STUDIES IN

ELEMENTARY PARTICLE THEORY

INVOLVING

THE PION-NUCLEON INTERACTION

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PREFACE

This thesis is the work done by the author during the period August 1958 to January 1963 under the guidance of Professor Alladi Ramakrishnan, formerly Professor of Physics at the University of Madras and now Director of the Institute of Mathematical Sciences, Madras, in the field of elementary particle theory.

The problems dealt with are divided into three parts. Part I is devoted ~~with~~ to the application of the Chew-Low formalism to some processes involving the production, scattering and absorption of pions. Part II uses the method of dispersion relations and Part III concerns some stochastic aspects in cascade theory and the proof of the equivalence of the field-theoretic and Feynman formalisms.

Seven papers which cover part of the subject-matter of this thesis have been published by the author in scientific journals, the available reprints being enclosed in the form of a booklet. The nature of the problems dealt with in these papers has necessitated collaboration either with my guide Professor Alladi Ramakrishnan or with my colleagues, Drs. S.K.Srinivasan and N.R.Ranganathan and Messrs. A. P. Balachandran and V. Devanathan. Due acknowledgment of this collaboration has been made in each chapter.

My grateful thanks are due to Professor Alladi Ramakrishnan for his constant guidance and encouragement throughout the course of this work. I am indebted to the University of Madras and the Institute of Mathematical Sciences, Madras for providing me with excellent facilities for research work and to the Atomic Energy Commission for the award of a Senior Research Fellowship during the period of study.

Madras,
January 1963 (K. Venkatesan)

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CHAPTER I.

Introduction

This thesis presents studies in some elementary particle interactions involving mainly the scattering, production and absorption of pions. While it has become almost conventional to speak of the inadequacy of the perturbation approach to the theory of strong interactions involving pions and nucleons, it is an undisputed fact that, no matter how long and how hard theoretical physicists have tried, the approximations even in intrinsically non-perturbative theories are still based on analogies and concepts derived from Feynman graphs representing the processes. Within the domain of field theory itself the main advance over the elementary form of perturbation theory has been to consider only 'physical' (and not 'bare') particle from the beginning. One such method which has had a good measure of success in explaining the low energy features of the pion-nucleon interaction is the Chew-Low formalism. The major part of the thesis is devoted to the application of this method to certain pion production processes, and the results obtained on this basis have been used to suggest that the concept of a specific interaction as given by a Lagrangian or Hamiltonian is useful at least in the sub-Bev region. Using, in addition, the impulse approximation, the scattering of pions from deuterons and the light nuclei have been studied.

The other main approach to the problem of strong interactions is the use of dispersion theory. Recently, with the advent of the double variable (or Mandelstam) representation, the possibility of a complete theory of strong interactions based on the analytically continued S-matrix has been envisaged[1] and thus the application of these methods to all possible problems is desirable even with drastic approximations. We have employed the 'strip approximation' to the Mandelstam representation in order to study some pion production processes. The pole approximation in single variable dispersion theory has also been used to study a decay process.

Parenthetically, we also deal with the mathematical problem of multiple production of particles within the frame work of cascade theory not with a view to computing cross sections; but assuming a knowledge of them to determine the mean number of particles at a certain thickness of matter. This is cited as an example where the study of a process 'in the gross' using classical stochastic methods can throw light on the cross-sections. ~~Other new aspect~~ This is perhaps reasonable as there exists no reliable theory of

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1) This approach is propounded in G.F.Chew's 'S-matrix Theory of Strong Interactions', W.A.Benjamin, Inc., New York (1961).

these processes based on the methods applicable to elementary particle interactions. Indeed it looks as if modern methods end when multiple production begins. Other new aspects of cascade theory are included.

The theory of the pion and its interaction with the nucleons is co-extensive with the theory of strong interactions which has undergone many vicissitudes and the difficulties associated with it have become part of the 'tradition' of strong interaction physics. The early application of weak coupling theory [1] failed to give correct results as expected and other methods like the Tamm-Dancoff approximation [2] were found unsuitable for one reason or other. As mentioned earlier, the Chew-Low method [3] was able to explain many of the features of low-energy pion physics. Low [4] showed how one could write down the matrix element for a process in terms of physical quantities (and not bare ones as in the Tamm-Dancoff method) the transition from the interaction to the Heisenberg representation being effected by using the relation between the physical and bare vacua derived earlier by Gell-Mann and Low [5]. The canonical

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1) See e.g., R.E.Marshak, 'Meson Physics', McGraw-Hill Book Co., New York (1952).
2) G.Bethe-de Hoffmann, 'Mesons and Fields', Vol. II, Row, Peterson and Co., New York (1955).
3) G.F.Zax Chew and F.E.Low, Phys. Rev., 101,1570,1579 (1956)
4) F.E.Low, Phys. Rev., 97, 1392 (1955)
5) M. Gell-Mann and F.E.Low, Phys. Rev., 84, 350 (1951)

commutation relations were used to convert the particles in the state vector into current operators, before the Gell-Mann Low theorem mentioned above, was applied. It is interesting to note that the result that the matrix element for a process can be successively reduced to matrix elements (or vacuum expectation values) of products of Heisenberg current operators can also be arrived at from an entirely different approach -- viz., the method of Lehmann, Symanzik and Zimmermann[1] in which there is no concept of a specific interaction and which is based on certain postulates which are of quite general validity like Lorentz invariance, microscopic causality and the asymptotic condition.

The matrix element derived by Low's method is exact, inprinciple, but unworkable in practice. One has to make the approximation of cutting off the intermediate states containing more than a few particles and use a static form of the interaction Hamiltonian. With these approximations, Chew and Low[2] showed that the crossing symmetry exhibited by Low's equation and the unitarity of the S-matrix are sufficient to establish the most striking feature of pion-nucleon scattering, namely, that there is a ~~resonance~~ resonance in the $T = 3/2$, $J = 3/2$ state. Their theory also gave a satisfactory explanation of the low energy features of the photoproduction of a pion from a nucleon.

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1) H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento, 1, 205 (1955)
2) G.F.Chew and F.E.Low, Loc. cit.

Part I of this thesis is in the main, an application of Low's method and the Chew-Low formalism to certain pion production, scattering and absorption processes. Extension of the Chew-Low method to energies higher than the first resonance in the pion-nucleon system (at about 300 Mev) where production of extra particles becomes feasible have given results in quantitative and in some cases qualitative agreement with experiment. The details of this part are as follows:

In Chapter II, the photoproduction of pion pairs from nucleons is studied using Low's method.[1] An integral equation for the process is set up. The inhomogeneous term arrived at on the assumption that the pion-pion interaction is not important for the process, for the photon energies considered (400-600 Mev) yields total cross sections in agreement with the experimental results of the Cocconi group[2]. This result taken in conjunction with that for pion production in pion-nucleon collisions where the pion-pion interaction is important even at low energies seems to imply that at these energies the pion-pion interaction acts essentially as an initial state interaction for capturing the incident pion and would therefore be naturally absent for the photoproduction processes. By making an angular momentum and isospin decomposition of the matrix element, the integral equation is solved using the Muskhelishvili-Omnes[3] method and the rescattering correction is shown

1) S.K. Srinivasan and K.Venkatesan, Nuclear Physics, 12, 418 (1959); 29, 335 (1962)
2) J.M. Sellen et al, Phys. Rev., 113, 306 (1959)
B.M.Chasan et al , Phys. Rev., 119, 811 (1960).
3) R. Omnes , Nuovo Cimento, 8, 316 (1958).

to be small. Finally the effect of a direct photon three pion interaction on the process is considered and it is shown that the same value of the coupling constant for the interaction cannot explain both this process and the electromagnetic structure of the nucleon.

Chapter III deals with the effect of pion-pion interaction on various pion production processes initiated by the photon and pion [1]. A qualitative argument is given to show that the interaction may not be important for the photoproduction of pion pairs. This is followed by a consideration of graphs in which the two final pions resonate or one of the pions resonates with a pion in the virtual cloud which leads to the result that the contributions from them is small. Next the contribution of the pion-pion interaction to (i) the photoproduction of triple pions and (ii) production of single and double pions in pion-nucleon collisions are estimated by reducing the matrix elements for these processes, by Low's method to what correspond to one-pion exchange graphs [2]. The values for the pion-pion coupling constant thus arrived at lie in the range given by Chew and Mandelstam [3] in their dispersion theoretic study of pion-pion scattering.

In Chapter IV, the effect of polarising both the incident particle (if it has spin) and the target, on the angular distributions and degree of polarization of the target

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1) S. K. Srinivasan and K. Venkatesan, Nuclear Physics, 29, 335 (1962)
2) F. Salzman and G. Salzman, Phys. Rev. Letters, 5, 377 (1960); S. Drell, Phys. Rev. Letters, 5, 342 (1960).
3) G. F. Chew and S. Mandelstam, Phys. Rev., 119, 467 (1960)

in the final state is studied for the processes of (i) single pion photoproduction from a nucleon [1], (ii) photoproduction of pion pairs [2] and (iii) electroproduction of pions. The necessity to polarize both the target and incident particle in order to obtain interesting results regarding angular distributions and polarizations is brought out.

The scattering of pions from light nuclei like the deuteron is the subject-matter of Chapter V [3]. The impulse approximation is used in conjunction with the Chew-Low matrix element for pion scattering from a nucleon to derive the cross-sections. Agreement with experiment is found upto a kinetic energy of 300 Mev. of the pion upto which only experimental data exist. The calculation at 300 Mev contradicts an earlier work [4] in which it is reported that the impulse approximation does not give correct results at this energy. The theory for scattering from light nuclei is also given.

In Chapter VI, the processes other than scattering which are possible when a pion is incident on the deuteron, - namely, pure and radiative absorption are considered in a unified way along with the other distintegration processes initiated by a photon and an electron. Low's method is employed

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1) Alladi Ramakrishnan, S.K. Srinivasan, N.R. Ranganathan and K. Venkatesan, Proc. Ind. Acad. Sci, 49, 302 (1959)
2) S.K. Srinivasan and K. Venkatesan, Proc. Ind. Acad. Sci. 50, 392 (1959)
3) Alladi Ramakrishnan, V. Devanathan and K. Venkatesan, Nuclear Physics, 29, 680 (1962)
4) L.S. Dulkova et al, Soviet Physics, JETP, 8, 217 (1959).

for deriving the matrix elements for these processes which brings out the similarity and interconnection of these processes. The angular distributions are studied by setting up the x 'angular operators'[1] for the processes.

In the last chapter of this part (Chapter VII), which is mainly expository in character, Low's method is employed to study the role in electrodynamics of the 'equal time commutator' term which always arises when the field variablex of the particle which is being converted into a 'current' appears in the interaction Hamiltonian in a bilinear or multilinear combination. The trouble such terms bring in their wake both by way of unrenormalized quantities and erroneous results (if proper care is not exercised in interpreting the matrix elements) is emphasized by considering the standard electrodynamic processes. Questions regarding symmetrization of the matrix element have been studied.

Part II deals with some applications of the dispersion relations, where the concept of an interaction as given by a Hamiltonian or a Lagrangian is given up and the analytic properties of the transition amplitudes are made use of for studying elementary particle interactions. In the single variable dispersion relations, (the variable being the energy or momentum transfer) the principle of microscopic causality used in conjunction with the Cauchy integral formula gives the dispersion relations which express the real part of the amplitude for real values

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1) V.I.Ritus, Soviet Physics - JETP, 5, 1249 (1957)

of the variable as an integral over the imaginary part and vice versa. Though proof for the existence of such relations could not be given in many cases (notably for nucleon-nucleon scattering) still these relations have found considerable application in determining coupling constants and relative parities and also in the choice of phase shifts [1]. An approximation which has been used in this connection is the pole approximation which corresponds to the renormalized Born approximation of the Lagrangian theory. The renormalized propagator has a pole with unit residue at a value of the square of the total centre of mass energy equal to the square of the mass of a single particle intermediate state which has the same quantum numbers as the initial or final state [2]. Further, the vertex functions (which appear in the numerator of the Born approximation) are analytic in the neighbourhood of this point [3]. Thus for a discussion of experiments close to a pole, the lowest order perturbation graphs may be reliable if renormalized quantities are used.

The single variable dispersion relations do not describe all the physical singularities and hence are insufficient to determine the S-matrix. Mandelstam's conjecture that the matrix element is simultaneously an analytic function of the

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1) See e.g., the article of M.J.Moravcsik, in 'Dispersion Relations', Edited by G.R.Screaton, Oliver and Boyd (1960).
2) A.O.G. Kallen in 'Relations de dispersion et particles elementaires', Hermann & Cie Paris (1960)
3) R. Omnes, ibid.

energy and momentum transfer tries to remedy this situation. The conjecture enabled him to generalize the substitution law (familiar in electrodynamics where the matrix element for e pair annihilation, for instance, is obtained from that for Compton scattering by suitable substitution) into the statement that there is a single analytic function of three variable only two of which are independent which describes all three channels corresponding to a given diagram representing two particles going over to two particles. The singularities (pole and branch poins) of this function are those allowed by unitarity in each of the three channels. This principle of maximum smoothness replaces, in a way, the causality condition here. A double application of Cauchy's theorem then gives the Mandelstam representation in which the comprehensive function (representing the matrix element for each of the three channels if the variables are properly chosen) is written as a sum of pole terms and of integrals over the double spectral functions which in turn are the discontinuities across the cuts in two of the three variables taken at a time. The representation cannot be derived from a general principle like causality, but has been verified to all orders in perturbation theory [1]. A generalization of Mandelstam's ideas to interactions involving more than two particles has been given by Landau [2] and Cutkosky [3].

1) R.J.Eden, Phys. Rev., 121, 1567 (1961); R.J.Eden, P.V.Landshoff, J.C.Polkinghorne and J.C.Taylor, Nuovo Cimento, 19, 939 (1961)
2) L.D.Landau, Nuclear Phys., 13, 181 (1959)
3) R.E.Cutkosky, J.Math. Phys. 1, 429 (1960).

In the earlier applications of the Mandelstam representation the double variable nature was not exploited though Mandelstam himself had indicated how the double spectral functions could be computed in the 'elastic' region and the matrix element built up by a series of iterations. But the 'strip' approximation recently suggested by Chew and Frautschi[1] (and independently by others) which states that the strips of the double spectral functions lying in the elastic strip between the thresholds for elastic and the first inelastic process involving production of an extra particle are all that we need to know to obtain a matrix element which will be satisfactory not only in the low energy region but for arbitrary energies with restricted momentum transfer.

We have applied these ideas to certain pion production processes in Part II. Chapter VIII which is the first chapter of this part deals with the evaluation of the double spectral functions for the problem of photoproduction of pions on the nucleons. Using the strip approximation in the sense mentioned above, integral equations have been set up, which can be the starting point for calculations involving energies higher than that of the first pion-nucleon resonance.

In Chapter IX, the photoproduction of pions on pions which is the simplest process one can think of as it involves

1) G.F.Chew and S.C.Frautschi, Phys. Rev. Lett., 5, 580 (1960); Phys. Rev., 123, 1478 (1961). Recently, however, they have given a re-definition of the 'strip' region in terms of the Regge Poles (G.F.Chew, S.C.Frautschi and S.Mandelstam, Phys. Rev., 126, 1202 (1962))

only one amplitude is studied in the strip approximation. Using the one dimensional solution for the problem, the inelastic contribution to the absorptive part is evaluated. In the limit of very high energy, the inelastic cross section is shown to fall off rapidly.

In Chapter X , the pole approximation is applied to the pion decay mode of the Ξ particle and a possible explanation for the curious pattern of the Σ decay asymmetries is given based on the idea of a parity clash.

In Part III of this thesis, certain new stochastic aspects of cascade theory are studied in Chapter XI.[2] An interesting relationship between the old and the new approach to cascade theory is established during the regeneration point method. An error in the usual formulation of the cascade theory of multiple particle production is pointed out. Application of the ambigenous stochastic process which enables us to consider problems involving back-scattering also, to the problem of lateral spread in extensive air showers is indicated.

Finally in Chapter XII, a comparative study of the well-known method of Wick[3] for proving the equivalence of the field theoretic and Feynman formalisms and the method

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1) A.P.Balachandran and K. Venkatesan, Progr. Theoret. Phys., 26, 792 (1961)
2) Alladi Ramakrishnan and K. Venkatesan, Proc. Cosmic Ray Symposium, Department of Atomic Energy, Government of India (1961).
3) G.C.Wick, Phys. Rev., 80, 268 (1950).

of Ramakrishnan et al[1] is made[3]. The use of the stochastic concept of realisation (or sample) is shown to lead to a proof of the equivalence which is straightforward and transparent.

1) A. Ramakrishnan et al , Journal of Mathematical Analysis and Applications,

2) A. Ramakrishnan, V. Devanathan and K. Venkatesan, Journal of Mathematical Analysis and Applications (in press)

CHAPTER II

PHOTO PRODUCTION OF PION PAIRS FROM NUCLEONS.[1)]

1. Introduction

There is a satisfactory theoretical description of the photo production of a single pion from a nucleon upto about 300 or 400 Mev which is the region where the first pion-nucleon resonance in the (3/2, 3/2) state is dominant. Using a cut-off and a value $f^2 = 0.08$ for the parameter f ~~which~~ which is the pseudovector renormalised unrationalised pion-nucleon coupling constant, Chew and Low[2)] were able to reproduce many of the broad features of photo pion production in this energy region. In addition, the Chew-Low theory led to the Kroll-Rudermann theorem viz., that the s -wave amplitude for charged photo pion production at threshold is simply related to f^2 and the value of f^2 thus obtained was found to be in close agreement with the value obtained from measurement of pion nucleon scattering. Further refinements to theory of photo production have been made by the application of dispersion relations.[3)]

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1) S.K.Srinivasan and K. Venkatesan, Nuclear Physics, 12, 418 (1959); ibid, 29, 335, (1962).
2) G.F.Chew and F.E.Low, Phys. Rev., 101, 1579 (1956).
3) See Chapter VIII

At energies above 400 Mev the photo production of pion pairs and higher multiplets becomes energetically possible. The process of pair production was first noted in experiments with a 500 Mev bremsstrahlung beam of the Caltech synchrotron[1] by observing the negative pions emitted from a hydrogen target. The negative pions are presumed to arise from the reaction

$$\gamma + p \longrightarrow \pi^- + \pi^+ + p \qquad (a)$$

More extensive measurements were carried out by Friedman and Crowe[2], Bloch and Sands[3], Sellen et al[4] and Chasan et al[5]. The last named authors investigated more than two hundred events of double-pion photo production and concluded that the total cross-section for this process is nearly a third of the cross-section for the photo production of a single pion at resonance.

In view of these data for the process, it is necessary to examine how far the observed cross-sections can be explained within the present frame-work of elementary particle theory. Cutkosky and Zachariasen[6] used the Chew-Low theory to study the process (a) assuming that the bombarding energy of the photon is low enough so that two p-wave mesons are not produced. Further assuming the pion-nucleon interaction to be predominantly

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1) M. Sands et al , Phys. Rev., 99, 652, (1955)
2) R.M.Friedman and K.M.Crowe, Phys. Rev., 105, 1369 (1957).
3) M.Bloch and M. Sands, Phys. Rev., 113, 305, (1959).
4) J.M.Sellen et al , Phys. Rev., 110, 779 (1958).
5) B.M.Chasan et al, Phys. Rev., 119, 811 (1960).
6) R.E.Cutkosky and F. Zachariasen, Phys. Rev., 103, 1108, (1958).

p -wave they argue that the cross-section for producing two s -wave mesons will depend only on recoil effects and hence are expected to be small. Thus they assume that only one p -wave and one s -wave pions are produced. The experimental fact that the energy spectrum of negative pions is peaked at low energy lent support to their view.

We have used Low's method for deriving an integral equation for the relativistic matrix element for process (a). Retaining in the first instance only the inhomogeneous terms and using the static approximation the total cross-sections have been evaluated for various values of the incident photon energy and these are found to be in agreement with the experimental values of Sellen et al and Chasan et al[1]. In a later section the integral equation is solved by making an angular momentum and isotopic spin analysis and utilising the experimental fact mentioned above, that the negative pion tends to come out with low energies for considerable energies of the incident photon.

In addition to its utility in deciding the validity of the static approximation at energies higher than the ones originally intended which would at the same time confirm certain characteristic features of the static theory such as the linearity of the interaction and the view-point that only low energy intermediate states are important for low energy processes, a study of process (a) can be useful for other reasons. The suggestion has been made by Chew[2] that the role of the pion-pion interaction in

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1) Refs. 5 and 6 , p.15

2) Remarks by G.F.Chew and G.Bernardini, Ninth International Annual Conference on High Energy Physics, Kiev (1959), pp.171-172.

pion-nucleon collisions is to capture the incident pion in the pion cloud surrounding the nucleon. The effect of the Yukawa interaction and especially the resonance in the (3/2, 3/2) state of the pion-nucleon system would enter essentially as a final state interaction. Now the absence of such an interaction in the photo production of pions, (if we ignore a possible direct photon-three pion interaction for the present) would mean that the Yukawa interaction between the pion and the nucleon along with the electromagnetic interaction by itself should be able to explain the experimental features of pion pair production upto a considerably energy. Ball's[1] results on single pion photo production seem to lend support to the view that the pion-pion interaction is not of much consequence in low-energy photo production of pions. A study of process (a) should throw further light on this matter.

It has alsox been suggested[2] that the application of the Chew-Low[3] extrapolation procedure to the process would enable us to determine the coupling constant for the direct photon-three pion interaction. If the differential cross-section for the process is extrapolated to the pion pole in the momentum transfer variable, it is related to the product of the pion-nucleon and photon-three pion vertices from which

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1) J.S.Ball, Phys. Rev., 124, 2014 (1961)
2) M. Gourdin and A. Martin : Nuovo Cimento, 16, 78 (1960).
3) G.F.Chew and F.E.Low, Phys. Rev., 113, 1640, (1959).

the coupling constant for the latter should be determinable (fig. 1). We have examined the effect of the photon-three pion interaction on the double-pion photo production employing a phenomenological interaction Hamiltonian.

2. Derivation of the matrix element

We use Low[1]'s method for deriving the matrix element for the process. The matrix element for the production of a pair of mesons of momenta q_1, q_2 and isobaric spin indices α, β from a nucleon due to a photon of momentum k and polarisation ϵ is given by[2]

$$\langle p', q_1\alpha, q_2\beta \,|S|\, k\epsilon\,; p\rangle = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \int dx_2 \cdots \int dx_n \cdot \left(\Phi_{p'}, a_\alpha(q_1)\, a_\beta(q_2), P\left(H(x_1)H(x_2)\cdots H(x_n) b^{(\epsilon)+}(k)\Phi_p \right) \right) \quad (1)$$

where p and p' are the initial and final momentum states of the nucleon. P is the chronological operators of Dyson and $H(x)$ the interaction Hamiltonian density given by

$$H(x) = i g \bar{\Psi}(x)\gamma_5 \tau_i \Psi(x)\varphi_i(x) - \delta m \bar{\Psi}(x)\Psi(x) - \tfrac{1}{2}\delta\mu^2 \varphi_i(x)\varphi_i(x) + ie\bar{\Psi}(x)\gamma_\mu A_\mu(x)\tfrac{1}{2}(1+\tau_3)\Psi(x) + \tfrac{1}{2} e \bar{\Psi}(x)\sigma_{\mu\nu} F_{\mu\nu}(x)\left[\tfrac{1}{2}(1+\tau_3)\mu_p + \tfrac{1}{2}(1-\tau_3)\mu_n\right]\Psi(x)$$

---

1) F.E.Low, Phys. Rev., 97, 1392, (1955).

2) Throughout this thesis, we shall write a four-vector x and a three-dimensional vector by an arrow placed above it. The scalar product of two four-vectors x and y will be denoted by $xy = \vec{x}\cdot\vec{y} - x_0 y_0$.

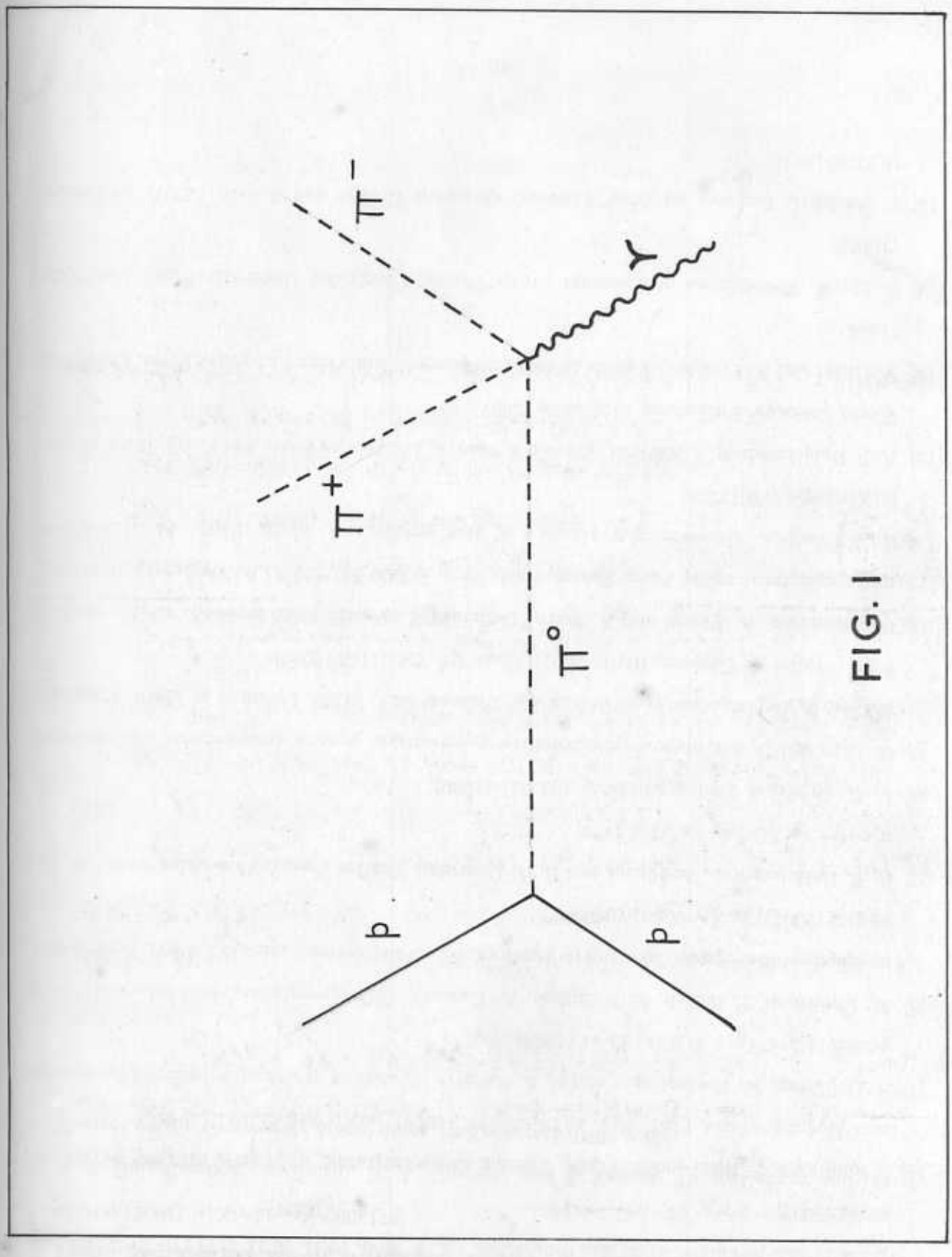

FIG. I

$$+ ie\left(\varphi^*(x)\frac{\partial\varphi(x)}{\partial x_\mu} - \frac{\partial\varphi^*(x)}{\partial x_\mu}\varphi(x)\right)A_\mu(x)$$

$$+ e^2 A_\mu(x) A_\mu(x) \varphi^*(x)\varphi(x)$$

$$+ 4\pi\lambda\left[\varphi_i(x)\varphi_i(x)\right]^2 \qquad (2)$$

Here the various terms on the right hand side inorder, are:

(1) the pion-nucleon interaction term,

(2) the nucleon mass renormalisation,

(3) the meson masses renormalisation,

(4) the photon-nucleon interaction through the charge,

(5) the magnetic moment interaction of the nucleon with the photon;

(6) and (7) the meson-photon interaction terms, and

(8) the direct pion-pion interaction.

The τ_i's are the usual iso-spin matrices of the nucleon and μ_p and μ_n are the magnetic moments of the proton and the neutron respectively; a is the annihilation operator of meson and $b^{(\epsilon)\dagger}$ is the creation operator for a photon of polarization type ϵ.

We ignore for the time being the last term in (2) which represents direct pion-pion interaction. We shall take the interaction into consideration in the next chapter. Commuting $a_\alpha(q_1)$ and $a_\beta(q_2)$ through to the right using the commutation relation

$$\left[a_j(q_\nu), \varphi_k(x)\right] = \delta_{kj} \frac{e^{-iq_\nu x}}{(2\omega(q_\nu))^{1/2}} \qquad (3)$$

and taking into consideration only the first five terms in (2), we obtain

$$\langle p'; q_1\alpha, q_2\beta | S | k\epsilon; p\rangle = (-i)^2 \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \cdot \int dx dy \frac{e^{-iq_1 x - iq_2 y}}{(4\omega(q_1)\omega(q_2))^{\frac{1}{2}}} \cdot \int dx_1 \int dx_2 \cdots \int dx_n \cdot \left(\Phi_{p'}, P\left[H(x_1) H(x_2) \cdots H(x_n) \cdot j_\alpha(x)\, j_\beta(y)\right] b^{(\epsilon)+}(k) \Phi_p\right) \qquad (4)$$

where j is the meson current operator defined by the equation of motion

$$(\Box^2 - \mu^2)\varphi_i(x) = ig\bar{\psi}(x)\gamma_5 \tau_i \psi(x) - \delta\mu^2 \varphi_i(x)$$

where μ on the left is the renormalised pion mass.[1)]

We now convert this matrix element which is defined in terms of quantities in the interaction representation into one in the Heisenberg representation by using the relation

$$\left(\Phi_{p'}, \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \int dx_2 \cdots \int dx_n \cdot P\left[H(x_1) \cdots H(x_n) A_1^I(x) A_2^I(y) \cdots\right] \Phi_p\right)$$

$$= \langle p' | P\left[A_1^H(x) A_2^H(y) \cdots\right] | p\rangle \qquad (5)$$

1) In the following the pion mass will be set equal to unity.

where A^{I} 's are interaction and A^{H}'s Heisenberg operators and $|p\rangle$ represents the exact single physical nucleon eigen state of the total Hamiltonian. This relation can be easily derived on making use of the Gell-Mann Low Theorem which gives the relationship between the "bare" and the "physical" vacuum.

$$c\,\psi_0 = \frac{U(0, \pm\infty)\,\Phi_0}{(\Phi_0, U(0, \pm\infty)\,\Phi_0} \qquad (6)$$

where U is the usual unitary operator. Thus the matrix element can be written as

$$\langle p'; q_1\alpha, q_2\beta\,|S|\,k\epsilon; p\rangle = -\int dx \int dy \cdot \frac{e^{-iq_1x - iq_2y}}{(4\omega_1\omega_2)^{\frac{1}{2}}} \langle p'\,|P(J_\alpha(x)J_\beta(y))|\,k\epsilon; p\rangle \qquad (7)^{1)}$$

where for brevity we have written ω_1 and ω_2 for $\omega(q_1)$ and $\omega(q_2)$ respectively and the J's are the pion current operators in the Heisenberg representation.

If we had taken the operators $a_\alpha(q_1)$ and $a_\beta(q_2)$ through the sixth term of (2) and followed the above procedure, then upto first order inthe electromagnetic coupling constant e, we should have obtained a matrix element which apart from factors would be

$$\langle p'\,|\,A_\mu(x)\,|\,k\epsilon; p\rangle \qquad (8)$$

Using the equation of motion for the photon-nucleon interaction

$$\Box A_\mu(x) = ie\,\bar{\psi}(x)\,\gamma_\mu\,\psi(x) = j_\gamma(x) \qquad (9)$$

---

1) This result can also be arrived at following the procedure of Lehmann et al , using the asymptotic condition to convert the particles(pions) in the state vector into currents.
H. Lehmann et al , Nuovo Cimento, 1 , 205, (1955)

and using the translational invariance for any Heisenberg operator

$$F(x) = e^{-iPx} F(0) e^{iPx} \tag{10}$$

where P is the energy-momentum operator, we could write (8) as

$$- \frac{\langle p' | j_\gamma(x) | k\epsilon ; p \rangle}{(k+p-p')^2} \tag{11}$$

The matrix element in the numerator represents Compton scattering from a nucleon. Since the cross-section for this is small above the threshold for pion production, we shall not consider this term any further. Similarly we can ignore the seventh term in (2) since it is already of second degree in e.

Using translational invariance, we obtain, on performing the x integration in (7),

$$\langle p' ; q_1\alpha , q_2\beta | S | k\epsilon ; p \rangle = -\delta(p+k-p'-q_1-q_2) \cdot \frac{1}{(4\omega_1\omega_2)^{1/2}} \int dy e^{-iq_2 y} \langle p' | P(J_\alpha(0) J_\beta(y)) | k\epsilon ; p \rangle \tag{12}$$

Introducing a complete set of intermediate states and performing the y integration we obtain

$$\langle p' ; q_1\alpha , q_2\beta | T | k\epsilon ; p \rangle = \frac{1}{(4\omega_1\omega_2)^{1/2}} \times \left[\sum_n \langle p' | J_\alpha(0) | n ; \vec{p} = \vec{k} + \vec{p} - \vec{q}_2 \rangle \right.$$
$$\left. \langle n ; \vec{p}_n = \vec{k} + \vec{p} - \vec{q}_2 | J_\beta(0) | k\epsilon ; p \rangle \right.$$
$$(\omega_2 + E_n - p_0 - k_0 - i\varepsilon)^{-1}$$

$$- \sum_n \langle p' | J_\beta(0) | n ; \vec{p}_n = \vec{p}' + \vec{q}_2 \rangle$$

$$\times \langle n ; \vec{p}_n = \vec{p}' + \vec{q}_2 | J_\alpha(0) | k\epsilon ; p \rangle (\omega_2 + p_0' - E_n + i\epsilon)^{-1} \Big] \quad (13)$$

where the T- matrix is defined by

$$\langle p' ; q_1\alpha, q_2\beta | S | k\epsilon ; p \rangle = i\delta(p + k - p' - q_1 - q_2)$$

$$\times \langle p' ; q_1\alpha, q_2\beta | T | k\epsilon ; p \rangle \quad 14)^{1)}$$

Next we make the usual "Tamm-Dancoff" approximation, restricting the ' n ' states to one nucleon,(ii) one nucleon + one photon and (iii) one nucleon + one pion states. This procedure yields an integral equation for the double pion photoproduction amplitude. The inhomogeneous part consists of two terms. The term arising from the single nucleon intermediate state contains the product of the pion nucleon vertex and the single pion photoproduction amplitude. The term arising from the single nucleon + single photon intermediate state includes the product of the amplitude for the photo-production of a single pion and the amplitude for the photo-production of a pion and photon from a nucleon. The third term arising from the single nucleon + single pion intermediate state is the integral equation term the kernel being related to the matrix element for pion-nucleon scattering.

---

1) It will be noticed that we are consistently ignorning all factors of π . These will be taken into account correctly when the cross-section is evaluated, by putting in the proper factors in the density of states.

Instead of ~~x~~ "contracting" on both the mesons in the state vector we could have converted the incident photon and one of the final pair of pions into current operators. Then the numerators of the two terms of the double-pion photoproduction matrix element would be, in the single nucleon approximation

$$\langle p' ; q_2 \beta | J_\alpha(0) | N \rangle \langle N | J_\gamma(0) | p \rangle$$

and

$$\langle p' ; q_2 \beta | J_\gamma(0) | N \rangle \langle N | J_\alpha(0) | p \rangle$$

respectively where $|N\rangle$ denotes the single nucleon state and $J_\gamma(0)$ is the complete "photon" current operator consisting of a part due to the nucleon current and a part due to the pion current. These two terms represent respectively,

(i) the product of pion-nucleon scattering amplitude and the electromagnetic vertex of the nucleon, and

(ii) the photo production of a single pion X the pion-nucleon vertex.

The kernel of the integral equation term which arises from the single-nucleon + single photon intermediate state contains the matrix element for Compton scattering from the nucleon.

