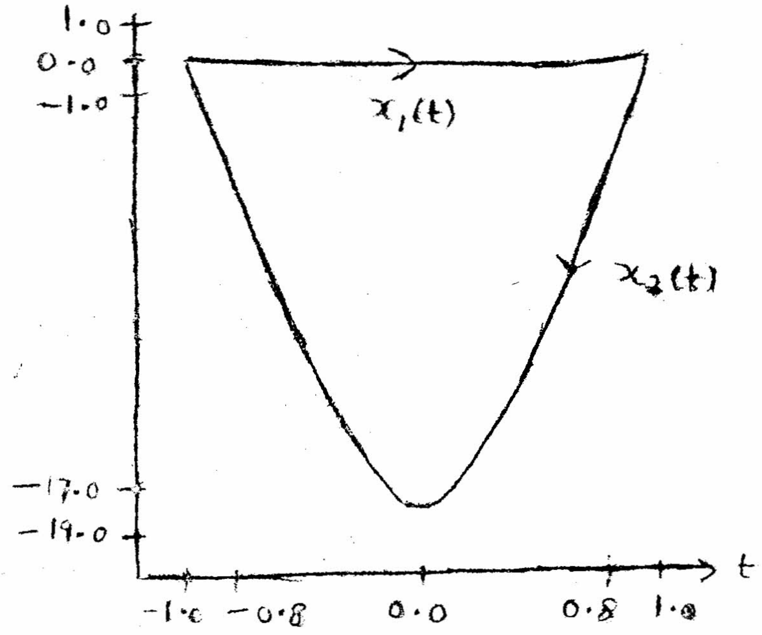
By sharpening the estimates employed in his iteration procedure he later [7] obtained the inequality

$$\frac{1}{8} L_0 h^2 + \frac{1}{2} L_1 h < 1,$$

The BVP (1.8) has two solutions (Fig.3) $x_1(t)$ lies in $[0, 1]$ whereas $x_2(t)$ drops below upto -17.

The problem (1.8) is equivalent to the following problem

$$w''(t) - \left(\frac{5}{4} + \frac{3}{2}\, w(t) + \frac{1}{4}\, w^2(t)\right) = 0 \qquad (1.9)$$
$$w(-1) = w(1) = 0.$$



(Fig.3)

2. Second Order Equations.

The BVP

$$x''(t) + f(t, x(t), x'(t)) = 0 \qquad (2.1)$$

$$\begin{cases} x(a_1) = A \\ x(a_2) = B, \quad a_1 < a_2 \end{cases} \qquad (2.2)$$

which was further improved by Lettenmeyer [8] to

$$\frac{1}{\pi^2} L_0 h^2 + \frac{4}{\pi^2} L_1 h < 1 .$$

In the special case when $f(t, x, x')$ is linear in x and x' and satisfies $f(t,0,0) = 0$, Opial [9], obtained the condition

$$\frac{1}{\pi^2} L_0 h^2 + \frac{2}{\pi^2} L_1 h < 1$$

and showed that it is best possible in the sense that the coefficients $\frac{1}{\pi^2}$ and $\frac{2}{\pi^2}$ cannot be replaced by smaller ones. But the claim is not true, if $L_1 \neq 0$ see [10].

For the equation (2.1), Bailey et all [5] have also treated the boundary conditions

$$x(a_1) = A, \quad x'(a_2) = B \tag{2.4}$$

or

$$x'(a_1) = A, \quad x(a_2) = B \tag{2.5}$$

and obtained the best possible result for each of the BVP (2.1), (2.2), (2.1), (2.4), (2.1), (2.5).

<u>Theorem 2.1</u> [5]. Assume that $f \in C[[a_1, a_2] \times R^2, R]$ and satisfies the Lipschitz condition. Let $u(t)$ be any solution of $u'' + L_1 u' + L_0 u = 0$ which vanishes at $t = a_1$ and let $\alpha(L_1, L_0)$ be the first unique number such that $u'(t) = 0$ for $t = a_1 + \alpha(L_1, L_0)$. If (1) $a_2 - a_1 < \alpha(L_1, L_0)$ then problem (2.1), (2.4) or (2.1), (2.5) (2) $a_2 - a_1 < 2\alpha(L_1, L_0)$ then problem (2.1), (2.2) has one and only one solution. This result is best possible.

3. Green's Functions

Let there be given the equation

$$L[x] = x^{(n)} + p_1(t)\, x^{(n-1)} + \cdots + p_n(t)\, x = f(t) \quad (3.1)$$

where $p_i(t)$ $(i = 1, 2, \ldots, n)$, $f(t)$ are assumed continous on a segment $[a,b]$.

For equation (3.1), we shall consider boundary conditions of the form

$$l_i[x] = \sum_{k=0}^{n-1} C_{ik}\, x^{(k)}(a_i) = A_i \qquad i = 1, 2, \ldots, n \quad (3.2)$$

$$a \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq b.$$

The coincidence of several a_i means that at a single point several functions are given, which are assumed to be linearly independent.

The existence and uniqueness of solutions of (3.1), (3.2) is equivalent to the corresponding homogeneous problem

$$L[x] = 0 \tag{3.3}$$

$$l_i[x] = 0 \qquad i = 1, 2, \ldots, n \tag{3.4}$$

has no nontrivial solution, i.e. if $y_1(t), y_2(t), \ldots, y_n(t)$ is any fundamental system of solutions of (3.3), then since

$$l_i\left[\sum_{j=1}^{n} \alpha_j y_j\right] = \sum_{j=1}^{n} \alpha_j \, l_i[y_j]$$

we should have

$$\begin{vmatrix} l_1[y_1] & l_1[y_2] & \cdots & l_1[y_n] \\ l_2[y_1] & l_2[y_2] & \cdots & l_2[y_n] \\ \cdots & \cdots & \cdots & \cdots \\ l_n[y_1] & l_n[y_2] & \cdots & l_n[y_n] \end{vmatrix} \neq 0. \tag{3.5}$$

In what follows we shall assume that condition (3.5) is satisfied. We shall denote the square $a \leq t, s \leq b$ by k; the same square with straight lines of the form $s = a_i$ rejected from it we shall denote by k_0; k_0 with rejected diagonal $t = s$ by k_1.

Let $x_1(t),\ x_2(t),\ \ldots\ldots,\ x_n(t)$ be a fundamental system of solutions of equation (3.3) satisfying the conditions

$$l_i[x_j] = \delta_{ij}$$
$$i, j = 1, 2, \ldots, n.$$

The existence of such a system is assured by condition (3.5).

We shall denote by $D_i(t)$ $(i = 1, 2, \ldots, n)$ the algebraic complement of the element $x_i^{(n-1)}(t)$ in the Wronskian

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) & & x_n(t) \\ x_1'(t) & x_2'(t) & & x_n'(t) \\ & & & \\ & & & \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & & x_n^{(n-1)}(t) \end{vmatrix}.$$

For convenience we shall write

$$a = a_0,\ b = a_{n+1},\ x_0(t) \equiv x_{n+1}(t) \equiv D_0(t) \equiv D_{n+1}(t) \equiv 0$$

We shall denote the Green's function of the problem (3.3), (3.4) as the function of two variables G(t,s) defined in k and satisfying the following conditions:

1. $G(t,s),\ G_t'(t,s),\ \ldots\ldots,\ G_{t^{n-2}}^{(n-2)}(t,s)$ are continuous with respect to the set of variables in K_0.

2. $G^{(n-1)}_{t^{n-1}}(t,s)$ is continuous with respect to the set of variables in K_1 and on the diagonal $t=s$ undergoes a discontinuity equal to unity:

$$G^{(n-1)}_{t^{n-1}}(t+0,t) - G^{(n-1)}_{t^{n-1}}(t-0,t) = 1 .$$

$$(t \neq a_i, \quad i = 1, 2, \ldots, n).$$

3. $G(t,s)$ as a function of t satisfies in K_1 conditions (3.3), (3.4).

In the usual manner it is established that the solution of the problem (3.1), (3.2) is given by the formula

$$x(t) = \int_a^b G(t,s) f(s)\, ds + \sum_{i=1}^{n} A_i\, x_i(t) . \tag{3.6}$$

The operator L for boundary conditions (3.2) thus possesses an entirely continuous inverse. In fact, any solution of (3.1) can be written as

$$x(t) = \sum_{i=1}^{n} \alpha_i\, x_i(t) + \int_a^t \frac{1}{W(s)} \sum_{i=0}^{n+1} D_i(s)\, x_i(t) f(s)\, ds .$$

From (3.2), we find

$$\alpha_i = A_i - \int_a^{a_i} \frac{1}{W(s)} D_i(s) f(s)\, ds .$$

Thus, it follows that

$$x(t) = \sum_{i=1}^{n} A_i\, x_i(t) - \sum_{i=0}^{n+1} \int_a^{a_i} \frac{1}{W(s)}\, D_i(s)\, x_i(t)\, f(s)\, ds$$

$$+ \int_a^t \frac{1}{W(s)} \sum_{i=0}^{n+1} D_i(s)\, x_i(t)\, f(s)\, ds$$

$$= \sum_{i=1}^{n} A_i\, x_i(t) - \sum_{k=0}^{n} \int_{a_k}^{a_{k+1}} \frac{1}{W(s)} \sum_{i=k+1}^{n+1} D_i(s)\, x_i(t)\, f(s)\, ds$$

$$+ \int_a^t \frac{1}{W(s)} \sum_{i=0}^{n+1} D_i(s)\, x_i(t)\, f(s)\, ds$$

and this is same as (3.6), where

$$G(t,s) = \begin{cases} -\dfrac{1}{W(s)} \displaystyle\sum_{i=k+1}^{n+1} D_i(s)\, x_i(t) & \text{for } t \leq s \\[2ex] \dfrac{1}{W(s)} \displaystyle\sum_{i=0}^{k} D_i(s)\, x_i(t) & \text{for } t > s \end{cases} \qquad (3.7)$$

$$a_k < s < a_{k+1}, \quad k = 0, 1, \ldots, n.$$

In order that G(t,s) be defined in the entire square k it is necessary to complete its definition on the straight lines $s = a_i$, $i = 0, 1, \ldots, n+1$. It will be convenient to complete the definition of G(t,a) by its continuity with respect to s from the right; similarly for G(t,b) by its continuity with respect to s from the left. As regards lines of the form $s = a_i$, where a_i is any interior point G(t,s) undergoes a discontinuity with respect to s on these lines, i.e. the identity $G(t, a_i - 0) = G(t, a_i + 0)$

does not hold in the general case. The method of completing the definition of G(t,s) on these straight lines can be arbitrarily chosen; for definiteness let it be the discontinuity with respect to s from the right.

For the problem: equation (3.3), with the boundary conditions

$$x(a_i) = x'(a_i) = \cdots = x^{(k_i)}(a_i) = 0, \quad 1 \leq i \leq r$$

$$a \leq a_1 < a_2 < \cdots < a_r \leq b, \quad 0 \leq k_i, \quad n - r = \sum_{i=1}^{r} k_i \qquad (3.8)$$

the following result is known [11] - [13] :

<u>Lemma 3.1</u>: Let G(t,s) be the Green's function of the problem (3.2), (3.8) and

$$P(t) = \prod_{i=1}^{r} (t - a_i)^{k_i+1} \qquad (3.9)$$

then, $G(t,s)/P(t) > 0$ for $a_1 < s < a_r, \; a_1 \leq t \leq a_r$.

For the Green's function g(t,s) of the problem

$$x^{(n)} = 0 \qquad (3.10)$$

conditions (3.8)

Beesack [14] derived the inequality

$$|g(t,s)| \leq \frac{1}{(n-1)!\,(a_r - a_1)} \, |P(t)| . \qquad (3.11)$$

In [15] Nehari gave a short proof of the same when $r = n$.

Lemma 3.2 [16] Let $g(t,s)$ be the Green's function of the problem (3.10), then

$$\int_{a_1}^{a_r} |g(t,s)|\, ds = \frac{1}{n!} |P(t)| . \tag{3.12}$$

Proof: The proof of the identity (3.12) follows from lemma 2.1 and the observation

$$\int_{a_1}^{a_r} g(t,s)\, ds = \frac{1}{n!} P(t) \tag{3.13}$$

where the right hand side of (3.13) is the unique solution of $x^{(n)}(t) = 1$ satisfying the boundary conditions (3.8).

Das and Vatsala in [17] have proved a similar result to that of lemma 3.2 for a particular case $r = n$, using complicated algebra.

Lemma 3.3 [18] The Green's function of the BVP

$$-x^{(n)}(t) = 0 \tag{3.14}$$

$$\begin{cases} x^{(i)}(a_1) = 0 , & i = 0, 1, \dots, n-2 \\ x^{(p)}(a_2) = 0 , & (0 \le p \le n-1) \end{cases} \tag{3.15}$$

and all its derivatives with respect to t upto order p are non-negative.

<u>Proof</u>: It can easily be verified that the Green's function of the problem (3.14) (3.15) is

$$k_1(t,s) = \frac{1}{(n-1)!} \begin{cases} (t-a_1)^{n-1}\left(\dfrac{a_r - s}{a_r - a_1}\right)^{n-k-1} - (t-s)^{n-1} & \text{for } a_1 \leq s \leq t \leq a_r \\[2ex] (t-a_1)^{n-1}\left(\dfrac{a_r - s}{a_r - a_1}\right)^{n-p-1} & \text{for } a_1 \leq t \leq s \leq a_r \end{cases} \tag{3.16}$$

Hence $\dfrac{\partial^k k_1(t,s)}{\partial t^k} \geq 0$ for $k \leq p$, provided $(t-a_1)^{n-k-1} \times \left(\dfrac{a_r - s}{a_r - a_1}\right)^{n-p-1} \geq (t-s)^{n-k-1}$ when $a_1 \leq s \leq t \leq a_r$. Since it is true if $t = s$ we consider only $a_1 \leq s < t \leq a_r$.

Because

$$\frac{t-a_1}{t-s} \geq 1, \qquad \frac{a_r - a_1}{a_r - s} \geq 1$$

and

$$\frac{t-a_1}{t-s} \geq \frac{a_r - a_1}{a_r - s}$$

we have

$$\left(\frac{t-a_1}{t-s}\right)^{n-k-1} \geq \left(\frac{a_r - a_1}{a_r - s}\right)^{n-p-1}$$

and hence the result.

Lemma 3.4 [18] . The Green's function of the BVP

$$
\begin{aligned}
&-x^{(n)}(t)=0\\
&x^{(p)}(a_1)=0,\ (0\le p\le n-1)\\
&x^{(i)}(a_2)=0,\ i=0,1,\dots,n-2
\end{aligned}
\tag{3.17}
$$

and its kth derivative $(0\le k\le p)$ with respect to t is nonnegative if $n+k$ is even and nonpositive if $n+k$ is odd.

Proof. The Green's function of the BVP (3.14) (3.17), is

$$
h_2(t,s)=\frac{(-1)^n}{(n-1)!}
\begin{cases}
(a_2-t)^{n-1}\left(\dfrac{s-a_1}{a_2-a_1}\right)^{n-p-1} & \text{for } a_1\le s\le t\le a_2\\[2ex]
(a_2-t)^{n-1}\left(\dfrac{s-a_1}{a_2-a_1}\right)^{n-p-1}-(s-t)^{n-1} & \text{for } a_1\le t\le s\le a_2.
\end{cases}
\tag{3.18}
$$

The proof is same as in Lemma 3.3.

4. Generalized Boundary Conditions

Here, we shall consider the differential equation (2.1) together with the boundary conditions

$$
\begin{aligned}
\alpha_0\, x(a_1)-\alpha_1\, x'(a_1)&=r_1\\
\beta_0\, x(a_2)+\beta_1\, x'(a_2)&=r_2
\end{aligned}
\tag{4.1}
$$

where f is a real valued continuous function on $[a_1, a_2] \times R^2$, $\alpha_0, \alpha_1, \beta_0, \beta_1, r_1, r_2$ are real constants, and $\gamma = \alpha_0\beta_0(a_2 - a_1) + \alpha_0\beta_1 + \beta_0\alpha_1 \neq 0$. We shall denote

$$K_1 = \frac{1}{8|\gamma|} Q(a_2 - a_1)\Big\{|\alpha_0\beta_0|(a_2 - a_1)^2 + 8|\alpha_1\beta_1| + 4\big[|\alpha_0\beta_1| + |\beta_0\alpha_1|\big](a_2 - a_1)\Big\}$$

$$K_2 = \frac{1}{2|\gamma|} Q(a_2 - a_1)\Big\{|\alpha_0\beta_0|(a_2 - a_1) + 2\max\big(|\alpha_1\beta_0|, |\beta_1\alpha_0|\big)\Big\}$$

and

$$\gamma^* = |\alpha_0\beta_0|(a_2 - a_1) + |\alpha_0\beta_1| + |\beta_0\alpha_1|.$$

Theorem 4.1 [19] Let $M > 0$, $N > 0$ be given real numbers and Q be the maximum of $|f(t, x, x')|$ on the compact set

$$\{(t, x, x') : a_1 \leq t \leq a_2,\ |x| \leq 2M,\ |x'| \leq 2N\}$$

Then, if

$$K_1 \leq M \tag{4.2}$$

and

$$K_2 \leq N \tag{4.3}$$

the BVP (2.1), (4.1) has a solution of class $C^{(2)}$ on $[a_1, a_2]$ provided that

$$\frac{1}{|\gamma|}\,|r_1 \beta_0 (a_2 - a_1) + r_1 \beta_1 + r_2 \gamma_1| \le M$$

$$\frac{1}{|\gamma|}\,|r_2 \alpha_0 (a_2 - a_1) + r_2 \alpha_1 + r_1 \beta_1| \le M$$

$$\frac{1}{|\gamma|}\,|r_2 \alpha_0 - r_1 \beta_0| \le N.$$

Proof: The set

$$B[a_1, a_2] = \{x(t) \in C^{(1)}[a_1, a_2],\ \|x\| \le 2M,\ \|x'\| \le 2N\}$$

where $\|\cdot\| = \max_{a_1 \le t \le a_2} |\cdot(t)|$, is a closed convex subset of the Banach space $C^{(1)}[a_1, a_2]$. The mapping $T : C^{(1)}[a_1, a_2] \to C^{(2)}[a_1, a_2]$ defined by

$$(Tx)(t) = -\int_{a_1}^{a_2} G(t,s)\, f(s, x(s), x'(s))\, ds + \ell(t)$$

is completely continuous, where G(t,s) is the Green's function for the BVP

$$x''(t) = 0, \quad \alpha_0\, x(a_1) - \alpha_1\, x'(a_1) = 0$$

$$\beta_0\, x(a_2) + \beta_1\, x'(a_2) = 0;$$

$$G(t,s) = \begin{cases} \frac{1}{\gamma}(\beta_0 t - \beta_0 a_2 - \beta_1)(\alpha_0 s - \alpha_0 a_1 + \alpha_1) & s \le t \\ \frac{1}{\gamma}(\beta_0 a_2 - \beta_0 s + \beta_1)(\alpha_0 a_1 - \alpha_0 t - \alpha_1) & t \le s \end{cases}$$

and $\ell(t)$ is defined as

$$\ell(t) = \frac{1}{\gamma}\left[r_1 \beta_0 (a_2 - t) + r_1 \beta_1 + \alpha_1 r_2 + r_2 \alpha_0 (t - a_1)\right].$$

For $x \in B[a_1, a_2]$, we have

$$\begin{aligned}|(Tx)(t)| \leq M + Q(a_2 - a_1)\Big\{ & |\alpha_0 \beta_0| (a_2 - a_1)^2 A B \\ & + 2|\alpha_1 \beta_1| + \left[2(|\alpha_1 \beta_0| + |\alpha_0 \beta_1|) A B \right. \\ & \left. + |\alpha_0 \beta_1| A^2 + |\alpha_1 \beta_0| B^2\right](a_2 - a_1)\Big\} (2|\gamma|)^{-1}\end{aligned}$$

where

$$A = \frac{1}{2\gamma^*}\left[2|\alpha_0 \beta_1| + |\alpha_0 \beta_0| (a_2 - a_1)\right]$$

$$B = \frac{1}{2\gamma^*}\left[2|\beta_0 \alpha_1| + |\alpha_0 \beta_0| (a_2 - a_1)\right].$$

Since $A + B = 1$

$$|(Tx)(t)| \leq M + K_1.$$

Also, we have

$$|(Tx)'(t)| \leq N + K_2.$$

Thus, conditions (4.2) and (4.3) imply that T maps into itself. It then follows from the Schauder Fixed-Point theorem that T has a fixed point in $B[a_1, a_2]$. The fixed point is a solution of the stated BVP.

Corollary 4.2. If there are constants $h > 0$ and $k > 0$ such that

$$|f(t, x, x')| \leq h + k|x|^{\theta}, \quad 0 \leq \theta \leq 1$$

for $a_1 \leq t \leq a_2$, $|x| + |x'| < \infty$, then the BVP (2.1),(4.1) has a solution for all r_1, r_2 .

Theorem 4.3 [19] . Let $f(t, x, x')$ is continuous and satisfies (2.3) on $[a_1, a_2] \times R^2$. Then, if

$$\frac{K_1}{Q} L_0 + \frac{K_2}{Q} L_1 < 1 \tag{4.4}$$

the BVP (2.1), (4.1) has one and only one solution.

Proof: The proof consists of a standard application of contraction Mapping Principle.

Corollary 4.4. Let $f(t, x, x')$ is continuous and satisfies (2.3) on $[a_1, a_2] \times R^2$. Then if

$$\frac{1}{8} L_0 (a_2 - a_1)^2 + \frac{1}{2} L_1 (a_2 - a_1) < 1 \tag{4.5}$$

the BVP (2.1), (2.2) has one and only one solution.

Also, if

$$\frac{1}{2} L_0 (a_2 - a_1)^2 + L_1 (a_2 - a_1) < 1 \tag{4.6}$$

the BVP (2.1), (2.4) or (2.1), (2.5) has one and only one solution.

In the boundary conditions (4.1), we take $\alpha_1 = -\alpha_1$ and call the equations as $(4.1)^*$. In the $x - x'$ plane these boundary conditions represent straight lines; the first as the initial line and the second as the terminal line. We shall represent these lines as $\ell_1(a_1) = r_1$ and $\ell_2(a_2) = r_2$. Also, when there is no ambiguity the a_1 and a_2 will be omitted and we will write $\ell_1 = r_1$ etc.

Consider

$$x'' + L\, x' + k\, x = 0 \tag{4.7}$$

$$\alpha_0\, x(0) + \alpha_1\, x'(0) = 0, \text{ or } \ell_1 = 0. \tag{4.8}$$

Define $\alpha(L, k, \ell)$ and $\beta(L, k, \ell)$ as the time (t-value) of the next and of the preceding zero of $x'(t)$ for a solution of (4.7) and (4.8) if such exist, and $+\infty$ and $-\infty$, respectively otherwise. If $\ell = 0$ is the x axis, both $\alpha(L, k, \ell)$ and $\beta(L, k, \ell)$ are taken to be zero. In the $x - x'$ plane this is just time to traverse the angle between the line $\ell = 0$ and x axis. Since the equation is linear $\alpha(L, k, \ell)$ and $\beta(L, k, \ell)$ are independent of the initial position on $\ell = 0$ and since the equation has constant coefficients these quantities can be computed explicitly and are independent of the starting time $t = 0$.

<u>Theorem 4.5.</u> [20] . Let $f(t, x, x')$ be continuous satisfy

$$L_1(x_2' - x_1') \le f(t, x, x_2') - f(t, x, x_1') \le L_2(x_2' - x_1'), \quad x_2' \ge x_1' \tag{4.9}$$

$$f(t, x_2, x') - f(t, x_1, x') \le k(x_2 - x_1), \quad x_2 \ge x_1 \tag{4.10}$$

and suppose solutions of the initial value problem for (2.1) at $t = a_1$ exist on $[a_1, a_2]$ and are unique. If $\alpha_0 \geq 0$, $\alpha_1 \leq 0$, $|\alpha_0| + |\alpha_1| \neq 0$, $\beta_0 \geq 0$, $\beta_1 \geq 0$, $|\beta_0| + |\beta_1| \neq 0$ and if $a_2 - a_1 < \alpha(L_2, k, \ell_1) + \beta(L_1, k, \ell_2)$ then there exists a unique solution of the BVP (2.1), (4.1)*.

It is clear that $|\alpha_0| + |\beta_0| \neq 0$, since otherwise $\alpha(L_1, k, \ell_1) = 0$ and $\beta(L_2, k, \ell_2) = 0$ and we obtain $a_2 - a_1 < 0$.

Remark 4.6: Both the cases of Theorem 2.1 are included in Theorem 4.5. Also there is no sign condition on the constants L_1, L_2 and k thus the result is more useful in applications. From Theorem 4.5, for the equation $x'' - x = 0$ with boundary conditions (2.2) there exists a unique solution on all finite intervals $[a_1, a_2]$ whereas Theorem 2.1 requires $a_2 - a_1 < \pi$.

Theorem 4.6 [21] If $f(t, x, x')$ satisifes

(i) $f(t, x, x')$ is a continuous real valued function defined on

$$S = \{(t, x, x') \mid a_1 \leq t \leq a_2,\ |x| + |x'| < \infty\}$$

(ii) $f(t, x, x')$ is nonincreasing on S with respect to x

(iii) $|f(t, x, x_1') - f(t, x, x_2')| \leq M|x_1' - x_2'|$ on S

and if $\alpha_0, \alpha_1, \beta_0, \beta_1 \geq 0$, $\alpha_0 + \beta_0 > 0$, $\alpha_0 + \alpha_1 > 0$, $\beta_0 + \beta_1 > 0$ then the BVP (2.1), (4.1) has a unique solution for any η_1 and η_2.

For several other results for BVP (2.1), (4.1) see [19], [22] - [27], and for the particular cases like BVP (2.1), (2.2); (2.1), (2.4) or (2.5) see [28]-[35].

5. Some Interpolation Theory.

In interpolation theory, the following inequalities are well known

Theorem 5.1: [36] Let $x(t) \in C^{(n)}[a, b]$ satisfying (3.8). Then

$$|x^{(k)}(t)| \leq C_{n,k}\, \mu\, (b-a)^{n-k} \qquad k = 0, 1, \ldots, n-1 \tag{5.1}$$

where $\mu = \max\limits_{a \leq t \leq b} |x^{(n)}(t)|$ and

$$C_{n,k} = \frac{1}{(n-k)!}. \tag{5.2}$$

The proof follows from osculatory interpolation formula

$$x(t) = \frac{1}{n!} P(t)\, x^{(n)}(p)$$

where p is in (a,b) and $P(t)$ is defined in (3.9). The kth derivative of $x(t)$ has at least n-k zeros in (a,b). The constant $C_{n,k}$ in (5.1) are obviously the best possible.

If we consider only the segment $[a_1, a_k]$, which corresponds to interpolation in the exact sense of the word, then the inequality (5.1) can be improved.

Theorem 5.2: let $x(t) \in C^{(n)}[a_1, a_2]$ satisfying (3.8). Then

$$|x^{(k)}(t)| \le C^*_{n,k}\, m\,(a_2 - a_1)^{n-k} \qquad k = 0, 1, \ldots, n-1 \tag{5.3}$$

where $m = \max\limits_{a_1 \le t \le a_2} |x^{(n)}(t)|$ and

$$C^*_{n,0} = \frac{(n-1)^{n-1}}{n!\; n^n}, \quad C^*_{n,k} = \frac{k}{(n-k)!\; n} \qquad k = 1, 2, \ldots, n-1. \tag{5.4}$$

This theorem has been proved in two different ways one using a theorem due to Krein and Milman concerning extremal points and the second using a suitable integral representation of $x(t)$. Hukuhara [37] indicates that Tumura [38] proved this result, this result has also been mentioned in [39] - [41]. The constants $C^*_{n,k}$ $(k = 0, 1, \ldots, n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (t-a_1)^{n-1}(a_2 - t), \quad x_2(t) = (t-a_1)(a_2 - t)^{n-1}$$

and only for these functions, upto a constant factor. Naturally the constants $C^*_{n,k}$ are free from any nature of multiplicity at the points a_i, $1 \le i \le r$. If we assume $\alpha = \min(k_1, k_2)$, then we obtain the following

Theorem 5.3 [42]. Let $x(t) \in C^{(n)}[a_1, a_2]$ satisfying (3.8). Then

$$|x^{(k)}(t)| \le C^{**}_{n,k} \; m \, (a_2 - a_1)^{n-k} \qquad k = 0, 1, \ldots, n-1 \tag{5.5}$$

where $m = \max\limits_{a_1 \le t \le a_2} |x^{(n)}(t)|$, and

$$C^{**}_{n,k} = \frac{1}{(n-k)!} \, \frac{(n-\alpha-1)^{n-\alpha-1}}{(n-k)^{n-k}} \, (\alpha-k+1)^{\alpha-k+1} \qquad k = 0, 1, \ldots, \alpha \tag{5.6}$$

$$C^{**}_{n,\alpha+k} = \frac{k}{(n-\alpha)\,(n-\alpha-k)!} \qquad k = 1, 2, \ldots, n-\alpha-1.$$

The constants $C^{**}_{n,k}$ are smaller than $C^{*}_{n,k}$.

