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Proceedings of the Conference on MATHEMATICAL METHODS IN PHYSICS [Differential Equations]

MYSORE
August 28-31, 1978

MATSCIENCE
THE INSTITUTE OF MATHEMATICAL SCIENCES
Madras-600 020 (India)
March 1979.

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C O N T E N T S

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FOREWORD

These volumes contain the Proceedings of the MATSCIENCE Conference on 'Mathematical Methods in Physics' held at Mysore from 28th to 31st August 1978. The third day of the conference was devoted to Differential Equations - theory, methods and applications.

The conference evoked tremendous response from the scientific community in India and due to budget limitations only 42 participants were invited to the conference from all over the country.

Due to the large number of papers presented it was found appropriate to publish the proceedings into two parts:

PART A : Theoretical Physics

PART B : Differential Equations.

Professor Alladi Ramakrishnan, Director, Matscience introduced a new concept called 'Activity' in evolutionary stochastic processes and gave a conjecture regarding its behaviour as time evolves. He also stressed the importance of such conferences where novel mathematical techniques could be discussed in dealing with current problems.

The organisers thank all the participants for their enthusiastic cooperation in making the conference a success. Thanks are due to the Institution of Engineers (India), Mysore City Sub-Centre for providing conference hall facilities and the Regional College of Education, Mysore for providing slide and overhead projectors.

We wish to thank Mr. N.S. Sampath, Mr. R. Jayaraman and Mr. D. Varadarajan for the help rendered in organising the conference as well as in bringing out the proceedings.

R. SRIDHAR

R.P. AGARWAL

MADRAS-600 020
Date: 14.3.1979

A MULTIPOINT BOUNDARY VALUE PROBLEM

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In interpolation theory, the following inequalities are well known:

THEOREM 1. If the n times continuously differentiable function $x(t)$ has in $[a,b]$ n zeros

$$x(a_i) = x'(a_i) = \cdots = x^{(k_i)}(a_i) = 0 \qquad 1 \le i \le r \tag{1}$$

$$a \le a_1 < a_2 < \cdots < a_r \le b, \quad 0 \le k_i, \quad \sum_{i=1}^{r} k_i + r = n.$$

Then

$$|x^{(k)}(t)| \le \frac{1}{(n-k)!}\, \mu\, (b-a)^{n-k}; \quad k = 0, 1, \ldots, n-1 \tag{2}$$

where

$$\mu = \max_{a \le t \le b} |x^{(n)}(t)|. \tag{3}$$

The proof follows from osculatory interpolation formula

$$x(t) = \frac{1}{n!} \prod_{i=1}^{r} (t-a_i)^{k_i+1}\, x^{(n)}(p)$$

where p is in (a,b), and the observation that the k^{th} derivative of $x(t)$ has at least $n-k$ zeros in (a,b). The constants $\frac{1}{(n-k)!}$ in (2) are obviously the best possible. However, if we consider only the segment $[a_1, a_r]$, which corresponds to interpolation in the exact sense of the word, then the inequality (2) can be improved.

THEOREM 2. If (1) and (3) are satisfied, then the inequalities

$$|x^{(k)}(t)| \le C_{n,k}\, m\, (a_r - a_1)^{n-k} \qquad k = 0, 1, \ldots, n-1 \tag{4}$$

hold, where

$$C_{n,0} = \frac{(n-1)^{n-1}}{n!\, n^n}, \quad C_{n,k} = \frac{k}{(n-k)!\, n} \qquad k = 1, 2, \ldots, n-1 \tag{5}$$

$$m = \max_{a_1 \le t \le a_r} |x^{(n)}(t)|. \tag{6}$$

Hukuhara [4] indicates that Tumura [6] has proved the above result. This result is also mentioned in [2], [3], [5]. The constants $C_{n,k}$ $(k = 0, 1, \ldots, n-1)$ are the best possible, as they are exact for the functions

$$x_1(t) = (t - a_1)^{n-1}(a_r - t), \quad x_2(t) = (t - a_1)(a_r - t)^{n-1}$$

and only for these functions, up to a constant factor. Naturally the estimates (5) are free from any nature of multiplicity at the points a_i, $1 \le i \le r$. If we assume $\alpha = \min(k_1, k_r)$ then we obtain the following:

THEOREM 3. If (1) and (6) are satisfied, then the inequalities (4) will $C^*_{n,k}$ hold, where

$$C^*_{n,k} = \frac{1}{(n-k)!}\, \frac{(n-\alpha-1)^{n-\alpha-1}}{(n-k)^{n-k}}\, (\alpha-k+1)^{\alpha-k+1} \qquad k = 0, 1, \ldots, \alpha \tag{7}$$

$$C^*_{n,\alpha+k} = \frac{k}{(n-\alpha)(n-\alpha-k)!} \qquad k = 1, 2, \ldots, n-\alpha-1.$$

The estimates (7) are better than (5).

Pr of. First we shall prove for $k = 0, 1, \ldots, \alpha$. Since $k_1 + 1$ and $k_r + 1$ is the multiplicity of zeros at a_1 and a_r respectively we find that $x^{(k)}(t)$ will have at least $(n - k_1 - k_r + k - 2)$ zeros (counting with multiplicity) in (a_1, a_r) and (k_1-k+1) at a_1 also (k_r-k+1) at a_r. also (k_r-k+1) at a_r.
Now using osculatory interpolation formula, we obtain

$$|x^{(k)}(t)| \leq m \frac{1}{(n-k)!} (t-a_1)^{k_1-k+1} (a_r-t)^{k_r-k+1} \times \prod_{i=1}^{n-k_1-k_r+k-2} |t - a_{k,i}| \qquad k = 0, 1, \ldots, \alpha.$$

Now to prove (7), suppose $a_{k,j} < t < a_{k,j+1}$, then we obtain

$$(t-a_1)^{k_1-k+1} (a_r-t)^{k_r-k+1} \prod_{i=1}^{n-k_1-k_r+k-2} |t-a_{k,i}| = P(t)$$

$$\leq (t-a_1)^{k_1-k+1} (t-a_1)^{j} (a_r-t)^{k_r-k+1} (a_r-t)^{n-k_1-k_r+k-j-2}$$

$$= (t-a_1)^{k_1-k+j+1} (a_r-t)^{n-k_1-j-1}$$

$$\leq \begin{cases} (t-a_1)^{n-\alpha-1} (a_r-t)^{\alpha-k+1} = f(t), & t-a_1 > a_r-t \\ (t-a_1)^{\alpha-k+1} (a_r-t)^{n-\alpha-1} = g(t), & t-a_1 \leq a_r-t, \end{cases}$$

where in obtaining $f(t)$, we have used $n-k_1-j-\alpha+k-2 \geq 0$ (since $j \leq n-k$, $k_r - 2$ and $k_r \geq \alpha$ and $g(t)$ follows from $k_1-\alpha+j \geq 0$.
Now an absolute maximum of $f(t)$ is at

$$t = a_1 + \frac{n-\alpha-1}{n-k}(a_n - a_1)$$

and of $g(t)$ at

$$t = a_1 + \frac{(\alpha-k+1)}{n-k}(a_n - a_1)$$

also, an absolute maximum value of $f(t)$ and $g(t)$ is same which is

$$\frac{(n-\alpha-1)^{n-\alpha-1}}{(n-k)^{n-k}}(\alpha-k+1)^{\alpha-k+1}$$

this proves (7).

To, show that (7) is better than (5), we note that

$$f(t) \leq \frac{n-k}{n}(t-a_1)^{n-k-1}(a_n-t) + \frac{k}{n}(t-a_1)^{n-k} = f^*(t)$$

$$g(t) \leq \frac{k}{n}(a_n-t)^{n-k} + \frac{n-k}{n}(t-a_1)(a_n-t)^{n-k-1} = g^*(t)$$

and for $k = 0$, $f^*(t)$ has an absolute maximum at $t = \frac{(n-1)a_n + a_1}{n}$ and $g^*(t)$ has an sbsolute maximum at $t = \frac{(n-1)a_1 + a_n}{n}$, also in both the cases the absolute maximum value is $\frac{(n-1)^{n-1}}{n^n}(a_n - a_1)^n$ this proves for $k = 0$. For $\alpha \geq k \geq 1$, $f^*(t)$ has an absolute maximum at $t = a_r$ and $g^*(t)$ has an absolute maximum at $t = a_1$, also in both the cases the absolute maximum value is k/n.

Now to prove for $k = \alpha+1, \alpha+2, \ldots, n-1$, we observe that $x^{(\alpha)}(t)$ has one zero at a_1 (or a_r) and at a_r (or a_1) may be more than one zero and at least $n-k_1-k_r+\alpha-2$ in (a_1, a_r), counting with multiplicity. Now, we define $h(t) = x^{(\alpha)}(t)$, then on using theorem 2, we obtain

$$|h^{(k)}(t)| \leq \frac{k}{(n-\alpha)(n-\alpha-k)!} m(a_r-a_1)^{n-\alpha-k}$$

$$k = 1, 2, \ldots, n-\alpha-1$$

which proves (1.2.8). Also, it is easy to show that $C^*_{n,\alpha+k} > C^*_{n,\alpha+k}$. This completes the proof of the theorem.

<u>Remarks, 1.</u> The inequality (7) for $k = 0, 1, \ldots, \alpha$ can be obtained using the identity obtained in [1], which deals with the Green's function.

2. The constant $C^*_{n,o}$ is the best possible, as this is exact for the functions

$$x_1(t) = (t-a_1)^{n-\alpha-1}(a_r-t)^{\alpha+1}, \quad x_2(t) = (t-a_1)^{\alpha+1}(a_r-t)^{n-\alpha-1}$$

and only for these functions, up to a constant factor.

Besides questions connected with interpolation, theorem 3 finds applications in the theory of multipoint boundary value problems for ordinary differential equations. We shall consider the following multipoint boundary value problem

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$$

$$x(a_i) = A_{1,i},\ x'(a_i) = A_{2,i}, \ldots, x^{(k_i)}(a_i) = A_{k_i,i} \qquad (8)$$

$$1 \leq i \leq r$$

$$a_1 < a_2 < \ldots < a_r,\ 0 \leq k_i,\ \sum_{i=1}^{r} k_i + r = n.$$

Problem (8) is equivalent to the solution of the integral equation

$$x(t) = P(t) + \int_{a_1}^{a_r} G(t,s)\, f(s, x(s), \ldots, x^{(n-1)}(s))\, ds \tag{9}$$

where $P(t)$ is a polynomial of the $(n-1)$st degree satisfying the boundary conditions of (8), and $G(t,s)$ is the Green's function of the homogeneous problem.

We shall denote M as the set of functions n times continuously differentiable on $[a_1, a_r]$ and satisfying the boundary conditions of (8). We define on M an operator B by the formula

$$(Bx)(t) = P(t) + \int_{a_1}^{a_r} G(t,s)\, f(s, x(s), \ldots, x^{(n-1)}(s))\, ds. \tag{10}$$

This operator evidently maps M into itself if f is continuous on $[a_1, a_r] \times R^n$, which we shall assume without mention.

THEOREM 4. Let $K_i > 0$, $i = 0, 1, \ldots, n-1$ be given real numbers and let Q be the maximum of $|f(t, u_0, u_1, \ldots, u_{n-1})|$ on the compact set

$$\{(t, u_0, u_1, \ldots, u_{n-1}) : a_1 \le t \le a_r,\ |u_i| \le 2K_i,\ i = 0, 1, \ldots, n-1\}.$$

Then, if

$$(a_r - a_1) \le \left(\frac{K_i}{Q\, C^*_{n,i}} \right)^{1/n-i}, \quad i = 0, 1, \ldots, n-1 \tag{11}$$

and $\max_{a_1 \le t \le a_r} |P^{(i)}(t)| \le K_i$, the boundary value problem (8) has a solution.

Proof. The set

$$M^*[a_1,a_n] = \{x(t) \in M : \|x\| \le 2k_0, \|x'\| \le 2k_1, \dots, \|x^{(n-1)}\| \le 2k_{n-1}, \|x^{(n)}\| \le Q\}$$

is a closed convex subset of the Banach space M. The operator B defined by (10) is completely continuous, and $(Bx)(t) - P(t)$ satisfies conditions (1). Also, for $x(t) \in M^*$, $(Bx)^{(n)}(t) - P^{(n)}(t) = f(t, x(t), \dots, x^{(n-1)}(t))$, hence

$$\|(Bx)^n\| \le Q.$$

Now using theorem 3, for $x(t) \in M^*$, we find

$$|(Bx)^{(i)}(t) - P^{(i)}(t)| \le Q\, C^*_{n,i} (a_n - a_1)^{n-i}$$

and hence

$$|(Bx)^{(i)}(t)| \le \max_{a_1 \le t \le a_n} |P^{(i)}(t)| + Q\, C^*_{n,i} (a_n - a_1)^{n-i}$$

$$i = 0, 1, \dots, n-1.$$

Thus, condition (11) implies that B maps M* into itself. It then follows from the Schauder's Fixed - Point theorem that B has a fixed point in M*. The fixed point is a solution of (3).

We shall denote

$$q = \sum_{i=0}^{n-1} C^*_{n,i} L_i (a_n - a_1)^{n-i} < 1. \tag{12}$$

THEOREM 5. Let $f(t, u_0, \dots, u_{n-1})$ satisfy the condition

$$|f(t, u_0, \dots, u_{n-1})| \le L + L_0|u_0| + \dots + L_{n-1}|u_{n-1}| \tag{13}$$

for all $(t, u_0, \dots, u_{n-1}) \in [a_1, a_n] \times R^n$, where L is any number, and let $L_0, \dots, L_{n-1}$ satisfy the inequality (12). Then the problem (8) has at least one solution.

Proof. Let

$$l = \max_{a_1 \le t \le a_n} \sum_{i=0}^{n-1} L_i \, |p^{(i)}(t)| .$$

If we introduce in M the metric

$$\rho(x, y) = \max_{a_1 \le t \le a_n} | x^{(n)}(t) - y^{(n)}(t) | \qquad (x, y \in M)$$

then M becomes a complete metric space. We shall show that the operator B maps a sphere of radius $(L+l)/(1-q)$ of the space M into itself. Indeed, if $y \in M$ and $\rho(y, p) \le (L+l)/(1-q)$, then

$$\begin{aligned}
\rho(By, p) &= \max_{a_1 \le t \le a_n} | f(t, y, \dots, y^{(n-1)}) | \\
&\le L + \max_{a_1 \le t \le a_n} \sum_{i=0}^{n-1} L_i \, | (y-p)^{(i)} + p^{(i)} | \\
&\le L + l + \max_{a_1 \le t \le a_n} \sum_{i=0}^{n-1} L_i \, | (y-p)^{(i)} | \\
&\le L + l + q \max_{a_1 \le t \le a_n} | y^{(n)} | \\
&\le L + l + q \cdot \frac{L+l}{1-q} = \frac{L+l}{1-q} .
\end{aligned}$$

Thus, it follows by Schauder's Fixed point theorem, B has at least one fixed point. The problem (8) has therefore at least one solution x(t) satisfying the condition

$$|x^{(n)}(t)| \leq \frac{L+l}{1-q} \quad (a_1 \leq t \leq a_n).$$

Hence, from theorem 3, we obtain the inequalities

$$|x^{(i)}(t) - P^{(i)}(t)| \leq C^*_{n,i} \frac{L+l}{1-q}(a_n - a_1)^{n-i}$$
$$i = 0, 1, \ldots, n-1 \quad (a_1 \leq t \leq a_n).$$

THEOREM 6. Let $f(t, u_0, \ldots, u_{n-1})$ satisfy the condition

$$|f(t, u_0, \ldots, u_{n-1}) - f(t, v_0, \ldots, v_{n-1})| \leq \sum_{i=0}^{n-1} L_i |u_i - v_i| \tag{14}$$

for all $(t, u_0, \ldots, u_{n-1})$, $(t, v_0, \ldots, v_{n-1}) \in [a_1, a_n] \times R^n$, where $L_0, \ldots, L_{n-1}$ satisfy the inequality (12). Then the boundary value problem (8) has a unique solution, for any $A_{i,k}$.

<u>Proof.</u> We shall show that the operator B becomes a contraction operator on the metric space M (metric defined in Theorem 5). Indeed, we find that for x, y ∈ M

$$\rho(Bx, By) = \max_{a_1 \leq t \leq a_n} |f(t, x(t), \ldots, x^{(n-1)}(t)) - f(t, y(t), \ldots, y^{(n-1)}(t))|$$

$$\leq \max_{a_1 \leq t \leq a_n} \sum_{i=0}^{n-1} L_i |x^{(i)}(t) - y^{(i)}(t)|$$

$$\leq \max_{a_1 \leq t \leq a_n} |x^{(n)}(t) - y^{(n)}(t)| \sum_{i=0}^{n-1} L_i C^*_{n,i} (a_n - a_1)^{n-i}$$

$$= q\, \rho(x, y).$$

Thus the operator B in M, has one fixed point, and this is equivalent to the existence and uniqueness of a solution for the problem (8).

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MHD FLOW THROUGH A POROUS STRAIGHT CHANNEL

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Abstract

The author studies the unsteady two-dimensional incompressible viscous flow through a straight channel with porous flat walls distant h apart in the presence of a uniform transverse magnetic field. He obtains the velocity distribution both in the unsteady and steady cases. The coefficients of skin friction at both the walls which are subjected to injection and suction are evaluated and the effect of magnetic field on velocity distribution is investigated.

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1. Introduction:

Fluid flow through porous media is of fundamental importance to a wide range of disciplines in various branches of natural science and technology. Civil engineers, Mining engineers, Petroleum engineers and hydrogeologists are interested in seepage problems in rock mass, sand beds and subterranean acquifers. Civil and agricultural engineers are interested in the same phenomenon for efficient layout of drainage system for irrigation and recovery of swampy area. The nuclear engineer is interested in fluid flow through reactors to maintain a uniform temperature throughout the bed. The textile technologist is interested in water movement through plant roots and through and out of the cells of living systems.

Verma and Mathur[5] have studied magnetohydrodynamic flow between two parallel plates, one in uniform motion and the other at rest with uniform suction at the stationary plate. They have observed that the coefficient of skin friction decreases with the increase of Hartmann number. Satyaprakash[3] has obtained the exact solution of the problem of unsteady viscous flow through a porous straight channel. He has obtained the result that the velocity increases with time and tends ultimately towards the steady state at both the points, as should have been the case in the presence of a pressure gradient which remains constant for all times.

In this paper the flow of a viscous incompressible slightly conducting fluid through a porous straight channel under a uniform transverse magnetic field is considered. The pressure gradient is taken as constant quantity as a special case. Under the constant pressure gradient the case of steady flow is obtained by taking the time since the start of the motion to be infinite.

2. Formulation and solution of the problem:

Unsteady two-dimensional incompressible viscous flow through a straight channel with porous flat walls distant h apart in the presence of a uniform transverse magnetic field is considered. The lower plate is taken as x-axis and a straight line perpendicular to that as y-axis. It is assumed that the fluid injected into the channel through the wall at $y=0$ and sucked through the wall at $y=h$. Let u and v be the velocity components of the fluid at a point (x,y) in the direction of axes of coordinates respectively. It is assumed that the fluid is of small electrical conductivity with magnetic Reynolds number much less than unity so that the induced magnetic field can be neglected in comparison with the applied magnetic field (Sparrow and Cess[4], Bathaiah et al[1]).

The equations of motion of an incompressible, viscous slightly conducting fluid, in the absence of input electric field, are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma \mu_e H_0^2}{\rho} u \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

where ρ is the density of the fluid, ν the coefficient of kinematic viscosity, t the time measured since the start of the motion, p the pressure at a point (x,y), σ the electrical conductivity, μ_e the magnetic permeability and H_0 the uniform applied magnetic field.

The initial and boundary conditions are:

$$\begin{aligned} &\text{when } t \leq 0,\ u = 0 \text{ and } v = 0 \text{ for } 0 \leq y \leq h \\ &\text{when } t > 0.\ u = 0 \text{ and } v = v_0 = \text{constant} > 0 \text{ for } y = 0, h. \end{aligned} \tag{2.4}$$

From the initial and boundary conditions (2.4) we may say that the velocity distribution is independent of x.

Hence $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} = 0$ (2.5)

On substituting $\frac{\partial u}{\partial x} = 0$ and using (2.4) the equation (2.3) yields $v = v_0$.

Substituting $v = v_0$ in equations (2.1) and (2.2), we obtain

$$\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma \mu_e^2 H_0^2}{\rho} u \tag{2.6}$$

and

$$0 = \frac{\partial p}{\partial y} \tag{2.7}$$

Let V_0 be the characteristic velocity, h the characteristic length ρV_0^2 the characteristic pressure and h/v_0 the characteristic time. We define the dimensionless quantities u', x', y', p' and t' as follows:

$$u' = \frac{u}{V_0}, \quad x' = \frac{x}{h}, \quad y' = \frac{y}{h}$$
$$p' = \frac{p}{\rho V_0^2}, \quad t' = \frac{t}{(h/V_0)} \tag{2.8}$$

In view of equation (2.8), equations (2.5) to (2.7) reduce to

$$\frac{\partial u'}{\partial x'} = 0 \tag{2.9}$$

$$\frac{\partial u'}{\partial t'} + \frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{R}\frac{\partial^2 u'}{\partial y'^2} - k u' \tag{2.10}$$

and

$$0 = \frac{\partial p'}{\partial y'} \tag{2.11}$$

where

$$K = \frac{\sigma \mu_e^2 H_o^2 h}{\rho V_o} \quad = \text{Magnetic parameter}$$

$$R = \frac{V_o h}{\nu} \quad = \text{Reynolds number}$$

The initial and boundary conditions (2.4) in view of (2.8) lead to

$$\begin{aligned} &\text{when } t' \leq 0,\ u' = 0 \quad \text{for } 0 \leq y' \leq 1 \\ &\text{when } t' > 0,\ u' = 0 \quad \text{for } y' = 0,1 \end{aligned} \tag{2.12}$$

It is observed that u' is independent of x', from equation (2.9). Hence u' is a function of y' and t' only. From the equation (2.11) we may say that p' is independent of y'. Therefore it follows from the equation (2.10) that $\frac{\partial p'}{\partial x'}$ is a function of t' only.

We assume that

$$\frac{\partial p'}{\partial x'} = -f(t') \tag{2.13}$$

Then equation (2.10) becomes

$$\frac{\partial u'}{\partial t'} + \frac{\partial u'}{\partial y'} = f(t') + \frac{1}{R}\frac{\partial^2 u'}{\partial y'^2} - Ku' \tag{2.14}$$

We define Laplace Transform as

$$\bar{u}' = \int_0^\infty u' e^{-\lambda t'} dt' \tag{2.15}$$

where $R_e(\lambda) > 0$ and γ is greater than the real part of all the singularities of $\bar{u}'$.

We denote

$$\bar{f}(\lambda) = \int_0^\infty f(t') e^{-\lambda t'} dt' \tag{2.17}$$

In view of equations (2.15) and (2.17) the equation (2.14) is transformed into

$$\frac{d^2\bar{u}'}{dy'^2} - R\frac{d\bar{u}'}{dy'} - R(k+\lambda)\bar{u}' = -R\bar{f}(\lambda) \qquad (2.18)$$

The transformed boundary conditions are

$$\bar{u}' = 0 \text{ for } y' = 0,1 \qquad (2.19)$$

In view of the conditions (2.19), the solution of equation (2.18) is

$$\bar{u}' = -\frac{\bar{f}(\lambda)}{(k+\lambda)\sinh\left(\frac{\sqrt{R^2+4R(k+\lambda)}}{2}\right)}\left[e^{-(1-y')\frac{R}{2}}\sinh\left(\frac{\sqrt{R^2+4R(k+\lambda)}}{2}\right.\right.$$

$$\left.+e^{Ry'/2}\sinh\left\{\frac{\sqrt{R^2+4R(k+\lambda)}}{2}(1-y')\right\}\right] + \frac{\bar{f}(\lambda)}{(k+\lambda)}$$

$$(2.20)$$

<u>A Special Case</u>

Let us assume that the pressure gradient is a constant quantity. Hence let

$$\frac{\partial p'}{\partial x'} = -f(t') = -P \qquad (2.21)$$

where P is a positive constant

$$\bar{f}(\lambda) = \int_0^\infty P e^{-\lambda t'} dt' = \frac{P}{\lambda} \qquad (2.22)$$

By taking the inversion of equation (2.20) (Carslaw and Jaeger[2]) we obtain velocity distribution

$$u' = -\frac{P}{K \sinh \frac{1}{2}(R^2+4RK)^{1/2}}\Big[e^{-(1-y')\frac{R}{2}} \sinh \tfrac{1}{2}\{(R^2+4RK)^{1/2} y'\}$$

$$+ e^{Ry'/2} \sinh \tfrac{1}{2}\{(R^2+4RK)^{1/2}(1-y')\}\Big] + \frac{P}{K}$$

$$+ \frac{Pe^{-Kt}}{K}\Big[e^{-(1-y')R/2} \sinh \tfrac{1}{2}(Ry') + e^{Ry'/2} \sinh \tfrac{1}{2}\{R(1-y')\} - 1\Big]$$

$$+ 32PR\pi \sum_{n=0}^{\infty}\Big[\frac{n e^{Ry'/2} \sin(n\pi y')\{e^{-R/2}(-1)^n - 1\}}{(R^2+4n^2\pi^2+4RK)(R^2+4n^2\pi^2)}$$

$$\times \exp\{-\frac{t'}{4R}(R^2+4n^2\pi^2+4RK)\}\Big] \qquad (2.23)$$

<u>Steady State</u>

The flow becomes steady after a lapse of infinite time since the start of the motion. Under a constant pressure gradient - P, the steady flow may be deduced from the equation (2.23). In the case of steady state the velocity distribution is

$$u' = -\frac{p}{K \operatorname{Sinh} \frac{1}{2}(R^2+4RK)^{1/2}}\Big[e^{-(1-y')R/2} \operatorname{Sinh} \tfrac{1}{2}\{(R^2+4RK)^{1/2}y'\}$$

$$+ e^{Ry'/2} \operatorname{Sinh} \tfrac{1}{2}\{(R^2+4RK)^{1/2}(1-y')\}\Big] + \frac{P}{K} +$$

(2.24)

Now let us find the steady state solution directly from the equation of motion.

After substituting P for f(t') for steady state the equation (2.14) becomes

$$\frac{d^2u'}{dy'^2} - R\frac{du'}{dy'} - KRu' = -PR \tag{2.25}$$

The boundary conditions are same as those given in (2.19). In view of the boundary conditions (2.19), the solution of equation (2.25) is

$$u' = -\frac{p}{K \operatorname{Sinh} \frac{1}{2}(R^2+4RK)^{1/2}}\Big[e^{-(1-y')R/2} \operatorname{Sinh} \tfrac{1}{2}\{(R^2+4RK)^{\frac{1}{2}}y'\}$$

$$+ e^{Ry'/2} \operatorname{Sinh} \tfrac{1}{2}\{(R^2+4RK)^{1/2}(1-y')\}\Big] + \frac{P}{K}$$

(2.26)

The results (2.24) and (2.26) are identitcal.

Skin Friction

The shearing stress at the wall y'=0 is

$$\tau_o = \frac{\mu V_o}{h}\left(\frac{\partial u'}{\partial y'}\right)_{y'=0}$$

and the coefficient of skin friction is given by

$$C_f = \frac{2\tau_o}{\rho V_o^2} = \frac{2}{R}\left(\frac{\partial u'}{\partial y'}\right)_{y'=0} \tag{2.27}$$

The coefficient of skin friction at the wall y'=0 is

$$C_f = -\frac{P}{k \sinh \frac{1}{2}(R^2+4RK)^{1/2}}\left[\frac{(R^2+4RK)^{1/2}}{R}\left\{e^{-R/2} + \cosh\frac{1}{2}(R^2+4RK)^{1/2}\right\} + \sinh\frac{1}{2}(R^2+4RK)^{1/2}\right]$$

$$+ \frac{Pe^{-kt'}}{k\sinh\frac{R}{2}}\left[e^{-R/2} - \cosh\frac{R}{2} + \sinh\frac{1}{2}R\right] +$$

$$+ 64P\pi^2\sum_{n=0}^{\infty}\left[\frac{n^2\{e^{-R/2}(-1)^n - 1}{(R^2+4n^2\pi^2+4RK)(R^2+4n^2\pi^2)}\right.$$

$$\left.\times \exp\left\{-\frac{t'}{4R}(R^2+4n^2\pi^2+4RK)\right\}\right] \tag{28}$$

3. Conclusions

We obtain the velocity distribution in unsteady and steady cases. When the magnetic parameter K tends to zero, our results coincide with those of Satyaprakash[3]. We plot a graph taking magnetic parameter K against velocity distribution u' (Figure 1). It is observed that the velocity increases with the increase of magnetic parameter. We draw another graph (Figure 2) taking velocity distribution against time. We find that the velocity increases with time and tends ultimately towards the steady state at both the points, as should have been the case in the presence of a pressure gradient which remains constant for all time. It is also observed that the steady state is obtained earlier than in the non-magnetic case. From Figure 2 we conclude that the velocity at the point near the wall which is subjected to injection is less than that at the point near the wall which is subjected to suction. Verma and Mathur[5] have observed that the coefficient of skin friction decreases as the magnetic field strength increases at the stationary wall which is subjected to suction. But we observe from the figures 3 and 4 that the coefficients of skin friction increase as the magnetic parameter increases both at the walls which are subjected to injection and suction. From figures 3 and 4 it is also observed that the coefficient of skin friction at the point near the wall which is subjected to suction is less than that at the point near the wall which is subjected to injection.

I thank Professor K. Sitaram for his constant encouragement.