We have preferred to work with the first form (equation (13)) for the double pion photoproduction matrix element. We shall study the following processes with its help:

(a) $\gamma + p \longrightarrow p + \pi^+ + \pi^-$

(b) $\gamma + p \longrightarrow n + \pi^+ + \pi^\circ$

(c) $\gamma + p \longrightarrow p + \pi^\circ + \pi^\circ$

3. $\gamma + p \rightarrow p + \pi^+ + \pi^-$

In this case we note that a non-vanishing contribution to the amplitude comes from only one of the two terms in (13) since a proton cannot emit a π^-. The complete matrix element for the process in the approximation of retaining the intermediate states mentioned in the previous section will be

$$\langle p'; q_{1-}, q_{2+}|T|k\epsilon; p\rangle (4\omega_1\omega_2)^{\frac{1}{2}}$$

$$= \langle p'|J_-(0)|\vec{p}+\vec{k}-\vec{q}_2\rangle \langle \vec{p}+\vec{k}-\vec{q}_2|J_+(0)|k\epsilon; p\rangle$$

$$\times\left[\omega_2 + E(\vec{p}+\vec{k}-\vec{q}_2) - p_0 - k_0 - i\varepsilon\right]^{-1}$$

$$+\Sigma\int d^3k' \langle p'|J_-(0)|k'\epsilon'; \vec{p}+\vec{k}-\vec{q}_2-\vec{k}'\rangle$$

$$\times\langle k'\epsilon'; \vec{p}+\vec{k}-\vec{q}_2-\vec{k}'|J_+(0)|k\epsilon; p\rangle$$

$$\times\left[(\omega_2 + k' + E(\vec{p}+\vec{k}-\vec{q}_2-\vec{k}') - p_0 - k_0 - i\epsilon)\right]^{-1}$$

$$+\Sigma\int dp''\int dq' \langle p'|J_-(0)|q'\alpha'; p''\rangle$$

$$\times\langle p''; q'\alpha'|J_+(0)|k\epsilon; p\rangle\ \delta(\vec{p}''+\vec{q}'-\vec{p}-\vec{k}+\vec{q}_2)$$

$$\times\left[p_0'' + \omega' + \omega_2 - p_0 - k_0 - i\varepsilon\right]^{-1}$$

$$+\Sigma\int dp''\int dq' \langle p'|J_+(0)|q'\alpha'; p''\rangle$$

$$\times\langle p''; q'\alpha'|J_-(0)|k\epsilon; p\rangle\ \delta(\vec{q}_2+\vec{p}'-\vec{p}''-\vec{q}')$$

$$\times\left[\omega_2 + p_0' - p_0'' - \omega' + i\varepsilon\right]^{-1} \qquad (15)$$

The summation sign in the second, third and fourth terms on the right hand side indicates a summation over the spin, isotopic spin and polarization of the particles in the intermediate state. We observe that in a consistent single nucleon approximation, it is necessary to include a part of the second term on the right hand side of (15) (which arises from the intermediate state containing a single nucleon and a photon) since this part contributes a term which is of the same order of magnitude as the 'n' state containing a single nucleon. This is because the second half of the matrix element of this term represents the matrix element for scattering of a photon with emission of a pion. In the lowest order the photon remains unscattered while a pion is emitted by a nucleon. In this case, there is no integration over k' since $\vec{k}' = \vec{k}$. The unscattered photon now interacts with the nucleon in the intermediate stategiving rise to the production of a pion. Thus the matrix element from this term is the same as the one arising from the first term on the right hand side of (15) except that the particles π^+ and π^- have switched ends, the pion-nucleon vertex involving the π^+ occurring earlier than the photo-production of π^- . The first term of (15) on the other hand involves the photo production of a π^+ which appears earlier than the pion-nucleon vertex involving the π^- . In the further calculations inthis section, we shall be taking into account only these two terms deferring the solution of the integral equation to a subsequent section.

We have to use some approximate forms for the single photo production matrix elemmat and the pion-nucleon vertex. It is generally believed[1] that many of the phenomena in the sub Gev region can be studied successfully on the basis of a cut-off theory suitably modified by the pion-pion interaction. We shall use the form for the pion-nucleon vertex in the static approximation given by Low[2] and the single photo-production matrix element (as given by Chew et al)[3] in which we retain only the significant terms, namely, the magnetic dipole (or phase shift) term, the electric dipole term and the pion current f term containing all multipoles. The matrix element for double-pion photo-production can then be written as

$$\langle p'; q_{1-}, q_{2+} | T | k\epsilon ; p \rangle = \frac{(4\pi)^{3/2} e f^2}{(8\omega_1 \omega_2 k)^{1/2}} |M|^2$$

where M is given by

$$M = \frac{i(\mu_p - \mu_n)}{6\omega_2 m f^2} \frac{e^{i\delta(q_1)} \sin\delta(q_1)}{q_1^3} \cdot \left\{ 2\left(\vec{q}_1 \cdot (\vec{k}\times\vec{\epsilon})\right)\vec{\sigma}\cdot\vec{q}_2 - \vec{\sigma}\cdot\left(\vec{q}_2 \times [\vec{q}_1 \times (\vec{k}\times\vec{\epsilon})]\right) - i\vec{q}_2\cdot\left(\vec{q}\times(\vec{k}\times\vec{\epsilon})\right)\right\}$$

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1) M.Cini and S.Fubini, Nuclear Physics, 13, 352 (1960)

2) F.E.Low, loc cit

3) G.F.Chew et al , loc cit . Here and in the rest of the thesis we shall be using a square cut off in the static expressions for vertices and matrix elements so that factors like $v(q_1)$ or $v(q_2)$ representing the cut off factors will not be explicitly written.

$$+\frac{2}{\omega_2\left(1+\frac{\omega_1}{m}\right)}\left\{\vec{q}_2\cdot\vec{\epsilon}+i\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{q}_2)\right.$$

$$\left.+\frac{2\vec{q}_1\cdot\vec{\epsilon}}{(\vec{k}-\vec{q}_1)^2+1}\left[(\vec{k}-\vec{q}_1)\cdot\vec{q}_2+i\vec{\sigma}\cdot(\vec{k}-\vec{q}_1)\times\vec{q}_2\right]\right\}$$

$$-\frac{i(\mu_p-\mu_n)}{6\omega_1 m f^2}\;\frac{e^{i\delta(q_2)}\sin\delta(q_2)}{q_2^3}$$

$$\times\left\{2\vec{q}_2\cdot(\vec{k}\times\epsilon)\,\vec{\sigma}\cdot\vec{q}_1+\vec{\sigma}\cdot\left[\vec{q}_1\times(\vec{q}_2\times(\vec{k}\times\vec{\epsilon}))\right]\right.$$

$$\left.-i\,\vec{q}_1\cdot(\vec{q}_2\times(\vec{k}\times\vec{\epsilon}))\right\}$$

$$-\frac{2}{\omega_1\left(1+\frac{\omega_2}{m}\right)}\left\{\vec{q}_1\cdot\vec{\epsilon}+i\vec{\sigma}\cdot\vec{q}_1\times\vec{\epsilon}\right.$$

$$\left.+\frac{2\vec{q}_2\cdot\vec{\epsilon}}{(\vec{k}-\vec{q}_2)^2+1}\left[\vec{q}_1\cdot(\vec{k}-\vec{q}_2)+i\vec{\sigma}\cdot\vec{q}_1\times(\vec{k}-\vec{q}_2)\right]\right\} \qquad (17)$$

Here $\delta(q)$ represents the phase shift for the 33 state corresponding to the momentum q and f^2 is the renormalised unrationalized pseudovector coupling constant which has a value 0.08.

Squaring (17), averaging over the initial spin and polarization directions and summing over the final spin states, we obtain for the differential cross-section for the process, the expression

$$\frac{d^3\sigma^{+-}}{d\omega_1\,d\Omega_+\,d\Omega_-}=(2\pi)^{-5}\;\frac{(4\pi)^3 e^2 f^4\omega_1\omega_2 q_1 q_2}{(8\omega_1\omega_2 k)^{1/2}}$$

$$\times|M_a+M_b|^2 \qquad (18)$$

where M_a and M_b are the sums of the first two and the last two terms of (17) respectively.

$$|Ma|^2 = \frac{(\mu_p-\mu_n)^2 k^2 q_2^2}{72 m^2 f^4 \omega_2^2 q_1^4} \sin^2\delta(q_1)\,(2+3\sin^2\theta_1)$$

$$+ \frac{4 q_2^2}{\left(1+\frac{\omega_1}{m}\right)^2 \omega_2^2}\left[1 - \frac{2 q_1^2 \sin^2\theta_1}{(k^2+q_1^2-2kq_1\cos\theta_1+1)^2}\right.$$

$$+ \frac{2}{3}\,\frac{(\mu_p-\mu_n) k q_2^2 \sin\delta(q_1)\cos\delta(q_1)}{q_1^2 \omega_2 m f^2}$$

$$\left. - \left[\frac{k q_1 \sin^2\theta_1}{k^2+q_1^2-2kq_1\cos\theta_1+1} - \cos\theta_1\right]\right] \qquad (19)$$

$$|M_b|^2 = \frac{(\mu_p-\mu_n)^2 k^2 q_1^2 \sin^2\delta(q_2)}{72 m^2 f^4 \omega_1^2 q_2^4}\,(2+3\sin^2\theta_2)$$

$$+ \frac{4 q_1^2}{\left(1+\frac{\omega_2}{m}\right)^2 \omega_1^2}\left[1 - \frac{2 q_2^2 \sin^2\theta_2}{(k^2+q_2^2-2kq_2\cos\theta_2+1)^2}\right.$$

$$+ \frac{2}{3}\,\frac{(\mu_p-\mu_n) k q_1^2 \sin\delta(q_2)\cos\delta(q_2)}{q_2^2 \omega_1 m f^2}$$

$$\left. \cdot \left[\frac{k q_2 \sin^2\theta_2}{k^2+q_2^2-2kq_2\cos\theta_2+1} - \cos\theta_2\right]\right] \qquad (20)$$

$$2\,\mathrm{Re}\,M_a^* M_b = \frac{1}{18}\,\frac{(\mu_p-\mu_n)^2 k^2}{\omega_1\omega_2 m^2 f^4 q_1 q_2}\,\sin\delta(q_1)\sin\delta(q_2)$$

$$\cdot\cos(\delta(q_1)-\delta(q_2))\Big[-\frac{3}{2}\sin\theta_1\sin\theta_2\cos\varphi$$

$$\times(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi)+\sin^2\theta_2+\sin^2\theta_2$$

$$+\sin^2\theta_1\cos^2\theta_2+\cos^2\theta_1\sin^2\theta_2-2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\cos\varphi\Big]$$

$$+\frac{(\mu_p-\mu_n)\sin 2\delta(q_1)kq_2}{3\omega_1\omega_2 m f^2 q_1}\Big[\frac{1}{2}\cos\theta_1(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi)$$

$$-\sin\theta_1\sin\varphi(\cos\theta_2\sin\theta_1\sin\varphi)-\sin\theta_1\cos\varphi$$

$$\times(\cos\theta_1\sin\theta_2+\cos\theta_2\sin\theta_1\cos\varphi)$$

$$+\frac{1}{2}(\sin^2\theta_1\cos\theta_2-\sin\theta_1\cos\theta_1\sin\theta_2\cos\varphi)$$

$$+\frac{q_2\sin\theta_2}{(\vec{k}-\vec{q}_2)^2+1}\Big(2k\sin^2\theta_1\sin\theta_2\sin^2\varphi$$

$$-(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi)$$

$$\times\Big\{k\sin\theta_1\cos\varphi+q_2(\cos\theta_1\sin\theta_2-\sin\theta_1\cos\theta_2\cos\varphi)\Big\}\Big)$$

$$+\frac{q_2\sin\theta_2}{(\vec{k}-\vec{q}_2)^2+1}\cos\theta_2\sin\theta_1\cos\varphi\Big\{k\cos\theta_1-q_1(\cos\theta_1\cos\theta_2$$

$$+\sin\theta_2\sin\theta_2\cos\varphi)\Big\}\Big]$$

$$+\frac{(\mu_p-\mu_n)\sin 2\delta(q_2)kq_1}{3\omega_1\omega_2 m f^2 q_2}\Big[\frac{1}{2}\cos\theta_1$$

$$+\sin\theta_2(\cos\theta_1\sin\theta_2-\sin\theta_1\cos\theta_2\cos\varphi)\Big]$$

$$+\frac{k}{(\vec{k}-\vec{q}_1)^2+1}\Big[q_1(\cos\varphi-\sin\varphi)\Big\{\sin\theta_1\cos\theta_2$$

$$\times(\sin\theta_2\cos\theta_1+\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi)\Big\}$$

$$-2\sin^2\theta_1\sin^2\theta_2\sin^2\varphi$$

$$-q\sin^2\theta_1(\cos^2\theta_2+\sin^2\theta_2\cos\varphi\sin\varphi)\Big]$$

$$-\frac{8q_1q_2\cos\theta_1\cos\theta_2}{\omega_1\omega_2}$$

$$+\frac{16q_1^2q_2^2\sin\theta_1\sin\theta_2\cos\varphi}{\omega_1\omega_2\Big[(\vec{k}-\vec{q}_1)^2+1\Big]\Big[(\vec{k}-\vec{q}_2)^2+1\Big]}$$

$$\times\Big\{k(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi)(q_1\cos\theta_1+q\cos\theta_2)$$

$$+2k^2\cos\theta_1\cos\theta_1-k(q_1\cos\theta_2+q_2\cos\theta_1)+q_1q_2\Big\}$$

$$+\frac{4q_1^2q_2\sin\theta_2}{(\vec{k}-\vec{q}_1)^2+1}\Big\{(k\cos\theta_1-q_1)\sin\theta_2\cos\varphi$$

$$+q_1\sin\theta_1(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi)\Big\}$$

$$+\frac{4q_2^2q_1\sin\theta_2}{(\vec{k}-\vec{q}_2)^2+1}\Big\{(k\cos\theta_2-q_2)\sin\theta_1\cos\varphi$$

$$+q_2\sin\theta_2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\varphi)\Big\} \qquad (21)$$

In the above θ_2 is the angle that the positive pion makes with the direction of the photon while θ_1 and φ are the spherical polar angles of the direction in which the negative pion with momentum $\vec{q}_1$ is emitted.

On performing the integration over the angles of both π^+ and π^- and also over the energy ω_1, we obtain the total crossection as

$$\sigma_{tot}^{(+-)}(k) = \int (F_D + F_I)\, d\omega_1 \qquad (22)$$

where

$$F_D = \frac{e^2 f^4 q_1 q_2}{k} \left\{ \frac{2(\mu_p - \mu_n)^2 k^2 q_1^2 q_2^2}{9 m^2 f^4} \right.$$

$$\times \left[\frac{\sin^2 \delta(q_1)}{\omega_1^2 q_1^6} + \frac{\sin^2 \delta(q_2)}{\omega_2^2 q_2^6} \right]$$

$$+ 16 \left(\frac{q_2^2}{\omega_2^2} + \frac{q_1^2}{\omega_1^2} \right) - 16 q_1^2 q_2^2 \left[\frac{J(q_1) - I(q_1)}{\omega_2^2} \right.$$

$$\left. \left. + \frac{J(q_2) - I(q_2)}{\omega_1^2} \right] \right\} \qquad (23)$$

$$F_I = \frac{e^2 f^4 q_1 q_2}{k} \left\{ \frac{2}{8\pi} \frac{(\mu_p - \mu_n)^2 \sin\delta(q_1) \sin\delta(q_2) \cos[\delta(q_1) - \delta(q_2)]}{\omega_1 \omega_2 m^2 f^4 q_1 q_2} \right.$$

$$-\frac{8}{\omega_1\omega_2}\,k^2 q_1^2 q_2^2\, I(q_1)\, I(q_2)$$

$$-\frac{8(\mu_p-\mu_n)k^2}{9\,\omega_1\omega_2 mf^2}\left(\frac{q_1^2}{q_2} I(q_1)\sin 2\delta(q_2) - \frac{q_2^2}{q_1} I(q_2)\sin 2\delta(q_1)\right.$$

$$+\frac{2}{3}\frac{(\mu_p-\mu_n)k^2}{mf^2}\left(\frac{q_1^2 \sin 2\delta(q_2)}{q_2\omega_1^2} I(q_2) + \frac{q_2^2 \sin 2\delta(q_1)}{q_1\omega_2^2} I(q_1)\right)$$

$$\left. -16\, q_1^2 q_2^2\left(\frac{I(q_1)}{\omega_2^2} + \frac{I(q_2)}{\omega_1^2}\right)\right\} \qquad (24)$$

Here

$$I(x) = \frac{k^2+x^2+1}{2k^2x^2} + \frac{(k^2+x^2+1)^2+4k^2x^2}{8k^3x^3} \times \log\frac{k^2+x^2-2kx+1}{k^2+x^2+2kx+1} \qquad (25)$$

$$J(x) = -\frac{1}{k^2x^2} - \frac{k^2+x^2+1}{4k^3x^3}\log\frac{k^2+x^2+1-2kx}{k^2+x^2+1+2kx} \qquad (26)$$

The integration over ω_1 was done numerically using desk calculating machines for three values of the incident photon energy k. The results are summarised in Table I.

TABLE I (a)

Total cross-sections σ_{tot}^{+-} for various values of the incident photon energy K

K in Mev.	σ_D in μb	σ_I in μb	σ_{tot}^{+-} in μb	σ_{tot}^{+-} (Experimental) of the Cocconi group[1]
420	7	7	14	≈ 10
490	37	15	52	≈ 50
560	51	26	77	≈ 77

1) B.M.Chasan et al, loc. cit;
J.M.Sellen et al, loc. cit.

TABLE I (b)

Differential cross-sections in μb/sr for a π^- kinetic energy of 72 Mev.

K. in Mev.	π^+ Scattering angle			
	0°	30°	60°	90°
420	0.07	0.088	ø.201	0.204
500	0.209	0.367	0.591	0.635
550	0.38	0.502	0.726	0.801
600	0.451	0.481	0.631	0.519
700	0.542	0.302	0.355	0.267

TABLE I (c)

Differential cross-sections in μb/sr for a π^- kinetic energy of 120 Mev.

K. in Mev.	π^+ Scattering angle			
	0°	30°	60°	90°
420	0.133	0.13	0.133	0.133
500	0.056	0.178	0.49	0.635
550	0.204	0.251	0.85	0.898
600	0.689	0.8	0.90	1.17
700	0.698	0.63	0.61	0.543

σ_D and σ_I represent the contributions to the total cross-section σ_{tot}^{+-} from F_D and F_I respectively, on integration over ω_1. The results are plotted along with the experimental data (in the form of a histogram) of Chasen et al.[1] (Fig. 2). As can be seen, the experimental cross section rises rapidly between 400 and 500 Mev and reaches a maximum value of about 80 microbarns at about 600 Mev. Our theoretical results compare favourably with these data. However the agreement ceases to be good at higher energies indicating that the neglect of the pion-pion interaction may not be justified at higher energies.

4. The processes (a) $\gamma + p \longrightarrow n + \pi^+ + \pi^\circ$ and (b) $\gamma + p \longrightarrow p + \pi^\circ + \pi^\circ$

Using the same procedure as in the last section we can write down the matrix elements for these processes in the one-nucleon approximation. The main difference is that for these cases both terms of (13) contribute. The complete matrix element for process (a) is given by

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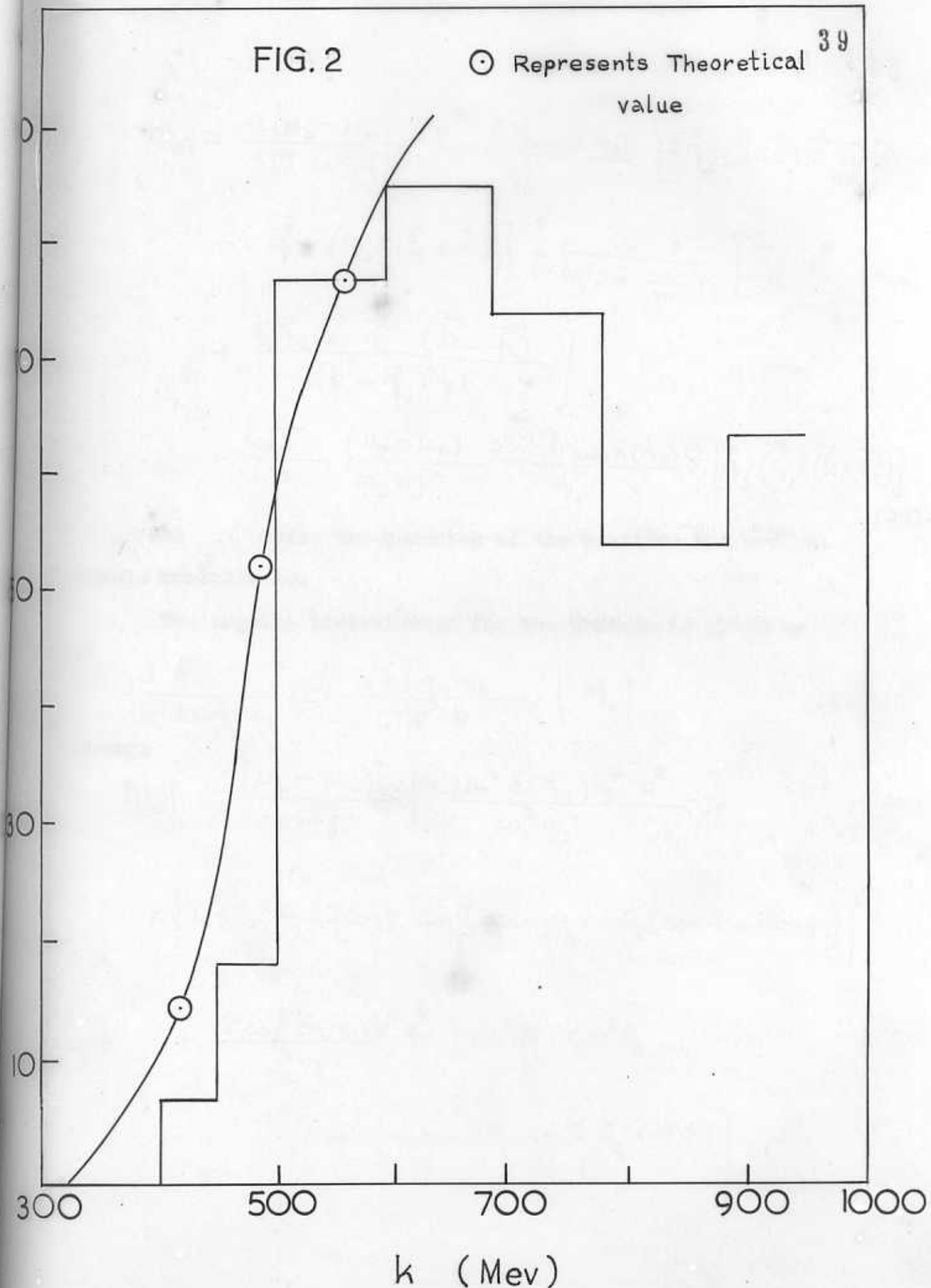
1) B.M.Chasen et al , Phys. Rev., 119 , 811 (1960). The total cross sections are presented in Table I(a). Tables I(b) and I(c) give the differential cross sections for the π^+ for fixed energies of the π^-. These differential cross-sections were obtained on retaining only the third and fourth terms on the right hand side of equation (17) in which case the π^- is emitted isotropically.

FIG. 2

⊙ Represents Theoretical value



$$M_c = \frac{-i(\mu_p - \mu_n)}{3\sqrt{2}\,\omega_1 m f^2}\,\frac{e^{i\delta(q_2)}\sin\delta(q_2)}{q_2^3}\left\{2\vec{q}_2\cdot(\vec{k}\times\vec{\epsilon})\,\vec{\sigma}\cdot\vec{q}_1 - i\,\vec{q}_1\cdot(q_2\times(\vec{k}\times\vec{\epsilon}))\right\} - \frac{4}{\omega_1\left(1+\frac{\omega_2}{m}\right)}\left\{\vec{q}_1\cdot\vec{\epsilon} + \frac{2\vec{q}_2\cdot\vec{\epsilon}\;\vec{q}_1\cdot(\vec{k}-\vec{q}_2)}{(\vec{k}-\vec{q}_2)^2+1}\right\}$$

$$-\frac{i\sqrt{2}}{3}\,\frac{(\mu_p-\mu_n)}{\omega_2 m f^2}\,\frac{e^{i\delta(q_1)}\sin\delta(q_2)}{q_1^3}\,\vec{\sigma}\cdot\left[\vec{q}_2\times(\vec{q}_1\times(\vec{k}\times\vec{\epsilon}))\right] \qquad (27)$$

q_2 and q_1 being the momentum of the positive and neutral pions respectively.

The angular distribution for the process is given by

$$\frac{d^3\sigma^{+o}}{d\omega_1\,d\Omega_+\,d\Omega_o} = \frac{e^2 f^4\, q_1 q_2}{4\pi^2 k}\cdot\left|M_c\right|^2 \qquad (28)$$

where

$$\left|M_c\right|^2 = \frac{(\mu_p-\mu_n)^2}{36 m^2 f^4}\left[\frac{\sin^2\delta(q_2)\,q_1^2\,k^2}{\omega_1^2 q_2^4}\times\right.$$

$$\times\left\{4\sin^2\theta_2 + \left[\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2(\sin\varphi+\cos\varphi)\right]\right\}$$

$$+\frac{8\sin^2\delta(q_1)\,k^2 q_2^2}{\omega_2^2 q_1^4}\left(\cos^2\theta_1\cos^2\theta_2\right.$$

$$\left.\left.+\frac{1}{4}\sin 2\theta_1\sin 2\theta_2\cos\varphi\right)\right]$$

$$+\frac{16}{\omega_1^2\left(1+\frac{\omega_2}{m}\right)^2}\left\{q_1^2\sin^2\theta_1+\frac{4q_2^2\sin^2\theta_2}{\left[(\vec{k}-\vec{q}_2)^2+1\right]^2}\times\right.$$

$$\left.\times\left(kq_1\cos\theta_1-\vec{q}_1\cdot\vec{q}_2\right)^2\right\}$$

$$-\frac{\sqrt{2}}{3}\frac{(\mu_p-\mu_n)}{\omega_1^2\left(1+\frac{\omega_2}{m}\right)}\frac{\sin 2\delta(q_2)\,q_1}{mf^2q_2^2}\left[\left(q_1\sin\theta_1\cos\theta_2\right.\right.$$

$$-\cos\theta_1\sin\theta_2\cos\varphi)+q_2\sin\theta_2(\sin\theta_1\cos\theta_2\cos\varphi$$

$$\left.-\cos\theta_1\sin\theta_2)\times\frac{(kq_1\cos\theta_1-\vec{q}_1\cdot\vec{q}_2)}{(\vec{k}-\vec{q}_2)^2+1}\right]$$

$$+\frac{2k^2(\mu_p-\mu_n)^2}{q\omega_1\omega_2m^2f^4q_1q_2}\cos\left[\delta(q_1)-\delta(q_2)\right]\sin\delta(q_2)\sin\delta(q_2)$$

$$\times\sin\theta_2\left[\sin\theta_2-\sin\theta_1\cos\varphi(\cos\theta_1\cos\theta_2\right.$$

$$\left.+\sin\theta_1\sin\theta_2\cos\varphi)\right] \qquad (29)$$

The matrix element for process (b) is

$$M_d=\frac{i(\mu_p-\mu_n)}{3\omega_2mf^2}\frac{e^{i\delta(q_1)}\sin\delta(q_2)}{q_1^3}$$

$$\times\left\{\vec{\sigma}\cdot\left[\vec{q}_2\times(\vec{q}_1\times(\vec{k}\times\vec{\epsilon})\right]\right\}$$

$$-\frac{i(\mu_p-\mu_n)}{3\omega_1mf^2}\frac{e^{i\delta(q_2)}\sin\delta(q_2)}{q_2^3}$$

$$\times\left\{\vec{\sigma}\cdot\left[\vec{q}_1\times(\vec{q}_2\times(\vec{k}\times\vec{\epsilon}))\right]\right\} \qquad (30)$$

The differential cross-section for the process is given by

$$\frac{d^3\sigma^{oo}}{d\omega_1 d\Omega_1 d\Omega_2} = \frac{e^2 f^4 q_1 q_2}{4\pi^2 k} \left| M_d \right|^2 \qquad (31)$$

where

$$\left|M_d\right|^2 = \frac{(\mu_p - \mu_n)^2 k^2}{9 m^2 f^4} \left[\frac{\sin^2\delta(q_1) q_2^2}{\omega_2^2 q_1^4} \times \right.$$

$$\times \left\{ \sin^2\theta_2 + \cos^2\theta_1 \cos^2\theta_2 + \frac{\sin 2\theta_1 \sin 2\theta_2 \cos\varphi}{4} \right\}$$

$$+ \frac{\sin^2\delta(q_2) q_1^2}{\omega_1^2 q_2^4} \left\{ \sin^2\theta_1 + \cos^2\theta_1 \cos^2\theta_2 \right.$$

$$\left. + \frac{\sin 2\theta_1 \sin 2\theta_2 \cos\varphi}{4} \right\}$$

$$- \frac{2}{q_1 q_2} \cos\left[\delta(q_1) - \delta(q_2)\right] \sin\delta(q_1) \sin\delta(q_2)$$

$$\times \left\{ \sin^2\theta_1 \sin^2\theta_2 \cos^2\varphi + 2\cos^2\theta_1 \cos^2\theta_2 \right.$$

$$\left. \left. + \frac{3}{4} \sin 2\theta_1 \sin 2\theta_2 \cos\varphi \right\} \right] \qquad (32)$$

There is no experimental data for these reactions at the energies we are considering. So we content ourselves with giving in Table II the angular distribution of the π^+ if it is emitted with an energy required for forming a resonant state with the proton. A point to be noted regarding the matrix element of process (b) is that it is obtained already symmetrized with respect to the exchange of the two identical neutral pions. This is apparent from equation (13) which gives the matrix element before the static approximation is made. Remembering the conservation laws for energy and momentum, the expression (13) is seen to be symmetric under the exchange

$$\omega_1 \longleftrightarrow \omega_2 \quad \text{and} \quad \vec{q}_1 \longleftrightarrow \vec{q}_2$$

as is required for bosons. This feature will persist even if we restrict the intermediate states arbitrarily. The fact that the matrix elements obtained by Low's method are already symmetrized according to the nature of the particles involved is further substantiated in a later chapter (see Chapter VII).

5. Angular Momentum and Isotopic spin analyses of the matrix element for double pion photoproduction[1)]

As a preliminary to solving the integral equation (15) for the matrix element for the double pion photoproduction, we have to expand the production matrix in terms of submatrices corresponding to the relevant angular momentumand isospin states.

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1) The angular momentum analysis for the problem has been done by Peierls, (R.F.Peierls, Phys. Rev. 111, 1373 (1958) and the isotopic spin analysis by Carruthers (P.Carruthers, Phys. Rev., 122, 1949 (1961), but both these were done without reference to any dynamical theory.

Table II.

$\frac{d^2\sigma^+}{d\Omega_2 d\omega_1}$ in μb / sterad / Mev

k in Mev \ θ_2	0°	30°	60°	90°	120°	150°	180°
490	0.153	0.418	0.762	0.553	0.516	0.264	0.153
560	0.774	1.933	2.894	2.737	1.956	1.126	0.774

Let us consider the angular momentum decomposition first. If J and M denote respectively the total angular momentum and its Z-component for the initial (i) or final (f) state, we can write the T -matrix as

$$\langle q_1 q_2 ; \mu' | T | k\epsilon ; \mu \rangle = \sum_{J'M'\pi'\nu'} \sum_{JM\pi\nu} a^{fi}_{J'M'\pi'\nu', JM\pi\nu}$$

$$\times | J'M'\pi'\nu', f \rangle \langle JM\pi\nu, i | \qquad (33)$$

Here μ and μ' denote the spins of the initial and final nucleons and π the parity of the state. a^{fi} are functions of the energy of the system and of its isotopic spin. ν represents collectively the vector addition of the orbital angular momentum and spin of the photon and the orbital angular momentum itself. ~~its z~~ -component. The prime indicates the corresponding quantities in the final state. Since there are three particles in the final state the angular momentum addition to arrive at the total angular momentum J' can be done in two ways, (i) by adding the orbital angular momentum of one of the pions (l_1 or l_2) with the spin of the nucleon and adding the orbital angular momentum of the other pion to the resultant or (ii) by adding the orbital angular momenta of the two pions first and then adding the spin of the nucleon. The coupling scheme is to be dictated by the expected dynamical behaviour of the system. In view of the well-known resonance in the $(p\pi^+)$ system, the natural choice is to add the system relative orbital angular momentum l_2 (z-component m_2) of the π^+ to the spin μ' of the proton and add the relative

orbital angular momentum l_1 (z-component m_1) of the π^- to ~~ke~~ ~~xe~~ the resultant L (z -component m)

In view of the invariance of T under rotations and reflections, the matrix element of T is diagnol in J , M and π and does not depend on M . Therefore

$$\langle q_1 q_2 ; \mu' | T | k\epsilon ; \mu \rangle = \sum_{J\pi\nu'\nu} a^{fi}_{J\pi\nu'\nu} \times \sum_M | J M \pi \nu ; f \rangle \langle J M \pi \nu ; i | \qquad (34)$$

The quantities $\sum_M | J M \pi \nu' ; f \rangle \langle J M \pi \nu ; i |$ which are functions of the angular variables and are matrices with respect to the spin variables have been called angular operators or polynomials of the reaction by Ritus[1]. They correspond to transitions with definite total angular momentum and parity and of other quantum numbers inthe initial and final states and they completely determine the angular distributions and polarizations of particles in these transitions.