Proof. First, we shall prove for $k = 0, 1, \ldots, \alpha$. Since $k_1 + 1$ and $k_2 + 1$ is the multiplicity of zeros at a_1 and a_2 respectively we find that $x^{(k)}(t)$ will have at least $(n - k_1 - k_2 + k - 2)$ zeros (counting with multiplicity) in (a_1, a_2) and $(k_1 - k + 1)$ at a_1 and $(k_2 - k + 1)$ at a_2. Define, $h(t) = x^{(k)}(t)$ then we have

$$\begin{aligned} h(a_1) &= h'(a_1) = \cdots = h^{(k_1-k)}(a_1) = 0 \\ h(a_{k,i}) &= 0, \quad i = 1, 2, \ldots, n - k_1 - k_2 + k - 2 \\ h(a_2) &= h'(a_2) = \cdots = h^{(k_2-k)}(a_2) = 0 \end{aligned} \tag{5.7}$$

where $a_{k,i}$ denotes the point where $x^{(k)}(t)$ vanishes in (a_1, a_2) for two different i, $a_{k,i}$ may be same. Now, using lemma 3.2 we find

$$h(t) = \int_{a_1}^{a_2} g_k(t,s)\, h^{(n-k)}(s)\, ds$$

or

$$|h(t)| \le \max_{a_1 \le t \le a_2} |h^{(n-k)}(t)| \int_{a_1}^{a_2} |g_k(t,s)|\, ds$$

$$= m \frac{1}{(n-k)!} Q(t)$$

where

$$Q(t) = (t-a_1)^{k_1-k+1} (a_2-t)^{k_2-k+1} \prod_{i=1}^{(n-k_1-k_2+k-2)} |t - a_{k,i}|$$

$$k = 0, 1, \ldots, \alpha$$

and $g_k(t,s)$ is the Green's function for the BVP $h^{(n-k)}(t) = 0$ satisfying (5.7).

Now to prove (5.6), suppose $a_{k,j} < t < a_{k,j+1}$, then we obtain

$$Q(t) \le (t-a_1)^{k_1-k+j+1} (a_2-t)^{n-k_1-j-1}$$

$$\le \begin{cases} (t-a_1)^{n-\alpha-1} (a_2-t)^{\alpha-k+1} = \phi(t), & t-a_1 \ge a_2-t \\ (t-a_1)^{\alpha-k+1} (a_2-t)^{n-\alpha-1} = \psi(t), & t-a_1 \le a_2-t \end{cases}$$

where in obtaining $\phi(t)$, we have used $n-k_1-j-\alpha+k-2 \geq 0$ (since $j \leq n-k_1-k_2-2$ and $k_2 \geq \alpha$) and $\psi(t)$ follows from $k_1-\alpha+j \geq 0$. Now an absolute maximum of $\phi(t)$ is at

$$t = a_1 + \frac{n-\alpha-1}{n-k}(a_2-a_1)$$

and of $\psi(t)$ is at

$$t = a_1 + \frac{\alpha-k+1}{n-k}(a_2-a_1)$$

also, an absolute maximum value of $\phi(t)$ and $\psi(t)$ is same which is

$$\frac{(n-\alpha-1)^{n-\alpha-1}}{(n-k)^{n-k}}(\alpha-k+1)^{\alpha-k+1}$$

this proves (5.6).

To, show that $C^{**}_{n,k}$ are smaller than $C^{*}_{n,k}$, we note that

$$\phi(t) \leq \frac{n-k}{n}(t-a_1)^{n-k-1}(a_2-t) + \frac{k}{n}(t-a_1)^{n-k} = \phi^*(t)$$

$$\psi(t) \leq \frac{k}{n}(a_2-t)^{n-k} + \frac{n-k}{n}(t-a_1)(a_2-t)^{n-k-1} = \psi^*(t)$$

and for $k = 0$, $\phi^*(t)$ has an absolute maximum at $t = \frac{(n-1)a_2 + a_1}{n}$ and $\psi^*(t)$ has an absolute maximum at $t = \frac{(n-1)a_1 + a_2}{n}$ also in both the cases the absolute maximum value is $\frac{(n-1)^{n-1}}{n^n}(a_2 - a_1)^n$; this proves for $k = 0$. For $\alpha \geq k \geq 1$, $\phi^*(t)$ has an absolute maximum at $t = a_2$ and $\psi^*(t)$ has an absolute maximum at $t = a_1$ also in both the cases the absolute maximum value is k/n.

Now, to prove $k = \alpha+1, \alpha+2, \ldots, n-1$; we observe that $x^{(\alpha)}(t)$ has one zero at a_1 (or a_2) and at a_2 (or a_1) may be more than one zero, also at least $n - k_1 - k_2 + \alpha - 2$ in (a_1, a_2) counting with multiplicity. Now, we define $h(t) = x^{(\alpha)}(t)$ then on using theorem 5.2, we obtain

$$|h^{(k)}(t)| \leq \frac{k}{(n-\alpha)\ (n-\alpha-k)!}\ m\ (a_2 - a_1)^{n-\alpha-k}$$

$$k = 1, 2, \ldots, n-\alpha-1$$

which proves (5.6). Also, it is easy to see that $C^*_{n,\alpha+k} > C^{**}_{n,\alpha+k}$. This completes the proof of the Theorem.

Remark 5.4. The constants $C^{**}_{n,k}$, $k = 0, 1, \ldots, \alpha$ can be obtained using osculatory interpolation formula see [43].

The constant $C^{**}_{n,0}$ is the best possible, as this is exact for the functions

$$x_1(t) = (t-a_1)^{n-\alpha-1}\ (a_2 - t)^{\alpha+1}$$

$$x_2(t) = (t-a_1)^{\alpha+1}\ (a_2 - t)^{n-\alpha-1}$$

and only for these functions, upto a constant factor.

For a particular case: $n = 4$, $\lambda = 2$, $\alpha = 1$, i.e. $x(a_1) = x'(a_1) = x(a_2) = x'(a_2) = 0$ we find the following comparison between $C^*_{n,k}$ and $C^{**}_{n,k}$

	$C^*_{n,k}$	$C^{**}_{n,k}$
k = 0	9/2048	1/384
k = 1	1/24	2/81
k = 2	1/4	1/6
k = 3	3/4	2/3

Theorem 5.5 [42] . Let $x(t) \in C^{(n)}[a_1, a_2]$ satisfying (3.15) or (3.17). Then

$$|x^{(k)}(t)| \leq \alpha_{n,k}\, m\, (a_2 - a_1)^{n-k} \qquad (5.8)$$

$$k = 0, 1, \cdots, n-1$$

where $m = \max\limits_{a_1 \leq t \leq a_2} |x^{(n)}(t)|$ and

$$\alpha_{n,k} = \begin{cases} \dfrac{(n-k-1)^{n-k-1}}{(n-k)!\,(n-k)^{n-k}} & \text{if } n-1 > p = k \\ 1 & \text{if } n-1 = p = k \\ \dfrac{(p-k)}{(n-p)\,(n-k)!} & \text{if } n-1 \geq p \geq k+1 \end{cases} \qquad (5.9)$$

$$\alpha_{n,\,p+k} = \frac{k}{(n-p)\,(n-p-k)!} \quad \text{if } k = 1, 2, \cdots, n-p-1.$$

Proof. First, we shall prove for $0 \leq k \leq p$. The function $x(t)$ satisfying (3.15) can be written as

$$x(t) = -\int_{a_1}^{a_2} h_1(t,s)\, x^{(n)}(s)\, ds$$

where $h_1(t,s)$ is defined in (3.16), and hence using lemma 3.3, we find

$$|x^{(k)}(t)| \leq \max_{a_1 \leq t \leq a_2} |x^{(n)}(t)| \int_{a_1}^{a_2} \frac{\partial^k h_1(t,s)}{\partial t^k}\, ds$$

$$= m \frac{1}{(n-k-1)!} (t-a_1)^{n-k-1} \left[\frac{a_2-a_1}{n-p} - \frac{t-a_1}{n-k} \right]$$

$$= m\, \phi_k(t) \text{ (say)}.$$

Similarly, for the function $x(t)$ satisfying (3.17), we obtain on using lemma 3.4

$$|x^{(k)}(t)| \leq m \frac{1}{(n-k-1)!} (a_2-t)^{n-k-1} \left[\frac{a_2-a_1}{n-p} - \frac{a_2-t}{n-k} \right]$$

$$= m\, \psi_k(t) \text{ (say)}.$$

Now, the result follows from the observation that $\phi_k(t)$ attains an absolute maximum at $t = a_1 + \frac{(n-k-1)}{(n-p)}(a_2-a_1)$ if $k = p \leq n-1$ and at $t = a_2$, if $k+1 \leq p \leq n-1$.

Also $\psi_k(t)$ attains an absolute maximum at $t = a_2 - \frac{(n-k-1)}{(n-p)}(a_2-a_1)$ if $k = p \leq n-1$ and at $t = a_1$ if $k+1 \leq p \leq n-1$.

To prove for $k = p+1, p+2, \ldots, n-1$ we note that for the function $x(t)$ satisfying (3.15), $x^{(p)}(t)$ will have n-p-1 zeros at a_1 and 1 zero at a_2 (counting with multiplicity). Hence, if we define $h(t) = x^{(p)}(t)$ then on using Theorem 5.2, we obtain

$$\left| h^{(k)}(t) \right| \leq \frac{k}{(n-p)\ (n-p-k)!}\, m\, (a_2 - a_1)^{n-p-k}$$

which proves the result. For the function $x(t)$ satisfying (3.17 the result follows analogously. This completes the proof.

The constants $\alpha_{n,k}$ $(k = 0, 1, \ldots, n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (t-a_1)^{n-1}\left[\frac{a_2-a_1}{n-p} - \frac{t-a_1}{n}\right]$$

$$x_2(t) = (a_2-t)^{n-1}\left[\frac{a_2-a_1}{n-p} - \frac{a_2-t}{n}\right].$$

and only for these functions, up to a constant factor.

The following lemma is an easy consequence of Rolle's theorem

Lemma 5.6. Suppose $x(t) \in C^{(n-1)}[a, b]$ and has at least n zeros on $[a,b]$. Then we can find points $a_0, a_1, \ldots, a_{2n-}$ such that

$$a \leq a_0 \leq a_1 \leq \cdots \leq a_{2n-2} \leq b$$

and

$$0 = x(a_0) = x'(a_1) = \cdots = x^{(n-2)}(a_{n-2})$$
$$= x^{(n-1)}(a_{n-1}) = x^{(n-2)}(a_n)$$
$$= \cdots = x'(a_{2n-3}) = x(a_{2n-2}).$$

<u>Theorem 5.7</u>. [44] Let $x(t) \in C^{(n)}[a, b]$ and satisfy the condition

$$x(a_0) = x'(a_1) = \cdots = x^{(n-1)}(a_{n-1}) = 0 \qquad (5.10)$$

where $a_0, a_1, \ldots, a_{n-1}$ are certain points in the interval $[a,b]$. Then the estimate

$$|x(t)| \leq C_n (b-a)^n \max_{a \leq t \leq b} |x^{(n)}(t)| \qquad (5.11)$$

is valid on $[a,b]$, where the numbers $C_1, \ldots, C_n$ are defined by the expansion

$$\tan t + \sec t = 1 + \sum_{k=1}^{\infty} C_k t^k \quad \left(|t| < \frac{\pi}{2}\right).$$

This result has also been mentioned in [41], also in [45] Levin express the constants C_k in terms of the Bernoulli numbers and Euler numbers.

Theorem 5.8. [13], [41], [45], [46] Let $x(t) \in C^{(n)}[a,b]$ satisfying (5.10) where $a = a_0 \le a_1 \le \cdots \le a_{n-1} \le b$. Then

$$|x(t)| \le m \frac{(b-a)^n}{n\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!} \tag{5.12}$$

where $m = \max\limits_{a \le t \le b} |x^{(n)}(t)|$.

Proof: From the identity

$$x^{(k)}(t) \equiv \int_{a_k}^{t}\int_{a_{k+1}}^{t_1}\cdots\int_{a_{n-1}}^{t_{n-k-1}} x^{(n)}(t_{n-k})\, dt_{n-k}\, dt_{n-k-1} \cdots dt_1$$

we obtain for $t \le a_k$

$$|x^{(k)}(t)| \le \int_{t}^{b}\int_{t_1}^{b}\cdots\int_{t_{n-k-1}}^{b} |x^{(n)}(t_{n-k})|\, dt_{n-k}\, dt_{n-k-1} \cdots dt_1$$

$$\le m \frac{(b-t)^{n-k}}{(n-k)!}. \tag{5.13}$$

In particular,

$$|x(t)| \le m \frac{(b-a)^n}{n!} \quad \text{for } a \le t \le a_0 .$$

Similarly from

$$x(t) = \int_{a_0}^{t} \int_{a_1}^{t_1} \cdots \int_{a_{k-1}}^{t_{k-1}} x^{(k)}(t_k)\, dt_k\, dt_{k-1} \cdots dt_1$$

we obtain for $t \geq a_{k-1}$

$$|x(t)| \leq \int_a^t \int_a^{t_1} \cdots \int_a^{t_{k-1}} |x^{(k)}(t_k)|\, dt_k\, dt_{k-1} \cdots dt_1$$

$$\leq \int_a^t |x^{(k)}(s)| \frac{(t-s)^{k-1}}{(k-1)!}\, ds. \qquad (5.14)$$

In particular

$$|x(t)| \leq m \int_a^b \frac{(b-s)^{n-1}}{(n-1)!}\, ds$$

$$= m \frac{(b-a)^n}{n!} \quad \text{for } a_{n-1} \leq t \leq b.$$

From (5.13) and (5.14) we obtain for $a_{k-1} \leq t \leq a_k$

$$|x(t)| \leq m \int_a^t \frac{(b-s)^{n-k}}{(n-k)!} \frac{(t-s)^{k-1}}{(k-1)!}\, ds$$

$$\leq m \int_a^b \frac{(b-s)^{n-1}}{(n-k)!\,(k-1)!}\, ds$$

$$= m \frac{(b-a)^n}{n\,(n-k)!\,(k-1)!} = m \begin{bmatrix} n-1 \\ n-k \end{bmatrix} \frac{(b-a)^n}{n!}.$$

When k runs through the values $1, 2, \ldots, n$ the binomial coefficient $\begin{bmatrix} n-1 \\ n-k \end{bmatrix}$ takes its maximum value for $k = \left[\frac{n+1}{2}\right]$ and hence (5.12) follows.

The inequalities (5.11) and (5.12) are the best possible. In fact in (5.12) from the proof it is clear that the equality cannot hold unless x(t) is a polynomial of degree n.

Hereafter for the function $x(t) \in C^{(n)}[a, b]$ we shall denote the conditions

$$x(a_i) = A_{1,i}, \quad x'(a_i) = A_{2,i}, \quad \ldots, \quad x^{(k_i)}(a_i) = A_{k_i+1}, \tag{*}$$

$$a \le a_1 < a_2 < \cdots < a_r \le b, \quad 0 \le k_i$$

$$\sum_{i=1}^{r} k_i + r = n$$

and

$$\begin{aligned} x^{(i)}(a_1) &= A_i, \quad i = 0, 1, \ldots, n-2 \\ x^{(p)}(a_r) &= B_p, \quad (0 \le p \le n-1) \end{aligned} \tag{**}$$

also

$$\begin{aligned} x^{(p)}(a_1) &= A_p, \quad (0 \le p \le n-1) \\ x^{(i)}(a_n) &= B_i, \quad i = 0, 1, \ldots, n-2. \end{aligned} \tag{***}$$

The function $x(t)$ can be written as

$$x(t) = l_j(t) + \int_{a_1}^{a_r} G_j(t, s)\, x^{(n)}(s)\, ds. \tag{5.15}$$

where $l_j(t)$ is polynomial of degree (n-1) satisfying (*) for $j = 1$; (**) for $j = 2$ and (***) for $j = 3$; also $G_1(t,s) = g(t,s)$, $G_2(t,s) = -k_1(t,s)$, $G_3(t,s) = -k_2(t,s)$.

Thus, (5.15) is the polynomial interpolation formula for the function x(t) with the error term

$$E(t) = \int_{a_1}^{a_2} G_j(t,s)\, x^{(n)}(s)\, ds .$$

The function $E(t) \in C^{(n)}[a,b]$ with $E^{(n)}(t) = x^{(n)}(t)$ for $j = 1$ and $-x^{(n)}(t)$ for $j = 2$ or 3 satisfies the hypothesis of Theorem 5.3 for $j = 1$ and Theorem 5.5 for $j = 2$, or 3 for $a_1 = a$, $a_2 = b$. Hence, we find that

$$|E^{(k)}(t)| = |x^{(k)}(t) - l_j^{(k)}(t)|$$

$$\leq \left\{ \begin{array}{ll} C^{**}_{n,k} & \text{if } j = 1 \\ \alpha_{n,k} & \text{if } j = 2 \text{ or } 3 \end{array} \right\} (a_2 - a_1)^{n-k}\, m$$

$$k = 0, 1, \ldots, n-1.$$

This gives error bounds in numerical differentiation in terms of only nth derivative of $x(t)$ and not in terms of higher order derivatives.

6. <u>Existence and Uniqueness</u>

Here, we shall consider the following nth order differential equation

$$x^{(n)} = f(t, x, x', \ldots, x^{(q)}) \tag{6.1}$$

and the boundary conditions (*) or (**) or (***) ; equivalently the integral equation

$$x(t) = l_j(t) + \int_{a_1}^{a_n} G_j(t, s)\, f(s, x(s), \ldots, x^{(q)}(s))\, ds \tag{6.2}$$

where $l_j(t)$ and $G_j(t, s)$ are defined in (5.15). The function f we shall assume continuous on $[a, b] \times R^{q+1}$ throughout without mention.

<u>Theorem 6.1</u>. [42], [43] . Let $K_i > 0,\ i = 0, 1, \ldots, q$ be given real numbers and let Q be the maximum of $|f(t, u_0, u_1, \ldots, u_q)|$ on the compact set

$$\{(t, u_0, u_1, \ldots, u_q) : a \le t \le b,\ |u_i| \le 2K_i,\ i = 0, 1, \ldots, q\}.$$

Then, if

1. $\max\limits_{a \le t \le b} |l_1^{(i)}(t)| \le K_i$ and $(b-a) \le \left(\dfrac{K_i}{Q\, C_{n,i}}\right)^{1/n-i}$ (6.3)

$i = 0, 1, \ldots, q$ the BVP (6.1), (*) has a solution

2. in case $a_1 = a$; $a_2 = b$

$$\max_{a_1 \le t \le a_2} |l_1^{(i)}(t)| \le K_i \text{ and } (a_2 - a_1) \le \left(\frac{K_i}{Q\, C^{**}_{n,i}} \right)^{1/n-i} \tag{6.4}$$

$i = 0, 1, \ldots, \nu$ the BVP (6.1), (*) has a solution.

3. in case $a_1 = a$, $a_2 = b$

$$\max_{a_1 \le t \le a_2} |l_j^{(i)}(t)| \le K_i \text{ and } (a_2 - a_1) \le \left(\frac{K_i}{Q\, \alpha_{n,i}} \right)^{1/n-i} \tag{6.5}$$

$(j = 2, 3)$

$i = 0, 1, \ldots, \nu$ the BVP (6.1) (**) (or (***)) has a solution.

Proof. We shall prove 2 and for 1 and 3 it will follow analogously. The set

$$B[a_1, a_2] = \left\{ x(t) \in C^{(\nu)}[a_1, a_2] : \|x^{(j)}\| \le 2K_j \right., \\ \left. j = 0, 1, \ldots, \nu \right\}$$

where $\|x^{(j)}\| = \max_{a_1 \le t \le a_2} |x^{(j)}(t)|$ is a closed convex subset of the Banach space $C^{(\nu)}[a_1, a_2]$. The mapping $T : C^{(\nu)}[a_1, a_2] \to C^{(n)}[a_1, a_2]$ defined by

$$(Tx)(t) = l_1(t) + \int_{a_1}^{a_2} g(t, s)\, f(s, x(s), \ldots, x^{(\nu)}(s))\, ds \tag{6.6}$$

is completely continous. Also, $(Tx)(t) - l_1(t)$ satisfies conditions (3.8) and $(Tx)^{(n)}(t) - l_1^{(n)}(t) = f(t, x(t), \ldots, x^{(\nu)}(t))$ hence $\|(Tx)^{(n)} - l_1^{(n)}\| = \|(Tx)^{(n)}\| \le Q$ over $B[a_1, a_2]$.

Now, using theorem 5.3 for : $x(t) \in B[a_1, a_2]$, we find

$$|(Tx)^{(i)}(t) - l_1^{(i)}(t)| \leq Q\, C_{n,i}^{**} (a_2 - a_1)^{n-i}$$

and hence

$$|(Tx)^{(i)}(t)| \leq \max_{a_1 \leq t \leq a_2} |l_1^{(i)}(t)| + Q\, C_{n,i}^{**} (a_2 - a_1)^{n-i}$$

$$i = 0, 1, \ldots, q.$$

Thus, condition (6.4) implies that T maps $B[a_1, a_2]$ into itself It then follows from the Schauder's fixed point theorem that T has a fixed point in $B[a_1, a_2]$. The fixed point is a solution of (6.1), (*).

Corollary 6.2. Assume that the function $f(t, u_0, u_1, \ldots, u_q)$ satisfies the following condition

$$|f(t, u_0, u_1, \ldots, u_q)| \leq C_0 + \sum_{j=0}^{q} C_{j+1} |u_j|^{\alpha(j)}$$

where $0 < \alpha(j) < 1$ for $j = 0, 1, \ldots, q$. Then, each of the BVPs (6.1), (*); (6.1), (**); (6.1), (***) has a solution.

Next, we shall give another existence theorem for the BVP (6.1), (*); and for (6.1), (**), (6.1), (***) it will follow analogously.

We shall denote

$$\theta = \sum_{i=0}^{q} C_{n,i}^{**} L_i (a_2 - a_1)^{n-i} < 1. \qquad (6.7)$$

Theorem 6.3. [42]. Let $f(t, u_0, u_1, \ldots, u_q)$ satisfy the condition

$$|f(t, u_0, u_1, \ldots, u_q)| \leq L + \sum_{i=0}^{q} L_i |u_i| \tag{6.8}$$

for all $(t, u_0, u_1, \ldots, u_q) \in [a_1, a_2] \times R^{q+1}$, where L is any number, and let L_i $(i = 0, 1, \ldots, q)$ satisfy the inequality (6.7). Then, the BVP (6.1), (*) has at least one solution for any $A_{i,k}$.

Proof: Let

$$l = \max_{a_1 \leq t \leq a_2} \sum_{i=0}^{q} L_i \, |l_1^{(i)}(t)|.$$

Define M as the set of functions n times continuously differentiable on $[a_1, a_2]$ and satisfying the boundary conditions (*). If we introduce in M the metric

$$\rho(x, y) = \max_{a_1 \leq t \leq a_2} |x^{(n)}(t) - y^{(n)}(t)| \qquad (x, y \in M)$$

then, M becomes a Banach space. Define the mapping $T: M \rightarrow M$ as in (6.6). We shall show that the mapping T maps a sphere of radius $\frac{L+l}{1-\theta}$ of the space M into itself. Indeed, if $x \in M$ and $\rho(x, l_1) \leq \frac{L+l}{1-\theta}$, then

$$\rho(Tx, l_1) \leq \max_{a_1 \leq t \leq a_2} |f(t, x(t), x'(t), \cdots, x^{(q)}(t))|$$

$$\leq L + \max_{a_1 \leq t \leq a_2} \sum_{i=0}^{q} L_i |(x(t) - l_1(t))^{(i)} + l_1^{(i)}(t)|$$

$$\leq L + l + \max_{a_1 \leq t \leq a_2} \sum_{i=0}^{q} L_i |(x(t) - l_1(t))^{(i)}|$$

$$\leq L + l + \theta \max_{a_1 \leq t \leq a_2} |x^{(n)}(t)|$$

$$\leq L + l + \theta \frac{L+l}{1-\theta}$$

$$= \frac{L+l}{1-\theta}.$$

Then, it follows by Schauder's fixed point theorem, T has at least one fixed point. The problem (6.1), (*) has therefore at least one solution $x(t)$ satisfying the condition

$$|x^{(n)}(t)| \leq \frac{L+l}{1-\theta} \quad (a_1 \leq t \leq a_2).$$

Hence, from Theorem 5.3, we obtain the inequalities

$$|x^{(i)}(t) - l_1^{(i)}(t)| \leq C_{n,i}^{**} \frac{L+l}{1-\theta} (a_2 - a_1)^{n-i}$$

$$i = 0, 1, \cdots, n-1 \quad (a_1 \leq t \leq a_2).$$

The following particular cases of theorem 6.1 are of independent interest (see [47] - [49]):

1. The BVP

$$x''' = f(t, x, x', x'') \tag{6.9}$$

$$x(a_1) = A, \quad x(a_2) = B, \quad x(a_3) = C \tag{6.10}$$

or

$$x(a_1) = A, \quad x(a_2) = B, \quad x'(a_2) = C \tag{6.11}$$

or

$$x(a_1) = A, \quad x'(a_1) = B, \quad x(a_2) = C \tag{6.12}$$

has a solution, provided

$$|\ell_j(t)| \leq K_0, \quad |\ell_j'(t)| \leq K_1, \quad |\ell_j''(t)| \leq K_2, \quad j = 1, 2, 3$$

$$(a_2 - a_1) \leq \min\left\{\left(\frac{81\, K_0}{9\, Q}\right)^{1/3}, \left(\frac{6\, K_1}{Q}\right)^{1/2}, \frac{3\, K_2}{2\, Q}\right\}$$

where $\ell_j(t)$ is the second degree polynomial satisfying (6.10) for $j = 1$, (6.11) for j = 2 and (6.12) for j = 3.

2. The BVP: equation (6.9) with

$$x(a_1) = A, \quad x'(a_1) = B, \quad x'(a_2) = C \tag{6.13}$$

or

$$x'(a_1) = A, \quad x(a_2) = B, \quad x'(a_2) = C \tag{6.14}$$

has a solution, provided

$$|\ell_j(t)| \le K_0, \quad |\ell_j'(t)| \le K_1, \quad |\ell_j''(t)| \le K_2; \; j = 1, 2$$

$$(a_2 - a_1) \le \min\left\{ \left(\frac{12 K_0}{Q}\right)^{1/3}, \left(\frac{8 K_1}{Q}\right)^{1/2}, \frac{2 K_2}{Q} \right\}$$

where $\ell_j(t)$ is the second degree polynomial satisfying (6.13) for j = 1 and (6.14) for j = 2.

3. The BVP: equation (6.9) with

$$x(a_1) = A, \quad x'(a_1) = B, \quad x''(a_2) = C \tag{6.15}$$

or

$$x''(a_1) = A, \quad x(a_2) = B, \quad x'(a_2) = C \tag{6.16}$$

has a solution, provided

$$|\ell_j(t)| \le K_0, \quad |\ell_j'(t)| \le K_1, \quad |\ell_j''(t)| \le K_2; \; j = 1, 2$$

$$(a_2 - a_1) \le \min\left\{ \left(\frac{3 K_0}{Q}\right)^{1/3}, \left(\frac{2 K_1}{Q}\right)^{1/2}, \frac{K_2}{Q} \right\}$$

where $\ell_j(t)$ is the second degree polynomial satisfying (6.15) for j = 1 and (6.16) for j = 2.