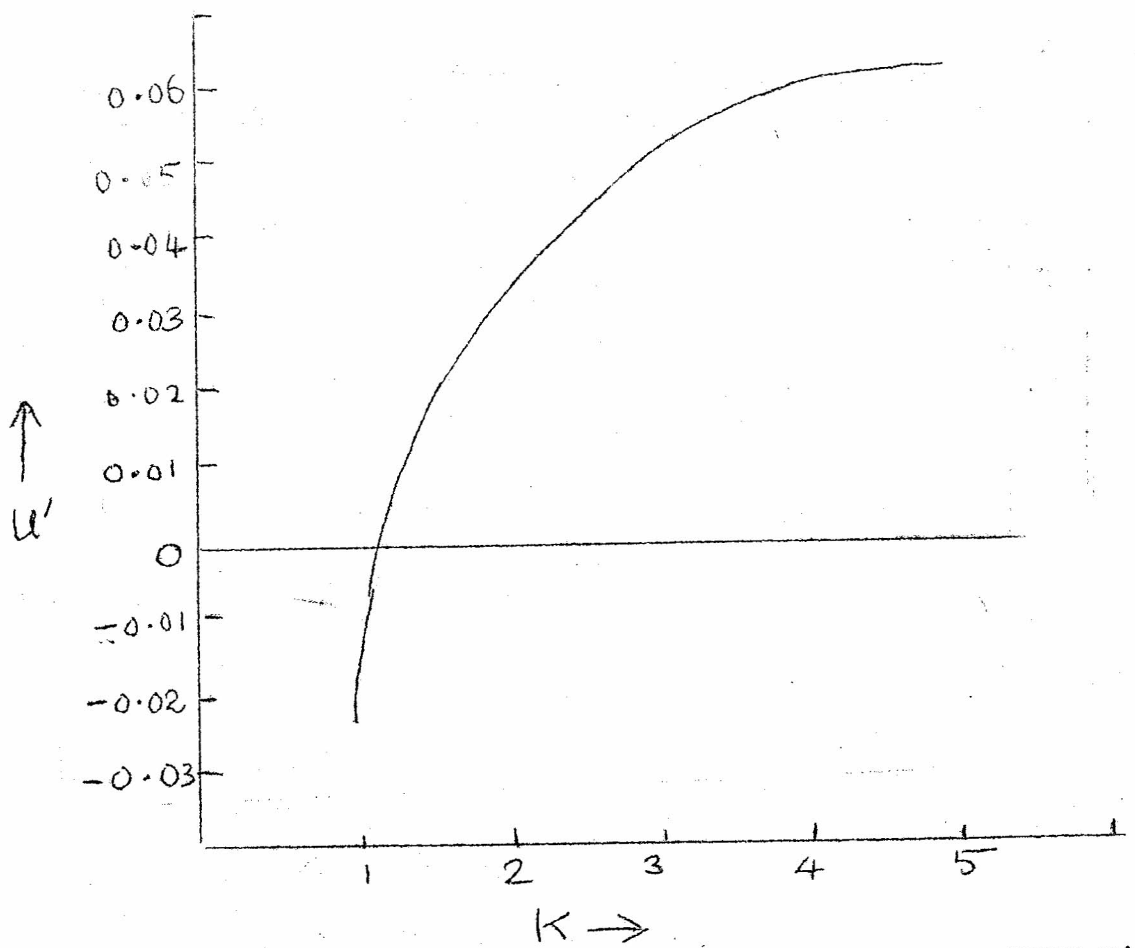

Fig.1. MAGNETIC PARAMETER K VERSUS VELOCITY U'

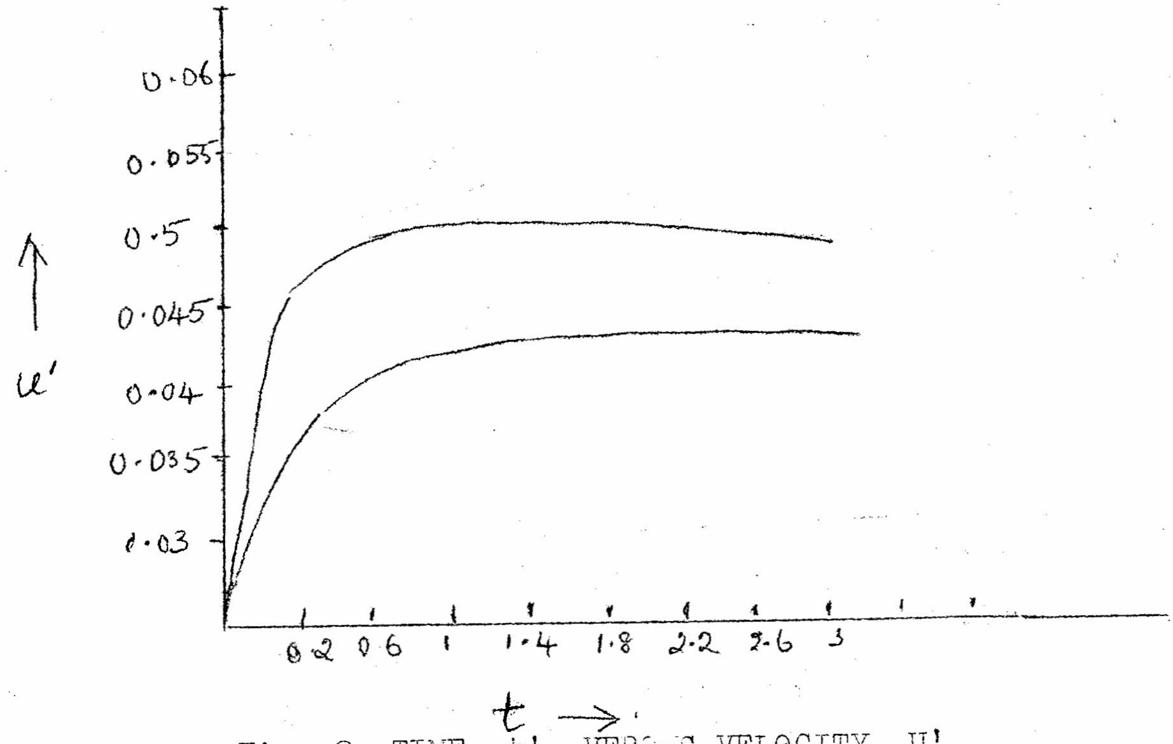

Fig. 2. TIME t' VERSUS VELOCITY U'

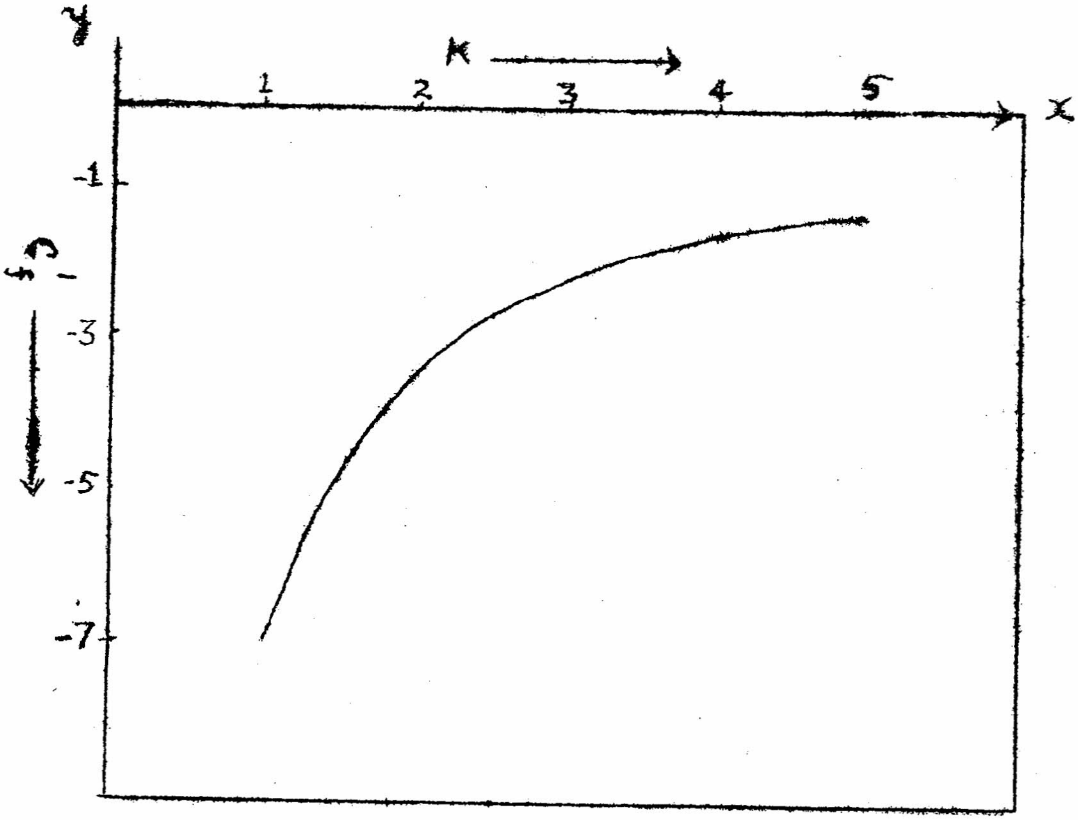

Fig 3. Magnetic parameter K versus Coefficient of skin friction C_f' at the wall $y' = 1$.

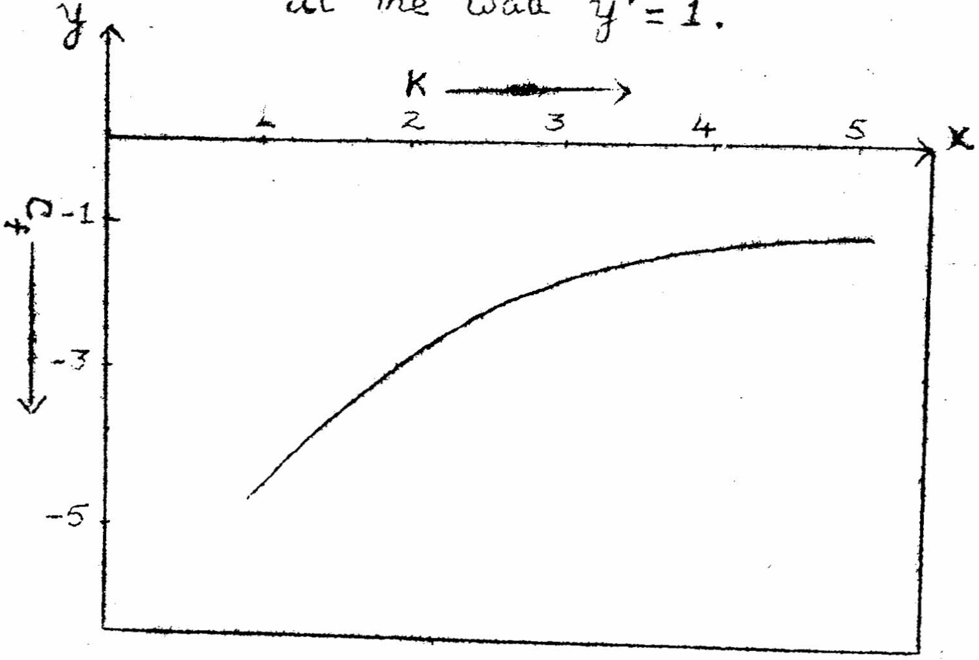

Fig 4. Magnetic parameter K versus Coefficient of skin friction C_f at the wall $y' = 0$.

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ON THE UNIQUENESS OF SOLUTIONS OF HYPERBOLIC DELAY DIFFERENTIAL EQUATIONS

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The theory of partial differential equations with deviating arguments has been very weakly worked out. The only such equations encountered in applications have deviations with respect to only one variable [1] - [3]. In this paper we have given some sufficient conditions of uniqueness for the solutions of the hyperbolic delay differential equations. The results obtained here are based on those given in [4] and [5] for the equations without deviating arguments.

We shall consider a general uniqueness theorem of Perron type for the hyperbolic delay differential equation

$$u_{xy} = f(x, y, u(x,y), u(g_1(x),y), u_x(x,y), u_x(g_2(x),y), u_y(x,y), u_y(g_3(x),y)) \quad (1)$$

subject to the conditions

$$u(x,0) = \phi(x), \; u(0,y) = \psi(y), \; \phi(0) = \psi(0) = u_0 \quad (2)$$

the functions $\phi(x)$ and $\psi(y)$ being continuously differentiable on $0 \le x \le a$, $0 \le y \le b$ respectively.

In what follows, we shall always denote R_0 as the rectangle

$$R_0 = [0 \le x \le a, 0 \le y \le b].$$

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*Paper presented by E. Thandapani.

THEOREM 1. Assume that

(i) $g_i(x)$ $(i = 1,2,3,)$ are continuous and map $[0,a]$ into itself,

(ii) $f \in C[R_0 \times R^6, R]$ and

$$|f(x,y,q_1,q_2,q_3,q_4,q_5,q_6) - f(x,y,\bar{q}_1,\bar{q}_2,\bar{q}_3,\bar{q}_4,\bar{q}_5,\bar{q}_6)|$$

$$\leq g(x,y,|q_1-\bar{q}_1|,|q_2-\bar{q}_2|,|q_3-\bar{q}_3|,|q_4-\bar{q}_4|, |q_5-\bar{q}_5|,|q_6-\bar{q}_6|)$$

where $g \in C[R_0 \times R_+^6, R_+]$, $g(x,y,0,0,0,0,0,0) \equiv 0$, $g(x,y,q_1,\ldots,q_6)$ is monotonic nondecreasing in $q_1, q_2, \ldots, q_6$ and bounded,

(iii) $Z(x,y) \equiv 0$ is the only solution of the hyperbolic delay differential equation

$$Z_{xy} = g(x,y,Z(x,y),Z(g_1(x),y),Z_x(x,y), Z_x(g_2(x),y), Z_y(x,y), Z_y(g_3(x),y)), \tag{3}$$

such that

$$Z(0,0) = 0,\ Z(x,0) \equiv 0,\ Z(0,y) \equiv 0. \tag{4}$$

Then, there is at most one solution for (1), (2).

<u>Proof</u>. Let us assume that there are two solutions $u(x,y)$, $v(x,y)$ for (1), (2) on R_0. We define

$$A(x,y) = |u(x,y) - v(x,y)|$$

$$B(x,y) = |u_x(x,y) - v_x(x,y)|$$

$$C(x,y) = |u_y(x,y) - v_y(x,y)|$$

Since we have

$$u(x,0) = v(x,0) = \phi(x),\ u_x(x,0) = v_x(x,0) = \phi'(x),\ 0 \leq x \leq a$$

$$u(0,y) = v(0,y) = \psi(y),\ u_y(0,y) = v_y(0,y) = \psi'(y),\ 0 \leq y \leq b$$

THEOREM 1. Assume that

(i) $g_i(x)$ (i = 1,2,3,) are continuous and map $[0,a]$ into itself,

(ii) $f \in C[R_0 \times R^6, R]$ and

$$|f(x,y,q_1,q_2,q_3,q_4,q_5,q_6) - f(x,y,\bar{q}_1,\bar{q}_2,\bar{q}_3,\bar{q}_4,\bar{q}_5,\bar{q}_6)|$$

$$\leq g(x,y,|q_1-\bar{q}_1|,|q_2-\bar{q}_2|,|q_3-\bar{q}_3|,|q_4-\bar{q}_4|, |q_5-\bar{q}_5|,|q_6-\bar{q}_6|)$$

where $g \in C[R_0 \times R_+^6, R_+]$, $g(x,y,0,0,0,0,0,0) \equiv 0$, $g(x,y,q_1,\cdots,q_6)$ is monotonic nondecreasing in $q_1, q_2, \cdots, q_6$ and bounded,

(iii) $Z(x,y) \equiv 0$ is the only solution of the hyperbolic delay differential equation

$$Z_{xy} = g(x,y,Z(x,y),Z(g_1(x),y),Z_x(x,y), Z_x(g_2(x),y), Z_y(x,y), Z_y(g_3(x),y)), \tag{3}$$

such that

$$Z(0,0) = 0,\ Z(x,0) \equiv 0,\ Z(0,y) \equiv 0. \tag{4}$$

Then, there is at most one solution for (1), (2).

<u>Proof.</u> Let us assume that there are two solutions $u(x,y)$, $v(x,y)$ for (1), (2) on R_0. We define

$$A(x,y) = |u(x,y) - v(x,y)|$$

$$B(x,y) = |u_x(x,y) - v_x(x,y)|$$

$$C(x,y) = |u_y(x,y) - v_y(x,y)|$$

Since we have

$$u(x,0) = v(x,0) = \phi(x),\ u_x(x,0) = v_x(x,0) = \phi'(x),\ 0 \leq x \leq a$$

$$u(0,y) = v(0,y) = \psi(y),\ u_y(0,y) = v_y(0,y) = \psi'(y),\ 0 \leq y \leq b$$

it follows that

$$A(0,0) = 0, B(x,0) \equiv 0, C(0,y) \equiv 0.$$

Furthermore, by conditions (i), (ii), we obtain

$$A(x,y) \le \int_0^x \int_0^y g(s,t,A(s,t),A(g_1(s),t),B(s,t),B(g_2(s),t), C(s,t),C(g_3(s),t))\,ds\,dt$$

$$B(x,y) \le \int_0^y g(x,t,A(x,t),A(g_1(x),t),B(x,t),B(g_2(x),t), C(x,t),C(g_3(x),t))\,dt$$

$$C(x,y) \le \int_0^x g(s,y,A(s,y),A(g_1(s),y),B(s,y),B(g_2(s),y), C(s,y),C(g_3(s),y))\,ds.$$

Let us define the sequence of successive approximations to the solution of (3), (4) as follows

$$\alpha_0(x,y) = A(x,y),\ \beta_0(x,y) = B(x,y),\ \gamma_0(x,y) = C(x,y)$$

and, for $n \geq 0$

$$\alpha_{n+1}(x,y) = \int_0^x \int_0^y g(s,t,\alpha_n(s,t),\alpha_n(g_1(s),t),\beta_n(s,t), \beta_n(g_2(s),t),\gamma_n(s,t),\gamma_n(g_3(s),t))\,ds\,dt$$

$$\beta_{n+1}(x,y) = \int_0^y g(x,t,\alpha_n(x,t),\alpha_n(g_1(x),t),\beta_n(x,t), \beta_n(g_2(x),t),\gamma_n(x,t),\gamma_n(g_3(x),t))\,dt$$

$$\gamma_{n+1}(x,y) = \int_0^x g(s,y,\alpha_n(s,y),\alpha_n(g_1(s),y),\beta_n(s,y), \beta_n(g_2(s),y),\gamma_n(s,y),\gamma_n(g_3(s),y))\,ds.$$

We note that $A(x,y)$, $B(x,y)$, $C(x,y)$ are continuous.

Since $\alpha_0(x,y) \le \alpha_1(x,y)$, $\beta_0(x,y) \le \beta_1(x,y)$, $\gamma_0(x,y) \le \gamma_1(x,y)$ we find in using (i) $\alpha_0(g_1(x),y) \le \alpha_1(g_1(x),y)$, $\beta_0(g_2(x),y) \le \beta_1(g_2(x),y)$, $\gamma_0(g_3(x),y) \le \gamma_1(g_3(x),y)$. Now on using nondecreasing property of G, it follows by induction that

$$\alpha_n(x,y) \le \alpha_{n+1}(x,y),\ \beta_n(x,y) \le \beta_{n+1}(x,y),\ \gamma_n(x,y) \le \gamma_{n+1}(x,y).$$

Also, the functions $\alpha_n(x,y)$, $\beta_n(x,y)$, $\gamma_n(x,y)$ are uniformly bounded in view of the fact that g is assumed to be bounded. Hence, we get

$$\lim_{n\to\infty} \alpha_n(x,y) = \alpha(x,y),\quad \lim_{n\to\infty} \beta_n(x,y) = \beta(x,y)$$

$$\lim_{n\to\infty} \gamma_n(x,y) = \gamma(x,y)$$

On using Lebesgue's monotone convergence theorem $\alpha(x,y)$ is a solution of (3), (4). Hence, we have

$$A(x,y) \le \alpha(x,y),\ B(x,y) \le \beta(x,y),\ C(x,y) \le \gamma(x,y)$$

Now, by assumption (iii) $\alpha(x,y) \equiv 0$, $\beta(x,y) \equiv 0$, $\gamma(x,y) \equiv 0$ and this proves $A(x,y) \equiv 0$, $B(x,y) \equiv 0$, $C(x,y) \equiv 0$. This completes the proof of the theorem.

3. Here we shall give some other sufficient conditions of uniqueness for the solutions of

$$u_{xy} = f(x,y,u(x,y),u(g(x),y)) \tag{5}$$

subject to the conditions

$$u(x,0) = \phi(x),\ u(0,y) = \psi(y),\ \phi(0) = \psi(0) = u_0 \tag{6}$$

where the functions $\phi(x)$ and $\psi(y)$ are same as defined earlier,

THEOREM 2. Assume that

(i) $g(x)$ is continuous and $g(x) \leq x$ for all $x \in [0, a]$, also $\min g(x) = 0$,

(ii) $f \in C[R_0 \times R^2, R]$ and bounded, and it satisfies in addition the following

$$|f(x, y, \phi_1, \phi_2) - f(x, y, \bar{\phi}_1, \bar{\phi}_2)|$$
$$\leq \frac{k}{xy}\left[a_1|\phi_1 - \bar{\phi}_1| + b_1|\phi_2 - \bar{\phi}_2|\right]$$
$$k > 0, \; a_1 + b_1 \leq 1$$

$$|f(x, y, \phi_1, \phi_2) - f(x, y, \bar{\phi}_1, \bar{\phi}_2)| \leq \frac{c}{x^\beta y^\beta}\left[a_2|\phi_1 - \bar{\phi}_1|^\alpha + b_2|\phi_2 - \bar{\phi}_2|^\beta\right] \tag{7}$$
$$c > 0, \; a_2 + b_2 \leq 1$$

with $0 < \alpha < 1$, $\beta < \alpha$ and $k(1-\alpha)^2 < (1-\beta)^2$ for all $(x, y, \phi_1, \phi_2) \in R_0 \times R^2$. Then, there exists at most one solution for (5), (6).

<u>Proof.</u> Let $M = \sup_{R_0 \times R^2} |f(x, y, \phi_1, \phi_2)|$ and assume that $u(x, y)$, $v(x, y)$ are two solutions for (1), (2) on R_0. Then, we have

$$|u(x, y) - v(x, y)| \leq 2Mxy, \; (x, y) \in R_0$$

also, on using (i) we find,

$$|u(g(x), y) - v(g(x), y)| \leq 2M\, g(x)\, y$$
$$\leq 2Mxy,$$

From (7) it follows that

$$|u(x,y)-v(x,y)| \leq \int_0^x \int_0^y |f(s,t,u(s,t),u(g(s),t)) - f(s,t,v(s,t),v(g(s),t))| \, ds\,dt$$

$$\leq C \int_0^x \int_0^y [a_2 (2M)^{\alpha} (st)^{\alpha-\beta} + b_2 (2M)^{\alpha} (st)^{\alpha-\beta}] \, ds\,dt$$

$$\leq \frac{C (2M)^{\alpha} (xy)^{(1-\beta)+\alpha}}{[(1-\beta)+\alpha]^2}$$

$$\leq C (2M)^{\alpha} (xy)^{(1-\beta)+\alpha}.$$

Since $(1-\beta)+\alpha > 1$ on using (i) we find

$$|u(g(x),y)-v(g(x),y)| \leq C (2M)^{\alpha} (g(x))^{(1-\beta)+\alpha} y^{(1-\beta)+\alpha}$$

$$\leq C (2M)^{\alpha} (xy)^{(1-\beta)+\alpha}.$$

Now it follows by induction that

$$|u(x,y)-v(x,y)| \leq C^{1+\alpha+\cdots+\alpha^m} (2M)^{\alpha^{m+1}} (xy)^{(1-\beta)(1+\alpha+\cdots+\alpha^m)+\alpha^{m+1}}$$

for $m = 1,2,\ldots$. Therefore, we obtain the following estimate

$$|u(x,y)-v(x,y)| \leq C^{1/1-\alpha} (xy)^{(1-\beta)/1-\alpha}. \tag{8}$$

Now for $k>0$, we disuss separately the following two possiblities (i) $k\geq 1$ and (ii) $k<1$.

Let $k\geq 1$ and define the function

$$Q(x,y)=(xy)^{-\sqrt{k}}\,|u(x,y)-v(x,y)|$$

for $xy>0$. Then from (8) it follows that

$$0\leq Q(x,y)=Q(s)\leq C^{1/1-\alpha}(xy)^{(1-\beta)-\sqrt{k}(1-\alpha)/1-\alpha}.$$

Hence, we have $\lim\limits_{s\to s_0} Q(s)=0$, where $s\in R_0$ and $s_0\in D=\{s: s\in R_0 \text{ and } x=0 \text{ or } y=0\}$. Clearly Q is continuous in D if we define $Q(s_0)=0$ for $s_0\in D$.

We shall show that $Q(x,y)\equiv 0$. If not, then there exists a point $(\bar{x},\bar{y})$ such that

$$0<r=Q(\bar{x},\bar{y})=\sup_{R_0} Q(x,y).$$

On the other hand, if we use (9) and $k\geq 1$, we obtain

$$\begin{aligned} r=Q(\bar{x},\bar{y}) &\leq k(\bar{x}\bar{y})^{-\sqrt{k}}\int_0^{\bar{x}}\int_0^{\bar{y}}\Big[a_1(st)^{\sqrt{k}-1}Q(s,t) \\ &\qquad + b_1\frac{(g(s)t)^{\sqrt{k}}}{st}Q(g(s),t)\Big]ds\,dt \\ &\leq k(\bar{x}\bar{y})^{-\sqrt{k}}\int_0^{\bar{x}}\int_0^{\bar{y}}\Big[a_1(st)^{\sqrt{k}-1}Q(s,t) \\ &\qquad + b_1(st)^{\sqrt{k}-1}Q(g(s),t)\Big]ds\,dt \\ &< k(\bar{x}\bar{y})^{-\sqrt{k}}\, r\int_0^{\bar{x}}\int_0^{\bar{y}}(st)^{\sqrt{k}-1}ds\,dt \\ &= r \end{aligned}$$

which is the desired contradiction.

Since in the other case $k \leq 1$, the conditions of theorem 2 can be relaxed, we prove the following

THEOREM 3. Assume that

(i) $g(x)$ is continuous and $g(x) \leq x$ for all $x \in [0,a]$ also $\min g(x) = 0$;

(ii) $f \in C[R_0 \times R^2, R]$ and bounded, and it satisfies in addition the following:

$$|f(x,y,\phi_1,\phi_2) - f(x,y,\bar\Phi_1,\bar\Phi_2)| \leq \frac{k}{xy}\left[a_1|\phi_1 - \bar\Phi_1| + b_1|\phi_2 - \bar\Phi_2|\right]_{a_1+b_1\leq 1} \tag{9}$$

with $k \leq 1$ for all $(x,y,\phi_1,\phi_2) \in R_0 \times R^2$. Then, there exists at most one solution for (5), (6)..

Proof. Assume $u(x,y)$ and $v(x,y)$ are two solutions of (5), (6). Define $B(x,y) = \dfrac{|u(x,y) - v(x,y)|}{xy}$ for $xy > 0$. Since f is continuous, we note that

$$|f(x,y,u(x,y),u(g(x),y)) - f(x,y,v(x,y),v(g(x),y))| \leq M_{xy}$$

where M_{xy} tends to zero as x or y tends to zero, or both. Therefore it follows that $B(x,y) \geq 0$ for all $(x,y) \in D$ and $\lim_{s \to s_0} B(s) = 0$ where $s = (x,y) \in R_0$ and $s_0 \in D$. .

Since $B(x,y)$ is now continuous over R_0 it attains its maximum at some point (x_0,y_0). Using (9), we observe

$$\begin{aligned} B(x_0,y_0) &\leq k(x_0y_0)^{-1}\int_0^{x_0}\int_0^{y_0}[a_1B(s,t) + b_1B[g(s),t]]\,ds\,dt \\ &< k(x_0y_0)^{-1}B(x_0,y_0)\,x_0y_0 \\ &\leq B(x_0,y_0) \end{aligned}$$

which is the desired contradiction.

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COSSERAT FLUID MOTIONS

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Abstract

We obtain here a variational algorithm for the stability of the flow of cosserat fluids in an arbitrary domain in space. The algorithm can be used for sharpening the estimate of the critical Reynolds number below which the flow is stable. A theorem concerning the existence of stable, periodic solutions of the Cosserat fluid flow equations is next established. It is also observed that the kinetic energy of Cosserat fluid motion over a domain bounded by rigid walls decays faster than the exponential.