Returning to our matrix element, we see that the initial state consists of the nucleon represented by its spin wave function and the photon , the angular part of which ~~xi~~ is given by a spherical vector $D_{j\, M-\mu}(\hat{k})$ where $\hat{k}$ is a unit vector in the direction $\vec{k}$. The latter is a linear combination of three spherical vectors. $D_{j\, l\, m}(\hat{k})$ (where $l = j , j \pm 1$) defined by

1) V.I. Ritus, Soviet Physics, JETP , 5, 1249 (1957); Also S. Ciulli and J. Fisher, Nuovo Cimento, 12, 264 (1959).

$$\vec{D}_{jlm}(\hat{k}) = \sum_{m''} C(l\,|\,j;m'm'')\,Y_{lm'}(\hat{k})\,\chi_{sm''} \qquad (35)$$

where $Y_{lm'}$ and the $\chi_{sm''}$ are the usual spherical harmonics and the spin wave function of the photon respectively. The spherical vector $\vec{D}_{jM-\mu}(\hat{k})$ must satisfy the transversality condition

$$\vec{D}_{jM-\mu}(\hat{k}) \cdot \vec{k} = 0 \qquad (36)$$

Therefore in the expansion of $\vec{D}_{jM-\mu}(\hat{k})$ as a sum of three spherical vectors only two of the coefficients of expansion are linearly independent and thus there are only two states of the photon with given quantum numbers j and $M-\mu$ which we can denote by $\lambda = 0$, 1 . Thus among the three mutually perpendicular spherical vectors, $\vec{D}_{jM-\mu}(\lambda = 0, \pm 1)(\hat{k})$ we can choose $\vec{D}^{(-)}_{jM-\mu}(\hat{k})$ to be longitudinal (i.e. along the direction of $\vec{k}$) and $\vec{D}^{(0)}_{jM-\mu}(\hat{k})$ and $\vec{D}^{(1)}_{jM-\mu}(\hat{k})$ as the transverse vectors with

$$\vec{D}^{(0)}_{jM-\mu}(\hat{k}) = \vec{D}_{jjM-\mu}(\hat{k}) \; ; \; \vec{D}^{(1)}_{jM-\mu}(\hat{k}) = i\left[\left(\vec{D}_{jjM-\mu}(\hat{k})\right) \times \hat{k}\right] \qquad (37)$$

If $\vec{D}^{(1)}_{jM-\mu}(\hat{k})$ is obviously perpendicular to $D^{(0)}_{jM-\mu}(\hat{k})$ and $\hat{k}$ the states with $\lambda = 1$ and 0 are called respectively the electric and magnetic states since the emission of a photon in the corresponding state is determined by the electric or magnetic moment of the system. The parities of these states are $(-1)^j$ and $(-1)^{j+1}$ respectively as can be seen from the equation (35). In terms of the spherical tensors $\vec{D}_{jM-\mu}(\hat{k})$, the function $D^{(1)}_{jM-\mu}(\hat{k})$ can be written as

$$D^{(1)}_{jM-\mu}(\hat{k}) = \sqrt{\frac{j}{2j+1}}\; D_{j,\,j+1,\,M-\mu}(\hat{k}) + \sqrt{\frac{j+1}{2j+1}}\; D_{j,\,j-1,\,M-\mu}(\hat{k}) \qquad (38)$$

Finally, it can be shown that

$$D^{(0)}_{jM-\mu}(\hat{k}) = -\frac{i}{\sqrt{j(j+1)}}\left[\hat{k} \times \frac{\partial}{\partial \hat{k}}\right] Y_{jM-\mu}(\hat{k}) \qquad (39)$$

This can be seen as follows. If $\vec{L}$ is the angular momentum operator, $\vec{L} = -i\left[\hat{k} \times \frac{\partial}{\partial \hat{k}}\right]$. Now using the angular momentum addition rules

$$\vec{L}\, Y_{jM-\mu} = \sum_{\nu} (-1)^{\nu} L_{\nu} Y_{jM-\mu}\, \vec{e}_{-\nu}$$

$$= \sum_{\nu} (-1)^{\mu} C(j|j;M-\mu,\nu)\left[j(j+1)\right]^{\frac{1}{2}} Y_{j,M-\mu+\nu}\, \vec{e}_{-\nu}$$

$$= \sum_{\nu} (-1)^{\nu} C(j|j,-(M-\mu+\nu),\nu)\,(-1)^{1+\nu} \times \left[j(j+1)\right]^{\frac{1}{2}} Y_{j,M-\mu+\nu}\, \vec{e}_{-\nu}$$

$$= \sum_{\nu} C(j|j;M-\mu+\nu,-\nu)\, Y_{jM-\mu+\nu}\, \vec{e}_{-\nu}\left[j(j+1)\right]^{\frac{1}{2}}$$

$$= \left[j(j+1)\right]^{\frac{1}{2}} \vec{D}_{jjM-\mu} \qquad (40)$$

In the above $\vec{e}$ represents the unit vector in the spherical basis, i.e.

$$\vec{e}_0 = \vec{e}_z \; ; \; \vec{e}_1 = -\frac{1}{\sqrt{2}}(\vec{e}_x + i\vec{e}_y) ;$$

$$\vec{e}_{-1} = \frac{1}{\sqrt{2}}(\vec{e}_x - i\vec{e}_y) \qquad (41)$$

The last step of (40) follows from the definition of $\vec{D}_{jjM-\mu}$, equation (35) . In the intermediate steps of (40), the properties of the Clebsch-Gordan coefficients have been made use of. It follows from (39) that

$$D^{(1)}_{jM-\mu}(\hat{k}) = \frac{1}{\sqrt{j(j+1)}}\left(\frac{\partial}{\partial \hat{k}} - \hat{k}\left(\hat{k}\cdot\frac{\partial}{\partial \hat{k}}\right)\right) Y_{jM-\mu}(\hat{k}) \qquad (42)$$

It also follows that

$$\left(\vec{\epsilon}\cdot\vec{D}^{(1)}_{jM-\mu}(\hat{k})\right) = \frac{1}{\sqrt{j(j+1)}}\left(\vec{\epsilon}\cdot\frac{\partial}{\partial \hat{k}}\right) Y_{jM-\mu}(\hat{k}) \qquad (43a)$$

$$\left(\vec{\epsilon}\cdot\vec{D}^{(0)}_{jM-\mu}(\hat{k})\right) = \frac{i}{\sqrt{j(j+1)}}\left[(\hat{k}\times\vec{\epsilon})\cdot\frac{\partial}{\partial \hat{k}}\right] Y_{jM-\mu}(\hat{k}) \qquad (43b)$$

We see from (43) that the effect of changing over from a particle with spin zero represented by an angular momentum function $Y_{jM-\mu}(\hat{k})$ to one with spin 1 (as in the case of the photon) is to operate on the wave function by the operators

$$\frac{1}{\sqrt{j(j+1)}}\left(\vec{\epsilon}\cdot\frac{\partial}{\partial \hat{k}}\right) \quad \text{and} \quad \frac{i}{\sqrt{j(j+1)}}\left[(\vec{k}\times\vec{\epsilon})\cdot\frac{\partial}{\partial \hat{k}}\right]$$

for the electric and magnetic states respectively. We shall denote these modified wave functions by $Y^{(e \text{ or } m)}_{jM-\mu}(\hat{k})$

The initial state of our reaction can thus be written as

$$|k\epsilon;\mu\rangle = \sum_{Jj M} C(j\tfrac{1}{2}J; M-\mu,\mu) Y^{*(e\text{ or }m)}_{jM-\mu}(\hat{k})|JjM\rangle \tag{44}$$

The final state is given by

$$|\vec{q}_1,\vec{q}_2;\mu'\rangle = \sum_{JL\ell_1\ell_2 m_1 m_2} C(\ell_1 L J; m_1 m)$$
$$\times C(\ell_2 \tfrac{1}{2} L; m_2 \mu') Y^{*}_{\ell_2 m_2}(\hat{q}_2)$$
$$\times Y^{*}_{\ell_1 m_1}(\hat{q}_1)\,|J L \ell_1 \ell_2 M\rangle \tag{45}$$

Thus the T-matrix for our process is given by

$$\langle \vec{q}_1,\vec{q}_2;\mu'|T|k\epsilon;\mu\rangle = \sum_{\{J\}m} C(\ell_1 L J; m_1 m)$$
$$\times C(\ell_2 \tfrac{1}{2} L m_2 \mu') C(j\tfrac{1}{2}J; M-\mu,\mu)$$
$$\times Y_{\ell_2 m_2}(\hat{q}_2)\, Y_{\ell_1 m_1}(\hat{q}_1)\, Y^{*(e\text{ or }m)}_{jM-\mu}(\hat{k})\, T^{(e\text{ or }m)}_{\{J\}}(\omega_1,\omega_2,k)$$
$$= \sum_{\{J\}} b^{\mu\mu'(e\text{ or }m)}_{\{J\}}(\hat{q}_1,\hat{q}_2,\hat{k})\; T^{(e\text{ or }m)}_{\{J\}}(\omega_1,\omega_2,k) \tag{46}$$

where $\{J\}$ stands for the set of quantum numbers $(JL\ell_1\ell_2 j)$ and the $b_{\{J\}}$ are ~~positiv~~ precisely the matrix elements between nucleon spin states of the angular operators mentioned above.

The isotopic spin analysis of the matrix element for the double pion photoproduction can be performed in a similar manner. Since we are dealing with an electromagnetic interaction isotopic spin is good only in parts, the allowed change in the total isotopic spin being given by $\Delta I = 0, \pm 1$. Thus we have to specify the I_2 values of the states. In the final state we again use the combination $(\pi^+ p) - \pi^-$ in the addition of the isospic spins. The result is

$$T^{(e\,or\,m)}_{\{J\}}(\omega_1,\omega_2,k) = \sum C(t|T;\beta+\gamma',\alpha)\, C(1\tfrac{1}{2}t;\beta\gamma)\, \delta_{\alpha+\beta+\gamma',T_3} \times T^{(e\,or\,m)}_{\{J\}(Tt)}(\omega_1,\omega_2,k)$$

$$= \sum_{Tt} a_{(Tt)}\, T^{(e\,or\,m)}_{\{J\}(Tt)}(\omega_1,\omega_2,k) \qquad (47)$$

where T and t are the ~~initial~~ final and intermediate isotopic spin The coefficients a_{Tt} for various possible photoproduced pion pairs from an initial nucleon (proton or neutron) are given in Tables III and IV.

Combining the angular momentum and isotopic spin decompositions, we can finally write our matrix element as

$$\langle \vec{q}_1, \vec{q}_2; \mu' | T | \vec{k}\vec{\epsilon}; \mu \rangle = \sum_{\{J\}m} \sum_{Tt} b^{\mu\,\mu'\,(e\,or\,m)}_{\{J\}}(\hat{q}_1,\hat{q}_2,\hat{k}) \times a_{Tt}\, T^{(e\,or\,m)}_{\{J\}(Tt)}(\omega_1,\omega_2,k) \qquad (48)$$

We shall also be interested in the angular momentum and

Table III

Initial state = (γ, p)

$a_{T t}$	$p + \pi^+ + \pi^-$	$p + \pi^\circ + \pi^\circ$	$n + \pi^+ + \pi^\circ$
$a_{3/2,\,3/2}$	$\sqrt{2/5}$	$\sqrt{2/45}$	$\sqrt{1/45}$
$a_{3/2,\,1/2}$	0	$-\sqrt{2/3}$	$2/3$
$a_{5/2,\,3/2}$	$\sqrt{1/10}$	$\sqrt{2/5}$	$\sqrt{1/5}$
$a_{1/2,\,1/2}$	0	$-(1/3)$	$\sqrt{2/3}$
$a_{1/2,\,3/2}$	$\sqrt{1/2}$	$-\sqrt{2/3}$	$-(1/3)$

Table IV

Initial state $= (\gamma, n)$

$a_{\pi t}$	$n+\pi^+_+\pi^-$	$n+\pi^\circ_+\pi^\circ$	$p+\pi^-_+\pi^\circ$
$a_{3/2,\,3/2}$	$\frac{2\sqrt{2}}{3\sqrt{5}}$	$-\sqrt{2/45}$	$-\sqrt{1/45}$
$a_{3/2,\,1/2}$	$\sqrt{2/3}$	$\sqrt{2/3}$	$-(2/3)$
$a_{5/2,\,3/2}$	$\sqrt{1/15}$	$\sqrt{2/5}$	$\sqrt{1/5}$
$a_{1/2,\,1/2}$	$2/3$	$-(1/3)$	$\sqrt{2/3}$
$a_{1/2,\,3/2}$	$\frac{1}{3\sqrt{2}}$	$-\sqrt{2/3}$	$-(1/3)$

isotopic spin decompositions of the pion-nucleon scattering and single pion photoproduction matrix elements. For the former we have

$$\langle l_2 \mu' | R | l_n \mu'' \rangle = \sum C(l_2 \tfrac{1}{2} J'; \mu' - \mu'', \mu') \times C(l_n \tfrac{1}{2} J', M' - \mu'', \mu'') Y_{l_2 m_2}(\hat{q}_2) \times Y^*_{l_n M' - \mu''}(\hat{n}) \; R_{J' l_2 T'}(\omega_2; \omega_n) \times \delta_{l_2 l_n} \; C(1 \tfrac{1}{2} T'; \alpha' T') \times C(1 \tfrac{1}{2} T'; \alpha T') \qquad (49)$$

The Kronecker delta $\delta_{l_2 l_n}$ represents parity conservation in the process. The b labels l_2 and l_n for the angular momentum will become clear in the next section.

The angular momentum analysis of the single photo-production of the π^- which we shall be requiring later gives

$$\langle \vec{q}_1, \mu'' | T | k\epsilon; \mu \rangle = \sum C(l_1 \tfrac{1}{2} J'', M'' - \mu'', \mu'') \times C(l \tfrac{1}{2} J''; M'' - \mu, \mu) Y_{l_1 m_1}(\hat{q}_1) \times Y^{*(e \text{ or } m)}_{j, M'' - \mu}(\hat{k}) \; T^{(e \text{ or } m)}_{\{J\}}(k, \omega_1) \qquad (50)$$

$l_1 = j \pm 1$ for the electric transitions and

$l_1 = j$ for the magnetic transitions.

6. Solution of the integral equation for the process and the rescattering correction

We are now in a position to introduce the above decomposition in angular momentum and isotopic spin into the integral equation for the matrix element for the $(\pi^+\pi^-)$ pair production equation (15) . The numerator of the ~~second~~ third term on the right hand side of (15) represents the product of matrix elements for $\pi^- - p$ scattering and pion pair production by a photon. The denominator of this term in the static approximation is $\omega' - \omega_1$. Similarly the numerator of the ~~third~~ fourth term is the product of $\pi^+ p$ scattering and double pion photoproduction matrix elements and the denominator is $\omega' - \omega_2$ in the static approximation. Now we introduce the expansion (48), (49) and (50) into equation (15). Using the orthogonality of the spherical harmonics the numerator of the second term on the right hand side of (15) gives on integrating over the solid angle $d\Omega_n$ corresponding to the direction of the pion ω_n ,

$$\sum C(l_n L J; m_n m)\, C(l_2 \tfrac{1}{2} L; m_2 \mu'')$$

$$\times C(j \tfrac{1}{2} J; M-\mu, \mu)\, C(l_1 \tfrac{1}{2} J; M-\mu', \mu')$$

$$\times C\left(\ell_n \tfrac{1}{2} J;\ M-\mu'', \mu''\right)\ a'\ Y_{\ell_2 m_2}(\hat{q}_2)$$

$$\times Y_{\ell_1 m_1}(\hat{q}_1)\ \ Y^{*\,(e\,or\,m)}_{j\ M-\mu}(k)$$

$$\times R^{+}(\omega_1, \omega_n)\ T^{(e\,or\,m)}_{\{J\}Tt}(\omega_n, \omega_2, k) \qquad (51)$$

Because of parity conservation in the scattering process, $\ell_n = \ell_1$

. Also the orthogonality property of the Clebsch-Gordon coefficients gives

$$\sum C\left(\ell_1 \tfrac{1}{2} J;\ M-\mu', \mu'\right) C\left(\ell_1 \tfrac{1}{2} J;\ M-\mu'', \mu''\right) = \delta_{\mu'\mu''} \qquad (52)$$

Thus the result of the angle integration is to reduce the co-efficient of $R^{+}(\omega_1, \omega_n) T^{(e\,or\,m)}_{\{J\}Tt}(\omega_n, \omega_2, k)$ in the expression (51) to the same product of Clebsch-Gordon coefficients, spherical harmonics and spherical vectors as is found in $b^{(e\,or\,m)}_{\{J\}Tt}$. In the above a' represents the product of the Clebsch-Gordans arising from the isotopic spin channels.

Considering the second term also in a similar way we finally arrive at the integral equation for the matrix element of the process in a definite angular momentum and isotopic spin channel

$$T_\alpha^{(e\,or\,m)}(\omega_1,\omega_2,k) = \Phi_\alpha^{(e\,or\,m)}(\omega_1,\omega_2,k) - \int_0^\infty \frac{n^2 dn}{(2\pi)^3}$$

$$\times \left[\frac{R^+_{l_1 l_1 t}(\omega_1,\omega_n) T_\alpha^{(e\,or\,m)}(\omega_n,\omega_2,k)}{\omega_1 - \omega_n - i\epsilon} \right.$$

$$\left. + \frac{R^+_{l_2 l_2 t}(\omega_2,\omega_n) T_\alpha^{(e\,or\,m)}(\omega_n,\omega_1,k)}{\omega_n - \omega_2 - i\epsilon} \right] \tag{53}$$

where n is the magnitude of the momentum of the intermediate pion and α is an abbreviation for $\{J\}Tt$ $\Phi_\alpha^{(e\,or\,m)}$ includes in addition to contributions from all genuine one-nucleon intermediate states also the effect of the pion-pion interaction. But since the considerations presented in the next chapter seem to indicate that the pion-pion interaction may not be important for the energies of the photon we are considering, we shall omit it and take Φ to represent only the one nucleon terms used in section 3. Finally writing

$$R^+_{Jt}(\omega,\omega_n) = R^+_{3/2,3/2}(\omega,\omega_2)$$

$$= -8\pi^2 q^2 \frac{h^*(\omega)}{\omega} \tag{54}$$

where

$$h(\omega) = \frac{e^{i\delta(\omega)} \sin\delta(\omega)}{q^3} \tag{55}$$

and

$$T_\alpha^{(e\,or\,m)}(\omega_1,\omega_2,k) = \frac{(4\pi)^3 i q_1 q_2 k\, \Phi_\alpha^{(e\,or\,m)}(\omega_1;\omega_2;k)}{(8\omega_1\omega_2 k)^{1/2}} \qquad (56)$$

$$\Phi_\alpha^{(e\,or\,m)}(\omega_1,\omega_2,k) = \frac{(4\pi)^3 i q_1 q_2 k\, \Phi_\alpha'^{(e\,or\,m)}(\omega_1,\omega_2,k)}{(8\omega_1\omega_2 k)^{1/2}} \qquad (57)$$

we obtain the integral equation

$$\Phi_\alpha^{(e\,or\,m)}(\omega_1\omega_2 k) = \Phi_\alpha'^{(e\,or\,m)}(\omega_1,\omega_2,k) + \frac{1}{\pi}\int_1^\infty d\omega' \frac{h^*(\omega')\,\Phi_\alpha^{(e\,or\,m)}(\omega',\omega_2,k)}{\omega'-\omega_1-i\varepsilon} + \frac{1}{\pi}\int_1^\infty d\omega' \frac{h^*(\omega')\,\Phi_\alpha^{(e\,or\,m)}(\omega',\omega_1,k)}{\omega'-\omega_2-i\varepsilon} \qquad (58)$$

Now we make use of the experimental information that upto 600-700 Mev the π^--meson is emitted essentially in an s- state.[1] This means that we can neglect the second term which involves the scattering of the π^- (predominantly in the p -state) compared to the third term which involves the scattering of the π^+ meson which can be emitted in the p- state. Thus the problem reduces to solving the equation

$$\Phi_\alpha^{(e\,or\,m)}(\omega_1,\omega_2,k) = \Phi_\alpha'^{(e\,or\,m)}(\omega_1,\omega_2,k) + \frac{1}{\pi}\int_1^\infty d\omega' \frac{h^*(\omega')\,\Phi_\alpha^{(e\,or\,m)}(\omega',\omega_1,k)}{\omega'-\omega_2-i\varepsilon} \qquad (59)$$

Such equations can be solved by the Mushelishvili - Omnes[2] method. Denoting the solution corresponding to taking the

1) M. Bloch and M. Sands, loc. cit.

2) R. Omnes, Nuovo Cimento, 8 , 316 (1958)

(3/2 , 3/2) enhanced state in all the amplitudes by $\varphi_{\frac{1}{2}}$, we have

$$\varphi_{\frac{1}{2}}(\omega_1,\omega_2,k) = e^{i\delta(\omega_2)}\cos\delta(\omega_2)\;\varphi_{\frac{1}{2}}^{\prime\,(e\,or\,m)}(\omega_1,\omega_2,k)$$

$$+\exp\left[\rho(\omega_2)+i\delta(\omega_2)\right]$$

$$\times\frac{P}{\pi}\int_1^\infty d\omega'\;\frac{\varphi_{\frac{1}{2}}^{\prime\,(e\,or\,m)}(\omega',\omega_1,k)\sin\delta(\omega')}{\omega'-\omega_2}\exp\left[-\rho(\omega')\right] \tag{60}$$

where

$$\rho(\omega)=\frac{\omega}{\pi}P\int_1^\infty\frac{\delta(\omega')}{\omega'(\omega'-\omega)}\,d\omega' \tag{61}$$

P denotes the principal value of the integral. A subtraction has been performed in the expression (61) . This is in order to take care of the fact that the integral $\int_1^\infty\frac{\delta(\omega')}{\omega'-\omega}\,d\omega'$ does not converge for $\delta(\infty)=\pi$. Thus a subtraction has to be made at $\omega=0$.

We can consider the difference $\varphi_{\frac{1}{2}}-\varphi_{\frac{1}{2}}^{\prime}$ as the effect of rescattering.

$$\varphi_{\frac{1}{2}}^{(e\,or\,m)}(\omega_1,\omega_2,k)-\varphi_{\frac{1}{2}}^{\prime\,(e\,or\,m)}(\omega_1,\omega_2,k)$$

$$=ie^{i\delta(\omega_2)}\cos\delta(\omega)\;\varphi_{\frac{1}{2}}^{\prime\,(e\,or\,m)}(\omega_1,\omega_2,k)$$

$$+\exp\left[i\delta(\omega_2)\right]\frac{P}{\pi}\int_1^\infty d\omega'\;\frac{\varphi_{\frac{1}{2}}^{\prime\,(e\,or\,m)}(\omega',\omega_1,k)}{\omega'-\omega_2}$$

$$\times\sin\delta(\omega')\exp\left[\rho(\omega)-\rho(\omega')\right] \tag{62}$$

Now, following Carruthers[1], we can derive an expression for $\rho(\omega)$. The starting point is the reciprocal function of Chew and Low[2] defined by

$$g(\omega) = \lambda/\omega k(\omega) \quad ; \quad \lambda = \frac{4}{3} f^2$$

where $k(\omega)$ is related to the T-matrix for pion-nucleon scattering in the 33 state by the relation ( ). $g(\omega)$ is connected with the phase shift by the relations

$$\mathrm{Re}\, g = \lambda q^3 \frac{\cot \delta}{\omega} \quad ; \quad \mathrm{Im}\, g = -\lambda \frac{q^3}{\omega} \tag{63}$$

$$\tan \delta = -\frac{\mathrm{Im}\, g}{\mathrm{Re}\, g} \tag{64}$$

Considering $g(\omega)$ as function of a complex variable which in the limit $\omega + i\epsilon \longrightarrow \omega$ becomes the actual matrix element, and using the property $g(\omega) = g^*(\omega')$ we can write (64) in the equivalent form

$$\delta(\omega) = \frac{1}{2} i \log \left[g(\omega) / g^*(\omega) \right] \tag{65}$$

From the defining equation (61), we see that $\rho(\omega)$ is the real part of the integral

$$I = -\frac{\omega}{2\pi i} \int_1^\infty \frac{\log[g(\omega')/g^*(\omega')]\, d\omega'}{\omega'(\omega' - \omega - i\epsilon)} \tag{66}$$

$$= \rho(\omega) + i\delta(\omega)$$

Represented as a contour integral, I can be written as

$$I = -\frac{\omega}{2\pi i} \int_{C_1} \frac{\log g(\omega')}{\omega'(\omega' - \omega)}\, d\omega' \tag{67}$$

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1) P. Carruthers, Annals of Physics, 14, 229 (1961)

2) G.F.Chew and F.E.Low, Phys. Rev., 101, 1570 (1956)

the contour C_1 being indicated in Fig. 3. The contour consists of the paths C_1 and C_3 around the right-and-left hand branch cuts and the two semicircles C_2. Since we are interested only in the right-hand cut, we can omit C_3 from consideration since the 33 phase shift which dominates the integral is small on the negative real axis. The integral over the infinite semicircle also turns out to be small. Thus the integral we are interested namely the one over the contour C_1 is identical with the integral over the whole contour C which is given by

$$\int_C = 2\pi i \left[\frac{\log g(\omega)}{\omega} + \frac{\log g(0)}{-\omega} \right] = \frac{2\pi i}{\omega} \log g(\omega) \tag{68}$$

since $g(0) = 1$ (by definition).

Thus

$$I = -\log g(\omega) = -\log |g(\omega)| + i\delta(\omega) \tag{69}$$

$$\rho(\omega) = \log \left| \frac{\omega h(\omega)}{\lambda} \right| = \log \left[\frac{\omega \sin\delta}{\lambda q^3} \right] \tag{70}$$

$$\exp[\rho + i\delta] = e^{i\delta} \sin\delta \left[\omega/\lambda q^3 \right] \tag{71}$$

Substituting this in the expression for the rescattering correction we have

$$\varphi_\alpha - \varphi'_\alpha = e^{i\delta(\omega_2)} \sin\delta(\omega_2) \left\{ i\varphi'_\alpha + \frac{\omega_2}{q_2^3} \frac{P}{\pi} \int_1^\infty \frac{q^3 \varphi'_\alpha(\omega', \omega_1, k)}{\omega'(\omega' - \omega_2)} \right\} \tag{72}$$

Now the genuine one-nucleon terms in the matrix element for double pion photo production are given by the first two terms on the right hand side of equation (15). We see that

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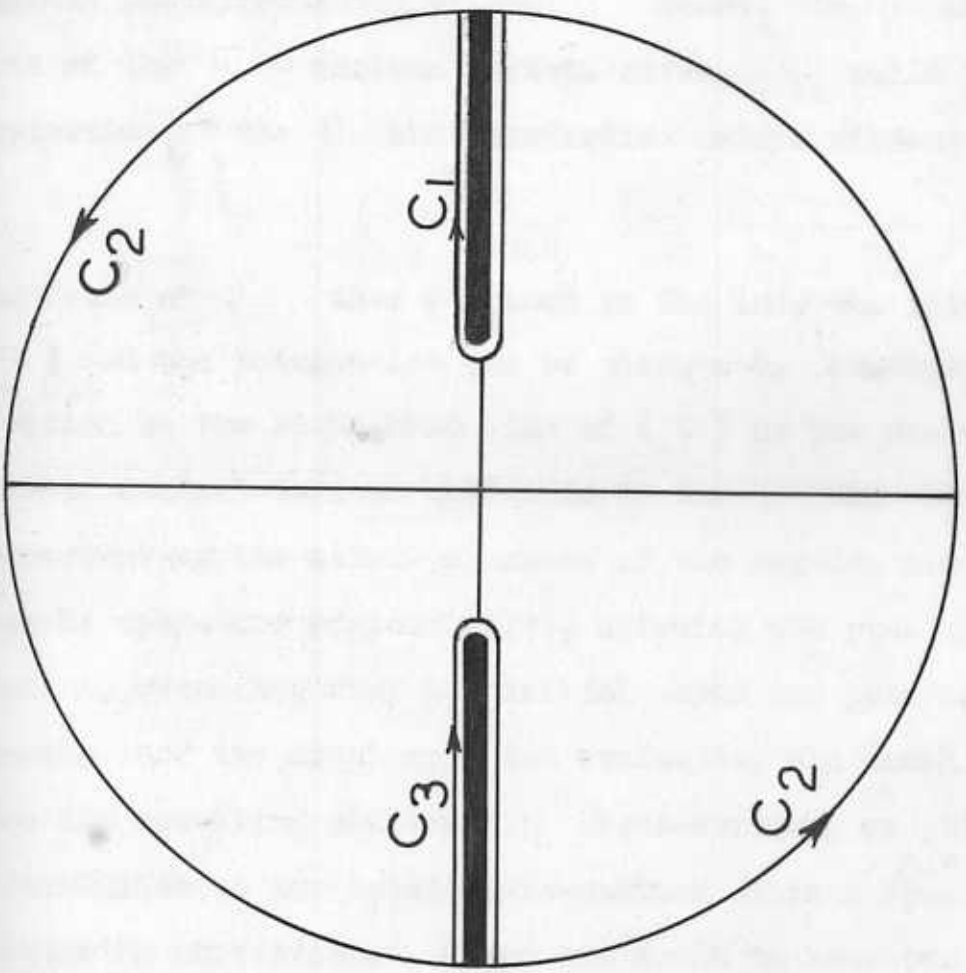

FIG. 3

the requirement that the π^- meson appear only in an s -state with the π^+ meson being emitted in a p -wave state can be met only by retaining the last second of these terms in (15) which involves the product of the π^+ – nucleon vertex and the matrix element for the photoproduction of the π^- meson. The p- wave projection of the π^+ – nucleon vertex, gives $\frac{1}{3} q_{12}$ while the s -wave projection of the π^- photoproduction matrix element gives

$$1 - \frac{1}{2}\left(1 + \frac{1}{2kq_1} \log \frac{\omega_1 - q_1}{\omega_1 + q_1}\right)$$

The value of φ'_2 thus obtained is fed into the integral in (72) and the integration can be performed. Multiplying the expression on the right hand side of (72) by the proper combination of Clebsch-Gordans contained in the b and a coefficients (representing the matrix elements of the angular momentum and isospin operators respectively), squaring the resulting matrix element, averaging over the initial spin and polarisation and summing over the final spin and evaluating the total cross-section from the resulting differential cross-section, we obtain the contribution to the total cross-section arising from the rescattering correction. These are found to have the value 1.817×10^{-31} cm^2 for the photon energy $k = 3.5$ and the value 1.976×10^{-30} cm^2 for $k = $ 4.0 . Remembering that at the latter value for k , the total cross-section for the process is about 70 microbarns, the rescattering correction is seen to be negligible. But this result is dependent on the assumption made on the basis of experimental facts that the π^--meson is produced .. essentially in an s -wave state.

7. The effect of the photon-three pion direct interaction on the photoproduction of pion pairs

It is conceivable that if there is a direct photon-three pion interaction as seems to be required ~~by~~ ~~ax~~ to explain the large isoscalar form factor for the nucleon, then the double pion photoproduction can go through such an interaction as shown in Fig. #. There is no field theoretic basis for such an interaction[1] for, while the four-pion vertex with four internal nucleon lines diverge, so that it becomes necessary to introduce into the Lagrangian a corresponding conter-term which introduces the pion-pion coupling constant, a similar vertex with one photon and three pion external lines converges so hat there is no need for new new counter terms and a new coupling constant. But following Bosco and De Alfaro[2], we could assume a phenomenological interaction Hamiltonian involving three pion fields. The current density operator then is

$$j_{int}(x) = ie\lambda' \varepsilon_{\alpha\beta\gamma\delta} \frac{\partial \varphi_1(x)}{\partial x_\beta} \frac{\partial \varphi_2(x)}{\partial x_\gamma} \frac{\partial \varphi_3(x)}{\partial x_\delta}$$

where $\varphi_i(x)$ are real pion fields and $\varepsilon_{\alpha\beta\gamma\delta}$ is a completely antisymmetric tensor of rank 4 . The contribution from such a term to the double pion photoproduction process can be estimated in the lowest order of the photon-three pion-coupling constant as follows.

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1) L.D. Solovev' , Soviet Physics -JETP 13,418 (1961)
2) B. Bosco and V. De Alfaro, Phys. Rev., 115 , 215 (1959)

Writing

$$\varphi_1 = \frac{\varphi + \varphi^*}{\sqrt{2}} \quad ; \varphi_2 = \frac{i(\varphi^* - \varphi)}{\sqrt{2}}; \quad \varphi_3 = \varphi_3$$

where φ and φ^* for the positive and negative pions respectively represent the creation operators we can obtain the matrix element for double pion photo production by the taking the final pion operators through the same current density operator with the same space time label (for only then will the matrix element be of the lowest order in λ' . It is given by

$$\langle p'; q_1, q_2 | T | k\epsilon; p \rangle$$

$$= \frac{-ie\lambda'}{4(2\omega_1\omega_2 k)^{1/2}} \epsilon_{\alpha\beta\gamma\delta}\left[q_{1\gamma} q_{2\beta} - q_{1\beta} q_{2\gamma}\right]$$

$$\times \frac{(k - q_1 - q_2)_\delta}{(\vec{k} - \vec{q}_1 - \vec{q}_2)^2 + 1} \langle p' | J_0(0) | p \rangle \qquad (74)$$

where the neutral pion-nucleon vertex $\langle p' | J_0(0) | p \rangle$ is given in the static approximation by

$$\langle p' | J_0(0) | p \rangle = -(4\pi)^{1/2} \sigma \cdot \vec{\epsilon}(\vec{q} + \vec{q}_2 - \vec{k})$$

On averaging over the initial spin and polarisations and summing over x the final spin of the nucleon, the differential cross-section is given by

$$\frac{d^3\sigma^{+-}}{d\Omega_+ d\Omega_- d\omega_1} = \frac{e^2\lambda'^2 f^2 q_1 q_2}{128\pi^4 k} \cdot \frac{(\vec{k} - \vec{q}_1 - \vec{q}_2)}{\left[(\vec{k} - \vec{q}_1 - \vec{q}_2)^2 + 1\right]}$$

$$\times\Big[(q_{12}\,\omega_2 - q_{22}\,\omega_1)^2\Big\{(\vec{k}-\vec{q}_1-\vec{q}_2)_x^2+(\vec{k}-\vec{q}_1-\vec{q}_2)_y^2\Big\}$$

$$+\Big\{(q_{1x}\,\omega_2 - q_{2x}\,\omega_1)^2+(q_{1y}\,\omega_2 - q_{2y}\,\omega_1)^2\Big\}$$

$$\times\,(\vec{k}-\vec{q}_1-\vec{q}_2)_z^2\Big] \qquad (76)$$

An approximate calculation of the total cross-section after performing the angle integration in (76) at a value $k=4$ gives a cross-section of $\sim$ 35 λ'^2 microbarns which would give a value of $\lambda' \approx$ 1.4. Bosco and de Alfaro conclude on the basis of the observed mean squared radius of the nucleon that λ' should lie between 5.4 and 130. Thus it appears that the same value of λ' is not able to fit the photoproduction data and the electromagnetic structure of the nucleon.

8. Conclusion

In this chpter we have evaluated the total cross-section for the photoproduction of a charged pair of pions from a nucleon and find that the experimental values can be fitted for energies of the photon upto 600 Mev by assuming a pure Yukawa interaction and retaining all the genuine "one-nucleon" terms in the matrix element. The effect of the pion-pion interaction seems to be negligible at these energies. The fact that this is so whereas the pion-pion interaction is dominant at much lower energies for pion production in pion nucleon collisions (see next chapter)

seems to show that its role is essentially that of an initial state interaction which will, of course, be absent for the photoproduction problem. Our considerations also show that if there is a direct photon-three pion interaction which is not necessitated by field theory, then itis not possible to explain both the electromagnetic form factors of the nucleon and pion pair production cross sections with the same value for the photon-three pion coupling constant, λ' .

CHAPTER III

THE PION-PION INTERACTION AND MULTIPLE PION PRODUCTION IN THE SUB-GEV REGION

1. Introduction

With the advent of the double variable dispersion relations (see Chapter), it has become clear that a knowledge of the pion-pion interaction is a prerequisite for a complete understanding of all reactions involving the pion and the nucleon. If we consider pion-nucleon scattering for instance, one of the crossed channel will be

$$\pi + \pi \longrightarrow N + \bar{N} \qquad (1)$$

In the unphysical region this process is dominated by pion-pion scattering. Similarly for photo pion production from a pion or a nucleon or for nucleon-nucleon scattering the pion-pion interaction plays an important role in the crossed channel (for the first mentioned process in the main channel also) and hence affects the reaction in the main channel also. So a good deal of interest has been evinced in its study. Since pion targets are not available for experiment, the process of pion-pion scattering has to be studied only indirectly. One way suggested for obtaining the cross-sections for the process by Chew and Low[1] is to extrapolate the differential cross-section

1) G.F.Chew and F.E.Low, Phys. Rev., 113, 1640 (1959).

for the process

$$\pi + N \longrightarrow N + \pi + \pi \qquad (2)$$

to a value of the square of the momentum transfer equal to the square of the mass of the pion, there being a pole in the matrix element for process (2) at this value. At the extrapolated point, the residue of the pole will involve the product of the pion-nucleon vertex and the total crossx-section for pion-pion scattering. The analyses of Bonsignori and Selleri[1] and Derado[2] have indicated evidence for the pion-pion interaction. The experiments of Perkins et al[5] on process (2) showed that the relevant cross-sections are too large in comparison with what one should expect on the basis of a pure Yukawa type of pion-nucleon interaction and this has been construed to mean firm evidence for a pion-pion interaction. Further, the work of Frazer and Fulco[6] on the isovector part of the nucleon magnetic moment has indicated that the experimental data for this quantity could be fitted only by assuming a pion-pion scattering resonance in the $I = 1$, $J = 1$ state. Recent experiments[3] have indeed given clear evidence for such a resonance.

On the theoretical side the pion-pion problem has been considered by Chew and Mandelstam[4] using an effective range

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1) F.Bonsignori and F.Selleri : Nuovo Cimento, 15, 465, (1960)
2) I.Derado: Nuovo Cimento, 15, 253 (1960).
3) A. Erwin et al , Phys. Rev. Letters, 6, 628 (1961); E.Pickup et al , Phys. Rev. Letters, 7, 192 (1961)
4) G.F.Chew and S. Mandelstam, Phys. Rev., 119, 467 (1960).
5) W. A. Perkins et al, Phs. Rev. 118, 1364 (1960)
6) W.R. Frazer and J.R. Fulco, Phs. Rev. 117, 1609 (1960)

approach in which they have derived a possible range of values for the pion-pion coupling constant λ .