<u>Definition</u>. The function $f(t, u_0, u_1, \ldots, u_q)$ is said to be of class Lipschitz, if for all $(t, u_0, u_1, \ldots, u_q)$, $(t, v_0, v_1, \ldots, v_q) \in [a_1, a_2] \times R^{q+1}$, the following is satisfied

$$|f(t, u_0, u_1, \ldots, u_q) - f(t, v_0, v_1, \ldots, v_q)| \tag{6.17}$$

$$\le \sum^{q} L_i |u_i - v_i|.$$

<u>Theorem 6.4</u>: [42] Let $f(t, u_0, u_1, \ldots, u_q)$ satisfy the Lipschitz condition (6.17). Then, .1. if $\theta < 1$, the BVP (6.1), (*) has a unique solution, for any $A_{i,k}$ .2. if

$$\alpha = \sum_{i=0}^{q} \alpha_{n,i} L_i (a_2 - a_1)^{n-i} < 1 \tag{6.18}$$

each of the BVPs (6.1), (**); (6.1), (***) has a unique solution, for any A_i and B_i.

<u>Proof</u>. We shall prove 1 and 2 follows analogously. We shall show that the mapping T defined on the metric space M (see theorem 6.3) is contracting. Indeed, we find that for $x_1, x_2 \in M$

$$\begin{aligned} \rho(Tx_1, Tx_2) &= \max_{a_1 \le t \le a_2} \left| f(t, x_1(t), x_1'(t), \ldots, x_1^{(q)}(t)) - f(t, x_2(t), x_2'(t), \ldots, x_2^{(q)}(t)) \right| \\ &\le \max_{a_1 \le t \le a_2} \sum_{i=0}^{q} L_i \left| x_1^{(i)}(t) - x_2^{(i)}(t) \right| \\ &\le \theta \max_{a_1 \le t \le a_2} \left| x_1^{(n)}(t) - x_2^{(n)}(t) \right| \\ &= \theta \rho(x_1, x_2). \end{aligned}$$

Thus, the mapping T in M, has one fixed point, and this is equivalent to the existence and uniqueness of the solution for the problem (6.1), (*).

The fact that T is in M a contraction mapping means, among other things, that under the conditions of theorem 6.4 for the existence of a unique solution of the BVP under consideration, the method of successive approximations can be applied. The rate of convergence of the solution will be not less than the rate of convergence of a geometric progression with common multiplier Θ for (6.1), (*) and α for (6.1), (**) or (***).

The following particular cases of theorem 6.4 one can find in [48] .

1. Each of the BVPs (6.9), (6.10), (6.9), (6.11); (6.9), (6.12) has a unique solution, provided

$$\frac{2}{81} L_0 (a_2-a_1)^3 + \frac{1}{6} L_1 (a_2-a_1)^2 + \frac{2}{3} L_2(a_2-a_1) < 1. \tag{6.19}$$

2. Each of the BVPs (6.9), (6.13); (6.9), (6.14) has a unique solution, provided

$$\frac{1}{12} L_0 (a_2-a_1)^3 + \frac{1}{8} L_1 (a_2-a_1)^2 + \frac{1}{2} L_2 (a_2-a_1) < 1. \tag{6.20}$$

3. Each of the BVPs (6.9), (6.15); (6.9), (6.16) has a unique solution, provided

$$\frac{1}{3} L_0 (a_2-a_1) + \frac{1}{2} L_1 (a_2-a_1)^2 + L_2 (a_2-a_1) < 1. \tag{6.21}$$

Theorem 6.5 [45]. Let $f(t, u_0, u_1, \ldots, u_q)$ satisfy the Lipschitz condition (6.17) on $[a, b] \times R^{q+1}$. Also, assume that

$$\beta = \sum_{k=n-q}^{n} \frac{1}{2^k\, k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} L_{n-k}(b-a)^k \leq 1. \tag{6.22}$$

Then, the problem (6.1), (*) has almost one solution, for any $\Lambda_{\cdot,k}$.

Proof. Suppose on the contrary that the problem (6.1), (*) has two solutions $x(t)$ and $y(t)$, then if we define $h(t) = x(t) - y(t)$ satisfies conditions (3.8). Then, by Lemma 5.6 with $a_{n-1} = c$ satisfies conditions (5.10) on a subinterval $[a, c]$ and the same conditions with the inequalities by $a_0, a_1, \ldots, a_{n-1}$ reversed on the complementary subinterval $[c, b]$. One of these two subintervals, say $[a, c]$ has length at most $\frac{(b-a)}{2}$. Moreover the interval $[a, c]$ is non-degenerate, since $h(t)$ cannot have a zero of multiplicity n. Applying theorem 5.8 to this interval we obtain

$$|h^{(n-k)}(t)| \leq \frac{\mu\,(b-a)^k}{2^k\, k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \tag{6.23}$$

where $\mu = \max_{a \leq t \leq c} |h^{(n)}(t)|$. But for some $\tau \in [a, c]$.

$$\mu = |h^{(n)}(\tau)| = |f(\tau, x(\tau), \ldots, x^{(q)}(\tau)) - f(\tau, y(\tau), \ldots, y^{(q)}(\tau))|$$
$$\leq \sum_{k=n-q}^{n} L_{n-k}\, |h^{(n-k)}(\tau)| \leq \mu\beta.$$

Evidently $\mu > 0$, since otherwise $h(t)$ would coincide on $[a,c]$ with a polynomial of degree $m < n$ and $h^{(m)}(t)$ would not vanish on $[a,c]$. Hence $\beta \geq 1$. It only remains to exclude the possibility of equality. At least one of the numbers $L_0, \ldots, L_q$ is different from zero, since otherwise $h(t)$ would be a polynomial of degree less than n and could not have n zeros. Thus if $\beta = 1$ then equality must hold in (6.23) for at least one value of k. This is possible only if h(t) coincides on $[a,c]$ with a polynomial of degree n. But we can take τ to be any point of $[a,c]$, and $|h^{(n-k)}(\tau)|$ is not constant on $[a,c]$ for any $k = n-q, n-q+1, \ldots, n$. Therefore also in this case we have $\beta > 1$.

For some more results similar to that of Theorem 6.5, see [5(

The next result is for the linear differential equation

$$x^{(n)} + p_1(t)\, x^{(n-1)} + \cdots + p_n(t)\, x = f(t) \tag{6.24}$$

with the boundary conditions (*). This corrects an error of the result given in [51], see also Hartman [46]. In (6.24), functions $p_1(t), p_2(t), \ldots, p_n(t)$ and f(t) are continuou over $[a,b]$.

Theorem 6.6 The problem (6.24), (*) has at most one solutio for any $A_{i,k}$ provided $\max(Q_1, Q_2) \leq 1$, where

$$Q_1 = \left(\exp \int_a^{(a+b)/2} |p_1(s)|\, ds\right) \sum_{k=1}^{n-1} \frac{(b-a)^k \int_a^{(a+b)/2} |p_{k+1}(s)}{2^k\, k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!}$$

$$Q_2 = \left(\exp \int_{\frac{(a+b)}{2}}^{t} |p_1(s)|\,ds\right) \sum_{k=1}^{n-1} \frac{(b-a)^k \int_{(a+b)/2}^{b} |p_{k+1}(s)|\,ds}{2^k\, k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!}$$

Proof. Define the function h(t) and the subintervals $[a, c]$, $[c, b]$ as in the proof of Theorem 6.5. The resulting differential equation in h(t) can be written as

$$\left(h^{(n-1)}(t) \exp\left(\int_a^t p_1(s)\,ds\right)\right)' = -\sum_{k=1}^{n-1} p_{k+1}(t)\, h^{(n-k-1)}(t) \exp\left(\int_a^t p_1(s)\,ds\right)$$

Let $M = \max |h^{(n-1)}(t)|$ on $[a, c]$ and let $|x^{(n-1)}(r)| = M$, $a \leq r < c$. An integration over the interval $[r, c]$ gives

$$M \exp\left(\int_a^r p_1(s)\,ds\right) < \sum_{k=1}^{n-1} \max_{a \leq t \leq c} |h^{(n-k-1)}(t)| \times$$
$$\int_r^c \left(\exp \int_a^t p_1(s)\,ds\right) |p_{k+1}(t)|\,dt.$$

By theorem 5.8 and

$$\exp\left(\int_r^t p_1(s)\,ds\right) \leq \exp\left(\int_a^c |p_1(s)|\,ds\right) \text{ for } a \leq r \leq t \leq c$$

we get

$$1 < \left(\exp \int_a^c |p_1(s)|\, ds\right) \sum_{k=1}^{n-1} \frac{(c-a)^k}{k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \int_a^c |p_{k+1}(t)| \, dt \tag{6.25}$$

Similarly,

$$1 < \left(\exp \int_c^b |p_1(s)|\, ds\right) \sum_{k=1}^{n-1} \frac{(b-c)^k}{k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \times \int_c^b |p_{k+1}(t)| \, dt. \tag{6.26}$$

Since, either $c \le \frac{a+b}{2}$ or $c \ge \frac{a+b}{2}$ that is $(c-a) \le \frac{b-a}{2}$ or $(b-c) \le b-a/2$, (6.25) and (6.26) imply that either $Q_1 > 1$ or $Q_2 > 1$.

7. Lipschitz condition over a compact region

In general, if the function $f(t, x, \ldots, x^{(\nu)})$ satisfies the Lipschitz condition was over a compact region, then the BVP under consideration may not have a unique solution. See examples 1.7-1.9. Here we shall show that the function f need not satisfy Lipschitz condition on $[a_1, a_2] \times R^{\nu+1}$ but it is sufficient if it satisfies Lipschitz condition over a proper compact set (defined in the result).

Lemma 7.1 [52] Let T map a ball $B = \{w : \|w - y_0\| \le \mu\}$ of a complete normed linear space (Banach space) S into S. If there is an $\alpha \in (0, 1)$ such that for $u, v \in B$

$$\|Tu - Tv\| \le \alpha \|u - v\| \tag{7.1}$$

and if

$$\| T y_0 - y_0 \| \leq \mu (1-\alpha). \tag{7.2}$$

Then T has a unique fixed point y in B. If T maps the ball B into itself, then the condition (7.2) can be omitted.

In the following theorem, without loss of generality we shall consider only the zero boundary conditions i.e. all the constants $A_{i,k}$, A_i, B_i appearing in the boundary conditions (*), (**), (***) are zero.

<u>Theorem 7.2</u> [42] Let the function $f(t, x, x', \ldots, x^{(q)})$ satisfy Lipschitz condition

1. on

$$D_1 = \Big\{ (t, x(t)) ;\ a_1 \leq t \leq a_2 ,\ x(t) \in C^{(q)}[a_1, a_2], \tag{7.3}$$
$$| x^{(j)}(t) | \leq N \frac{C^{**}_{n,j}}{C^{**}_{n,0}\,(a_2-a_1)^j} ,\ j = 0, 1, \ldots, q \Big\}$$

where N satisfies either

$$m (a_2 - a_1)^n C^{**}_{n,0} \leq N(1-\theta) \tag{7.4}$$

if $m = \max | f(t, 0, 0, \ldots, 0) |$ for $a_1 \leq t \leq a_2$ or, merely

$$M (a_2 - a_1)^n C^{**}_{n,0} \leq N \tag{7.5}$$

if $M = \max_{D_1} |f(t, x(t), \ldots, x^{(q)}(t))|$, θ is defined in (6.7). Then, the BVP (6.1), (3.8) has one and only one solution $x(t) \in D_1$.

2. on

$$D_2 = \Big\{ (t, x(t)) ;\ a_1 \le t \le a_2 ,\ x(t) \in C^{(q)}[a_1, a_2], \quad |x^{(j)}(t)| \le N \frac{\alpha_{n,j}}{\alpha_{n,0}\,(a_2-a_1)^j} ;\ j = 0, 1, \ldots, q \Big\} \tag{7.6}$$

where N satisfies either

$$m\,(a_2-a_1)^n \alpha_{n,0} \le N\,(1-\alpha) \tag{7.7}$$

or

$$M\,(a_2-a_1)^n \alpha_{n,0} \le N \tag{7.8}$$

where m and M are defined above except M here defined as the maximum of $|f|$ taken over D_2 and α is defined in (6.18). Then, each of the BVPs (6.1), (3.15); (6.1), (3.17) has one and only one solution $x(t) \in D_2$.

Proof. We shall prove 1 and 2 will follow analogously. Let the space S consist of q times continuously differentiable functions on $[a_1, a_2]$ with the norm

$$\|x\| = \max_{0 \le j \le q} \left\{ \frac{C^{**}_{n,0}\,(a_2-a_1)^j}{C^{**}_{n,j}} \max_{a_1 \le t \le a_2} |x^{(j)}(t)| \right\}.$$

We shall show that the mapping $T: S \to C^{(n)}[a_1, a_2]$ satisfies the conditions of Lemma 7.1, where

$$(Tx)(t) = \int_{a_1}^{a_2} g(t,s)\, f(s, x(s), \ldots, x^{(q)}(s))\, ds. \quad (7.9)$$

Let $x_0(t) \equiv 0$ and B be the ball $\{w \in S : \|w\| \le N\}$. Then, if $x_1(t), x_2(t) \in B$, we have on using theorem 5.3

$$|(Tx_1)^{(j)}(t) - (Tx_2)^{(j)}(t)| \le C^{**}_{n,j}\, (a_2 - a_1)^{n-j} \times$$

$$\max_{a_1 \le t \le a_2} |f(t, x_1(t), \ldots, x_1^{(q)}(t)) - f(t, x_2(t), \ldots, x_2^{(q)}(t))|$$

$$\le C^{**}_{n,j}\, (a_2 - a_1)^{n-j} \max_{a_1 \le t \le a_2} \sum_{i=0}^{q} L_i\, |x_1^{(i)}(t) - x_2^{(i)}(t)|$$

$$\le C^{**}_{n,j}\, (a_2 - a_1)^{n-j} \sum_{i=0}^{q} L_i \frac{C^{**}_{n,i}}{C^{**}_{n,0}\, (a_2 - a_1)^{i}} \times \|x_1 - x_2\|$$

and hence

$$\frac{C^{**}_{n,0}\, (a_2 - a_1)^{j}}{C^{**}_{n,j}} |(Tx_1)^{(j)}(t) - (Tx_2)^{(j)}(t)|$$

$$\le \sum_{i=0}^{q} L_i\, C^{**}_{n,i}\, (a_2 - a_1)^{n-i} \|x_1 - x_2\|$$

$$j = 0, 1, \ldots, q$$

from which it follows that

$$\| Tx_1 - Tx_2 \| \leq \theta \| x_1 - x_2 \| .$$

To apply Lemma 7.1, we need to show that (7.2) holds. Let (7.4) hold, then we have

$$|(Tx_0)^{(j)}(t)| \leq C^{**}_{n,j} (a_2 - a_1)^{n-j} \max_{a_1 \leq t \leq a_2} | f(t, 0, 0, \ldots,$$

$$\leq C^{**}_{n,j} (a_2 - a_1)^{n-j} m$$

or

$$\frac{C^{**}_{n,0} (a_2 - a_1)^j}{C^{**}_{n,j}} |(Tx_0)^{(j)}(t)| \leq (a_2 - a_1)^n \, m \, C^{**}_{n,0}$$

$$j = 0, 1, \ldots, q.$$

Hence, we have

$$\| Tx_0 - x_0 \| \leq N (1 - \theta) .$$

Next, let (7.5) hold, then for any $x(t) \in B$, we have

$$| x^{(j)}(t) | \leq \frac{C^{**}_{n,j}}{C^{**}_{n,0} (a_2 - a_1)^j} N$$

hence, by the hypothesis $M = \max_{D_1} |f(t, x(t), \ldots, x^{(\nu)}(t$

and it follows that

$$|(Tx)^{(j)}(t)| \leq C^{**}_{n,j} (a_2 - a_1)^{n-j} M$$

or

$$\frac{C^{**}_{n,0} (a_2 - a_1)^{j}}{C^{**}_{n,j}} |(Tx)^{(j)}(t)| \leq (a_2 - a_1)^n M \, C^{**}_{n,0}$$

$$j = 0, 1, \ldots, \nu.$$

Thus

$$\|Tx\| \leq N.$$

This completes the proof of the Theorem 7.2.

In example 1.8 $(\lambda = 1)$, we have $C^{**}_{n,0} = \frac{1}{8}$ and hence

$$D_1 = \{(t, x(t)) : 0 \leq t \leq 1, \; x(t) \in C[0,1], \; |x(t)| \leq N\}$$

and, we find $L_0 = e^N$, $m = 1$, $M = e^N$. Thus, from Theorem 7.2 if

$$\theta = \frac{1}{8} e^N < 1 \quad \text{or} \quad N < \log 8 \simeq 2.079 \tag{7.10}$$

and

$$\frac{1}{8} \leq N \left(1 - \frac{1}{8} e^N\right) \tag{7.11}$$

or

$$\frac{1}{8} e^{N} \leq N \tag{7.12}$$

there exists a unique solution. Clearly (7.10) and (7.12) are satisfied if $|x| \ll 2.079$, this is also clear from Fig.2.

In example 1.7 for the problem (1.6), $C^{**}_{n,0}$ and D_1 remains same as above, but we have

$$L_0 = 3 \max_{D_1} |w(t) + 4 - 3t| = 3(N+4).$$

Thus $\theta = \frac{3}{8}(N+4) < 1$, which is not true and hence Theorem 7.2 cannot be used.

In example 1.9 for the problem (1.9), $C^{**}_{n,0} = \frac{1}{8}$ and

$$D_1 = \{(t, x(t)) : -1 \leq t \leq 1,\ x(t) \in C[-1,1],\ |x(t)| \leq N\}$$

and, we find $a_2 - a_1 = 2$, $L_0 = \frac{3}{2} + \frac{1}{2}N$, $m = 5/4$

$M = \frac{5}{4} + \frac{3}{2}N + \frac{1}{4}N^2$.

Thus, from Theorem 7.2, if

$$\frac{1}{8}\left(\frac{3}{2} + \frac{1}{2}N\right)4 < 1 \text{ or } N < 1 \tag{7.13}$$

and

$$\frac{5}{4} \times 4 \times \frac{1}{8} \leq N\left(1 - \frac{3}{4} - \frac{1}{4}N\right) \text{ or } \frac{5}{2} \leq N(1-N) \tag{7.14}$$

or

$$\frac{1}{2}\left(\frac{5}{4} - \frac{3}{2}N + \frac{1}{4}N^2\right) \leq N$$

$$\text{or} \quad \frac{5}{2} + \frac{1}{2}N^2 \leq N \tag{7.15}$$

there exists a unique solution. But if (7.13) is satisfied (7.14) as well as (7.15) cannot be satisfied. Hence theorem 7.2 cannot be used here.

8. Weight Function Technique.

Here we shall employ weight function technique which is previously used by Collatz [53], and show that in some particular cases where the explicit form of the Green's function and some of its properties are known then one can find better results than obtained in previous results. In some cases one can find best possible results also.

Theorem 8.1 [42]. Let the function $f(t, x, \ldots, x^{(q)})$ satisfy Lipschitz condition on $[a_1, a_2] \times R^{q+1}$. Then, if

1. $$\frac{1}{W_j(t)} \int_{a_1}^{a_2} \left\{ \left| \frac{\partial^j g(t,s)}{\partial t^j} \right| \sum_{i=0}^{q} L_i W_i(s) \right\} ds \leq \theta_1 < 1 \tag{8.1}$$

$$j = 0, 1, \ldots, q$$

the BVP (6.1), (*) has a unique solution for any $A_{i,k}$.

2. $$\frac{1}{W_j(t)} \int_{a_1}^{a_2} \left\{ \left| \frac{\partial^j k_1(t,s)}{\partial t^j} \right| \sum_{i=0}^{q} L_i W_i(s) \right\} ds \leq \alpha_1 < 1 \tag{8.2}$$

$$j = 0, 1, \ldots, q$$

the BVP (6.1), (**) has a unique solution for any A_i and B_p.

3.

$$\frac{1}{W_j(t)} \int_{a_1}^{t} \left\{ \left| \frac{\partial^j k_2(t,s)}{\partial t^j} \right| \sum_{i=0}^{\infty} L_i W_i(s) \right\} ds \leq \alpha_2 < 1 \qquad (8.3)$$

$$j = 0, 1, \ldots, q$$

the BVP (6.1), (***) has a unique solution for any A_p and B_i .

The functions $W_i(t)$, $(i = 0, 1, \ldots, q)$ are positive or possibly non-negative continuous functions on $[a_1, a_2]$.

<u>Proof</u>. Again, we shall prove 1 and 2,3 follow analogously. Define the space S as in the previous Theorem 7.2 with the norm

$$\|x\| = \max_{0 \leq j \leq q} \left[\max_{a_1 \leq t \leq a_2} \frac{|x^{(j)}(t)|}{W_j(t)} \right].$$

We have from the mapping T defined as in (6.6) on this S

$$|(Tx_1)^{(j)}(t) - (Tx_2)^{(j)}(t)| \leq \int_{a_1}^{a_2} \left\{ \left| \frac{\partial^j g(t,s)}{\partial t^j} \right| \times \sum_{i=0}^{q} L_i W_i(s) \frac{|x_1^{(i)}(s) - x_2^{(i)}(s)|}{W_i(s)} \right\} ds$$

$$\leq \|x_1 - x_2\| \int_{a_1}^{a_2} \left\{ \left| \frac{\partial^j g(t,s)}{\partial t^j} \right| \sum_{i=0}^{q} L_i W_i(s) \right\} d$$

and hence

$$\|Tx_1 - Tx_2\| \leq \theta_1 \|x_1 - x_2\|.$$

Since, $\theta_1 < 1$ we can apply the contraction mapping principle, which proves the result.

Remark 8.2. With the proper choice of the weight function the inequality (4.5) for the BVP (2.1), (2.2) can be improved to

$$\frac{\sqrt{3}-1}{4\sqrt{3}} L_0 (a_2-a_1)^2 + \frac{1}{3} L_1 (a_2-a_1) < 1 .$$

The inequality (6.19) for the BVPs (6.9), (6.11); (6.9), (6.12) can be improved to

$$\frac{3}{160} L_0 (a_2-a_1)^3 + \frac{33}{320} L_1 (a_2-a_1)^2 + \frac{3}{8} L_2 (a_2-a_1) < 1 .$$

The inequality (6.20) for the BVPs (6.9), (6.13); (6.9), (6.14) can be improved to

$$\frac{7}{120} L_0 (a_2-a_1)^3 + \frac{\sqrt{3}-1}{4\sqrt{3}} L_1 (a_2-a_1)^2 + \frac{1}{3} L_2 (a_2-a_1) < 1$$

The inequality (6.21) for the BVPs (6.9), (6.15); (6.9), (6.16) can be improved to

$$\frac{1}{3} L_0 (a_2-a_1)^3 + \frac{1}{2} L_1 (a_2-a_1)^2 + \frac{1}{2} L_2 (a_2-a_1) < 1 .$$

For the proof see Agarwal [48], Agarwal et al [18], [54].

Remark 8.3 . If, we consider the BVP (6.1), (**) for $q \le p$, then on using Lemma 3.9 (8.2) can be replaced by to

$$\frac{1}{W_j(t)} \int_{a_1}^{a_2} \left\{ \frac{\partial^j f_1(t,s)}{\partial t^j} \sum_{i=0}^{q} L_i W_i(s) \right\} ds \le \alpha_1 < 1 \qquad (8.4)$$

$$j = 0, 1, \cdots, q.$$

We attempt to find suitable $W_j(t)$ $(j = 0, 1, \cdots, q)$ by requiring equality in (8.4). For this we choose $W_j(t) = W^{(j)}(t)$ and $W(t)$ to satisfy

$$W^{(n)}(t) + \frac{1}{\alpha_1} \sum_{j=0}^{q} L_j W^{(j)}(t) = 0 \qquad (8.5)$$

with $W^{(p)}(t) > 0$, $t \in (a_1, a_2]$ so that $W^{(i)}(t) > 0$, $t \in (a_1, a_2]$ for all $i = 0, 1, \cdots, p$, and if $p = n-1$, $W^{(p)}(t) > 0$, $t \in [a_1, a_2]$.

Now, let $u(t)$ be the solution of

$$u^{(n)}(t) + \sum_{j=0}^{q} L_j u^{(j)}(t) = 0 \qquad (8.6)$$

satisfying $u^{(i)}(a_1) = 0$ $(i = 0, 1, \cdots, n-2)$; $u^{(p)}(t) > 0$ on $(a_1, a_2]$; and if $p = n-1$ then $u^{(n-1)}(t) > 0$ on $[a_1, a_2]$.

Since the solution of (8.5) depend continuously on α_1 , we choose α_1 sufficiently close to but less than 1, so that (8.5) has a solution $W(t)$ which satisfies $W^{(i)}(a_1)=0 \ (i=0,1,\ldots,q)$ and whose p-th derivative is arbitrarily close to the p-th derivative of $u(t)$. Since $u^{(p)}(t)>0$ on $(a_1, a_2]$, and $u^{(p)}(t)>0$ on $[a_1, a_2]$ if $p=n-1$, $W^{(p)}(t)$ can be taken to be strictly positive on $(a_1, a_2]$; and if $p=n-1$ we can take on $[a_1, a_2]$. With such α_1 and $W_j(t)=W^{(j)}(t) \ (j=0,1,\ldots,q)$ equality holds in (8.4) theorem 8.1 ensures the existence of a unique solution of the problem (6.1), (**) for $q \le p$.

From the above observation, if we denote $a_1 + l_p(L_0, L_1, \ldots, L_q)$ as the first point where p-th derivative of u(t) vanishes after a_1, then if $a_2 - a_1 < l_p(L_0, L_1, \ldots, L_q)$ there exists a unique solution of (6.1), (**) for $q \le p$ and this result is best possible. For, if equality holds then there exists trivial as well as nontrivial solution of (8.6). Also, if $u^{(p)}(a_1 + l_p(L_0, \ldots, L_q)) \ne 0$ there is no solution.

For n = 2, see the theorem 2.1 and for n = 3 see Agarwal [55].

9. Initial Value Problem.

We shall denote $w(t)$ as the solution of the following initial value problem

$$\begin{aligned} &D_n w(t) = w^{(n)}(t) + \sum_{i=0}^{q} b_i \, w^{(i)}(t) = 0 \\ &w^{(i)}(a) = 0, \quad i = 0, 1, \ldots, n-2 \\ &w^{(n-1)}(a) = 1 \end{aligned} \tag{9.1}$$

where $b_i \; (i = 0, 1, \ldots, q)$ are constants. It is well known that $w(t-s) \, (a \le s \le t)$ can be written as

$$w(t-s) = w_1(t,s)$$

$$= \sum_{i=1}^{n} \frac{e^{(t-s)\lambda_i}}{p'(\lambda_i)}$$

$$\left(p(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) \right)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the distinct roots of the auxiliary equation

$$\lambda^n + \sum_{i=0}^{q} b_i \lambda^i = 0$$

also in case the roots are $\lambda_1^*, \ldots, \lambda_k^*$ having multiplicity $r_1, \ldots, r_k$ respectively, with $n = \sum_{i=1}^{k} r_i$, then if we assume $\lambda_1^* = \lambda_1 = \lambda_2 = \cdots = \lambda_{r_1}$; $\lambda_2^* = \lambda_{r_1+1} = \lambda_{r_1+2} = \cdots = \lambda_{r_1+r_2}$; $\ldots$; $\lambda_k^* = \lambda_{n-r_k+1} = \lambda_{n-r_k+2} = \cdots = \lambda_n$

we can write

$$w(t-s) = w_2(t,s) = \lim_{\lambda_2 \to \lambda_1^*} \lim_{\lambda_3 \to \lambda_1^*} \cdots$$

$$\lim_{\lambda_{r_1} \to \lambda_1^*} \lim_{\lambda_{r_1+2} \to \lambda_2^*} \cdots \lim_{\lambda_n \to \lambda_k^*} w_1(t,s).$$

The following properties of $w(t-s)$ are immediate.