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Introduction

The equations governing the flow of Cosserat fluids are

$$\frac{d\rho}{dt} + \operatorname{div} \bar{q} = 0 \tag{1.1}$$

$$\rho \frac{d\bar{q}}{dt} = \rho \bar{f} + \frac{1}{2} \operatorname{curl}(\rho \bar{c}) + \operatorname{DIV}(\tau) + \frac{1}{2} \operatorname{curl} \operatorname{DIV}(M) \tag{1.2}$$

where $\frac{d}{dt}$ denotes the material time-derivate and ρ, $\bar{q}$ are respectively the density and flow velocity of the fluid. The vector $\bar{f}$ and the second order tensor τ are the body-force and force-stress tensor. The presence of the vector $\bar{c}$ and the second order tensor M

distinguishes the Cosserat material from the classical one and these denote the body couple vector and the couple-stress tensor. Across any surface bounding a volume in the material, besides the force $\tau_n = n.\tau$ there is also exertion of the couple $m_n = n.M$ on account of the operation of the couple stress in the medium. The force-stress vector τ_n and the body force $\bar{f}$ are polar vectors whereas the couple-stress vector $\bar{m}_n$ and the body couple vector $\bar{c}$ are axial vectors. The antisymmetric part τ^A of the force-stress tensor does not contribute to the momentum balance. Also, only the deviator part M^D of the couple-stress tensor contributes to the momentum balance. For Cosserat fluids the governing constitutive equations are linear and are expressible in the form ([1], [2], [3])

$$\tau^S = -pI + \lambda(\mathrm{div}\, q)I + 2\mu(\mathrm{grad}\, \bar{q})S \tag{1.3}$$

$$M^d = 2\eta\, \mathrm{grad}(\mathrm{curl}\, \bar{q}) + 2\eta'\{\mathrm{grad}(\mathrm{curl}\, \bar{q})\}^T . \tag{1.4}$$

In the above $\bar{I}$ denotes the unit matrix, grad $\bar{q}$ is the matrix gradient of the velocity vector and the superscript T denotes the transpose of the matrix in question. The superscripts S and d denote the symmetric part and the deviatoric part. The coefficients $\lambda, \mu, \eta, \eta'$ are material constants and conform to the following restrictions. $\mu > 0,\ 3\lambda + 2\mu > 0.\ \eta > 0,\ |\eta'| < \eta$. (1.5)

At points on a boundary the condition to be satisfied by the field vector $\bar{q}$ is the hyperstick or super-adherence condition which in effect means that the velocity $\bar{q}$ and the spin $\bar{W} = \frac{1}{2}$ curl $\bar{q}$ can be so prescribed as to match with their values on the boundary. At a rigid and fixed boundary the vectors $\bar{q}$ and $\frac{1}{2}$ curl $\bar{q}$ would then have to vanish.

In section 2 our aim is to obtain a variational algorithm for the mean stability of Cosserat flow in an arbitrary domain in space. In the next section we prove the existence of periodic solutions of the Cosserat flow equation. In the last section we evaluate the rate of decay of the kinetic energy of a Cosserat fluid motion in a domain bounded by rigid walls. Throughout the paper the field vector $\bar{q}$ and the other field functions that occur are presumed to be smooth enough to allow the conversion of volume integrals into surface integrals by Gauss's divergence theorem. Also the conventional infinitesimal volume or surface area element in the integrals is omitted throughout .

2. Variational algorithm for the stability of Cosserat fluid.

The equations governing the flow of incompressible Cosserat fluid are

$$\operatorname{div} \bar{q} = 0 \tag{2.1}$$

$$\rho \frac{d\bar{q}}{dt} = - \operatorname{grad} p - \mu \operatorname{curl} \operatorname{curl} \bar{q} - \eta \operatorname{curl} \operatorname{curl} \operatorname{curl} \operatorname{curl} \bar{q} \quad \ldots. \quad (2.2)$$

and on a boundary vectors $\bar{q}$ and $\frac{1}{2}$ curl $\bar{q}$ are prescribed. If $\bar{q}(x,y,z,t)$ is the velocity specifying a flow over the domain $D(t)$ and $\bar{q}^*(x,y,z,t)$ is the velocity of another possible flow conforming to the same boundary conditions as the former and $\bar{u}(x,y,z,t)$ is the difference velocity $\bar{q}^* - \bar{q}$, we see easily that

$$\operatorname{div} \bar{u} = 0 \tag{2.3}$$

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho (\bar{q}^*.\operatorname{grad}) \bar{u} + (\bar{u}\cdot\operatorname{grad}) \bar{q}$$
$$= - \operatorname{grad} (p^* - p) - \mu \operatorname{curl} \operatorname{curl} \bar{u} - \eta \operatorname{curl} \operatorname{curl} \operatorname{curl} \operatorname{curl} \bar{u} \quad (2.4)$$

over the domain $D(t)$ and that on the boundary $\partial D(t)$

$$\bar{u} = \tfrac{1}{2}\,\mathrm{curl}\,\bar{u} = \bar{0} \tag{2.5}$$

We may adopt the kinetic energy of the difference motion $\bar{u}$, viz.,

$$T = \tfrac{1}{2}\int \rho\,(\bar{u})^2 \tag{2.6}$$

(the integral extends over the volume of the domain $D(t)$ as the Liapunoff measure of the stability.

From (2.3) - (2.6) we see that

$$\frac{dT}{dt} = -\int \rho\bar{u}.D.\bar{u} - \mu\int(\mathrm{curl}\,\bar{u})^2 - \eta\int(\mathrm{curl}\ \mathrm{curl}\,\bar{u})^2 \tag{2.7}$$

where D is the rate of deformation matrix for the unstarred or primary flow. The equation (2.7) from which the criterior for the mean stability of the flow is readily deduced, is effectively the same as the equation (2.9) in [3] by M.Shahimpoor and G.Ahmadi. The relation (2.7) can also be rendered into the alternative form

$$\frac{dT}{dt} = \int \rho\, u.\,(\mathrm{grad}\,\bar{u}).\,\bar{q} - \mu\int(\mathrm{curl}\,\bar{u})^2 - \eta\int(\mathrm{curl}\ \mathrm{curl}\,\bar{u})^2 \qquad \ldots \quad (2.8)$$

We introduce the symbols;

d = diameter of a ball in which the bounded domain $D(t)$ is embeddable.

V_0 = maximum speed of the unstarred or primary flow $\bar{q}(\bar{x},\bar{t})$ over the time interval $(0,t)$

$-m$ = lower bound for the eigenvalues of the rate of deformation matrix D of the unstarred flow over $(0,t)$

$R_1 = \rho md^2/\mu$ = Reynolds number of the primary flow

$R_2 = \rho V_0 d/\mu$ = Reynolds number of the primary flow

$C_0 = \eta/\mu d^2$ = Cosserat number of the fluid

$\epsilon_1 = 2m(80 + 6400\,C_0 - R_1)/R_1$

$\epsilon_2 = \mu(80 + 12800\,C_0 - R_2^2)/\rho d^2$

From (2.7) one can deduce as in [3] that

$$\frac{dT}{dt} \leq -\epsilon_1 T \tag{2.9}$$

and so $T(t) \downarrow_0$ as $t \uparrow \infty$ if $\epsilon_1 > 0$. Similarly from (2.8) we can find that

$$\frac{dT}{dt} \leq -\epsilon_2 T \tag{2.10}$$

and so $T(t) \downarrow_0$ as $t \uparrow \infty$ if $\epsilon_2 > 0$.

The equation (2.7) can be cast into the non-dimensional form

$$\frac{dT}{dt} = -\frac{1}{R}\left\{ \int (\text{curl}\,\bar{u})^2 + C_0 \int (\text{curl curl}\,\bar{u})^2 \right\} - \int \bar{u}.\bar{D}.\bar{u} \tag{2.11}$$

On adopting the following scheme of non-dimensionalisation and setting $R_2 = R$;

$$\bar{q} \to V_0\,\bar{q}\,,\ \bar{u} \to V_0\,\bar{u},\ \bar{r} \to d\,\bar{r}$$

$$t \to \frac{d}{V_0} t,\ T \longrightarrow \rho V_0^2\, T. \tag{2.12}$$

The mean stability of the basic flow is assured as long as $T(t) \to 0$ as $t \to \infty$ and this in turn is assured as long as

$$\frac{1}{R}\left\{ \int (\text{curl}\ u)^2 + C_0 \int (\text{curl curl}\ u)^2 \right\} + \int \bar{u}.D.\bar{u} > 0 \tag{2.13}$$

for all vectors u that are divergence-free over the domain and have the property that $\bar{u} = \frac{1}{2}$ curl $\bar{u} = \bar{0}$ on the boundary $\partial D(t)$.

Let

$$I_1(\bar{u}) = \int \bar{u}.D.\bar{u} \tag{2.14}$$

$$F(\bar{u}) = \int (\text{curl } \bar{u})^2 + C_o \int (\text{curl curl } \bar{u})^2$$

We consider the following variational problem:

$$\text{Mximum} \left\{ -I(\bar{u}) \right\} \tag{2.15}$$

subject to the constraints:

$$\text{div } \bar{u} = 0 \quad \text{over } D(t) \tag{2.16}$$

$$F(\bar{u}) = 1 \tag{2.17}$$

and the boundary conditions

$$\bar{u} = 0, \ \text{curl } \bar{u} = 0 \ \text{on } \partial D(t). \tag{2.18}$$

To obtain the solenoidal vector $\bar{u}$ that maximizes the functional $\left\{ -I(\bar{u}) \right\}$ we seek the variation of the functional

$$\delta \left\{ I_1(\bar{u}) - \int P \text{ div } \bar{u} + \frac{1}{R} F(\bar{u}) \right\}. \tag{2.19}$$

The Euler-Lagrange equations of the above variational problem lead to the equation

$$2\,\bar{u}.D - \text{grad } P + \frac{2}{R} \text{curl curl } \bar{u} + \frac{2 C_o}{R} \text{curl curl curl curl } \bar{u} = 0 \tag{2.20}$$

Scalar product of the equation (2.20) with $\bar{u}$ and integration over the domain yields the result

$$I_1(u) + \frac{1}{R} F(\bar{u}) = 0. \tag{2.21}$$

Thus a solution vector $\bar{u}$ of the variational problem satisfies the relation

$$- I_1(\vec{u}) = + \frac{1}{R} \tag{2.22}$$

From the calculus of variations it is known that [4] there exist maximizing functions u which solve the variational problem (2.15)-(2.18) and that these functions are also eigenfunctions of the variational equation (2.20) (cf. [5]) with the eigenvalue

$$\frac{1}{\tilde{R}} = \left\{ - I_1(\bar{u}) \right\} = \operatorname{Max} \left\{ - I_1(\bar{u}) \right\}. \tag{2.3}$$

It follows from (2.22) and (2.23) that

$$\tilde{R} \leq R \tag{2.24}$$

The variational problem generates a complete set of eigenfunctions $(\bar{u}_i)$ with a corresponding set of eigenvalues (R_i) and for any admissible solution u, we have

$$- \left\{ I_1 \ (\bar{u}) \right\} \leq \text{l.u.b.}\left(\frac{1}{R_i} \right) \leq \frac{1}{\tilde{R}} \tag{2.25}$$

The left hand side can be made arbitrarily close to its maximum value by a suitable choice of $\bar{u}$ and hence it follows that

$$\frac{1}{\tilde{R}} = \text{l.u.b.}\left(\frac{1}{R_i} \right) \tag{2.26}$$

Hence the following theorem:

Theorem: Let $\bar{u}$ be solution of the variational problem (2.15) - (2.18) for fixed values of the Cosserat number C_0 and let

$$\text{Max-}\left\{-I_1(\bar{u})\right\} = \frac{1}{\tilde{R}} \tag{2.27}$$

Then the eigen value problem (2.15), (2.16),(2.18), (2.20) has a least eigen value R and the unstarred flow $\bar{q}(\bar{x},t)$ is stable if the Reynolds number $R_2 < \tilde{R}$. Further, given a complete set of eigen functions $(\bar{u}_i)$ and the corresponding eigenvalues R_i, we have $\tilde{R} = \text{g.l.b.}\ (R_i)$.

The variational problem seen above is constructed to obtain a critical estimate of the Reynolds number R of the primary flow flow for ensuring its universal stability by accentuating its role and prescribing the value of the Cosserat number. In principle it is conceivable to accentuate the role of the Cosserat number j; however this does not compel our attention for the reason that it is a number characteristic of the material and not of the flow under consideration.

3. Existence of periodic solutions.

Let the velocity $\bar{q}(\bar{x},t)$ and spin $\bar{w}(\bar{x},t) = \frac{1}{2}$ curl $\bar{q}(\bar{x},t)$ be prescribed at each point of the boundary $\partial D(t)$ and let the domain D(t) as well as the quantities prescribed on $\partial D(t)$ be periodic in t. We assume that to every continuous initial distribution of the velocity over the domain there exists a solution of the Cosserat flow equations valid for all time $t \geq 0$ and satisfying the prescribed conditions on the boundary. Further, we also presume that there is a solution for which the Reynolds number R_2 is such that ϵ_2 is positive and that this solution is equicontinuous in $\bar{x} = (x,yz)$ for all t.

Then there exists a unique, stable, periodic solution $\bar{q}(\bar{x},t)$ of the Cosserat flow equations in $D(t)$ which takes the prescribed values on the boundary $D(t)$.

Let $q(x,t)$ be the velocity vector field of the flow guaranteed by one of the conditions of the theorem and without any loss of generality, let the period of the assigned boundary values be one. The sequence of vector functions

$$\bar{f}_n(\bar{x}) = \bar{q}(\bar{x},n) \quad (n = 1,2,\ldots)$$

is bounded and equicontinuous in $\bar{x}$. Hence by Arzela's theorem [5] this sequence contains a subsequence which converges uniformly to a continuous vector function $\bar{A}(\bar{x})$ in the domain $D(t)$. We shall see that the <u>entire</u> sequence $\bar{q}(\bar{x},n)$ converges to $\bar{A}(\bar{x})$. If this is not true, there will be another subsequence converging uniformly to the continuous vector function $\bar{B}(\bar{x})$. Set

$$\bar{q}'(x,t) = \bar{q}(\bar{x},t+m-n), \quad t \geq 0 \tag{3.1}$$

and let $m > n$. The vector function $\bar{q}'(x,t)$ is a solution of the Cosserat flow equations and satisfies the prescribed boundary conditions. Let the kinetic energy of the flow $\bar{q}(\bar{x},t)$ be

$$T\left\{\bar{q}(\bar{x},t)\right\} = \frac{1}{2}\int \rho\,(\bar{q}(\bar{x},t))^2 \tag{3.2}$$

From the definitions of $\bar{q}(\bar{x},t)$ and $\bar{q}'(\bar{x},t)$ and a condition assumed in the beginning of the present section both the above flows are such that ϵ which stands for ϵ_2 is positive and from the discussion in Section (2) it follows that

$$T\left\{\bar{q}'(\bar{x},t) - \bar{q}(\bar{x},t)\right\} \leq T(0)\exp(-\epsilon t) \tag{3.2}$$

where $T(0) = T\left\{\bar{q}'(\bar{x},0) - \bar{q}(\bar{x},0)\right\}$

and this is bounded above by constant $\Lambda = \frac{8\pi}{3} d^3 V_0^2$. On putting t = n (3.3) we see that

$$T\left\{\bar{f}_m(x) - \bar{f}_n(x)\right\} \leq \Lambda \exp(-n\epsilon) \tag{3.4}$$

Allowing n to infinity in (3.4) and taking limits we see that

$$\lim_{m,n \to \infty} T\left\{\bar{f}_m - \bar{f}_n\right\} = 0. \tag{3.5}$$

The domain of integration in (3.4) and (3.5) is D(n) = D90).
If we allow m, n to infinity through sequences of integers such that $\bar{f}_m(\bar{x}) \to \bar{B}(\bar{x})$ and $\bar{f}_n(\bar{x}) \to \bar{A}(\bar{x})$ we see that (3.5) contradicts the earlier assumption that the vector functions $\bar{A}(\bar{x})$ and $\bar{B}(\bar{x})$ are different. Hence the assertion that the entire sequence $\bar{q}(\bar{x},n)$ converges to the continuous function $\bar{A}(\bar{x})$. By an assumption made earlier there exists then a flow $\bar{q}^*(x,t)$ such that

$$\bar{q}^*(\bar{x},0) = \bar{A}(\bar{x}) \tag{3.6}$$

We shall see that the solution $\bar{q}^*(\bar{x},t)$ is periodic and stable. Put

$$\bar{q}''(\bar{x},t) = \bar{q}(\bar{x}, t + n) \tag{3.7}$$

By (3.3)

$$T\left\{\bar{q}^*(x,t) - \bar{q}''(\bar{x},t)\right\} \leq T_1 \exp(-\epsilon t) \tag{3.8}$$

where

$$T_1 = T\left\{\bar{A}(\bar{x}) - \bar{q}(\bar{x},n)\right\} = T\left\{\bar{A}(\bar{x}) - \bar{f}_n(\bar{x})\right\} \tag{3.9}$$

On putting t = 1 in (3.9) we get

$$T\left\{\bar{q}^*(\bar{x},1) - \bar{f}_{n+1}(\bar{x})\right\} \leq T_1 \exp(-\epsilon) < T_1 \tag{3.10}$$

If we allow n to infinity in (3.10) we get

$$T\left\{\bar{q}^*(\bar{x},1) - \bar{A}(\bar{x})\right\} = 0 \qquad (3.11)$$

since T_1 as defined in (3.9) tends to zero when $n \to \infty$. From (3.11) we conclude that

$$\bar{q}^*(\bar{x},1) = \bar{A}(\bar{x}) = \bar{q}^*(\bar{x},0) \qquad (3.12)$$

and the function $\bar{q}^*(\bar{x},t)$ is periodic. We know that

$$T\left\{\bar{q}^*(\bar{x},t) - \bar{q}(\bar{x},t)\right\} \to 0 \text{ as } t \to \infty. \qquad (3.13)$$

Since both the functions $\bar{q}^*(\bar{x},t)$ and $\bar{q}(\bar{x},t)$ are equicontinuous, we have

$$\bar{q}^*(\bar{x},t) - \bar{q}(\bar{x},t) \to 0 \text{ as } t \to \infty \qquad (3.14)$$

and so

$$\max_{(\bar{x})} \bar{q}^*(\bar{x},t) - \max_{(\bar{x})} \bar{q}(\bar{x},t) \leq 0\ (t) \qquad (3.15)$$

Since $\bar{q}(\bar{x},t)$ is periodic, we see from the above that

$$\max_{\bar{x}} \bar{q}^*(\bar{x},t) \leq \max_{\bar{x}} \bar{q}(\bar{x},t) \qquad (3.16)$$

Hence the Reynolds numbers R for the flows are such that

$$R_2^2(\bar{q}^*) \leq R_2^2(\bar{q}) \qquad (3.17)$$

Since $\epsilon_2 > 0$ we conclude that the flow $\bar{q}^*(x,t)$ is stable. The above result is an extension of the result established for classical fluids by James Serrin [6] and for micropolar fluids by Lakshmana Rao [7].

It must be noted however that the boundary conditions prescribed must be compatible with a flow for which $\epsilon > 0$. This entails that on the boundary the prescribed values are sufficiently low. When the assigned conditions on the boundary are steady, the theorem assures the existence of a unique, stable, time-independent solution of the Cosserat flow equations, taking the prescribed boundary values. The condition regarding the existence for all time $t \geq 0$ of a solution corresponding to every continuous initial distribution is mathematically stringent, though we required this condition only for those initial data for which $\epsilon > 0$.

4. Decay of Kinetic Energy

Consider the motion of incompressible Cosserat fluid over a domain D bound by rigid walls D. The kinetic energy of the flow is

$$T = \tfrac{1}{2} \int \rho \, (q)^2 \tag{4.1}$$

and on the boundary we have

$$\bar{q} = \operatorname{curl} \bar{q} = 0 \tag{4.2}$$

From the momentum balance we can arrive at the equation

$$\rho \, \bar{q} \cdot \frac{\partial \bar{q}}{\partial t} = - \bar{q} \cdot \operatorname{grad} \left(p + \tfrac{1}{2} \rho (\bar{q})^2\right) - \mu \, \bar{q} \cdot \operatorname{curl} \operatorname{curl} \bar{q} - \eta \, \bar{q} \cdot \operatorname{curl} \operatorname{curl} \operatorname{curl} \operatorname{curl} \bar{q} \tag{4.3}$$

and hence by integration and use of divergence theorem we can arrive at the equation

$$\frac{dT}{dt} = -\mu \int (\operatorname{curl} \bar{q})^2 - \eta \int (\operatorname{curl} \operatorname{curl} \bar{q})^2 \tag{4.4}$$

In view of the inequalities ([8] , [9] , [10])

$$\int (\text{curl } \bar{q})^2 \geqslant \frac{80}{d^2} \int (q)^2 \tag{4.5}$$

$$\int (\text{curl curl } \bar{q})^2 \geqslant \frac{6400}{d^4} \int (q)^2 \tag{4.6}$$

we find that

$$\frac{dT}{dt} \leq -\frac{160\mu}{\rho d^2} (1 + 80\ C_0)\ T \tag{4.7}$$

This substantially proves that the kinetic energy of the flow of a Cosserat fluid over a rigid container decreases faster than the exponential and that the decay constant

$$\alpha = \frac{160\mu(1 + 80\ C_0)}{\rho d^2} \tag{4.8}$$

is larger for Cosserat fluids than for the classical Navier-Stokes fluids. The result is an extension of a classical result due to J.Kampe de Feriet [11] and R.Berker [12] and corresponding results in micropolar fluids have been noticed by Lakshmana Rao [13], [10].

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FOUR POINT BOUNDARY VALUES PROBLEMS - EXISTENCE AND UNIQUENESS

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1. INTRODUCTION

This paper gives a guarantee for the existence and uniqueness of solutions of four-point boundary value problems associated with the differential equation

$$y^{(iv)} = f(x, y, y', y'', y''') \tag{1.1}$$

where $f(x, y, z, v, w)$ is assumed to be continuous on a subset of R^5, initial value problems associated with (1.1) exists and are unique and extend throughout a fixed subinterval of R.

In section 2, a monotonicity restriction of f, insures that the following two-point boundary value problems

$$y^{(iv)} = f(x, y, y', y'', y'') \tag{1.2$_i$}$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y^{(i)}(x_2) = m,\ y'''(x_2) = k\ (i = 1, 2)$$

$$y^{(iv)} = f(x, y, y', y'', y''') \tag{1.3$_i$}$$

$$y(x_2) = y_2,\ y^{(i)}(x_2) = m,\ y'''(x_2) = k,\ y(x_3) = y_3 \quad (i = 1, 2)$$

$$y^{(iv)} = f(x, y, y', y'', y''') \tag{1.4$_i$}$$

$$y(x_2) = y_2,\ y^{(i)}(x_3) = p,\ y'''(x_2) = k,\ y(x_3) = y_3$$

$$y^{(iv)} = f(x, y, y', y'', y''') \tag{1.5$_i$}$$

$$y(x_3) = y_3,\ y^{(i)}(x_3) = p,\ y'''(k_3) = k_1,\ y(x_4) = y_4,$$

have atmost one solution, and with added hypothesis that solutions exist to (1.2_i), (1.3_i), (1.4_i) and (1.5_i) $(i = 1,2)$, a unique solution to the four-point boundary value problem is constructed. This is done by 'matching' the solutions of the problems (1.2_2), (1.3_2), (1.4_2) and (1.5_2).

Section 3, investigates the conditions under which solutions exist to the problems (1.2_i), (1.3_i), (1.4_i) and (1.5_i) $(i = 1,2)$. Length of interval estimates for the existence and uniqueness of solutions of two-point boundary value problems are derived.

2. Existence and uniqueness of solutions of four-point boundary Value Problems

In this section we are concerned with the criteria under which solutions of (1.1) which satisfy boundary conditions at point boundary value problems. Theorem 2.1 displays the technique of 'matching' solutions of two-point boundary value problem to obtain a unique solution of four-point boundary problems.

THEOREM 2.1. Let $f : [x_1, x_4] \times R^4 \to R$ and let $y_1, y_2, y_3, y_4, h, k, \in R$ with $x_1 < x_2 < x_3 < x_4$ and suppose that

(i) For each $m \in R$ there exist solutions of (1.2_i), (1.3_i), (1.4_i) and (1.5_i) $(i = 1,2)$.

(ii) For each $m \in R$ and each t there exists atmost one solution of each of the following boundary value problems

$$y^{(iv)} = f(x, y, y', y'', y''')$$

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y''(t) = m, \quad y'''(x_2) = k \tag{2.1}$$

When $t \in (x_1, x_2]$

$$y^{(iv)} = f(x, y, y', y'', y''')$$

$$y(x_2) = y_2,\ y''(t) = m,\ y'''(x_2) = k,\ y(x_3) = y_3 \tag{2.2}$$

When $t \in [x_2, x_3]$

$$y^{(iv)} = f(x, y, y', y'', y''') \tag{2.3}$$

$$y(x_3) = y_3,\ y'''(x_3) = k_1,\ y''(t) = p,\ y(x_4) = y_4$$

When $t \in [x_3, x_4]$

Then there exists unique solution of the problem

$$y^{(iv)} = f(x, y, y', y'', y''')$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) = y_4. \tag{2.4}$$

__Proof.__ Let $\phi(\cdot, m)$ denote a solution of (1.2_2) with second derivative m at $x = x_2$. It will be first shown that $\varphi'(x_2, m)$ is an increasing function of m in the interval $(x_1, x_2]$. By definition if $m_2, m_1, \varphi''(x_2, m_2)\ \varphi''(x_2, m_1)$ Furthermore, it is claimed that $\varphi''(x, m_2), \varphi''(x, m_1)\ \forall x \in (x_1, x_2]$ To the contrary suppose there exists an $x \in (x_1, x_2)$ such that $\varphi''(x, m_2), \varphi''(x, m_1)$. Since $\varphi''(x, m_2)$ and $\varphi''(x, m_1)$ are continuous functions, there exists a point $\lambda \in (x, x_2)$ such that $\varphi''(\lambda, m_2)$ $\varphi''(\lambda, m_1)$. Hence $\varphi(\cdot, m_2)$ and $\varphi(\cdot, m_1)$ are two distinct solutions of (2.1) which is a contradiction. Since $\varphi(x_1, m_1) = \varphi(x_1, m_2)$ and $\varphi(x_2, m_2) = \varphi(x_2, m_1)$ there exists and (x_1, x_2) such that $\varphi'(\cdot, m_2) = \varphi'(\cdot, m_1)$. This together with the above claim implies that $\varphi'(x, m_2) = \varphi'(x, m_1)$ for all $x \in (x_1, x_2]$ Thus $\varphi'(x_2, m)$ is a strictly increasing function of m.

Let $\varphi(x,m)$ denote the solution of (1.3_2) with second derivative m at $x = x_2$. A similar reasoning given above demonstrates that $\varphi(x,m)$ is a strictly decreasing function of m alone in the interval $[x_2, x_3)$.

Since $\varphi'(x_2,\cdot)$ is a function of m alone $\varphi''(x_2,m_2)\,\varphi''(x_2,m_1)$ when m_2, m_1 and range of $\varphi'(x_2,\cdot)$ is R it follows that $\varphi'(x_2,\cdot)$ is a strictly increasing continuous function of m alone with range R. Similarly $\varphi(x_2,\cdot)$ is a strictly decreasing continuous function of m alone with range R. Therefore there exists a unique m_0 such that $\varphi'(x_2,m_0) = \varphi'(x_2, m_0)$.

Let $(\cdot, p)$ and $(\cdot, p)$ be solutions of (1.4_2) and (1.5 (1.5_2) respectively. Then in a similar way it can be concluded that $\varphi'(\cdot, p)$ is an increasing function of alone in the interval $(x_2, x_3]$ and $(\cdot, p)$ is a decreasing funtion of p alone in the interval (x_3, x_4). Hence as above there exists a unique p_0 such that $\theta'(x_3, p_0) = \eta'(x_3, p_0)$.

Claim: $\psi(x, m_0) = \theta(x, p_0)$ for all $x \in [x_2, x_3]$.

$\psi'\psi p$. We have $\psi(x_2, m_0) = \theta(x_2, p_0)$, $\psi''(x_2, m_0) = \theta'''(x_2, m_0)$ and $\psi(x_3, m_0) = \theta(x_3, p_0)$. If $\theta''(x_2, p_0) \neq \psi''(x_2, m_0)$ then θ and ψ are two distinct solutions of (2.2) which is a contradiction. Thus $\psi(x) \equiv \theta(x)\ \forall\ x \in [x_2, x_3]$.

By definition of $\theta(x,m_0)$, $\psi(x, m_0)$ and $\eta(x, p_0)$

$\chi(x)$ defined by

$$\chi(x) = \begin{cases} \phi(x, m_0), & x_1 \le x \le x_2 \\ \psi(x, m_0) & x_2 \le x \le x_3 \\ \eta(x, p_0) & x_3 \le x \le x_4 \end{cases}$$

is a solution of (2.4).

To establish uniqueness, suppose Φ and Ψ be solutions of (2.4). Then $\Phi(x_1) = \Psi(x_1)$, $\Phi(x_2) = \Psi(x_2)$, $\Phi(x_3) = \Psi(x_3)$ and $\Phi(x_4) = \Psi(x_4)$ follows that there exists $\alpha \in (x_1, x_2)$, $\beta \in (x_2, x_3)$ and $\gamma \in (x_3, x_4)$ such that $\Phi'(\alpha) = \Psi'(\alpha)$, $\Phi'(\beta) = \Psi'(\beta)$ and $\Phi'(\gamma) = \Psi'(\gamma)$, which again imply that there exists $\xi \in (\alpha, \beta)$ and $\zeta \in (\beta, \gamma)$ such that $\Phi''(\xi) = \Psi''(\xi)$ and $\Phi''(\zeta) = \Psi''(\zeta)$, ... imply that there exists ... such that ... and ... which is a contradiction to (ii). Thus uniqueness is established.

3. Length of Interval Estimates for two-point Boundary value problems

In this section results are established giving conditions for the validity of hypothesis (i) in Theorem 2.1. Length of interval estimates for the existence and uniqueness of solutions of two-point boundary value problems are derived. We assume throughout that f satisfies the Lipschitz condition

$$|f(x, y_1, z_1, v_1, w_1) - f(x, y_2, z_2, v_2, w_2)| \leq k_0|y_1 - y_2| + k_1|z_1 - z_2| + k_2|v_1 - v_2| + k_3|w_1 - w_2|. \quad (3.1)$$

Theorem 3.1. Let $y_1, y_2, y_3, y_4, x_2, k, k_1 \in R$, $x_1 < x_2 < x_3 < x_4$ and suppose that there exists a constant $N > 0$ such that f is continuous and $|f(x,y,z,v,w)| \leq N$ for all $x \in [x_1, x_4]$

Then there exit solutions of (1.2_i), (1.3_i), (1.4_i) and (1.5_i) $(i = 1, 2)$.