In the conventional Lagrangian description of elementary particle interactions the pion-pion interaction enters naturally if one considers the fourth order diagram involving ~~fxix~~ four external pion lines. Unlike in the case of fourth order photon-photon scattering where due to gauge invariance, the contribution from the square diagram is convergent, the square diagram yields a divergent result both in scalar and pseudoscalar meson theory. This divergence has to be removed by a suitable renormalization and for this it is necessary to introduce a term of the type

$$4\pi\lambda\,(\vec{\phi}\cdot\vec{\phi})^2 \qquad (3)$$

into the original Lagrangian with a suitably chosen coefficient λ which must contain an infinite part which will compensate for the infinity introduced by the fourth-order pion-pion scattering and may also contain a finite part. Indeed the considerations of Chew and Mandelstam lend support to this view though in their case the coupling constant enters through a subtraction term in the matrix element (the pole term which usually yields the coupling constant being absent in this case as there is no single stable particle apart from the photon which has all the ~~xxxxxxx~~ quantum numbers the same as the two pion system).

In this chapter we have examined the role of the pion-pion s interaction defined through (3) in various pion-production process in the sub-Gev region and made estimates of the contribution of the interaction to these processes.

2. Photoproduction of pions pairs from a nucleon.

(a) We assume to begin with that the reaction proceeds completely through interaction (3) which we can rewrite as

$$4\pi\lambda\left[2\varphi\varphi^{*}+\varphi_3^2\right]^2 \qquad (4)$$

Let us consider the reaction

$$\gamma + p \longrightarrow p + \pi^{+} + \pi^{-} \qquad (5)$$

On "contracting" on the final pair of pions of momenta q_{1} and q_{2}, but taking the operator corresponding to one of them through the current operator arising from the other, we obtain, using the usual procedure of Low, the matrix element for the process as

$$\langle p' q_1 q_2 | S | pk \rangle = -\frac{16\pi\lambda}{(4\omega_1\omega_2)^{\frac{1}{2}}}\,\delta(p+k-p'-q_1-q_2)$$

$$\cdot\left[\langle p' | \left\{ T\left(\varphi(0)\varphi^{*}(0)\right) + T\left(\varphi_3(0)\varphi_3(0)\right\}\right| pk\rangle\right] \qquad (6)$$

Now using translational invariance and the equation of motion, we can write, on introducing a complete set of "incoming" states and retaining only the single nucleon intermediate state,

$$\langle p'|\varphi(0)|N\rangle = - \frac{\langle p'|j_+(0)|N\rangle}{[(p-p_N)^2+1]} \qquad (7)$$

Thus the numerator of each of the terms on the right hand side of (6) will be a product of a pion-nucleon vertex and the single pion photo production matrix element, the whole multiplied by the pion-pion coupling constant λ . From the range of values for λ given by Chew and Mandelstam and also from the considerations of the next sections, the value of λ could be taken as of the order of 10^{-1} so that the term on the right hand side would be a factor 10 smaller than the corresponding matrix elements for double pion photoproduction using the Yukawa interaction only. This fact coupled with the pion propagators in the denominators shows that the contribution of the pion-pion interaction to the double pion photoproduction is small compared to that of the Yukawa interaction = if we retain only terms of the lowest order in λ . This lends again support to the view of Chew mentioned in the last chapter, that the role of the pion-pion interaction, at least in the low energy region is to act as an initial state interaction which will not be present in the case of photo-production. Of course at higher energies ($\gtrsim$ 750 Mev) the final pair of pions may resonate in the $I = 1$, $J = 1$

state and thus the pion-pion interaction may make itself felt as a final state interaction. We shall study the effect of the pion-pion resonance more closely in the next sub-section.

(b) We shall study the effect of the pion-pion resonance in the (1, 1) state on the photoproduction of pion pairs from nucleons by considering the situation when both the final pions produced resonate as well as the case when one of the pions produced resonates with a pion in the virtual cloud. To derive the matrix element for the former case, we start from the S -matrix element

$$< p; q_{\nu_1}, q_{\nu_2} | S | p; k \epsilon >$$

Taking the photon creation operator through the part of the interaction Hamiltonian corresponding tothe photon-meson interaction viz.,

$$\left(\varphi(x) \frac{\partial \varphi^*(x)}{\partial x_\mu} - \varphi^*(x) \frac{\partial \varphi(x)}{\partial x_\mu} \right) A_\mu(x)$$

and the operators corresponding to one of the final pions with four momentum q_{ν_2} through the part of the interaction Hamiltonian corresponding to pion-pion interaction viz.,

$$4\pi\lambda \left(\vec{\varphi} \cdot \vec{\varphi} \right)^2$$

we obtain on going through the usual Low's procedure the T -matrix element

$$\langle p'; q_1, q_2 | T | p; k\epsilon \rangle = \frac{-\epsilon_\mu}{\sqrt{4\omega_2 k}}$$

$$\cdot \left[\frac{\langle q_1 p' | J_{\pi\pi}(0) | n \rangle \langle n | j_\gamma(0) | p \rangle}{\omega' + p_0' - k - E_n + i\varepsilon} \right.$$

$$\left. - \frac{\langle q_1 p' | j_\gamma(0) | n \rangle \langle n | J_{\pi\pi}(0) | p \rangle}{E_n - k - p_0 - i\varepsilon} \right] \qquad (8)$$

We shall consider the first term only and take the ' n ' state to be the one corresponding to a nucleon with momentum p' (the same as that of the final nucleon) plus two pions with four-momenta q_1' and q_2' , . Taking the operator corresponding to the pions of momentum q_2' through the photon current operator $j_\gamma(0)$ we obtain finally

$$\langle p'; q_1, q_2 | T | p; k\epsilon \rangle = -\frac{\sqrt{2}\vec{q}_2' \cdot \vec{\epsilon}}{(4\omega_2 k)^{\frac{1}{2}}} \int \frac{dq_2'}{(2\omega_2')^{1/2}} \langle p' q_1 | J_{\pi\pi}(0) | p'; q_1', q_2' \rangle$$

$$\times \langle p' q_1' | j_\pi(0) | p \rangle \left[(\omega_1 - \omega_2' - \omega_2' - k)\left((\vec{k} - \vec{q}_2')^2 + 1 \right) \right]^{-1} \qquad (9)$$

where $j_\pi(0) = i g \bar{\psi}(0) \gamma_5 \psi(0)$ is the meson current operator. The right hand side contains the product of the pion-pion scattering matrix element, $\langle q_1 | J_{\pi\pi}(0) | q_1' q_2' \rangle$ and the pion nucleon scattering matrix element $\langle p' q_1' | j_\pi(0) | p \rangle$. For the former we assume the interaction to proceed through a resonance in the (1, 1) state.

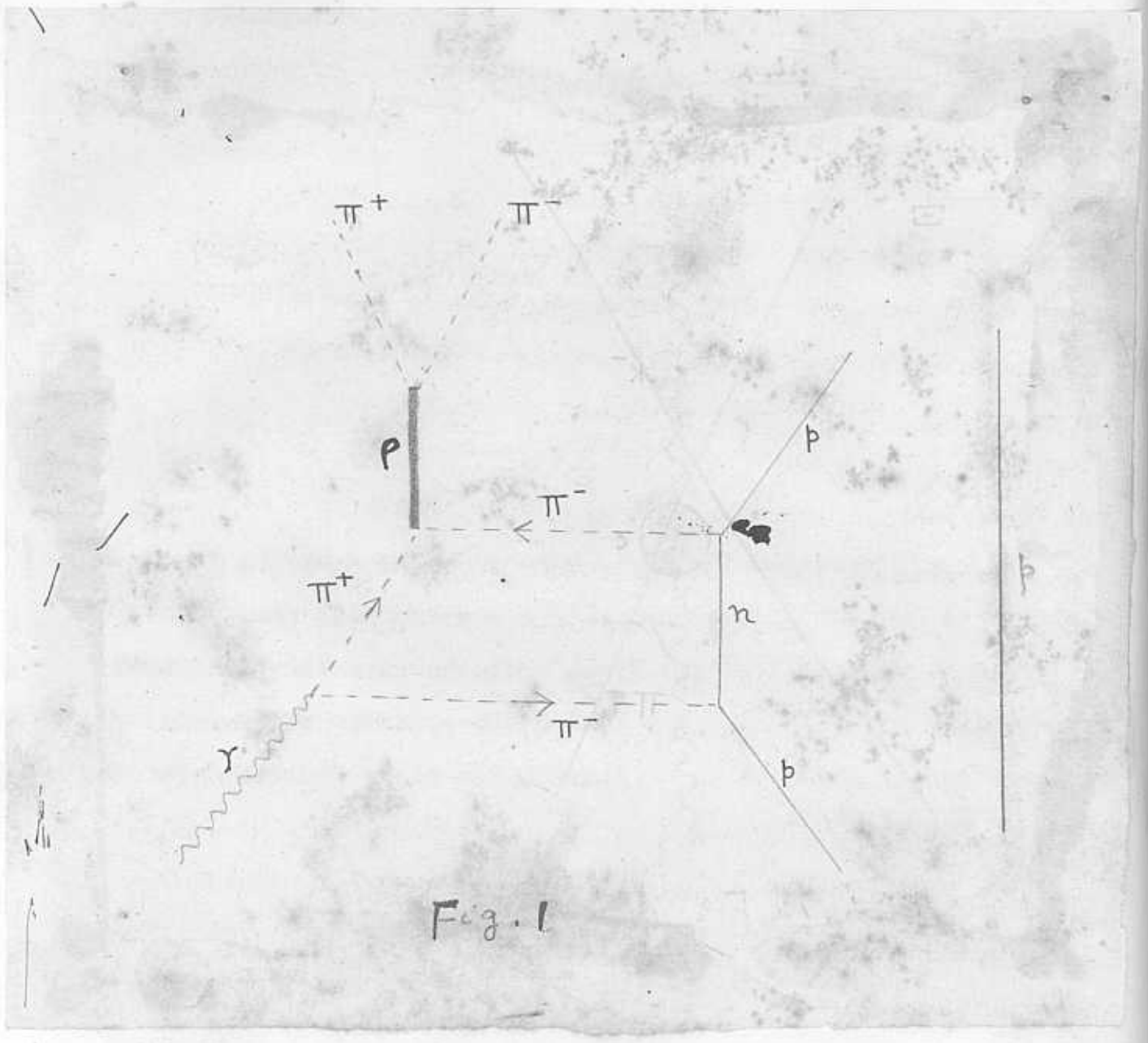

Fig. 1

This is simulated by taking a 'ρ' particle intermediate state, the $\rho\pi\pi$ interaction being given by

$$\frac{1}{\sqrt{2}} f_{\rho\pi\pi} \sum_{\alpha,\beta,\gamma} \partial_\mu \varphi^\alpha(x)\, \varphi^\beta(x)\, B_\mu^\gamma(x)\, \epsilon_{\alpha\beta\gamma}$$

where $B_\mu(x)$ is the ρ -meson field. The resulting matrix element for the process of photoproduction of pion pairs will then correspond to the diagram (1). For the $\rho\pi\pi$ coupling constant $f_{\rho\pi\pi}$ we use the experimental value[1)]

$$f^2_{\rho\pi\pi}/4\pi \simeq 4$$

The differential cross-section $d\sigma/d\omega_1$ for various values of k, the photon energy and ω_1, the energy of the negative pion is given in Table I. As can be seen the cross_section increases with increasing k as well as increasing ω_1, but the contribution tends to be small for values of the photon energy below the value corresponding to the pion-pion resonance viz., $\sim$ 750 Mev.

The ~~design~~ diagram corresponding to the situation where one of the final pions with four-momentum q_1 resonates with a pion in the virtual state is given by Fig. 2. A numerical estimate of the contribution from this graph again indicates that the effect of the pion-pion resonance on the photoproduction of pion pairs tends to be small for values of the photon energy $<\sim$ 750 Mev.

Thus we get the result which was used in the previous chapter viz., that the effect of the pion-pion interaction on the photoproduction of pion pairs from nucleons is negligible for moderate energies of the incident photon.

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1) J.A.Anderson et al, phys. Rev. Letters, 6, 365, (1961); C.R.Erwin et al, ibid, 628, (1961).

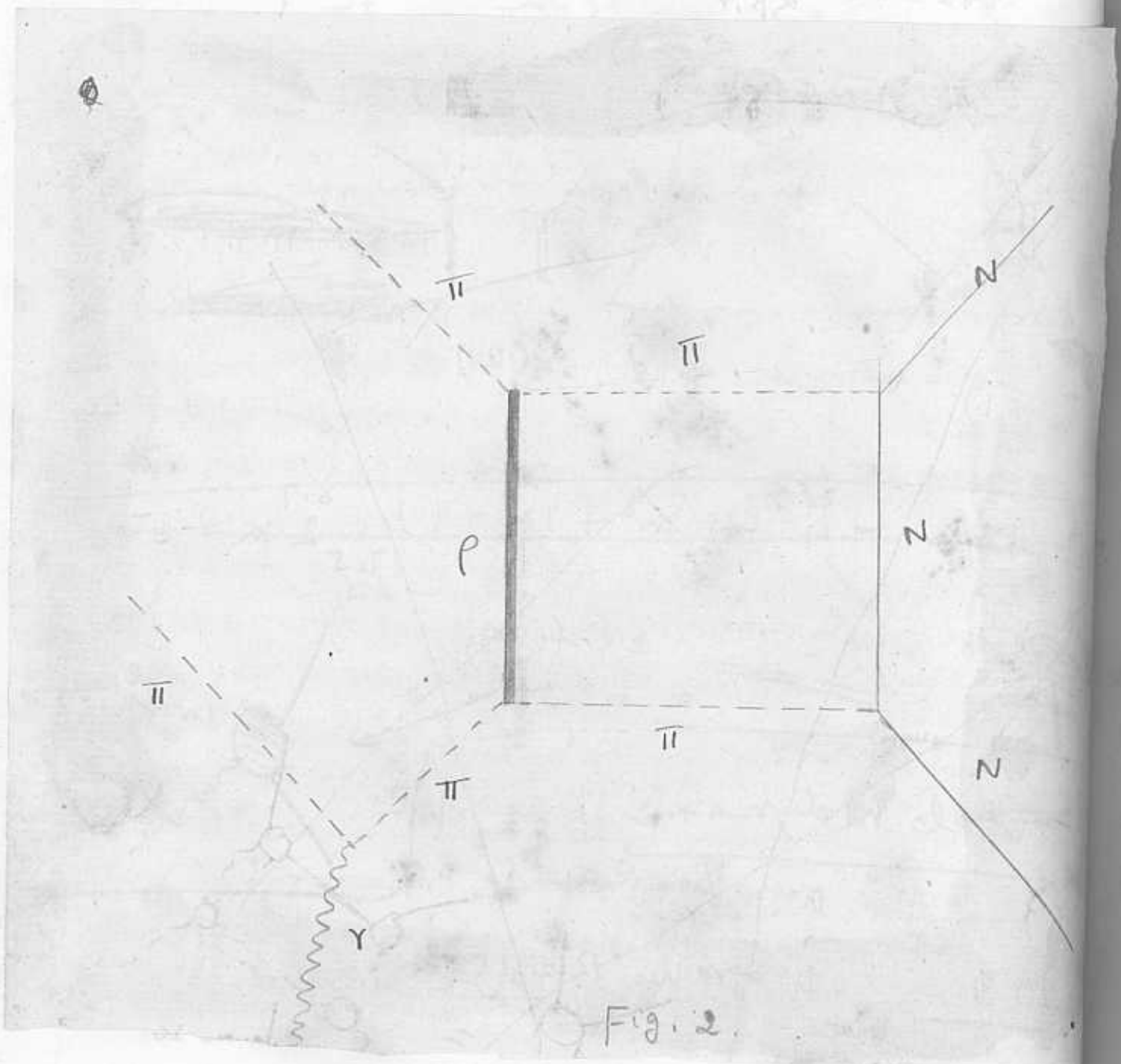

Fig. 2.

Table I

Results corresponding to Fig. I.

	$(d\sigma/d\omega_1) \times 10^{33}$	
ω_1	k = 3	k = 4
1.281	0.479	---
1.6	1.108	3.781
1.887	1.84	12.54
2.4	---	28.06

Table II

Result corresponding to Fig. 2

k = 4

ω_1	$(d\sigma/d\omega_1) \times 10^{32}$
1.281	.072
1.6	27
1.887	52.4

UNIT OF ENERGY = PION MASS = 1

3. Photoproduction of triple pions

We shall estimate the contribution arising fromthe pion-pion interaction to this process. To be more precise let us consider the reaction

$$\gamma + p \longrightarrow p + \pi^+ + \pi^- + \pi^\circ \qquad (10)$$

Then using Low's procedure and the interaction (4) the relevant matrix element for the production of one positive pion of momentum q_{v_1}, one negative pion of momentum q_{v_2} andone neutral pion of momentum q_{v_3} is given by

$$\langle p', q_{v_1+}, q_{v_2-}, q_{v_3\circ} | S | k\epsilon ; p \rangle$$

$$= -\frac{i}{(2\omega_1)^{\frac{1}{2}}} \int dx\, e^{-iq_{v_1}x} \langle p', q_{v_2-}, q_{v_3\circ} | J^+_{\pi\pi}(x) | k\epsilon; p \rangle \qquad (11)$$

where $J^+_{\pi\pi}$ is the current density operator due to the pion-pion coupling interaction.

The matrix element (9) can be evaluated in the lowest order of the pion-pion coupling constants λ by taking both the operators corresponding to the pions of momenta q_{v_2} and q_{v_3} through $J^+_{\pi\pi}(x)$ itself when the matrix element reduces to

$$\langle p', q_{v_1+}, q_{v_2-}, q_{v_3\circ} | S | k\epsilon; p \rangle = -\frac{32 i \pi \lambda}{(8\omega_1\omega_2\omega_3)^{\frac{1}{2}}}$$

$$\times \delta(p+k-p'-q_{v_1}-q_{v_2}-q_{v_3}) \langle p' | \varphi_0(0) | k\epsilon; p \rangle \qquad (12)$$

The matrix element on the right is connected with that for the photo-production for a single neutral pion by the equation

$$\langle p' | \varphi_0(0) | k\epsilon ; p \rangle = - \frac{\langle p' | j_0(0) | k\epsilon ; p \rangle}{\left[(q_{v_1} + q_{v_2} + q_{v_3})^2 + 1 \right]} \qquad (13)$$

Using the complete amplitude for the photoproduction of neutral pion given by Chew and Low[1] we obtain the differential cross section for the process as

$$\frac{d^5\sigma^{+-0}(k)}{d\Omega_1 d\Omega_2 d\Omega_3 d\omega_2 d\omega_3} = |M|^2 (2\pi)^{-8} q_{v_1} q_{v_2} q_{v_3} \omega_1 \omega_2 \omega_3 \qquad (14)$$

where

$$|M|^2 = \frac{256\pi^4 \lambda^2 e^2}{\omega_1 \omega_2 \omega_3 k} \cdot \frac{1}{\left[(\vec{q}_{v_1} + \vec{q}_{v_2} + \vec{q}_{v_3})^2 - k^2 + 1 \right]^2}$$

$$\cdot \left[\frac{f^2 (\mu_p + \mu_n)^2}{m^2 \omega^2} \left\{ (\vec{q}_{v_1} + \vec{q}_{v_2} + \vec{q}_{v_3}) \times (\vec{k} \times \vec{\epsilon}) \right\}^2 \right.$$

$$+ \frac{(\mu_p - \mu_n)^2}{36 f^2 m^2 (k-1)^6} \left\{ \left[(\vec{q}_{v_1} + \vec{q}_{v_2} + \vec{q}_{v_3}) \times (\vec{k} \times \vec{\epsilon}) \right] \left| (a_1 + 4a_2 + 4a_3) \right|^2 \right.$$

$$+ \left[(\vec{q}_{v_1} + \vec{q}_{v_2} + \vec{q}_{v_3}) \times (\vec{k} \times \vec{\epsilon}) \right]^2$$

$$\left. \left. \times \left| (a_1 + a_2 - 2a_3) \right|^2 \right\} \right] \qquad (15)$$

Here

$$a_i = \frac{e^{i\delta_i(q)} \sin \delta_i(q)}{q^3} \qquad (16)$$

the subscripts 1, 2 and 3 referring to the 11, 13 (or 31) and 33 states of the pion-nucleon respectively. The phase

1) G.F.Chew and F.E.Low, Phys. Rev., 101, 1579 (1956)

shifts for these states given by Salzman and Salzman[1] were used and the angular and energy integrations were done numerically to obtainthe total cross section for the process. Table (III) gives the total cross sections for two values of λ and for two values of the incident photon energy k.

The experimental values of Chasan et al[2] (which are the only ones that seem to be available at the energies considered here) for the combined crosssection $\sigma(\pi^+\pi^-\pi^0) + \sigma(\pi^+\pi^+\pi^-)$ at energies 700 and 840 Mev of the incident photon are about 7 and 14 microbarns respectively. In our method of derivation, the matrix element for the process

$$\gamma + p \longrightarrow p + \pi^+ + \pi^+ + \pi^-$$

will be

$$-\frac{64 i \pi \lambda}{(8\omega_1\omega_2\omega_3)^{\frac{1}{2}}}\,\delta(p+k-p'-q_1-q_2-q_3)$$

$$\cdot \langle p' | \varphi^*(0) | p\epsilon\,; k\rangle \qquad (17)$$

We notice that externally there is a factor 2 more than in the case of $\pi^+\pi^-\pi^0$ production. Further if we retain only the contribution of the isovector part of the current for the

1) G. Salzman and F. Salzman, Phys. Rev., 108, 1619 (1957).
2) B.M.Chasan et al , Phys. Rev., 119, 811 (1960).

charged pion photoproduction the matrix element for which occurs on the right hand side of (17) $\sigma(\pi^+\pi^+\pi^-)$ will be roughly twice the contribution of $\sigma(\pi^+\pi^-\pi^\circ)$ From Table III , we notice that a value of $\lambda \sim$ 0.1 will fit these data.

4. Pion production in pion-nucleon collisions

As mentioned in the introduction, this is the process which has shed much light on the pion-pion interaction. To be specific , we confine ourselves to the process

$$\pi^- + p \longrightarrow \pi^+ + \pi^- + n$$

The matrix element for this process can be evaluated in the lowest order in λ in exactly the same way as in the previous sections. It is given by

$$\langle p', q_{1+}, q_{2-} | S | p, q \rangle = -\delta(p' + q_1 + q_2 - p - q) \cdot \frac{64 i \pi \lambda}{(8\omega_1\omega_2\omega)^{\frac{1}{2}}} \langle p' | \varphi^*(0) | p \rangle \qquad (18)$$

where q is four-momentum of the incident pion and q_1 and q_2 those of the final pions. In the non-relativistic limit

$$\langle p' | \varphi(0) | p \rangle = \frac{i(4\pi)^{\frac{1}{2}} \cdot \sqrt{2}\, f \vec{\sigma} \cdot (\vec{p}' - \vec{p})}{(p' - p)^2 + 1} \qquad (19)$$

Using (18) and (19) , we obtain the differential cross-section as

Table III

Calculated values of $\sigma^{+-0}(k)$ in μb

$k = E_\gamma$ (MeV)	λ	
	0.1	0.15
700	2.5	5.6
840	4.3	9.7

$$\frac{d^3\sigma^{+-}}{d\Omega_+ d\Omega_- d\omega_1} = \frac{512\lambda^2 f^2 q_1 q_2}{(2\pi)^2 q} \cdot \frac{(\vec{q} - \vec{q}_1 - \vec{q}_2)^2}{\left[(\vec{q} - \vec{q}_1 - \vec{q}_2)^2 + 1\right]^2} \tag{20}$$

We shall choose $\vec{q}$ along the z-axis and $\vec{q}_1$ to lie in the zox plane. Let θ_1 be the angle between $\vec{q}$ and $\vec{q}_1$ and (θ_2, φ) be the angular co-ordinates of $\vec{q}_2$. Then the total cross section can be written as

$$\sigma^{+-}(q) = 128\lambda^2 f^2 J \tag{21}$$

where

$$J = \frac{2}{q}\iiint \Bigg[\frac{1}{F + 2q_1 q_2 \sin\theta_1 \sin\theta_2} + \frac{1}{F - 2q_1 q_2 \sin\theta_1 \sin\theta_2} - \frac{1}{(F + 2q_1 q_2 \sin\theta_1 \sin\theta_2)^2} - \frac{1}{(F - 2q_1 q_2 \sin\theta_1 \sin\theta_2)^2} \Bigg] \times q_1 q_2 \sin\theta_1 \sin\theta_2 \, d\theta_1 \, d\theta_2 \, d\omega_1 \tag{22}$$

$$F = q^2 + q_1^2 + q_2^2 - 2qq_1\cos\theta_1 - 2qq_2\cos\theta_2 + 2q_1 q_2 \cos\theta_1 \cos\theta_2 + 1 \tag{23}$$

The integration over θ_1 , θ_2 and φ as in other cases considered in this chapter can be performed numerically. We have performed the integration for q_V = 3.0, 3.5 and 6.0 pion masses using desk calculating machines. The theoretical cross-sections for two values of λ and the experimental values reported by Perkins et al from Berkeley are given in Table IV .

From the Table, it is seen that λ^2 lies between 0.01 and 0.005 which is consistent with the reported results of Rodberg[1] . However, we wish to remark that our fit for λ^2 is made on the basis of pion-pion interaction alone, the contribution from the Yukawa interaction being negligible.

5. Production of pion pairs in pion nucleon collisions.

The matrix elements for the processes

$$\pi^+ + p \longrightarrow \pi^+ + \pi^+ + \pi^- + p \qquad \text{(i)}$$

$$\longrightarrow \pi^+ + \pi^\circ + \pi^\circ + p \qquad \text{(ii)}$$

$$\longrightarrow \pi^+ + \pi^+ + \pi^\circ + n \qquad \text{(iii)}$$

can be evaluated, assuming the interaction (4). For process (i) the matrix element is given by

$$\langle p', q_{3-}, q_{2+}, q_{1+} | S | p, q \rangle =$$

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1) L.S. Rodberg, phys. Rev Letts , 3 , 58 (1959)

TABLE IV

Total cross section for pion production in mb.

W Mev	Theoretical cross section		Berkeley[1]
	$\lambda^2 = 0.01$	$\lambda^2 = 0.005$	value.
305	1.13	0.56	0.75 - 0.2
370	3.10	1.55	2.0 - 0.5
700	3.78.	1.89	——

1) W.A. Perkin's et al, loc. cit.

$$= -\frac{64 i\pi\lambda}{(8\omega_1\omega_2\omega)^{\frac{1}{2}}}\,\delta(p+q-p'-q_1-q_2-q_3)$$

$$\cdot\ \frac{\langle p' q_3|\, j_-(0)\,|p\rangle}{(\vec{p}'+\vec{q}_3-\vec{p})^2+1-\omega_3^2} \qquad (24)$$

where $\langle p' q_3|\, j_-(0)\,|p\rangle$ represents the matrix element for the scattering of a negative pion. We use the static approximation of this matrix element given by Chew and Low[1]. If θ_1, (θ_2, φ_2) and (θ_3, φ_3) are the spherical (polar) angles of the directions $\vec{q}_1$, $\vec{q}_2$ and $\vec{q}_3$ respectively with $\vec{q}$ along the z-axis, integration over φ, φ_2 and φ_3 yields the differential cross section

$$\frac{d^5\sigma}{d\cos\theta_1\, d\cos\theta_2\, d\cos\theta_3\, d\omega_2 d\omega_3} = \frac{A}{(a^2-b^2)^{3/2}}\left[\frac{2}{3}q_3^2(a^2-b^2+aq_3^2)\right.$$
$$\left. +2ac+ad+ae-bf\right] \qquad (25)$$

where

$$A = \frac{192\lambda^2}{\pi}\cdot\frac{q_1 q_2 \sin^2\delta_{33}}{q\, q_3^5}$$

$$a = q^2+q_1^2+q_2^2-q_3^2-2qq_1\cos\theta_1-2qq_2\cos\theta_2$$
$$+2q_1q_2\cos\theta_1\cos\theta_2$$

$$b = 2q_1q_2\sin\theta_1\sin\theta_2$$

$$c = q_3^2q_2^2\cos^2\theta_3+q_3^2q_1^2\cos^2\theta_1\cos^2\theta_3$$
$$+q_3^2q_2^2\cos^2\theta_2\cos^2\theta_3-2q_3^2qq_1\cos^2\theta_3\cos\theta_1$$
$$-2q_3^2qq_2\cos^2\theta_3\cos\theta_2+2q_1^3q_1q_2\cos^2\theta_3\cos\theta_1\cos\theta_2$$

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1) G. F. Chew and F.E. Low, loc. cit.

$$d = q_{V_3}^2 q_{V_1}^2 \sin^2\theta_3 \sin^2\theta_1$$

$$e = q_{V_3}^2 q_{V_2}^2 \sin^2\theta_3 \sin^2\theta_2$$

$$f = 2 q_{V_3}^2 q_{V_1} q_{V_2} \sin^2\theta_3 \sin\theta_1 \sin\theta_2 \qquad (26)$$

Experimental data on double pion production in pion-nucleon collisions in the low energy region is scanty. Willis[1] gives on the basis of a single event a value of 30 μb for the total cross section for process (i) at an incident pion energy of 500 Mev . A rough calculation using the fact that the values of the momenta q_{V_1} , q_{V_2} , and q_{V_3} are small compared to q_V yields a cross-section of about $\simeq$ 40 μb for a value of $\lambda^2 = 0.01$.

6. Conclusion

In this chapter we have made estimates of the value of the pion-pion coupling constant assuming that the processes of double and triple pion production in pion nucleon and photo-nucleon collisions proceed only through the pion-pion interaction. It has been found that whereas the interaction

1) W.J.Willis, Phys. Rev., 116, 753 (1959).

may not be important for double pion photo-production, it is otherwise as for the other processes considered. From the previous section, it will be seen that a value of $\lambda^2 \approx 0.01$ seems able to give the cross-sections for the various processes considered. Further the variations in the value of λ fall within the limits given by Chew and Mandelstam[1] in their effective range approach involving a single real parameter which we can identify with λ. Assuming that there is no pion-pion bound state, they obtain the range of variation of λ as

$$-0.36 \lesssim \lambda \lesssim 0.3$$

It may also be mentioned that a value of $\lambda^2 \approx 0.15$ has been obtained by Khuri and Treiman[2] in their study of the effect of pion-pion interaction on τ -decay.

1) G.F.Chew and S. Mandelstam, loc. cit.
2) N.N.Khuri and S.B.Treiman, Phys. Rev., 119, 1115, (1960).

CHAPTER IV

POLARIZATION EFFECTS IN SOME PION PRODUCTION PROCESSES

1. Introduction

It is customary in the evaluation of the differential cross-section for a process to average over the initial states and sum over the final spin states and polarizations as it was experimentally difficult to produce spin-aligned targets or polarized beams of projectiles till recently. The summing over the final spins and polarizations is in case one is not interested in the spins and polarizations of the final particles. Fermi[1] had pointed out in an interesting paper in which he considered the scattering of a pion by a nucleon that even if the initial nucleon is unpolarized the final state nucleon is polarized in a direction perpendicular to the scattering plane. This shows that in averaging and summing over the spins and polarizations we are losing many interesting aspects of a process. The calculation of angular distributions for polarized targets and projectiles is of interest now. It is, for instance possible, as shown by the experiments by Goldhaber et al [2], to produce high energy bremsstrahlung which is almost completely circularly polarized using polarized electrons originating in β -decay,.

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1) E. Fermi, Phys. Rev., 91, 947 (1953)
2) M. Goldhaber et al , Phys. Rev., 106, 826 (1957)

In this chapter we have examined the questionof whether the angular distributions of pions produced through various agencies from a nucleon are sensitive to the state of polarization of the nucleon as well as that of the incident projectile (if it has spin). We have studied this problem with reference to (1) the photo-production of a single pion from a nucleon, (2) photoproduction of pion pairs from a nucleon and (3) electropion production from a nucleon.

2. Photoproduction of single mesons from polarized nucleons[1]

Several features concerning photoproduction of pions by nucleons have been well understood on the basis of the single variable dispersion theory of Chew, Goldberger, Low and Nambu (C G L N)[2] for the process. As referred to ~~earlier~~ in Chapter , the calculations of Ball[3] show that the C G L N matrix elements for single pion photoproduction give a good account of the process upto about 400 Mev incident energy of the photon. We wish to suggest that the study of polarization phenomena (i.e. the angular distribution of the pion when the incident photon is polarized and the target nucleon, has its spin aligned and the degree of polarization of the final nucleon) can also be used to check the validity of the C G L N formula.

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1) Alladi Ramakrishnan, S.K.Srinivasan, N.R.Ranganathan and K. Venkatesan, Proc. Ind. Acad. Sci. 49 , 302, 1959; and unpublished material.
2) G.F.Chew,M.Goldberger, F.Low and Y. Nambu, Phys. Rev. 106 , 1345 (1957)
3) J.S.Ball, loc. cit.

a) Photoproduction of charged pions

The complete matrix element for photo-production of a positively charged pion as given by C G L N is

$$M(\gamma+p\to\pi^{+}+n)=\frac{2\sqrt{2}\,\pi e f}{\sqrt{\omega k}}\left[\frac{1}{1+\frac{\omega}{m}}\right.$$

$$\cdot\left\{i\vec{\sigma}\cdot\vec{\epsilon}+\frac{2i\vec{\sigma}\cdot(\vec{k}-\vec{q}_v)(\vec{q}_v\cdot\vec{\epsilon})}{[\vec{k}-\vec{q}_v)^2+1]}\right\}$$

$$+i\vec{\sigma}\cdot\vec{q}_v\times(\vec{k}\times\vec{\epsilon})\lambda h^{(--)}$$

$$+\vec{q}_v\cdot(\vec{k}\times\vec{\epsilon})\lambda h^{(-+)}$$

$$+i\vec{\sigma}\cdot\vec{\epsilon}\left[i\left(\frac{2}{3}\delta_1+\frac{1}{3}\delta_3\right)F_s+\omega^2N^{(-)}\right]$$

$$+\frac{1}{3}ie^{i\delta_{33}}\sin\delta_{33}\left[\left(F_q-\frac{1}{3}F_M\right)i\vec{\sigma}\cdot\vec{\epsilon}\,\vec{q}_v\cdot\vec{k}\right.$$

$$\left.+\left(F_q+\frac{1}{3}F_M\right)i\vec{\sigma}\cdot\vec{k}\,\vec{q}_v\cdot\vec{\epsilon}\right]$$

$$-\frac{2}{9}i\vec{q}_v\cdot(\vec{k}\times\vec{\epsilon})e^{i\delta_{33}}\sin\delta_{33}F_M$$

$$-i\vec{\sigma}\cdot\vec{\epsilon}\,\frac{\mu_p+\mu_n}{2m}\omega$$

$$\left.-i\vec{\sigma}\cdot\vec{q}_v\times(\vec{k}\times\vec{\epsilon})\frac{\mu_p+\mu_n}{2m\omega}\right] \qquad (1)$$

Here μ_p and μ_n are the magnetic moments of the proton and the neutron respectively; m the mass of the nucleon, $\vec{k}$ and $\vec{q}$ are the momenta of the photon and pion respectively; ω is the pion energy, ϵ the polarization vector of the photon and σ the spin operator for the nucleon and

$$\lambda = \frac{\mu_p - \mu_n}{4mf^2} \; ; \qquad h = \frac{e^{i\delta}\sin\delta}{q^3}$$

$$h^{(--)} = \frac{1}{3}(h_{11} - h_{13} - h_{31} + h_{33})$$

$$h^{(-+)} = \frac{1}{3}(h_{11} + 2h_{13} - h_{31} - 2h_{33})$$

the subscripts of h referring to the isotopic spin and angular momentum states for which h is the amplitude

$$F_S = 1 - \frac{1}{2}\left(1 + \frac{1-v^2}{2v}\log\frac{1-v}{1+v}\right)$$

$$F_Q = \frac{1}{\omega^2}\left[1 - \frac{3}{4v^2}\left(1 + \frac{1-v^2}{2v}\log\frac{1-v}{1+v}\right)\right]$$

$$F_M = \frac{3}{4q^2}\left(1 + \frac{1-v^2}{2v}\log\frac{1-v}{1+v}\right),$$

v being the velocity of the pion.