1. If we denote $t-s = X$ then $w(t-s)$ and its derivatives with respect to t can be written as a function of X alone, and we shall denote this by $w(X)$.

2.
$$w(0) = \frac{\partial w(0)}{\partial t} = \cdots = \frac{\partial^{n-2} w(0)}{\partial t^{n-2}} = 0, \quad \frac{\partial^{n-1} w(0)}{\partial t^{n-1}} = 1.$$

3. Let $X_0, X_1, \ldots, X_{n-1}$ be first right positive roots of $w(X), \frac{\partial w(X)}{\partial t}, \ldots, \frac{\partial^{n-1} w(X)}{\partial t^{n-1}}$ then $X_0 \geq X_1 \geq \cdots \geq X_{n-1}$.

4. If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$, then

$$w(X) = \frac{1}{(n-1)!} e^{\lambda X} X^{n-1}$$

and hence $X_0 = \infty$.

10. Some Comparison Results.

Theorem 10.1 [42] Let $u(t), v(t)$ be two functions satisfying

$$D_n u(t) \geq 0 \qquad (10.1)$$

$$D_n v(t) \leq 0 \qquad (10.2)$$

on $(a_1, a_2]$ with

$$u^{(i)}(a_1) = v^{(i)}(a_1); \quad i = 0, 1, \ldots, n-1.$$

Then

$$u^{(i)}(t) \geq v^{(i)}(t) \text{ for } t \in [a_1, a_1 + X_i]. \tag{10.3}$$

Proof. Inequalities (10.1) and (10.2) with some $\phi_1(t) \geq 0$, $\phi_2(t) \geq 0$ can be written as

$$D_n u(t) - \phi_1(t) = 0 \tag{10.4}$$

$$D_n v(t) + \phi_2(t) = 0 \tag{10.5}$$

and hence if $u_0(t)$ is the solution of the homogeneous equation $D_n u_0(t) = 0$ with $u_0^{(i)}(a_1) = u^{(i)}(a_1) = v^{(i)}(a_1)$ for $i = 0, 1, \ldots, n-1$; then we can write the solutions of (10.4) and (10.5) as

$$u(t) = u_0(t) + \int_{a_1}^{t} w(t-s)\phi_1(s)\,ds$$

$$v(t) = u_0(t) - \int_{a_1}^{t} w(t-s)\phi_2(s)\,ds.$$

Thus,

$$u^{(i)}(t) - v^{(i)}(t) = \int_{a_1}^{t} \frac{\partial^i w(t-s)}{\partial t^i}\left[\phi_1(s) + \phi_2(s)\right] ds$$

But since $\dfrac{\partial^i w(t-s)}{\partial t^i} \geq 0$ as long as $a_1 \leq s \leq t \leq a_1 + X_i$ $(i = 0, 1, \ldots, n-1)$ the result follows immediately.

Lemma 10.2 Let h(t) be a function n-1 times continuously differentiable on $[a_1, a_2]$. Then, if $h^{(n-1)}(a_2) > 0$, $h^{(i)}(a_2) = 0 \quad i = 0, 1, \ldots, n-2$ there is some point $t^* < a_2$ such that $(-1)^{n-i+1} h^{(i)}(t) > 0$, $t \in (t^*, a_2)$; $i = 0, 1, \ldots, n-1$.

Proof. Let $t^* < a_2$ be the first point where $h^{(n-1)}(t) = 0$, then we have $h^{(n-1)}(t) > 0, t \in (t^*, a_2]$. For this t^*, the conclusion follows.

We shall denote $Y_0, Y_1, \ldots, Y_{n-1}$ as the distance between the point 0 and the first left zeros (positive or negative according as $n-i+1$ is even or odd) of $w(x)$, $\frac{\partial w(x)}{\partial t}$, $\ldots$, $\frac{\partial^{n-1} w(x)}{\partial t^{n-1}}$ then we have from Lemma 10.2,

$$Y_0 \geq Y_1 \geq Y_2 \geq \cdots \geq Y_{n-1}.$$

Theorem 10.3 [42] Let $u(t)$, $v(t)$ be two functions satisfying (10.1) and (10.2) on $[a_1, a_2)$ with $u^{(i)}(a_2) = v^{(i)}(a_2)$, $i = 0, 1, \ldots, n-1$. Then

$$(-1)^{n-i+1} u^{(i)}(t) \leq (-1)^{n-i+1} v^{(i)}(t); \; t \in [a_2 - Y_i, a_2]$$

Proof: As in Theorem 10.1, we have

$$u(t) = u_0(t) - \int_t^{a_2} w(t-s)\, \phi_1(s)\, ds$$

$$v(t) = u_0(t) + \int_t^{a_2} w(t-s)\, \phi_2(s)\, ds$$

where $u_0(t)$ is the solution of $D_n u_0(t) = 0$ with $u_0^{(i)}(a_2) = u^{(i)}(a_2) = v^{(i)}(a_2)$. Thus

$$u^{(i)}(t) - v^{(i)}(t) = -\int_t^{a_2} \frac{\partial^i w(t-s)}{\partial t^i}\left[\phi_1(s) + \phi_2(s)\right] ds$$

But since $(-1)^{n-i+1} \dfrac{\partial^i w(t-s)}{\partial t^i} \geq 0$ (from lemma 10.2) as long as $a_2 - \gamma_i \leq t \leq s \leq a_2 \quad (i = 0, 1, \ldots, n-1)$ the result follows immediately.

11. Uniqueness Results

Here, we shall assume that the function $f(t, x, x', \ldots, x^{(q)})$ satisfies the following condition

$$\begin{aligned} M_j(x_j - y_j) \leq\ & f(t, x_0, x_1, \ldots, x_j, \ldots, x_q) \\ & - f(t, x_0, x_1, \ldots, y_j, \ldots, x_q) \\ \leq\ & K_j(x_j - y_j) \\ & x_j \geq y_j, \; j = 0, 1, \ldots, q. \end{aligned} \tag{11.1}$$

The condition (11.1) is equivalent to the Lipschitz condition but more informative, particularly since there are no sign restrictions on the constants.

Since the function f is continuous and satisfies (11.1) all initial value problems for (6.1) have unique solutions throughout the interval $[a_1, a_2]$ and are continously dependent on the initial conditions and parameters .

We shall denote $X_i(M_0, M_1, \ldots, M_q)$ in short $X_i(M)$; $i = 0, 1, \ldots, n-1$ as the first right positive zero of $\frac{\partial^i w(x)}{\partial t^i}$ replacing b_j by $-M_j$ $(j = 0, 1, \ldots, q)$ in (9.1).

Theorem 11.1 [42] Let $f(t, x, x', \ldots, x^{(q)})$ satisfy (11.1). Then, the BVP (6.1), (**) for $q \leq p$ has at most one solution if $0 < a_2 - a_1 < X_p(M)$.

Proof. Let $x_1(t)$, $x_2(t)$ be two solutions of (6.1) (**) and that they are distinct. Then, since solutions of IVPs are unique we must have $x_1^{(n-1)}(a_1) \neq x_2^{(n-1)}(a_1)$. So we can assume without loss of generality that $x_1^{(n-1)}(a_1) > x_2^{(n-1)}(a_1)$. Let $h(t) = x_1(t) - x_2(t)$ and t^* be the first possible right zero when $h^{(n-1)}(t) = 0$ so that $h^{(n-1)}(t) > 0$, $t \in [a_1, t^*]$. Since $h^{(n-2)}(a_1) = 0$, $h^{(n-2)}(t) > 0$, $t \in (a_1, t^*)$ and so $h^{(p)}(t) > 0$ $t \in (a_1, t^*)$ Let t_p be the first right positive zero of $h^{(p)}(t)$, i.e. where $h^{(p)}(t) = 0$, then naturally $a_1 < t_p \leq a_2$. Without loss of generality we may assume that $t_p = a_2$ then $h^{(i)}(t) \geq 0$, $t \in [a_1, a_2]$; $i = 0, 1, \ldots, q$ and hence $h(t)$ satisfies

$$h^{(n)}(t) + \sum_{j=0}^{q} - M_j \, h^{(j)}(t) \geq 0$$

$$h^{(i)}(a_1) = 0, \quad i = 0, 1, \ldots, n-2$$

$$h^{(p)}(a_2) = 0$$

and $h^{(p)}(t) > 0, \; t \in (a_1, a_2)$.

Let $u(t)$ be the function defined by

$$u^{(n)}(t) + \sum_{j=0}^{q} -M_j \, u^{(j)}(t) = 0$$

$$u^{(i)}(a_1) = 0, \quad i = 0, 1, \ldots, n-2$$

$$u^{(n-1)}(a_1) = h^{(n-1)}(a_1) .$$

By Theorem 10.1, we have $h^{(p)}(t) \geq u^{(p)}(t) \geq 0$; $t \in [a_1, a_1 + X_p(M)]$. Since $h^{(p)}(a_2) = 0$ and $a_2 < a_1 + X_p(M)$, it follows that $u^{(p)}(t) = 0$ for some t in $[a_1, a_2]$. But this contradicts the fact that $0 < a_2 - a_1 < X_p(M)$; that is $a_2 - a_1$ is so small that no nontrivial solution of (11.1) with $u^{(i)}(a_1) = 0, \; i = 0, 1, \ldots, n-2$ can have $u^{(p)}(a_2) = 0$. Hence the proof of the theorem 11.1 is complete.

In the next result for (1) n is even and p is odd, or (2) n is even and p is even we shall denote Y_p^{*} by $Y_p(M_0, K_1, M_2, K_3, \ldots, M_q$ (or K_q) according as q is even or odd) on replacing b_j by $-M_j$ (or ...) according as in (9.1) according as j is even or odd. Also for (3) n is odd and p is even, or (4) n is odd and p is odd we shall denote Y_p^{**} by $Y_p(K_0, M_1, K_2, M_3, \ldots, K_q$ (or M_q) according as q is even or odd) on replacing b_j by $-K_j$ (or $-M_j$) in (9.1) according as j is even or odd.

<u>Theorem 11.2</u> [42] Let $f(t, x, x', \ldots, x^{(q)})$ satisfy (11.1). Then, the BVP (6.1), (***) for $q \leq p$ has at most one solution if

1. for the cases (1) and (2), $0 < a_n - a_1 < Y_p$
2. for the cases (3) and (4), $0 < a_n - a_1 < Y_p^{**}$.

<u>Proof</u>. Let $x_1(t)$, $x_2(t)$ be two solutions of (6.1), (***) and that they are distinct. Then, since solutions of IVPs are unique we must have $x_1^{(n-1)}(a_n) \neq x_2^{(n-1)}(a_n)$. So, we can assume without loss of generality that $x_1^{(n-1)}(a_n) > x_2^{(n-1)}(a_n)$ Let, $h(t) = x_1(t) - x_2(t)$, this h(t) satisfies the conditions of Lemma 10.2 and hence $(-1)^{n-i+1} h^{(i)}(t) > 0$, $t \in (t^*, a_n)$. Let t_p be the first left zero of $h^{(p)}(t)$ (positive or negative according as $n-p+1)$ is even or odd) then naturally $a_1 \leq t_p < a_n$. Without loss of generality we may assume that $t_p = a_1$, then

$$(-1)^{n-i+1} h^{(i)}(t) \geq 0, \quad t \in [a_1, a_n]; \quad i = 0, 1, \ldots, q$$

and hence h(t) satisfies

$$h^{(n)}(t) \leq \sum_{\substack{j = \text{even} \\ \leq q}} M_j h^{(j)}(t) + \sum_{\substack{j = \text{odd} \\ \leq q}} k_j h^{(j)}(t) \qquad \text{in case (1) and (2)}$$

$$h^{(n)}(t) \leq \sum_{\substack{j = \text{even} \\ \leq q}} k_j h^{(j)}(t) + \sum_{\substack{j = \text{odd} \\ \leq q}} M_j h^{(j)}(t) \qquad \text{in case (3) and (4)}$$

$$u^{(i)}(a_n) = 0, \quad i = 0, 1, \ldots, n-2$$

$$h^{(p)}(a_1) = 0$$

and $(-1)^{n-p+1} h^{(p)}(t) > 0 \ , \ t \in (a_1, a_2)$.

Let u(t) be the function defined by

$$u^{(n)}(t) = \sum_{\substack{j = \text{even} \\ \leq q}} M_j u^{(j)}(t) + \sum_{\substack{j = \text{odd} \\ \leq q}} k_j u^{(j)}(t) \qquad \text{in case (1) and (2)}$$

$$u^{(n)}(t) = \sum_{\substack{j = \text{even} \\ \leq q}} k_j u^{(j)}(t) + \sum_{\substack{j = \text{odd} \\ \leq q}} M_j u^{(j)}(t) \qquad \text{in case (3) and (4)}$$

$$u^{(i)}(a_2) = 0, \quad i = 0, 1, \ldots, n-2$$

$$u^{(n-1)}(a_2) = h^{(n-1)}(a_2).$$

By theorem 10.3, we have $(-1)^{n-p+1} h^{(p)}(t) \geq (-1)^{n-p+1} u^{(p)}(t) \geq$ for $t \in [a_2 - Y_1^*, a_2]$ in case (1) and (2) and $t \in [a_2 - Y_p^{**}, a_2]$ in case (3) and (4). Hence, in case (1), we find $h^{(p)}(t) \geq u^{(p)}(t) \geq 0$, in case (2) $h^{(p)}(t) \leq u^{(p)}(t) \leq 0$ for $t \in [a_2 - Y_p^*, a_2]$ and in case (3) $h^{(p)}(t) \geq u^{(p)}(t) \geq 0$ in case (4) $h^{(p)}(t) \leq u^{(p)}(t) \leq 0$ for $t \in [a_2 - Y_p^{**}, a_2]$.

Since $h^{(p)}(a_1) = 0$ and $a_2 < a_1 + Y_p^*$ for (1) and (2) also $a_2 < a_1 + Y_p^{**}$ for (3) and (4), it follows that $u^{(p)}(t) = 0$ for some t in $[a_1, a_2)$. But this is a contradiction. Hence the proof is complete.

12. Best Possible Results.

In what follows, we shall put $f(t, 0, 0, \dots, 0) = \phi_q(t)$ and $X_i(k)$ for $X_i(M)$ replacing M_j to k_j ($j = 1, \dots, q$).

Lemma 12.1 [42]. Suppose that $a_1 < T < a_1 + X_p^*$ where $X_p^* = \min\{X_p(M), X_p(k)\}$ and for $a_1 \le t \le T$ the functions $u_1(t)$, $u_2(t)$ and their derivatives upto order p are nonnegative and satisfy

$$\begin{aligned}
&u_1^{(n)}(t) - \sum_{j=0}^{q} M_j\, u_1^{(j)}(t) - \phi_q(t) = 0\\
&u_2^{(n)}(t) - \sum_{j=0}^{q} k_j\, u_2^{(j)}(t) - \phi_q(t) = 0\\
&u_1^{(i)}(a_1) = u_2^{(i)}(a_1) = 0, \quad i = 0, 1, \dots, n-2 \qquad (12.1)\\
&u_1^{(n-1)}(a_1) = u_2^{(n-1)}(a_1) > 0
\end{aligned}$$

and that x(t) satisfies

$$\begin{aligned}
&x^{(n)}(t) = f(t, x, x', \dots, x^{(q)})\\
&x^{(i)}(a_1) = 0, \quad i = 0, 1, \dots, n-2 \qquad (12.2)\\
&x^{(n-1)}(a_1) = u_1^{(n-1)}(a_1) = u_2^{(n-1)}(a_1)
\end{aligned}$$

where f satisfies (11.1). Then,

$$\begin{aligned}
&u_1^{(j)}(t) \le x^{(j)}(t) \le u_2^{(j)}(t)\\
&\qquad j = 0, 1, \dots, p. \qquad (12.3)
\end{aligned}$$

Proof: As in Theorem 10.1, if x(t) and $u_i(t)$ $(i = 1, 2)$ satisfy the same initial conditions, then

$$x(t) - u_1(t) = \int_{a_1}^{t} w_M(t-s)\Big[-\sum_{j=0}^{q} M_j\, x^{(j)}(s)$$

$$+ f(s, x(s), \ldots, x^{(q)}(s)) - \phi_q(s)\Big]\, ds,$$

and

$$x(t) - u_2(t) = \int_{a_1}^{t} w_K(t-s)\Big[-\sum_{j=0}^{q} K_j\, x^{(j)}(s) +$$

$$f(s, x(s), \ldots, x^{(q)}(s)) - \phi_q(s)\Big]\, ds$$

where $w_M(t-s)$ is $w(t-s)$ replacing b_j to $-M_j$ and $w_K(t-s)$ is $w(t-s)$ replacing b_j to $-K_j$.

But since $\dfrac{\partial^i w(t-s)}{\partial t^i}$ $(i = 0, 1, \ldots, q)$ are each non-negative for $a_1 \leq s \leq t \leq a_1 + \gamma_p^*$, it follows from (11.1).

$x^{(i)}(t) - u_1^{(i)}(t) \geq 0$ as long as $x^{(i)}(t) \geq 0$; $i = 0, 1, \ldots, p$

$x^{(i)}(t) - u_2^{(i)}(t) \leq 0$ as long as $x^{(i)}(t) \geq 0$; $i = 0, 1, \ldots, p$

Thus (12.3) hold as long as $x^{(i)}(t) \geq 0$ and obviously $x^{(i)}(t) \geq 0$ as long as (12.3) hold. This proves the Lemma.

Remark 12.2: In Lemma 12.1, $X_p^* = X_p(M)$ if we consider only the left half inequality, i.e. $u_1^{(j)}(t) \leq x^{(j)}(t)$, $j = 0, 1, \ldots, p$.

Remark 12.3. From (11.1)

$$f(t, x, \ldots, x^{(v)}) - \phi_v(t) \leq \sum_{j=0}^{v} M_j \, x^{(j)}(t)$$

as long as $x^{(j)}(t) \leq 0$, $(j = 0, 1, \ldots, v)$. Thus, if $u_1(t)$ and all its derivatives up to order p are nonpositive for $a_1 \leq t \leq T$ where $a_1 < T < a_1 + X_p(M)$ and satisfy (12.1) with $u_1^{(n-1)}(a_1) < 0$ and that x(t) satisfies (12.2), then $x^{(j)}(t) \leq u_1^{(j)}(t)$, $j = 0, 1, \ldots, p$.

Lemma 12.4. [42] If $0 < a_2 - a_1 < X_p(M)$ there is a unique solution to the problem

$$u^{(n)}(t) - \sum_{j=0}^{v} M_j \, u^{(j)}(t) - \phi_v(t) = 0 \tag{12.4}$$

$$\begin{cases} u^{(i)}(a_1) = 0, \; i = 0, 1, \ldots, n-2 \\ u^{(p)}(a_2) = m \in R. \end{cases} \tag{12.5}$$

Proof. Any solution of (12.4) can be written as

$$u(t) = A\, u_0(t) + \int_{a_1}^{t} \omega_M(t-s)\, \phi_v(s)\, ds$$

where $u_0(t)$ is the solution of

$$u_0^{(n)}(t) - \sum_{j=0}^{v} M_j \, u_0^{(j)}(t) = 0$$

$$u_0^{(i)}(a_1) = 0, \quad i = 0, 1, \ldots, n-2$$

$$u_0^{(n-1)}(a_1) = 1.$$

In fact, the only solution of (12.4), (12.5) is

$$u(t) = \left[u_0^{(p)}(a_2) \right]^{-1} \left[m - \int_{a_1}^{a_2} \frac{\partial^p w_M(a_2 - s)}{\partial t^p} f_q(s)\, ds \right] u_0(t)$$

$$+ \int_{a_1}^{t} w_M(t-s) f_q(s)\, ds$$

since $u_0^{(p)}(a_2) > 0$.

Using this lemma we can take $m_2 > m$ and sufficiently large and positive so that $u^{(i)}(t) \geq 0$ on $[a_1, a_2]$ and $u^{(n-1)}(a_1) > 0$, also $m > m_1$ and sufficiently large and negative so that $u^{(i)}(t) \leq 0$ on $[a_1, a_2]$ and $u^{(n-1)}(a_1) < 0$ for all $i = 0, 1, \ldots, p$.

Theorem 12.5 [42] Let $f(t, x, \ldots, x^{(r)})$ satisfy (11.1). Then, there exists a unique solution of the BVP (6.1), (**) for $r \leq p$ if $0 < a_2 - a_1 < X_p(\cdot)$. This result is best possible.

Proof. In theorem 11.1, we have already proved the uniqueness part so now we must prove at least one part only. We assume for simplicity that $A_i = 0$, $i = 0, 1, \ldots, n-2$.

Let $m_2 > m > B_p$ be sufficiently large in the sense of Lemma 12.4, so that there is a unique solution $u(t, m_2)$ of (12.4), (12.5). As a result of remark 12.2 and standard theorem on the continuation of solutions of differential equations, the solution $x_2(t)$ of (6.1) with the initial conditions $x_2^{(i)}(a_1) = 0$, $i = 0, 1, \ldots, n-2$; $x_2^{(n-1)}(a_1) = u^{(n-1)}(a_1, m_2)$

can be continued to $t = a_r$ and has

$$\beta_p < m < m_2 \le x_2^{(p)}(a_r).$$

Likewise if $m_2 < m < \beta_p$ and is sufficiently large and negative in the sense of the proof of Lemma 12.4, so that there is a unique solution $u(t, m_1)$ of (12.4), (12.5). By remark 12.3., the solution $x_1(t)$ of (6.1) with the initial conditions

$$x_1^{(i)}(a_1) = 0, \ i = 0, 1, \ldots, n-2; \ x_1^{(n-1)}(a_1) = u^{(n-1)}(a_1, m_1)$$

exists as far as $t = a_r$ and has

$$x_1^{(p)}(a_r) \le m_2 < m < \beta_p$$

Now let $x(t)$ be any solution of (6.1) satisfying $x^{(i)}(a_1) = 0$, $i = 0, 1, \ldots, n-2$ and

$$x_1^{(n-1)}(a_1) < x^{(n-1)}(a_1) < x_2^{(n-1)}(a_1),$$

From the uniqueness of solutions of the BVP (6.1), (**) it follows that

$$x_1^{(i)}(t) < x^{(i)}(t) < x_2^{(i)}(t), \quad i = 0, 1, \ldots, p$$

for $a_1 < t < a_r$, which guarantees that $x(t)$ exists as far as $t = a_r$. Now, by standard theorems on continuity with respect to initial conditions and the intermediate value theorem, it follows that there exists a unique solution of (6.1), (**).

From the definition of $X_p(M)$, when

$$f(t, x, \dots, x^{(V)}) = \sum_{j=0}^{V} M_j x^{(j)}$$

$a_2 - a_1 = X_p(M)$, $x^{(p)}(a_2) = 0$, the problem (6.1), (**) has a nontrivial as well as the trivial solution also if $x^{(p)}(a_2) \neq 0$, this problem fails to have a solution. Hence the result is best possible.

<u>Theorem 12.6</u>. Let $f(t, x, \dots, x^{(q)})$ satisfy (11.1). Then there exists a unique solution of the BVP (6.1), (***) for $q \leq p$ if

1. for the cases (1) and (2), $0 < a_2 - a_1 < Y_p^*$.
2. for the cases (3) and (4), $0 < a_2 - a_1 < Y_p^{**}$

where the cases (1)-(4) and Y_p^*, Y_p^{**} are defined in theorem 11.2. This result is best possible.

<u>Proof</u>. The proof is similar to that of theorem 12.5.

<u>Remarks 12.7</u> (1). Theorem 12.5 only requires a one-sided condition (11.1), if it is assumed that solutions of IVPs for (6.1) extend to $[a_1, a_2]$. (2) Let $f(t, x)$ be continuous and satisfy (11.1) with $M_0 = 0$ i.e. nondecreasing in x). Then, there exists a unique solution of the problem (6.1), (**) The proof follows from the property (4) of $w(t-s)$. This, result was previously obtained by Schrader et al [56].
(3) In defining $X_p(M)$, Y_p^*, Y_p^{**} there is no necessity to say positive or negative, here in the results we have mentioned this to understand proofs more clearly.

13. Convergence of Successive Approximation

Let $u(t)$ be the solution of (8.6) satisfying $u^{(i)}(a_1)=0$, $i=0,1,\ldots,n-2$; $u^{(n-1)}(a_1)=1$ and $v(t)$ be the solution of

$$v^{(n)}(t)+\mu\sum_{j=0}^{q}L_j\,v^{(j)}(t)=0\quad(\mu>1)$$

satisfying $v^{(i)}(a_1)=0$, $i=0,1,\ldots,n-2$; $v^{(n-1)}(a_1)=1$ then as the notations used in remark 8.3 $v^{(p)}(t)$ vanishes at $a_1+\ell_p(\mu L_0,\ldots,\mu L_q)$. Hence we have $v^{(p)}(t)>0$; $t\in(a_1,\ a_1+\ell_p(\mu L_0,\ldots,\mu L_q))$ and if $p=n-1$ then $v^{(p)}(t)>0$, $t\in[a_1,a_1+\ell_p(\mu L_0,\ldots,\mu L_q))$.

Thus $v^{(i)}(t)\geq 0$, $i=0,1,\ldots,p$; $t\in[a_1,\ a_1+\ell_p(\mu L_0,\ldots,\mu L_q)]$.

We shall show that $\ell_p(\mu L_0,\ldots,\mu L_q)<\ell_p(L_0,\ldots,L_q)$.

If not, then obviously $v^{(i)}(t)\geq 0$, $i=0,1,\ldots,p$; $t\in[a_1,a_1+\ell_p(L_0,\ldots,L_q)]$

and we have

$$v^{(n)}(t)+\sum_{j=0}^{q}L_j\,v^{(j)}(t)\leq 0.$$

Now, using theorem 10.1, we find $u^{(p)}(t)\geq v^{(p)}(t)$; $t\in[a_1,\ a_1+\ell_p(L_0,\ldots,L_q)]$. Hence $v^{(p)}(t)$ must vanish before $a_1 + \ell_p(L_0, \ldots, L_q)$ since $u^{(p)}(t)$ vanishes at $a_1 + \ell_p(L_0, \ldots, L_q)$. This is a contraction and hence thesult follows. If $a_2 - a_1 < \ell_p(L_0, \ldots, L_q)$ as in the remark 8.3 we find on using the above fact and the uniqueness of the solutions of IVPs that for some $\mu > 1$, $\ell_p(\mu L_0, \ldots, \mu L_q) = a_2 - a_1$. Thus the problem

$$\begin{aligned} &\psi^{(n)}(t) + \mu \sum_{j=0}^{q} L_j \psi^{(j)}(t) = 0 \quad (\mu > 1) \\ &\psi^{(i)}(a_1) = 0, \quad i = 0, 1, \ldots, n-2 \\ &\psi^{(p)}(a_2) = 0 \end{aligned} \tag{13.1}$$

has a nontrivial solution with $\psi^{(p)}(t) > 0$, $t \in (a_1, a_2)$ and if $p = n-1$ then on $[a_1, a_2)$. To fix the choice of we also require $\psi^{(n-1)}(a_1) = 1$. Then it is always possible to choose C to be the smallest possible constant such that

$$\int_{a_1}^{a_2} \frac{\partial^j R_1(t,s)}{\partial t^j}\, ds \le C\left(\frac{1}{\mu}\right) \psi^{(j)}(t) \tag{13.2}$$

$$j = 0, 1, \ldots, q.$$

In the following theorem, without loss of generality we shall consider $A_i = 0$ $(i = 0, 1, \ldots, n-2)$, $B_p = \omega$ appearing in (**).

Theorem 13.1 [18] Let $f(t, x, \ldots, x^{(q)})$ satisfy Lipschitz condition (6.17) on D

$$D = \Big\{ (t, x(t)) : a_1 \leq t \leq a_2,\ x(t) \in C^{(q)}[a_1, a_2],$$
$$|x^{(j)}(t)| \leq m\, C\, \frac{1}{\mu} \Big(1 - \frac{L}{\mu}\Big)^{-1} \psi^{(j)}(t)$$
$$j = 0, 1, \ldots, q \Big\},$$

where $m = \max\limits_{a_1 \leq t \leq a_2} |f(t, 0, \ldots, 0)|$. Then an infinite sequence $\{x_n(t)\}$ $(n = 0, 1, \ldots)$ can be obtained in D by the successive approximations

$$x_0(t) \equiv 0$$
$$x_{n+1}(t) = -\int_{a_1}^{a_2} g_1(t, s)\, f(s, x_n(s), \ldots, x_n^{(q)}(s))\, ds \tag{13.3}$$
$$n = 0, 1, \ldots$$

and as $n \to \infty$ it converges to a unique solution of the BVP (6.1) (3.15) in D.