Proof. The proof of existence of a solution of the boundary value problem (1.3_2) will be given. Similar proofs will establish existence of solutions of the other boundary value problems. Let $\phi(x, \alpha)$ denote a solution of the initial value problem:

$$y^{(iv)} = f(x, y, y', y'', y''')$$

$$y(x_2) = y_2, \quad y'(x_2) = k, \quad y''(x_2) = \alpha, \quad y'''(x_2) = m.$$

Write
$$\phi(x_3, \alpha) = \phi(x_2 + (x_3 - x_2), \alpha)$$
$$= \phi(x_2, \alpha) + (x_3 - x_2)\phi'(x_2, \alpha) + \frac{(x_3 - x_2)^2}{\lfloor 2}\phi''(x_2, \alpha)$$
$$+ \frac{(x_3 - x_2)^3}{\lfloor 3}\phi'''(x_2, \alpha) + \frac{(x_3 - x_2)^4}{\lfloor 4}\phi^{4}(x_2 + \theta(x_3 - x_2), \alpha)$$
$$0 < \theta < 1$$

Therefore
$$\phi(x_3, \alpha) \leq y_2 + (x_3 - x_2)k + \frac{(x_3 - x_2)^2}{\lfloor 2}\alpha + \frac{(x_3 - x_2)^2}{\lfloor 3}m$$
$$+ \frac{(x_3 - x_2)^4}{\lfloor 4}N.$$

From which it can be observed that there exists α_0, such that

$$\phi(x_3, \alpha_0) < y_3$$

If the above proceedure is repeated using the lower bound on f, it follows that

$$\phi(x_3, \alpha) > y_3.$$

Since the solution of an initial value problem continuously depends upon the initial conditions, there exists α_2, such that

$$\phi(x_3, \alpha_2) = y_3.$$

Thus $\phi(x, \alpha)$ is a solution of (1.3_2).

Theorem 3.2. Let $f : [x_1, x_2] \times R^3 \to R$ with $x_1 < a < b < x_2$.

If $f(x, y, z, v, w)$ satisfies a Lipschitz condition with positive real constants k_0, k_1, k_2 and k_3 and $\frac{k_0 h^4}{24} + \frac{2k_1 h^3}{81} + \frac{k_2 h^2}{4} + \frac{2 k_3 h}{3} < 1$ where $h = (b-a)$ and $t_1, t_2, y_1, y_2, y_3, y_4 \in R$ with $a \le t_1 \le t_2 \le b$ then there exists a unique solution of the boundary value problem (1.3_2).

Proof. The proof is analogous as in [2]

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ON SIGNAL PROPAGATION VELOCITY AND PHASE VELOCITY

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One of the most common and simple methods to study wave propagation in a continuum physical system is to linearise the governing equations and to derive the dispersion relation

$$f(\omega, \underline{k}) = 0 \quad \text{or} \quad \omega = \Omega\,(\underline{k}) \tag{1}$$

connecting the frequency ω and the wave number vector $\underline{k}$. We can even write the solution of the general initial or boundary value problem by superposition of harmonic waves with ω and $\underline{k}$ satisfying the dispersion relation (1). The phase and group velocities of the harmonic waves is given by

$$v_{ph} = \frac{\omega}{\underline{k}} \;, \quad V_{\underline{g}} = \text{grad}\,\Omega \tag{2}$$

respectively, where in the second expression the gradient of Ω is taken over the space of the wave number vector. The phase velocity represents the velocity of an individual periodic sinusoidal wave perpendicular to the wave crest and the group velocity represents the energy propagation velocity (Lighthill (1965), Hayes (1970), Bhatnagar (1978)). Therefore, it turns out that the bulk of the wave moves with the group velocity, which is, therefore, more important than the phase velocity. The question arises, what does the phase velocity represent ? It represents the velocity of propagation of the individual harmonic waves in a direction normal to the wave-crest. Does the

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phase velocity give a measure of the signal propagation velocity ? By signal propagation velocity we mean the velocity with which a disturbance, which is initially non-zero only in a closed bounded domain, spreads at subsequent times. Since an arbitrary perturbation can be expressed as a superposition of harmonic waves with the help of the Fourier integral theorem, it appears that the signal propagation velocity in the system can be obtained by taking the maximum of the phase velocity with respect to the components of the wave number. We shall show that the phase velocity does not give even this information. Infact, for the surface water waves, the phase velocity is bounded but the signal propagation v velocity is infinite, a result which seems to be wrong at first sight.

Let us consider the surface water waves under the action of gravity assuming the medium to be incompressible and assuming that the motion is two dimensional. The motion of water is governed by the equation

$$\phi_{xx} + \phi_{yy} = 0 , \quad -\infty < x < \infty, \quad -h < y < o \qquad (3)$$

with the boundary conditions

$$\phi_{tt} + g\phi_y = 0 \quad \text{on} \quad y = 0 \qquad (4)$$

and

$$\phi_y = 0 \qquad \text{on} \quad y = -h \qquad (5)$$

where ϕ is the velocity potential, g the acceleration due to gravity in the negative y direction and h the depth of the undisturbed water assumed to be constant. It is well known that this system gives surface water waves moving in the horizontal direction along the x-axis. The linear theory gives the following dispersion relation see Whitham (1974)

$$\omega^2 = gk \tanh (kh) \tag{6}$$

Therefore, the phase velocity satisfies

$$v_{ph}^2 = \left(\frac{\omega}{k}\right)^2 = \frac{g \tanh (kh)}{k} \tag{7}$$

The right hand side of (7) is an even function of k, decreases monotonically as k varies from o to ∞ and attains the maximum value gh at $k = 0$. The phase velocity is bounded and satisfies $|v_{ph}| < \sqrt{gh}$.

However, the signal propagation velocity of surface water waves (assuming the medium to be incompressible) is infinite. This follows from the fact that the irrotational motion of the water is governed by the Laplace equation in the variables x and y. Therefore, if we have a non-zero confined disturbance on the surface of water, it will affect the entire flow region for all x and $-h < y < o$. Viewing this as an initial value problem, we note that an initially confined disturbance on the surface of water is felt every where in the flow field at the same instant of time. The disturbance ceases to be confined and the signal propagation velocity is infinite. This is clearer from the following considerations. The function $f(x,t) = \phi(x,y = 0,t)$, the value of the velocity potential at the surface, satisfies the pseudo-partial differential equation [Ravindran and Prasad (1978)]

$$\left[\frac{\partial^2}{\partial t^2} + g\, P \tanh (hP) \right] f(x,t) = 0 \tag{8}$$

where $P = -i \frac{\partial}{\partial x}$. For the equation (8) we can prescribe initial values in the form

$$f(x,o) = f_o(x), \quad f_t(x,o) = f_1(x) \tag{9}$$

The Fourier transform (with respect to x) $\hat{f}(\xi,t)$ of the solution when $f_o(x) \equiv 0$ is given by

$$\hat{f}(\xi,t) = \hat{f}_1(\xi) \frac{\sin \left\{ t \sqrt{g\, \xi \tanh (h\xi)} \right\}}{\sqrt{g\, \xi \tanh (h\xi)}} \tag{10}$$

where $\hat{f}_1(\xi)$ is the Fourier transform of $f_1(x)$. If the function $f_1(x)$ is of compact support, the function $\hat{f}_1(\xi)$, when ξ is replaced by a complex variable ζ, is necessarily an entire analytic function of ζ [Hormander, (1964)] . However, if we replace ξ by ζ in the rest of the factors on the right hand side of (10), due to the presence of the term $\tanh (h\zeta)$ the function $\hat{f}(\zeta, t)$ is not an entire analytic function of ζ . Consequently the support of $f(x,t)$ is not bounded in x for any value of $t > 0$. The signal propagation velocity is, therefore, not finite. We conclude that the phase velocity is the velocity of propagation of a periodic wave of a permanent form and has no relation to the signal propagation velocity.

If we take the compressibility of the liquid into account, we expect a signal in water waves to travel with a finite speed.

The governing equation of motion of the water for the velocity potential ϕ is the wave equation

$$\phi_{tt} - a^2(\phi_{xx} + \phi_{yy}) = 0 \tag{11}$$

where a is the speed of sound in water. Taking the boundary conditions (4) and (5) into account, we get the following dispersion relation for a harmonic wave propagating along the x-axis:

$$\omega^2 - g\sqrt{k^2 - \omega^2/a^2}\ \tanh\left\{ h\sqrt{k^2 - \omega^2/a^2} \right\} = 0, \quad \text{for } \omega^2 < a^2 k^2 \tag{12}$$

and

$$\omega^2 + g\sqrt{(\omega^2/a^2) - k^2}\ \tanh\left\{ h\sqrt{(\omega^2/a^2) - k^2} \right\} = 0 \quad \text{for } \omega^2 > a^2 k^2 \tag{13}$$

However, it is simpler to show the finiteness of the speed of propagation from the theory of a hyperbolic equation applied to (11) in (x,yt)-space. The mathematical problem is to find the solution of (11) in the semi-infinite domain $t > 0$ between the two planes $y = 0$ and $v = -h$ satisfying the homogeneous boundary conditions (4) and (6) on the planes $y = 0$ and $y = -h$ respectively and the initial conditions prescribed at $t = 0$

$$\begin{aligned} (x,y,o) &= \phi_o(x,y) && \text{in } -\infty < x < \infty, \\ {}_t(x,y,o) &= \phi_1(x,y) && -h < y < o \end{aligned} \tag{14}$$

When ϕ_o and ϕ_1 are of compact support in x, say they are non-zero only in $x_1 < x < x_2$, then the solution will also be non-zero only in the domain which is bounded by the characteristic planes of (11) passing through the lines $x = x_1$, $t = 0$ and $x = x_2$, $t = 0$. These planes are $x + at = x_1$ and $x - at = x_2$ respectively. This shows that the surface water waves propagating along the x-axis in the plane $y = 0$ have a finite singal propagation velocity when compressibility of water is included. We cannot infer about it by analysing the phase valocity v_{ph}.

The motion of the fluid is governed by a hyperbolic partial differential equation in the physical space, i.e., (x,y,t)-space but the surface wave, even with the inclusion of the compressibility, is governed by an operator which consists of the equation (11) along with the two boundary conditions (4) and (5). This leads to an operator equation in the propagation space which is (x,t)-space here.

For a hyperbolic partial differential equation, the finiteness of the speed of propagation follows immediately [Courant and Hilbert (1962)]. Conversely if the state of a physical system is characterised by n variables, if the motion of the system is uniquely determined by the knowledge of these variables at a given time and if the signal propagation velocity is finite, then it follows that the system is governed by a hyperbolic partial differential equation [Lax (1963)]. It is clear from Lax's article that if the governing equation is not a hyperbolic partial differential equation, then one of the two conditions must necessarily break down, i.e. either 1) the signal propagation velocity is finite or 2) motion of the system is not uniquely determined by the knowledge of the n-variables at a given time. Which of these conditions breaks down must be separately investigated in each case.

In any system sustaining wave propagation and governed by a differential or a more general operator equation, we have a dispersion relation and it is this that we we will have to exploit in order to test if the signal propagation velocity is finite. If the dispersion relation is algebraic both in ω and k and the power of ω is the same as the degree of the polynomial relation, then following the usual theory of hyperbolic equations [John (1977)] we can show that the signal propagation velocity is finite if the Garding conditions is satisfied. Following the same procedure, we can show that if the dispersion relation is algebraic in ω, but not necessarily in k (as in the case of surface water waves), then the signal propagation velocity will be finite provided a Garding-type condition is satisfied for complex k, i.e., $\mathrm{Im}\,\omega = O(|\mathrm{Im}k| + 1)$. Still more complicated is the case when the dispersion relation is not algebraic in ω (as in the case of surface water waves with compressibility). An analysis of this would be rewarding.

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ABSTRACT

This paper analyses the implications of phase velocity, dispersion relation and signal propagation velocity in surface water waves and examines if these are interrelated. The effect of compressibility on the signal propagation velocity in the case of surface water waves is also investigated.

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GRONWALL-BELLMAN TYPE INEQUALITIES TO DISTRIBUTIONS

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Differential and integral inequalities play a vital role in the study of stability, boundedness and various other qualitative properties of solutions of differential and integral equations (cf. [6,9]). Therefore, the development of such inequalities needs no further explanation. The basic inequality that is prominent in this direction is the following

LEMMA [Gronwall, 3].

Let $f(t)$ and $g(t)$ be nonnegative, continuous functions on $0 \leq t \leq T$, for which the inequality

$$f(t) \leq \eta + \int_0^t g(s)\, f(s)\, ds, \quad t \in [0,T]$$

holds, where $\eta \geq 0$ is a constant. Then

$$f(t) \leq \eta \exp \int_0^t g(s)\, ds \qquad t \in [0,T] .$$

Due to various motivations linear, nonlinear and discrete generalizations of this lemma have been obtained and used considerably in various contexts (cf. [4] and the reference therein). Gronwall's Lemma has been extended in [10], the modified Stieltjes integrals (which does not hold in general). Also in [2] a few generalizations have been obtained and are used to study the stability properties of solutions of measure differential equations.

Now we present analogues of this lemma for a class of distributions. We assume some elementary concepts of analysis and distribution theory [1] and give the following auxiliary result which is crucially employed in proving our main results.

THEOREM 1.

Let f and g be two real-valued functions on the real line R such that both are of bounded variation on every compact subinterval of R. Then fg defines a distribution and the derivative of fg in the sense of distributions is equal to the locally summable function $(fg)'$ given by

$$f'(x)\ g(x) + f(x)\ g'(x) \quad \text{for almost all } x.$$

That is

$$D(fg) = (Df)\,g + f(Dg),$$

where 'D' denotes the derivative of a function in the sense of distributions.

THEOREM 2.

Let $y(t)$ and $u(t)$ be nonnegative functions of bounded variation with $u(t)$ increasing and $k(t)$ be a nonnegative integrable function with respect to $u(t)$ on $0 \leq t \leq T$, for which the inequality

$$y(t) \leq \beta + \int_0^t k(s)\ y(s)\ du(x), \qquad 0 \leq t \leq T$$

holds, where $\beta \geq 0$ is a constant. Then

$$y(t) \leq \beta \left[1 + \int_0^t k(s)\left[\exp\left(\int_s^t k(\eta)\, du(\eta)\right)\right] du(s),\right]$$

for $0 \leq t \leq T$.

Now we generalize Theorem 2 in the following way.

THEOREM 3.

Let $y(t)$, $f(t)$ and $u(t)$ be functions of bounded variation, with $u(t)$ increasing and $k(t)$ be a nonnegative function integrable with respect to $u(t)$ over $0 \leq t \leq T$. Further let the inequality

$$y(t) \leq f(t) + \int_0^t k(s)\, y(s)\, du(s), \quad t \in [0,T]$$

hold. Then

$$y(t) \leq f(t) + \int_0^t k(s)\, f(s) \left[\exp\left(\int_0^t k(\eta)\, du(\eta)\right)\right] du(s),$$

for all $t \in [0,T]$.

It is interesting to note that if we suppose u is absolutely continuous on $[0,T]$, where $u'(t)$, which exists a.e. is nonnegative or in particular $u(t) \equiv t$, our results reduce to many celebrated inequalities of this type (cf. [2-5, 10]).

Consider the differential equation

$$Dx = F(t,x)\, Du \tag{1}$$

where $F : [0,\infty) \times R^n \longrightarrow R^n$, R^n the Euclidean n-space. $u : [0,\infty) \longrightarrow R$ is a right continuous function of bounded variation on compact subsets of $[0,\infty)$ and Du denotes the distributional derivative of the function u. Here Du will be identified with Stieltjes measure and will have the effect of instantaneously changing the state of the system at the discontinuities of u.

Let S be an open connected set in R^n and I be an interval with left end point $t_o \geq 0$. A function $x(.) = x(.,t_o,x_o)$ is said to be a solution of (1) through (t_o,x_o) on the interval I if $x(.)$ is right continuous function $\in BV(I,S)$, $x(t_o) = x_o$ and the distributional derivative of $x(.)$ on (t_o,α) for any arbitrary $\alpha \in I$ satisfies (1). Assume that for each $x(.) \in BV(I,S)$, $F(t, x(t))$ is integrable with respect to the Lebesgue-Stieltjes measures du. Then, as in [7], $x(.)$ is a solution of (1) through (t_o,x_o) on $J = [t_o, t_o + b]$ if and only if satisfies the integral equation

$$x(t) = x_o + \int_{t_o}^{t} F(s, x(s))\, du(s) \qquad (2)$$

for $t \in J$.

Now, as an application to our results we shall prove the uniqueness of solution of (1) under the assumption that F is Lipschitzian. Let, if possible, $x(t)$ and $y(t)$ be two solutions of (1) through the same point (t_o,x_o), i.e., $x(t_o) = x_o = y(t_o)$.

Let $Z(t) = |x(t) - y(t)|$. Clearly $Z(t_o) = 0$, and from (2) we see that $Z(t) \leq L \int_{t_o}^{t} Z(s)\, dv_u(s)$, where v_u is the total variation function of u, and L is the Lipschitz constant. Now Theorem 2, with $\beta \equiv 0$ and $k \equiv L$, completes the proof.

Details and proofs of the above three theorems, as well as further results, will appear in [8] .

Finally, I would like to express my deep gratitude to Professor P. Suryanarayana for his advice and interest.

This work is dedicated to my parents Smt. V. Satyavathi and (late) Sri V. Seshagiri Rao.

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DIFFERENTIAL INEQUALITIES AND SINGULAR PERTURBATION PROBLEMS

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ABSTRACT :

This article deals with singularly perturbed linear, quasi-linear boundary and initial value problems for vector second and first order ordinary differential eeuqations. It also considers singular perturbation problems for systems of partial differential equations of elliptic and parabolic type. Making use of the theory of differential inequalities, estimates for solution or/and its derivatives are obtained.

1. Introduction+

The present article is aimed at describing our work on the use of differential inequalities in the discussion of certain aspects of solutions of singularly perturbed linear, quasi-linear and non-linear boundary and initial value problems for vector second and first order ordinary differential equations (ODEs) and systems of partial differential equations (PDEs) of elliptic and parabolic type.

+Throughout this paper we shall use the notation f : = g to mean f is defined by g.

* Paper presented by U.N. Srivastava

The article is divided into two parts. Part A contains results concerning singularly perturbed linear, quasi-linear and non-linear boundary and initial value problems for vector second and first first order ODEs. In part B, we consider singular perturbation probelms for systems of PDEs of elliptic and parabolic types. In the following we are concerned only with weakly coupled systems [1, p.189] . Infact, in part A, we focus our attention to the following boundary and initial value problems.

I. Linear Boundary Value Problems.

$$(1.1)\quad \begin{cases} P_i y := \varepsilon y_i'' + \alpha_i(t)\, y_i' - \sum_{j=1}^{N} \beta_{ij}(t)\, y_j = \gamma_i(t), \quad t \in D := (a, b), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1(1)N, \\ R_i y := \begin{pmatrix} y_i(a) \\ y_i(b) \end{pmatrix} \text{ or } \begin{pmatrix} y_i(a) \\ y_i'(b) \end{pmatrix} \text{ or } \begin{pmatrix} y_i(a) - \varepsilon y_i'(a) \\ y_i'(b) \end{pmatrix} = \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $y = (y_1, \dots, y_N)$ and a 'prime' denotes differentiation with respect to 't' .

II. Quasi-linear Boundary Value Problems.

$$(1.2)\quad \begin{cases} P_i y := \varepsilon^{h_i} y_i'' + \alpha_i(t, y, \varepsilon) y_i' - \sum_{j=1}^{N} \beta_{ij}(t, y, \varepsilon) y_j = \gamma_i(t, y, \varepsilon), \quad t \in D, \\ R_i y := \begin{pmatrix} y_i(a) \\ y_i(b) \end{pmatrix} \text{ or } \begin{pmatrix} y_i(a) \\ y_i'(b) \end{pmatrix} = \begin{pmatrix} A_i(\varepsilon) \\ B_i(\varepsilon) \end{pmatrix}, \quad i = 1(1)N. \end{cases}$$

III. Nonlinear Boundary Value Problems.

$$(1.3)\quad \begin{cases} P_i y := \varepsilon y_i'' - f_i(t, y_1, \dots, y_N, y_i') = 0, \quad t \in D, \\ R_i y := \begin{pmatrix} y_i(a) \\ y_i(b) \end{pmatrix} \text{ or } \begin{pmatrix} y_i(a) \\ y_i'(b) \end{pmatrix} \text{ or } \begin{pmatrix} y_i(a) - \varepsilon y_i'(a) \\ y_i'(b) \end{pmatrix} = \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \quad i = 1(1)N. \end{cases}$$

IV. Linear Initial Value Problems.

$$(1.4)\quad \begin{cases} \varepsilon y'' + \alpha(t,\varepsilon)y' + \beta(t,\varepsilon)y = \gamma(t,\varepsilon), \; t \in D := (a,b], \\ y(a,\varepsilon) = A(\varepsilon), \quad y'(a,\varepsilon) = B(\varepsilon). \end{cases}$$

V. Nonlinear Initial Value Problems.

$$(1.5)\quad \begin{cases} \dfrac{dx_i}{dt} = u_i(t, x_1, \dots, x_N, y_1, \dots, y_M, \varepsilon), \; x_i(a,\varepsilon) = A_i(\varepsilon), \\ \qquad i = 1(1)N, \quad t \in D, \\ \varepsilon \dfrac{dy_j}{dt} = v_j(t, x_1, \dots, x_N, y_1, \dots, y_M, \varepsilon), \; y_j(a,\varepsilon) = B_j(\varepsilon), \\ \qquad j = 1(1)M, \quad t \in D. \end{cases}$$

In part B, we treat the following elliptic and parabolic problems.

VI. Linear Elliptic Boundary Value Problems.

$$(1.6)\quad \begin{cases} P_i u := \varepsilon^{p_i} P_{i2} u + P_{i1} u = f_i(x), \; x \in D \subset R^n, \\ \qquad D \text{ being a bounded domain}, \quad p_i \geqslant 0, \\ R_i u := u_i(x) = g_i(x), \quad x \in \partial D \text{ (the boundary of D)}, \end{cases}$$

where

$$(1.7)\quad \begin{cases} P_{i2} u := \sum\limits_{j=1}^{N} a_{ij} u_j + \sum\limits_{k=1}^{n} b_{ik} u_{i,k} - \sum\limits_{k,l=1}^{n} c_{ikl} u_{i,kl}, \\ P_{i1} u := \sum\limits_{j=1}^{N} \alpha_{ij} u_j + \sum\limits_{k=1}^{n} \beta_{ik} u_{i,k}, \quad i = 1(1)N, \\ u = (u_1, \dots, u_N), \; u_{i,k} := \partial u_i / \partial x_k, \; u_{i,kl} := \partial^2 u_i / \partial x_k \partial x_l. \end{cases}$$

VII. Linear Parabolic Boundary Value Problems.

$$(1.8)\quad \begin{cases} P_i u := \partial u_i/\partial t + \varepsilon^{p_i} P_{i2} u + P_{i1} u = f_i(t,x),\ (t,x) \in G_p, \\ R_i u := \begin{cases} u_i(t,x) = g_i(t,x),\ (t,x) \in R_p - R_\nu, \\ \partial u_i/\partial\nu + \sum_{j=1}^{N} d_{ij} u_j = h_i(t,x) \text{ on } R_\nu,\ i = 1(1)N, \end{cases} \end{cases}$$

where P_{i1} and P_{i2} are given by (1.7), $G_p := (0,T] \times D$, $D \subset R^n$ being a bounded domain, is the parabolic domain [2] , R_p is the parabolic boundary and R_ν is a part of the boundary $(0,T] \times \partial D$, $\partial()/\partial\nu$ is the outer normal derivative [2] .

VIII. Nonlinear Elliptic Boundary Value Problems.

$$(1.9)\quad \begin{cases} P_i u := F_i(x, u, u_{i,j}, \varepsilon^{p_i} u_{i,jk}) = f_i(x),\ x \in D, \\ R_i u := u_i(x) = g_i(x),\ x \in \partial D,\ i = 1(1)N. \end{cases}$$

IX. Nonlinear Parabolic Boundary Value Problems.

$$(1.10)\quad \begin{cases} P_i u := \partial u_i/\partial t + F_i(t, x, u, u_{i,j}, \varepsilon^{p_i} u_{i,jk}) = f_i(t,x), \\ \qquad (t,x) \in G_p, \\ R_i u := \begin{cases} u_i(t,x) = g_i(t,x),\ (t,x) \in R_p - R_\nu, \\ \partial u_i/\partial\nu + H_i(t,x,u) = h_i(t,x) \text{ on } R_\nu, \\ i = 1(1)N. \end{cases} \end{cases}$$

The boundary value problems (1.1) - (1.3) do not appear to have been studied using the theory of differential inequalities, although the scalar BVP has been treated extensively in the literature. For example, Dorr, Parter and Shampine [3] obtained results (estimates) for solutions of linear, quasi-linear and nonlinear two point scalar boundary value problems for second order ODEs by making use of the maximum principle. We extend a few results of [3] from scalar to vector problems and also obtain some new results. Chang [4] has obtained, using some other methods, results on the limiting behaviour of solution and its derivative of the vector BVP:

$$(1.11)\quad \begin{cases} \varepsilon y'' + A(t,y,\varepsilon)y' = f(t,y,\varepsilon), & 0 < t < 1, \\ y(0,\varepsilon) = \alpha(\varepsilon), \quad y(1,\varepsilon) = \beta(\varepsilon), \end{cases}$$

where $\varepsilon > 0$ is a small parameter, y, f, α, β are n-dimensional vector functions, A is an n x n matrix function. Habets [5] also considered the BVP:

$$(1.12)\quad \begin{cases} \varepsilon y'' + f(t,y,y',\varepsilon) = 0, & 0 < t < 1, \\ y(0) = \alpha(\varepsilon), \quad y(1) = \beta(\varepsilon), \end{cases}$$

where $y \in R^n, \quad 0 < \varepsilon \leq \varepsilon_0,$

and proved the existence of solution obtaining its form by the use of Schauder's theorem under appropriate assumptions.

Several authors [6,7,8] have studied singular perturbation problems via the maximum principle but only authors like Howes [9,10,11] and Harrris [12] exploited the theory of differential inequalities in the study of such problems. Subfunctions and solutions of differential inequalities have been used in their analysis in dealing with existence and properties of solutions of second order non-linear ODEs. It is only after going through this article carefully that one should be able to find the difference between the work of Howes and Harris and that of ours.

As far as authors' knowledge goes, it appears that much attention has not been devoted to systems (1.1) - (1.3) with the boundary conditions (BCs)

$$(1.13)\quad \begin{pmatrix} y_i(a) \\ y_i'(b) \end{pmatrix} = \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \quad i = 1(1)N,$$

and

$$(1.14)\quad \begin{pmatrix} y_i(a) - \varepsilon\, y_i'(a) \\ y_i'(b) \end{pmatrix} = \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \quad i = 1(1)N.$$

In fact, explicit estimates for solutions and their derivatives of the above mentioned problems are not given in the literature (using the theory of differential inequalities) even in the case of a single differential equation. We shall see in the following analysis that derivative of solutions of these problems feature non-uniform behaviour as a function of the independent variable and the small parameter appearing there where as solutions themselves do not. Here lies the difference between the BVPs of the first kind where solution itself shows nonuniform behaviour and the BVPs of the second and third kind where the non-uniformity occurs only in the derivative of solutions. Considerations on the BVPs (1.1)-(1.3) with the boundary conditions (1.14) have been motivated by physical problems arising in the field of chemical engineering. Several authors have discussed these problems and we mention in particular the work of O'Malley [13] and further references given there. Also, the paper by Freeman and Houghton [14] considers these types of works. We, in the following analysis, discuss these types of problems and obtain a priori bounds for solutions and their derivatives of the BVPs (1.1) - (1.3) with the BCs (1.13) and (1.14).

These results are entirely new in the literature and are derived by making use of the theory of differential inequalities.

Weinstein and Smith [15], using comparison techniques, have obtained a few estimates for solution of the IVP (1.4) under the assumption that the parameter ε satisfies the overdamping condition

$$4\varepsilon\beta_1 < \alpha_0^2, \quad \beta(t,\varepsilon) \le \beta_1, \quad \alpha(t,\varepsilon) \ge \alpha_0 > 0. \tag{1.15}$$

We, in the present article, obtain similar results for the IVP (1.4) not only by an entirely different approach but also without assuming the overdamping condition (1.15)

The nonlinear IVP (1.5) has been treated by Happensteadt [16] and Vasileva [17] by using some other methods. Making use of the matched asymptotic expansions, O'Malley [18] discussed the properties of solutions of the IVP (1.5) in the case when i = j = 1. From the standard theory of first-order differential inequalites [19], Howes [20] deduced the existence and the asymptotic behaviour of solution of the IVP for scalar first order ODEs. We here, using the theory of differential inequalities [2], discuss the IVP (1.5) through estimates of its solution.

Van Harten [21,22] has considered the BVP's (1.6) and (1.9) in the case when N = 1. Supposing that a formal approximation Z to the solution u of the BVP (1.6) or (1.9) (N = 1) is given such that the differential equation and the BC are satisfied upto the order $\chi(\varepsilon) = o(1)^{\dagger}$ in some norm, he has shown that under certain conditions the difference between the exact solution and the formal approximation can be made small. His proofs are based

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† o and O stand for Landau order symbols.

On the generalised maximum principle for linear and nonlinear elliptic equations and a contraction principle in a suitable Banach space. We, in the part B of this article, consider the vector BVPs (1.6) - (1.10) and obtain similar results.

The unique feature of this article is the uniformity in approach in the discussion of the above problems.