To simplify the kinematics, let us choose the direction of propagation of the photon as the z -axis and the x-z plane is defined as the plane which contains $\vec{k}$ and $\vec{q}$. Let θ denote the angle between $\vec{k}$ and $\vec{q}$. It is to be noted that the above matrix element is a matrix in the spin space

of the nucleon. We shall now work out the polarization effects from (1) by prescribing that the target nucleon be polarized along the direction of the propagation of the photon . Here as well as in the next sections we shall use the superscripts 1 and 2 to denote the no-spin flip and spin flip cross sections respectively. We note that with our choice of the co-ordinate system, the right circular and left circular polarization of the photons are $\frac{\epsilon_x + i\epsilon_y}{\sqrt{2}}$ and $\frac{\epsilon_x - i\epsilon_y}{\sqrt{2}}$. Making use of these expressions for ϵ , we can obtain the relevant matrix elements when positively charged pions are produced by right circularly polarized and left circularly polarized photons. Since we require the spin of the nucleon to be oriented along the z -axis, i.e. in the direction of propagation of the photon, the operator $\sigma_- = \frac{\sigma_x - i\sigma_y}{2}$ causes the spin-flip of the nucleon while $\sigma_+ = (\sigma_x + i\sigma_y)/2$ acting on this spin state of the nucleon yields zero eigenvalue. From the amplitudes thus obtained for the various cases of right and left circularly polarized photons and spin-flip and no-spin flip for the nucleon, we can calculate the angular distributions and they are given by

$$\frac{d\sigma^{(1)}_{r.c.}(\pi^+)}{d\Omega} = \frac{2e^2f^2q}{k}\left[\frac{2(k-q\cos\theta)^2q^2\sin^2\theta}{(1+\frac{\omega}{m})^2[(\vec{k}-\vec{q})^2+1]^2}\right.$$

$$+\frac{1}{2}\frac{\lambda^2k^2}{q^4}\sin^2\delta_{33}\sin^2\theta$$

$$+\frac{2\lambda(k-q\cos\theta)k\sin^2\theta\cos\delta_{33}\sin\delta_{33}}{q(1+\frac{\omega}{m})[(\vec{k}-\vec{q})^2+1]}$$

$$+\frac{1}{18}\sin^2\delta_{33}(F_Q+F_M)^2k^2q^2\sin^2\theta$$

$$+\frac{(\mu_p+\mu_n)^2}{8m^2\omega^2}k^2q^2\sin^2\theta$$

$$-\frac{2}{3}\frac{1}{(1+\frac{\omega}{m})}\sin^2\delta_{33}\frac{(F_Q+F_M)}{[(\vec{k}-\vec{q})^2+1]}kq^2\sin^2\theta$$

$$-\frac{(\mu_p+\mu_n)}{m\omega(1+\frac{\omega}{m})}kq^2\sin^2\theta(k-q\cos\theta)$$

$$-\frac{\lambda(\mu_p+\mu_n)}{2m\omega}\frac{k^2\sin^2\theta}{q}\cos\delta_{33}\sin\delta_{33}$$

$$\left.+\frac{\mu_p+\mu_n}{6m\omega}k^2q^2\sin^2\theta\sin^2\delta_{33}(F_Q+F_M)\right] \qquad (2)$$

$$\frac{d\sigma^{(2)}_{r.c.}(\pi^+)}{d\Omega} = \frac{2e^2f^2q}{k}\left\{\frac{2q^4\sin^4\theta}{(1+\frac{\omega}{m})^2[(\vec{k}-\vec{q})^2+1]^2}\right\}$$

$$= \frac{2e^2f^2q}{k}J^{(2)}_{r.c.}(\pi^+) \qquad (3)$$

Here the suffix $\pi.c.$ denotes that the photon beam is right circularly polarized. In a similar way, the angular distributions when the photon beam is left circularly polarized aregiven below. In the above as well as in what follows all the phase shifts other than the dominant 33 phase shift are omitted.

$$\frac{d\sigma^{(1)}_{l.c.}(\pi^+)}{d\Omega} = \frac{2e^2 f^2 q}{k}\left[\frac{2(k-q\cos\theta)^2 q^2\sin^2\theta}{(1+\frac{\omega}{m})^2\left[(\vec{k}-\vec{q})^2+1\right]^2}\right.$$

$$+\frac{\lambda^2 k^2\sin^2\theta\sin^2\delta_{33}}{18q^4}$$

$$-\frac{2(k-q\cos\theta)\lambda k\sin^2\theta\cos\delta_{33}\sin\delta_{33}}{3q(1+\frac{\omega}{m})\left[(\vec{k}-\vec{q})^2+1\right]}$$

$$+\frac{1}{18}k^2q^2\sin^2\theta(F_Q-\tfrac{1}{3}F_M)^2\sin^2\delta_{33}$$

$$+\frac{(\mu_p+\mu_n)^2}{8m^2\omega^2}k^2q^2\sin^2\theta$$

$$-\frac{2(k-q\cos\theta)kq^2\sin^2\theta(F_Q-\frac{1}{3}F_M)}{3(1+\frac{\omega}{m})\{(\vec{k}-\vec{q})^2+1\}}$$

$$\left.-\frac{(\mu_p+\mu_n)}{\omega(\omega+m)}\frac{(k-q\cos\theta)kq^2\sin^2\theta}{(\vec{k}-\vec{q})^2+1}\right.$$

$$+\frac{\lambda}{6m\omega}\,\frac{(\mu_p+\mu_n)\cos\delta_{33}\sin\delta_{33}k^2\sin^2\theta}{q}$$

$$+\frac{1}{6m\omega}(\mu_p+\mu_n)\sin^2\delta_{33}k^2q^2\sin^2\theta\left(F_Q-\frac{1}{3}F_M\right)\Big] \qquad (4)$$

$$\frac{d\sigma^{(2)}_{l.c.}(\pi^+)}{d\Omega}=\frac{2e^2f^2q}{k}\left[\frac{2}{(1+\frac{\omega}{m})^2}\left\{1-\frac{q^2\sin^2\theta}{[(\vec{k}-\vec{q})^2+1]}\right\}\right]^2$$

$$+\frac{2\lambda^2k^2}{9q^4}\cos^2\theta\sin^2\delta_{33}$$

$$-\frac{4\lambda k\sin\delta_{33}\cos\delta_{33}\cos\theta}{3q^2(1+\frac{\omega}{m})}\cdot\left\{1-\frac{q^2\sin^2\theta}{[(\vec{k}-\vec{q})^2+1]}\right\}$$

$$+2\left(\frac{2}{3}\delta_1+\frac{1}{3}\delta_3\right)^2F_S^2+\left(\frac{\mu_p+\mu_n}{2m^2}\right)^2\omega^2$$

$$+\frac{2}{9}\sin^2\delta_{33}\left(F_Q-\frac{1}{3}F_M\right)^2k^2q^2\cos^2\theta$$

$$+\frac{(\mu_p+\mu_n)}{2m^2\omega^2}k^2q^2\cos^2\theta$$

$$-\frac{4\sin^2\delta_{33}\left(F_Q-\frac{1}{3}F_M\right)kq\cos\theta}{3(1+\frac{\omega}{m})}\cdot\left\{1-\frac{q^2\sin^2\theta}{[(\vec{k}-\vec{q})^2+1]}\right\}$$

$$- \frac{4\lambda \sin^2\delta_{33} k \cos\theta}{3q^2} \cdot \left(\frac{2}{3}\delta_1 + \frac{1}{3}\delta_3\right) F_S$$

$$+ \frac{4}{3}\lambda \cos\delta_{33} \sin\delta_{33} \left(F_Q - \frac{1}{3}F_M\right)\left(\frac{2}{3}\delta_1 + \frac{1}{3}\delta_3\right) F_S \, kq \cos\theta$$

$$+ \frac{2(\mu_p + \mu_n) kq \cos\theta}{\omega(\omega + m)} \cdot \left\{1 - \frac{q^2 \sin^2\theta}{\left[(\vec{k} - \vec{q})^2 + 1\right]}\right\}$$

$$+ \frac{2(\mu_p + \mu_n)\omega}{(\omega + m)} \left\{1 - \frac{q^2 \sin^2\theta}{\left[(\vec{k} - \vec{q})^2 + 1\right]}\right\}$$

$$- \frac{2}{3}\lambda \frac{(\mu_p + \mu_n)}{m\omega} \frac{k^2 \cos^2\theta}{q^2} \cos\delta_{33} \sin\delta_{33}$$

$$+ \frac{2}{3} \frac{\lambda(\mu_p + \mu_n)}{m} \cdot \frac{\omega k \cos\theta}{q^2} \cos\delta_{33} \sin\delta_{33}$$

$$- \frac{2}{3} \frac{(\mu_p + \mu_n)}{m\omega} k^2 q^2 \sin^2\delta_{33} \cos^2\theta \left(F_Q - \frac{1}{3}F_M\right)$$

$$+ \frac{2}{3} \frac{(\mu_p + \mu_n)}{m} \omega k q \cos\theta \sin^2\delta_{33} \left(F_Q - \frac{1}{3}F_M\right)$$

$$\left. - \frac{(\mu_p + \mu_n)^2}{m^2} k q \cos\theta \right]$$

(5)

We notice that except for the case of right circularly polarized photon with spin-flip of the nucleon, the expressions for the angular distributions are cumbersome. But if we take only the first four terms of (1) into consideration we will be retaining the dominant terms. The first term which is the "interaction current" term in the static theory of Chew and Low[1] is due to the electric dipole interaction of the photon. The second term is the "pion current" term which includes all multipole moments. The third and fourth terms arising from magnetic dipole interaction of the photon with the static nucleon introduces the pion-nucleon phase shift into the photo-production problem. Retaining only these terms of the matrix element (1), we will have, for $\frac{d\sigma^{(1)}_{\pi.c.}}{d\Omega}$ only the first three terms of (2) which we shall write as $\frac{2e^2f^2q}{k} J^{(1)}_{\pi.c.}(\pi^+)$; $\frac{d\sigma^{(2)}_{\pi.c.}}{d\Omega}$ as it is; for $\frac{d\sigma^{(1)}_{l.c.}(\pi^+)}{d\Omega}$ only the first three terms of (4) which we shall write as $\frac{2e^2f^2q}{k} J^{(2)}_{l.c.}(\pi^+)$; and for $\frac{d\sigma^{(2)}_{l.c.}(\pi^+)}{d\Omega}$ only the first three terms in (5) which we shall write as $\frac{2e^2f^2q}{k} J^{(2)}_{l.c.}(\pi^+)$. The angular distributions $\frac{d\sigma^{(1)}_{\pi.c.}(\pi^+)}{d\Omega}$, $\frac{d\sigma^{(2)}_{\pi.c.}(\pi^+)}{d\Omega}$, $\frac{d\sigma^{(1)}_{l.c.}(\pi^+)}{d\Omega}$, and $\frac{d\sigma^{(2)}_{l.c.}(\pi^+)}{d\Omega}$ have been calculated for the simplified case for various values of the photon energy. The results are given in Tables (I to IV) respectively and also drawn as curves in Figures (1- 4) As can be seen even from the expressions (2), (3), (4), (5), the angular distributions are entirely different for the four cases considered.

1) G.F.Chew and F.E.Low, Phys. Rev., 101 , 1579 (1956)

Table I

$\frac{d\sigma^{(1)}_{r.c}(\pi^+)}{d\Omega}$ in microbarns/sterad

k \ $\theta_{c.m.}$	0°	30°	60°	90°	120°	150°	180°
1.5	0	2.25	6.75	9.0	6.75	2.25	0
2	0	9.48	28.43	37.9	28.43	9.48	0
2.5	0	12.36	37.1	49.43	37.1	12.36	0

Table II

$\frac{d\sigma^{(2)}_{r.c}(\pi^+)}{d\Omega}$ in microbarns/sterad

k \ $\theta_{c.m.}$	0°	30°	60°	90°	120°	150°	180°
1.5	0	0.16	0.68	0.60	0.20	0.02	0
2	0	0.33	3.21	2.14	0.63	0.05	0
2.5	0	0.70	5.34	3.14	0.87	0.07	0

Table XIII

$\frac{d\sigma^{(1)}_{l.c}(\pi^+)}{d\Omega}$ in microbarns/sterad

k \ $\theta_{c.m.}$	0°	30°	60°	90°	120°	150°	180°
1.5	0	0.07	0.20	0.26	0.20	0.07	0
2	0	0.50	1.49	1.98	1.49	0.50	0
2.5	0	1.47	4.42	5.90	4.42	1.47	0

Table XIV

$\frac{d\sigma^{(2)}_{l.c}(\pi^+)}{d\Omega}$ in microbarns/sterad

k \ $\theta_{c.m.}$	0°	30°	60°	90°	120°	150°	180°
1.5	0 7.42	0x07 7.75	0x08 7.21	0x09 12.57	0x06 21.98	0x05 31.37	0 35.8
2	11.19	6.56	4.86	11.61	29.89	46.95	56.59
2.5	29.45	15.47	8.02	9.87	22.31	33.97	40.77

FIG. 1

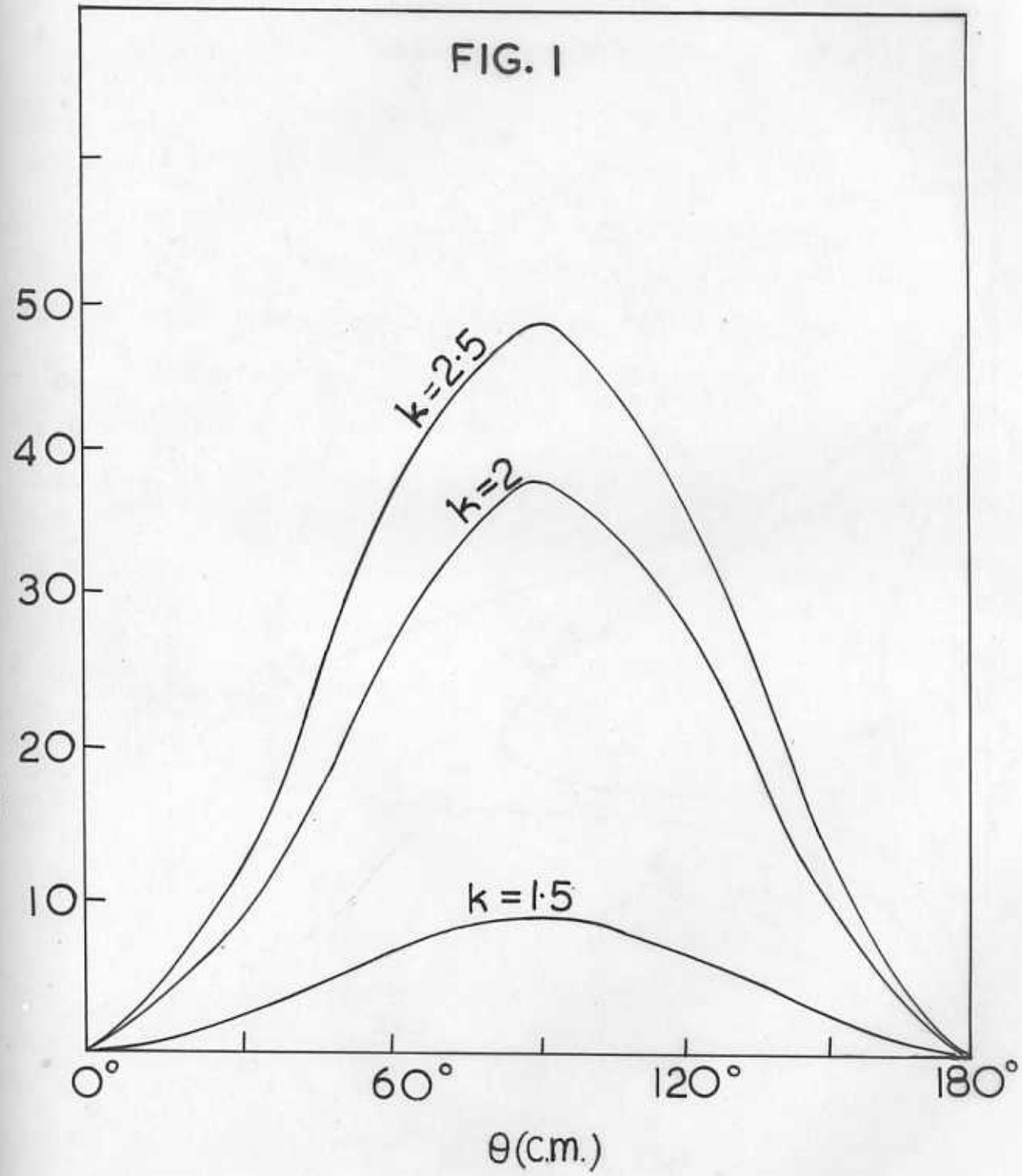
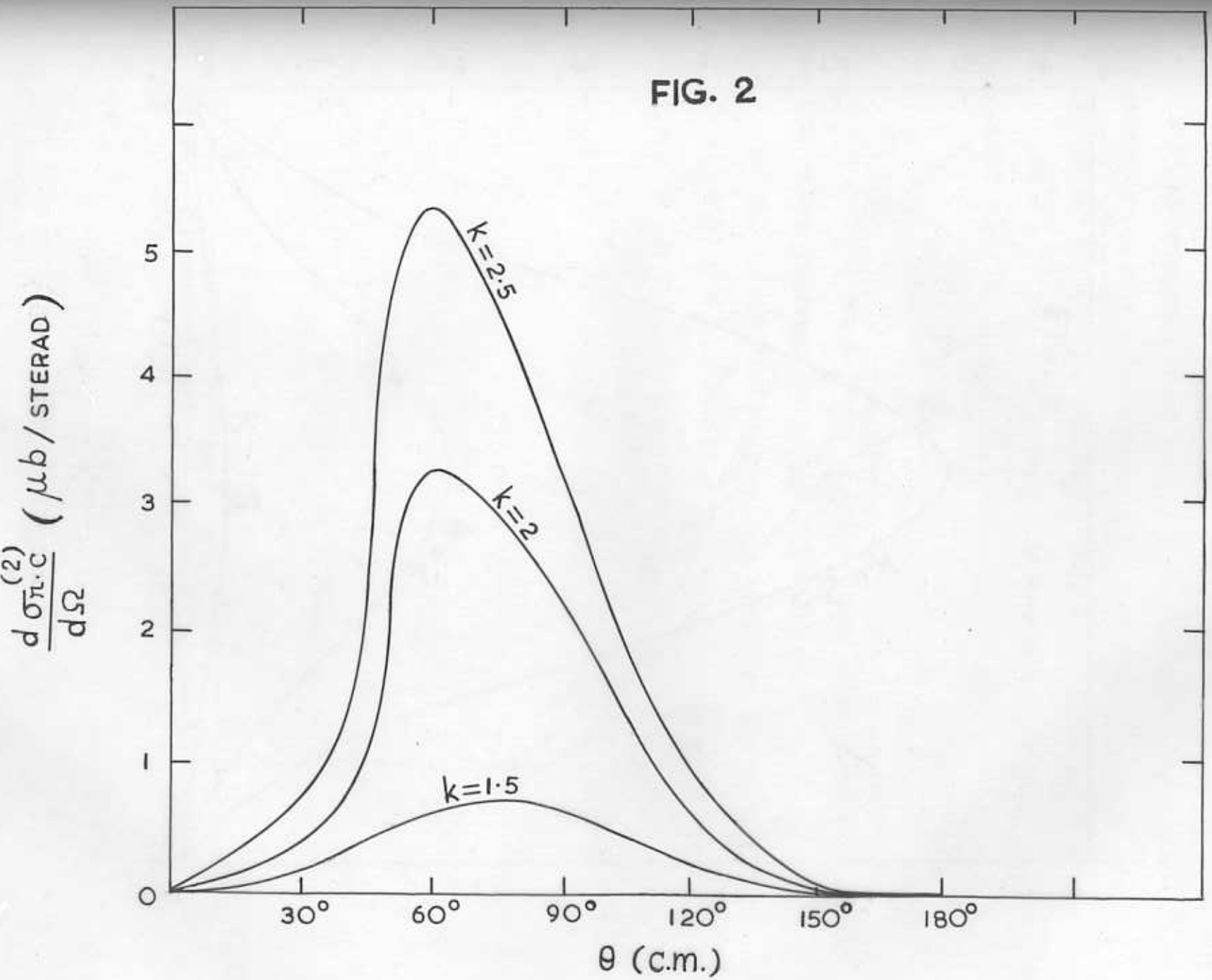


FIG. 2


$\frac{d\sigma_{\pi\cdot c}^{(2)}}{d\Omega}$ (μb/STERAD)
0
1
2
3
4
5
k=2·5
k=2
k=1·5
30°
60°
90°
120°
150°
180°
θ (c.m.)

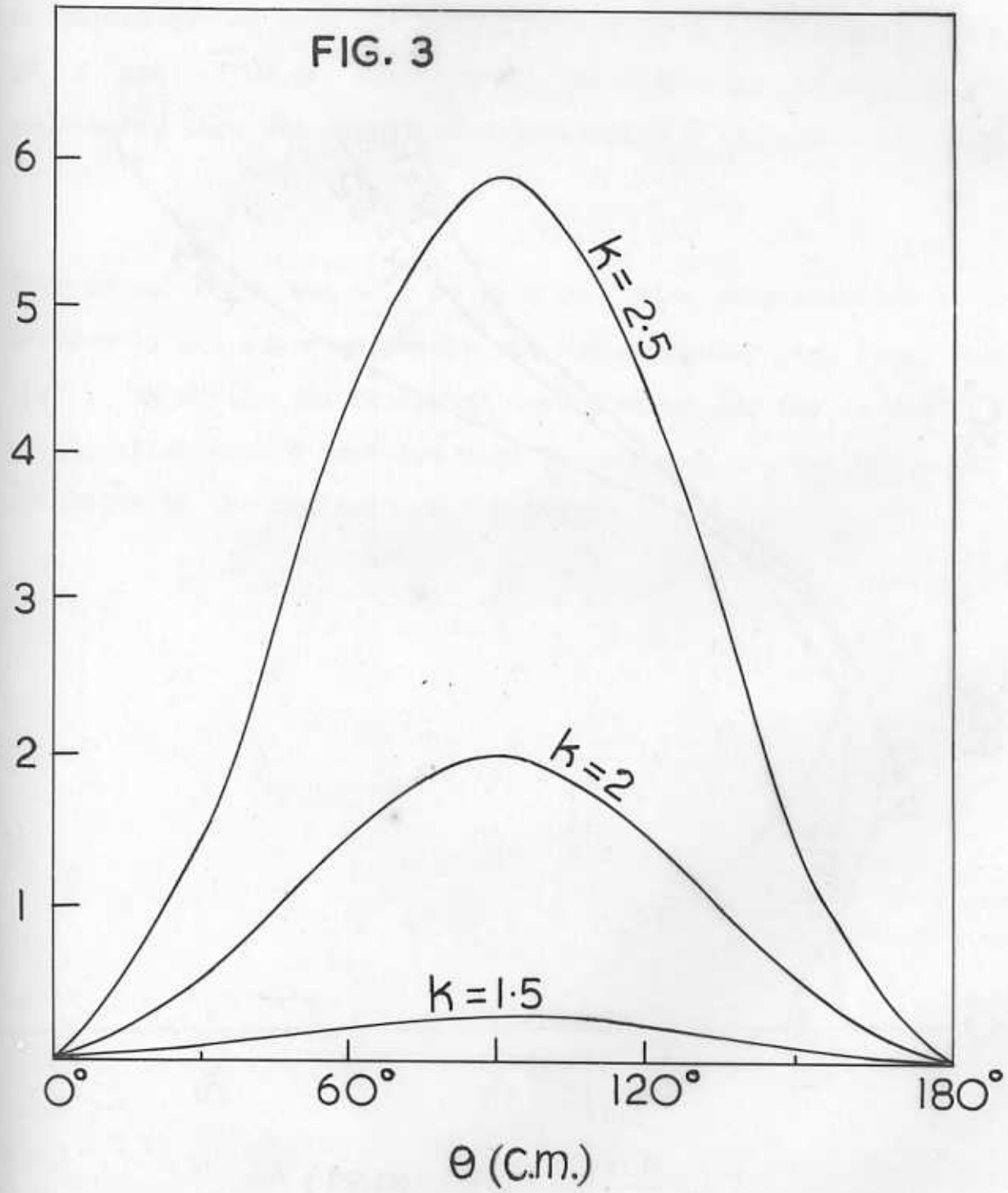


FIG. 3
6
5
4
3
2
1
K=2·5
K=2
K=1·5
0°
60°
120°
180°
θ (C.m.)
dσ/dΩ (μb/sterad)

60
50
40
30
20
10
0°
60°
120°
180°
θ(C.m.)
k=2
k=1.5
k=2.5
k=2
k=2.5
k=1.5
$\frac{d\sigma_{l.c.}^{(2)}}{d\Omega}$ (μb/sterad)

FIG. 4.

For the simplified case we can also calculate the degree of polarization of the nucleon in the final state since we are in possession of both the spin-flip and no spin-flip amplitudes. If A and (i B) are respectively the spin-flip and no spin-flip amplitude, then the degree of polarization P is given by

$$P = 2\,\frac{\mathrm{Im}\,(A\,B^{*})}{|A|^{2}+|B|^{2}} \qquad (6)$$

This polarization can only be in a direction perpendicular to the plane defined by the incoming photon and the outgoing pion (see, Fermi 1953). We obtain the following experessions for the degree of polarization when a positive pion is produced for the two polarizations of the incident photon beams

$$P_{r.c}(k,\theta) = \frac{-2}{[J^{(1)}_{r.c}+J^{(2)}_{r.c}]}\left[\frac{k\lambda\sin^{3}\theta\cos\delta_{33}\sin\delta_{33}}{(1+\frac{\omega}{m})^{2}\left[(\vec{k}-\vec{q})^{2}+1\right]}\right.$$

$$\left. + \frac{2q^{3}\sin^{3}\theta\,(k-q\cos\theta)}{(1+\frac{\omega}{m})^{2}\left[(\vec{k}-\vec{q})^{2}+1\right]^{2}}\right]$$

$$P_{l.c}(k,\theta) = \frac{2}{[J^{(1)}_{l.c}+J^{(2)}_{l.c}]}\left[\frac{1}{q}\,\frac{\lambda^{2}k^{2}\sin^{2}\delta_{33}}{q^{4}}\cos\theta\sin\theta\right.$$

$$+ \frac{2(k-q\cos\theta)\,q\sin\theta}{(1+\frac{\omega}{m})^{2}\left[(\vec{k}-\vec{q})^{2}+1\right]}\left\{1-\frac{q^{2}\sin^{2}\theta}{(\vec{k}-\vec{q})^{2}+1}\right\}$$

$$\left. - \frac{2\lambda k\cos\delta_{33}\sin\delta_{33}(k-q\cos\theta)\cos\theta\sin\theta}{3q\,(1+\frac{\omega}{m})\left[(\vec{k}-\vec{q})^{2}+1\right]}\right]$$

$P(k,\theta)$ denotes the degree of polarization of the final nucleon when the pion is emitted at an angle θ to the direction of the photon momentum $\vec{k}$. Here again we notice a noticeable difference in the degrees of polarization when right circularly polarized and left cicularly polarized photons are used, a difference which can be confronted with experimental data when they become available.

b) Photoproduction of neutral pions

The complet matrix element for this case as given by Chew et al is

$$M(\gamma+p\rightarrow p+\pi^{\circ}) = \frac{2\pi e f}{\sqrt{\omega k}}\left[\vec{q}_{\gamma}\cdot(\vec{k}\times\vec{\epsilon})\lambda h^{(++)} + i\vec{\sigma}\cdot(\vec{q}_{\gamma}\times(\vec{k}\times\vec{\epsilon}))\lambda h^{(+-)}\right.$$

$$+ i\vec{\sigma}\cdot\vec{\epsilon}\cdot\left\{\frac{2}{3}\,i(\delta_1-\delta_3)F_S + \omega N^{(+)}\right\}$$

$$- i\vec{\sigma}\cdot\vec{\epsilon}\;\frac{\mu_p+\mu_n}{2m}\,\omega + \frac{2}{3}\,e^{i\delta_{33}}\sin\delta_{33}$$

$$\times\left\{\left(F_Q-\frac{1}{3}F_M\right)\vec{\sigma}\cdot\vec{\epsilon}\,\vec{q}_{\gamma}\cdot\vec{k} + \left(F_Q+\frac{1}{3}F_M\right)\vec{\sigma}\cdot\vec{k}\,\vec{q}_{\gamma}\cdot\vec{\epsilon}\right\}$$

$$+ \frac{4}{q}\,\vec{q}_{\gamma}\cdot(\vec{k}\times\vec{\epsilon})\,ie^{i\delta_{33}}\sin\delta_{33}F_M$$

$$\left.- i\vec{\sigma}\cdot\vec{q}_{\gamma}\times(\vec{k}\times\vec{\epsilon})\,\frac{\mu_p+\mu_n}{2m\omega} + \frac{i\vec{\sigma}\cdot\vec{q}_{\gamma}\,\vec{q}_{\gamma}\cdot\vec{\epsilon}}{m\omega}\right] \qquad (9)$$

Here $h^{(++)} = \frac{1}{3}(h_{11} + 2h_{13} + 2h_{31} + 4h_{33})$

$h^{(+-)} = \frac{1}{3}(h_{11}^{*} - h_{13} + 2h_{31} - 2h_{33})$

and the other quantities have the same significance as in (1).

The angular distributions for the various cases are given by

$$\frac{d\sigma_{\pi.c.}^{(1)}(\pi^0)}{d\Omega} = \frac{e^2 f^2 q}{k}\Big[\frac{2\lambda^2 k^2 \sin^2\theta \sin^2\delta_{33}}{q^4}$$

$$+ \frac{2}{q} q^2 k^2 \sin^2\delta_{33}(F_M + F_Q)^2 \sin^2\theta$$

$$+ \frac{(\mu_p+\mu_n)^2}{8m^2\omega^2} k^2 q^2 \sin^2\theta + \frac{1}{2}\frac{q^4\cos^2\theta\sin^2\theta}{m^2\omega^2}$$

$$+ \frac{1}{3}\frac{\lambda(\mu_p+\mu_n)}{m\omega}\cos\delta_{33}\sin\delta_{33}\frac{k^2}{q}\sin^2\theta$$

$$- \frac{2}{3}\frac{\lambda k}{m\omega}\cos\delta_{33}\sin\delta_{33}\cos\theta\sin^2\theta$$

$$- \frac{1}{3}\frac{(\mu_p+\mu_n)}{m\omega} k^2 q^2 \sin^2\delta_{33}\sin^2\theta(F_Q+F_M)$$

$$- \frac{1}{2}\frac{(\mu_p+\mu_n)}{m^2\omega^2} k q^3 \cos\theta\sin^2\theta$$

$$+ \frac{2kq^3}{3m\omega}\sin^2\delta_{33}\cos\theta\sin^2\theta(F_Q+F_M)\Big] \qquad (10)$$

$$\frac{d\sigma^{(2)}_{r.c.}(\pi^0)}{d\Omega} = \frac{e^2 f^2 q^5}{2k m^2 \omega^2} \sin^4\theta \qquad (11)$$

$$\frac{d\sigma^{(1)}_{l.c.}(\pi^0)}{d\Omega} = \frac{e^2 f^2 q}{k}\left[\frac{2}{9}\frac{\lambda^2 k^2 \sin^2\theta \sin^2\delta_{33}}{q^4}\right.$$

$$+\frac{2}{9} k^2 q^2 \sin^2\delta_{33}\sin^2\theta\left(F_Q - \frac{1}{3}F_M\right)^2$$

$$+\frac{(\mu_p+\mu_n)^2}{8m^2\omega^2} k^2 q^2 \sin^2\theta$$

$$+\frac{1}{2m^2\omega^2} q^4 \cos^2\theta \sin^2\theta$$

$$+\frac{1}{3}\lambda\frac{(\mu_p+\mu_n)}{m\omega}\frac{k^2}{q}\cos\delta_{33}\sin\delta_{33}\sin^2\theta$$

$$-\frac{2\lambda}{3m\omega} k \cos\delta_{33}\sin\delta_{33}\cos\theta\sin^2\theta$$

$$+\frac{(\mu_p+\mu_n)}{3m\omega} k^2 q^2 \sin^2\delta_{33}\sin^2\theta\left(F_Q - \frac{1}{3}F_M\right)$$

$$\left.-\frac{4}{27} k^2 q^2 \sin\delta_{33}\cos\delta_{33}\sin^2\theta F_Q F_M\right.$$

$$- \frac{4kq^3}{3m\omega} \sin^2\delta_{33} \cos\theta \sin^2\theta \left(F_Q - \frac{1}{3}F_M\right)$$

$$- \frac{\mu_p+\mu_n}{2m^2\omega^2} kq^2 \cos\theta \sin^2\theta \Bigg] \qquad (12)$$

$$\frac{d\sigma^{(\omega)}_{l.c.}(\pi^\circ)}{d\Omega} = \frac{e^2 f^2 q}{k} \Bigg[\frac{8}{9} \frac{\lambda^2 k^2 \cos^2\theta \sin^2\delta_{33}}{q^4}$$

$$+ \frac{8}{9} k^2 q^2 \sin^2\delta_{33} \cos^2\theta \left(F_Q - \frac{1}{3}F_M\right)^2$$

$$+ \frac{(\mu_p+\mu_n)^2}{2m^2\omega^2} k^2 q^2 \cos^2\theta + \frac{1}{2m^2\omega^2} q^4 \sin^4\theta$$

$$+ \frac{1}{2m^2} (\mu_p+\mu_n)^2 \omega^2 + \frac{8}{9}(\delta_1 - \delta_3)^2 F_S^2$$

$$+ \frac{4\lambda}{3m\omega} \cos\delta_{33} \sin\delta_{33} \frac{k\cos^2\theta}{q^2}$$

$$+ \frac{4\lambda}{3m\omega} \cos\delta_{33} \sin\delta_{33} k\cos\theta \sin^2\theta$$

$$- \frac{4\lambda(\mu_p+\mu_n)}{3m} \frac{\omega k}{q^2} \cos\delta_{33} \sin\delta_{33} \cos\theta$$

$$
\begin{aligned}
&- \frac{8\lambda(\delta_1-\delta_3)F_S}{q}\cos\delta_{33}\sin\delta_{33}\,\frac{k\cos\theta}{q^2} \\
&+ \frac{8\lambda(\delta_1-\delta_3)F_S}{q}\sin^2\delta_{33}\,\frac{k\cos\theta}{q^2} \\
&- \frac{8}{q}(\delta_1-\delta_3)F_S\cos\delta_{33}\sin\delta_{33}\,kq\cos\theta\left(F_Q-\tfrac{1}{3}F_M\right) \\
&- \frac{8}{q}(\delta_1-\delta_3)F_S\sin^2\delta_{33}\,kq\cos\theta\left(F_Q-\tfrac{1}{3}F_M\right) \\
&- \frac{4}{3}(\delta_1-\delta_3)F_S\,\frac{(\mu_p+\mu_n)}{m\omega}\,kq\cos\theta \\
&- \frac{4}{3}(\delta_1-\delta_3)F_S\,\frac{1}{m\omega}\,q^2\sin^2\theta \\
&+ \frac{4}{3}(\delta_1-\delta_3)F_S\,\frac{(\mu_p+\mu_n)\omega}{m} \\
&+ \frac{4}{3}\,\frac{(\mu_p+\mu_n)}{m\omega}\left(F_Q-\tfrac{1}{3}F_M\right)k^2q^2\cos^2\theta\sin^2\delta_{33} \\
&+ \frac{8}{3m\omega}\left(F_Q-\tfrac{1}{3}F_M\right)kq^3\sin^2\theta\cos\theta\sin^2\delta_{33} \\
&- \frac{4}{3}\,\frac{\mu_p+\mu_n}{m}\,\omega\left(F_Q-\tfrac{1}{3}F_M\right)kq\cos\theta\sin^2\delta_{33}\Big]
\end{aligned}
\tag{13}
$$

There is an enormous simplification if we retain only the dominant magnetic dipole ***x (phase shift) term in the matrix element (the first two terms of (9)). The various cross-sections then look as follows:

$$\frac{d\sigma^{(1)}_{r.c.}(\pi^0)}{d\Omega} = \frac{2\lambda^2 e^2 f^2 k \sin^2\theta \sin^2\delta_{33}}{q^3} \tag{14}$$

$$\frac{d\sigma^{(2)}_{r.c.}(\pi^0)}{d\Omega} = 0 \tag{15}$$

$$\frac{d\sigma^{(1)}_{l.c.}(\pi^0)}{d\Omega} = \frac{2\lambda^2 e^2 f^2 k \sin^2\theta \sin^2\delta_{33}}{9q^3} \tag{16}$$

$$\frac{d\sigma^{(2)}_{l.c.}(\pi^0)}{d\Omega} = \frac{8\lambda^2 e^2 f^2 k \cos^2\theta \sin^2\delta_{33}}{9q^3} \tag{17}$$

Thus the angle dependence of the differential cross-sections is particularly simple. The differential cross-section for a right circular photon is proportional to $\sin^2\theta$ while that for a left circular photon is proportional to $\sin^2\theta$ for the no-spin-flip case and $\cos^2\theta$ for spin-flip. Further in this approximation there is no spin-flip for the nucleon if the incident photon is right circularly polarized. These features in the angular distribution can perhaps be checked with experiment. Also due to the vanishing of the spin flip

amplitude in the case of a right circular photon, the final nucleon cannot be polarised for this case, i.e. $P^{(k,\theta)}_{r.c.} = 0$. We also note that if we have the nucleon polarized opposite to the direction of propagation of the photon it is the right circular photon that will yield a non-zero spin flip amplitude while the spinflip amplitude arising from a left-circular photon is zero which means that $P^{(k,\theta)}_{l.c.} = 0$ in this case. This remark also brings out the need for polarising the target nucleon. Finally we wish to remark that recently considerable interest on the polarization of the recoil nucleon in π^0 production at higher energies has been evinced with a view to determining the parity of higher resonances other than the 3/2, 3/2 resonance[1].