Proof. First we shall prove by induction that an infinite sequence $\{x_n(t)\}$ $(n = 0, 1, \ldots)$ can be obtained in D by the successive approximations (13.3). Since $x_0(t) \in D$, let us assume that $x_1(t), x_2(t), \ldots, x_{n-1}(t)$ have been obtained in D. Then $x_n(t)$ is obtained by (13.3) since $x_{n-1}(t) \in D$. In order to complete the induction it is sufficient to prove $x_n(t) \in D$. Now by (13.3) and (13.2), we have

$$\begin{aligned} |x_1^{(j)}(t) - x_0^{(j)}(t)| &\leq \int_{a_1}^{a_2} \frac{\partial^j g_1(t, s)}{\partial t^j} |f(s, 0, \ldots, 0)|\, ds \\ &\leq m \int_{a_1}^{a_2} \frac{\partial^j g_1(t, s)}{\partial t^j}\, ds \\ &\leq m\, C\, \frac{1}{\mu} \psi^{(j)}(t), \quad j = 0, 1, \ldots, q. \end{aligned} \tag{13.4}$$

Hence $x_1(t) \in D$.

Recall that the problem (13.1) is equivalent to

$$\psi^{(j)}(t) = \mu \int_{a_1}^{a_2} \left\{ \frac{\partial^j k_1(t,s)}{\partial t^j} \sum_{i=0}^{q} L_i \psi^{(i)}(s) \right\} ds$$

$$j = 0, 1, \ldots, q.$$

Hence, by (13.3) and (13.4), we have

$$|x_2^{(j)}(t) - x_1^{(j)}(t)| \leq \int_{a_1}^{a_2} \left\{ \frac{\partial^j k_1(t,s)}{\partial t^j} \sum_{i=0}^{q} L_i |x_1^{(i)}(s) - x_0^{(i)}(s)| \right\} ds$$

$$\leq m C \left(\frac{1}{\mu}\right) \int_{a_1}^{a_2} \left\{ \frac{\partial^j k_1(t,s)}{\partial t^j} \sum_{i=0}^{q} L_i \psi^{(i)}(s) \right\} ds$$

$$= m C \left(\frac{1}{\mu}\right)^2 \psi^{(j)}(t)$$

$$j = 0, 1, \ldots, q.$$

Continuing this way, we get

$$|x_n^{(j)}(t) - x_{n-1}^{(j)}(t)| \leq m \left(\frac{1}{\mu}\right)^n C \psi^{(j)}(t) \tag{13.5}$$

$$j = 0, 1, \ldots, q.$$

Then from these inequalities, we successively have

$$|x_n^{(j)}(t) - x_0^{(j)}(t)|$$

$$\le \sum_{i=0}^{n-1} |x_{n-i}^{(j)}(t) - x_{n-i-1}^{(j)}(t)|$$

$$\le \sum_{i=0}^{n-1} m\, C \left(\frac{1}{\mu}\right)^{n-i} \psi^{(j)}(t)$$

$$\le m\, C \left(\frac{1}{\mu}\right) \left(1 - \frac{1}{\mu}\right)^{-1} \psi^{(j)}(t)$$

$$j = 0, 1, \ldots, \nu$$

and hence $x_n(t) \in D$. This completes the induction. Also, since $\mu > 1$ estimates (13.5) ensure that the sequence $\{x_n(t)\}$ converges to a limit, say $x(t)$. Hence, we have proved that the BVP (6.1), (3.15) has at least one solution in D. Since it is easy to prove the uniqueness part it is left to the reader.

Remark 13.2. From

$$|x^{(j)}(t) - x_n^{(j)}(t)| \le \sum_{k=n}^{\infty} |x_{k+1}^{(j)}(t) - x_k^{(j)}(t)|$$

and (13.5), we find the error estimates as

$$|x^{(j)}(t) - x_n^{(j)}(t)| \le m\, C \left(\frac{1}{\mu}\right)^{n+1} \left(1 - \frac{1}{\mu}\right)^{-1} \psi^{(j)}(t) \qquad (13.6)$$

$$j = 0, 1, \ldots, \nu.$$

For $n = 2$, the same results are obtained by Bailey et al [5 and for $n = 3$ Agarwal [57] . The error bound obtained in (13.6) satisfies the same boundary conditions as the original problem.

14. Maximal Solution and comparison Result.

Here we shall consider the differential equation

$$x^{(n)} + F(t, x, x', \ldots, x^{(\nu)}) = 0 \tag{14.1}$$

with the boundary conditions (**), also we shall assume F is continuous on $[a_1, a_2] \times R^{\nu+1}$ and $0 \leq \nu \leq p$, $\nu \leq n-2$

The $\ell_2(t)$ i.e. the polynomial of degree (n-1) satisfying (**) is

$$\ell_2(t) = \sum_{r=0}^{n-2} \frac{A_r}{r!}(t-a_1)^r + \left[B_p - \sum_{k=0}^{n-p-2} \frac{A_{p+k}}{k!}(a_2-a_1)^k\right] \times \frac{(n-p-1)!}{(n-1)!} \frac{(t-a_1)^{n-1}}{(a_2-a_1)^{n-p-1}} . \tag{14.2}$$

Let B denote the Banach space of q times continuously differential functions on $[a_1, a_2]$ with the norm

$$\|x\| = \max_{0 \leq j \leq q} \left\{ \max_{a_1 \leq t \leq a_2} |x^{(j)}(t)| \right\}$$

and let $K \subset B$ denote the closed, convex cone of functions having non-negative derivatives upto order ν . Let S denote these elements of K which satisfy the boundary conditions (**). Further we define

$$S_\rho = \{ x \in S \ : \ \|x\| \leq \rho \}.$$

Theorem 14.1 [18]. Let the following assumptions hold

1. $G(t, u_0, u_1, \ldots, u_q)$ is continuous non-negative and satisfies

$$G(t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_q) \geq G(t, u_0, u_1, \ldots, u_q)$$

for all $(t, u_0, u_1, \ldots, u_q), (t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_q)$

$\in [a_1, a_2] \times R_+^{q+1}$ such that

$\bar{u}_i \geq u_i \; ; \; i = 0, 1, \ldots, q$

2. $\overset{*}{A}_j \geq 0, \; j = 0, 1, \ldots, n-2 \; ; \; \overset{*}{B}_p \geq \sum_{k=0}^{n-p-2} \frac{\overset{*}{A}_{p+k}}{k!} (a_2 - a_1)^k$

3. there exists a $\rho > 0$ such that for all $t \in [a_1, a_2]$

$$\overset{*}{\ell}{}_2^{(j)}(t) + \int_{a_1}^{a_2} \frac{\partial^j k_1(t, s)}{\partial t^j} G(s, \rho, \ldots, \rho) \leq \rho$$

$$j = 0, 1, \ldots, q.$$

for $t \in [a_1, a_2]$ define the iterates

$$x_0(t) = \int_{a_1}^{a_2} k_1(t, s) \, G(s, \rho, \ldots, \rho) \, ds + \overset{*}{\ell}_2(t)$$

$$x_n(t) = \int_{a_1}^{a_2} k_1(t, s) \, G(s, x_{n-1}(s), x'_{n-1}(s), \ldots x_{n-1}^{(q)}(s)) \, ds + \ell_2^*(t)$$

$$n = 1, 2, \ldots .$$

Then, the sequence $\{x_n(t)\}$ converges to the maximal solution $x(t)$ of the BVP

$$\begin{aligned} & x^{(n)} + G(t, x, x', \ldots, x^{(q)}) = 0 \\ & x^{(i)}(a_1) = \overset{*}{A}_i, \quad i = 0, 1, \ldots, n-2 \\ & x^{(p)}(a_2) = \overset{*}{B}_p \end{aligned} \tag{14.3}$$

in S_ρ i.e. y(t) is anyx other solution in S_ρ then

$x^{(j)}(t) \geq y^{(j)}(t)$, $j = 0, 1, \ldots, q$.

Here $\overset{*}{l}_2(t)$ is same as $l_2(t)$ replacing A_i to $\overset{*}{A}_i$ $(i = 0, 1, \ldots, n-2)$ and B_p to $\overset{*}{B}_p$.

<u>Proof</u>. The fixed points of $Tx = x$ where T is defined on and

$$T x(t) = \int_{a_1}^{a_2} k_1(t,s)\, G(s, x(s), \ldots, x^{(q)}(s))\, ds + \overset{*}{l}_2(t) \tag{14.4}$$

are solutions of the BVP (14.3).

Now clearly $x_0(t) \in S_\rho$, since $x_0^{(j)}(t) \leq \rho$, $j = 0, 1, \ldots, q$. Using assumptions 1-3, we find

$$\begin{aligned} x_1^{(j)}(t) &= \int_{a_1}^{a_2} \frac{\partial^j k_1(t,s)}{\partial t^j}\, G(s, x_0(s), \ldots, x_0^{(q)}(s))\, ds + \overset{*}{l}{}_2^{(j)}(t) \\ &\leq \int_{a_1}^{a_2} \frac{\partial^j k_1(t,s)}{\partial t^j}\, G(s, \rho, \ldots, \rho)\, ds + \overset{*}{l}{}_2^{(j)}(t) \\ &\leq x_0^{(j)}(t), \quad j = 0, 1, \ldots, q. \end{aligned}$$

Using an inductive argument it is easily seen that

$$\rho \geq x_0^{(j)}(t) \geq x_1^{(j)}(t) \geq \ldots \geq x_n^{(j)}(t)$$

Next, from the uniform continuity of $\dfrac{\partial^j k_1(t,s)}{\partial t^j}$ on $[a_1, a_2] \times [a_1, a_2]$; $0 \leq j \leq q$ given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that for $|t_1 - t_2| < \delta(\epsilon)$,

$$\| k_1(t_1, s) - k_1(t_2, s) \| < \frac{\epsilon}{2(a_2 - a_1) \sup\limits_{a_1 \leq t \leq a_2} G(t, \rho, \ldots, \rho)}$$

and for the same $\epsilon > 0$, there exists a $\delta_2(\epsilon) > 0$ such that

$$|t_1 - t_2| < \delta_2(\epsilon) \Rightarrow \| \overset{*}{l}_2(t_2) - \overset{*}{l}_2(t_2) \| \leq \epsilon/2 .$$

Hence for all $n = 0, 1, \ldots$

$$|x_n^{(j)}(t_1) - x_n^{(j)}(t_2)| \leq \int_{a_1}^{a} \left| \frac{\partial^j k_1(t_1, s)}{\partial t^j} - \frac{\partial^j k_1(t_2, s)}{\partial t^j} \right| \times$$

$$G(s, \rho, \ldots, \rho)\, ds + | l_2^{(j)}(t_1) - l_2^{(j)}(t_2)|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Whenever $|t_1 - t_2| < \min(\delta_1, \delta_2)$ uniformly in n for all $j = 0, 1, \ldots, q$. This shows that $\{x_n(t)\}$ is equicontinuous. Hence by Arzela's theorem $\{x_n(t)\}$ is precompac and being monotonic, converges uniformly to some $x(t) \in S_\rho$. From the continuity of G, x(t) is a solution of the operator equation (14.4) or equivalently of (14.3).

Now let y(t) be any other solution of (14.3) in S_ρ. Then

$$x_0^{(j)}(t) - y^{(j)}(t) = \int_{a_1}^{a_2} \frac{\partial^j R_1(t,s)}{\partial t^j} \left[G(s, \rho, \ldots, \rho) - G(s, y(s), \ldots, y^{(q)}(s))\right] d$$

$$\geq 0.$$

In fact, induction shows that for each n

$$x_n^{(j)}(t) - y^{(j)}(t) \geq 0, \quad j = 0, 1, \ldots, q.$$

Therefore, in the limit, $x^{(j)}(t) - y^{(j)}(t) \geq 0$. This completes the proof of theorem 14.1.

Theorem 14.2 [18] Let the assumptions of Theorem 14.1 hold. Suppose Z(t) is a solution of BVP (14.1) in B with $\|Z\| \leq \rho$, and let

$$|F(t, u_0, u_1, \ldots, u_q)| \leq G(t, |u_0|, |u_1|, \ldots, |u_q|)$$

for all $(t, u_0, u_1, \ldots, u_q)$ in $[a_1, a_2] \times R^{q+1}$. Also we assume

$$|A_i| \le \overset{*}{A}_i, \quad i = 0, 1, \ldots, n-2$$

$$\left| B_p - \sum_{k=0}^{n-p-2} \frac{A_{p+k}}{k!}(a_2 - a_1)^k \right| \le \overset{*}{B}_p - \sum_{k=0}^{n-p-2} \frac{\overset{*}{A}_{p+k}}{k!} \times (a_2 - a_1)^k.$$

Then $|z^{(j)}(t)| \le x^{(j)}(t)$, $j = 0, 1, \ldots, q$; $t \in [a_1, a_2]$ where x(t) is the maximal solution of (14.3).

Proof: For $n = 1, 2, \ldots,$ set

$$y_n(t) = \int_{a_1}^{a_2} k_1(t, s)\, G(s, y_{n-1}(s), \ldots, y_{n-1}^{(q)}(s))\, ds + \overset{*}{l}_2(t)$$

with $y_0(t) = |z(t)|, \ldots, y_0^{(q)}(t) = |z^{(q)}(t)|$. Then

$$\begin{aligned} y_1^{(j)}(t) &= \int_{a_1}^{a_2} \frac{\partial^j k_1(t,s)}{\partial t^j} G(s, |z(s)|, \ldots, |z^{(q)}(s)|)\, ds + \overset{*}{l}_2(t) \\ &\ge \int_{a_1}^{a_2} \frac{\partial^j k_1(t,s)}{\partial t^j} |F(s, z(s), \ldots, z^{(q)}(s))|\, ds + |l_2(t)| \\ &\ge \left| \int_{a_1}^{a_2} \frac{\partial^j k_1(t,s)}{\partial t^j} F(s, z(s), \ldots, z^{(q)}(s))\, ds + l_2(t) \right| \\ &= |z^{(j)}(t)|, \quad j = 0, 1, \ldots, q. \end{aligned}$$

Inductively, it can readily be seen that

$$|z^{(j)}(t)| \leq y_1^{(j)}(t) \leq y_2^{(j)}(t) \leq \cdots \leq y_n^{(j)}(t) \leq \rho.$$

As in Theorem 14.1, $\{y_n(t)\}$ is equicontinuous and, in view of the monotonicity, converges uniformly to a solution $y(t) \in S_\rho$ of (14.3). But the maximality of x(t) completes the proof of theorem 14.2.

Similar results to that of theorems 14.1 and 14.2 have been obtained for a particular case $n = 2, p = 0$ in [58], [59]. Also in [59] F is taken as a positive function whereas here it is not necessary. Also see [60].

15. <u>More General Problem</u>

Here we shall consider the following BVP

$$L[x] = x^{(n)} + p_1(t)\,x^{(n-1)} + \cdots + p_n(t)\,x = f(t, x, x', \ldots, x^{(q)}) \quad (q \leq n-1) \tag{15.1}$$

$$x^{(i)}(t_j) = x_{ij}$$

where $p_i(t),\ i = 1, 2, \ldots, n$ are continuous on the interval I (I may be open, closed etc.). We assume L is disconjugate on I, i.e., any nontrivial solution x of $L[x] = 0$ has at most n-1 zeros on I, counting multiplicities. The function $f : I \times R^{q+1}$ is a continuous function and let x_{ij}, $0 \leq i \leq i_j - 1$, $j = 1, 2, \ldots, k$, $\sum_{j=1}^{k} i_j = n$ be given real numbers and let $\{t_i\}_{i=1}^{k} \subset I$ be k points in I so that $t_1 < t_2 < \cdots < t_k$. Recall that the operator L is same as defined in (3.1) and the boundary conditions (15.2) same as (*) with different notations.

Lemma 15.1. The equation

$$L[x] = 0 \tag{15.3}$$

has a unique solution $\phi(t)$ satisfying the boundary condition (15.2).

Lemma 15.2. There exists a Green's function $g(t,s)$, $t_1 \le s, t \le t_k$, such that the BVP (15.1), (15.2) is equivalent to the integral equation

$$x(t) = \int_{t_1}^{t_k} g(t,s) f(s, x(s), x'(s), \ldots, x^{(q)}(s))\,ds + \phi(t) \tag{15.4}$$

where $\phi(t)$ is as in Lemma 15.1.

Lemma 15.3. If $f(t, u_0, \ldots, u_q)$ is bounded on $[t_1, t_k] \times R^{q+1}$, then (15.4) has a solution.

The proof of lemmas 15.1 and 15.2 follows from section 3 and Lemma 15.3 follows as an elementary consequence of the Schauder-Tychonoff fixed point theorem (see Corollary 6.2 for a particular case).

Lemma 15.4 Let $\psi \in C^{(n)}[I]$ be such that ψ satisfies (15.2) with $x_{ij} = 0$, and let $L[\psi] > 0$ on I. Then

$$\begin{aligned} \operatorname{sgn} \psi(t) &= (-1)^n, & t < t_1 \\ &= (-1)^{n+s(j)}, & t_j < t < t_{j+1}; j = 1, \ldots, k-1 \\ &= 1, & t > t_k \end{aligned} \tag{15.5}$$

where $s(j) = \sum_{p=1}^{j} i_p$ and $sgn\ \psi(t) = +1\ (-1)$ whenever $\psi(t) > 0\ (< 0)$. Moreover, ψ has at most n zeros on I, counting multiplicity.

Proof. From the inequality $L[\psi] > 0$ on I it follows that there exists some function $p(t) > 0$ on I such that $L[\psi] = p(t)$ Since L is disconjugate $\psi(t)$ can have at most n zeros on I, counting multiplicity. Now from Lemma 15.2 it follows that

$$\psi(t) = \int_{t_1}^{t_k} g(t, s)\, p(s)\, ds$$

Hence from Lemma 3.1 it follows that $sgn\ \psi(t) = sgn\ g(t, s)$ $= sgn\ P(t)$, where $P(t) = \prod_{j=1}^{k} (t - t_j)^{i_j}$ and hence (15.5) follow

Lemma 15.5 Let $\psi \in C^{(n)}(I)$ be such that $L[\psi] > 0$ on I and satisfies (15.2) with $x_{i_j} = 0$ at the points $t_2, \ldots, t_{k-1}$ and

$$\psi^{(i)}(t_j) = 0, \quad 0 \leq i \leq i_j - 2, \quad j = 1 \cdot k \qquad (15.6)$$

$$(-1)^{n+i_1}\, \psi^{(i_1-1)}(t_1) \geq 0, \qquad (15.7)$$

$$\psi^{(i_k-1)}(t_k) \leq 0. \qquad (15.8)$$

Then

$$\operatorname{sgn} \psi(t) = (-1)^{n+s(j)}, \quad t_j < t < t_{j+1}$$
$$j = 1, 2, \ldots, k-1. \tag{15.9}$$

The conditions above are to be interpreted that (15.6) is absent for $j = 1$ or k if $i_1 = 1$ or $i_k = 1$, respectively.

<u>Proof</u>. If equality holds in (15.7) and (15.8) there is nothing to prove (Lemma 15.4). If not then assume the inequality is strict in (15.7), the proof being similar if the inequality is strict in (15.8). Assume therefore that (15.9) fails on (t_1, t_2). If equality holds in (15.8) then there exist $\bar{t}_1 \in (t_1, t_2)$ such that $\psi(\bar{t}_1) = 0$ and by Lemma 15.4 $\bar{t}_1$ must be a simple zero, and ψ can have no other zeros. We may therefore apply lemma 15.4 relative to the set of points $t_1, \bar{t}_1, t_2, \ldots, t_k$ zeros of ψ of multiplicity $i_1 - 1, 1, i_2, \ldots, i_k$, respectively. Thus by Lemma 15.4 $\operatorname{sgn} \psi(t) = (-1)^{n+s(1)-1}$, $t_1 < t < \bar{t}_1$. On the other hand it follows from (15.7) that $\operatorname{sgn} \psi(t) = (-1)^{n+s(1)}$; $t_1 < t < \bar{t}_1$, a contradiction.

Thus the strict inequality must hold in (15.8). If $\bar{t}_1$ is a double zero, or ψ has another zero in (t_1, t_k) aside from those listed, we again may apply Lemma 15.4 to obtain that $\operatorname{sgn} \psi(t) = (-1)^{n+s(1)-1}$ in a right hand neighborhood of t_1; this however is impossible by (15.7). Thus it must happen that $\bar{t}_1$ is a simple zero; ψ has zeros at $t_1, t_2, \ldots, t_k$ of multiplicity $i_1 - 1, i_2, \ldots, i_{k-1}, i_k - 1$, respectively.

We thus conclude that $\operatorname{sgn} \psi(t) = (-1)^{i_k - 1}$, $t_{k-1} < t < t_k$.

It follows from Taylor's theorem that

$$\operatorname{sgn} \psi(t) = \operatorname{sgn} \psi^{(i_k - 1)}(t_k) \operatorname{sgn} (t - t_k)^{i_k - 1} \qquad (t_{k-1} < t < t_k)$$

$$= -1 (-1)^{i_k - 1}$$

which again is a contradiction. To consider the case where (15.9) fails on (t_j, t_{j+1}), $j > 1$, but holds on all intervals (t_i, t_{i+1}), $i < j$, we proceed as above to obtain a contradiction.

<u>Theorem 15.6.</u> [61] Let there exist functions $\alpha, \beta \in C^{(n)}[t_1, t_k]$ so that α and β satisfy the boundary conditions (15.2) at $t_2, t_3, \ldots, t_{k-1}$. Further let

$$\alpha^{(i)}(t_j) = x_{ij} = \beta^{(i)}(t_j), \quad 0 \le i \le i_j - 2, \; j = 1, k \tag{15.10}$$

$$(-1)^{n + i_1 - 1} \left(\alpha^{(i_1 - 1)}(t_1) - x_{i_1 - 1\, 1} \right) \le 0 \le (-1)^{n + i_1 - 1} \left(\beta^{(i_1 - 1)}(t_1) - x_{i_1 - 1\, 1} \right) \tag{15.11}$$

$$\operatorname{sgn} (\alpha(t) - \beta(t)) = (-1)^{n + s(j)}, \quad t_j < t < t_{j+1}, \; j = 1, \ldots, k-1 \tag{15.12}$$

$$\alpha^{(i_k - 1)}(t_k) \le x_{i_k - 1\, k} \le \beta^{(i_k - 1)}(t_k) \tag{15.13}$$

$$L[\alpha(t)] - f(t, x) \ge 0 \ge L[\beta(t)] - f(t, x) \tag{15.14}$$

for all $(t, x) \in W$, where W is given by

$$W = \{(t,x): \alpha(t) \leq x \leq \beta(t)\} \cup \{(t,x): \alpha(t) \geq x \geq \beta(t)\}. \tag{15.15}$$

Then the BVP (15.1), (15.2) for $\gamma \leq 0$ has a solution such that $(t, x(t)) \in W$.

Proof. Define $\bar{x}(t)$ in the following way

$$\bar{x}(t) = \begin{cases} \max\{\beta(t), \alpha(t)\}, & \text{if } x > \max\{\beta(t), \alpha(t)\} \\ x & \text{if } \min\{\beta(t), \alpha(t)\} \leq x \leq \max\{\beta(t), \alpha(t)\} \\ \min\{\beta(t), \alpha(t)\}, & \text{if } x < \min\{\beta(t), \alpha(t)\}, \end{cases} \tag{15.16}$$

and let

$$F(t,x) = f(t, \bar{x}(t)). \tag{15.17}$$

The function F so defined is bounded and continuous on $[t_1, t_k] \times R$. Using lemma 15.3 we conclude that there exists a solution $x(t)$ of

$$L[x] = F(t,x) \tag{15.18}$$

satisfying the boundary conditions (15.2). We next show that $(t, x(t)) \in W$, $t_1 \leq t \leq t_k$ and hence conclude that x(t) is a solution of (15.1) for $\gamma = 0$. To this end let $P(t)$ be the solution of $L[x] = 1$ satisfying the boundary conditions (15.2) with $x_{ij} = 0$ and let $\epsilon > 0$ be given. It follows from

Lemma 15.4 that $\epsilon P(t)$ satisfies the sign condition

$$\operatorname{sgn} (\epsilon P(t)) = (-1)^{s(j)}, \quad t_j < t < t_{j+1} \qquad (15.19)$$
$$j = 1, \ldots, k-1.$$

We let

$$W_\epsilon = \{(t, x) : \beta(t) - \epsilon P(t) \leq x \leq \alpha(t) + \epsilon P(t)\} \cup$$
$$\{(t, x) : \alpha(t) + \epsilon P(t) \leq x \leq \beta(t) - \epsilon P(t)\}.$$

It follows from (15.10)-(15.13) and (15.19) that $W \subset W_\epsilon$ and that $\bigcap_{\epsilon > 0} W_\epsilon = W$. To obtain our desired conclusion we need to show that $(t, x(t)) \in W_\epsilon$ for arbitrary $\epsilon > 0$.

Let $z(t) = \alpha(t) + \epsilon P(t) - x(t)$. Then

$$L[z(t)] = L[\alpha(t)] + \epsilon - L[x(t)]$$
$$\geq f(t, \bar{x}(t)) + \epsilon - F(t, x(t)) = \epsilon.$$

Further $z(t)$ satisfies the conditions of Lemma 15.5 and hence

$$\operatorname{sgn} z(t) = (-1)^{n + s(j)}, \quad t_j < t < t_{j+1}, \quad j = 1, \ldots, k-1.$$

Similarly one shows that

$$\operatorname{sgn} (\beta(t) - \epsilon P(t) - x(t)) = - \operatorname{sgn} z(t)$$

and hence that $(t, x(t)) \in W_\epsilon$, completing the proof.

Remark 15.7. In interpreting the conditions of Theorem 15.6 we follow the convention as in Lemma 15.5. The result of theorem 15.6 is true if $q \neq 0$ also, provided that the differential inequalities (15.14) hold for arbitrary choices of $(x', \ldots, x^{(q)})$ and $|f(t, x, x', \ldots, x^{(q)})| \leq g(t, x)$, where g is continuous.

Corollary 15.8. Consider the equation

$$x''' = f(t, x) \tag{15.20}$$

and let there exist functions $\alpha, \beta \in C^{(3)}[a, b]$ such that

$$\alpha(t) \geq \beta(t), \quad a \leq t \leq c, \quad \alpha(t) \leq \beta(t), \quad c \leq t \leq b \tag{15.21}$$

where $a < c < b$ and assume that

$$\alpha'''(t) - f(t, x) \geq 0 \geq \beta'''(t) - f(t, x) \tag{15.22}$$

for all $(t, x) \in W$. Then there exists a solution $x(t)$ of (15.20) such that $(t, x(t)) \in W$ and $x(a) = A$, $x(b) = B$ for any A, B with $\beta(a) \leq A \leq \alpha(a)$, $\alpha(b) \leq B \leq \beta(b)$.

This is an improvement of Theorem 9 of Klaasen [62].

Theorem 15.9. Assume $f(t, x)$ ($\nu = 0$) in (15.1) is continuous on $I \times R$ and that $f(t, x) \leq M$ for $t \in I$ and $x \in R$. Also, let $f(t, x) \geq k$ for t in (t_i, t_{i+1}) and x such that $(-1)^{n+s(i)} x \leq 0$. Then (15.1),(15.2) for $\nu = 0$ has a solution for all choices of $x_{ij} \in R$.