At this stage, it may be proper to mention that the discussion on singular perturbation problems for systems of PDEs of elliptic type described in part B has been accepted for publication in the Journal of Mathematical Analysis and Applications. Also, a part of the material of part A has been submitted for publication elsewhere. The rest of the present article, when expanded suitably, will be published elsewhere.

In order to save space and time, we present here only the main results obtained by us without going into details of the proof.

2. PRELIMINARIES

In this section we state (without proof) a few fundamental comparison theorems which shall be used frequently to obtain results in the rest of the sections. The basic ideas involved in proving these theorems are the same as in [2?,24,25] but the form of the test function, yet to be defined, seems to be slightly different from that given there. After reading the manuscript carefully, one should be able to see the difference between the test function introduced here and the function appearing in the generalized maximum principle [1].

THEOREM 2.1. Consider the nonlinear elliptic vector BVP (1.9) and assume that :

(i) $F_i(x, u, p, q)$, for fixed i, is weakly monotone decreasing with respect to n x n matrix 'q' in D [2, p.304] . This, in fact, is the ellipticity condition,

(ii) $F_i(x, u, p, q)$ is monotone decreasing with respect to u_j $j \neq i$, $i, j = 1(1)N$ (quasi-monotonicity condition),

(iii) there exists a vector function $v = (v_1, \ldots, v_N)$, known as a test function, such that

$$
(2.1) \begin{cases} v_i(x) > 0 \text{ on } \overline{D},\ v_i \in U := C^2(D) \cap C(\overline{D}), \\ P_i[u + cv] - P_i u > 0,\ i = 1(1)N,\ cv = (cv_1, \ldots, cv_N) \\ \text{for all } c \in R^+ \text{ (the set of all positive real numbers)}, \\ \text{for all } u,\ u_i \in U;\ i = 1(1)N. \end{cases}
$$

Then the following implication (2.2) is true for all

$z = (z_1, \ldots, z_N)$, $y = (y_1, \ldots, y_N)$, $z_i, y_i \in U$, $i = 1(1)N$.

$$
(2.2) \begin{cases} P_i z \leq P_i u \leq P_i y,\ R_i z \leq R_i u \leq R_i y,\ i = 1(1)N \\ \Longrightarrow \\ z_i(x) \leq u_i(x) \leq y_i(x) \text{ on } \overline{D},\ i = 1(1)N. \end{cases}
$$

In the case of the nonlinear parabolic BVP (1.10) the above theorem is still valid but with (iii) replaced by the condition that there exists a vector function $v = (v_1, \ldots, v_N)$, known as a test function, such that

$$(2.3)\quad \begin{cases} \vartheta_i = \vartheta_i(t,x) > 0 \text{ on } \bar{G} := G_p \cup R_p,\ \vartheta_i \in U := C^{(1,2)}(G_p) \cap \\ P_i[u + c\vartheta] - P_i u > 0,\ R_i[u + c\vartheta] - R_i u > 0, \\ c\vartheta = (c\vartheta_1, \ldots, c\vartheta_N),\ i = 1(1)N,\ c \in R^+, \text{ for all} \\ u,\ u_i \in U,\ i = 1(1)N. \end{cases}$$

If we consider the linear BVPs (1.6) and (1.8) the conditions (i) and (ii) of theorem 2.1 are identical with the following conditions:

(i) for fixed i, the matrix with elements $c_{ik\ell}$ is symmetric and positive definite with respect to k and ℓ,

(ii) $\varepsilon^{p_i} a_{ij} + \alpha_{ij} \leq 0,\ j \neq i,\ i,j = 1(1)N.$

THEOREM 2.2. Let $y = (y_1, \ldots, y_N)$ be a solution of the BVP (1.2) and assume that

(i) $\beta_{ij}(t, y, \varepsilon) \leq 0,\ j \neq i,\ i,j = 1(1)N.$

(ii) there exists a vector function $\vartheta = (\vartheta_1, \ldots, \vartheta_N)$, known as a test function, such that

$$(2.4)\quad \begin{cases} \vartheta_i(t) > 0 \text{ on } \bar{D},\ \vartheta_i \in Y := C^2(D) \cap C(\bar{D}) \text{ or} \\ \qquad C^2(D) \cap C^1(\bar{D}), \\ \tilde{P}_i \vartheta < 0,\quad R_i \vartheta > 0,\quad i = 1(1)N, \end{cases}$$

where

$$(2.5)\quad \tilde{P}_i(\,) := \varepsilon^{p_i}(\,)_i'' + \alpha_i(t,y,\varepsilon)(\,)_i' - \sum_{j=1}^{N} \beta_{ij}(t,y,\varepsilon)(\,)_j,$$

$$i = 1(1)N.$$

Then for all $x=(x_1,\dots,x_N)$, $z=(z_1,\dots,z_N)$, $x_i, z_i \in Y$, $i=1(1)N$, the following implication is true,

$$(2.6)\quad \begin{cases} \tilde{P}_i x \le \tilde{P}_i y \le \tilde{P}_i z,\ R_i z \le R_i y \le R_i x,\ i=1(1)N, \\ \Longrightarrow \\ z_i(t) \le y_i(t) \le x_i(t) \text{ on } \bar{D},\ i=1(1)N. \end{cases}$$

To give an idea as to how the proof of theorem 2.1 goes about, we consider the simpler case defined by the problem

$$(2.7)\quad \begin{cases} Pu := -u'' + a(t)u' + b(t)u = c(t),\ a<t<b, \\ Ru := \begin{pmatrix} u(a) \\ u(b) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}. \end{cases}$$

In other words we shall prove the following statement. Assuming that there exists a function $v(t)$ such that

$$(2.8)\quad v(t) > 0 \text{ on } \bar{D},\quad Pv > 0 \text{ on } D := (a,b),$$

We have for all $u, w \in Y := C^2(D) \cap C(\bar{D})$,

$$(2.9)\quad Pu \le Pw,\ Ru \le Rw \Longrightarrow u(t) \le w(t) \text{ on } \bar{D}.$$

Supposing that the implication (2.9) is not true for the BVP (2.7). Then there exists a point $t_1 \in D$ such that

$$u(t_1) - w(t_1) > 0.$$

Let

$$\alpha_0 = \max_{t\in\bar{D}} \left[[u(t)-w(t)]/v(t) \right] \in R^+,$$

and hence there exists a point $t = t_0$ such that

$$\begin{cases} \alpha_0 v(t_0) = u(t_0) - w(t_0),\quad t_0 \in D, \\ \alpha_0 v(t) \ge u(t) - w(t) \text{ on } \bar{D}. \end{cases}$$

That is, $\alpha_0 v - u + w$ attains absolute minimum at $t = t_0 \in D$.

Therefore,

$$[\alpha_0 v - u + w]'(t_0) = 0, \quad [\alpha_0 v - u + w]''(t_0) \geq 0.$$

Further at $t = t_0$ we have

$$0 < \alpha_0 Pv + Pw - Pu = -[\alpha_0 v + w - u]''(t_0) + a(t_0)[\alpha_0 v + w - u]'(t_0) + b(t_0)[\alpha_0 v + w - u](t_0) \leq 0.$$

It is a contradiction and hence the claim.

From the above discussion, we understand that it is enough that $a(t)$ and $b(t)$ appearing in (2.7) need exist but not necessarily continuous and bounded. Here lies the difference between the generalised maximum principle [1] and the result proved here.

We now give test functions for some of the BVPs (1.1)-(1.10) under suitable conditions.

(A) Consider the BVP (1.1) with the conditions $(y_i(a), y_i(b))^T$ prescribed and assume that

(i) $\beta_{ij}(t) \leq 0, \; j \neq i, \quad i, j = 1(1)N,$

(ii) $\sum_{j=1}^{N} \beta_{ij}(t) \geq 0, \quad i = 1(1)N,$

(iii) $\alpha_i(t) \leq \alpha_0 \in R$ (the set of all real numbers).

The test function $v = (v_1, \ldots, v_N)$ is given by

$$v_i'(t) = \operatorname{Exp}[-k(t-a)], \; t \in \overline{D}, \; k \in R^+, \; i = 1(1)N.$$

(B) Consider the BVP (1.1) with the conditions $(y_i(a), y_i(b))^T$ prescribed and assume that

(i) $\beta_{ij}(t) \leq 0$, $j \neq i$, $i,j = 1(1)N$,

(ii) $\alpha_i(t) \geq \alpha_0 > 0$, $i = 1(1)N$,

(iii) $\sum_{j=1}^{N} \beta_{ij}(t) \geq \beta_0 \in R$, $i = 1(1)N$.

Then a test function $v = (v_1, \ldots, v_N)$ for $0 < \varepsilon \leq \varepsilon_1$ where ε_1 is sufficiently small, is given by

$$v_i(t) = \operatorname{Exp}[-k(t-a)] \text{ on } \bar{D},\ k \in R^+,\ i = 1(1)N.$$

(C) Consider the parabolic BVP (1.8) and assume that

(i) $\varepsilon^{k_i} a_{ij} + \alpha_{ij} \leq 0$, $j \neq i$, $i,j = 1(1)N$,

(ii) $\sum_{j=1}^{N} [\varepsilon^{k_i} a_{ij} + \alpha_{ij}] \geq \alpha_0 \in R$, $i = 1(1)N$,

(iii) $\sum_{j=1}^{N} d_{ij} \geq d_0 > 0$, $i = 1(1)N$.

Then a test function $v = (v_1, \ldots, v_N)$ is given by

$$v_i(t) = \operatorname{Exp}[\eta t],\ \eta \in R^+,\ t \in [0, T],\ i = 1(1)N.$$

PART A.

In this part we consider singularly perturbed linear, quasi-linear and nonlinear boundary and initial value problems (1.1)-(1.5). Based on the theorems of the previous section we now give estimates and asymptotic behaviour of solution or/and its derivative of the stated problems. The results are given in the form of theorems.

In the following analysis, M_i, L_i, $i = 1, 2, \ldots,$ are constants independent of ε.

3. <u>Boundary Value Problems.</u>

THEOREM 3.1. Consider the BVP (1.2) and let $y(t) = y(t,\varepsilon)$ be its solution. Further assume that :

(i) $\alpha_i(t, y, \varepsilon) \leq -\alpha_0 < 0$,

(ii) $\sum_{j=1}^{N} \beta_{ij}(t, y, \varepsilon) \geq \beta_0 \in R$,

(iii) $h_i > 0$,

where $i = 1(1)N$.

Then we have for $0 < \varepsilon \leq \varepsilon_1$, where ε_1 is sufficiently small,

$$(3.1)\quad |y_i(t,\varepsilon)| \leq L_1 \max_{j=1(1)N}\left[|A_j(\varepsilon)|, |B_j(\varepsilon)|\right] + M_2 \|r\|_\infty/\alpha_0, \quad t \in \bar{D},\ i = 1(1)N,$$

where

$$\|r\|_\infty := \max_{\substack{t \in \bar{D} \\ i=1(1)N}} \left[|r_i(t, y(t), \varepsilon)|\right].$$

It should be noted that the estimate (3.1) is true for both the B.V.P s (1.2).

THEOREM 3.2. Let $y(t) = y(t,\varepsilon)$ be a solution of the BVP (1.2) with $r_i(t, y, \varepsilon) \equiv 0$, $i = 1(1)N$. Further assume that

(i) $h_i = h > 0$, (iii) $\left|\sum_{j=1}^{N} \beta_{ij}(t, y, \varepsilon)\right| \leq \beta_0 \in R^+$,

(ii) $\alpha_i(t, y, \varepsilon) \leq -\delta < 0$, (iv) $A_i(\varepsilon) \equiv 0$,

where $i = 1(1)N$.

Then we have for $0 < \varepsilon \leq \varepsilon_1$, where ε_1 is sufficiently small, the following results. With $(y_i(a), y_i(b))^T$ prescribed, we have

$$(3.2) \quad |y_i(t,\varepsilon)| \leq \left[\max_{j=1(1)N} |B_j(\varepsilon)|\right] \mathrm{Exp}\left[-\delta(b-t)/2\varepsilon^h\right],$$
$$t \in \bar{D}, \; i = 1(1)N,$$

where as with $(y_i(a), y_i'(b))^T$ prescribed, we have

$$(3.3) \quad |y_i(t,\varepsilon)| \leq (2\varepsilon^h/\delta)\left[\max_{j=1(1)N} |B_j(\varepsilon)|\right] \mathrm{Exp}\left[-\delta(b-t)/2\varepsilon^h\right],$$
$$t \in \bar{D}, \; i = 1(1)N,$$

and

$$(3.4) \quad |y_i'(t,\varepsilon)| \leq |B_i(\varepsilon)| \, \mathrm{Exp}\left[-\delta(b-t)/2\varepsilon^h\right] +$$
$$M_2 \varepsilon^h \left[\max_{j=1(1)N} |B_j(\varepsilon)|\right]\left[1 - \mathrm{Exp}\left[-\delta(b-t)/2\varepsilon^h\right]\right],$$
$$t \in \bar{D}, \; i = 1(1)N.$$

Using the theorems 3.1 and 3.2 one can prove the following results. The following result has been proved in [3] for scalar BVP (1.1) with Dirichlet's BCs under the assumption that the coefficient of dependent variable is non-positive. An another important point to be observed here is that we give results for all BVPs (1.1).

THEOREM 3.3. Let $y=(y_1,\ldots,y_N)$ be the solution of the BVP (1.1). Further assume that :

(i) $\beta_{ij}(t) \leq 0,\ j \neq i$ (quasi-monotonicity condition),

(ii) $\left|\sum_{j=1}^{N} \beta_{ij}(t)\right| \leq \beta_0 \in R^+$, (iii) $\alpha_i(t) \leq -\delta < 0$,

where $i, j = 1(1)N$.

Let $u=(u_1,\ldots,u_N)$ be the solution of the following IVP (3.5) such that $\|u\|_\infty \leq M_3$.

(3.5) $$\begin{cases} \alpha_i(t)\,u_i'(t) - \sum_{j=1}^{N} \beta_{ij}(t)\,u_j(t) = \gamma_i(t), \\ u_i(a) = A_i, \quad i = 1(1)N. \end{cases}$$

Then we have,

with $(y_i(a),\ y_i(b))^T$ prescribed

(3.6) $$\lim_{\varepsilon\to 0+} y_i(t,\varepsilon) = u_i(t),\quad a \leq t < b,\quad i = 1(1)N,$$

and with $(y_i(a),\ y_i'(b))^T$ or

$(y_i(a) - \varepsilon\, y_i'(a),\ y_i'(b))^T$ prescribed

(3.7) $$\lim_{\varepsilon\to 0+} y_i(t,\varepsilon) = u_i(t),\quad a \leq t \leq b,\quad i = 1(1)N,$$

and

(3.8) $$\lim_{\varepsilon\to 0+} y_i'(t,\varepsilon) = u_i'(t),\quad a \leq t < b,\quad i = 1(1)N.$$

The conclusions (3.6) - (3.8) are made after finding appropriate estimates for $|y_i(t,\varepsilon) - u_i(t)|$ or/and $|y_i'(t,\varepsilon) - u_i'(t)|$, $i = 1(1)N$.

In the above theorem, we have assumed that $\alpha_i(t) \leq -\delta < 0$. When $\alpha_i(t) \geqslant \delta > 0$ and Dirichlet's BCs are prescribed for the BVP (1.1), we can show that

$$\lim_{\varepsilon \to 0+} y_i(t,\varepsilon) = u_i(t), \quad a < t \leq b, \; i = 1(1)N, \tag{3.9}$$

where $u = (u_1, \ldots, u_N)$ is the solution of the following terminal value problem (3.10) with $\|u\|_\infty \leq M_4$.

$$\begin{cases} \alpha_i(t)\, u_i'(t) - \sum_{j=1}^{N} \beta_{ij}(t)\, u_j(t) = \gamma_i(t), \\ u_i(b) = B_i, \quad i = 1(1)N. \end{cases} \tag{3.10}$$

For other BCs with $\alpha_i(t) \geqslant \delta > 0$, we are not able to conclude any thing. When the coefficients α_i, β_{ij}, γ_i depend also on, the results (3.6) - (3.9) will still be valid.

THEOREM 3.4. Assume that all the conditions of theorem 3.3 except the quasimonotonicity condition

$$\beta_{ij}(t) \leq 0, \; j \neq i, \quad i, j = 1(1)N, \tag{3.11}$$

are satisfied. Then also the conclusions (3.6) - (3.8) are valid for the BVP (1.1).

To obtain the results stated in this theorem, one has to introduce an auxiliary problem as done in [23,25,26] which satisfies the quasimonotonicity condition. This approach has not yet become very popular and therefore we urge the reader to note it carefully. For example, consider the BVP (1.1) with $(y_i(a), y_i(b))^T$ prescribed.

Set

$$(3.12)\quad \begin{cases} \beta_{ij}^{+} := \beta_{ij}, \text{ if } \beta_{ij} \geq 0, & \text{and '0' otherwise,} \\ \beta_{ij}^{-} := \beta_{ij} - \beta_{ij}^{+}, & i,j = 1(1)N. \end{cases}$$

Then the required auxiliary problem is defined by

$$(3.13)\quad \begin{cases} \hat{P}_i \hat{y} := \varepsilon \hat{y}_i'' + \alpha_i(t)\hat{y}_i' - \beta_{ii}\hat{y}_i - \sum\limits_{\substack{j=1 \\ j\neq i}}^{N} \left[\beta_{ij}^{+}\hat{y}_j - \beta_{ij}^{-}\hat{y}_{j+N}\right] = -\gamma_i(t),\ t\in D, \\ \hat{P}_{i+N}\hat{y} := \varepsilon \hat{y}_{i+N}'' + \alpha_i(t)\hat{y}_{i+N}' - \beta_{ii}\hat{y}_{i+N} - \sum\limits_{j=1, j\neq i}^{N}\left[\beta_{ij}^{+}\hat{y}_{j+N} - \beta_{ij}^{-}\hat{y}_j\right] = \gamma_i(t),\ t\in D, \\ \hat{R}_i\hat{y} := \begin{pmatrix} \hat{y}_i(a) \\ \hat{y}_i(b) \end{pmatrix} = \begin{pmatrix} -A_i \\ -B_i \end{pmatrix},\ \hat{R}_{i+N}\hat{y} := \begin{pmatrix} \hat{y}_{i+N}(a) \\ \hat{y}_{i+N}(b) \end{pmatrix} = \begin{pmatrix} A_i \\ B_i \end{pmatrix}, \end{cases}$$

where $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_{2N})$, $i = 1(1)N$.

It can be seen readily that the above system is quasi-monotone. Also, if $(y_1, \ldots, y_N)$ is a solution of the BVP (1.1) with BCs $(y_i(a), y_i(b))^T$, then $(-y_1, \ldots, -y_N, y_1, \ldots, y_N)$ is a solution of the BVP (3.13).

In this sequence we finally give results for nonlinear two point BVPs (1.3) by making use of theorem 2.1. Using sub-functions method [27], Howes [9,10,11] obtained estimates for solutions of the BVP (1.3) in the case when N = 1 and $(y_i(a), y_i(b))^T$ is prescribed.

THEOREM 3.5. Consider the nonlinear two point BVPs (1.3) and assume that there are constants $\delta > 0$ and $K_{ij} \geq 0$ such that

$$(3.14)\quad |f_i(t, y_1, \ldots, y_N, y_i') - f_i(t, x_1, \ldots, x_N, y_i')| \leq \sum_{j=1}^{N} K_{ij}|y_j - x_j|,$$

$$(3.15)\quad f_i(t, y_1, \ldots, y_N, y_i') - f_i(t, y_1, \ldots, y_N, x_i') \geq \delta(y_i' - x_i'),$$

$$\delta > 0, \quad y_i' \geq x_i', \quad i = 1(1)N, \quad x_i, y_i \in Y.$$

Then we have,

with $(y_i(a), y_i(b))^T$ prescribed,

$$(3.16)\quad \lim_{\varepsilon \to 0+} y_i(t,\varepsilon) = u_i(t), \quad a \leq t < b, \quad i = 1(1)N,$$

where as with $(y_i(a), y_i'(b))^T$ or

$(y_i(a) - \varepsilon y_i'(a), \; y_i'(b))^T$ prescribed

$$(3.17)\quad \lim_{\varepsilon \to 0+} y_i(t,\varepsilon) = u_i(t), \quad a \leq t \leq b, \quad i = 1(1)N,$$

and

$$(3.18)\quad \lim_{\varepsilon \to 0+} y_i'(t,\varepsilon) = u_i'(t), \quad a \leq t < b, \quad i = 1(1)N,$$

where $y = (y_1, \ldots, y_N)$ and $(u_1, \ldots, u_N)$ are solutions of the BVP (1.3) and the following IVP (3.19) respectively.

$$(3.19)\quad f_i(t, u_1, \ldots, u_N, u_i') = 0, \quad u_i(a) = A_i, \quad i = 1(1)N,$$

where $\|u\|_\infty \leq M_5$.

The same analysis can be carried out in the case when f_i, A_i and β_i appearing in the BVP (1.3) depend also on the parameter ε.

4. Initial Value Problems

In the present section we obtain results for the IVPs (1.4) and (1.5). The approach given for the IVP (1.4) is entirely different from that of [15] and [28].

THEOREM 4.1. Consider the IVP (1.4) and assume that $c_1 \leq c(t,\varepsilon) \leq 0$, and $b(t,\varepsilon) \geq b_0 > 0$. Then we have

(4.1) $\lim\limits_{\varepsilon \to 0+} y(t,\varepsilon) = u(t) \qquad a \leq t \leq b,$

(4.2) $\lim\limits_{\varepsilon \to 0+} y'(t,\varepsilon) = u'(t) \qquad a < t \leq b < +\infty,$

where y and u are solutions of the IVPs (1.4) and (4.3) respectively.

(4.3) $$\begin{cases} b_0(t)u' + c_0(t)u = d_0(t), & a < t \leq b, \\ u(a) = A(0), \text{ where } b(t,\varepsilon) = b_0(t) + O(\varepsilon), \text{ etc.} \end{cases}$$

THEOREM 4.2. Assume that all the conditions of theorem 4.1. except the quasimonotonicity condition i.e., $c(t,\varepsilon) \leq 0$ are satisfied. Then also the results (4.1) and (4.2) are valid.

To obtain the results (4.1) and (4.2) in theorem 4.1, one can make use of results available in [2] by reducing the second order equation in (1.4) into a system of first order equations, and for theorem 4.2 an auxiliary problem as introduced in (3.13) can be made use of. Also, using modified version of theorems available in section 12 of [2], we have the following results for the IVP (1.5).

THEOREM 4.3. Consider the IVP (1.5) and let $x=(x_1,\dots,x_N)$, $y=(y_1,\dots,y_M)$ and $X=(X_1,\dots,X_N)$, $Y=(Y_1,\dots,Y_M)$ be the solutions of the IVPs (1.5) and (4.4), (4.5).

$$(4.4)\quad X_i' - u_i(t, X_1,\dots,X_N, Y_1,\dots,Y_M, 0) = 0\,,\ X_i(a) = A_i(0),\quad i = 1(1)N,$$

$$(4.5)\quad v_j(t, X_1,\dots,X_N, Y_1, Y_2,\dots,Y_M, 0) = 0\,,\ j = 1(1)M.$$

Further assume that:

H1. $u_i(t, x, y, \varepsilon) = u_i(t, x, y, 0) + O(\varepsilon)\,,\ i = 1(1)N,$

$v_j(t, x, y, \varepsilon) = v_j(t, x, y, 0) + O(\varepsilon)\,,\ j = 1(1)M,$

H2. $(u, v) = (u_1,\dots,u_N, v_1,\dots,v_M)$

is quasi-monotone increasing in $(x,y) = (x_1,\dots,x_N, y_1,\dots,y_M)$

H3. there exists a vector function $Y = (Y_1,\dots,Y_M)$. satisfying (4.5) such that after substituting this Y the resulting non-linear IVP defined by (4.4) has a unique solution $X = (X_1,\dots,X_N)$ on $\bar{D}$ and that

$$(4.6)\quad v_j(t, x, y+\bar{\chi}, \varepsilon) - v_j(t, x, y, \varepsilon) \leq -k\chi\,,\quad k > 0\,,\ \bar{\chi} = (\chi,\dots,\chi)\,,\ \chi(t,\varepsilon) \geq 0\,,\ j = 1(1)M,$$

and

H4. that for some constant $\ell > 0$,

$$(4.7)\quad \begin{cases} u_i(t, x+\bar{\psi}, y+\bar{\chi}, \varepsilon) - u_i(t, x, y, \varepsilon) \leq \ell[\psi + \chi], \\ \bar{\psi} = (\psi,\dots,\psi),\quad \psi(t,\varepsilon) \geq 0,\quad i = 1(1)N, \end{cases}$$

$$(4.8)\quad v_j(t, x+\bar{\psi}, y, \varepsilon) - v_j(t, x, y, \varepsilon) \leq \ell\,\psi,\ j = 1(1)M.$$

Then we have

$$(4.9)\quad |x_i(t,\varepsilon) - X_i(t)| \leq \varepsilon M_6 e^{m(t-a)} + \varepsilon M_7 \max_{j=1(1)M} |B_j(\varepsilon) - Y_j(a)| \left[e^{m(t-a)} - e^{-k(t-a)/\varepsilon} \right],$$
$$t \in \overline{D},\quad i = 1(1)N,$$

$$(4.10)\quad |y_j(t,\varepsilon) - Y_j(\varepsilon)| \leq \varepsilon M_8 e^{m(t-a)} + \varepsilon M_9 \max_{j=1(1)M} |B_j(\varepsilon) - Y_j(a)| e^{m(t}$$
$$+ \max_{j=1(1)M} |B_j(\varepsilon) - Y_j(a)| e^{-k(t-a)/\varepsilon},\quad t \in \overline{D},\quad j = 1(1)M.$$

In other words,

$$(4.11)\quad \lim_{\varepsilon \to 0+} x_i(t,\varepsilon) = X_i(t),\quad a \leq t \leq b,\quad i = 1(1)N,$$

$$(4.12)\quad \lim_{\varepsilon \to 0+} y_j(t,\varepsilon) = Y_j(t),\quad a < t \leq b,\quad j = 1(1)M.$$

PART B.

5. Elliptic Boundary Value Problems

Making use of theorem 2.1, we obtain estimates for solutions of the BVPs (1.6) and (1.9).

Constants C_i, $i = 1, 2, \ldots$ appearing in the present and the following section are independent of the parameter ε.

THEOREM 5.1. Consider the linear elliptic BVP (1.6) and assume that

(i) $\varepsilon^{p_i} a_{ij} + \alpha_{ij} \leq 0,\quad j \neq i,\quad i,j = 1(1)N,$

(quasi-monotonicity condition)

(ii) there exists a vector function $w = (w_1, \ldots, w_N)$ such that $w_i(x) > 0$ on $\bar{D}$, $w_i \in U := C^2(D) \cap C(\bar{D})$, $P_i w \geq c_1 > 0$, $i = 1(1)N$.

Then we have

(5.1) $|u_i(t,\varepsilon)| \leq c_2 \{ \|f\| + \|g\|' \}$, $c_2 \in R^+$, where

$$\|f\| := \sup_{x \in D} \left\{ \sum_{i=1}^{N} |f_i(x)|^2 \right\}^{1/2}, \text{ and}$$

$$\|g\|' := \sup_{x \in \partial D} \left\{ \sum_{i=1}^{N} |g_i(x)|^2 \right\}^{1/2}.$$

THEOREM 5.2. Assume that all the conditions of theorem 5.1 are satisfied. Further let Z be a formal approximation to the solution u of the BVP (1.6) such that

(5.2) $P_i(u - Z) = r_i$ in D, $R_i(u - Z) = s_i$ on ∂D, $i = 1(1)N$,

where $\|r\| = O(\delta(\varepsilon)) = o(1)$, $\|s\|' = O(\delta'(\varepsilon)) = o(1)$,

and δ, δ' are order functions.

Then we have

(5.3) $\|u - Z\| \leq O(\max(\delta, \delta'))$.

THEOREM 5.3. If the system (1.6) is not quasimonotone, we introduce an auxiliary problem defined as in (5.5) below and assume that there exists a vector function $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_{2N})$ such that $\hat{w}_i(x) > 0$ on $\bar{D}$, $\hat{w}_i \in U$, $i = 1(1)2N$,

$\hat{P}_i \hat{w} \geq c_3$, $\hat{P}_{i+N} \hat{w} \geq c_3 > 0$, $i = 1(1)N$.

Further let Z be a formal approximation to the solution u of the BVP (1.6) and satisfying (5.2). Then we have

(5.4) $\|u - Z\| \leq O(\max(\delta, \delta'))$.

$$
(5.5)\quad \begin{cases}
\hat{P}_i \hat{u} := (\varepsilon^{p_i} a_{ii} + \alpha_{ii})\hat{u}_i - \sum_{j=1, j\neq i}^{N} [(\varepsilon^{p_i} a_{ij} + \alpha_{ij})^{+} \hat{u}_{j+N} - \\
\quad (\varepsilon^{p_i} a_{ij} + \alpha_{ij})^{-} \hat{u}_j] + \sum_{k=1}^{n} [\varepsilon^{p_i} b_{ik} + \beta_{ik}] \hat{u}_{i,k} - \\
\quad \sum_{k,l=1}^{n} \varepsilon^{p_i} c_{ikl} \hat{u}_{i,kl} = -f_i(x), \ x \in D, \\
\hat{P}_{i+N} \hat{u} := (\varepsilon^{p_i} a_{ii} + \alpha_{ii})\hat{u}_{i+N} - \sum_{j=1, j\neq i}^{N} [(\varepsilon^{p_i} a_{ij} + \alpha_{ij})^{+} \hat{u}_j - \\
\quad (\varepsilon^{p_i} a_{ij} + \alpha_{ij})^{-} \hat{u}_{j+N}] + \sum_{k=1}^{n} [\varepsilon^{p_i} b_{ik} + \beta_{ik}] \hat{u}_{i+N,k} - \\
\quad \sum_{k,l=1}^{n} \varepsilon^{p_i} c_{ikl} \hat{u}_{i+N,kl} = f_i(x), \ x \in D, \ i = 1(1)N, \\
\hat{R}_i \hat{u} := \hat{u}_i(x) = -g_i(x) \ \text{on } \partial D, \\
\hat{R}_{i+N} \hat{u} := \hat{u}_{i+N}(x) = g_i(x) \ \text{on } \partial D, \ i = 1(1)N,
\end{cases}
$$

where $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_{2N})$.