3. Photo production of pion pairs[2]

a) We shall next consider the polarization effects in process

$$\gamma + p \longrightarrow p + \pi^+ + \pi^-$$

, the matrix element for which is given by equation( ) ch.

Again fixing the spin of the nucleon along the direction of propagation of the photon, we can derive the angular distribution for the various cases. The angular distributions of the π^+ for the various cases obtained by integrating over the angles of π^- are given by

1) J.J.Sakurai, Phys. Rev. Letters, 1, 258 (1958)

2) S.K.Srinivasan and K. Venkatesan, Proc.Ind. Acad. 50, 392 (1959)

$$\frac{d^2\sigma^{(1)}_{\pi.c}}{d\Omega\, d\omega_1} = \frac{q_1 q_2 e^2 f^4}{2\pi k}\left\{\frac{(\mu_p-\mu_n)^2 k^2}{3m^2 f^4}\left[\frac{\sin^2\delta(q_1)\, q_2^2}{q_1^4\,\omega_2^2}\left(\frac{1}{2}-\frac{7}{18}\sin^2\theta_2\right)\right.\right.$$

$$\left.+\frac{\sin^2\delta_2(q_2)\, q_1^2}{4 q_2^4\,\omega_1^2}\sin^2\theta_2\right]+\frac{8 q_2^2}{\omega_2^2\left(1+\frac{\omega_1}{m}\right)^2}$$

$$\times\left[\left(2+q_1^4 A_3-2q_1^2 A_1\right)\sin^2\theta_2+\left(k^2 q_1^2 A_1+q_1^4 A_2-2k q_1^3 A_4\right)\cos^2\theta_2\right]$$

$$-\frac{(\mu_p-\mu_n)^2 k^2 \sin\delta(q_1)\sin\delta(q_2)\cos[\delta(q_1)-\delta(q_2)]}{9\omega_1\omega_2 m^2 f^4 q_1 q_2}\sin^2\theta_2$$

$$+\frac{(\mu_p-\mu_n) k q_2^2 \sin 2\delta(q_1)\left[2q_1 A_5\sin^2\theta_2+3(kA_6-q_1 A_5)\cos^2\theta_2\right]}{6 q_1\omega_2^2 m f^2\left(1+\frac{\omega_1}{m}\right)}$$

$$-\frac{(\mu_p-\mu_n) k q_1^2 \sin 2\delta(q_2)}{2 q_2\omega_1\omega_2\left(1+\frac{\omega_1}{m}\right) m f^2}A_5\sin^2\theta_2$$

$$+\frac{4(\mu_p-\mu_n)k}{3\omega_1 m f^2\left(1+\frac{\omega_2}{m}\right)(a+b\cos\theta_2)}\left[\frac{q_2^2\sin 2\delta(q_1)}{2 q_1\omega_2}\right.$$

$$\times\left(q_2\sin\theta_2\cos\theta_2-\frac{2}{3}(k-q_2\cos\theta_2)\sin^2\theta_2\right)$$

$$\left.+\frac{q_1^2\sin 2\delta(q_2)}{2 q_2\omega_1}(k-q_2\cos\theta_2)\sin^2\theta_2\right]$$

$$-\frac{1}{a+b\cos\theta_2}\cdot\frac{16 q_2^2 q_1^3 A_5\sin\theta_2(k\sin\theta_2+\cos\theta_2)}{\omega_1\omega_2\left(1+\frac{\omega_1}{m}\right)\left(1+\frac{\omega_2}{m}\right)}$$

$$+\frac{16q_1^2\sin^2\theta_2}{(a+b\cos\theta_2)^2}\left[k^2+q_2^2(1+\sin^2\theta_2)-2kq_2\cos\theta_2\right]\Bigg\}$$

$$\frac{d^2\sigma^{(2)}_{\pi.c}}{d\Omega_\pi d\omega_1}=\frac{q_1q_2e^2f^4}{2\pi k}\Bigg\{\frac{(\mu_p-\mu_n)^2k^2}{6m^2f^4}$$

$$\times\left[\frac{\sin^2\delta(q_1)q_2^2}{9\omega_2^2q_1^4}+\frac{\sin^2\delta(q_2)q_1^2}{\omega_1^2q_2^4}\right]\sin^2\theta_2$$

$$+\frac{8q_1^2q_2^2}{\omega_2^2\left(1+\frac{\omega_1}{m}\right)^2}\left[q_1^2A_3\cos^2\theta_2+\left(k^2A_1+q_1^2A_2-2kq_1A_4\right)\sin^2\theta_2\right]$$

$$+\frac{(\mu_p-\mu_n)^2k^2\sin\delta(q_1)\sin\delta(q_2)\cos\left[\delta(q_1)-\delta(q_2)\right]}{9\omega_1\omega_2q_1q_2m^2f^4}\sin^2\theta_2$$

$$-\frac{(\mu_p-\mu_n)kq_2^2\sin2\delta(q_1)}{3q_1\omega_2^2\left(1+\frac{\omega_1}{m}\right)mf^2}\left[kA_6-q_1A_5\right]\sin^2\theta_2$$

$$-\frac{(\mu_p-\mu_n)kq_1^2\sin2\delta(q_2)}{q_2\omega_1\omega_2\left(1+\frac{\omega_2}{m}\right)mf^2}\left[kA_6-q_1A_5\right]\sin^2\theta_2$$

$$-\frac{4}{3}\frac{(\mu_p-\mu_n)k(q_2\cos\theta_2-k)\sin^2\theta_2}{\omega_1\left(1+\frac{\omega_2}{m}\right)mf^2(a+b\cos\theta)}\times\left[\frac{q_2^2\sin2\delta(q_1)}{3\omega_2q_1}+\frac{q_1^2\sin2\delta(q_2)}{q_2\omega_1}\right]$$

$$+\frac{16q_1^2q_2^2\left[(kq_2A_6-q_1q_2A_5)\sin\theta_2\cos\theta_2+k(q_1A_5-kA_6)\sin\theta_2\right]}{\omega_1\omega_2\left(1+\frac{\omega_1}{m}\right)\left(1+\frac{\omega_2}{m}\right)(a+b\cos\theta_2)}$$

$$+\frac{16q_1^2q_2^2}{3\omega_1^2\left(1+\frac{\omega_2}{m}\right)(a+b\cos\theta_2)^2}\left[2k^2+q_2^2(1+\cos^2\theta_2)-4kq_1\cos\theta_2\right]\sin^2\theta_2\Bigg\}$$

$$\frac{d^2\sigma^{(1)}_{l.c}}{d\Omega_2 + d\omega_1} = \frac{q_1 q_2 e^2 f^4}{2\pi k} \left\{ \frac{(\mu_p - \mu_n)^2 k^2}{108 m^2 f^4} \right.$$

$$\times \left[\frac{2\sin^2\delta(q_1) q_2^2 \cos^2\theta_2}{\omega_2^2 q_1^4} + \frac{\sin^2\delta(q_2) q_1^2}{\omega_1^2 q_2^4} (1 + 7\cos^2\theta_2) \right]$$

$$+ \frac{8 q_1^2 q_2^2}{\omega_1^2 \left(1 + \frac{\omega_1}{m}\right)^2} \left[(k^2 A_1 + q_1^2 A_2 - 2k q_1 A_4) \cos^2\theta_2 + q_1^2 A_3 \sin^2\theta_2 \right]$$

$$+ \frac{32 q_1^2}{3\omega_1^2 \left(1 + \frac{\omega_2}{m}\right)^2} - \frac{4(\mu_p - \mu_n) q_2 \sin 2\delta(q_1) \cos\theta_2}{9\omega_1 \omega_2 q_1 \left(1 + \frac{\omega_2}{m}\right) m f^2}$$

$$+ \frac{16 q_1^2 q_2}{\omega_1 \omega_2 \left(1 + \frac{\omega_1}{m}\right)\left(1 + \frac{\omega_2}{m}\right)} \left[(k A_6 - q_1 A_5) \cos\theta_2 \right]$$

$$+ \frac{2(\mu_p - \mu_n)^2 k^2 \sin\delta(q_1) \sin\delta(q_2) \cos\left[\delta(q_1) - \delta(q_2)\right] \cos^2\theta_2}{27 \omega_1 \omega_2 m^2 f^4 q_1 q_2}$$

$$- \frac{(\mu_p - \mu_n) k^2 q_2^2 \sin 2\delta(q_1) \left[k A_6 - q_1 A_5\right] \cos^2\theta_2}{3\omega_2^2 q_1 \left(1 + \frac{\omega_1}{m}\right) m f^2}$$

$$- \frac{2(\mu_p - \mu_n) k q_1^2 \sin 2\delta(q_2) \left[k A_6 - q_1 A_5\right] \cos^2\theta_2}{3\omega_1 \omega_2 q_2 \left(1 + \frac{\omega_1}{m}\right) m f^2}$$

$$- \frac{8(\mu_p - \mu_n) k q_1^2 \sin 2\delta(q_2) \cos\theta_2}{9\omega_1^2 q_2 \left(1 + \frac{\omega_2}{m}\right) m f^2}$$

$$+ \frac{2(\mu_p - \mu_n) k}{9\omega_1 \left(1 + \frac{\omega_2}{m}\right) m f^2 (a + b\cos\theta_2)} \left[\frac{2 q_2^2 \sin^2\theta_2 \cos\theta_2 \sin 2\delta(q_1)}{q_1 \omega_2} \right.$$

$$\left. \left. - \frac{q_1^2 \sin 2\delta(q_2)}{\omega_1 q_2} \times \left[k - q_2 \cos\theta_2 - \frac{4}{3} q_2 \sin\theta_2 \cos\theta_2 \right] \right] \right]$$

$$- \frac{16 q_1^2 q_2^2 k (q_2 A_6 \cos\theta_2 - q_1 A_5) \sin^2\theta_2}{\omega_1 \omega_2 (1+\frac{\omega_1}{m})(1+\frac{\omega_2}{m})(a+b\cos\theta_2)} - \frac{64 q_1 q_2^2 \sin^2\theta_2}{a+b\cos\theta_2}$$

$$+ \frac{8 q_1^2 q_2^2 \sin^2\theta_2}{(a+b\cos\theta_2)^2} \left[k^2 + 2 q_2^2 (\sin^2\theta_2 - k q_2 \cos\theta_2 + 1) \right] \Bigg\}$$

$$\frac{d^2 \sigma^{(2)}_{l.c}}{d\Omega_+ \, d\omega_1} = \frac{q_1 q_2 e^2 f^4}{2\pi k} \left\{ \frac{(\mu_p - \mu_n)^2 k^2}{54 m^2 f^4} \right.$$

$$\times \left[\frac{2 \sin^2\delta(q_1) q_2^2}{\omega_2^2 q_1^4} (1 + 14 \sin^2\theta_2) \right.$$

$$\left. + \frac{\sin^2\delta(q_2) q_1^2}{\omega_1^2 q_2^4} (2 - \sin^2\theta_2) \right]$$

$$+ \frac{8 q_2^2}{\omega_2^2 (1+\frac{\omega_1}{m})^2} \left[(2 + q_1^4 A_3 - 2 q_1^2 A_6) \cos^2\theta_2 + (q_1^2 A_2 + k^2 A_1 - 2 k q_2) \times \sin^2\theta_2 \right]$$

$$- \frac{(\mu_p - \mu_n)^2 k^2 \sin\delta(q_1) \sin\delta(q_2) \cos[\delta(q_1) - \delta(q_2)] (1+\cos^2\theta_2)}{27 \omega_1 \omega_2 q_1 q_2 m^2 f^4}$$

$$+ \frac{4 (\mu_p - \mu_n) k q_2^2 \sin 2\delta(q_1)}{9 \omega_1 \omega_2 (1+\frac{\omega_2}{m}) m f^2 q_1} \cos\theta_2$$

$$+ \frac{2 (\mu_p - \mu_n) k q_1^2 \sin 2\delta(q_2) A_5 \cos\theta_2}{3 q_2 \omega_1 \omega_2 (1+\frac{\omega_1}{m}) m f^2} + \frac{8}{9} \frac{(\mu_p - \mu_n) k \sin^2\theta_2}{\omega_1 (1+\frac{\omega_2}{m}) m f^2 (a+b\cos\theta_2)}$$

$$\times \left[\frac{q_1^2 (2 q_1 \cos\theta_2 - k) \sin 2\delta(q_2)}{2 q_2 \omega_1} - \frac{q_2^2 (k - q_2 \cos\theta_2) \sin 2\delta(q_1)}{2 q_1 \omega_2} \right]$$

$$- \frac{16 q_1^2 q_2 \cos\theta_2}{\omega_1 \omega_2 (1+\frac{\omega_1}{m})(1+\frac{\omega_2}{m})} \left[\frac{A_5 q_1 q_2^2 \sin^2\theta_2 - q_1 A_5 (a+b\cos\theta_2)}{(a+b\cos\theta_2)} \right]$$

$$-\frac{16q_1^2 q_2 \cos\theta_2}{\omega_1\omega_2\left(1+\frac{\omega_1}{m}\right)\left(1+\frac{\omega_2}{m}\right)} \frac{\left[A_5\, q_1 q_2^2 \sin^2\theta_2 - q_1 A_5 (a+b\cos\theta_2)\right]}{(a+b\cos\theta_2)}$$

$$+\frac{16q_1^2 q_2 \sin^2\theta_2}{3\omega_1^2\left(1+\frac{\omega_2}{m}\right)^2 (a+b\cos\theta_2)^2}$$

$$\left[2q_2(k^2+2) - 2q_1 + 2q_2(q_2-4k)\cos^2\theta_2 + q_2^3\sin^2\theta_2\right.$$
$$\left. - 2q_1(a+b\cos\theta_2)\right]$$

Here

$$A_1 = \frac{-4}{b^2} + \frac{2a}{b^3}\log\frac{a+b}{a-b}$$

$$A_2 = \frac{4}{3b^2} - \frac{8a^2}{b^4} - \frac{2a(b^2-2a^2)}{b^5}\log\frac{a+b}{a-b}$$

$$A_3 = \frac{-16}{3b^2} + \frac{8a^2}{b^4} - \frac{4a(a^2-b^2)}{b^5}\log\frac{a+b}{a-b}$$

$$A_4 = \frac{6a}{b^3} + \log\frac{a+b}{a-b}\cdot\frac{b^2-3a^2}{b^4}$$

$$A_5 = \frac{4}{3b^2} - \frac{2a^2}{b^3} + \frac{a(a^2-b^2)}{b^4}\log\frac{a+b}{a-b}$$

$$A_6 = \frac{2a}{b^2} + \frac{b^2-a^2}{b^3}\log\frac{a+b}{a-b}$$

$$a = k^2 + \omega_1^2 \; ; \; b = -2kq_1$$

4. Electro-pion production from nucleons

We shall calculate the matrix elements for the emission of pions when polarized electrons collide with polarized nucleons. We can again investigate (1) whether the type of polarization of the projectile electron affects the angular distribution of pion emission even if the initial nucleon is unpolarized and (2) the degree of polarization of the final nucleon if initially it was unpolarized, in a given direction.

Fubini et al [1] and recently Dennery [2] have analysed the process by the method of dispersion relations, the former using the single variable dispersion relations and the latter, the Mandelstam representation. In both cases the matrix element for electro-pion production is reduced to that for the photo-production of a pion by a virtual photon. This can be seen as follows. The matrix element for the electron of four-momentum s_1 impinging on a nucleon of momentum p_1 by which it goes over into an electron of momentum s_2, producing a pion of momentum q, the final nucleon having the momentum p_2 is given by

$$\langle p_2, s_2, q | S | p_1, s_1 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \iiint \ldots dx_1 dx_2 \ldots \left(\Phi_{p_2} a(s_2) c(q), \right.$$

$$\left. \cdot T\left(H_{int}(x_1) \ldots H_{int}(x_n), a^{+}(s_1) \Phi_{p_1} \right) \right. \tag{23}$$

1) S. Fubini, Y. Nambu and V. Wataghin, Phys. Rev., 111, 329 (1958).

2) P. Dennery, Phys. Rev., 124, 2000 (1961).

The interaction Hamiltonian density is given by

$$H_{int}(x) = ie\,\bar{\psi}_e(x)\,\gamma_\mu\,\psi_e(x)\,A_\mu(x) + ie\,\bar{\psi}_N(x)\,\gamma_\mu\,\psi_N(x)\,A_\mu(x) + \text{renormalisation terms} \tag{24}$$

where ψ_N and ψ_e are the field operators of the nucleon and electron respectively. Commuting $a(s_2)$ to the right and $a^+(s_1)$ to the left but taking one of them through the current operator arising from the other (only in which case we will have a matrix element which is of the lowest order in the electromagnetic coupling constant e), we obtain, on going over to the Heisenberg representation,

$$\langle p_2, s_2, q\nu | S | p_1, s_1 \rangle = e\left(\frac{m^2}{E_1 E_2}\right)^{1/2} \cdot \bar{u}(s_2)\,\gamma_\mu\,u(s_1) \int dx\, e^{i(s_1 - s_2)x} \cdot \langle p_2\, q\nu | A_\mu(x) | p_1 \rangle \tag{25}$$

Performing the x integration, using translational invariance for $A_\mu(x)$ we can remove out a delta function $\delta(p_1 + s_1 - p_2 - s_2 - q\nu)$ corresponding to over-all energy momentum conservation. The matrix element $\langle p_2\, q\nu | A_\mu(0) | p_1 \rangle$ can be connected with $\langle p_2\, q\nu | j_\mu(0) | p_1 \rangle$ where $j_\mu(0)$ is the photon current operator, on differentiating by $\Box$.

Finally we get

$$\langle p_2, s_2, q | T | p_1, s_1 \rangle = -ie\left(\frac{m^2}{E_1 E_2}\right)^{\frac{1}{2}} \epsilon_\mu \langle p_2 q | j_\mu(0) | p \rangle \qquad (26)$$

The matrix element on the right corresponds to photo-pion production by a virtual photon of polarization, $\vec{\epsilon} = \frac{\bar{u}(s_2)\vec{\gamma}u(s_1)}{(s_1 - s_2)^2}$

The importance of a study of the process can be said to stem from the virtual nature of the photon since in the inhomogeneous terms of the dispersion relations, nuclear form factors which depend on the invariant momentum transfer $\lambda^2 = (s_1 - s_2)^2$ enter. These form factors are exactly the same as the ones obtained by Hofstadter[1] in elastic electron-nucleon scattering. Since the electro-pion production involves both the proton and neutron form factor effects it can offer an independent method, different from the electron-deutron scattering or electro-distintegration of the deuteron for determining the neutron form factors.

Let us now consider the electro-production of a ~~neutral~~ charged pion from a nucleon, the matrix element of which is given in the static approximations and on retaining, only the dominant magnetic moment term by

$$M = \frac{f}{\sqrt{\omega}}\left[\frac{1}{1+\frac{\omega}{m}}\left\{e\,(\lambda^2)\, i\vec{\sigma}\cdot\vec{\epsilon} + ie\frac{(\vec{k}-\vec{q})(2\vec{q}-\vec{k})\cdot\vec{\epsilon}}{\lambda^2 - 2k\cdot q}\right.\right.$$

$$\left.\left. -\frac{1}{3}\left\{2\vec{q}\cdot(\vec{k}\times\vec{\epsilon}) - i\vec{\sigma}\cdot\vec{q}\times(\vec{k}\times\vec{\epsilon})\right\}\right\}\frac{\mu\,(\lambda^2)\, e^{i\delta_{33}}\sin\delta_{33}}{2f^2 q^3} \qquad (27)$$

where f is the pesudovector renormalized rationalized pion-nucleon coupling constant, q and k the momenta of the pion and virtual photon respectively, δ_{33} is the pion-nucleon phase shift for the 33 resonant state and $\mu^v(\lambda^2)$

1) R. Hofstadter, Revs. Modern Phys., 28, 214 (1956)

the isovector magnetic moment form factor which is given in terms of the magnetic moment form factors of the protons $(F_2^{\,p})$ and neutron, $(F_2^{\,n})$, by

$$\mu^{v}(\lambda^2) = \mu_p F_2^{\,p}(\lambda^2) - \mu_n F_2^{\,n}(\lambda^2)$$

where μ_p and μ_n are the magnetic moments of the proton and neutron. $\vec{\epsilon}$ is the polarization vector of the virtual photon and is equal to $\dfrac{ie\bar{u}\vec{\gamma}u}{(s_1 - s_2)^2} = \dfrac{ie\bar{u}\gamma u}{k^2}$

Equation (27) contains only the matrix elements for the case where the polarization vector ϵ has only space components; the matrix element for the time component of the four-current can be obtained by using the continuity equation

$$k_0 j_0 - \vec{k} \cdot \vec{j} = 0 \qquad (28)$$

From (27) we can now find the matrix elements for different polarization states of the nucleon and electron. We designate the various combinations of the polarisation of the particles by M ~~and~~ ~ where i represents the polarisation of the incident electron and j that of the outgoing electron, k that of the initial nucleon and l that of the final nucleon. Each of these indices can take two values designated by F and B . In the case of the electron (which we take to be completely relativistic since at the energies considered, namely $\sim$ 600 Mev , the rest energy of the electron is negligible compared to its kinetic energy) F and B refers respectively

to spin in the direction of and opposite to the electron momentum. In the case of the nucleon which is static, F and B refer to spin along the z -axis, which we take to be the direction of the incident electron and the negative z - axisx respectively. The spinors corresponding to spin parallel and antiparallel to the directions of the incident and final electrons of momenta $\vec{s}_1$ and $\vec{s}_2$ respectively are required in order to compute $\vec{\epsilon}$. Since we are interested in production matrix elements when the initial and final electron are with their spins in either 'forward' or 'backward' direction with respect to their momenta it is clear that we should use spinors which also satisfy the equation

$$\frac{\vec{\sigma}\cdot\vec{s}}{|\vec{s}|}\, u(\vec{s}) = \pm u(\vec{s}) \qquad (29)$$

The two eigen functions will then describe respectively the electron whose spin is parallel or antiparallel to its direction of motion. We give below the spinors we shall be using in our calculations in which we have made the approximation that s_z equals $|\vec{s}|$ whenever they occur together and the spinors are normalized as

$$\bar{u}(\vec{s})\, u(\vec{s}) = 2m_e \qquad (30)$$

1. Electron spin parallel to its momentum

$$u(s) = \sqrt{E+m_e} \begin{pmatrix} 1 \\ s_+/(2|\vec{s}|) \\ |\vec{s}|/(E+m_e) \\ s_+/2(E+m_e) \end{pmatrix} \quad (31)$$

2. Electron spin antiparallel to its momentum

$$u(s) = \sqrt{E+m_e} \begin{pmatrix} -s_-/(2|\vec{s}|) \\ 1 \\ (s_-)/2(E+m_e) \\ -|\vec{s}|/(E+m_e) \end{pmatrix} \quad (32)$$

Using the above we can calculate $\vec{\epsilon}$ for the four different cases of polarisation states of the two electrons. Similarly there will be four different types of matrix elements corresponding to the various nucleon states. Hence we arrive at the following sixteen different matrix elements for the electroproduction of neutral pions from nucleons.

1. $M_{FFFF} = iA\epsilon_z + iB(k_z - q_{1z})\alpha_1 + C[i(k_z\eta_1 - \beta\epsilon_z) + 2\delta_1]$

2. $M_{FFBB} = -iA\epsilon_z - iB(k_z - q_{1z})\alpha_1 - C[i(k_z\eta_1 - \beta\epsilon_z) - 2\delta_1]$

3. $M_{FFFB} = iA\epsilon_+ + iB(\vec{k} - \vec{q})_+ \alpha_1 + iC[k_+\eta_1 - \beta\epsilon_+]$

4. $M_{FFBF} = iA\epsilon_- + iB(\vec{k} - \vec{q})_- \alpha_1 + iC[k_-\eta_1 - \beta\epsilon_-]$

5. $M_{BBFF} = iA\epsilon_z^* + iB(k_z - q_{2z})\alpha_2 + C[i(k_z\eta_2 - \beta\epsilon_z^*) + 2\delta_2]$

6. $M_{BBBB} = -iA\epsilon_z^* - iB(k_z - q_{2z})\alpha_2 - C[i(k_z\eta_2 - \beta\epsilon_z^*) - 2\delta_2]$

7. $M_{BBFB} = iA\epsilon_- + iB(\vec{k}-\vec{q})_+ \alpha_2 + iC\left[k_+\eta_2 - \beta\epsilon_-\right]$

8. $M_{BBBF} = iA\epsilon_+ - iB(\vec{k}-\vec{q})_- \alpha_2 + iC\left[k_-\eta_2 - \beta\epsilon_+\right]$

9 to 16. $M_{FBFF} = M_{FBBB} = M_{FBFB} = M_{FBBF} = M_{BFFF} = M_{BFBB} = M_{BFFB} = M_{BFBF} = 0$

In the above

$$A = \frac{f}{\sqrt{\omega}} \cdot \frac{1}{1+\frac{\omega}{m}}\, e^V(\lambda^2);$$

$$e^V(\lambda^2) = e\left[F_1^p(\lambda^2) - F_1^n(\lambda^2)\right]$$

$$B = \frac{f}{\sqrt{\omega}} \cdot \frac{1}{1+\frac{\omega}{m}}\, e \cdot \frac{1}{\lambda^2 - 2k.q}$$

$$C = \frac{f}{\sqrt{\omega}} \cdot \frac{\mu^V(\lambda^2)}{6f^2q^3} \cdot e^{i\delta_{33}} \sin\delta_{33}$$

$$\epsilon_x = \sqrt{E_1E_2}\left[\frac{s_{1+}}{E_1} + \frac{s_{2-}}{E_2}\right]$$

$$\epsilon_y = -i\sqrt{E_1E_2}\left[\frac{s_{1+}}{E_1} - \frac{s_{2-}}{E_2}\right]$$

$$\epsilon_z = 2\sqrt{E_1E_2}\left[1 - \frac{s_{1+}\, s_{2-}}{4E_1E_2}\right]$$

$$s_{\pm} = s_x \pm i s_y \quad ; \quad \epsilon_{\pm} = \frac{\epsilon_x \pm i \epsilon_y}{\sqrt{2}}$$

$$k_{\pm} = \frac{k_x \pm i k_y}{\sqrt{2}} \quad ; \quad q_{\pm} = \frac{q_x \pm i q_y}{\sqrt{2}}$$

$$\alpha_1 = 2\sqrt{E_1 E_2}\left[\frac{s_{2-}(2q_+ - k_+)}{2E_2} + \frac{s_{1+}(2q_- - k_-)}{2E_1} + (2q_z - k_z)\left(1 - \frac{s_{1+}s_{2-}}{4E_1E_2}\right)\right]$$

$$\eta_1 = 2\sqrt{E_1 E_2}\left[\frac{s_{2-}q_+}{2E_2} + \frac{s_{1+}q_-}{2E_1} + q_z\left(1 - \frac{s_{1+}s_{2-}}{4E_1E_2}\right)\right]$$

$$\delta = 4\sqrt{E_1 E_2}\left[-i(s_{1z} - s_{2z})\left\{\frac{s_{2+}q_+}{2E_2} - \frac{s_{1+}q_-}{2E_1}\right\}\right.$$

$$+ (s_{1x} - s_{2x})\left\{i q_z\left(\frac{s_{2-}}{2E_2} - \frac{s_{1+}}{2E_1}\right)\right\}$$

$$\left. + (s_{1y} - s_{2y})\left\{q_x\left(1 - \frac{s_{1+}s_{2-}}{4E_1E_2}\right) - q_z\left(\frac{s_{2-}}{2E_2} + \frac{s_{1+}}{2E_1}\right)\right\}\right]$$

$$\alpha_2 = \alpha_1^* \quad ; \quad \eta_2 = \eta_1^* \quad ; \quad \delta_2 = \delta_1^* \qquad (34)$$

To investigate whether the state of polarisation of the incident electron affects the pion emission cross-sections without regard to the polarizations of the other particles we find the total matrix element for electropion production with the incident electronpolarised in the forward direction

$$M_{I} = M_{FFFF} + M_{FFFB} + M_{FFBB} + M_{FFBF}$$

$$+ M_{FBFF} + M_{FBFB} + M_{FBBB} + M_{FBBF}$$

$$= i\sqrt{2}\left[A\epsilon_x + B(k_x - q_x)\alpha_1 + C(k_x\eta_1 - \beta\epsilon_x - i2\sqrt{2}\,\delta_1)\right] \quad (35)$$

Similarly the matrix element for pion production by an electron polarised with its spinxx antiparallel to its direction of motion is

$$M_{II} = M_{BBFF} + M_{BBFB} + M_{BBBF} + M_{BBBB}$$

$$+ M_{BFFF} + M_{BFFB} + M_{BFBF} + M_{BFBB}$$

$$= i\sqrt{2}\left[A\epsilon_x + B(k_x - q_x)\alpha_1^* + C(k_x\eta_1^* - \beta\epsilon_x - i2\sqrt{2}\,\delta_1^*)\right] \quad (36)$$

We see therefore that $M_I \cdot M_I^*$ is identically equal to $M_{II}\, M_{II}^*$ which implies that the cross-section for electro-pion production does not depend on the state of polarization of the incident electron if the nucleon is unpolarized. But if both the electron and the nucleon are polarized we see from the expression (33) themselves that the cross-sections will be different for the different cases.

We also compute the degree of polarization of the final nucleon if the initial nucleon is polarized with its spin up and the incident electron is polarized with its spin along the direction of motion. Using (33), this is given by

$$P = 2\,\mathrm{Im}\{i[A\epsilon_z + B(k_z - q_{1z})\alpha_1 + C(k_z\eta_1 - \epsilon_2\beta - 2i\delta_1)] \times [A\epsilon_+ + B(\vec{k}-\vec{q}_1)_+\alpha_1 + C(k+\eta_1 - \beta\epsilon_+)] \times [|M_{FFFF}|^2 + |M_{FFFB}|^2]\} \quad (37)$$

From (34) we see that this will in general be different from zero and can take in any value between 0 and 1.

Conclusion

We see from a study of the three processes considered in this chapter that the angular distributions are quite sensitive to the state of polarisation of the incoming particles. A measurement of the degree of polarization of the final nucleon when the particle incident on the initial nucleon is is also polarized may throw forther light on the correctness of any particular theory.

Recently Chamberlain et al[1] have produced a target containing polarized protons. Only one proton in thirty was polarized but with coincidence methods (e.g. for proton-proton scattering) even this was found to be useful. A much larger

1) E. Segre, Lectures delivered at the Institute of Mathematical Sciences, Madras. (November 1962)

degree of polarization is expected to be achieved with improvements. With polarized targets, it would be possible in addition to testing specific theories of interactions as we have shown above, to carry out other experiments in which, for instance, a polarized proton beam is scattered off a polarized target of protons, which would directly give the spin dependence of phase shifts. Also relative parities could be established by observing the polarization of the particles produced. For instance, by observing the polarization of Σ's produced by pions incident on polarized nucleons, the $(K\Sigma)$ relative parity could be established.

CHAPTER V

SCATTERING OF PIONS BY DEUTERONS AND LIGHT NUCLEI[1]

1. Introduction

A study of the interaction of pions with targets containing more than a single nucleon would be extremely complicated if we did not use some scheme of approximation in the calculations. An approximation which has been frequently used in the case of scattering and photoproduction from deuterons is the impulse approximation which assumes that the matrix element for each of these processes is the sum of the matrix elements for interaction with each of the two nucleons in the deutron assuming that during this interaction the other nucleon plays the role of a spectator. We have used this approximation and the Chow-Low matrix element for pion scattering from nucleons to study the scattering of pions from deuterons at various energies. In an appendix, we discuss the scattering of pions from light nuclei.

For the scattering of a positive pion from deuterium the following processes are possible:

(1) $\pi^+ + D \longrightarrow \pi^+ + D$ (Elastic Scattering)

(2) $\pi^+ + D \longrightarrow p + n + \pi^+$ (Inelastic Scattering)

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(1) Alladi Ramakrishnan, V.Devanathan and K.Venkatesan, Nuclear Physics, 29, 680 (1962).

3) $\pi^+ + D \rightarrow p + p + \pi^\circ$ (Charge Exchange Scattering)

4) $\pi^+ + D \rightarrow p + p$ (Pure Absorption)

5) $\pi^+ + D \rightarrow p + p + \gamma$ (Radiative Capture)

The absorption processes (4) and (5) will be considered in the next chapter. The early experiments of Ashkin et al[1] have shown that there is no great difference between the scattering of π^+ and π^- indicating that charge symmetry holds.

Fernbach, Green and Watson[2] and Rockmore[3] have studied earlier, scattering of pions by deuterons using the impulse approximation which involves the neglect of the internucleon potential, multiple scattering and off-energy shell scattering of the pion by a nucleon. The correction due to the internucleon potential was investigated by Rockmore who found it to be less than 10% . But there are discrepant views regarding the effect of multiple scattering. This effect was studied initially by Brueckner[4] who found a considerable reduction in the cross-section of about 50 % at 30° and at an incident energy of 135 Mev. However Rockmore points out that the neglect of off-energy shell matrix elements in multiple scattering is not valid. He has carried out the calculations at only one energy (85 Mev) using the impulse approximation and finds an

1) J. Ashkin et al , Phys. Rev., 96 , 1104 (1954)
2) S.Fernbach, T.A.Green and K.M.Watson, Phys. Rev., 80 , 1084, (1951).
3) R.M.Rockmore, Phys. Rev., 105 , 256 (1957)
4) K.A.Brueckner, Phys. Rev., 89 , 834 (1953); Ibid, 90 , 716 (1953).

excellent agreement with the experimental results of Rogers and Lederman[1]. The elastic scattering of pions by deuterons has also been studied by Bransden and Moorhouse[2] employing the variational method and they give numerical results for various incident energies from 85 to 378 Mev. In the expression they have derived they are able to identify the terms corresponding to multiple scattering and show that the inclusion of these terms alters the cross-section by less than 5 % even at the most favourable angles and energies for the range of energies considered.

In contrast to the above, Green[3] reported earlier an impulse approximation calculation for elastic scattering at 135 Mev where he found too large values for the cross-sections as compared with experimental results then available. As a result he has made some sceptical remarks about the validity of the impulse approximation at the energy 135 Mev. Further, in presenting their experimental data for elastic scattering of positive and negative pions from deuterons at 300 Mev, Dul'kova et al[4] give for comparison a set theoretical values supposed to have been derived on the basis of the impulse approximation which are in violent disagreement with the

1) K.C.Rogers and L.M.Lederman, Phys. Rev., 105, 247 (1957)
2) B.H.Bransden and R.G.Moorhouse, Nuclear Physics, 6, 310 (1958)
3) T.A.Green, Phys. Rev., 90, 161 (1953)
4) L.S.Dul'kova et al, Soviet Physics, JETP, 8, 217 (1959)

experimental values, - a disturbing feature since by the very nature of the approximation one might expect the impulse approximation to give good results at higher energies.