Proof. Without loss of generality we may assume that $M > 0$ and $k < 0$. Let $\phi(t)$ be the unique solution of $L[x] = 0$ satisfying the boundary conditions (15.2) (see Lemma 15.1). Let $v(t)$ be the solution of the differential equation $L[v] = M$ satisfying (15.2) with $x_{ij} = 0$, $0 \leq i \leq i_j - 1$,

$j = 1, 2, \ldots, k$. Then from Lemma 15.4 $\operatorname{sgn} v(t) = (-1)^{n+s(i)}$ i.e., $(-1)^{n+s(i)} v(t) > 0$

for $t_i < t < t_{i+1}$.

Choose $\alpha(t) = \phi(t) + v(t)$; consequently

$$(-1)^{n+s(i)} [\alpha(t) - \phi(t)] = (-1)^{n+s(i)} v(t) > 0$$

on $t_i < t < t_{i+1}$. Also $L[\alpha(t)] = L[\phi(t)] + L[v(t)]$ $= L[v(t)] = M \geq f(t, x)$ for all t, x with $t_1 \leq t \leq t_k$ and $x \in R$. We will now show that we can choose a function $\beta \in C^{(n)}[t_1, t_k]$ such that $\operatorname{sgn}[\alpha(t) - \beta(t)] = (-1)^{n+s(i)}$ on $t_i < t < t_{i+1}$ for $i = 1, \ldots, k-1$ and such that $L[\beta(t)] \leq f(t, x)$ for all $(t, x) \in W$ where

$$W = \bigcup_{i=1}^{k-1} \Big\{ (t, x) : t_i \leq t \leq t_{i+1},\ (-1)^{n+\delta(i)} \beta(t) \leq (-1)^{n+\delta(i)} x \leq (-1)^{n+\delta(i)} \alpha(t) \Big\}.$$

In fact we shall obtain a function $\beta \in C^{(n)}[t_1, t_k]$ such that $\operatorname{sgn}[\alpha(t) - \beta(t)] = (-1)^{n+\delta(i)}$ on $t_i < t < t_{i+1}$ and $L[\beta(t)] \leq f(t, x)$ for all (t, x) with $t_i \leq t \leq t_{i+1}$ and $(-1)^{n+\delta(i)} x \leq (-1)^{n+\delta(i)} \alpha(t)$ for $i = 1, \dots, k-1$.

By hypotheses $f(t, x) \geq k$ for $t_i \leq t \leq t_{i+1}$ and $(-1)^{n+\delta(i)} x \leq 0$. Let

$$C_i = \max \Big\{ (-1)^{n+\delta(i)} \alpha(t) : t_i \leq t \leq t_{i+1} \Big\}$$

and $C_i' = \max\{0, C_i\}$. Since $C_i' \geq 0$ there exists a constant say k_i' such that $f(t, x) \geq k_i'$ for $(t, x) \in D = \{ t_i \leq t \leq t_{i+1} \text{ and } 0 \leq (-1)^{n+\delta(i)} x \leq C_i' \}$ (in case $C_i' = 0$, $k_i' = k$) which is possible because D is compact and f is continuous. Let $k_i = \min\{k, k_i'\}$, then we have $f(t, x) \geq k_i$ for $t_i \leq t \leq t_{i+1}$, $(-1)^{n+\delta(i)} x \leq C_i'$. Define $k_0 = \min\{k_i : i = 1, \dots, k-1\}$

Let $u(t)$ be the solution of the differential equation $L[u] = k_0$ satisfying (15.2) with $x_{i_j} = 0$, $0 \leq i \leq i_j - 1$, $j = 1, 2, \ldots, k$. Then from Lemma 15.4 $\operatorname{sgn} u(t) = (-1)^{n+s(i)+1}$, i.e. $(-1)^{n+s(i)+1} u(t) > 0$ for $t_i < t < t_{i+1}$.

Let $\beta(t) = \phi(t) + u(t)$, so that we have $\operatorname{sgn}[\beta(t) - \phi(t)] = (-1)^{n+s(i)+1}$ and $(-1)^{n+s(i)}[\beta(t) - \phi(t)] < 0$ for $t_i < t < t_{i+1}$.
Thus $(-1)^{n+s(i)} \beta(t) \leq (-1)^{n+s(i)} \phi(t) \leq (-1)^{n+s(i)} \alpha(t)$ for $t_i \leq t \leq t_{i+1}$.
Now $L[\beta(t)] = L[\beta(t)] = L[\phi(t)] + L[u(t)] = k_0$ $\leq k_i \leq f(t, x)$ for $t_i \leq t_i \leq t_{i+1}$ and $(-1)^{n+s(i)} x \leq C_i'$.

The functions α and β satisfy the hypotheses of Theorem 15.6 so it follows that (15.1), (15.2) for $q = 0$ has a solution x such that $(t, x(t)) \in W$.

<u>Corollary 15.10</u>. Assume $f(t, x)$ $(q = 0)$ in (15.1) is continuous on $I \times R$ and that $f(t, x) \leq M$ for $t \in I$ and $x \in R$. Also, let $f(t, x_1) \geq f(t, x_2)$ for $t_i \leq t \leq t_{i+1}$ and $(-1)^{n+s(i)}(x_1 - x_2) \leq 0$. Then (15.1), (15.2) for $q = 0$ has a solution for all choices of $x_{i_j} \in R$.

<u>Proof.</u> If $t_i \leq t \leq t_{i+1}$ and $(-1)^{n+s(i)} x \leq 0$ then $f(t, x) \geq f(t, 0)$ so let $k = \min\{f(t, 0) ; t_1 \leq t \leq t_k\}$

Theorem 15.11. Assume $f(t,x)$ $(q=0)$ in (15.1) is continuous on $I \times R$ and that $f(t,x) \geq M$ for $t \in I$ and $x \in R$. Also, let $f(t,x) \leq K$ for t in (t_i, t_{i+1}) and x such that $(-1)^{n+s(i)} x \geq 0$. Then (15.1), (15.2), for $q = 0$ has a solution for all choices of $x_{ij} \in R$.

Proof. The proof of this theorem is similar to the proof of Theorem 15.9.

For a particular case of Theorems 15.9 and 15.11 see Schrader et al [56], Schrader [63].

Definition. The operator $L[x] = 0$ is said to be disfocal on I provided the Cauchy function $w(t,s)$ for $L[x]=0$ satisfies

$$w^{(i)}(t,s) > 0$$

for $i = 0, 1, \ldots, n-1$, $t > s$ with $t, s \in I$.

The operator $L[x]=0$ is said to be left disfocal on I provided

$$w^{(i)}(t,s) \neq 0$$

for $i = 0, 1, \ldots, n-1$, $t < s$ with $s, t \in I$.

Lemma 15.12 [64]. Assume

$$L[x] = q_1(t)\, x^{(n-1)} + \cdots + q_n(t)\, x \tag{15.23}$$

is disfocal {left disfocal} on I and $u(t)$ is a solution of $L[x] \geq q_1(t)x^{(n-1)} + \cdots + q_n(t)x$ on I. If $t_0 \in I$ and $x(t)$ is the solution of (15.23) satisfying the same initial conditions as $u(t)$ at t_0, then

$$u^{(i)}(t) \geq x^{(i)}(t), \quad t \geq t_0 \text{ in } I$$

$$\left\{(-1)^{n+i}u^{(i)}(t) \geq (-1)^{n+i}x^{(i)}(t), \ t \leq t_0 \text{ in } I\right\}$$

for $i = 0, 1, \cdots, n-1$.

Proof. We will prove only the first statement of the lemma. For $t \geq t_0$ in I set

$$h(t) = L[u(t)] - q_1(t)u^{(n-1)}(t) - \cdots - q_n(t)u(t) \geq 0.$$

Let $w(t, s)$ be the Cauchy function for (15.23), then, since $x(t)$ satisfies the same initial conditions as $u(t)$ at t_0,

$$u(t) = x(t) + \int_{t_0}^{t} w(t, s)\, h(s)\, ds.$$

It follows that

$$u^{(i)}(t) \geq x^{(i)}(t), \quad t \geq t_0 \text{ in } I.$$

Remark 15.13. Compare this result with Theorems 10.1 and 10.2.

Lemma 15.14. If (15.23) is disfocal $\{$left disfocal$\}$ on I and $q_i(t) \leq r_i(t)$ $\{(-1)^i q_i(t) \leq (-1)^i r_i\}$ on I, $i = 0, 1, \dots, n-$ then

$$L[x] = r_1(t)\, x^{(n-1)} + \cdots + r_n(t)\, x \qquad (15.24)$$

is disfocal $\{$ left disfocal$\}$ on I and if $t_0 \in I$, then

$$w^{(i)}(t, t_0; 15.24) \geq w^{(i)}(t, t_0;\ 15.23) > 0, \quad t > t_0 \text{ in } I$$

and

$$\{(-1)^{n+i+1} w^{(i)}(t, t_0; 15.24) \geq (-1)^{n+i+1} \times w^{(i)}(t, t_0; 15.23) > 0, \quad t < t_0 \text{ in } I$$

for $i = 0, 1, \dots, n-1$; , where $w(t, t_0; 15.23)$ and $w(t, t_0; 15.24)$ be the Cauchy functions at t_0 for equations (15.23) and (15.24) respectively.

Proof. Let $u(t) = w(t, t_0; 15.24)$ and $x(t) = w(t, t_0; 15.23)$ then the first inequality holds, by Lemma 15.12, therefore (15.24) is disfocal. The other claim of the Lemma is proved similarly.

Remark 15.15. Compare this result with Lemma 12.1.

Theorem 15.16 64 Assume $q = n-1$ and the function f in 15.1 satisfies the following Lipschitz condition

$$f(t, x_0, \ldots, x_i, \ldots, x_{n-1}) - f(t, x_0, \ldots, y_i, \ldots, x_{n-1}) \geq V_{n-i}(t)[x_i - y_i] \tag{15.25}$$

for $t \in I$, $x_i \geq y_i$, $i = 0, 1, \ldots, n-1$.

Let (15.23) be disfocal on $[a_1, a_2] \subseteq I$, then the $(n-1, 1)$ BVP (15.1).

$$x^{(i)}(a_1) = A_i$$

$$x(a_2) = B_1$$

has a solution. If, in addition, solutions of initial value problems for (15.1) are unique, then the above (n-1,1)-BVP has has a unique solution.

Proof. For $k = 1, 2, \ldots$, define the integral means $f_k = f_k(t, x_0, \ldots, x_{n-1})$ on $[a_1, a_2] \times R^n$ by

$$f_k = \frac{k^n}{2^n} \int_{x_{n-1}-\frac{1}{k}}^{x_{n-1}+\frac{1}{k}} \cdots \int_{x_0-\frac{1}{k}}^{x_0+\frac{1}{k}} f(t, s_0, \ldots, s_{n-1})\, ds_0 \ldots ds_{n-1}.$$

Note that $f_k \to f$ uniformly on compact subsets of $[a_1, a_2] \times R$ the functions f_k, $\frac{\partial f_k}{\partial x_i}$, $i = 0, 1, \ldots, n-1$; $k = 1, 2, \ldots$ are continuous on $[a_1, a_2] \times R^n$ and

$$\frac{\partial f_k}{\partial x_i} \geq V_{n-1}(t)$$

on $[a_1, a_2] \times R^n$, $i = 0, 1, \ldots, n-1$; $k = 1, 2, \ldots$.

Let $x_k(t, m)$ be the solution of the IVP

$$L[x] = f_k(t, x, \ldots, x^{(n-1)})$$

$$x^{(i)}(a_1) = A_i, \quad i = 0, 1, \ldots, n-2 \tag{15.26}$$

$$x^{(n-1)}(a_1) = m. \tag{15.27}$$

Assume $m > 0$, then by Theorem V-3.1 [28], there is an $\bar{m} \in (0, m)$ such that

$$x_k(a_2, m) - x_k(a_2, 0) = m \frac{\partial x_k}{\partial m}(a_2, \bar{m}) = m\, z_k(a_2, \bar{m})$$

where $z_k(t, \bar{m})$ is the solution of the IVP

$$L[z] = \sum_{i=0}^{n-1} \frac{\partial f_k}{\partial x_i}(t, x_k(t, \bar{m}), \ldots, x_k^{(n-1)}(t, \bar{m}))\, z^{(i)}$$

$$z^{(i)}(a_1) = 0, \quad i = 0, 1, \ldots, n-2$$

$$z^{(n-1)}(a_1) = 1.$$

By use of Lemma 15.14 we get that

$$x_k(a_2, m) - x_k(a_2, 0) \geq m\, w(a_2, a_1; 15.23)$$

where $w(t, a_1; 15.23)$ is the Cauchy function at a_1 for (15.23). It follows by succ ssive use of K mke's convergence Theorem [28, Theorem II-3.2] that for each $m > 0$;

$$x(a_r, m) - x(a_r, 0) \geq m\,w(a_r, a_1; 15.23),$$

where $x(t, m)$ is a fixed solution of (15.1), (15.26), (15.27 and $x(t, 0)$ is a solution of (15.1), (15.26), (15.27) (at this stage of the proof we are not assuming uniqueness of IVP's). Sinc
$w(a_r, a_1; 15.23) > 0$.

$$\lim_{t\to\infty} x(a_r, m) = \infty .$$

Similarly for each $m < 0$ there is a solution $x(t, m)$ of (15. (15.26)(15.27) such that

$$\lim_{t\to\infty} x(a_r, m) = -\infty .$$

Since $\{x(a_r, m); m \in R$ and $x(t, m)$ is now any solution of (15.1), (15.26), (15.27)$\}$ is connected the first statement of the theorem follows.

Now assume that solutions of IVPs for (15.1) are unique. Assume some (n-1,1) BVP has distinct solutions, then without loss of generality assume there are points $m_2 > m_1$ (since solutions of IVP's are unique) such that

$$x(a_r, m_2) = x(a_r, m_1), \tag{15.28}$$

It follows from (15.28) that there is an $t_1 > a_1$ such that

$$x^{(i)}(t, m_2) > x^{(i)}(t, m_1)$$

for $a_1 < t < t_1$, $i = 0, 1, \ldots, n-1$ and $x^{(n-1)}(t_1, m_2)$
$= x^{(n-1)}(t_1, m_1)$. Set

$$w(t) = \frac{x(t, m_2) - x(t, m_1)}{m_2 - m_1}.$$

By the one sided Lipschitz condition of f it is easy to see that

$$L[w(t)] \geq q_1(t)\, w^{(n-1)}(t) + \cdots + q_n(t)\, w(t)$$

on $[a_1, t_1]$.

Since (15.23) is disfocal on $[a_1, a_2]$ it follows from Lemma 15.12 that

$$w^{(n-1)}(t) = \frac{x^{(n-1)}(t, m_2) - x^{(n-1)}(t, m_1)}{m_2 - m_1}$$

$$\geq w^{(n-1)}(t, a_1; 15.23) > 0 \text{ on } [a_1, t_1).$$

This implies that $x^{(n-1)}(t_1, m_2) > x^{(n-1)}(t_1, m_1)$ which is a contradiction.

Theorem 15.17. Assume

$$(-1)^{n+i}\left[f(t, x_0, \ldots, x_i, \ldots, x_{n-1}) - f(t, x_0, \ldots, y_i, \ldots, x_{n-1})\right] \geq (-1)^{n+i} q_i(t)[x_i - y_i]$$

where $t \in [a_1, a_2]$, $x_i \geq y_i$, $i = 0, 1, \ldots, n-1$ and (15.23) is left disfocal on $(a_1, a_2]$. Then the $(1, n-1)$BVP (15.1)

$$x(a_1) = A_1$$

$$x^{(i)}(a_2) = B_i, \quad i = 0, 1, \ldots, n-2$$

has a solution. If, in addition, solutions of IVPs for (15.1) are unique, the above $(1, n-1)$ BVP has a unique solution.

The proof of the above theorem is similar to the proof of Theorem 15.6.

16. Three-Point Problems.

Theorem 16.1 [65] Suppose that

(H_1) for each $m \in R$ there exist solutions to each of the four BVPs (6.1) ($V = \ldots$) with boundary conditions

$$\begin{cases} x(a_1) = \lambda_1, \quad x^{(n-i)}(a_2) = m, \quad x^{(r-1)}(a_2) = \lambda_{n-1} \\ \qquad\qquad (i = 2, 3), \ n \geq 3 \\ x^{(j)}(a_2) = \lambda_{j+2} \ (j = 0, 1, \ldots, n-4) \text{ when } n > 3; \end{cases} \tag{16.1}$$

$$\begin{cases} x^{(n-1)}(a_2) = \lambda_{n-1}, \quad x^{(n-i)}(a_2) = m, \quad x(a_3) = \lambda_n \ (i = 2, 3), \\ \qquad\qquad n \geq 3 \\ x^{(j)}(a_2) = \lambda_{j+2} \ (j = 0, 1, \ldots, n-4) \text{ when } n > 3 \end{cases} \tag{16.2}$$

(H_2) for each $m \in R$ and each t_1 there exists at most one solution to each of the two BVPs defined by (6.1) (q=n-1) and

$$
\begin{aligned}
&x(a_1)=\lambda_1,\ x^{(n-2)}(t_1)=m,\ x^{(n-1)}(a_2)=\lambda_{n-1},\\
&\quad t_1\in(a_1,a_2],\ n\geq 3\\
&x^{(j)}(a_2)=\lambda_{j+2}\ (j=0,1,\cdots,n-4),\ n>3
\end{aligned}
\tag{16.3}
$$

$$
\begin{aligned}
&x^{(n-1)}(a_2)=\lambda_{n-1},\ x^{(n-2)}(t_1)=m,\ x(a_3)=\lambda_n\\
&\quad t_1\in[a_2,a_3),\ n\geq 3\\
&x^{(j)}(a_2)=\lambda_{j+2}\ (j=0,1,\cdots,n-4),\ n>3\ .
\end{aligned}
\tag{16.4}
$$

Then there exists a unique solution to the BVP (6.1) (q=n-1) with boundary conditions

$$
\begin{aligned}
&x(a_1)=\lambda_1,\ x^{(n-1)}(a_2)=\lambda_{n-1},\ x(a_3)=\lambda_3,\ n\geq 3,\\
&x^{(j)}(a_2)=\lambda_{j+2}\ (j=0,1,\cdots,n-4),\ n>3\ .
\end{aligned}
\tag{16.5}
$$

Proof. By taking $t_1=a_2$ in (H_2) we see that respective solutions $x_1(t,m)$ and $x_2(t,m)$ of the BVPs (6.1), (16.1) and (6.1), $(16.2)_2$ exist and are unique. We first show that $x_1^{(n-3)}(a_2,m)$ is continuous and a strictly increasing function of m and its range is the set of all real numbers.

Let $m_2>m_1$ and consider $z(t)=x_1(t,m_2)-x_1(t,m_1)$. Now $z^{(n-2)}(t)>0$ for all $t\in(a_1,a_2]$ sincd otherwise $z^{(n-2)}(p)=0$ for some $p\in(\alpha,\beta)$ which contradicts (H_2). Also $z(a_1)=0$, $z^{(n-2)}(t)>0,\ t\in(a_1,a_2]$ $n\geq 3$ and $z^{(j)}(a_2)=0\ (j=0,1,\cdots,n-4)$, $n>3$ imply that $z^{(n-3)}(a_2)>$. Hence $x_1^{(n-3)}(a_2,m)$ is a strictly increasing function of m.

Suppose $x_1^{(n-3)}(a_2, m)$ has a jump discontinuity at $m = m_1$ such that $x_1^{(n-3)}(a_2, m_1^-) = a$, $x_1^{(n-3)}(a_2, m_1) = b$ and $x_1^{(n-3)}(a_2, m_1^+) = c$, where monotonicity asserts that $a \leq b \leq c$, $a < c$. Let b_1 be a real number different from b such that $a < b_1 < c$, and consider the solution $X(t)$ of the problem (6.1), $(16.1)_3$ where $X^{(n-3)}(a_2) = b_1$. By (H_1), $X(t)$ and all its derivatives through the nth order exist and are well defined in $[a_1, a_2]$. In particular $X^{(n-2)}(a_2)$ exists and has a real value, say, k. Then $X(t)$ is identical with $x_1(t, k)$ of (6.1), $(16.1)_2$ with m = k and therefore $x_1^{(n-3)}(a_2, k) =$ which is impossible. Thus $x_1^{(n-3)}(a_2, m)$ is a strictly increasing continuous function of m.

To prove that $x_1^{(n-3)}(a_2, m)$ has as its rage the set of all reals, let us assume that for all real m, $x_1^{(n-3)}(a_2, m)$ is bounded above, that is, $x_1^{(n-3)}(a_2, m) \leq M < \infty$. From (H_1), the two-point problem (6.1), $(16.1)_3$ with $m = M+1$ has a solution $X(t)$ such that $X^{(n-3)}(a_2) = M+1$. If we set $X^{(n-2)}(a_2) = k$, we find, as before, that $x_1^{(n-3)}(a_2, k) = M+1$ which contradicts our assumption on the upper bound. Similarly, $x_1^{(n-3)}(a_2, m)$ is not bounded below either.

An exact parallel treatment shows that $x_2^{(n-3)}(a_2, m)$ is a strictly decreasing continuous function of m, the range being the set of all reals. Consequently, there exists a unique such that $x_1^{(n-3)}(a_2, m_0) = x_2^{(n-3)}(a_2, m_0)$.

Thus, $x(t)$ defined as

$$x(t) = x_1(t, m_0), \quad t \in [a_1, a_2]$$
$$= x_2(t, m_0), \quad t \in [a_2, a_3] \qquad (16.6)$$

where $x_1^{(n-2)}(a_2, m_0) = x_2^{(n-2)}(a_2, m_0) = m_0$, is a solution of the BVP (6-1), (16.5).

To establish uniqueness, suppose y(t) is another solution distinct from $x(t)$ in (16.6). Let the restrictions of $y(t)$ to the subintervals $[a_1, a_2]$ and $[a_2, a_3]$ be labelled $y_1(t)$ and $y_2(t)$ respectively. Then, from hypothesis (H_2), $y_1(t) \equiv x_1(t, m^*)$ and $y_2(t) \equiv x_2(t, m^*)$ where $m^* = x^{(n-2)}(a_2)$. If $m^* > m_0$, the preceding proof implies that

$$y_1^{(n-3)}(a_2) = x_1^{(n-3)}(a_2, m^*) > x_1^{(n-3)}(a_2, m_0)$$
$$= x_2^{(n-3)}(a_2, m_0) > x_2^{(n-2)}(a_2, m^*)$$
$$= y_2^{(n-3)}(a_2)$$

which is a contradiction. Thus m^* cannot be greater than m_0 and likewise m^* cannot be less than m_0. Hence $m^* = m_0$, that is, $y(t) = x(t)$, which proves the uniqueness of $x(t)$.

Theorem 16.2. [65] Let $\mu, \nu \in \{0, 1\}$. For specific values of μ and ν suppose that

(H_1) for each $m \in R$ there exist solutions to each of the BVPs defined by (6.1) (q = n-1) and the following boundary conditions:

$$x^{(\mu)}(a_1) = \lambda_1, \quad x^{(j)}(a_2) = \lambda_{j+2} \quad (j = 0, 1, \ldots, n-3),$$
$$x^{(n-i)}(a_2) = m \quad (i = 1, 2) \tag{16.7$_i$}$$

$$x^{(j)}(a_2) = \lambda_{j+2} \quad (j = 0, 1, \ldots, n-3), \quad x^{(n-i)}(a_2) = m \tag{16.8$_i$}$$
$$x^{(\nu)}(a_3) = \lambda_n \quad (i = 1, 2)$$

(H_2) for each $m \in R$ and each t_1 there exists at most one solution to each of the two BVPs given by (6.1) (q = n-1) and the conditions

$$x^{(\mu)}(a_1) = \lambda_1, \quad x^{(n-1)}(t_1) = m,$$
$$x^{(j)}(a_2) = \lambda_{j+2} \quad (j = 0, 1, \ldots, n-3), \quad t_1 \in (a_1, a_2] \tag{16.9}$$

$$x^{(j)}(a_2) = \lambda_{j+2} \quad (j = 0, 1, \ldots, n-3)$$
$$x^{(n-1)}(t_1) = m, \quad x^{(\nu)}(a_3) = \lambda_n, \quad t_1 \in [a_2, a_3). \tag{16.10}$$

Then, there exists a unique solution to the BVP (6.1)(q=n-1) with boundary conditions

$$x^{(\mu)}(a_1) = \lambda_1, \quad x^{(j)}(a_2) = \lambda_{j+2} \quad (j = 0, 1, \ldots, n-3),$$
$$x^{(\nu)}(a_3) = \lambda_n. \tag{16.11}$$

The proof of this theorem is similar to the proof of Theorem 16.1. Also for $\mu = \gamma = 0$, $n = 3$, theorem 16.2 covers theorem 5.1 of [66] as a special case.

Theorem 16.3 [65] Suppose that (A_1) for equation (6.1) (q = n-1) initial value problems have at most one solution on each of the subintervals $[a_1, a_2]$ and $[a_2, a_3]$, (A_2) the function $f(t, u_1, u_2, \ldots, u_n)$ satisfies the following monotonicity conditions:

$$u_{n-1} < v_{n-1}, \quad (-1)^{n-j}(u_j - v_j) \geq 0 \ (j = 1, 2, \ldots, n-2)$$
$$\text{implies} \tag{16.12}$$
$$f(t, u_1, u_2, \ldots, u_{n-1}, u_n) < f(t, v_1, v_2, \ldots, v_{n-1}, u_n)$$
$$t \in (a_1, a_2]$$

and

$$u_{n-1} < v_{n-1}, \quad u_j \leq v_j \ (j = 1, 2, \ldots, n-2)$$
implies

$$f(t, u_1, u_2, \ldots, u_{n-1}, u_n) <$$
$$f(t, v_1, v_2, \ldots, v_{n-1}, u_n), \quad t \in [a_2, a_3). \tag{16.13}$$

Then, for each $m \in R$ and for $\mu, \nu \in \{0, 1\}$, there is at most one solution to each of the eight problems (6.1), $(16.7)_i$ and (6.1), $(16.8)_i$ (i = 1,2) for specified values of μ and γ.

Proof. The uniqueness of solution to (6.1), $(16.7)_1$ under the above hypothesis will be proved. The remaining problems can be treated in an analogous manner.

Suppose, for a given value of μ and a fixed m, there exist two distinct solutions $x_1(t)$ and $x_2(t)$ to (3.1), $(16.7)_1$. Let $w(t) = x_1(t) - x_2(t)$. Then

$$w^{(\mu)}(a_1) = w^{(n-1)}(a_2) = w^{(j)}(a_2) = 0$$
$$(j = 0, 1, \ldots, n-3). \tag{16.14}$$

Since by hypothesis (A_1), $w^{(n-2)}(a_2) \neq 0$, let us assume, without loss of generality, that $w^{(n-2)}(a_2) > 0$. This, together with (16.14) implies that there exists $r \in [a_1, a_2)$ such that $w^{(n-2)}(r) = 0$ and $w^{(n-2)}(t) > 0$ for all $t \in (r, a_2]$. We can therefore find suitable p and q, $p \in [r, a_2)$ and $q \in (p, a_2]$ such that $w^{(n-1)}(q) = 0$ and $w^{(n-1)}(t) > 0$ whenever $t \in (p, q)$. Hence

$$w^{(n)}(q) = \lim_{t \to \bar{q}} \frac{w^{(n-1)}(t) - w^{(n-1)}(q)}{t - q} \leq 0.$$

Now, $w^{(n-2)}(t) > 0$ in $(r, a_2]$ implies $w^{(n-2)}(q) >$ and also that $w^{(n-3)}(t)$ increases in $(r, a_2]$. Since $w^{(n-3)}(a_2) = 0$, this yields $w^{(n-3)}(t) \leq 0$ for all $t \in (r, a_2]$ and, in particular, $w^{(n-3)}(q) \leq 0$. Continuing this reasoning, we see that

$$w^{(n-3)}(t) = 0, \quad w^{(m-2)}(q) > 0, \quad (-1)^{n-j} w^{(j)}(q) \geq 0$$
$$(j = 0, 1, \ldots, n-3).$$

By (16.12), this requires $w^{(n)}(\gamma) > 0$ which contradicts the earlier evaluation. Thus $w(t) \equiv 0$ which proves the uniqueness of the solution.