THEOREM 5.4. Consider the BVP (1.9) and assume that the implication (2.2) is true for the BVP(1.9). Further assume that:

(i) for each i, $P_i(Z + Z_0) - P_i Z \geq \ell \|Z_0\|$,

where $\ell \in R^{+}$, Z_0 is a nonnegative constant vector and Z is a formal approximation to the solution u of BVP (1.9) satisfying

$$F_i(Z) = f_i + \gamma_i \ \text{ in } D, \quad Z_i = g_i + \delta_i \ \text{ on } \partial D, \ i = 1(1)N$$

where $\|\gamma\| = O(\delta(\varepsilon)) = o(1)$, $\|\delta\|' = O(\delta'(\varepsilon)) = o(1)$.

Then

(5.6) $\|u - Z\| = O(\chi(\varepsilon)) = o(1), \quad \chi(\varepsilon) = \max(\delta, \delta')$.

6. Parabolic Boundary Value Problems

THEOREM 6.1. Consider the linear parabolic BVP (1.8) and assume that:

(i) $\varepsilon^{k} a_{ij} + \alpha_{ij} \leq 0, \quad d_{ij} \leq 0, \ j \neq i, \ i,j = 1(1)N,$

(ii) there exists a vector function

$w = (w_1, \ldots, w_N)$ such that

$$(6.1) \begin{cases} w_i(t,x) > 0 \text{ on } \overline{G}, \quad w_i \in U := C^{(1,2)}(G_p) \cap C(\overline{G}), \\ P_i w \geq C_4 \in R, \quad R_i w \geq C_5 > 0, \quad i = 1(1)N. \end{cases}$$

Then we have

$$|u_i(t,x,\varepsilon)| \leq C_6 \{\|f\| + \|g\|' + \|h\|''\} \text{ on } \overline{G}, \ C_6 > 0, \ i = 1(1)N$$

where $\|f\| := \sup_{(t,x)\in G_p} \left[\sum_{i=1}^{N} |f_i(t,x)|^2\right]^{1/2}$,

$$\|g\|' := \sup_{(t,x)\in R_p - R_{22}} \left\{\sum_{i=1}^{N} |g_i(t,x)|^2\right\}^{1/2}$$

and

$$\|h\|'' = \sup_{(t,x)\in R_\partial} \left\{ \sum_{i=1}^{N} |h_i(t,x)|^2 \right\}^{1/2},$$

$$f = (f_1, \dots, f_N) \qquad \text{etc.}$$

THEOREM 6.2. Assume that all the conditions of theorem 6.1 are satisfied. Further let Z be a formal approximation to the solution u of the BVP (1.8) such that

$$(6.2)\quad \begin{cases} P_i(u-Z) = q_i \text{ in } G_p, \; R_i(u-Z) = r_i \text{ on } R_p - R_\partial, \\ R_i(u-Z) = s_i \text{ on } R_\partial, \; i = 1(1)N, \end{cases}$$

where $\|q\| = O(\delta(\varepsilon)) = o(1)$, $\|r\|' = O(\delta'(\varepsilon)) = o(1)$,

$\|s\|'' = O(\delta''(\varepsilon)) = o(1)$.

Then we have

$$(6.3)\quad |u_i(t,x,\varepsilon) - Z_i(t,x,\varepsilon)| \le O(\max(\delta,\delta',\delta'')) \text{ on } \overline{G},\; i = 1(1)N.$$

THEOREM 6.3. Consider the linear BVP (1.8) and assume that

(i) $[\varepsilon^{h} a_{ii} + \alpha_{ii}] - \sum_{j=1, j\neq i}^{N} |\varepsilon^{h} a_{ij} + \alpha_{ij}| \ge c_7 \in R$, $i = 1(1)N$,

(ii) $d_{ii} - \sum_{j=1, j\neq i}^{N} |d_{ij}| \ge c_8 > 0$, $i = 1(1)N$.

Let Z be a formal approximation to the solution u of the BVP (1.8) and satisfy (6.2). Then the result (6.3) is valid.

The above theorem can be proved by introducing an auxiliary problem as in (5.5) since the quasimonotonicity condition is not assumed here.

THEOREM 6.4. Consider the nonlinear BVP (1.10) and assume that the implication (2.2) is true for the BVP (1.10). Further assume that :

(i) for each fixed i,

$$F_i(t, x, Z_1 + Z_0, Z_{i,j}, \varepsilon^{k_i} Z_{i,jk}) - F_i(t, x, Z, Z_{i,j}, \varepsilon^{k_i} Z_{i,jk}) \geq \lambda Z_{0i}, \quad i = 1(1)N,$$

(ii) for each fixed i,

$$H_i(t, x, Z + Z_0) - H_i(t, x, Z) \geq \ell' Z_{0i}, \quad i = 1(1)N,$$

where $\lambda \in R$, $\ell' > 0$ are constants, Z_0 is a vector function whose components are positive and functions of 't', and Z_1 is a formal approximation to the solution u of the BVP (1.10) satisfying

$$P_i u - P_i Z_1 = q_i \text{ in } G_p, \quad R_i u - R_i Z_1 = \tau_i \text{ on } R_p - R_b,$$

$$R_i u - R_i Z = s_i \text{ on } R_b, \quad i = 1(1)N,$$

with $\|q\| = O(\delta(\varepsilon)) = o(1)$, $\|\tau\|' = O(\delta'(\varepsilon)) = o(1)$,

and $\|s\|'' = O(\delta''(\varepsilon)) = o(1)$.

Then we have

$$|u_i(t, x, \varepsilon) - Z_i(t, x, \varepsilon)| = O(\chi(\varepsilon)) = o(1) \text{ on } \overline{G}, \quad i = 1(1)N,$$

where $\chi(\varepsilon) = \max(\delta, \delta', \delta'')$.

CONCLUSIONS

The present article deals with problems related to finite domains. Since results are available in the theory of differential inequalities corresponding to infinite domains [2], it should be possible to extend some of the results of the present paper to infinite domains. Also, it may be worthwhile investigating other types of boundary conditions than the ones discussed here. Turning point problems for scalar second order ODEs are discussed in [3] and [11]. Making use of our approach, it should be possible to discuss turning point problems for vector differential equations. Hyperbolic systems could be another area for application of our ideas.

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STABILITY OF GENERAL DYNAMICAL SYSTEM*

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A theory to study the stability behaviour of a solution for a system of n first order ordinary differential equations was due to A.M. Lyapunov [10], who introduced the concepts - stability, asymptotic stability and instability. His second method (known also as the direct method) gave a qualitative approach to the study of stability properties of a given solution. without the knowledge of the actual solutions of the given system. He used certain auxiallary functions for this purposes, and these functions have come to be known as Lyapunov functions for determining the stability behaviour.

A.A.Markov [11] and H.Whitney [13] independently observed that the familiar of curves can be studied qualitatively as one-parameter family of transformations group in an appropriate space, instead of considering a single solutions of an ordinary differential equation Barbashin [1] introduced the notion of a general dynamical system in a metric space as a two-parameter family of transformation of into itself. General control systems are special cases of general dynamical system (see Roxin [12]) Zubov [14] extended Lyapunov's direct method to study the stability properties of general dynamical systems in metric space. Kayande and Lakshmikantham 5 have used vector-Lyapunov functions to study conditional stability and boundedness of a general dynamical system on a locally compact separable metric space.

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The concepts of relative stability discussed in Bhatia and Szego [2] are similar to the concepts of conditional stability discussed in [5]. Other stability properties are also discussed in [2]. Lakshmikantham and Leela [8] have discussed general methods for construction of Lyapunov function giving necessary and sufficient conditions for stability of differential systems in terms of differential in equalities.

In this paper, the author considers a general dynamical system (g.d.s.) in a locally compact separable metric space E as a two parameter family of transformations of E into $A(E)$ the set of all subsets of E, and obtains the existence of a Lyapunov function defined on $[0, \infty) \times A(E)$ if reversibilty of the g.d.s. is assured. For this purpose the concepts of strict and asymptotic stability of an invariant set with respect to a g.d.s. are introduced and sufficient conditions in terms of a Lyapunov function defined on $[0, \infty) \times A(E)$ is obtained It is to be noted that eventhough such a definition of a Lykpunov function is most natural in view of the fact that a g.d.s. is defined in E into A(E), in the literature available, Lyapunov functions are defined on $[0, \infty) \times E$ [1,2,3,4,8,12,14]. Thus a Lyapunov function in its natural setting is obtained. The concept was introduced by Kayande jointly in [7].

In section 2, defining a g.d.s., the concept of reversibility of a g.d.s. stability of an invariant set A with respect to a g.d.s. and the strict and asymptotic stability are introduced. These are given in terms of Hausdorff metric in A(E). Also certain classes of monotonic functions (considered by Hahn [4] and used in the literature on stability of ordinary differential equations and dynamical systems [2,4,8,9] are brought in.

In Section 3 comparison theorems are introduced and sufficient conditins for stability of an invariant set A with respect to a g.d.s. in terms of Lyapunov functions defined on $[0,\infty) \times A(E)$ are obtained.

In Section 4, comparison techniques are used and sufficient conditions, involving the stability of a scalar differential equation and the existence of a Lyapunov function for the stability of A with respect to a g.d.s. are got.

In Section 5 converse theorems on the existence of Lyapunov functions for a reversible dynamical system (r.d.s.) are derived.

In Section 6 the concept of conditions invariancy [6] of a set B with respect to A in a g.d.s. in E and definitions of stability of a conditional invariant set B with respect to A are introduced. Sufficient conditions for stability of conditional invariant set B with respect to in a g.d.s. in terms of Lyapunov function are proved. Also converse theorems in some form for a r.d.s. are attempted and their relations to some theorems of section 5 are traced.

In Section 7 conditional stability of a compact set A with respect to a g.d.s. are defined and a general comparison technique involving a vector Lyapunov function and the notion of quasi-monotonicity, (which is developed in the section) is used to prove the sufficiency conditions for conditional stability of set A for set M.

In Section 8 two converse theorems for the existence of a vector Lyapunov function, in a r.d.s. are proved.

Section 2:

Definition 2.1: Let I denote the half-line $0 \leq t < \infty$ and $R_+ = [0, \infty)$. We consider a general dynamical system (g.d.s.) in a locally compact separable metric space E defined as follows:

(i) for each $P_0 \in E$ and $t_0 \in I$, there is a defined a set $F(t, t_0, P_0) \subset A(E)$ for all $t \geq t_0$, $t_0 \in I$ and $A(E)$ the set of all subsets of E.

(ii) $F(t_0, t_0, P_0) = \{P_0\}$ for all $(t_0, P_0) \in I \times E$, and

(iii) for any $p_1 \in F(t, t_0, P_0)$, there is defined the set $F(t, t_1, P_1)$ such that $F(t, t_0, P_0) = \bigcup\limits_{p_1 \in F(t, t_0, P_0)} F(t, t_1, P_1)$.

For a fixed $P_0 \in E$, $F(t, t_0, p)$ is called a motion while the set defined in (i) is called the trajectory of this motion.

For any $X_0 \in A(E)$, let $F(t, t_0, X_0) = \bigcup\limits_{P_0 \in X_0} F(t, t_0, P_0)$

With this notation, the properties (ii) and (iii) of a g.d.s. can be written as

(iv) $F(t_0, t_0, X_0) = X_0$, $t_0 \in I$, $X_0 \in A(E)$

(iii) $F(t, t_1, F(t_1, t_0, X_0)) = F(t, t_0, X_0)$ for all $t \geq t_1 \geq t_0$.

Definition 2.2: A reversible dynamical system (r.d.s.) in E is a g.d.s. in E such that the property (iii) is also satisfied when $t \in (t_0, t_1)$. In particular, therefore we have

$$F(t_0, t_1, F(t_1, t_0, X_0)) = F(t_0, t_0, X_0) = X_0.$$

Thus for a r.d.s.

$$F(t_0, t, F(t, t_0, x_0)) = x_0 \quad \text{for all } t \geq t_0.$$

The word reversible, in the context, is self-explanatory.

Let the set A be compact in E, in what follows.

Definition 2.3: A set $A \in A(E)$ is called (positively) invariant with respect to a g.d.s. in E if

$$F(t, t_0, A) \subset A \qquad \text{for all } t \geq t_0.$$

In the sequel, it is convenient to use the classes of monotonic functions introduced by Hahn [4].

Definition 2.4:

(i) A function $\phi = \phi(r) \in K$ if $\phi \in C[I, R_+]$ the class of continuous non-negative functions defined on I, such that $\phi(0) = 0$, strictly monotonic increasing in r and $\phi(r) \to \infty$ as $r \to \infty$.

(ii) A function $\sigma = \sigma(s) \in L$, if $\sigma \in C[I, R_+]$ strictly monotonic decreasing in s and $\sigma(s) \to 0$ as $s \to \infty$.

(iv)(...) A function $\sigma = \sigma(t, s) \in L^*$, if $L \in C[I \times R_+, R_+]$ and $\sigma \in L$ for each $t \in I$.

(iii)(..) A function $\phi = \phi(t, r) \in K^*$, if $\phi \in C[I \times R_+, R_+]$ and $\phi \in K$ for each $t \in I$.

We mention below a few useful results on monotonic functions

(i) If $b \in K$ then $b^{-1} \in K$.

(ii) If $\phi_1 \in K^*$ and $\phi_2 \in K^*$ then $\phi_3 = \phi_1(t, \phi_2(t, \eta)) \in K^*$

(iii) $\phi \in K^*$ and $b \in K$ imply $b^{-1}(\phi) \in K^*$

(iv) $\sigma \in L$ and $b \in K$ imply $b^{-1}(\sigma) \in L$.

(v) $a \in K$ and $b \in K$ imply $ab \in K$.

If d is the metric in E, for any two sets A and B $\in A(E)$ let

$$d^*(A, B) = \sup \{ d(a, B), a \in A \}$$

and

$$d^*(B, A) = \sup \{ d(b, A), b \in B \}$$

where

$$d(x, A) = \inf \{ d(x, y), y \in A \}.$$

Then the Hausdorff distance between A and B is defined to be

$$d(A, B) = \max (d^*(A, B), d^*(B, A))$$

For any set $M \in A(E)$, $S(M, \epsilon)$ and $\bar{S}(M, \epsilon)$ are given by

$$S(M, \epsilon) = \{ x : d(x, M) < \epsilon \}$$

and

$$\bar{S}(M, \epsilon) = \{ x : d(x, M) \leq \epsilon \}.$$

In what follows, the flow $F(t, t_0, X_0)$, $X_0 \in A(E)$ is assumed to be Hausdorff continuous in the triplet (t, t_0, X_0).

2.5. Stability definitions of the set A

In the following definitions, the inequalities hold for all $t \geq t_0 \in I$ and $X_0 \in S(A, \rho)$ for some $\rho > 0$. The set A is said to be

S_1: Equi-stable, if there exists a function $\phi \in K^*$ such that $d(F(t, t_0, X_0), A) \leq \phi(t_0, d(X_0, A))$

S_2: Equi-strict stable, if there exist functions $\phi_1, \phi_2 \in K^*$ such that

$$\phi_2(t_0, d(X_0, A)) \leq d(F(t, t_0, X_0), A) \leq \phi_1(t_0, d(X_0, A))$$

S_3: Uniform-stable, if there exists a function $\phi \in K$ such that $d(F(t, t_0, X_0), A) \leq \phi(d(X_0, A))$

S_4: Uniform-strict stable, if there exist functions ϕ_1 and $\phi_2 \in K$ such that

$$\phi_1(d(X_0, A)) \leq d(F(t, t_0, X_0), A) \leq \phi_2(d(X_0, A))$$

S_5: Equip-asymptotic stable, if there exist functions $\phi_1 \in K^*$ and $\sigma_1 \in L^*$ such that

$$d(F(t, t_0, X_0), A) \leq \phi_1(t_0, d(X_0, A))\, \sigma_1(t, t - t_0)$$

S_6: Equi-strict-asymptotic stable, if there exist functions $\phi_1, \phi_2 \in K^*$ and $\sigma_1, \sigma_2 \in L^*$ such that

$$\phi_2(t_0, d(X_0, A))\, \sigma_2(t, t - t_0) \leq d(F(t, t_0, X_0), A)$$

$$\leq \phi_1(t_0, d(X_0, A))\, \sigma(t_0, t - t_0)$$

S_7 : Uniform-asymptotic stable, if there exist functions $\phi \in K$ and $\sigma \in L$ such that

$$d(F(t,t_0,X_0),A) \leq \phi(d(X_0,A))\sigma(t-t_0)$$

S_8 : Uniform-strict-asymptotic stable, if there exist functions $\phi_1, \phi_2 \in K$ and $\sigma_1, \sigma_2 \in L$ such that

$$\phi_2(d(X_0,A))\sigma_2(t-t_0) \leq d(F(t,t_0,X_0),A) \leq \phi_1(d(X_0,A))\sigma_1(t-t_0)$$

Note : 1. Uniform....stability imply....stability of A.

ie $S_3 \Rightarrow S_1$; $S_4 \Rightarrow S_2$; $S_7 \Rightarrow S_5$ & $S_8 \Rightarrow S_6$.

2. $S_5 \Rightarrow S_1$ and $S_7 \Rightarrow S_3$

3. Strict stability imply stability of A

$S_2 \Rightarrow S_1$, $S_4 \Rightarrow S_3$, $S_6 \Rightarrow S_5$, $S_8 \Rightarrow S_7$

4. Strict stability precludes asymptotic stability of A. i.e. S_2 and S_4 exclude the possibility of S_5 to S_8

Thus the flow is in some tube-like domain (stability in tubelike domain).

Section 3:

In order to study the stability behaviour of a set A of a g.d.s. in E, we use Lyapunov functions defined over $I \times A(E)$. The following Lemmas are necessary for further work:

Lemma 3.1: Let there exists a function $V \in C[I \times A(E), R_+]$ and let

$$V^*(t, F(t, t_0, X_0)) = \lim_{h \to 0^+} \sup \frac{1}{h}\left[V(t+h, F(t+h, t_0, X_0)) - V(t, F(t, t_0, X_0))\right] \quad (3.1.0)$$

and

$$V^*(t, F(t, t_0, X_0)) \leq g(t, V(t, F(t, t_0, X_0))) \quad (3.1.1)$$

for all $t \geq t_0$, $t_0 \in I$; $X_0 \subset A(E)$ and $g \in C[I \times R_+, R_+]$

Assume that $r(t, t_0, r_0)$ is the maximal solution of the scalar differential equation

$$r' = g(t, r), \quad r(t_0) = r_0 \quad (3.1.2)$$

Then $V(t_0, X_0) \leq r_0$ imply that

$$V(t, F(t, t_0, X_0)) \leq r(t, t_0, X_0) \text{ for all } t \geq t_0 \quad (3.1.3)$$

Lemma 3.2: Let there exist a function $V \in C[I \times A(E), R_+]$ and let

$$V_*(t, F(t,t_0,X_0)) = \lim_{h \to 0^+} \inf \frac{1}{h}\Big[V(t+h, F(t+h, t_0, X_0)) - V(t, F(t,t_0,X_0))\Big] \quad (3.2.0)$$

and

$$V_*(t, F(t,t_0,X_0)) \geq p(t, V(t, F(t,t_0,X_0))) \quad (3.2.1.)$$

for all $t \geq t_0$, $t_0 \in I$, $X_0 \in A(E)$ and $p \in C[I \times R_+, R_+]$

Assume that $u(t, t_0, u_0)$ is the minimal solution of the scalar differential equation

$$u' = p(t, u) \quad u(t_0) = u_0 \quad (3.2.2)$$

Then $V(t_0, X_0) \geq u_0$ imply that

$$V(t, F(t, t_0, X_0)) \geq u(t, t_0, u_0) \quad (3.2.3)$$

Remark: If the functions g and p in (3.1.1) and (3.2.1) are both identically zero, then (3.1.3) and (3.2.3) are respectively equivalent to

$$V(t, F(t, t_0, X_0)) \leq V(t_0, X_0) \quad (3.1.4)$$

and

$$V(t, F(t, t_0, X_0)) \geq V(t_0, X_0) \quad (3.2.4)$$

Theorems on the stability of the set A with respect to the g.d.s. in E.

Theorem 3.3: Let there exist a function $V \in C[I \times A(E), R_+]$ such that (i) for all $(t, x) \in I \times A(E)$

$$b(d(x,A)) \le V(t,x) \le a(t, d(x,A)) \qquad (3.3.1)$$

where $b \in K$ and $a \in K^*$

(ii) the inequality (3.1.1) holds with $g = 0$.

Then the set A is equistable for the g.d.s. in E.

Proof: By (3.1.4), because of the condition (ii) above

$$V(t_0, x_0) \geq V(t, F(t, t_0, x_0))$$

and $V(t, F(t, t_0, x_0)) \geq b(d(F(t, t_0, x_0), A))$

Therefore by (3.3.1)

$$b(d(F(t, t_0, x_0), A)) \le V(t_0, x_0) \le a(t_0, d(x_0, A))$$

so that $d(F(t, t_0, x_0), A) \le (b^{-1}a)(t_0, d(x_0, A))$

Putting $\phi = b^{-1} a$, $\phi \in K^*$

Hence $d(F(t, t_0, x_0), A) \le \phi(t_0, d(x_0, A))$

so that A is equistable for the g.d.s. in E.

Theorem 3.4: Let there exist functions V_1 and V_2 such that V_1 satisfies the hypotheses of Theorem 3.3 and

(i) $V_2 \in C[I \times A(E), R_+]$ and for all $(t, x) \in I \times A(E)$

$$b_1(t, d(x, A)) \leq V_2(t, x) \leq a_1(d(x, A)) \qquad (3.41)$$

where $b_1 \in K^*$ and $a_1 \in K$.

(ii) the inequality (3.2.1) holds with $p = 0$

Then the set A is equi-strict stable for the g.d.s. in E.

Proof: Because V_1 satisfies all the conditions of the previous theorem, the conclusion of the previous theorem holds here.

i.e. $$d(F(t, t_0, x_0), A) \leq (b^{-1}a)(t_0, d(x_0, A)) \qquad (3.4.2)$$

when $(b^{-1}a) \in K^*$

Again, by (3.4.1), $V_2(t_0, x_0) \geq b_1(t_0, d(x_0, A))$

and $V_2(t, F(t, t_0, x_0)) \leq a_1(d(F(t, t_0, x_0), A)$

But, became of (ii), by (3.2.4)

$V_2(t, F(t, t_0, x_0)) \geq V_2(t_0, x_0)$ and

$V_2(t_0, x_0) \geq b_1(t_0, d(x_0, A))$

Hence

$$a_1(d(F(t, t_0, x_0)) \geq b_1(t_0, d(x_0, A))$$

so that

$$d(F(t, t_0, x_0), A) \geq (a_1^{-1}b_1)(t_0, d(x_0, A)) \qquad (3.4.3)$$

Putting $\phi_1 = b^{-1} a$ and $\phi_2 = a_1^{-1} b_1$ from (3.4.2) and (3.4.3) together then

$$\phi_2(t_0, d(x_0, A)) \leq d(F(t, t_0, x_0), A) \leq \phi_1(t_0, d(x_0, A))$$

where $\phi_1, \phi_2 \in K^*$, so that A is equistrict stable for the g.d.s. in E.

Note: Theorem (3.4) implies Theorem (3.3).

Theorem 3.5: Let the hypotheses of Theorem 3.3 hold, and let (3.3.1) be replaced by

$$b(d(x, A)) \leq V(t, x) \leq a(d(x, A)) \tag{3.5.1}$$

where $a, b \in K$. Then the set A is uniformly stable with respect to a g.d.s. in E.

Theorem 3.6. Let there exist functions V_i $(i = 1, 2)$

$V_i \in C[I \times A(E), R_+]$ and for all $(t, x) \in I \times A(E)$

$$b_i(d(x, A)) \leq V_i(t, x) \leq a_i(d(x, A)) \tag{3.6.1}$$

where $b_i, a_i \in K$ and the inequalities (3.1.1) and (3.2.1) hold with the functions g and p, identically zero. Then the set A is uniformly strict stable with respect to the g.d.s. in E.

Proof of these are parallel to those given already.

Section 4:

The results on asymptotic stability can be obtained conveniently by using comparison techniques. For this purpose, we state the following definitions for the stability properties of the comparison equations (3.1.2) and (3.2.2).

The trivial solution of (3.1.2) is said to be

S_1^*: equistable: If there exists a function $\phi_1 \in K^*$ and a number $\rho_1 > 0$ such that $r(t, t_0, r_0) \leq \phi_1(t, r_0)$ for all $t \geq t_0$, where $r(t, t_0, r_0)$ is any solution of (3.1.2) and the above inequality holds for $r_0 \geq \rho_1$.

S_3^*: uniformly stable: if their exist a function $\phi \in K$ and a number $\rho > 0$ such that $r(t, t_0, r_0) \leq \phi(r_0)$ for all $t \geq t_0$ where $r(t, t_0, r_0)$ is any solution of (3.1.2) and the inequality holds for $r_0 \leq \rho$.

S_5^*: equi-asymptotic stable: if there exist functions $\psi \in K^*$ and $\sigma \in L^*$ and a number $\rho > 0$ such that

$r(t, t_0, r_0) \leq \psi(t_0, r_0)\, \sigma(t_0, t - t_0)$ for all $t \geq t_0$

where $r(t, t_0, r_0)$ is any solution of (3.1.2) and $r_0 \leq \rho$.

S_7^*: uniform asymptotic stable: if there exist functions $\psi \in K$ and $\sigma \in L$ and a number $\rho > 0$, such that

$r(t, t_0, r_0) \leq \psi(r_0)\, \sigma(t - t_0)$ for all $t \geq t_0$

when $r(t, t_0, r_0)$ any solution of (3.1.2) and $r_0 \leq \rho$.

Note: These definitions corr. to S_1, S_3, S_5 and S_7 of 2.5

For proving the strict-results we require the following properties for the s l tions of he equation (3.2.2)

S_2^*: There exists a function $\phi_1 \in K^*$ such that, for any solution $u(t, t_0, u_0)$ of (3.2.2) with $u_0 \leq p$, $p > 0$

$$u(t, t_0, u_0) \geq \phi_1(t_0, u_0), \quad t \geq t_0$$

S_4^* There exists a function $\phi_2 \in K$ such that for any solution $u(t, t_0, u_0)$(3.2.2) with $u_0 \leq \delta$, $\delta > 0$

$$u(t, t_0, u_0) \geq \phi_2(u_0), \quad t \geq t_0$$

S_6^*: There exist functions $\phi \in K^*$ and $\sigma \in L^*$ such that for any solution

$u(t, t_0, u_0)$ of (3.2.2) with $u_0 \leq \delta$, $\delta > 0$

$$u(t, t_0, u_0) \geq \phi(t_0, u_0)\, \sigma(t_0, t - t_0), \quad t \geq t_0$$

S_8^* There exist functions $\phi \in K$ and $\sigma \in L$ such that for any solution

$u(t, t_0, u_0)$ of (3.2.2) with $u_0 \leq \delta$, $\delta > 0$,

$$u(t, t_0, u_0) \geq \phi(u_0)\, \sigma(t - t_0), \quad t \geq t_0.$$

Note: S_2^*, S_4^*, S_6^* and S_8^* do not reflect properties corresponding to S_2, S_4, S_4 and S_8 of 2.5.

Theorem 4.1: Assume the existence of a function $V \in C[I \times A(\epsilon), R_+]$ such that for $(t, x) \in I \times A(\epsilon)$

$$b(d(x, A)) \leq V(t, x) \leq a(t, d(x, A)) \tag{4.1.1}$$

where $b \in K$ and $a \in K^*$ and the inequality (3.1.1) holds. Then the equistability of the trivial solution of (3.1.2) implies that the set A is equistable.

Proof: Given , $t_0 \in I$ as the trivial solution of (3.1.2) is equistable, from S_1^* there exist $\phi_1 \in K^*$ and a positive number δ_1 such that $\delta_1 \geq r_0$ (4.1.2) implies that

$r(t, t_0, r_0) \leq \phi_1(t, r_0)$ (4.1.3) for all $t \geq t_0$

where $r(t, t_0, r_0)$ denote any solution of (2.1.2) through (t_0, r_0)

Due to the properties of the function a in (4.1.1) there exists a number $\delta = \delta(t_0, \delta_1) > 0$ such that $\delta(x_0, A) \leq \delta$ and $a(t_0, d(x_0, A)) \leq \delta_1$ (4.1.4) hold simultaneously.