In view of the above it was considered worthwhile to test the validity of the approximation by making use of the Chew-Low[1] amplitude for pion-nucleon scattering in order to study the elastic, inelastic and charge-exchange scattering of pions by deuterons at various incident pion energies. The numerical results that we have obtained for elastic and inelastic scattering of positive pions by deuterons at 85 Mev agree well with the experimental values of Rogers and Lederman[2]. The agreement in the case of charge exchange scattering at this energy (which is the only one which seems to be available) is not good as is also the case for elastic scattering at 140 Mev when compared with the experimental values of Pewitt et al[3] . But the value of Duk'ova et al[4] for elastic scattering at 300 Mev is fitted excellently by our computed values at this energy, which indicates that theoretical calculations reported in their paper is incorrect. The fact that there is a disagreement at 140 Mev may possibly be due to the fact that we are approaching the resonance region of the pion-nucleon system. Unfortunately there are no experimental data at or near the resonance energy ($\sim$ 190 Mev).

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1) G.F.Chew and F.E.Low, Phys. Rev., 101, 1570 (1956)
2) K.C.Rogers and R.G.Moorhouse, loc. cit.
3) F.G.Pewitt et al , Proc. of the 1960 Annual International Conference on High Energy Physics (Rochester) p.196, (1960)
4) L.S.Duk'ova et al, loc. cit.

2. The impulse series and the impulse approximation

In this section, we shall summarise the theory of the impulse approximation which was first introduced by Chew[1] and developed by Chew and Wick[2] and Chew and Goldberger[3]. We consider a particle incident on a nucleus, its interaction with the individual nucleons of the nucleus being represented by the potential V_k (for interaction with the k^{th} nucleon). If K is the total kinetic energy of the system and U the inter-nucleon potential then the total Hamiltonian for the system can be written as

$$H = K + U + V = H_0 + V \qquad (a)$$

where

$$V = \sum_k V_k$$

The T -matrix for the process is given by the equation

$$T = V + V \frac{1}{E_a + i\eta - H_0} VT$$

$$= V + V \frac{1}{E_a + i\eta - H_0 - V} V \qquad (b)$$

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1) G.F.Chew, Phys. Rev., 80, 196 (1950)
2) G.F.Chew and G.C.Wick, Phys. Rev., 85, 636 (1952)
3) G.F.Chew and M.L.Goldberger, Phys. Rev., 87, 778 (1952)

The first form for T in (b) is obtained from the S -matrix expansion of Dyson by doing the space and (time-ordered) time integrations separately. The term $+i\eta$ in the denominator represents the outgoing wave boundary condition. A limit of $\eta \to 0$ is implied in (b)

We define the two-particle scattering matrix

$$t_k = V_k \omega_k \qquad (c)$$

where

$$\omega_k = 1 + \frac{1}{E_\ell + i\eta - K - V_k} V_k \qquad (d)$$

Now if B and b are two operators defined by

$$B = \frac{1}{E_a + i\eta - H_o - V} A$$

$$b_k = \frac{1}{E_\ell + i\eta - K - V_k} A$$

where A is any operator, then

$$B = b_k + \frac{1}{E_a + i\eta - H_o - V} \left\{ [U, b_k] + (V - V_k) b_k \right\} \qquad (e)$$

This result follows on using the operator identity

$$\frac{1}{x-y} - \frac{1}{x} = \frac{1}{x-y} y \frac{1}{x}$$

We use (e) in (b) which we rewrite as

$$T = \sum_{k=1}^{N} \left\{ V_k + V \frac{1}{E_a + i\eta - H_o - V} V_k \right\} \qquad (b')$$

From (e)and (d) we have

$$\frac{1}{E_a+i\eta-H_0-V}\,V_k = (\omega_k-1) + \frac{1}{E_a+i\eta-H_0-V} \times\left\{\left[U,\omega_k\right]+(V-V_k)(\omega_k-1)\right\} \qquad (f)$$

Substituting (f) in (b') we finally obtain

$$T=\sum_{k=1}^{N}\left\{t_k + V\frac{1}{E_a+i\eta-H_0-V}\left[U,\omega_k\right] + \left[1+V\frac{1}{E_a+i\eta-H_0-V}\right](V-V_k)(\omega_k-1)\right\} \qquad (g)$$

which may be called the impulse series. The impulse approximation consists in retaining only the first term in (g) which represents a sum of two-body matrix elements. The second term will be negligible if the inter-nucleon potential U is negligible. The third term of (g) gives the effect of multiple scattering.

3. Elastic scattering of a π^+ from a deuteron

The impulse approximation requires that we should know the matrix elements for the scattering of a π^+ from a proton and a neutron. The Chew-Low amplitude[1] for pion

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1) G.F.Chew and F.E.Low, loc. cit

nucleon scattering in the static approximation is given by

$$t(2,1) = \frac{2\pi}{\omega} \sum_{\alpha=1}^{4} P_\alpha(2,1)\, h_\alpha \qquad (1)$$

where

$$h_\alpha = \frac{e^{i\delta_\alpha(q)} \sin\delta_\alpha(q)}{q^3} \qquad (2)$$

is the p -wave scattering amplitude and $P_\alpha(2,1)$ are the projection operator for the state α . α stands for any one of the four p -wave states 33 , 31 , 13 , 11 , where following Fermi's notation the first number in each case stands for twice the total isotopic spin and the second, twice the total angular momentum of the pion nucleon system. In a static theory the amplitudes 13 and 31 are identical so that we need consider only three states. Explicitly the P_α's are given by

$$P_1(2,1) = P_1(q_{v_2}, q_{v_1}) = \frac{1}{3}\tau_{q_{v_2}}\tau_{q_{v_1}}(\vec{\sigma}\cdot\vec{q}_{v_2})(\vec{\sigma}\cdot\vec{q}_{v_1})$$

$$P_2(2,1) = P_2(q_{v_2}, q_{v_1}) = \delta_{q_{v_2}q_{v_1}}(\vec{\sigma}\cdot\vec{q}_{v_2})(\vec{\sigma}\cdot\vec{q}_{v_1}) + \frac{1}{3}\tau_{q_{v_2}}\tau_{q_{v_1}}\left[3(\vec{q}_{v_2}\cdot\vec{q}_{v_1}) - 2(\vec{\sigma}\cdot\vec{q}_{v_2})(\vec{\sigma}\cdot\vec{q}_{v_1})\right]$$

$$P_3(2,1) = P_3(q_{v_2}, q_{v_1}) = \left[\delta_{q_{v_2}q_{v_1}} - \frac{1}{3}\tau_{q_{v_2}}\tau_{q_{v_1}}\right] \cdot \left[3(\vec{q}_{v_2}\cdot\vec{q}_{v_1}) - (\vec{\sigma}\cdot\vec{q}_{v_2})(\vec{\sigma}\cdot\vec{q}_{v_1})\right] \qquad (3)$$

The τ's are the isotopic spin matrices for the nucleon. In the matrix element (1), we have assumed that the cut-off factor $v(q) = 1$.

We apply the above amplitude to the case of the deuteron and follow essentially the approach of Chew and Lewis[1]. In theimpulse approximation, the matrix element for the process can be written as

$$Q = \langle f | T^{(1)} \exp(i \vec{D} \cdot \vec{r}_1) + T^{(2)} \exp(i D \cdot r_2) | i \rangle \qquad (4)$$

The subscripts 1 and 2 refer to the nucleons in the deuteron and they may represent a proton or a neutron, $\vec{D}$ represents the transfer of momentum to the nucleon. $\vec{r}_1$ and $\vec{r}_2$ are the position co-ordinates of the nucleons 1 and 2 . The initial and final state wave functions of the deuteron are isotopic spin singlets and spin triplets and are given by

$$|i\rangle = (2)^{-\frac{1}{2}} \left[p(1)\, n(2) - p(2)\, n(1) \right] {}^3\chi_m u_d(\rho)$$

$$|f\rangle = (2)^{-\frac{1}{2}} \left[p(1)\, n(2) - p(2)\, n(1) \right] \cdot {}^3\chi_m u_d(\rho) \exp(i \vec{D}.\vec{R})$$

where χ and u_d are the spin and space wave functions respectively. $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$. The transition operator $T^{(j)}$ in (4) can bewritten as

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1.) G.F.Chew and H.W.Lewis, Phys. Rev., 84, 779 (1951)

$$T^{(j)} = t_p^{(j)} \cdot \frac{1+T_3^{(j)}}{2} + t_n^{(j)} \cdot \frac{1-T_3^{(j)}}{2}$$

$$= \frac{t_p^{(j)} + t_n^{(j)}}{2} + \frac{t_p^{(j)} - t_n^{(j)}}{2} T_3^{(j)} \qquad (5)$$

Here T_3 is the third component of the isotopic spin vector of the nucleon and t_p and t_n are the matrix elements for pion scattering from proton and neutron respectively and are given by

$$t_p = t(p\pi^+ \longrightarrow p\pi^+)$$

$$= \frac{-2\pi}{\omega} \left[2\vec{q}_2 \cdot \vec{q}_1 - i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) \right] \cdot \frac{e^{i\delta_{33}} \sin\delta_{33}}{q^3}$$

$$t_n = t(n\pi^+ \longrightarrow n\pi^+)$$

$$= \frac{-2\pi}{\omega q^3} \left[\frac{1}{3} \left\{ 2\vec{q}_2 \cdot \vec{q}_1 - i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) \right\} \cdot e^{i\delta_{33}} \sin\delta_{33} \right.$$

$$+ \frac{1}{3} \left\{ 5\vec{q}_2 \cdot \vec{q}_1 - i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) \right\} \cdot e^{i\delta_{31}} \sin\delta_{31}$$

$$\left. + \frac{2}{3} \left\{ \vec{q}_2 \cdot \vec{q}_1 + i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) \right\} \cdot e^{i\delta_{11}} \sin\delta_{11} \right] \qquad (6)$$

where q_1 and q_2 are the momenta of the initial and final pions and $\omega_1(=\omega)$ and $\omega_2(=\omega)$ are their energies. The term containing $T_3^{(2)}$ in (5) does not connect the charge singlet initial and final states. Using the centre of mass and relative co-ordinates given by

$$r_1 = R + \frac{\rho}{2}; \qquad r_2 = R - \frac{\rho}{2} \tag{7}$$

the matrix element (4) finally reduces to

$$Q = \langle 3\chi_m | \tfrac{1}{2}\left[t_p^{(1)} + t_p^{(2)} + t_n^{(1)} + t_n^{(2)} \right] | 3\chi_m \rangle E \tag{8}$$

where E is given by

$$E = \int u_d^*(\rho) \exp(i\vec{k}_0 \cdot \vec{\rho})\, u_d(\rho)\, d\vec{\rho}$$
$$\vec{k}_0 = \tfrac{1}{2}\vec{D} = \tfrac{1}{2}(\vec{q}_1 - \vec{q}_2) \tag{9}$$

Squaring the matrix element (8) , summing over the final spin states of the deuteron and averaging over the initial spin state, we obtain the differential cross-section for elastic pion-deuteron scattering as

$$\left(\frac{d\sigma}{d\Omega}\right)_{el} = (2\pi)^{-2}\, \omega^2 \left[|(a+c)|^2 + \frac{2}{3}|(b+d)|^2 \right] E^2 \tag{10}$$

Here

$$|a+c|^2 = \frac{4\pi^2}{9\omega^2 q^2} \cos^2\theta \left[(8\cos\delta_{33}\sin\delta_{33} + 8\cos\delta_{31}\sin\delta_{31} + 2\cos\delta_{11}\sin\delta_{11})^2 \right.$$

$$+(8\sin^2\delta_{33}+8\sin^2\delta_{31}+2\sin^2\delta_{11})^2\,] \qquad (11)$$

$$|(b+d)|^2=\frac{16\pi^2\sin^2\theta}{9\omega^2q^2}\{(2\cos\delta_{33}\sin\delta_{33}$$

$$-\cos\delta_{31}\sin\delta_{31}-\cos\delta_{11}\sin\delta_{11})^2$$

$$+(2\sin^2\delta_{33}-\sin^2\delta_{31}-\sin^2\delta_{11})^2\} \qquad (12)$$

Here θ is the angle of scattering of the pion. The integral in equation (9) turns out to be

$$E=\frac{1}{1-\alpha\rho_1}\,\frac{2\alpha}{k_0}\left\{\tan^{-1}\left(\frac{k_0}{2\alpha}\right)+\tan^{-1}\left(\frac{k_0}{2\beta}\right)\right.$$

$$\left.-2\tan^{-1}\left(\frac{k_0}{\alpha+\beta}\right)\right\} \qquad (13)$$

if we use the Hulthen wave function for the deuteron. Here α , β and ρ_1 are constants, the last being the triplet effective range.

Using (10) , the differential cross-sections for elastic pion-deuteron scattering have been calculated numerically at incident pion energies of 85 , 140 , 195 and 250 and 300 Mev. It may be noted that the expression (10) has to be evaluated in the centre of mass system of the pion-nucleon system (and not the pion-deuteron system), since in the impulse approximation the incident pion "sees" only the individual

nucleon and the interaction is described as the sum of individual scattering amplitudes. The results are tabulated in Table I and have also been drawn for the case of 85, 140 and 300 Mev. The effect of including the p -wave phase-shifts other than the dominant δ_{33} phase shifts have been studied and as shown in figures (1) and (3), the effect is to reduce the cross-sections slightly.

At 85 Mev , there is a goodm agreement between the numerical results that we have obtained and the experimental data of Rogers and Lederman, (fig. 1) and at other energies with those of Pewitt et al[1] (fig.3) and Dul'kova et al (fig. 4)[2]. At small angles the theoretical cross-sections have to be corrected for Coulomb scattering, the effect of which is, as shown by Rockmore, to depress the cross-sections at small angles. The dotted lines in figs. (1) and (3) are obtained after including the small p -wave phase shifts.

3. Inelastic scattering

In this case the final proton-neutron system can be a charge singlet or a chargetriplet state and corresponding to each of these possibilities the spin warm wave functions can be symmetric (triplet) or antisymmetric (singlet). The corresponding space wave functions will be symmetric or antisymmetric as the

1) E.G.Pewitt et al , Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester (1960).
2) L.S.Dul'kova et al , loc. cit.

Table I

The differential cross-section $d\sigma/d\Omega$ for the elastic scattering of π^+ by deuteron in units of m.b / sterad

Pion kinetic energy in lab. (Mev) \ Lab angle	0°	30°	60°	90°	120°	150°	180°
85	12.62	7.568	1.747	0.636	1.039	1.388	1.495
140	45.19	23.66	4.342	1.340	1.974	2.503	2.641
195	68.71	31.17	4.737	1.311	1.660	2.008	2.186
250	38.44	15.13	1.480	0.476	0.611	0.713	0.735
300	22.79	7.954	0.854	0.201	0.243	0.279	0.285

Table IV

The differential cross section $\frac{d\sigma}{d\Omega}$ in laboratory system for the elastic scattering of charged pions from deuterons at 300 Mev in units of mb/sterad

Laboratory angle	0°	30°	60°	90°	120°	150°	180°
B.M.	10.44	2.72	0.25	0.03	0.02	0.03	0.03
D.S.S. (Theoretical)	--	18.00	4.00	0	0	0	0
D.S.S. (Experimental)	--	7.5 ± 3	1 ± 0.5	1 ± 0.5	0.5 ± 0.25	0.25 ± 0.1	-
P.C.	22.79	7.954	0.8543	0.2007	0.2426	0.2785	-

B.M. = The theoretical values of Bransden and Moorhouse given in ref. 2 p130 at energy 298 Mev.

D.S.S. (Theoretical) = The theoretical values at 300 Mev obtained on the impulse approximation as reported in ref. 4, p130. The values are taken from the curve given for the purpose of comparison with their experimental results.

D.S.S. (Experimental) = The experimental values at 300 Mev taken from the experimental curve given in ref. 4, p130

P.C. = The present calculation at 300 Mev based on the impulse approximation using the Chew-Low amplitude for the scattering of pions from free nucleons.

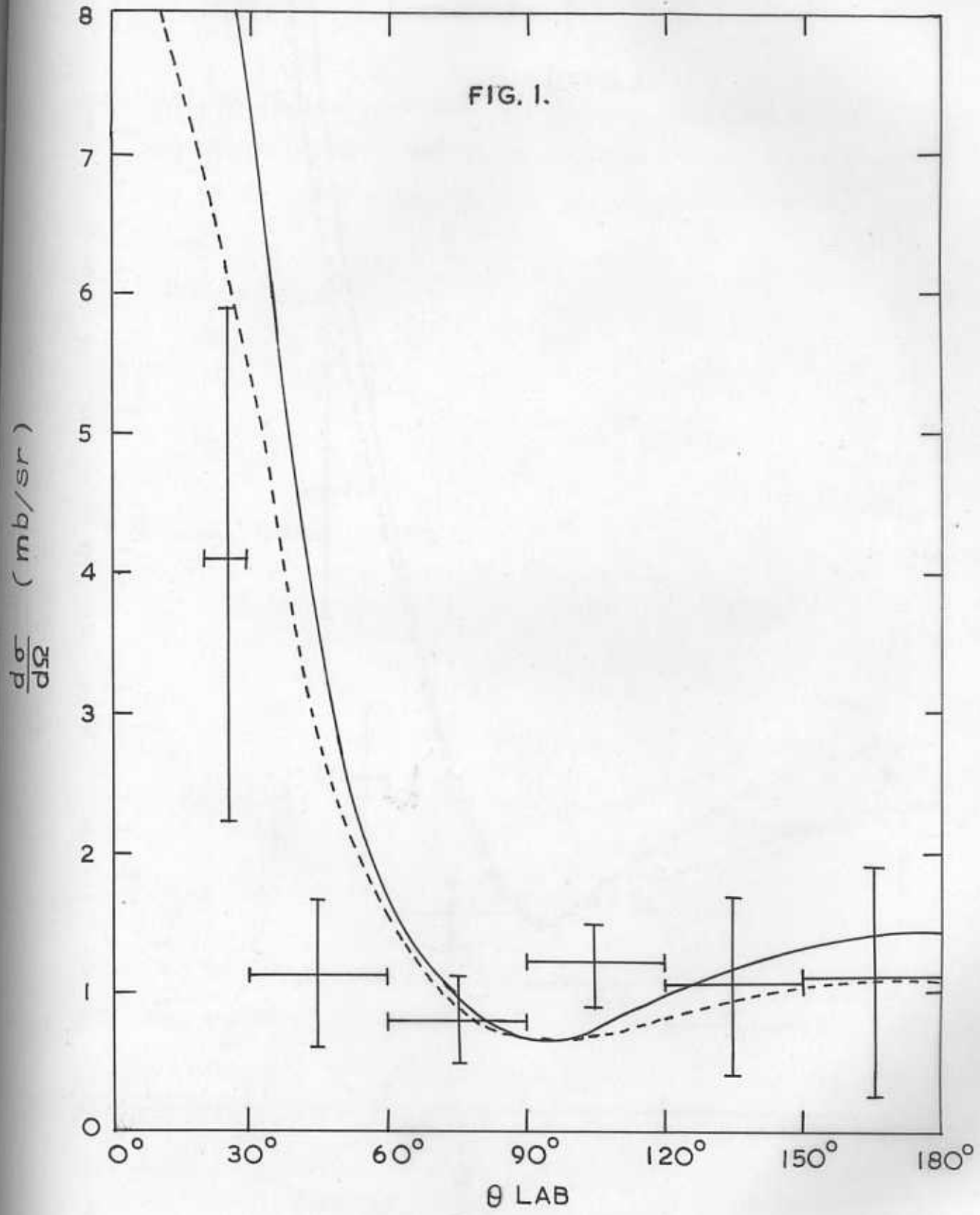


FIG. 1.
8
7
6
5
4
3
2
1
0
0°
30°
60°
90°
120°
150°
180°
θ LAB
dσ/dΩ (mb/sr)

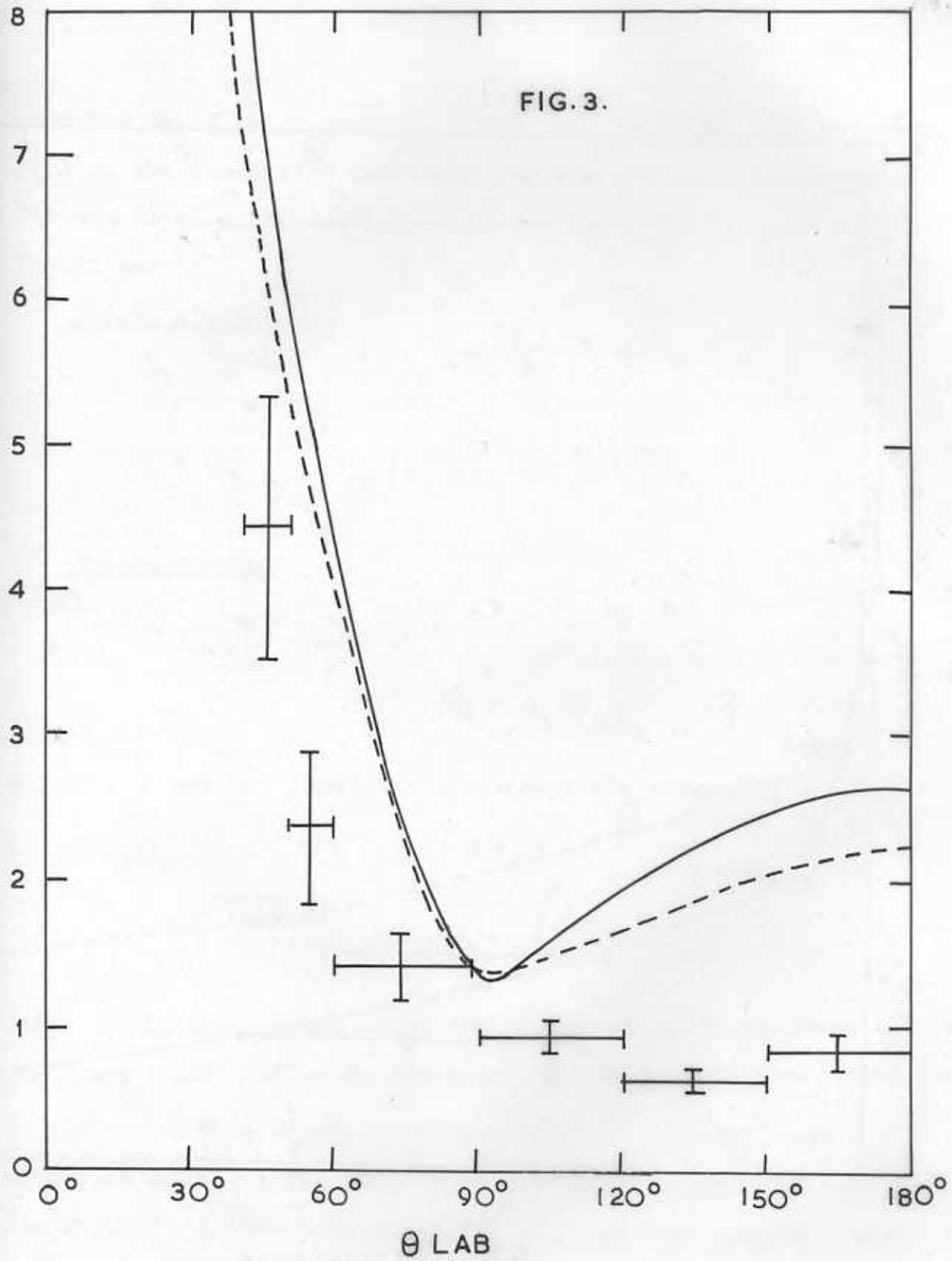


FIG. 3.
8
7
6
5
4
3
2
1
0
0°
30°
60°
90°
120°
150°
180°
θ LAB
dσ/dp (mb/sr)

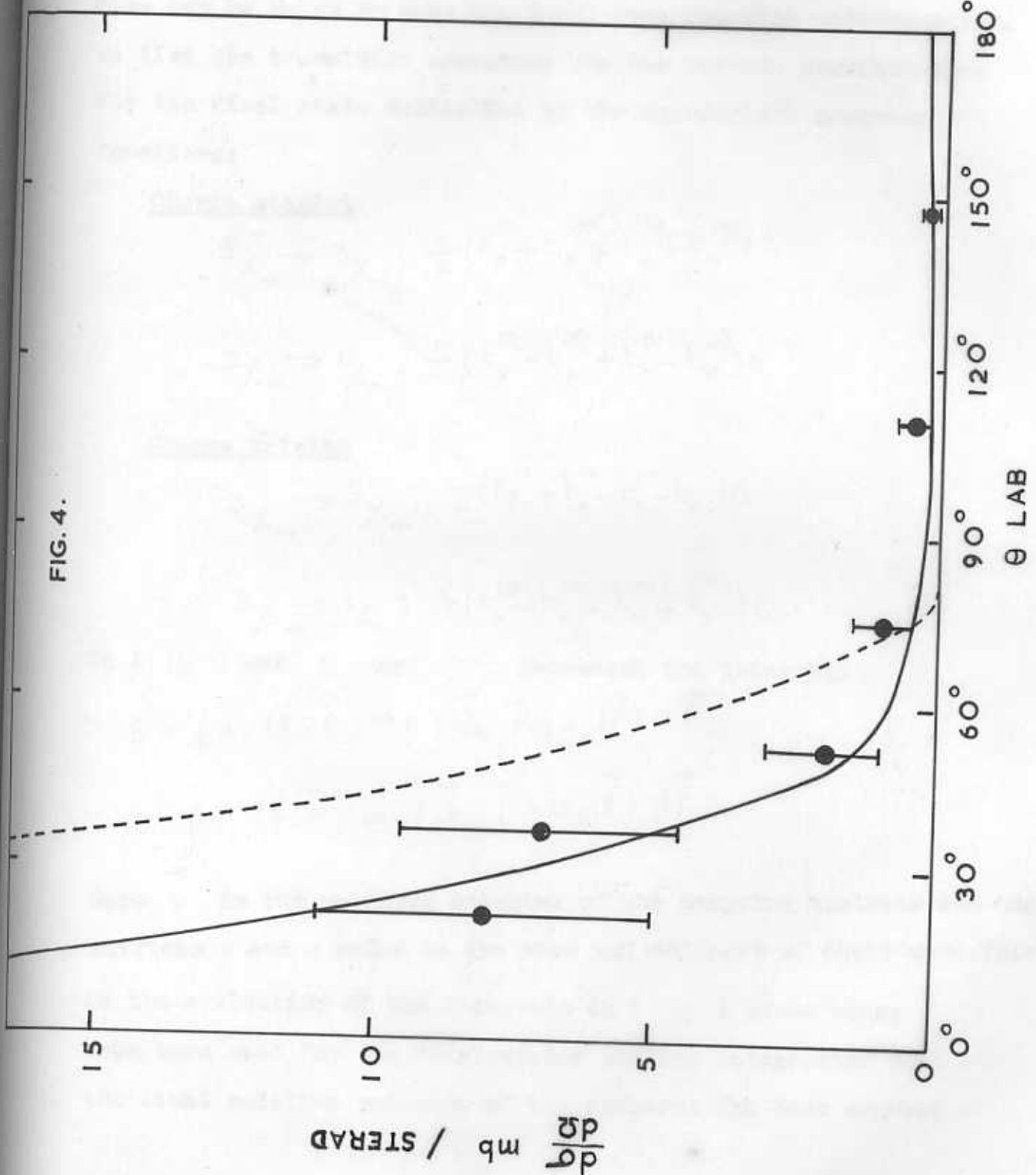


FIG. 4.

case may be so as to make the total wave function antisymmetric. We list the transition operators for the various possibilities for the final state, multiplied by the appropriate space-wave functions:

Charge singlet

$$ {}^3\chi_m \rightarrow {}^3\chi_m \; ; \; \tfrac{1}{2}\left(t_p^{(1)} + t_p^{(2)} + t_n^{(1)} + t_n^{(2)}\right)\varepsilon $$

$$ {}^3\chi_m \rightarrow {}^1\chi_o \; ; \; \tfrac{1}{2}\left(t_p^{(1)} - t_p^{(2)} + t_n^{(1)} - t_n^{(2)}\right)o $$

Charge triplet

$$ {}^3\chi_m \rightarrow {}^3\chi_m \; ; \; \tfrac{1}{2}\left(t_p^{(1)} + t_p^{(2)} - t_n^{(1)} - t_n^{(2)}\right)o $$

$$ {}^3\chi_m \rightarrow {}^1\chi_o \; ; \; \tfrac{1}{2}\left(t_p^{(1)} - t_p^{(2)} - t_n^{(1)} + t_n^{(2)}\right)\varepsilon \qquad (14) $$

In (14) and ε and o represent the integrals

$$ \varepsilon = \int u_e(\vec{k}\cdot\vec{\rho}) \exp(i\vec{k}_o\cdot\vec{\rho})\, u_d(\vec{\rho})\, d\vec{\rho} $$

$$ o = \int u_o(\vec{k}\cdot\vec{\rho}) \exp(i\vec{k}_o\cdot\vec{\rho})\, u_d(\vec{\rho})\, d\vec{\rho} \qquad (15) $$

Here $\vec{k}$ is the relative momentum of the outgoing nucleons and the suffices e and o refer to the even and odd part of their wave functions. In the evaluation of the integrals in (15) plane waves have been used for the final states and the integration over the final relative momentum of the nucleons has been carried

out using the closure approximation. This approximation implies the neglect of the energy of relative motion of the nucleons in the final state in the over-all energy conservation. The meson is assumed to have the characteristics of a free particle collision. Since in the complete set of final states the deuteron is also included, we obtain, on squaring and averaging, the sum of the differential cross-sections for elastic and inelastic scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{el+inel} = (2\pi)^{-2}\omega^2 \times \left[\left(|a|^2+|b|^2+|c|^2+|d|^2\right) + 2F\left\{\mathrm{Re}\,a\,\mathrm{Re}\,c + \mathrm{Im}\,a\,\mathrm{Im}\,c + \frac{1}{3}\left(\mathrm{Re}\,b\,\mathrm{Re}\,d + \mathrm{Im}\,b\,\mathrm{Im}\,d\right)\right\}\right] \tag{16}$$

where

$$a = \frac{2\pi}{\omega q}\cos\theta\left(2e^{i\delta_{33}}\sin\delta_{33} + e^{i\delta_{31}}\sin\delta_{31}\right)$$

$$b = \frac{2\pi}{\omega q}\sin\theta\left(e^{i\delta_{33}}\sin\delta_{33} - e^{i\delta_{31}}\sin\delta_{31}\right)$$

$$c = \frac{2\pi}{\omega q}\cos\theta\left(2e^{i\delta_{33}}\sin\delta_{33} + 5e^{i\delta_{31}}\sin\delta_{31} - 2e^{i\delta_{11}}\sin\delta_{11}\right)$$

$$d = \frac{2\pi}{3\omega q}\sin\theta\left(e^{i\delta_{33}}\sin\delta_{33} + e^{i\delta_{31}}\sin\delta_{31} - 2e^{i\delta_{11}}\sin\delta_{11}\right) \tag{17}$$

$$F = \frac{1}{1-\alpha\rho_1}\,\frac{\alpha}{k_0}\left\{\tan^{-1}\left(\frac{k_0}{\alpha}\right)\right.$$

$$\left. + \tan^{-1}\left(\frac{k_0}{\beta}\right) - 2\tan^{-1}\left(\frac{2k_0}{\alpha+\beta}\right)\right\} \qquad (18)$$

The differential cross-sections were calculated numerically for the same values of the incident energy as in the case of elastic scattering. The results are given in Table II and Figure (2) . Only the dominant phase shifts have been taken into account. The experimental points in Fig. (2) are those of Rogers and Lederman. As can be seen the agreement between theory and experiment is good at 85 Mev . No experimental data are available for other values of the incident energy.

4. Charge Exchange Scattering

For this process we require the matrix element for the process

$$\pi^+ + n \longrightarrow \pi^0 + p$$

which is given by

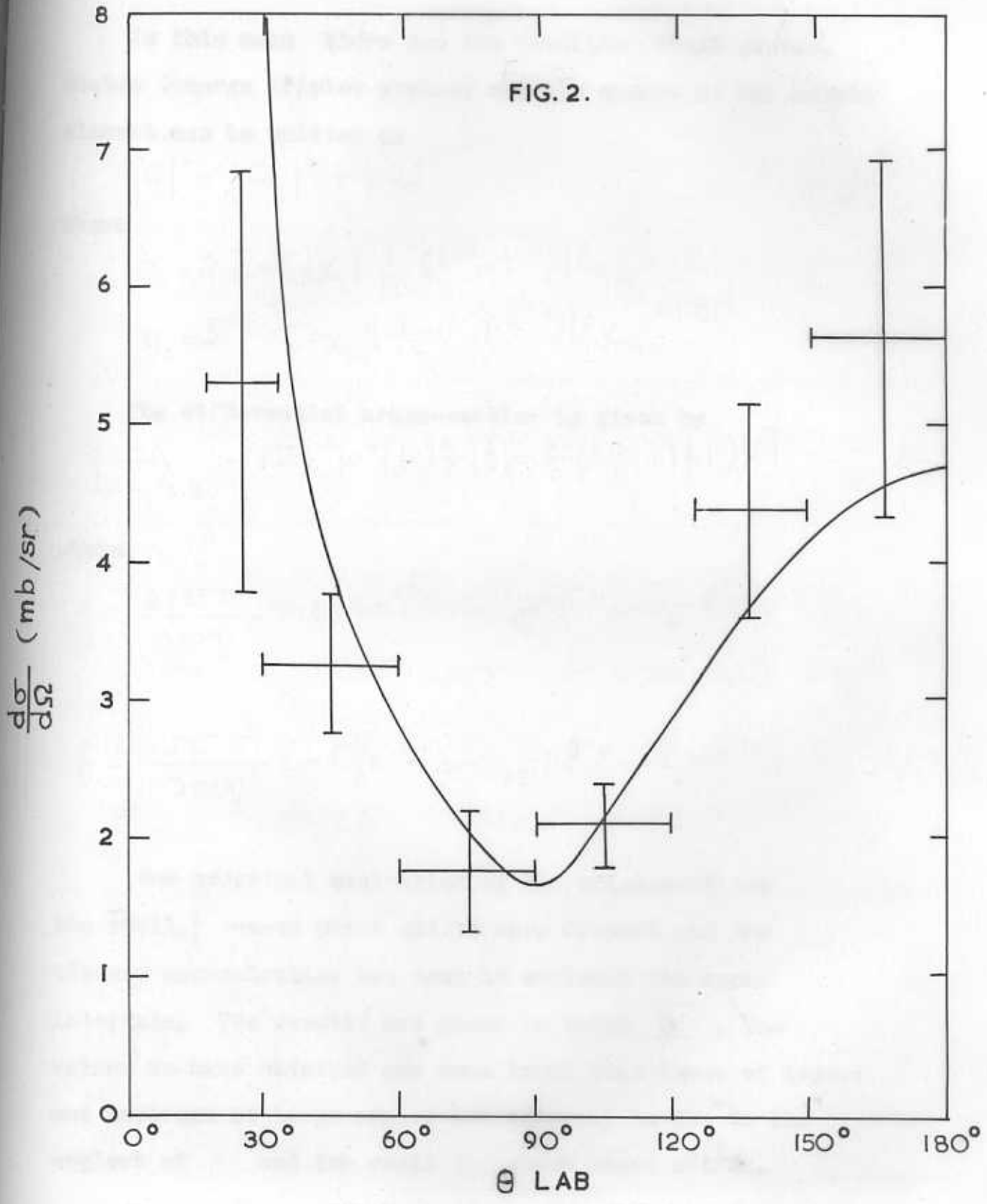
$$t(\pi^+ n \to \pi^0 p) = t = \frac{-2\pi}{\omega q^3}\left[\frac{-\sqrt{2}}{3}\left\{2\vec{q}_2\cdot\vec{q}_1 - i\vec{\sigma}\cdot(\vec{q}_2\times\vec{q}_1)\right\}e^{i\delta_{33}}\sin\delta_{33}\right.$$
$$+\frac{\sqrt{2}}{3}\left\{\vec{q}_2\cdot\vec{q}_1 - 2i\vec{\sigma}\cdot(\vec{q}_2\times\vec{q}_1)\right\}e^{i\delta_{31}}\sin\delta_{31}$$
$$\left.+\frac{\sqrt{2}}{3}\left\{\vec{q}_2\cdot\vec{q}_1 + i\vec{\sigma}\cdot(\vec{q}_2\times\vec{q}_1)\right\}e^{i\delta_{11}}\sin\delta_{11}\right] \qquad 19$$

Table II

$\frac{d\sigma}{d\Omega}$ for the sum of elastic and inelastic scattering of π^+ by deuterons in units of m.b./sterad

Pion energy (Mev) \ Lab. angle	0°	30°	60°	90°	120°	150°	180°
85	12.62	8.104	2.844	1.674	2.928	4.263	4.751
140	45.19	27.54	9.754	5.353	10.08	14.63	16.31
195	68.71	40.18	14.43	8.794	14.99	21.82	24.34
250	38.44	21.75	7.857	4.884	8.383	12.08	13.47

FIG. 2.