<u>Theorem 16.4</u> [65] Suppose that the hypotheses of Theorem 16.3 are satisfied. If, for specified values of μ and γ and for every $m \in R$ solutions exist for each of the four two-point problems (6.1), $(16.7)_i$; (6.1), $(16.8)_i$ (i = 1,2), then a unique solution exists for the three point problem defined by (6.1), (16.11).

<u>Proof</u>. For a given m, the hypotheses of the present theorem assures us the existence of solutions to problems with boundary conditions $(16.7)_i$, $(16.8)_i$ (i = 1,2), whereas theorem 16.3 states that the solution to each of these four problems is unique. Let $x_1(t, m)$ denote the solution corresponding to the boundary condition $(16.7)_1$ and let us define $w(t) = x_1(t, m_2) - x_1(t, m_1)$ where $m_2 > m_1$. Then,

$$\begin{gathered} w^{(\mu)}(a_1) = w^{(j)}(a_2) = 0 \quad (j = 0, 1, \cdots, n-3), \\ w^{(n-1)}(a_2) > 0 . \end{gathered} \tag{16.15}$$

Now two possibilities arise.

Case (i). Suppose $w^{(n-1)}(t) > 0$ for all $t \in [a_1, a_2]$. Then, $w^{(n-2)}(t)$ increases with t in $[a_1, a_2]$. Further, (16.15) implies that there exists $p \in [a_1, a_2)$ such that $w^{(n-2)}(p) = 0$. Therefore, $w^{(n-2)}(t) > 0$ for all $t \in (p, a_2]$.

Case (ii). There exists $q \in [a_1, a_2)$ such that $w^{(n-1)}(q)=0$ and $w^{(n-1)}(t)>0$ for all $t \in (q, a_2]$. We now show that there exists $p \in [q, a_2)$ such that $w^{(n-2)}(p) \geq 0$ and consequently $w^{(n-2)}(t) > 0$ for all $t \in (p, a_2]$. Suppose it were not true, that is, $w^{(n-2)}(t) < 0$ for $t \in [q, a_2)$. This would imply that $w^{(n-2)}(q) < 0$ and that $w^{(n-3)}(t)$ is decreasing in $[q, a_2)$ and, in particular, $w^{(n-3)}(q) > 0$.

Following this line of argument, we see that

$$w^{(n-1)}(q)=0, \quad w^{(n-2)}(q)<0, \quad (-1)^j w^{(n-j)}(q) < 0$$
$$(j = 0, 1, \ldots, n-3).$$

(16.12) now implies that $w^{(n)}(q)<0$ which contradicts
The monotonicity condition the
evaluation of $w^{(n)}(q)$ by the limit

$$w^{(n)}(q) = \lim_{t \to q^+} \frac{w^{(n-1)}(t) - w^{(n-1)}(q)}{t-q} \geq 0.$$

Thus, in either case there exists $p \in [a_1, a_2)$ such that $w^{(n-2)}(t) > 0$ for all $t \in (p, a_2]$. This results in $w^{(n-2)}(a_2) > 0$ and hence $x_1^{(n-2)}(a_2, m)$ strictly increases with m. Similarly, if $x_2(t, m)$ is the solution of the two-point problem (6.1), $(16.8)_1$, then $x_2^{(n-2)}(a_2, m)$ can be shown to be a strictly decreasing function of m. The rest of the proof follows exactly as that for Theorem 16.2.

Theorem 16.5 [65] Suppose that the hypotheses of Theorem 16.3 are satisfied. Also, let

(A_1) for all $m \in R$, solutions exist for the four BVPs (6.1), $(16.1)_i$; (6.1), $(16.2)_i$ $(i = 2, 3)$;

(A_2) for all $m \in R$, there exists at most one solution to each of the two BVPs (6.1), $(16.1)_2$; (6.1), $(16.2)_2$.

Then, there exists a unique solution to he problem (6.1), (16.5).

The proof of this theorem is similar to proof of theorem 16.4. Similar results for the third order systems have been obtained recently in [67].

Theorem 16.6. Let $x(t) \in C^{(n)}[a_1, a_3]$ satisfying

$$x'(a_1) = 0, \quad x^{(j)}(a_2) = 0, \quad j = 0, 1, \ldots, n-2 \tag{16.16}$$

or

$$x^{(j)}(a_2) = 0, \quad j = 0, 1, \ldots, n-2; \quad x'(a_3) = 0 \tag{16.17}$$

or

$$x'(a_1) = 0, \quad x^{(j)}(a_2) = 0, \quad j = 0, 1, \ldots, n-3; \quad x'(a_3) = 0. \tag{16.18}$$

Then

$$|x^{(k)}(t)| \le \alpha_{n,k}\, m_i\, h_i^{(n-k)} \quad (i = 1, 2, 3), \quad k = 0, 1, \ldots, n-1 \tag{16.19}$$

where $h_1 = a_2 - a_1$ for (16.16); $h_2 = a_3 - a_2$ for (16.17); $h_3 = a_3 - a_1$ for (16.18) and m_i is the maximum of $|x^{(n)}(t)|$

in the corresponding interval. Also

$$\alpha_{n,0} = \frac{1}{(n-1)\ \ n!}\ , \ \alpha_{n,1} = \frac{1}{(n-1)!}\ \frac{(n-2)^{n-2}}{(n-1)^{n-1}} \tag{16.20}$$

$$\alpha_{n,k+1} = \frac{k}{(n-1)\ (n-k-1)!}$$

$$k = 1, 2, \dots, n-2.$$

<u>Proof</u> . We shall prove only for (16.13) and for (16.16) and (16.17) it will follow similarly. We have as in the proof of Theorem 5.3

$$|x'(t)| \le m_3 \frac{1}{(n-1)!} (t-a_1)\,|t-a_2|^{n-3} (a_3-t).$$

Now integrating the above inequality from a_2 to t $(a_2 \le t)$ we find on using $x(a_2) = 0$

$$|x(t)| = \left| \int_{a_2}^{t} x'(s)\,ds \right| \le \int_{a_2}^{t} |x'(s)|\,ds$$

$$\le m_3 \frac{1}{(n-1)!} \int_{a_2}^{t} (s-a_1)\,(s-a_2)^{n-3} (a_3-s)\,ds$$

$$\le m_3 \frac{1}{(n-1)!} \int_{a_1}^{a_3} (s-a_1)^{n-2} (a_3-s)\,ds$$

and similarly for $(a_2 \ge t)$. This proves for $k=0$. Now for $k \ge 1$ define $f(t) = x'(t)$, then $f(t)$ satisfies the hypotheses of theorem 5.2 and the result follows.

The constants $\alpha_{n,k}$ $(k = 0, 1, \ldots, n-1)$ are the best possible, for (16.16) and (16.17), as they are exact for the functions

$$x_1(t) = (a_2 - t)^{n-1}\left[\frac{a_2 - a_1}{n-1} - \frac{a_2 - t}{n}\right]$$

$$x_2(t) = (t - a_2)^{n-1}\left[\frac{a_3 - a_2}{n-1} - \frac{t - a_2}{n}\right]$$

and only for those functions, upto a constant factor.

<u>Theorem 16.7</u>. Let $x(t) \in C^{(n)}[a_1, a_3]$ satisfying

$$x(a_1) = 0, \quad x^{(j)}(a_2) = 0, \quad j = 0, 1, \ldots, n-3;$$
$$x^{(n-1)}(a_2) = 0 \qquad (16.21)$$

or

$$x^{(j)}(a_2) = 0, \quad j = 0, 1, \ldots, n-3; \quad x^{(n-1)}(a_2) = 0,$$
$$x(a_3) = 0. \qquad (16.22)$$

Then

$$|x^{(k)}(t)| \leq \beta_{n,k}\, m_i\, h_i^{n-k} \quad (i = 1, 2)$$

where h_1, h_2, m_1 and m_2 are same as in Theorem 16.6. Also,

$$\beta_{n,0} = \frac{1}{n!}\,\frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}$$

$$\beta_{n,k} = \frac{k}{(n-1)(n-k-1)!}, \quad k = 1, 2, \ldots, n-3$$

$$\beta_{n,n-2} = \frac{1}{2} - \frac{1}{n(n-1)}, \quad \beta_{n,n-1} = 1. \qquad (16.23)$$

Proof. We shall prove for (16.22) and for (16.21) it will follow similarly. The function $x(t)$ satisfying (16.22) can be written as

$$x(t) = \frac{1}{(n-1)!}\left[\int_{a_2}^{t} - \left\{ \frac{(t-a_2)^{n-2}(a_3-s)^{n-1}}{(a_3-a_2)^{n-2}} - (t-s)^{n-1}\right\} \times\right.$$

$$\left. x^{(n)}(s)\,ds + \int_{t}^{a_3} - \frac{(t-a_2)^{n-2}(a_3-s)^{n-1}}{(a_3-a_2)^{n-2}}\, x^{(n)}(s)\,ds \right]$$

Thus, we find that

$$|x(t)| \le m_2 \frac{1}{n!}\left[(t-a_2)^{n-2}(a_3-a_2)^2 - (t-a_2)^n\right]$$

$$\le m_2 \frac{1}{n!}\,\frac{2}{n}\left(\frac{n-2}{n}\right)^{\frac{n-2}{2}}(a_3-a_2)^n.$$

Hence, the result follows for $k=0$, also for $k=n-1$ the result is immediate, so now we shall prove for $k=n-2$.
In fact, we have from the above integral representation

$$x^{(n-2)}(t) = \int_{a_2}^{t} - \left[\frac{(a_3-s)^{n-1}}{(n-1)(a_3-a_2)^{n-2}} - (t-s)\right] x^{(n)}(s)\,ds$$

$$+ \int_{t}^{a_3} - \frac{(a_3-s)^{n-1}}{(n-1)(a_3-a_2)^{n-2}}\, x^{(n)}(s)\,ds .$$

It can be shown by elementary calculations that $(a_3 - s)^{n-1} - (n-1)(a_3 - a_2)^{n-2}(t-s)$ is nonnegative if and has only one zero in $[a_2, t]$ if $t \leq a_2 + \frac{a_3 - a_2}{n-1}$. Hence, we find for $a_2 + \frac{a_3-a_2}{n-1} \leq$

$$|x^{(n-2)}(t)| \leq m_2\left[\frac{1}{n(n-1)}(a_3-a_2)^2 - \frac{1}{2}(t-a_2)^2\right]$$

$$\leq m_2\left[\frac{1}{2} - \frac{1}{n(n-1)}\right](a_3-a_2)^2.$$

If $t \geq a_2 + \frac{a_3 - a_2}{n-1}$, then we find

$$|x^{(n-2)}(t)| \leq m_2\left[\int_{a_2}^{t_1}\left[(t-s) - \frac{1}{(n-1)}\frac{(a_3-s)^{n-1}}{(a_3-a_2)^{n-2}}\right]ds\right.$$

$$+ \int_{t_1}^{t} - \left[(t-s) - \frac{1}{(n-1)}\frac{(a_3-s)^{n-1}}{(a_3-a_2)^{n-2}}\right]ds$$

$$\left.+ \int_{t}^{a_3}\frac{1}{(n-1)}\frac{(a_3-s)^{n-1}}{(a_3-a_2)^{n-2}}ds\right]$$

$$= m_2\left[\frac{2(a_3-t_1)^n}{n(n-1)(a_3-a_2)^{n-2}} - \frac{1}{n(n-1)}(a_3-a_2)^2\right.$$

$$\left. - (t-t_1)^2 + \frac{1}{2}(t-a_2)^2\right]$$

$$= m_2 f(t, t_1(t)) \text{ (say)}$$

where

$$\frac{(a_3 - t_1)^{n-1}}{(n-1)(a_3 - a_2)^{n-2}} = (t - t_1).$$

Now, it is easy to verify that the maximum value of $f(t, t_1(t))$ attains at $t = a_3$ and then $t_1 = a_3$, also the maximum value is $\beta_{n,n-2} (a_3 - a_2)^2$.

To prove for $k = 1, 2, \ldots, n-3$ we note that $x(t)$ satisfies $x^{(j)}(a_2) = 0$, $j = 0, 1, \ldots, n-3$; $x(a_3) = 0$, thus on using Theorem 5.2, we find

$$|x^{(k)}(t)| \leq \frac{k}{(n-1)\quad (n-k-1)!} (a_3 - a_2)^{n-k-1} \times$$

$$\max_{a_2 \leq t \leq a_3} |x^{(n-1)}(t)|.$$

Since $|x^{(n-1)}(t)| \leq (a_3 - a_2)\, m_2$, the result follows.

The constants $\beta_{n,k}$ $(k = 0, n-2, n-1)$ are the best possible as they are exact for the functions

$$x_1(t) = (a_2 - t)^{n-2} \left[(a_2 - a_1)^2 - (a_2 - t)^2\right]$$

$$x_2(t) = (a_2 - t)^{n-2} \left[(a_3 - a_2)^2 - (a_2 - t)^2\right].$$

<u>Theorem 16.8</u> (a) Let $x(t) \in C^{(n)}[a_1, a_3]$ satisfying

$$x'(a_1) = 0, \; x^{(j)}(a_2) = 0, \; j = 0, 1, \ldots, n-3; \qquad (16.24)$$
$$x^{(n-1)}(a_2) = 0$$

or

$$x^{(j)}(a_2) = 0, \; j = 0, 1, \ldots, n-3; \; x^{(n-1)}(a_2) = 0 \qquad (16.25)$$
$$x'(a_3) = 0.$$

Then,

$$|x^{(k)}(t)| \leq \gamma_{n,k} \, m_i \, h_i^{n-k} \quad (i = 1, 2) \qquad (16.26)$$

where

$$\gamma_{n,0} = \frac{2}{(n-2)\; n!}, \quad \gamma_{n,1} = \frac{(n-3)^{n-3}}{(n-2)!\;(n-2)^{n-2}}$$

$$\gamma_{n,k} = \frac{k}{(n-2)\;(n-k-2)!}, \quad k = 2, 3, \ldots, n-3$$

$$\gamma_{n,n-2} = \frac{1}{2} - \frac{1}{(n-1)\,(n-2)} \text{ if } n > 3 \text{ and } \frac{1}{2} \text{ if } n = 3, \qquad (16.27)$$

$$\gamma_{n,n-1} = 1.$$

The proof of this theorem is same as of Theorem 16.7. The constants $\gamma_{n,k}$ $(k = 0,\ n-2,\ n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (a_2 - t)^{n-2}\left[(a_2 - t)^2 - \frac{n}{(n-2)}(a_2 - a_1)^2\right]$$

$$x_2(t) = (t - a_2)^{n-2}\left[(t - a_2)^2 - \frac{n}{(n-2)}(a_3 - a_2)^2\right]$$

and only for these functions, upto a constant factor.

<u>Theorem 16.8</u> (b) Let $x(t) \in C^{(n)}[a_1, a_3]$ satisfying

$$x(a_1) = 0,\quad x^{(n-2)}(a_3) = 0,\quad x^{(n-1)}(a_2) = 0$$
$$x^{(j)}(a_2) = 0,\quad (j = 0, 1, \cdots, n-4) \text{ when } n > 3$$

or

$$x^{(n-\ )}(a_2) = 0,\quad x^{(n-\ )}(a_2) = 0,\quad x(a_3) = 0$$
$$x^{(j)}(a_2) = 0,\quad (j = 0, 1, \cdots, n-4) \text{ when } n > 3.$$

Then

$$|x^{(k)}(t)| \leq \delta_{n,k}\, m_i\, h_i^{n-k} \quad (i = 1, 2)$$

where

$$\delta_{n,0} = \frac{3}{n\, n!}\left(\frac{n-3}{n}\right)^{\frac{n-3}{3}}$$

$$\delta_{n,k} = \frac{k}{2(n-2)\,(n-k-2)!}, \quad k = 1, 2, 3, \ldots, n-3 \tag{16.27)'$$

$$\delta_{n,n-2} = \frac{1}{2}, \quad \delta_{n,n-1} = 1.$$

The proof of this theorem is same as of theorem 16.7. The constants $\delta_{n,k}$ $(k = 0, n-2, n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (a_2 - t)^{n-3}\left[(a_2 - a_1)^3 - (a_2 - t)^3\right]$$

$$x_2(t) = (t - a_2)^{n-3}\left[(a_3 - a_2)^3 - (t - a_2)^3\right]$$

and only for these functions, upto a constant factor.

<u>Theorem 16.9</u>. Let $x(t) \in C^{(3)}[a_1, a_3]$ satisfying

$$x(a_1) = 0, \quad x'(t_1) = 0, \quad x''(a_2) = 0, \quad t_1 \in (a_1, a_2] \tag{16.28}$$

or

$$x''(a_2) = 0, \quad x'(t_1) = 0, \quad x(a_3) = 0, \quad t_1 \in [a_2, a_3). \tag{16.29}$$

Then

$$|x(t| \leq \frac{1}{6}\, m_i\, h_i^3, \; |x'(t)| \leq \frac{1}{2}\, m_i\, h_i^2 \qquad (16.30)$$

$$|x''(t)| \leq m_i\, h_i$$

where

$$m_1 = \max_{a_1 \leq t \leq a_2} |x'''(t)| \quad \text{and} \quad m_2 = \max_{a_2 \leq t \leq a_3} |x'''(t)|.$$

Proof. We shall prove only for (16.28) and for (16.29) it follows similarly. Any function x(t) satisfying (16.28) can be written as

$$x(t) = \int_{a_1}^{t_1} \frac{1}{2}(s-a_1)^2\, x'''(s)\, ds + \int_{t_1}^{a_2} - \Big[\frac{1}{2}(t-a_1)^2$$

$$- (t-a_1)(t_1-a_1)\Big]\, x'''(s)\, ds - \frac{1}{2}\int_{t}^{t_1} (s-t)^2\, x'''(s)\, ds.$$

Now, it follows that for $a_1 \leq t \leq t_1$

$$|x(t)| \leq m_1 \Big[\frac{1}{6}(t_1-a_1)^3 - \frac{1}{6}(t_1-t)^3 + (t-a_1)(t_1-a_1)(a_2-$$

$$- \frac{1}{2}(t-a_1)^2 (a_2-t_1)\Big]$$

$$\leq \frac{m_1}{6}\Big[(t-a_1)^2 + 3(a_2-a_1)(a_2-t)\Big](t-a_1) \qquad (16,31)$$

$$\leq \frac{m_1}{6}(a_2-a_1)^3. \qquad (16.32)$$

Similarly, if $t_1 \leq t \leq 2t_1 - a_1 \leq a_2$ we get the same estimate as (16.31) and hence (16.32). If $2t_1 - a_1 \leq t \leq a_2$ then we find

$$|x(t)| \le m_1 \Big[\int_{a_1}^{t_1} \tfrac{1}{2}(s-a_1)^2\, ds + \int_{t_1}^{t} \Big| -\tfrac{1}{2}(t-a_1)^2$$

$$+ (t-a_1)(t_1-a_1) + \tfrac{1}{2}(s-t)^2 \Big|\, ds$$

$$+ \int_{t}^{a_2} \Big[\tfrac{1}{2}(t-a_1)^2 - (t-a_1)(t_1-a_1) \Big]\, ds \Big]$$

$$\le m_1 \Big[\tfrac{1}{6}(t_1-a_1)^3 + \tfrac{1}{6}(t-t_1)^3 + \tfrac{1}{2}(t-a_1)(t-2t_1+a_1) \times$$

$$(a_2+t_1-2t^*) - \tfrac{1}{3}(t-a_1)(t-2t_1+a_1)(t-t^*) \Big]$$

where

$$t^* = t - \sqrt{(t-a_1)(t-2t_1+a_1)} \; .$$

Thus, we find

$$|x(t)| \le m_1 \frac{(t-a_1)}{6} \Big[(t_1-a_1)^2 + (t-2t_1+a_1) \times$$

$$(3a_2+2t_1-4t^*-t) \Big]$$

$$\le m_1 \frac{(t-a_1)^2}{6} (3a_2-2a_1-t) \le m_1 \frac{(a_2-a_1)^3}{6}$$

and from this (16.32) follows for all $t \in [a_1, a_2]$.

The following estimates are easy to determine and proves (16.30),

$$|x'(t)| \le \frac{m_1}{2}(a_2-t)^2, \text{ if } a_1 \le t \le t_1$$

$$|x'(t)| \le \frac{m_1}{2}(t-a_1)(2a_2-t-a_1), \text{ if } t_1 \le t \le a_2$$

$$|x''(t)| \le m_1(a_2-t), \text{ if } a_1 \le t \le a_2 .$$

<u>Theorem 16.10.</u> Let $x(t) \in C^{(3)}[a_1, a_3]$ satisfying

$$x'(a_1) = 0, \quad x''(t_1) = 0, \quad x(a_2) = 0, \; t_1 \in (a_1, a_2] \tag{16.33}$$

or

$$x(a_2) = 0, \quad x''(t_1) = 0, \quad x'(a_3) = 0, \; t_1 \in [a_2, a_3). \tag{16.34}$$

Then

$$|x(t)| \leq \frac{1}{3} m_i h_i^3, \; |x'(t)| \leq \frac{1}{2} m_i h_i^2, \; |x''(t)| \leq m_i h_i \tag{16.35}$$

where m_1 and m_2 are defined in Theorem 16.9.

<u>Theorem 16.11.</u> Let $x(t) \in C^{(3)}[a_1, a_3]$ satisfying

$$x(a_1) = 0, \quad x''(t_1) = 0, \quad x(a_2) = 0, \; t_1 \in (a_1, a_2] \tag{16.36}$$

or

$$x(a_2) = 0, \quad x''(t_1) = 0, \quad x(a_3) = 0, \; t_2 \in [a_2, a_3). \tag{16.37}$$

Then

$$|x(t)| \leq \frac{\sqrt{3}}{27} m_i h_i^3, \; |x'(t)| \leq \frac{1}{3} m_i h_i^2, \quad |x''(t)| \leq m_i h_i \tag{16.38}$$

where m_1 and m_2 are defined in Theorem 16.9.

The proof of the above two theorems is same as of Theorem 16.9.

<u>Theorem 16.12</u>: Let $f(t, u_0, u_1, \ldots, u_{n-1})$ satisfy the Lipschitz condition (6.17). Then each of the BVPs (6.1), $(16.7)_2$ $(\mu = 1)$, (6.1), $(16.8)_2$ $(\gamma = 1)$ has a unique solution provided

$$\sum_{j=0}^{n-1} \alpha_{n,j} L_j h_i^{n-j} < 1. \quad i = 1, 2 \tag{16.39}$$

The $\alpha_{n,j}$ are defined in (16.20).

<u>Theorem 16.13</u>: Let $f(t, u_0, u_1, \ldots, u_{n-1})$ satisfy the Lipschitz condition (6.17). Then each of the BVPs (6.1), $(16.1)_3$; (6.1), $(16.2)_3$; (6.1), $(16.7)_1$ $(\mu = 0)$ (6.1), $(16.8)_1$ $(\gamma = 0)$ has a unique solution provided

$$\sum_{j=0}^{n-1} \beta_{n,j} L_j h_i^{n-j} < 1 \quad i = 1, 2. \tag{16.40}$$

The $\beta_{n,j}$ are defined in (16.23).

<u>Theorem 16.14</u>: Let $f(t, u_0, \ldots, u_{n-1})$ satisfy the Lipschitz condition (6.17). Then each of the BVPs (6.1), $(16.7)_1$ $(\mu = 1)$; (6.1), $(16.8)_1$ $(\gamma = 1)$ has a unique solution provided

$$\sum_{j=0}^{n-1} \gamma_{n,j} L_j h_i^{n-j} < 1. \quad i = 1, 2 \tag{16.41}$$

The $\gamma_{n,j}$ are defined in (16.27).

Theorem 16.15 Let $f(t, u_0, u_1, \ldots, u_{n-1})$ satisfy the Lipschitz condition (6.17). Then each of the BVPs (6.1), $(16.1)_2$; has a unique solution provided

$$\sum_{j=0}^{n-1} \delta_{n,j} L_j h_i^{n-j} < 1 . \qquad i = 1, 2 \tag{16.42}$$

The $\delta_{n,j}$ are defined in (16.27)'.

The proof of all the above results is similar to the proof of Theorem 6.4.

Theorem 16.16: Let the hypotheses (A_2) of Theorem 16.3 is satisfied and let $f(t, u_0, u_1, \ldots, u_{n-1})$ satisfy (6.17). Then there exists a unique solution to the BVP (6.1), (16.5) provided

$$\sum_{j=0}^{n-1} \delta_{n,j} L_j h_i^{n-j} < 1 . \qquad i = 1, 2 \tag{16.43}$$

Proof: First we note that $\beta_{n,k} \leq \delta_{n,k}$ and hence from theorems 16.13 and 16.15 each of the BVPs (6.1), $(16.1)_i'$; (6.1), $(16.2)_i'$ (i = 2,3) have at most one solution provided (16.43) is satisfied. Thus Theorem 16.5 proves the theorem.

Theorem 16.17: Let the hypotheses (A_2) of Theorem 16.3 is satisfied and let $f(t, u_0, u_1, \ldots, u_{n-1})$ satisfy (6.17). Then there exists a unique solution to the BVP (6.1), (16.11) provided.

$$\sum_{j=0}^{n-1} \beta_{n,j} L_j h_i^{n-j} < 1 \qquad (\mu = 0, \nu = 0) \quad i = 1, 2 \tag{16.44}$$

$$\sum_{j=0}^{n-1} \gamma_{n,j} L_j \ell_i^{n-j} < 1 \tag{16.45}$$

$(\mu = 1, \nu = 1$ or $\mu = 1, \nu = 0$ or $\mu = 0, \nu = 1)$
$i = 1, 2$

The proof of this theorem is an application of Theorem 16.4.

It should be noted that theorems 4.1-4.8 obtained in [65] are the particular cases of Theorem 16.4 and Theorems 16.12-16.17 In the case n = 3 which was considered in [65] in fact the condition (A_2) required in theorems 16.16-16.17 is not necessary, and hence theorems 4.6-4.8 of [65] can be modified.

Theorem 16.18: Let $f(t, u_0, u_1, u_2)$ satisfy (6.17). Then there exists a unique solution to the BVP: equation

$$x''' = f(t, x, x', x'') \tag{16.46}$$

with boundary conditions (16.28) or (16.29) (not necessarily zero conditions) provided

$$\frac{1}{6} L_0 \ell_i^3 + \frac{1}{2} L_1 \ell_i^2 + L_2 \ell_i < 1 \tag{16.47}$$

with boundary conditions (16.33) or (16.34) (not necessarily zero conditions) provided

$$\frac{1}{3} L_0 \ell_i^3 + \frac{1}{2} L_1 \ell_i^2 + L_2 \ell_i < 1 \tag{16.48}$$

with boundary conditions (16.36) or (16.37) (not necessarily zero conditions) provided.

$$\frac{\sqrt{3}}{27} L_0 \ell_i^3 + \frac{1}{3} L_1 \ell_i^2 + L_2 \ell_i < 1 . \qquad (16.49)$$
$$i = 1, 2$$

The proof is similar to that of Theorem 6.4 with the applications of Theorems 16.9-16.11.

Now an application of theorems 16.1 and 16.2 proves the following

Theorem 16.19: Let $f(t, u_0, u_1, u_2)$ satisfy (6.17). Then there exists a unique solution to the BVP (a) (16.46), (16.5) ($n=3$) provided (16.48) is satisfied (b) (16.46), (16.11) ($n=3$, $\mu = 1, \nu = 1$ or $\mu = 1, \nu = 0$ or $\mu = 0, \nu = 1$) provided (16.48) is satisfied (c) (16.46), (16.11) ($n = 3$, $\mu = 0 = \nu$) provided (16.49) is satisfied.

In fact for the case (c) we have shown recently that condition (16.49) can be improved further to

$$\frac{7}{160} L_0 \ell_i^3 + \frac{1}{6} L_1 \ell_i^2 + \frac{1}{2} L_2 \ell_i < 1 .$$

One can use a similar iterative procedure to improve the results in several other cases. It will be interesting to find whether condition (A_2) can be relaxed for the class of functions satisfying Lipschitz condition (6.17) in order to prove for any n.

17. Quasilinearisation

Quasilinearisation is a practical method to construct the solution of the nonlinear problems in an iterative way, the nonlinear problem is being reduced to solving a sequence of linear problems. This method has attracted considerable attention in recent years, for example see Bellman et al [69] Lee [70] Bernfeld et al [35], Kalaba [33], also for the systems and component-wise analysis see Agarwal [71].