Choose $d(x_0, A) \leq \delta$. and let $v(t_0, x_0) \leq a(t_0, d(x_0, A)) = r_0$. Then (4.1.4) implies that (4.1.2) is verified; thus (4.1.3) holds. Similarly, the choice of r_0 and the lemma 3.1 show that

$$v(t, F(t, t_0, x_0)) \leq r_{max}(t, t_0, r_0)$$

$$\leq r_{max}(t, t_0, a(t_0, d(x_0, A)))$$

Hence from (4.1.1) and (4.1.3)

$$b(d(F(t, t_0, x_0), A)) \leq v(t, F(t, t_0, x))$$

$$\leq r_{max}(t, t_0, a(t_0, d(x_0, A))$$

$$\leq \sim \phi_1(t_0, a(t_0, d(x_0, A))$$

i.e. $d(F(t, t_0, x_0), A) \leq b^{-1} \phi_1(t_0, a(t_0, d(x_0, A)))$

Putting $\Phi_2 = b^{-1}\varphi_1$

$$d(F(t,t_0,x_0),A) \leq \Phi_2(t_0, a(t_0, d(x_0,A)))$$
$$\leq \Phi_3(t_0, d(x_0,A))$$

when $\Phi_3 \in K^*$. Hence A is equistable

Theorem 4.2: Assume that the hypotheses of Theorem 4.1 hold with

$$b(d(x,A)) \leq V(t,x) \leq a(d(x,A)), a, b \in K \tag{4.2.1}$$

in place of (4.1.1). Then the uniform stability of the trivial solution of (3.1.2) implies the uniform stability of the set A.

<u>Proof</u>: The proof is parallel to that of theorem 4.1 and is independent of t_0.

<u>Theorem 4.3</u>: Assume that the conditions of Theorem 4.1 are satisfied. Let there exist also a function V_2 satisfying the hypothesis of Lemma 3.2 and (3.4.1). Then S_2^* together with S_1^* (i.e. the equi-stability of the trivial solution of (3.1.2)) imply the equi-strict stability of the set A.

<u>Proof</u>: The conditions of theorem 4.1 are all satisfied. Hence the conclusion of the theorem 4.1 is valid. Thus, for $d(x_0,A) \leq \delta_1$

$$d(F(t,t_0,x_0),A) \leq \Phi_2(t_0, d(x_0,A)) \tag{4.3.1}$$

where $\Phi_2 \in K^*$

As S_2^* holds, there exists a number $\alpha > 0$ such that $u_0 < \alpha$,

$$u(t,t_0,u_0) \geq \Phi_3(t_0, u_0) \tag{4.3.2}$$

where $\phi_3 \in K^*$ and $t \geq t_0$, $u(t, t_0, u_0)$ being any solution of (3.2.2). From (3.4.1) and the properties of the function b_1, there exists a number $\alpha_1 = \alpha_1(t_0, \alpha) > 0$ such that $d(x_0, A) \leq \alpha_1$ and $b_1(t_0, d(x_0, A)) \leq \alpha$ hold simultaneously. Define $\delta_2 = \min(\delta_1, \alpha_1)$. Then (4.3.1) holds for all x_0 such that $d(x_0, A) \leq \delta_2$. Choose u_0 so that

$$V_2(t_0, x_0) \geq b_1(t_0, d(x_0, A)) = u_0$$

Then from Lemma 3.2 (3.2.3)

$$\begin{aligned} V_2(t, F(t, t_0, x_0)) &\geq u_{min}(t, t_0, u_0) \\ &\geq u_{min}(t, t_0, b_1(t_0, d(x_0, A))) \end{aligned} \qquad (4.3.3)$$

It follows from (3.4.1), (4.3.3) and (4.3.2) that

$$\begin{aligned} a_1(d(F(t, t_0, x_0), A)) &\geq V_2(t, F(t, t_0, x_0)) \\ &\geq \phi_3(t_0, b_1(t_0, d(x_0, A))) \\ &= \phi_4(t_0, d(x_0, A)) \end{aligned}$$

Thus

$$d(F(t, t_0, x_0), A) \geq a_1^{-1} \phi_4(t_0, d(x_0, A)) = \phi_5(t_0, d(x_0, A)) \qquad (4.3.4)$$

when $\phi_5 = \phi_4 a_1^{-1} \in K^*$

(4.3.1) and (4.3.4), by definition S_2, mean the equi-strict stability of A.

<u>Theorem 4.4</u>: Let the assumptions of Theorems 4.2 and 4.3 hold with condition (3.4.1) replaced by

$$b_3(d(x, A)) \leq V_2(t, x) \leq a_2(d(x, A)) \qquad (4.4.1)$$

where $a_2, a_3 \in K$. Then S_4^* together with S_3^* imply the uniform-strict stability of the set A. (Uniform stability of the trivial solution of 3.1.2)

Proof: The proof is on the same lines as for Theorem 4.3 except that δ_2 is independent of t.

Theorem 4.5: Let the assumptions of theorem 4.1 hold. Then the equi-asymptotic stability of the trivial solution of (3.1.2) implies the equi-asymptotic stability of the set A.

Proof: Let $t_0 \in I$ be given. As the trivial solution of (3.1.2) is equi-asymptotic stable, there exist functions $\varphi_1 \in K^*$ and $\sigma_1 \in L^*$ and a number $\rho > 0$ such that

$$r_0 \leq \rho \tag{4.5.1}$$

implies that

$$r(t, t_0, r_0) \leq \varphi_1(t_0, r_0)\, \sigma_1(t_0, t - t_0) \tag{4.5.2}$$

where $r(t, t_0, r_0)$ is a solution of (3.1.2).

As in the proof of the earlier theorems we can determine a number $\rho_1 = \rho_1(t_0, \rho) > 0$ such that $d(X_0, A) \leq \rho_1$ and $a(t_0, d(X_0, A)) \leq \rho$ hold simultaneously.

Let X_0 be such that $d(X_0, A) \leq \rho_1$ and choose r_0 such that $V(t_0, X_0) = r_0 \leq a(t_0, d(X_0, A))$. Then (4.5.1) holds and hence (4.5.2) follows. From Lemma 3.1, (4.5.2) and (4.1.1) it follows that

$$b(d(F(t,t_0,x_0),A)) \leq V(t,F(t,t_0,x_0))$$
$$\leq r(t,t_0,a(t_0,d(x_0,A)))$$
$$\leq \psi_1(t_0,a(t_0,d(x_0,A)))\sigma_1(t_0,t-t_0)$$
$$\leq \psi_2(t_0,d(x_0,A))\sigma_1(t_0,t-t_0)$$

Hence

$$d(F(t,t_0,x_0),A) \leq b^{-1}[\psi_2(t_0,d(x_0,A))\sigma_1(t_0,t-t_0)] \tag{4.5.3}$$

Now $\sigma_1 \in L^*$ Hence $\sigma_1(t_0,t-t_0) \leq \sigma_1(t_0,0)$

Thus from (4.5.3)

$$d(F(t,t_0,x_0),A) \leq b^{-1}[\psi_2(t_0,d(x_0,A))\sigma_1(t_0,0)]$$
$$\leq b^{-1}\psi_3(t_0,d(x_0,A))$$
$$\leq \psi_4(t_0,d(x_0,A)) \tag{4.5.4}$$

where $\psi_4 \in K^*$

Also from (4.5.3) and $d(x_0,A) \leq \rho_1$ and $\psi_2 \in K^*$

$$d(F(t,t_0,x_0),A) \leq b^{-1}[\psi_2(t_0,\rho_1)\sigma(t_0,t-t_0)]$$
$$\leq b^{-1}(\sigma_2(t_0,t-t_0))$$
$$\leq \sigma_3(t_0,t-t_0) \tag{4.5.5}$$

where $\sigma_3 \in L^*$

Combining (4.5.4) and (4.5.5)

$$d(F(t,t_0,X_0),A) \leq \left[\psi_4(t_0,d(X_0,A))\,\sigma(t_0,t-t_0)\right]^{\frac{1}{2}}$$
$$\leq \psi_5(t_0,d(X_0,A))\,\sigma_4(t_0,t-t_0) \qquad (4.5.6)$$

where $\psi_5 \in K^*$ and $\sigma_4 \in L^*$ being square root of ψ_4 and σ_3 respectively. Hence the set A is equi-asymptotic stable by definitions.

Theorem 4.6. Let the assumptions of theorem 4.3 hold. Then S_6^* together with S_5^* (the equi-asymptotic stability of the trivial solution of (3.1.2)) imply equi-strict-asymptotic stability of the set A.

Proof: The assumptions of Theorem 4.3 include those of Theorem 4.1. Thus the theorem 4.5 holds. Hence for $d(X_0,A) \leq \rho_1$ (4.5.6) holds. As S_6^* hold, there exists number $\rho_2 > 0$ such that with $\phi_5 \in K^*$ and $\sigma_5 \in L^*$

$u_0 \leq \rho_2$ implies

$$u(t,t_0,u_0) \geq \phi_5(t_0,u_0)\,\sigma_5(t_0,t-t_0) \qquad (4.6.1)$$

As before, from (3.4.1) and the properties of the function b_1, we determine a number $\rho_3 = \rho_3(\rho_2,t_0) > 0$ such that $d(X_0,\rho) \leq \rho_3$ and $b_1(t_0,d(X_0,A)) \leq \rho_2$ hold simultaneously. Let $\rho_4 = \min(\rho_1,\rho_3)$. Thus for $d(X_0,A) \leq \rho_4$, (4.5.6) and (4.6.1) are satisfied.

Now choose u_0 such that

$$V_2(t_0, x_0) = u_0 \leq b_1(t_0, d(x_0, A))$$

Then from Lemma 3.2, (4.6.1) and (4.5.6) it follows that

$$\begin{aligned} a_1(d(F(t, t_0, x_0), A)) &\geq V(t, F(t, t_0, x_0)) \\ &\geq u(t, t_0, b_1(t_0, d(x_0, A))) \\ &\geq \phi_5(t_0, d(x_0, A)\sigma_5(t_0, t-t_0) \end{aligned}$$

$$d(F(t, t_0, x_0), A) \geq a^{-1}[\phi_5(t_0, d(x_0, A)\sigma_5(t_0, t-t_0)] \tag{4.6.2}$$

(4.5.6) together with (4.6.2) complete the proof of the theorem.

<u>Theorem 4.7</u>. Let the assumptions of Theorem 4.2 hold. Then the uniform asymptotic stability of the trivial solution of (3.2) imply the uniform asymptotic stability of the set A.

<u>Theorem 4.8</u>. Let the assumptions of theorem 4.4 hold. Then S_8^* together with S_7^* (the uniform asymptotic stability of the trivial solution of (3.1.2)) imply uniform-strict-asymptotic stability of the set A.

Both these theorems can be proved easily following the argument in the proofs of theorems 4.5 and 4.6 respectively.

<u>Note</u>: While proving the results in this section, we have used the Lyapunov's functions preceding theorems. This does not require that the Lyapunov functions should use the same functions. Similar comments hold for the comparison functions of classes K, K^*, L and L^*

Section 5:

In this section we prove some theorems on the existence of the Lyapunov functions in the case of the reversible dynamical system. Thus the converse theorems are proved for r.d.s. in E.

Theorem 5.1: If the set A is equi-strict stable with respect to a r.d.s. in E, then there exists a function V satisfying the assumptions of theorem 3.3.

Proof: Define V as follows. Let $t_0 \in I$

Then $V(t,x) = d(F(t, t_0, x), A)$

$V \in C[I \times A(E), R_+]$ follows from the continuity of flow F. By the reversibility condition of the r.d.s.

$$x = F(t, t_0, F(t, t_0, x_0))$$ In fact

$$x_0 = F(t_0, t, x) \Longleftrightarrow x = F(t, t_0, x_0)$$

By the equi-strict stability of the set A, there exist functions ϕ_1 and $\phi_2 \in K^*$ such that

$$\phi_2(t_0, d(x_0, A)) \le d(F(t, t_0, x_0), A) \le \phi_1(t_0, d(x_0, A))$$
$$\le \phi_1(t_0, d(F(t_0, t, x), A))$$

(5.1.1)

With the definition of V, we have

$$\phi_2(t_0, V(t,x)) \leq d(x,A) \leq \phi_1(t_0, V(t,X))$$

It follows that $\phi_1^{-1}(t_0, d(x,A)) \leq V(t,x)$

$$\leq \phi_2^{-1}(t_0, d(x,A)),$$ when $\phi_1^{-1}, \phi_2 \in K^*$

Thus the inequality (3.4.1) is now verified.

Now for $h > 0$

$$V(t+h, F(t+h, t_0, x_0)) = d(F(t_0, t+h, F(t+h, t_0, x_0), A)$$ by the reversibility.

$$= d(x_0, A)$$

$$\frac{V(t+h, F(t+h, t_0, x_0)) - V(t, F(t, t_0, x_0))}{h} = 0$$

It follows that the inequalities (3.1.1) and (3.2.1) are both satisfied by V, when g and p identically vanish.

Theorem 5.2. If the set A is uniformly strict-stable for the r.d.s. in E, then there exists functions V_1 and V_2 satisfying the assumptions of Theorem 3.4.

Proof. Define the function V by

$$V(t,x) = d(F(0,t,x),A)$$

Following the arguments in the proof of Theorem 5.1, it can be easily seen that

$$\phi_1^{-1}(d(x,A)) \leq V(t,x) \leq \phi_2^{-1}(d(x,A))$$

when $\phi_1^{-1}, \phi_2^{-1} \in K$.

Other conditions are similarly proved.

<u>Theorem 5.3</u>: (a converse of Theorem 4.4)

Assume that

(i) The set A is uniformly-strictly stable, so that

for some $\delta > 0$, $d(x_0, A) \leq \delta$ imply that

$$\beta_1(d(x_0, A)) \leq d(F(t, 0, x_0), A) \leq \beta_2(d(x_0, A)) \quad (5.3.1)$$

where $\beta_1, \beta_2 \in K$.

(ii) Let $g \in C[I \times R_+, R_+]$, $g(t, 0) = 0$ and that the trivial solution of $r' = g(t, r)$ is uniformly-strictly stable, so that for $u_0 \leq \alpha$, $\alpha > 0$:

$$r_1(u_0) \leq u(t, 0, u_0) \leq r_2(u_0) \quad (5.3.2)$$

when $r_1, r_2 \in K$ and $u(t, 0, u_0)$ is any solution of $r' = g(t, r)$ with $u(0) = u_0$ (5.3.3).

Then there exist a function $V = V(t, x)$ such that

(1) $V \in C[I \times S(A, \delta), R_+]$

(2) There exist functions a, b, $\in$ K such that for $(t, x) \in I \times S(A, \delta)$

$$b(d(x, A)) \leq V(t, x) \leq a(d(x, A))$$

(3) $V^*(t, F(t, t_0, x_0)) = g(t, V(t, F(t, t_0, x_0))) = V_*(t, F(t, t_0, x_0))$

for all $t \geq t_0$

for which $F(t, t_0, x_0) \in S(A, \delta)$.

Proof: Due to the reversibility of the dynamical system, if we denote by $x_0 = F(0, t, x) \Longleftrightarrow x = F(t, 0, x_0)$

Choose any function $\mu \in C[S(A, \varepsilon), R_+]$ such that

$$\alpha_1(d(x, A)) \le \mu(x) \le \alpha_2(d(x, A)) \tag{5.3.4}$$

when $\alpha_1, \alpha_2 \in K$.

Define

$$V(t, x) = u(t, 0, \mu(F(0, t, x))) \tag{5.3.5}$$

when $u(t, 0, \omega)$ is a solution of (5.3.3). Due to the continuity of function in (5.3.5), (1) holds.

Also for $(t, x) \in I \times S(A, \delta)$, we have

$$\begin{aligned}
V(t, x) &= u(t, 0, \mu(F(0, t, x))) \\
&\le \gamma_2(\mu(F(0, t, x))) && \text{from (5.3.2)} \\
&\le \gamma_2 \alpha_2 d(F(0, t, x), A) && \text{from (5.3.4)} \\
&\le \gamma_2 \alpha_2 \beta_1^{-1}(d(x, A)) && \text{from (5.3.1)} \\
&\le a(d(x, A)) && \text{where } a = \gamma_2 \alpha_2 \beta_1^{-1} \in K
\end{aligned}$$

Again,

$$\begin{aligned}
V(t, x) &= u(t, 0, \mu(F(0, t, x))) \\
&\ge \gamma_1 \mu(F(0, t, x))) && \text{from (5.3.2)} \\
&\ge \gamma_1 \alpha_1 d(F(0, t, x), A)) && \text{from (5.3.4)} \\
&\ge \gamma_1 \alpha_1 \beta_2^{-1}(d(x, A)) && \text{from (5.3.1)} \\
&\ge b(d(x, A)) && \text{when } b = \gamma_1 \alpha_1 \beta_2^{-1} \in K
\end{aligned}$$

Then $b(d(x, A)) \le V(t, x) \le a(d(x, A))$

which is (2)

Finally, so long as $F(t, t_0, X_0) \in S(A, \delta)$

we have $V(t, F(t, t_0, X_0)) = u(t, 0, \mu(F(0, t, F(t, t_0, X_0))))$

Hence for small $h > 0$

$$V(t+h, F(t+h, t_0, X_0)) = u(t+h, 0, \mu(F(0, t+h, F(t+h, t_0, X_0)))$$

$$= u(t+h, 0, \mu(F(0, t, F(t, t_0, X_0)))$$

Consequently,

$$\frac{V(t+h, F(t+h, t_0, X_0)) - V(t, F(t, t_0, X_0))}{h}$$

$$= \frac{u(t+h, 0, \mu(F(0, t, F(t, t_0, X_0))) - u(t_0, 0, \mu(F(0, t, F(t, t_0, X_0)))}{h}$$

$= u'(t, 0, \mu(F(0, t, F(t, t_0, X_0)))$ due to the differentiability of u as $h \to 0$.

From left hand side taking the limit $\sup$ and $\liminf h \to 0^+$

(3) is verified.

Hence the theorem.

Theorem 5.4: Suppose that the following conditions are satisfied

(i) The set A is strictly uniformly - asymptotically stable for a r.d.s. in E i.e. there exist functions $\beta_1, \beta_2 \in K$ and $\sigma_1, \sigma_2 \in L$ such that

$$\beta_1(d(x_0, A))\sigma_1(t) \leq d(F(t,0,x)), A) \leq \beta_2(d(x_0, A))\sigma_2(t) \tag{5.4.1}$$

for all $t \in I$ and $x_0 \in S(A, \delta)$

(ii) $g \in C[I \times R_+, R_+], g(t,0) = 0$ ensure existence uniqueness and continuous dependence of solutions of $r' = g(t, r)$ on initial conditions and the trivial solution of $r' = g(t,r)$ is uniformly strictly asymptotically stable, i.e. then exist functions $\gamma_1, \gamma_2 \in K$, $\eta_1, \eta_2 \in L$ such that

$$\gamma_1(u_0)\eta_1(t) \leq u(t, 0, u_0) \leq \gamma_2(u_0), t \in I \tag{5.4.2}$$

when $u(t, 0, u_0)$ is a solution of $r' = g(t,r)$ through $(0, u_0)$

(iii) γ_1 is differentiable and $\gamma_1'(r) \geq \lambda > 0$ for all $r \in R_+$

(iv) $\eta_1(t) = \lambda, \sigma_2(t) \quad \lambda_1 > 0$ for all $t \in I$

Then there exists a function $V(t, x)$ such that

(1) $V \in C[I \times S(A, \delta), R_+]$ $\delta_1 = \beta_1(\delta)\sigma_1(0)$

(2) for $(t, x) \in I \times S(A, \delta)$

$$b(d(x,A)) \leq V(t,x) \leq a(t, d(x,A)) \; t \in I$$

when b and a are defined on $[0, \delta]$ and belong to K and K* respectively on this domain.

(3) $$\limsup_{h\to 0^+} \frac{V(t+h, F(t+h, t_0, x_0)) - V(t, F(t, t_0, x_0))}{h} = g(t, V(t, F(t, t_0, x_0)))$$

(4) $$\liminf_{h\to 0^+} \frac{V(t+h, F(t+h, t_0, x_0)) - V(t, F(t, t_0, x_0))}{h} = g(t, V(t, F(t, t_0, x_0)))$$

where (3) and (4) hold for all $t \geq t_0$ for which

$F(t, t_0, x_0) \in S(A, \delta_1)$

Proof: Let $X \in S(A, \delta_1)$ and denote by X_0 the set $F(0, t, X)$. Then $X = F(t, 0, X_0)$ by reversibility condition. By 5.4.1, $X_0 \in S(A, \delta) \Rightarrow X \in S(A, \delta_1))$

Choose any function $\mu \in C[S(A, \delta_1), R_+]$ such that

$$\beta_2(d(X, A)) \leq \mu(X) \leq \alpha_2(d(X, A)) \qquad (5.4.4)$$

when $\alpha_2 \in K$ and β_2 is defined in (5.4.1).

Define the Lyapunov function by

$$V(t, X) = U(t, 0, \mu(F(0, t, X))) \qquad (5.4.5)$$

As in Theorem 5.3, V satisfies (1) and (3). The verification of (2) alone remains to be done.

$$F(0, t, X) = X_0$$

Hence from (5.4.1)

$$\beta_1(d(x_0, A)) \leq d(x,A) \leq \beta_2(d(x,A))\,\sigma_2(t)$$

so that $\beta_2^{-1}\left(\dfrac{d(x,A)}{\sigma_2(t)}\right) \leq d(x_0, A)$

$$\leq \beta_1^{-1}\left(\frac{d(x,A)}{\sigma_1(t)}\right) \qquad (5.4.6)$$

Using (5.4.2), (5.4.4) and (5.4.6) together with the assumptions (iii) and (iv) we have, since

$$V(t,x) = u(t, 0, \mu(F(0,t,x))) \qquad \text{by (5.4.3)}$$

$$\geq \gamma_1(\mu(F(0,t,x)))\,\eta_1(t)$$

$$\geq \gamma_1(\beta_2(d(F(0,t,x),A)))\,\eta_1(t)$$

$$\geq \gamma_1(\beta_2(d(x_0,A)))\,\eta_1(t)$$

$$\geq \gamma_1\left(\beta_2\beta_2^{-1}\left(\frac{d(x,A)}{\sigma_2(t)}\right)\right)\eta_1(t)$$

$$= \gamma_1\left(\frac{d(x,A)}{\sigma_2(t)}\right)\eta_1(t) = \lambda_1\gamma_1(d(x,A))$$

$$= b(d(x,A)) \text{ where } b = \lambda_1\gamma_1 \in K$$

Similarly,

$$
\begin{aligned}
V(t,x) &= U(t, c, \mu(f(0,t,x))) \\
&\leq \gamma_2(\mu(f(0,t,x)))\,\eta_2(t) \\
&\leq \gamma_2(\alpha_2\, d(x_0, A))\,\eta_2(t) \\
&\leq \gamma_2\left(\alpha_2\, \beta_1^{-1}\left(\frac{d(x,A)}{\sigma_2(t)}\right)\right)\eta_2(t) \\
&\leq a(t, d(x,A))
\end{aligned}
$$

where $a \in K^*$

This completes the proof of the theorem.

Remarks :

(1) While proving the sufficiency criteria in theorems 3.4, 3.6 and 4.4 we used two functions $(\gamma_1 \;\&\; \gamma_2)$ because of the nature of the inequalities.

However, in Theorems 5.1, 5.2 and 5.3 a stronger result, in the form of a single Lyapunov function satisfying both conditions of theorems 3.4, 3.6 and 4.4 is proved.

(2) Theorems 5.1, 5.2 and 5.3 are converse theorems for a r.d.s. in E where in the existence of a single Lyapunov function is established. But theorem 5.4 is not a converse of theorem 4.6 or theorem 4.8, for, assuming uniform-strict-asymptotic stability for the set A, we have got a Lyafamov function that yields only equi-strict-asymptotic stability . Thus a weather result is obtained.

Section 6:

In this section, earlier results are extended to stability of conditional invariant set B with respect to A for a g.d.s. in E. The concept of conditional invariancy for ordinary differential equations was introduced by Kayande and Lakshmi Kantham [6]. See also [8].

Definition: The set B is said to be conditionally invariant with respect to the set A for a g.d.s. in E if $F(t, t_0, A) \subset B$ for all $t \geq t_0$.

Note (1): If B is conditionally invariant with respect to A for a g.d.s. in E and $C \supset B$, then C is also conditionally invariant with respect to D.

Note (2): Recalling the definition of A invariant for a g.d.s. in E((i.e.) $F(t, t_0, A) \subset A$ for all $t \geq t_0$) it follows that A is self invariant and DCA then A is conditionally invariant with respect to DCA.

Let B be conditionally invariant with respect to A for a g.d.s. in E in what follows:

Definitions:

6.1: B is said to be equi-stable with respect to A for a g.d.s. in E if there exists a function $\varphi \in K^*$ such that

$$d(F(t, t_0, x_0), B) \leq \varphi(t_0, d(x_0, A)), \ t \geq t_0 \tag{6.1.1}$$

6.2: B is said to be equi-strict-stable with respect to A for a g.d.s. in E if there exist functions φ_1 and $\varphi_2 \in K^*$ such that

$$\varphi_2(t_0, d(x_0, B)) \leq d(F(t, t_0, x_0), B) \leq \varphi_1(t_0, d(x_0, A)) \quad t \geq t_0 \tag{6.2.1}$$

6.3 B is said to be uniform stable with respect to A for a g.d.s. in E if there exists a function $\phi \in K$ such that

$$d(F(t,t_0,x_0),B) \leq \phi(d(x_0,A)),\ t \geq t_0 \tag{6.3.1}$$

6.4: B is said to be uniform-strict-stable with respect to A for a g.d.s. in E if there exist functions ϕ_1 and $\phi_2 \in K$ such that

$$\phi_2(d(x_0,B)) \leq d(F(t,t_0,x_0),B) \leq \phi_1(d(x_0,A)) \tag{6.4.1}$$

6.5: B is said to be equi-asymptotic stable with respect to A for a g.d.s. in E if there exist functions $\phi \in K^*$ and $\sigma \in L^*$ such that

$$d(F(t,t_0,x_0),B) \leq \phi(t_0, d(x_0,A))\,\sigma(t_0, t-t_0) \tag{6.5.1}$$
$$t \geq t_0$$

6.6 B is said to be equistrict-asymptotic stable with respect to A for a g.d.s. in E if there exists functions $\phi_i \in K^*$ and $\sigma_i \in L^*$ $(i=1,2)$ such that

$$\phi_2(t_0, d(x_0,B))\,\sigma_2(t_0, t-t_0) \leq d(F(t,t_0,x_0),B)$$
$$\leq \phi_1(t_0, d(x_0,A))\,\sigma_1(t_0,t-t_0) \tag{6.6.1}$$
$$t \geq t_0$$

6.7: B is said to be uniform asymptotic stable with respect to A for a g.d.s. in E if there exist functions $\phi \in K$ and $\sigma \in L$ such that

$$d(F(t,t_0,x_0),A) \leq \phi(d(x_0,A))\,\sigma(t-t_0) \tag{6.7.1}$$
$$t \geq t_0.$$

6.8. B is said to be uniform-strict-asymptotic stable with respect to A for a g.d.s. in E if there exist $\phi_i \in K$ and $\sigma_i \in L$ (i = 1,2) such that

$$\phi_2(d(X_0,B))\sigma_2(t-t_0) \leq d(F(t,t_0,X_0),A) \leq \phi_1(d(X_0,A)) \times \sigma_1(t-t_0),\ t \geq t_0 \tag{6.8.1}$$

Remark: In the definitions above, the distance d is given by $d(A,B) = \sup[d(a,B),\ a \in A]$ where

$$d(a,B) = \inf d(a,b),\ b \in B$$

instead of the Hausdorff distance, for the following reasons. Suppose that we define (6.1) with d as Hausdorff distance. Then $X_0 = A$ implies $d(F(t,t_0,A),B) = 0$ due to $\phi \in K^*$ and $d(X_0,A) = 0$. In particular, at $t = t_0$ this shows that A = B. Thus the equi-stability condition (6.1.1) with Hausdorff distance d implies the equality of the sets A and B.

Moreover our definition for conditional invariancy is in terms of the 'subset of or contain' relation. Now in the Hausdorff distance, there is no way of inferring subset relationship between two sets. On the other hand $d(A,B) = 0$ implies $A \subset B$.

We have the following theorems for conditional invariancy:

Theorem 6.1: Let the assumptions of the Theorem 3.3 be satisfied except that the condition (3.3.1) is replaced by

$$b(d(X,B)) \leq V(t,X) \leq a(t,d(X,A)) \tag{6.1.1}$$

when $a \in K^*$ and $b \in K$. Then the conditionally invariant set B is equistable with respect to A.

Theorem 6.2: Let the assumptions of Theorem 3.4 hold except that the conditions (3.3.1) and (3.4.) are replaced by

$$b_1(d(x,B)) \leq V_1(t,x) \leq a_1(t, d(x,A)) \tag{6.2.1}$$

and

$$b_2(t, d(x,B)) \leq V_2(t,x) \leq a_2(d(x,B)) \tag{6.2.2}$$

where $a_1, b_2 \in K^*$ and $a_2, b_1 \in K$. Then the conditionally invariant set B is equi-strict-stable with respect to A.

Theorem 6.3: Let the assumptions of Theorem 6.1 be satisfied except that the condition (6.1.1) is replaced by

$$b(d(x,B)) \leq V(t,x) \leq a(d(x,A)) \tag{6.3.1}$$

where $a, b \in K$. Then the conditionally invariant set B is uniform stable with respect to A.

Theorem 6.4: Let the assumptions of Theorem 6.2 be satisfied except that the conditions (6.2.1) and (6.2.2) are replaced by

$$b_1(d(x,B)) \leq V_1(t,x) \leq a_1(d(x,A)) \tag{6.4.1}$$

and

$$b_2(d(x,B)) \leq V_2(t,x) \leq a_2(d(x,B)) \tag{6.4.2}$$

where a_i, b_i $(i=1,2) \in K$. Then the conditionally invariant set B is uniform-strict stable with respect to A.

Above theorems can be proved following the arguments in Theorems (3.3) to (3.6) only the proof of Theorem 6.4 is given below.

Proof of Theorem 6.4: As the assumptions of Theorem (6.2) hold, we have

$$V_1(t, F(t, t_0, x_0)) \leq V_1(t_0, x_0) \tag{6.4.3}$$

and

$$V_2(t, F(t, t_0, x_0)) \geq V_2(t_0, x_0) \tag{6.4.4}$$

for all $t \geq t_0$

By (6.4.1) and (6.4.3)

$$b_1(d(F(t, t_0, x_0), B)) \leq V_1(t, F(t, t_0, x_0)) \leq a_1(d(x_0, A))$$

Hence

$$d(F(t, t_0, x_0), B) \leq b_1^{-1} a_1(d(x_0, A)) \tag{6.4.5}$$

Similarly, by (6.4.2) and (6.4.4) we obtain

$$d(F(t, t_0, x_0), B) \geq a_2^{-1} b_2(d(x_0, B)) \tag{6.4.6}$$

Putting $q_1 = b_1^{-1} a_1 \in K$ and $q_2 = a_2^{-1} b_2 \in K$ (6.4.5) and (6.4.6) together yield

$$q_2(d(x_0, B)) \leq d(F(t, t_0, x_0), B) \leq q_1(d(x_0, A))$$

Hence the conditionally invariant set B is uniform-strict-stable with respect to A.

Comparison Theorems corresponding to Theorems 4.1 to 4.8

Theorem 6.5: Let the assumptions (i) and (ii) of theorem 4.1 hold except that the condition (4.1.1) is replaced by (6.1.1). Then

(i) equi-stability of the trivial solution of (3.1.2) implies equi-stability of the conditionally invariant set B with respect to A.

and

(ii) equi-asymptotic-stability of the trivial solution of (3.1.2) implies equi-asymptotic stability of the conditionally invariant set B with respect to A.

Theorem 6.6: Assume that the conditions (i) and (ii) of Theorem 4.1 hold, with the condition (4.1.1) replaced by (6.3.1) Then

(i) Uniform-stability of the trivial solution of (3.2) implies the uniform stability of conditionally invariant set B with respect to A.

and

(ii) Uniform-asymptotic-stability of the trivial solution of (3.1.2) implies uniform-asymptotic-stability of the conditionally invariant set B with respect to A.

Theorem 6.7: Let the assumptions of Theorem 4.3 hold, except that (4.1.1) and (4.3.1) are replaced by (6.2.1) and (6.2.2). Then

(i) S_2^* and S_1^* imply equi-strict stability

and

(ii) S_6^* and S_5^* imply equi-strict-asymptotic stability of the conditionally invariant set B with respect to A.

Theorem 6.8: Assume that the conditions of Theorem 4.3 hold with the conditions (4.1.1) and (4.3.1) replaced by (6.4.1) and (6.4.2) respectively. Then

(i) S_4^* and S_3^* imply uniform strict stability

and

(ii) S_8^* and S_7^* imply uniform strict asymptotic stability of conditional invariant set B with respect to A.

Note: Theorems 6.5 (i) and (ii), 6.6 (i) and (ii) 6.7 (i) and (ii) and (6.8) (i) and (ii) correspond to theorems 4.1, 4.5; 4.2, 4.7; 4.3, 4.6 and 4.4, 4.8, in order.

A theorem that corresponds to Theorem 5.2 and is a converse for Theorem 6.4:

Theorem 6.9: If B is uniformly-strictly-stable with respect to A for a reversible dynamical system, then there exist functions V_1 and V_2 satisfying the hypotheses of Theorem 6.4.

Proof: Define the functions V_1 and V_2 as follows:

$$V_1(t,x) = \inf_{0 \le \sigma \le t} d(F(\sigma, t, x), A) \tag{6.9.1}$$

and

$$V_2(t,x) = \sup_{0 \le \sigma \le t} d(F(\sigma, t, x), B) \tag{6.9.2}$$

Because of uniform strict stability of the conditionally invariant set B with respect to A, we have the existence of two functions $\varphi_1, \varphi_2 \in K$ such that

$$\varphi_2(d(x_0,B)) \le d(F(t,t_0,x_0),B) \le \varphi_1(d(x_0,A)), \; t \ge t_0$$

Thus for $\sigma \leq t$, by the reversibility condition,

$X = F((t, \sigma, X(\sigma))$ we have

$$\phi_2(d(X(\sigma), B)) \leq d(X, B) \leq \phi_1(d(X(\sigma), A)) \quad (6.9.3)$$

when $X(\sigma) = F(\sigma, t, x)$

Hence, for each σ, $0 \leq \sigma \leq t$

$$d(X(\sigma), A) \geq \phi_1^{-1}(d(X, B)) \quad (6.9.4)$$

so that

$$V_1(t, x) = \inf_{0 \leq \sigma \leq t} d(X(\sigma) A) \geq \phi_1^{-1}(d(X, B)) \quad (6.9.5)$$

Also trivially,

$$V_1(t, x) \leq d(X, A) \quad (6.9.6)$$

(6.9.5) and (6.9.6) together gives

$$\phi_1^{-1}(d(X, B)) \leq V_1(t, x) \leq d(X, A) \quad \text{for } 0 \leq \sigma \leq t$$

Thus (6.4.1) is verified.

Similarly, $V_2(t, x) \geq d(X, B)$ (6.9.7)

Also from (6.9.3)

$$d(X(\sigma), B) \leq \phi_2^{-1}(d(X, B))$$

so that

$$V_2(t, x) = \sup_{0 \leq \sigma \leq t} d(X(\sigma), B) \leq \phi_2^{-1}(d(X, B)) \quad (6.9.8)$$

(6.9.7) and (6.9.8) together given

$$d(x,B) \leq V_2(t,x) \leq \varphi_2^{-1}(d(x,B))$$

which verifies (6.4.2).

Now to show that the inequalities (3.1.1) and (3.2.1) are satisfied with $g = 0$ and $p = 0$

$$V_1(t, F(t, t_0, x_0)) = \inf_{0 \leq \sigma \leq t} d(F(\sigma, t, x), A)$$

$$= \inf_{0 \leq \sigma \leq t} d(x(\sigma), A)$$

Also for $h > 0$

$$V_1(t+h, F(t+h, t_0, x_0)) = \inf_{0 \leq \sigma \leq t+h} d(x(\sigma), A)$$

$$V_1(t+h, F(t+h, t_0, x_0)) \leq V_1(t, F(t, t_0, x_0))$$

so that $V^{+}(t, F(t, t_0, x_0)) \leq 0$ This verifies (3.1.1) with $g = 0$. Also

$$V_2(t+h, F(t+h, t_0, x_0)) = \sup_{0 \leq \sigma \leq t+h} d(x(\sigma), B)$$

$$\geq \sup_{0 \leq \sigma \leq t} d(x(\sigma), B)$$

$$\geq V_2(t, F(t, t_0, x_0))$$

so that

$$V_{*}(t, F(t, t_0, x_0)) \geq 0$$

This verifies (3.2.1) with .

Hence V_1 and V_2 satisfy all the conditons of Theorem 6.4.

Remark 1. The function of Theorem (6.9) is a Lyapunov function for Theorem 6.3. 2. The Theorem 6.9 is a converse of Theorem 4.4 giving two different functions V_1 and V_2 unlike Theorem 5.2, if B = A and d is Hausdorff distance.

Section 7: Conditional (or relative) stability of a compact set A with respect to a general dynamical system in E.

Let the set $A \in A(E)$ be compact on E and M be a subset of E. $A \subset M \subset E$. We state the following definitions of conditional stability of the set A with respect to a g.d.s. in E and the Lyapunov (vector) function defined on $I \times A(E)$ to determine the sufficient conditions (i.e. those conditions implying) for conditional stability of A with respect to a g.d.s. in E. The conditional stability and boundedness of the set A is discussed Kayande and Lakshmi Kantham [5] and the concept of relative stability discussed in Bhatia and Szego [2] is identical with the concept of conditional stability.

Definitions:

7.1: The set A is said to be conditionally equi-stable for the set M with respect to a g.d.s. in E, if there exists a function $\phi \in K^*$ such that

$$d(F(t, t_0, x_0), A) \leq \phi(t_0, d(x_0, A)), \; t \geq t_0$$

whenever $x_0 \subset M \cap S(A, \delta)$ for some $\delta > 0$.

7.2: The set A is said to be conditionally uniformly stable for the set M with respect to a g.d.s. in E, if there exists a function $\phi_1 \in K$ such that

$$d(F(t, t_0, x_0), A) \le \phi_1(d(x_0, A)), \quad t \ge t_0$$

whenever $x_0 \in M \cap \bar{S}(A, \delta)$ for some $\delta > 0$.

7.3: The set A is said to be conditionally equi-asymptotic stable for the set M with respect to a g.d.s. in E, if there exist functions $\phi_2 \in K^*$ and $\sigma_2 \in L^*$ such that

$$d(F(t, t_0, x_0), A) \le \phi_2(t_0, d(x_0, A))\,\sigma_2(t_0, t - t_0) \quad t \ge t_0$$

whenever $x_0 \in M \cap \bar{S}(A, \delta)$ for some $\delta > 0$.

7.4: The set A is said to be conditionally uniformly asymptotically stable for the set M with respect to a g.d.s. in E, if there exist functions $\phi_3 \in K$ and $\sigma_3 \in L$ such that

$$d[F(t, t_0, x_0), A) \le \phi_3(d(x_0, A))\,\sigma_3(t - t_0), \quad t \ge t_0$$

whenever $x_0 \in M \cap \bar{S}(A, \delta)$ for some $\delta > 0$.

Notes: (1) If $M \subseteq E$ the above definitions are the same as S_1, S_3, S_5 and S_7 respectively.

(2) These definitions are similar to the ones given in [5]. We are expressing them in terms of the monotone functions.

(3) If M is equal to a neighborhood of A, then the result in Note - (1) above holds.

In order to obtain sufficient conditions for the conditional stability properties of the set A, we use the comparison technique based on vector Lyapunov function.

Definition: The function $W = W(t, r)$ a vector function with components $W_1, W_2, \ldots, W_n$ on $r = (r_1, \ldots, r_n)$ is said to possess quasi-monotone property in r for each fixed $t \in I$, if for each $i = 1, 2, \ldots, n$ the ith component $W_i(t, r)$ is monotonic decreasing in r_j, $j \neq i$ for each j. Readily $W \in C[I \times R^n, R^n]$

If W has the quasi-monotone property in , then the differential system

$$r' = W(t, r) \quad \left(' = \frac{d}{dt}\right) \tag{7.1}$$

has the maximal (in the sense of component in majorisation) solution existing to the right of t_0. We will assume that is smooth enough to ensure that this maximal solution exists for all $t \in [t_0, \infty)$

Let V be a n-vector and $V \in [I \times A(E), R_+^n]$ when R_+^n consists of real n-vectors with all components non-negative.

In what follows, the vector inequalities are to be interpreted as being satisfied component wise.

Let

$$V^*(t, X(t)) = \limsup_{h \to 0^+} \left[\frac{V(t+h, X(t+h)) - V(t, X(t))}{h}\right] \tag{7.2}$$

for $(t, X(t)) \in I \times A(E)$.

Lemma 7.1: Let there exist a vector functional V defined above such that V^+ defined in (7.9) satisfy the vector inequality

$$V^+(t, x(t)) \leq W(t, V(t, x(t)), \quad t \geq t_0 \tag{7.3}$$

when W is smooth having quasimonotone property.

Let $r(t, t_0, r_0)$ be the maximal solution of the differential system (7.1) existing to the right of t_0. Then

$$V(t, x(t)) \leq r(t, t_0, r_0) \tag{7.4}$$

whenever

$$V(t_0, x(t_0)) \leq r(t_0, t_0, r_0) = r_0 \tag{7.5}$$

Let $m(t) = V(t, x(t))$. Then (7.2) and (7.3)

$$\Rightarrow \limsup_{h \to 0^+} \left[\frac{m(t+h) - m(t)}{h} \right] \leq W(t, m(t))$$

Using the notion of the maximal solution for a differential system and the quasimonotone property of W, the Lemma is established following the arguments in [8].

We take $V = (V_1, V_2)$, $W = (W_1, W_2)$ and the quasi-monotone property of W_1 is now equivalent to W_1 being non-decreasing in r_2 and W_2 non-decreasing in r_1.

Let $r = (r_1, r_2) \in R_+^2$

Let $|V| = V_1 \dot{\underset{\wedge}{+}} V_2$ and $|Y| = Y_1 + Y_2$

(these make sense as the component V_1, V_2, Y_1, Y_2 are all non-negative)

Let $Y_0 = (Y_0, 0)$ (7.6)

i.e. Y_0 is a vector in R^2_+ with the second component identically zero. Then $|Y_0| \leq Y_1$.

Let $Y(t, t_0, Y_0)$ be the maximal solution of (7.1) with defined in (7.6). Corresponding to the definitions (7.1) to (7.4) we state the following properties for (7.1.1) Y_0 is supposed to satisfy (7.6)

7.1.5: There exists a number $\delta > 0$ such that

$$Y(t, t_0, Y_0) \leq \phi_1(t_0, Y_0), \; t \geq t_0, \; \phi_1 \in K^*$$

whenever $|Y_0| \leq \delta$

7.2.5: There exists a number $\delta > 0$ such that

$$Y(t, t_0, Y_0) \leq \phi_2(Y_0), \; t \geq t_0, \; \phi_2 \in K$$

whenever $|Y_0| \leq \delta$.

7.3.5: For a given number, $\delta > 0$ there exis functions $\phi_3 \in K^*$ and $\sigma_3 \in L^*$ such that $|Y_0| \leq \delta$ implies that

$$Y(t, t_0, Y_0) \leq \phi_3(t_0, Y_0)\, \sigma_3(t_0, t - t_0), \; t \geq t_0,$$

For a given number $\delta > 0$, there exist functions

$\phi_4 \in K$ and $\sigma_4 \in L$ such that,

$|r_0| \leq \delta$ implies that

$$r(t, t_0, r_0) \leq \phi_4(r_0)\,\sigma_4(t - t_0)$$

Let

$$M = \{x \in A(E) : \forall V_2(t, x) = 0\} \tag{7.7}$$

We also state the following conditions on V satisfied for all

$(t, x) \in I \times A(E)$

$$b(d(x, A)) \leq V(t, x) \leq a(t, d(x, A)) \tag{7.8}$$

and

$$b(d(x, A)) \leq V(t, x) \leq a(d(x, A)) \tag{7.9}$$

wherein (7.8) $a \in K^*$ and in (7.9) $a \in K$, while $b \in K$ in both conditions.

Theorem 7.2: Let the assumptions of Lemma 7.1 hold, together with (7.7) and (7.8). Then

(i) Condition (7.1.s) implies that the set A is conditionally equistable for the set M with respect to a g.d.s. in E (vide definition 7.1)

(ii) Condition (7.3.s) implies that the set A is conditionally equi-asymptotically stable for the set M with respect to g.d.s. in E.

In each of these cases M is given by (7.7)

Proof: Let $X(t) = F(t, t_0, X_0)$. By the properties of (7.8), there exists $\delta_1 > 0$, $\delta_1 = \delta_1 (t_0, \delta)$ such that $d(X_0, A) \leq \delta_1$ and $a(t_0, d(X_0, A)) \leq \delta$ hold simultaneously. Choose $r_0 = (r_1, 0)$ with $r_1 = V_1 (t_0, X_0)$ Let $X_0 \in M$. Then by (7.7), $V_2 (t_0, X_0) = 0$ if $X_0 \in \bar{S}(A, \delta_1)$, then $d(X_0, A) \leq \delta_1$ and the choice of δ_1 and r_0 show that

$$r_0 \leq a(t_0, d(X_0, A)) \leq \delta \tag{7.1.1}$$

Thus (7.5) is satisfied and $r_0 \in M \cap \bar{S}(A, \delta_1)$, we have

$$V(t, F(t, t_0, X_0)) \leq r(t, t_0, r_0), \; t \geq t_0 \tag{7.1.2}$$

This inequality, component in would mean

$$V_1 (t, F(t, t_0, X_0)) \leq r_1 (t, t_0, r_0) \tag{7.1.3}$$

and

$$V_2 (t, F(t, t_0, X_0)) \leq r_2 (t, t_0, r_0) \tag{7.1.4}$$

From these two and the definition of norm, it followws that

$$|V(t, F(t, t_0, X_0))| \leq |r(t, t_0, r_0)| \tag{7.1.5}$$

(7.1.5), (7.1.1), (7.1.5) and (7.8) imply

$$b(d(F(t,t_0,x_0),A) \leq |V(t,F(t,t_0,x_0))$$

$$\leq |r(t,t_0,x_0)|$$

$$\leq \phi_1(t_0,r_0)$$

$$\leq \phi_1(t_0,a(t_0,d(x_0,A)))$$

$$d(F(t,t_0,x_0),A) \leq b^{-1}\phi_1(t_0,a(t_0,d(x_0,A)))$$

$$\leq \phi_5(t_0,d(x_0,A))$$

Thus by definition (7.1), A is conditionally equistable for the set M with respect to a g.d.s. in E.

(ii) Proceeding on the same lines, because of (7.3.5), we get

$$b(d(F(t,t_0,x_0),A)) \leq \phi_3(t_0,\phi(t_0,d(x_0,A))\sigma_3(t-t)$$

$$\Rightarrow d(F(t,t_0,x_0),A)) \leq b^{-1}\phi_3(t_0,a(t_0,d(x_0,A))$$

$$\times\sigma(t_0,t-t_0)$$

$$\leq \phi_6(t_0,d(x_0,A))\sigma_5((t_0,t-t_0) \qquad (7.1.6)$$

Here $\delta_1 = \delta_1(t_0,\delta)$ such that $d(x_0,A) \leq \delta_1$

and $a(t_0,d(x_0,A)) \leq \delta$ hold simultaneously.

Then A is conditionally equiasymptotically stable for M with respect to a g.d.s. in E.

Theorem 7.3: Let the assumptions of Lemma 3 hold together with (7..) and (7.9). Then

(i) Condition (7.2.S) implies that the set A is conditionally uniformly stable for the set M.

and

(ii) Condition (7.4.5) implies that the set A is conditionally uniformly asymptotically stable for the set M, with respect to a g.d.s. on E.

The proof is similar to that for Theorem 7.2. The functions ϕ, σ belong to K and L respectively instead and δ_1 is independent of t_0.

Section 8

A converse theorem for conditional stability

Let $g = g(t, u)$, $u = (u_1, u_2)$ so that $g = g(t, u_1, u_2)$ be defined and continuous on $I \times R_+^2$ into R^2 and satisfy the quasi-monotone decreasing condition in . We also assume that g is smooth enough to ensure the existence, uniqueness and continuity of the solution with respect to initial conditions of the equations,

$$u' = g(t, u) \tag{8.1}$$

for all $t \in I$, $t_0 \in I$

Let $g(t, u) = (g_1(t, u), g_2(t, u))$

$$= (g_1(t, u_1, u_2), g_2(t, u_1, u_2))$$

Define

$$g^*(t,u) = (g_1(t,u_1,u_2),\ g_2(t,0,u_2))$$

As $u_1, u_2 \geq 0$, the quasi-monotone property of g shows that

$$g^*(t,u) \leq g(t,u) \tag{8.2}$$

Suppose $u^*(t,0,u_0)$ and $u(t,0,u_0)$ are solutions of $u' = g^*(t,u)$ (8.3) and (8.1) respectively through the same point $(0,u_0)$, $u_0 \in R_+^2$

Then we have

$$u^*(t,0,u_0) \leq u(t,0,u_0),\ t \in I \tag{8.4}$$

Suppose $P_1 = (u_{10},0) \in R_+^2$ and $P_2 = (u_{10}, u_{20}) \in R_+^2$

Then the solutions of (8.3) through $(0,p_1)$ and $(0,p_2)$ be denoted by $u_1^*(t,0,P_1)$ and $u_2^*(t,0,P_2)$ respectively.

Writing these equations component in

$$u_1^*(t,0,P_1) = (u_{11}^*(t,0,P_1),\ u_{12}^*(t,0,P_1))$$

and

$$u_2^*(t,0,P_2) = (u_{21}^*(t,0,P_2),\ u_{22}^*(t,0,P_2))$$

Thus we have

$$\left.\begin{array}{l} u_1^*(t,0,p_1) \le u_2^*(t,0,p_2) \\ u_{11}^*(t,0,p_1) \le u_{21}^*(t,0,p_2) \\ u_{12}^*(t,0,p_1) \le u_{22}^*(t,0,p_2) \end{array}\right\} \qquad (8.5)$$

Theorem 8.1: (i) Let the g.d.s. be r.d.s. and the flow $F(t,t_0,X_0)$, $X_0 \in A(E)$ be Hausdorff continuous in the triplet of its arguments.

(ii) Let there exist functions $\alpha, \beta \in K$ such that

$$\beta(d(X_0,A)) \le d(F(t,t_0,X_0),A) \le \alpha(d(X_0,A)) \qquad (8.1.1.)$$

(iii) Let $g \in C$ on $[I \times R_+^2, R^2]$, $g(t,0)=0$ and g has the properties mentioned for g in (8.1)

(iv) The solution $u(t,0,u_0)$ of (8.1) satisfy

$$u(t,0,u_0) \le r_2(|u_0|) \qquad (8.1.2)$$

where $u_0 = u_{20}$ as $u_{10} = 0$, where $u_0 = (u_{10}, u_{20})$

(v) The component $u_2^*(t,0,u_0)$ of the solution of (8.3) has the property

$$u_2(t,0,u_0) \geq r_1(|u_0|) = r_1(u_{20}) \qquad (8.1.3)$$

where u_0 satisfies the definition given in (iv) above.

Then there exists a function $V = V(t, X)$ with the following properties.

1. V is continuous on $I \times A(E)$ into R_+^2
2. $V^*(t, x) \leq g(t, V(t, x))$ for the flows X of r.d.s.
3. If $X \subset M$ then $V_1(t, x) = 0$
4. $b(d(x, A)) \leq |V(t, x)| \leq a(d(x, A))$

where a, b $\in$ K and $(t, x) \in I \times A(E)$

Proof: As the g.d.s. is r.d.s., $X = F(t, 0, X_0)$ implies that $X_0 = F(0, t, x)$ Choose any function $\mu \in C[A(E), R_+^2]$ such that

$$\alpha_1(d(x, A)) \leq \mu(\alpha) \leq \alpha_2(d(x, A)) \qquad (8.1.3)$$

and $(\mu_1(x), \mu_2(x))$ for $X \subset M$ (8.1.4)

$$\mu(x) = (\mu_1(x), \mu_2(x))$$

Let $u_1^*(t, 0, (\mu_1(X_0)_0)$ and $u_2^*(t, 0, (\mu_1(X_0), \mu_2(X_0))$

be the solutions of the equation (8.3), define

$$V_1(t, x) = u_{11}^*(t, 0, (\mu_1(F(0, t, x)), 0))$$

$$= u_{11}^*(t, 0, (\mu_1(X_0), 0))$$

and

$$V_2(t,x) = u^*_{22}(t,0,(\mu_1(F(0,t,x)),\mu_2(F(0,t,x))))$$

$$= u^*_{22}(t,0,(\mu_1(x_0),\mu_2(x_0)))$$

where u^*_{11} and u^*_{22} are respectively first and second components of u^*_1 and u^*_2. The continuity of V_1 and V_2 follow from the continuity of u^*_{11} and u^*_2 with respect to the initial conditions together with the continuity properties of μ and F with respect to their arguments.

Let $X(t) = F(t,t_0,X(t_0))$ so that

$$X(t+h) = F(t+h,t_0,X(t_0))$$

Also $F(0,t+h,X(t+h)) = F(0,t,X(t)) = x_0$

by the reversibility property.

Hence

$$V_1^+(t,X(t)) = \limsup_{h\to 0^+}\left[\frac{u^*_{11}(t+h,0,(\mu_1(x_0),0)) - u^*_{11}(t,0,(\mu_1(x_0),0))}{h}\right]$$

$$= u'_{11}(t,0,(\mu_1(x_0),0))$$

$$= g^*_1(t,u^*_{11}(t,0,(\mu_1(x_0),0)),u^*_{12}(t,0,(\mu_1(x_0),0)))$$

Similarly,

$$V_2^+(t, x(t)) = g_2^*(t, 0, u_{22}^*(t, 0, (\mu_1(x_0), \mu_2(x_0)))$$

With the definition of V_1 and V_2

$$V_1^+(t, x(t)) = g_1^*(t, v_1(t, x(t)), u_{12}^*(\mu_1(x_0), 0))) \quad (8.1.5)$$

and

$$V_2^+(t, x(t)) = g_2^*(t, 0, v_2(t, x(t))) \quad (8.1.6)$$

Now from the inequalities (8.5)

$$u_{12}(t, 0, (\mu_1(x_0), 0)) \leq u_{22}^*(t, 0, (\mu_1(x_0), \mu_2(x_0)))$$

$$\leq V_2(t, x(t)) \quad (8.1.7)$$

Also trivially, $\quad 0 \leq V_1(t, x(t)) \quad$ (8.1.8)

Hence by the quasi-monotone non-decreasing property of g_1 and g_2 we have

$$V_1^+(t, x(t)) \leq g_1^*(t, V_1(t, x(t)), V_2(t, x(t)))$$

$$\leq g_1(t, V_1(t, x(t)), V_2(t, x(t))) \quad (8.1.9)$$

and

$$V_2^+(t, x(t)) \leq g_2^*(t, 0, V_2(t, x(t)))$$

$$\leq g_2(t, 0, V_2(t, x(t)))$$

$$\leq g_2(t, V_1(t, x(t)), V_2(t, x(t))) \quad (8.1.10)$$

(8.1.7) and (8.1.8) verify the property (2). Property (3) follows from the property (8.1.2) and the fact that $V_1(t,x) = 0$ if $\mu_1(x) = 0$

Now

$$|V(t,x)| = V_1(t,x) + V_2(t,x)$$

$$= u_{11}^*(t,0,(\mu_1(x_0),0)) + u_{22}^*(t,0,(\mu_1(x_0),\mu_2(x_0)))$$

$$\leq u_{21}(t,0,(\mu_1(x_0),\mu_2(x_0))) + u_{22}(t,0,(\mu_1(x_0),\mu_2(x_0)))$$

i.e.

$$|V(t,x)| \leq \gamma_2(|\mu(x_0)|) \text{ by hypothesis (iv)}$$

$$\text{and } \mu(x_0) = 0 \implies x_0 \subset M$$

$$= \gamma_2(\alpha_2(d(x_0,A))) \quad \text{by} \quad (8.1.3)$$

$$= \gamma_2(\alpha_2(\beta^{-1}(d(x,A)))) \quad \text{from} \quad (8.1.1)$$

$$= a(d(x,A)) \quad \text{when} \quad a = \gamma_2\alpha_2\beta^{-1} \in K$$

(8.1.11)

$$|V(t,x)| = V_1(t,x) + V_2(t,x)$$
$$\geq V_2(t,x) = u_{22}(t,o,(\mu_1(x_0),\mu_2(x_0)))$$
$$\geq \gamma_1(\mu(x_0)) \text{ by hyp. (iv) and}$$
$$\geq \gamma_1(\alpha_1 d(x_0,A)) \quad \mu(x_0) = 0 \Rightarrow x_0 \subset M$$
$$\geq \gamma_1 \alpha_1 \alpha^{-1}(d(x_0,A)) \text{ by hyp (ii)}$$
$$= b(d(x,A)) \text{ When } \gamma_1 \alpha_1 \alpha^{-1} = b \in K$$

(8.1.12)

(8.1.11) and (8.1.12) verify the property 4.

Hence the proof.

Note: The theorem proved is not a converse of either Theorem 7.1 or Theorem 7.2 in the strict sense. We find that the hypothesis (ii) on the estimates for $d(F(t,o,x_0),A)$ imply strict conditional stability for the set A with respect to the set M. Similarly the condition (iv) corresponds to (7.1.s) for the equation (8.1), but we also require condition (v), which is compatible with the condition (7.1.s) similar remarks hold for the Theorem 8.2 below.

2. Using the notion of mini-max solutions for a system, we can get theorems that will give strict-conditional stability for the set A.

3. Theorem 8.1 can be considered on the extension of Theorem 4.5.1 of [9] on conditional stability of ordinary differential system to reversible dynamical system. The results (8.2), (8.4) and (8.5) are special cases of (4.5.2), (4.5.3), and (4.5.4) from the reference [9], pp. 285 .

We can also prove the following extension of Theorem (4.5.2) to reversible dynamical systems.

Theorem 8.2:- Let the assumptions (i) and (iii) of Theorem (8.1) hold. Assume further -

(a) there exist functions $\beta_1, \beta_2 \in K$, $\sigma_1, \sigma_2 \in L$ such that $X_0 \subset M$

$$\beta_1(d(x_0, A))\sigma_1(t) \leq d(F(t_0, x_0), A) \leq \beta_2(d(x_0, A))\sigma_2(t)$$

for $t \geq 0$

(b) the solution $u(t, 0, u_0)$ of (8.1) satisfies the condition

$$u(t, 0, u_0) \leq r_2(|u_0|)\eta_2(t), \; t \geq 0, \; r_2 \in K, \; \eta_2 \in L$$

when $u_0 = (u_{10}, u_{20})$ and $u_{10} = 0$

(c) The component $u^*_{22}(t, 0, u_0)$ of the equation (8.2) satisfies the condition $u^*_{22}(t, 0, u_0) \geq r_0(u_0)\eta_1(t)$

with u_0 satisfying conditions in (b) and $r_1 \in K$ and $\eta_1 \in L$

(d) $r_1(t)$ is differentiable and $r_1'(t) \geq \lambda \geq 0$

(d) η_1 and σ_2 are such that

$$\eta_1(t) \geq \lambda_1 \sigma_2(t), \; t \geq t_0, \; \lambda_1 > 0$$

There exists a function $V(t,x)$ with properties (1), (2) and (3) of Theorem (8.1) and

$$b(d(x,A)) \leq V(t,x) \leq a(t_0, d(x,A))$$

when $b \in K$ and $a \in K^*$

This theorem shows the existence of a Lyapunov function for asymptotic conditional stability.

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