Here, for the equation (6.1), we define an iterative scheme as follows

$$x_{m+1}^{(n)}(t) = f(t, x_m(t), \ldots, x_m^{(q)}(t)) + c \sum_{j=0}^{q} \left(x_{m+1}^{(j)}(t) - x_m^{(j)}(t) \right) p_j(t) \qquad m = 0, 1, \ldots \tag{17.1}$$

where

$$p_j(t) = \frac{\partial f(t, x_m(t), \ldots, x_m^{(q)}(t))}{\partial x_m^{(j)}(t)}$$

and c is any constant.

In (17.1), $x_0(t)$ is any function at least q times continuously differentiable and satisfy the boundary conditions (*) (we shall consider only this and for (**) or (***) we shall state the results directly). For each m the equation (17.1) is solved with the conditions that $x_{m+1}(t)$ satisfy (*). Thus, the problem (6.1), (*) is being reduced to solving the sequence of problems (17.1) satisfying (*), we shall denote this as $\{x_m(t)\}$.

It has been proved that the sequence $\{x_m(t)\}$ (even for more general boundary conditions) under certain conditions on f will actually exist and converge in a suitable space provided the length of the interval $a_r - a_1$ is sufficiently small. Here we shall give some lower bound on $a_r - a_1$.

We shall denote B as the Banach space $C^{(q)}[a_1, a_r]$ with the norm

$$\|x\| = \sum_{i=0}^{q} L_i \max_{a_1 \le t \le a_r} |x^{(i)}(t)| \tag{17.2}$$

and consider the closed, bounded subset B_1 of B such that $\|x - \ell_1\| \le 1$. In (17.2), the constants L_j $(j = 0, 1, \ldots, q)$ are defined as follows: for $t \in [a_1, a_r]$, $x(t) \in B_1$.

$$\left| \frac{\partial f(t, x(t), \ldots, x^{(q)}(t))}{\partial x^{(j)}(t)} \right| \le L_j \tag{17.3}$$

$$j = 0, 1, \ldots, q.$$

The maximum of $|f(t, x(t), \ldots, x^{(q)}(t))|$ over $[a_1, a_r] \times B$ we shall denote as L, also we will define $L^* = \max(L, 1)$.

<u>Theorem 17.1</u> [42] Let (i) $f(t, x(t), \ldots, x^{(q)}(t))$ be continuous over $[a_1, a_r] \times B_1$ and hence bounded by L

(ii) $\dfrac{\partial f}{\partial x^{(j)}(t)}(t, x(t), \ldots, x^{(q)}(t))$ exist and are continuous for all $j = 0, 1, \ldots, q$ on $[a_1, a_2] \times B_1$ and hence bounded by L_j. Then, if $k = (L^* + c)\theta / 1 - c\theta <$

$(1 > c\theta)$

where θ is defined in (6.7), the sequence generated by (17.1) with $\| x_0 - l_1 \| \leq 1$ converges uniformly in t to the unique solution of (6.1),(*) say x(t).

A bound on the error is given by

$$\| x_m - x \| \leq k^m (1-k)^{-1} \| x_1 - x_0 \| .$$

<u>Proof</u>: First we shall show that the sequence $\{ x_m(t) \}$ exists in B_1. We define an implicit operator T

$$Tx(t) = l_1(t) + \int_{a_1}^{a_2} g(t,s) \Big[f(s, x(s), \ldots, x^{(q)}(s)) + c \sum_{j=0}^{q} \big((Tx)^{(j)}(s) - x^{(j)}(s) \big) p_j(s) \Big] ds \qquad (17.4)$$

whose form is patterned on the integral equation representation of (17.1).

Since $x_0(t) \in B_1$, it is sufficient to show that, if $x(t) \in B_1$ then $Tx(t) \in B_1$ i.e. T is a mapping of B_1 into B_1. From (17.4) and theorem 5.3, we have

$$|(Tx)^{(k)}(t) - l_1^{(k)}(t)| \leq C^{**}_{n;k}\,(a_2-a_1)^{n-k} \times$$

$$\max_{a_1 \leq t \leq a_2}\Big[\,|f(t, x(t), \ldots, x^{(\nu)}(t))| + C\sum_{j=0}^{\nu}|(Tx)^{(j)}(t) - x^{(j)}(t)|\,|b_j(t)|\Big]$$

$$\leq C^{**}_{n,k}\,(a_2-a_1)^{n-k}\Big[L^* + C\{\|Tx - l_1\| + \|x - l_1\|\}\Big]$$

$$k = 0, 1, \ldots, \nu. \tag{17.5}$$

Multiplying (17.5) by L_k and summing over from $k=0$ to $k=\nu$ we obtain

$$\|Tx - l_1\| \leq \theta\,[\,L^* + C + C\,\|Tx - l_1\|\,]$$

and hence

$$\|Tx - l_1\| \leq \frac{(L^* + C)\,\theta}{1 - C\theta} = k.$$

Thus, the sequence $\{x_m(t)\}$ exists in B_1, provided $k \leq 1$.

Now, we shall show that $\{x_m(t)\}$ converges: For this, we have

$$x_{m+1}(t) - x_m(t) = \int_{a_1}^{a_2} g(t,s)\Big[f(s, x_m(s), \ldots, x_m^{(\nu)}(s)) + C\sum_{j=0}^{\nu}(x_{m+1}^{(j)}(s) - x_m^{(j)}(s))\,b_j(s) - f(s, x_{m-1}(s), \ldots$$

$$x_{m-1}^{(\nu)}(s)) - C\sum_{j=0}^{\nu}(x_m^{(j)}(s) - x_{m-1}^{(j)}(s))\,\nu_j(s)\Big]\,ds$$

where $v_j(s)$ is same as $p_j(s)$ replacing m by m-1.

Thus, using theorem 5.3, we obtain

$$|x_{m+1}^{(k)}(t) - x_m^{(k)}(t)| \leq C_{n,k}^{**}(a_2 - a_1)^{n-k} \times$$

$$\max_{a_1 \leq t \leq a_2} \Big[|f(t, x_m(t), \ldots, x_m^{(q)}(t)) - f(t, x_{m-1}(t),$$

$$\ldots, x_{m-1}^{(q)}(t))| + C \sum_{j=0}^{q} \Big\{ |x_{m+1}^{(j)}(t) - x_m^{(j)}(t)| \times$$

$$|p_j(t)| + |x_m^{(j)}(t) - x_{m-1}^{(j)}(t)| \, |v_j(t)| \Big\} \Big]$$

$$\leq C_{n,k}^{**}(a_2 - a_1)^{n-k} \Big[\|x_m - x_{m-1}\| +$$

$$C \|x_{m+1} - x_m\|$$

$$+ C \|x_m - x_{m-1}\| \Big] . \qquad (17.6)$$

Multiplying (17.6) by L_k and summing over from $k = 0$ to $k = q$, we obtain

$$\|x_{m+1} - x_m\| \leq \theta \Big[\|x_m - x_{m-1}\| (L^* + C) + C \|x_{m+1} - x_m\| \Big]$$

Hence, we find

$$\|x_{m+1} - x_m\| \leq k \|x_m - x_{m-1}\|$$

or

$$\|x_{m+1} - x_m\| \leq k^m \|x_1 - x_0\| .$$

Since , $k < 1$ the sequence $\{x_m(t)\}$ converges to the solution of (6.1), (*) say $x(t)$.

The error bound follows from the following triangular inequality

$$\|x_{m+p} - x_m\| \le \|x_{m+p} - x_{m+p-1}\| + \cdots + \|x_{m+1} - x_m\|$$

$$\le (k^{m+p-1} + k^{m+p-2} + \cdots + k^m)\|x_1 - x_0\|$$

$$\le k^m (1-k)^{-1} \|x_1 - x_0\|$$

and taking $p \to \infty$.

Remark 17.2: For the BVP (6.1), (**) or (6.1), (***) there is an analogous result to therorem 17.1. Actually the result follows on replacing $l_1(t)$ to $l_j(t)$ (j=2,3) in B_1 and θ to α in the Definition of k.

18. Equations with Deviating Arguments.

Here, we shall consider the following nth order differential equation with deviating arguments:

$$\begin{aligned} x^{(n)}(t) = f(t, x(t), x(t-\theta_0(t)), x'(t), \\ x'(t-\theta_1(t)), \ldots, x^{(q)}(t), x^{(q)}(t-\theta_q(t))) \end{aligned} \tag{18.1}$$

where f is a real-valued continuous function defined on $[a_1, a_2] \times R^{2q+2}$ and θ_i $(i = 0, 1, \cdots, q)$ are non-negative continuous functions with domain $[a_1, a_2]$.

Let, the initial function $\phi(t)$ be a $C^{(n-1)}$ function which is bounded together with its all derivatives upto order n-1 on $[a_0, a_1]$ where

$$a_0 = \min \left\{ \min_{a_1 \leq t \leq a_2} (t - \theta_i(t)),\ i = 0, 1, \cdots, q \right\}.$$

We now consider BVPs connected with equation (18.1) subject to boundary conditions (*) or (**) or (***). In all these problems we seek a function x(t), satisfying (18.1) and the concerned boundary conditions. For this, we define

$$l_j^*(t) = \begin{cases} \phi(t) & \text{if} \quad a_0 \leq t \leq a_1 \\ l_j(t) & \text{if} \quad a_1 \leq t \leq a_2 \end{cases} \tag{18.2}$$

and

$$G_j^*(t, s) = \begin{cases} G_j(t, s) & \text{if } a_1 \leq t \leq a_2 \\ 0 & \text{otherwise} \end{cases} \tag{18.3}$$

otherwise

where $l_j(t)$ and $G_j(t, s)$ are defined in (5.15).

Now, solving (18.1) (*) or (**) or (***) is equivalent to finding the solution of the following integral equation

$$x(t) = \overset{*}{l}_j(t) + \int_{a_1}^{a_r} G_j^*(t,s)\, f(s, \dot{x}(s), x(s-\theta_0(s)), \ldots, x^{(\nu)}(s), x^{(\nu)}(s-\theta_q(s)))\, ds. \qquad (18.4)$$

Theorem 18.1 [42] : Let $k_i > 0$, $i = 0, 1, \ldots, q$ be given real numbers and let Q be the maximum of $|f(t, u_0, u_1, \ldots, u_{2\nu+1})|$ on the compact set

$$\{(t, u_0, u_1, \ldots, u_{2\nu+1}) : a_1 \leq t \leq a_r, \; |u_i|, |u_{2i+1}| \leq 2k_i, \; i = 0, 1, \ldots, \nu\}.$$

Then, if

1. $\max\limits_{a_0 \leq t \leq a_r} |\overset{*}{l}{}_1^{(i)}(t)| \leq k_i$ and $(a_r - a_1) \leq \left(\dfrac{k_i}{Q\, C^{**}_{n,i}}\right)^{\frac{1}{n-i}}$

$i = 0, 1, \ldots, \nu$; the BVP (18.1), (*) has a solution.

2. $\max\limits_{a_0 \leq t \leq a_r} |\overset{*}{l}{}_j^{(i)}(t)| \leq k_i$, $j = 2$ (or 3) (18.6)

$$(a_r - a_1) \leq \left(k_i / Q\, \alpha_{n,i}\right)^{1/n-i}$$

$i = 0, 1, \ldots, q$ the BVP (18.1), (**) (or(***)) has a solution.

Proof. We shall prove 1, and for 2 it will follow analogously. Define $B[a_0, a_r]$ as in Theorem 6.1 and define a mapping

$$T : C^{(v)}[a_0, a_r] \to C^{(n-1)}[a_0, a_1] \cap C^{(n)}[a_1, a_r]$$

as follows

$$(Tx)(t) = \ell_1^*(t) + \int_{a_1}^{a_r} g^*(t,s)\, f(s, x(s), x(s-\theta_0(s)), \ldots, x^{(q)}(s), x^{(v)}(s-\theta_q(s)))\, ds. \tag{18.7}$$

The following properties of T may be easily established

(a) $(Tx)(t) = \phi(t)$, if $a_0 \le t \le a_1$

(b) $(Tx)(t)$ is n times continuously differentiable on $a_1 \le t \le a_r$

(c) $(Tx)^{(n)}(t) = f(t, x(t), x(t-\theta_0(t)), \ldots, x^{(q)}(t-\theta_q(t)))$, $\forall\, a_1 \le t \le a_r$

(d) $(Tx)(t) - \ell_1^*(t)$ satisfies conditions (3.8)

(e) fixed points of T are solutions of the BVP (18.1), (*)

(f) T is a continuous operator.

Now, we shall show that T maps $B[a_0, a_r]$ into itself. For this, if $a_0 \le t \le a_1$ then we have from the property (a) and (18.5) $\|(Tx)^{(i)}\| = \|\phi^{(i)}\| \le K_i$ and hence the conclusion. If $a_1 \le t \le a_r$, then from the property (c) we have $\|(Tx)^n - \ell_1^{(n)}\| = \|(Tx)^n\| \le Q$, also on using

Theorem 5.3, we find

$$|(Tx)^{(i)}(t)| \leq k_i + Q\, C^{**}_{n,i}\, (a_2 - a_1)^{n-i}$$
$$i = 0, 1, \ldots, q.$$

Thus, condition (18.5) implies that T maps $B[a_0, a_2]$ into itself. It then follows from the Schauder's fixed point theorem that T has a fixed point in $B[a_0, a_2]$.

Let $K_0, k_1, \ldots, k_q$ be given positive constants such that $|\phi^{(i)}(t)| \leq k_i$, $i = 0, 1, \ldots, q$ for all $a_0 \leq t \leq a_1$ and let D defined by

$$D = \{(u_0, u_1, \ldots, u_{2q+1}) : |u_{2i}|, |u_{2i+1}| \leq k_i$$
$$i = 0, 1, \ldots, q\}.$$

We shall assume that f satisfies the Lipschitz condition

$$|f(t, u_0, u_1, \ldots, u_{2q+1}) - f(t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{2q+1})|$$
$$\leq \sum_{i=0}^{2q+1} L_i |u_i - \bar{u}_i|$$

where L_i $(i = 0, 1, \ldots, 2q+1)$ are Lipschitz constants, for all $a_1 \leq t \leq a_2$ and $(u_0, u_1, \ldots, u_{2q+1})$, $(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{2q+1}) \in D$.

Theorem 18.2 [42] : Let $K_i > 0$, $i = 0, 1, \ldots, q$ be given positive constants such that $|\phi^{(i)}(t)| \le K_i$ for all $a_0 \le t \le a_1$. Let f satisfy the Lipschitz condition on $[a_1, a_2] \times D$. Then, if

1. $$\left(\sum_{i=0}^{2q+1} L_i\right) \max_{0 \le i \le q} \left\{ C^{**}_{n,i} (a_2 - a_1)^{n-i} \right\} < 1$$

the BVP(18.1), (*) has at most one solution x(t) with $|x^{(i)}(t)| \le K_i$, $i = 0, 1, \ldots, q$.

2. $$\left(\sum_{i=0}^{2q+1} L_i\right) \max_{0 \le i \le q} \left\{ \alpha_{n,i} (a_2 - a_1)^{n-i} \right\} < 1$$

the BVP (18.1), (**) as well as (18.1), (***) has at most one solution x(t) with $|x^{(i)}(t)| \le K_i$, $i = 0, 1, \ldots, q$.

Proof: We shall prove 1, and for 2 it follows analogously. Define M as the set of q times continuously differentiable functions on $[a_0, a_2]$ with the norm

$$\|x\| = \max_{0 \le i \le q} \left\{ \max_{a_0 \le t \le a_2} |x^{(i)}(t)| \right\}$$

Let us assume that there are two solutions $x_1(t)$ and $x_2(t)$ of the BVP (18.1), (*) with $|x_1^{(i)}(t)|, |x_2^{(i)}(t)| \leq k_i$, $i=0,1,\ldots,q$.

Then, we have

$$x_1(t) - x_2(t) = \int_{a_1}^{a_r} g^*(t,s)\Big[f(s, x_1(s), \ldots, x_1^{(q)}(s-\theta_q(s))) - f(s, x_2(s), \ldots, x_2^{(q)}(s-\theta_q(s)))\Big]\, ds.$$

Thus, $x_1(t) - x_2(t)$ satisfies the hypothesis of Theorem 5.3, and we have on using the Lipschitz condition over D

$$|x_1^{(k)}(t) - x_2^{(k)}(t)| \leq \Big[L_0 \max_{a_1 \leq t \leq a_r} |x_1(t) - x_2(t)| + L_1 \max_{a_1 \leq t \leq a_r} |x_1(t-\theta_0(t)) - x_2(t-\theta_0(t))| + \cdots + L_{2q+1} \max_{a_1 \leq t \leq a_r} |x_1^{(q)}(t-\theta_q(t)) - x_2^{(q)}(t-\theta_q(t))|\Big] \times C^{**}_{n,k}(a_r - a_1)^{n-k}$$

$$\leq C^{**}_{n,k}(a_r - a_1)^{n-k}\Big(\sum_{i=0}^{2q+1} L_i\Big)\|x_1 - x_2\|$$

$$k = 0,1,\ldots,q.$$

Thus, we have

$$\| x_1 - x_2 \| \leq \left(\sum_{i=0}^{2\nu+1} L_i \right) \max_{0 \leq k \leq \nu} \left\{ C^{**}_{n,k} (a_i - a_1)^{n-k} \right\} \times \| x_1 - x_2 \|,$$

which is same as $\| x_1 - x_2 \| < \| x_1 - x_2 \|$. This contradiction proves the result.

For n = 2 the results similar to that of Theorem 18.1 and 18.2 were obtained by Grimm et al [72] [73] Jain et al [74] and for n ≥ 3 see Agarwal [49] , [75] where the nested type of delays were also considered.

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1. L.Collatz, "The numerical treatment of differential equations", 3rd ed. Springer, Berlin, 1960.

2. G.H.Meyer, "Initial value methods for boundary value problems", Academic Press, New York, 1973.

3. W.Walter, "Differential and integral inequalities", Springer Berlin, 1970.

4. R.E.Kidder, Unsteady flow of gas through a semi-infinite porous medium, J. Appl. Mech. 27 (1957), 329-332.

5. P.Bailey, L.Shampine and P.Waltman, "Nonlinear two point boundary value problems", Academic Press, New York, 1968.

6. E.Picard, Sur l' application des methodes d'approximations successives a l'etude de certaines equations differentielles ordinaires. J. Math. 9 (1893), 217-271.

7. E.Picard, "Traite d' Analyse". Gauthier-Villars, Paris, 1929.

8. F.Lettenmeyer, Uber die von einem Punkt ausgehenden Integralkurven einer Differentialgleichung 2. Ordnung. Deutsche Math. 7 (1944), 56-74.

9. Z.Opial, Sur une inegalite' C. de la Vallee Poussin dans la theorie de l'equation differentielle du second ordre. Ann. Polon. Math. 6 (1959), 87-91.

10. P.Bailey and P.Waltman, On the distance between consecutive zeros for second order differential equations, J. Math. Anal. Appl. 14 (1966), 23-30.

11. A.Yu. Levin, Some problems bearing on the oscillation of solutions of linear differential equations. Soviet Math. Dokl. 4 (1963), 121-124.

12. Ju.V.Pokornyi, Some estimates of the Green's function of a multi-point boundary value problem. Mat. Zametki. 4 (1968), 533-540.

13. W.A.Coppel, "Disconjugacy". Lecture notes in mathematics, Springer, New York, 1971.

14. P.R.Beesqck, On the Green's function of an N-point boundary value problem, Pacific J. Math. 12 (1962), 801-812.

15. Z.Nehari, On an inequality of P.R.Beesack, Pacific J. Math. 14 (1964), 261-263

16. R.P.Agarwal, An identity for Green's function of multipoint boundary value problems, Proc. Tamil Nadu Acad. Sci. 2, (1978) 41-43.

17. K.M.Das and A.S.Vatsala, On Green's function of an n-point boundary value problem, Trans. AMS, 182 (1973), 469-480.

18. R.P.Agarwal and P.R.Krishnamoorthy, Boundary value problems for n-th order differential equations (to appear in Bulletin of the institute of mathematics academia Sinica, Republic of China, 1979)

19. R.P.Agarwal and U.N.Srivastava, Generalised two-point boundary value problems, J.Mathl. Phyl. Sci. 10 (1976), 367-373.

20. P.Waltman, A nonlinear boundary value problem, J. Diff. Equs. 4 (1968), 597-603.

21. J.W.Bebernes and R.Gaines, A generalised two-point boundary value problem. Proc. AMS. 19 (1968) 749-754.

22. ______________________, Dependence on boundary data and a generalised boundary value problem. J.Diff.Eqns. 4 (1968) 359-368

23. A.Lasota, Boundary value problems for second order differential equations, Seminar on Differential Equations and Dynamic systems II, Lecture Notes in Math. 144, Springer, New York (1970) 140-152

24. K.Schmitt, A nonlinear boundary value problem, J. Diff. Equs. 9 (1970), 527-537.

25. H.B.Keller, Existence theory for two point boundary value probelems, Bull. AMS. 72 (1966) 728-731.

26. R.Gaines, Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations, SIAM, J. Num. Anal.

27. H.Hethcote, Geometric existence proofs for nonlinear boundary value problems. SIAM Rev. 14 (1972) 121-129.

28. P.Hartman, Ordinary differential equations, Wiley, New York, 1964.

29. P.Bailey, L.Shampine and P.Waltman, Nonlinear second order boundary value problems, existence and regions of uniqueness, J. Math. Anal. Appl. 14 (1966), 433-444.

30. P.Bailey, L.Shampine and P.Waltman, The first and second boundary value problems for nonlinear second order differential equations, J. Diff. Equs. 2 (1966), 399-411.

31. W.Coles and T.Sherman, Convergence of successive approximations for nonlinear two point boundary value problems, SIAM J. Appl. Math. 15 (1967) 426-433.

32. M.Lees, A boundary value problem for nonlinear ordinary differential equations. J. Math. Mech. 10 (1961) 423-430.

33. R.Kalaba, On nonlinear differential equations, the maximum operations and monotone convergence, J.Math.Mech. 8 (1959) 518-574.

34. L.Jackson, Subfunctions and second order ordinary differential inequalities, Advances in Math. 2 (1968) 307-363.

35. S.R.Bernfeld and V.Lakshmikantham, An introduction to nonlinear boundary value problems, Academic Press, New York, 1974

36. E.Isaacson and H.B.Keller, 'Analysis of numerical methods', John Wiley, New York, 1966.

37. Masuo Hukuhara, On the zeros of solutions of linear ordinary differential equations, Sugaku, 15 (1963) 108-109, Math. Reviews 29 (1969), 709 No.3704.

38. M.Tumura, KOkai Zyobibunhoteisiki ni tuite, Kansu Hoteisiki, 30 (1941) 20-35

39. James Brink, Inequalities involving f_p and $f^{(n)}$ q for f with n zeros. Pacific J. Math. 42 (1972), 289-311.

40. G.A.Bessmertnyh and A.Ju. Levin, Some inequalities satisfied by differentiable functions of one variable. Soviet Math. Dokl. 3 (1962) 737-740.

41. A. Ju. Levin, A bound for a function with monotonely distributed zeros of successive derivatives. Mat. Sb. (N.S.) 64 (106) (1964) 396-409.

42. R.P.Agarwal, Boundary value problems for higher order differential equations (submitted for publication).

43. R.P.Agarwal, A multipoint boundary value problems, Proc. Matscience Conference on Mathematical Methods in Phys. (Differential equations) (1978) Matscience Report 96B, p.1-10.

44. S.N.Berstein, Collected works 2, Izdat. Akad. Nauk. SSSR, Moscow, 1954, article 100, p.497.

45. A.Yu. Levin, On some estimates of a differentiable function, Soviet Math. Dokl. 3 (1961), 523-524.

46. P.Hartman, On disconjugacy criteria, Proc. AMS. 24 (1970) 374-381.

47. R.P.Agarwal, Nonlinear two-point boundary value problems, Proc. Seventeenth Conference on Theoretical and Appl. Mech. (1972).

48. R.P.Agarwal, Nonlinear two-point boundary value problems, Indian J Pure and Appl. Math. 4 (1973), 757-769.

49. R.P.Agarwal, Boundary value problems for differential equations with deviating arguments. J. Math. Phyl. Sci. 6 (1972) 425-438.

50. Z.Opial, Linear problems for systems of nonlinear differential equations, J. Diff. Equs. 3 (1967) 580-594.

51. G.S.Zaiceva, A multipoint boundary value problem, Soviet Math. Dokl. 8 (1967) 1183-1185.

52. P.L.Falb and J.L.Jong, Soms successive approximation methods in control and oscillation theory', Academic Press, New York, 1969.

53. L.Collatz, Einige Anwendungen functionalanalytischer Methoden in der praktischen Analysis. Z.Angew. Math. Phys. 4 (1953) 327-357.

54. R.P.Agarwal and P.R.Krishnamurthy, On the uniqueness of solution of nonlinear boundary value problems, J.Math. Phyl. Sci . 10 (1976), 17-31.

55. R.P.Agarwal, Two-point problems for nonlinear third order differential equations. ibid . 8 (1974) 571-576.

56. K.Schrader and S.Umamaheswaram, Existence theorems for higher order boundary value problems, Proc. AMS. 47 (1975) 89-97.

57. R.P.Agarwal, Improved error bounds for the Picard iterates. ibid. 12 (1978) 45-48.

58. J.Eisenfeld and V.Lakshmikantham, On a boundary value problem for a class of differential equations with a deviating argument. J. Math. Anal. Appl. 51 (1975) 158-164.

59. Jagdish Chandra, A comparison result for a boundary value problem for a class of nonlinear differential equations with a deviating argument, J. Math. Anal. Appl. 47 (1975) 573-577.

60. P.R.Krishnamoorthy and R.P.Agarwal, Higher order boundary value problems for differential equations with deviating arguments (to appear in Math. Seminar Notes, Japan).

61. K.Schmitt, Boundary value problems and comparison theorems for ordinary differential equations. SIAM J. Appl. Math. 26 (1974) 670-678.

62. G.Klasen, Differential inequalities and existence theorems for second and third order boundary value problems, J. Diff. Equs. 10 (1971) 529-537.

63. K.Schrader, Boundary value problems for second-order ordinary differential equations, J. Diff. Equs. 3 (1967) 403-413.

64. A.C.Peterson, Existence-uniqueness for two-point boundary value problems for nth order nonlinear differential equations, Rocky Mountain J. Math. 7 (1977) 103-109.

65. V.R.G.Moori and J.B.Garner, Existence - uniqueness theorems for three-point boundary value problems for nth order nonlinear differential equations, J. Diff. Equs. 29 (1978) 205-213.

66. D.Barr and T.Sherman, Existence and uniqueness of solutions of three-point boundary value problems, J.Diff. Equs. 13 (1973), 197-212.

67. K.N.Murty, Three-point boundary value problems - existence and uniqueness, J. Math. Phyl. Sci. 11 (1977) 265-272.

68. R.P.Agarwal and P.R.Krishnamoorthy, Esistence and uniqueness of solutions of boundary value problems for third order differential equations (to appear in Proc. Indian Acad. Sci. A (1979)).

69. R.E.Bellman and R.K.Kalaba, Quasilinearisation and nonlinear boundary value problems, American Elsevier, New York, 1965

70. E.S.Lee, Quasilinarisation and Invariant imbedding, Academic Press, New York, 1968.

71. R.P.Agarwal, Component-wise convergence in quasilinarization, Proc. Indian Acad. Sci. 86 (1977), 519-529.

72. L.G.Grimm and K.Schmitt, Boundary value problems for delay-differential equations, Bull. Amer. Math. Soc. 74 (1968), 997-1000

73. L.G.Grimm and K.Schmit, Boundary value problems for differential equations with deviating arguments. Acquations Mathematical 3 (1969) 24-38.

74. R.K.Jain and R.P.Agarwal, Finite difference method for second order functional differential equations. J. Math. Phyl. Sci. 7 (1973) 301-306.

75. R.P.Agarwal, Existence and uniqueness for nonlinear functional differential equations, Indian J. Pure and Appl. Math. 7 (1976) 933-938.