In this case there are two possible final states. States (charge triplet states) and the square of the matrix element can be written as

$$|Q|^2 = |Q_e|^2 + |Q_o|^2 \qquad (20)$$

where

$$Q_e = 2^{-\frac{1}{2}} \langle |\chi_o| \frac{1}{\sqrt{2}} (t^{(2)} - t^{(1)}) |^3\chi_m \rangle |\mathcal{E}|$$

$$Q_o = 2^{-\frac{1}{2}} \langle {}^3\chi_m | \frac{1}{\sqrt{2}} (t^{(2)} + t^{(1)}) |^3\chi_m \rangle |0|$$

The differential cross-section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{c.e.} = (2\pi)^{-2} \omega^2 \left[|e|^2 + |f|^2 - \frac{1}{3}(3|e|^2 + |f|^2)F\right] \qquad (21)$$

where

$$e = \frac{2\sqrt{2}\,\pi}{3\omega q} \cos\theta \left(2e^{i\delta_{33}} \sin\delta_{33} - e^{i\delta_{31}} \sin\delta_{31} - e^{i\delta_{11}} \sin\delta_{11}\right)$$

$$f = \frac{2\sqrt{2}\,\pi}{3\omega q} \sin\theta \left(e^{i\delta_{33}} \sin\delta_{33} - 2e^{i\delta_{31}} \sin\delta_{31} + e^{i\delta_{11}} \sin\delta_{11}\right) \qquad (22)$$

For numerical evaluation of the cross-sections the small p -wave phase shifts were dropped and the closure approximation was used to evaluate the space integrals. The results are given in Table III . The values we have obtained are much lower than those of Rogers and Lederman at large angles and this may be due to the neglect of s and the small p -wave phase shifts.

Table III

$\frac{d\sigma}{d\Omega}$ for the charge exchange scattering of π^+ by deuteron in units of m.b. / sterad

Pion energy (Mev) \ Lab. angle	0°	30°	60°	90°	120°	150°	180°
85	0	.3865	.3708	.2826	.4489	.6488	.7252
140	0	1.878	1.471	1.058	1.740	2.530	2.829
195	0	3.417	2.371	1.644	2.760	3.990	4.460
250	0	2.153	1.397	.9315	1.563	2.277	2.544

5. Conclusion

At pion energies 85 and 300 Mev, there exists good agreement between the numerical results that we have obtained and the experimental data of Rogers and Lederman and Dul'kova et al . For charge exchange scattering however the values that we have obtained are much lower than those of Rogers and Lederman at large angles and this may be xxxx due to the neglect of s and other p -wave phase shifts. The agreement at 140 Mev between theory and experiment seems to be poor and the large cross-sections at backward angles cannot be explained away by the on-the-energy shell multiple scattering effects and the inclusion of the s-wave phase shifts and the D -state wave function. An exhaustive study of the elastic scattering at 142 Mev which includes the effect of various corrections has been made by Pendleton[1] using what he calls the form factor approximation; but his final result does not seem to differmuch from the numerical values we have obtained.

In Fig. 4 , the calculated cross-section for the elastic scattering at 300 Mev is presented together with the experimental results of Dul'kova et al. The agreement is excellent which seems to show that the complete disagreement between the theoretical values obtained on using the impulse approximation and the experimental values of Dul'kova et al

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1) Pendleton, private communication

reported by them is incorrect. It may be observed that the values of Bransden and Moorhouse are also in disagreement with the experimental results at 300 Mev and also with those of ours. We present in Table IV, the three sets of values for the purpose of comparison.

It is instructive to compare the cross-sections for the deuteron with the free nucleon cross-sections

$$\left(\frac{d\sigma}{d\Omega}\right)_{\pi^+ + p \rightarrow \pi^+ + p} = (2\pi)^{-2}\,\omega^2\left(|a|^2 + |b|^2\right) \qquad (23)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\pi^+ + n \rightarrow \pi^+ + n} = (2\pi)^{-2}\,\omega^2\left(|c|^2 + |d|^2\right) \qquad (24)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\pi^+ + n \rightarrow \pi^0 + p} = (2\pi)^{-2}\,\omega^2\left(|e|^2 + |f|^2\right) \qquad (25)$$

We find that in the case of the sum of elastic and inelastic scattering by deuteron, the cross-section is larger than the sum of the free-proton and the free-neutron cross-sections. In the case of charge exchange scattering to which only the neutron contributes, we find that the cross-section is reduced by the presence of the other particle (proton). This effect is to be attributed to the Pauli exclusion principle. In the former case (elastic and inelastic scattering by deuteron) this effect is shroudedby the presence of the interference terms. E and F are decreasing functions of both energy and angle. Hence at high energies and at large angles, the differential cross-section for the process (2) approaches the sum of the free-proton and the free-neutron cross-sections and for the process (3) the cross-section approaches that of the charge-exchange scattering of $\pi^{\pm}$ by free neutrons.

APPENDIX

In this appendix, we examine the scattering of pions from light nuclei like lithium, carbon, and aluminium for which experiments have been done[1]. Previous theoretical work on this subject mainly uses the optical model according to which one replaces the interaction between the incident pion and the nucleus by a complex potential well having the form

$$V_c(x) = V(\omega)\,\rho(x)$$

where $\rho(x)$ is the nuclear density and $V(\omega)$ is a complex quantity called the" well-depth" and is a function of ω), the energy of the pion. The procedure then is to obtain $V(\omega)$ for some $\rho(x)$ which best fits the scattering data at the energy ω. Alternatively the value of $V(\omega)$ is deduced from a knowledge of the elementary two-particle interactions. This has been done by Frank et al[2] who have evaluated both the real and imaginary parts of the potential from the knowledge of pion-nucleon scattering.

The usual optical model approach does not make use of the complete matrix element for the elementary pion-nucleon scattering process but only the forward scattering amplitude $f(0)$. Thus the predicted angular distribution of the scattering is determined by the nuclear density and the forward scattering amplitude $f(0)$

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1) E.g., W.F.Baker, J.Rainwater and R.R.Williams, Phys.Rev. 112, 1763(1958

2) R.M.Frank, J.L.Gammel and K.M.Watson, Phys.Rev. 101, 891 (1953).

for neutrons and protons. But recently Kisslinger[1] has developed a theory which takes some account of $f(\theta)$ for $\theta \neq 0$ also. The experimental data of Baker et al[2] seem to require a Kisslinger-type model which takes into account the p-wave nature of the basic pion-nucleon scattering process.

We shall develop in this appendix an alternative point of view based on the impulse approximation which uses the complete angular dependence of the pion-nucleon scattering modified by a form factor of the nucleus. In fact such an attempt was already made by Williams et al[3], who tried to fit their theoretical cross-sections obtaind by combining coherently the scattering amplitude for pion-nucleon interactions weighted by a form factor with their their experimental results for lithium at 78 Mev. But we wish to point out that Li^7 has a spin 3/2 and hence the spin-dependent terms of the pion-nucleon scattering amplitude will contribute along with the spin-independent term even for elastic scattering but Williams et al[3] neglect the spin-dependent part without justification. We outline below a rigorous method of evaluating the cross section

The pion-nucleon matrix element can be symbolically written

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1) L.S.Kisslinger, Phys.Rev. 98, 761 (1955)

2) W.W.Baker et al, loc. cit.

3) Williams, Rainwater and Fevener, Phys.Rev. 101, 412 (1956)

as $\vec{\sigma} \cdot \vec{K} + L$ of which the first term is the spin-dependent and the second the spin-independent parts. The coefficients $\vec{K}$ and L are functions of the pion energy and momentum and the pion-nucleon phase shifts. The matrix element for the "elastic" scattering of pions by a nucleus of mass number can bewritten as

$$Q_{M_i \to M_f} = \langle \psi^A (J M_f \; T M_T)$$

$$\sum_{i=1}^{A} \left(\frac{t_p^{(i)} + t_n^{(i)}}{2} + \frac{t_p^{(i)} - t_n^{(i)}}{2} \tau \right.$$

$$\left. \times e^{i \vec{R} \cdot \pi_i} \right| \psi^A (J M_i \; T M_T) \rangle \qquad (A\,1)$$

where τ is the isotopic spin operator τ_3 . The quantities t_p and t_n are the direct scattering amplitudes of the pion by the proton and neutron respectively J and M represent the total angular momentum of the nucleus and (t). T and M_T refer to similar quantities for isospin. i and z- component and f refer to the initial ad final states respectively writing

$$t_p = \vec{\sigma} \cdot K_p + L_p$$

and

$$t_n = \vec{\sigma} \cdot \vec{K}_n + L_n ,$$

we obtain

$$\frac{L_p + L_N}{2} = \vec{q} \cdot \frac{\vec{K}_p + \vec{K}_N}{2} + \frac{L_p + L_N}{2}$$

$$= \sum_{n=0,1} \vec{q}_n \cdot \vec{K}_{nm} \,, \quad m = 0$$

and

$$\frac{L_p - L_N}{2} = \vec{q} \cdot \frac{\vec{K}_p - \vec{K}_N}{2} + \frac{L_p - L_N}{2}$$

$$= \sum_{n=0,1} \vec{q}_n \cdot \vec{K}_{nm} \,; \quad m = 1$$

A (2)

where

$$K_{00} = \frac{L_p + L_N}{2} \,, \quad K_{01} = \frac{L_p - L_N}{2}$$

$$K_{10} = \frac{K_p + K_N}{2} \,, \quad K_{11} = \frac{K_p - K_N}{2}$$

In (A2) we are able to write $\vec{q} \cdot \vec{K} + L$ as $\sum_{n=0,1} \vec{q}_n \cdot \vec{K}_n$ and by defining the operator $\sigma_0, \lambda \tau_0$ in spin space. Expanding $e^{i \vec{k} \cdot \vec{r}}$ into spherical harmonics, we can write

$$(\vec{q} \cdot \vec{K} + L)\, e^{i \vec{k} \cdot \vec{r}} = 4\pi \sum_{n} \sum_{l,m} (i)^l \qquad (k)$$

$$\times (-1)^m \; Y_l^m (\hat{r}) \; Y_l^{-m} (\hat{k}) \sum (-1)^\mu \sigma_n^\mu \; K_n^{-\mu}$$

(A3)

where the symbol $\wedge$ indicates a unit vector. Thus we can write the matrix element as

$$\mathcal{M}_{M_i \to M_f} = 4\pi \sum_{i=1}^{A} \sum_{\ell,m,n} \sum_{\lambda,m_\lambda} (i)^{\ell} (-1)^{\ell+n-\lambda}$$

$$\times (-1)^{m_\lambda} \left(Y_\ell(\hat{k}) \times K_{nm} \right)_\lambda^{-m_\lambda}$$

$$\times \langle J M_f T M_T | \left(Y_\ell(\hat{r}_i) \times \sigma_n \right)_\lambda j_\ell(kr_i) \tau_m^{0(i)} |$$

$$J M_i T M_T \rangle \qquad (A4)$$

On using the orthogonality property of the Clebsch-Gordan coefficients and forming the tensor products. Using the Wigner-Eckart theorem that the independence on the magnetic quantum numbers is contained in a Clebsch-Gordan coefficient we can write

$$\langle J_f M_f | \left(Y_\ell(\hat{r}) \times \sigma_{nm} \right)_\lambda^{m_\lambda} | J_i M_i \rangle$$

$$= C(J_i \lambda J_f ; M_i M_\lambda)$$

$$\times \langle J_f \| \left(Y_\ell(\hat{r}) \times \sigma_{nm} \right)_\lambda \| J_i \rangle$$

$$(A5)$$

where the double bars in the matrix element of the right hand side represent the reduced matrix element (i.e. the part of the matrix element which does not depend on the magnetic quantum numbers) does not depend on the magnetic quantum

Squaring the matrix element and averaging and summing over the initial and final spin states, we obtain the differential cross-section

$$\frac{d\sigma}{d\Omega} = (2\pi)^{-2} \mu_0^2 |Q|^2 \qquad (A6)$$

where μ and μ_0 are themomentum and energy respectively of the pion (in the centre of mass of the pion-nucleon system) and $|Q|^2$ is given by

$$|Q|^2 = \frac{1}{[J]^2} \sum_{l,l'} \sum_{n,n'} \sum_{m,m'} , \sum_{\lambda,m_\lambda} 16\pi^2$$

$$\times (i)^{l-l'} (-1)^{l+l'+n+n'} \quad \frac{[J]^2}{[\lambda]^2}$$

$$\times \left(Y_l(\hat{R}) \times K_{nm} \right)_\lambda^{-m_\lambda} \left(Y_{l'}(\hat{R}) \times K_{n'm'} \right)_\lambda^{-m_\lambda^*}$$

$$\times \left\langle JT \| \sum_{i=1}^{A} \left(Y_l(\hat{r}_i) \times \sigma_n \right)_\lambda j_l(kr_i) \tau_m^{o(i)} \| JT \right\rangle$$

$$\times \left\langle JT \| \sum_{i=1}^{A} \left(Y_{l'}(\hat{r}_i) \times \sigma_{n'} \right)_\lambda j_{l'}(kr_i) \tau_{m'}^{o(i)} \| JT \right\rangle$$

(A7)

In the above $[J] = (2J+1)^{1/2}$ and $[\lambda] = (2\lambda+1)^{1/2}$

and ~~~

if there is only one closed shell or tos sum of similar terms if there are more shells. is the expectation value of and it is different for diferent shells. The second term in ( )

can be evaluated as shown by Devanathan and Ramachandran[1] by using the concept of fractional parentage coefficients. The calculation of this term is of curse a little tedious.

If we consider the elastic scattering of pions from nuclei for which both the proton and theneutron shells are closed e.g. carbon, then the problem becomes enormously simple. The square of the matrix element in the case of carbon is

The suffixes and denote that the expectation value of is taken between th corresponding states. is the spin-independent term and thus we see that only th spin independent part contributes for th elastic scattering of pions by carbon. Using harmonic oscillator wave functions, exp. ( ) can be evaluated.

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1) V. Devanathan and G. Ramachandran, Nuclear Physics, (In press).

$$|Q|^2 = |6|\,K_{oo}|^2\, e^{-k^2/3\beta} \left\{ 3 - \frac{k^2}{3\beta} \right\}^2 \qquad (A12)$$

where β is a parameter which depends on the oscillator potential well depth, the value of which can be determined by finding the expectation value $\langle r^2 \rangle$ and equating it with the experimentally determined value. Alternately, the experimentally data of the scattering of pions from nuclei can be used to determine the value of β which gives the best fit. Thus the scattering of pions from nuclei can be used as a tool to probe the structus of nuclei.

Preliminary calculations have been carried out in the case of carbon and the agreement is good at low angles but the theoretical values obtained are much smaller at large angles. In this connection, attention may be drawn to the experimental findings of Ba ker, Rainwater and Williams. The differential cross-sections they have obtaind agree well with the previously reported results at small angles but smalle at large angles by a factor vrying between two and ten. It may be pointed out that at large angle, the inelastic events predominate due tolarge momentum transfers; and it is very likely that some of thee evens with low loss of energy maybe counted as elastic since in almost all experiments, the elastic and inelastic events are distinguished only by measuring the energy of the scattered pions and the reliability of the experimental result thus largely depends on th energy resolution of the apparatus.

With the improvement in th experimental technique, the experiments register a lo er cross section at large angles andwith further improvement, itmay be expected that thelarge angle cross-sections may further decrease giving an agreement with our preliminary findings. On the other hand, in order to obtain a fit with the existing experimenal results for theelastic scattering at large angles, we have to take into account some of the inelastic events with low energy losses.

CHAPTER VI

DISINTEGRATION PROCESS IN DEUTERIUM

In the previous chapter we studied the various scattering process that a pion can undergo on the deuteron. In the present chapter we study the absorption process.

$$\pi^+ + D \rightarrow p + p \qquad \text{(Pure absorption)} \quad \text{I}$$

$$\pi^+ + D \rightarrow p + p + \gamma \qquad \text{(Radiative capture)} \quad \text{II}$$

and also the disintegration process initiated by a photon and an electron, namely

$$\gamma + D \rightarrow p + n \qquad \text{(Photo-disintegration)} \quad \text{(III)}$$

$$e + D \rightarrow e' + p + n \qquad \text{(Electro-disintegration)} \quad \text{(IV)}$$

These processes are studied in a unified way using low's method which brings out the similarity and inter-connection between the matrix elements for these processes. The general features of the angular distributions have been studied.

Chew et al[1] considered process I under the impulse approximation. They made the observation that to explain the observed angular distribution, it is necessary to include the D-state wave function of the deuteron Later Rockmore[2] calculated the contribution of the pion absorption process to the total scattering cross-section of the pion on deuterium. He used the model for pion absorption due to Brueckner and Watson which treats the absorption

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1) G.F.Chew, M.L.Goldberger, J.L.Steinberger and C.N. Yang, Phys. Rev. 84 , 581 (1951)

2) R.M.Rockmore, Phys. Rev. 105 , 256 (1957)

as a co-operative process between the two nucleons. He obtained an order of magnitude accuracy for the cross-section which, according to the experiment of Rogers and Lederman[1] are about 7 millibarns at a pion kinetic energy of 85 Mev. The differential cross-section is of the form

$$\frac{d\sigma}{d\Omega} \propto \frac{A + \cos^2\theta}{A + 1/3}$$

so that if one observes the outgoing nucleon at an angle θ for which $\cos\theta = \frac{1}{\sqrt{3}}$, the differential and total cross-sections will be independent of A and differ only by a factor 4π. The total cross-section for the radiative capture at 85 Mev as given by Rogers and Lederman is 1 millibarn which is not inconsiderable.

The photo - and electro - disintegration of the deuteron have been studied by different approaches by various workers. Pearlstein and Klein[2] have suggested that it is convenient to divide the analysis of the photo-disintegration problem into two parts corresponding to two energy regions. In one, the region below 100 Mev, the experimental results can be completely understood within the framework of conventional quantum mechanics which gives the result that the process essentially proceeds through a dipole transition in this energy region. In the region above 100 Mev, however, there is evidence for the presence of virtual meson currents which make themselves felt with increasing photon energy. Mixing a covariant formalism for deriving a formally exact expression for the matrix element for the process with a phenomenological procedure they

1) K.C.Rogers and L.M.Lederman, Physics Review, 105, 247 (1957)
2) L.D.Pearlstein and A.Klein, Physics Review, 108, 193 (1960), References to earlier literature is given in this paper.

show that such a division has a theoretical basis and that contributions from the virtual mesons are negligible below 100 Mev. Retaining only the on-meson exchange effect they connect the matrix element for dipole transition with the p-wave meson-nucleon coupling constant and the amplitude for photo-production off the energy shell.

As regards the electro-disintegration process it has been most useful in studying the form factors of the nucleon. The basic non-relativistic theory of the process was given by Jankus[1] who calculated the differential cross-section using a Hulthen wave function of the outgoing nucleons. The final state interaction between the outgoing nucleons was treated approximately using a central force model for the interaction. Durand[2] made a more complete formulation of the problem of calculating final state effects and studying the influence of the D-state component of the deuteron wave function on the process. He has also discussed the relativistic corrections using the methods of dispersion relations and has pointed out that the calculation of the final state interactions would require an examination of the double spectral function in the Mandelstam representation[3] for the transition matrix element. The importance of the final state interaction and mesonic corrections near the threshold for deuteron break-up was also pointed out.

We have studied the above processes in a unified way using Low's technique. The deuteron which is of composite structure is always kept in the state vector and only the other particles are converted

1) V.Z. Jankus, Physics Review 102, 1586 (1956)

2) Durand III, Physics Review 115, 1020 (1959); ibid, 123, 1393 (1961)

3) S.Mandelstam, Physics Review- JETP,5,1249 (1957) 112, 1344 (1958)

into current operators when necessary and so the matrix elements in each case will include a term involving the D-p-n vertex which has a pole the residue of which is the normalization factor of the deuteron wave function. Integral equations are derived for the matrix elements for the various processes which are interrelated. An angular momentum analysis of these processes which would be helpful in the solution of these equations is made using the angular operators of Ritus[1]. The angular distributions are discussed on this basis and relations between the energy dependent coefficients are given.

2. THE MATRIX ELEMENTS

Using Low's method the matrix element for process I when a pion of four-momentum is q absorbed by the deuteron of momentum d, the outgoing protons having momenta p_1 and p_2 can be written as

$$\langle p_1 p_2 | T | q d \rangle = - \frac{1}{\sqrt{4 E_{p_2} \omega_q}} \bar{u}(p_2)$$

$$\times \left\{ \sum_{\vec{n} = \vec{p_1} + \vec{p_2}} \frac{\langle p_1 | J_N(0) | n \rangle \langle n | J_\pi(0) | d \rangle}{E_{p_2} + E_{p_1} - E_{p_n} + i\varepsilon} \right.$$

$$- \sum_{\vec{n} = \vec{d} - \vec{p_2}}{}^{n} \frac{\langle p_1 | J_\pi(0) | n \rangle \langle n | J_N(0) | d \rangle}{E_{p_2} + E_{p_n} - E_d - i\varepsilon}$$

$$\left. + i g \gamma_5 \gamma_i \frac{\langle p_1 | J_N(0) | d \rangle}{i \gamma_\mu (d - p_1)_\mu - m} \right\} \qquad (1$$

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1) V.I. Ritus, Soviet Physics - JETP, 5, 1249 (1957)

where J_N and J_π are the nucleon and meson currents respectively.

$$J_N = i g \gamma_5 \tau_i \phi(x) \psi(x) + \delta m \psi(x)$$

$$J_\pi = i g \bar{\psi} \gamma_5 \tau_i \psi(x)$$

The lowest intermediate state that contributes to the first term is the two nucleon state so that this term will give rise to an integral equation for the matrix element for the process, the kernel being the matrix element for nucleon-nucleon scattering, which, in our procedure, represents the final state interaction. Diagrammatically this term is given by Fig. (2). The second term gives the "pole" or Born approximation term if we take the single nucleon intermediate state only. The numerator is then the product of the pion-nucleon vertex and the deuteron-proton-neutron (D-p-n) vertex The latter on the energy shell of the intermediate nucleon is just the normalization of the deuteron wave function, which is equal to $\left[2\alpha / 1-\alpha\rho\right]^{\frac{1}{2}}$ where $\alpha = m\epsilon$, m being the mass of the nucleon, and ϵ the binding energy of the deuteron. ρ is the triplet effective range. In terms of the dispersion graphs, this term would, if we assume it to be written in terms of the propagator (instead of the energy denominator) by addition of a term containing a suitable higher particle intermediate state (see next chapter), correspond to Fig. (1a). Fig. (1b) corresponds to the third term of (1) if we rewrite it such that the denominator becomes $\left((\alpha-p_1)^2+m^2\right)$ However, by this procedure, the first factor in the numerator, namely the pion-nucleon vertex, would then be written in terms of bare quantities whereas the second factor which is the D-p-n vertex is given in the Heisenberg representation. A suitable renormalization procedure has to be gone through before identification with the dispersion graphs (1) is made.

However, by this procedure, the first factor in the numerator, namely, the pion-nucleon vertex, wouldthen be written in terms of bare quantities whereas the second factor which is the D-p-n vertex is given in the Heisenberg representation. A suitable renormalisation procedure has to be gone through before identification with the dispersion graphs (1) is made.

The lowest two particle intermediate in the second term of (1) corresponds to a one pion plus one nucleon state, thus giving rise to the product of matrix elements mt for those for pion-nucleon scattering and process I itself. This term is represented by Fig. 3.

The matrix element for the photo-disintegration of the deuteron (process III) is similar to (1) if we replace the pion-current operator J_π by the photon current operator, J_γ , the only difference being an additional pole term.

$$\frac{\langle p_1 | J_N(0) | d' \rangle \langle d' | J_\gamma(0) | d \rangle}{E_{p_2} + E_{p_1} - E_{d'} + i\varepsilon} \qquad (3)$$

obtained by taking the deuteron intermediate state. This term is absent for meso-disintegration because of the requirement of conservation of isotopic spin. The pole terms are represented by Fig. (4) and the term representing final state interaction by Fig. (5)

Fig.(6) which corresponds to Fig.(3) for process I involves a matrix element connected with that for photoproduction of a pion from a nucleon and represents the mesonic contribution to the process, referred to in the introduction.

Using the procedure adopted for deriving the matrix element for electro-pion production (chapter IV, Sec. 4), the matrix element for electro-disintegration of the deuteron can be reduced to that for photodisintegration by a virtual photon with plarization vector $\frac{ie\,\bar{u}(s_2)\,\gamma_\mu\,u(s_1)}{(s_1-s_2)^2}$

where s_1 and s_2 are the four-momenta of the initial and final electrons respectively. An important difference is in the fact that the electromagnetic form factors for the nucleon charge (F_1) and magnetic moment (F_2) appearing in the expression for the electromagnetic vertex of the nucleon

$$\langle p \mid J_\gamma(0) \mid n \rangle = \frac{ie}{(4p_0 n_0)^{1/2}} \times \bar{u}(p)\left[\gamma_\mu F_1(q^2) + \frac{\mu}{2m} F_2(q^2)\,\sigma_{\mu\nu}(p-n)_\nu\right] u(n) \tag{4}$$

and the electromagnetic form factors of the deuteron charge (F_{1d}), magnetic moment (F_{2d}) and the electric quadrupole moment (F_{3d}) appearing in the expression for the electromagnetic vertex of the deuteron.

$$\langle d' \mid J_\gamma(0) \mid d \rangle = \frac{ie}{(4d_0 d_0')^{1/2}}\Big[F_{1d}(q^2)\,\xi'^{*}\cdot\xi\,(d'+d)_\mu + F_{2d}(q^2)\,(\xi_\mu \xi'^{*}\cdot q - \xi'^{*}_\mu \xi\cdot q) + F_{3d}(q^2)\,\frac{(\xi'^{*}\cdot q)(\xi\cdot q)}{m^2}(d+d')_\mu\Big] \tag{5}$$

are no longer constants (since the square of the momentum transfer q^2 which is the square of the momentum of the virtual photon need not be zero) but will depend on q^2. In (5) ξ is a complex vector representing the polarization vector of the deuteron. As

in the case of photodisintegration, the mesonic contribution arising from Fig. (6) is likely to become important for energies corresponding to and above the threshold for pion production.

Finally the matrix element for the radiative capture of the pion of momentum q by a deuteron of momentum, d, (process II), with emission of a photon of momentum k is given by

$$\langle p_1 p_2 k|T|qd\rangle = \frac{1}{\sqrt{4\omega_q k}}\left[\frac{\langle p_1 p_2|J_\pi(0)|d'\rangle\langle d'|J_\gamma(0)|d\rangle}{\omega_q + E_{p_1} + E_{p_2} - E_{d'} + i\varepsilon}\right.$$

$$+\iint \frac{\langle p_1 p_2|J_\pi(0)|p'n'\rangle\langle p'n'|J_\gamma(0)|d\rangle}{\omega_q + E_{p_1} + E_{p_2} - E_{p'} - E_{n'} + i\varepsilon}$$

$$\times\ \delta(\vec{p}' + \vec{n}' - \vec{p}_1 - \vec{p}_2 - \vec{q})\, d\vec{p}'\, d\vec{n}'$$

$$\left.+\iint \frac{\langle p_1 p_2|J_\gamma(0)|p'p''\rangle\langle p'p''|J_\pi(0)|d\rangle}{\omega_q + E_{p'} + E_{p''} - E_d - i\varepsilon}\right.$$

$$\times\ \delta(\vec{p}' + \vec{p}'' - \vec{d} - \vec{q})\, dp'\, dp'' \qquad (6)$$

if we retain only one and two-particle intermediate states. In the above, the incoming pion and the outgoing photon are converted into current operators. The first term on the right hand side of (6) is represented in Fig. (7a). The numerator represents the product of the matrix element for pure absorption in deuterium (process I) and the electromagnetic vertex of the deuteron. If instead of "contracting" on the pion and photon we had converted the initial pion and the final nucleon into currents we would have obtained the term represented by Fig. (7b) which involves the product of the matrix element for radiative capture of the pion by a nucleon and the N-p-n vertex. It is interesting to note that this is precisely the term we should expect on the basis of the

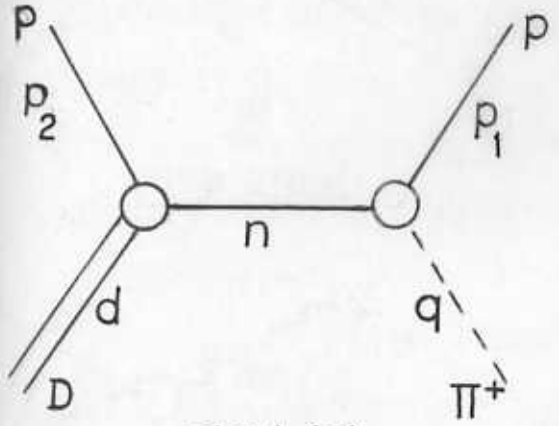

FIG. 1 (a)

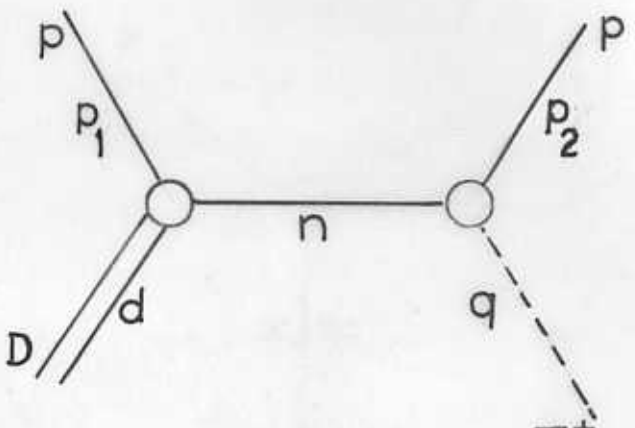

FIG. 1 (b)

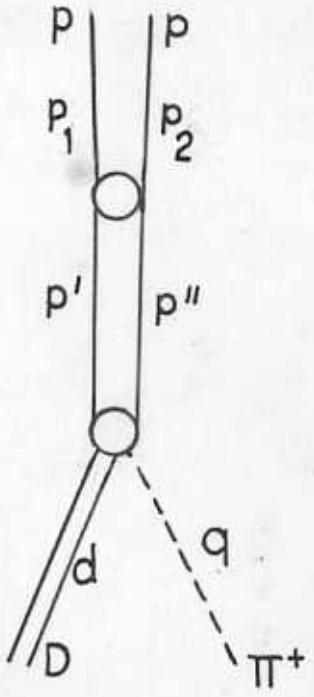

FIG.2

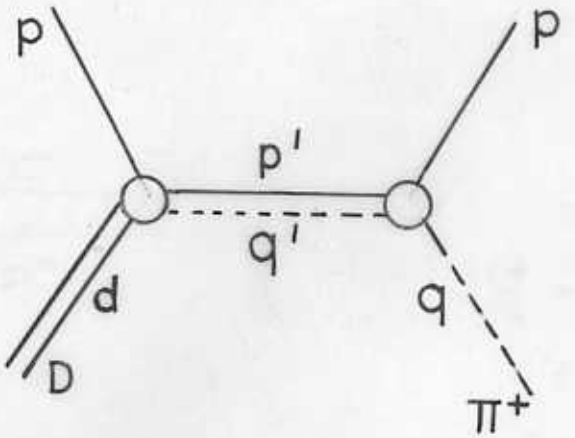

FIG. 3.

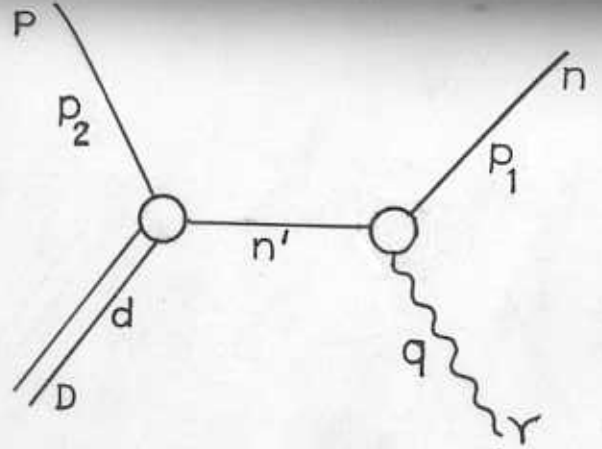

FIG. 4 (a)

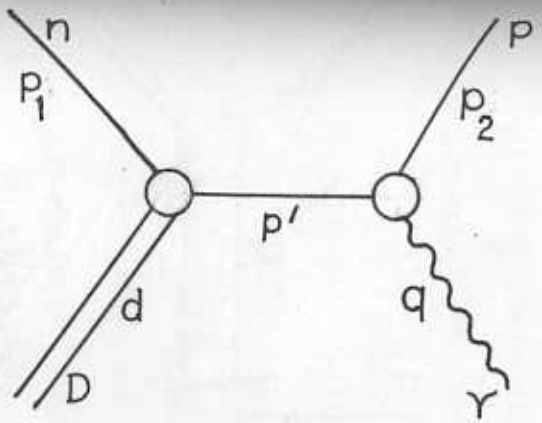

FIG. 4 (b)

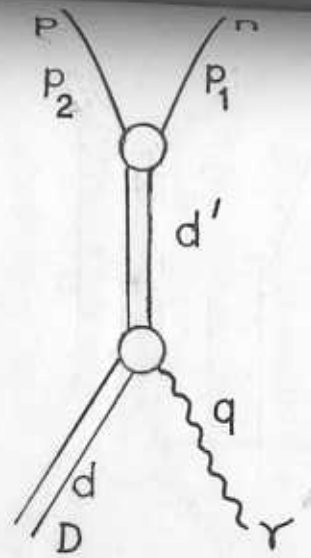

FIG. 4 (C)

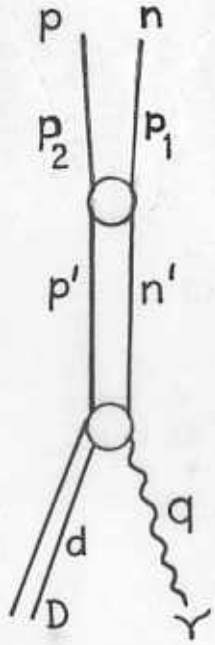

FIG. 5

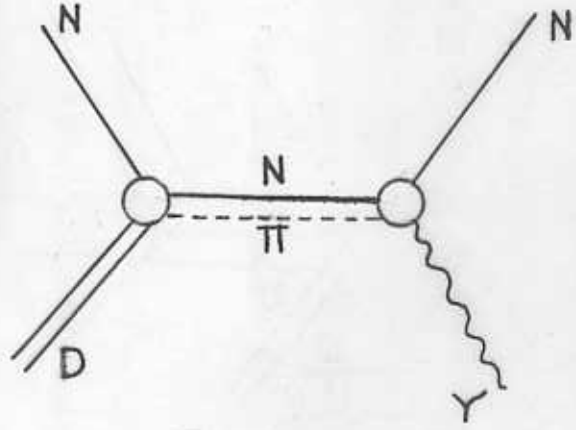

FIG. 6

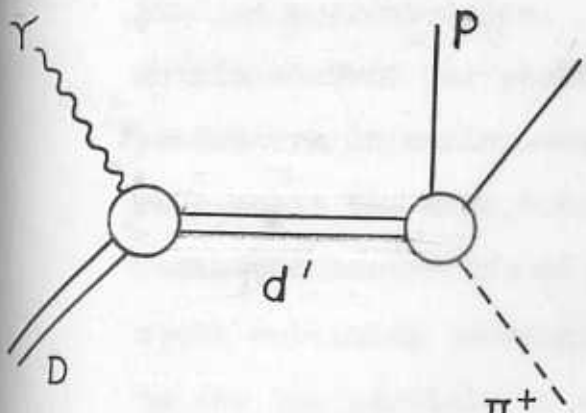

FIG. 7 (a)

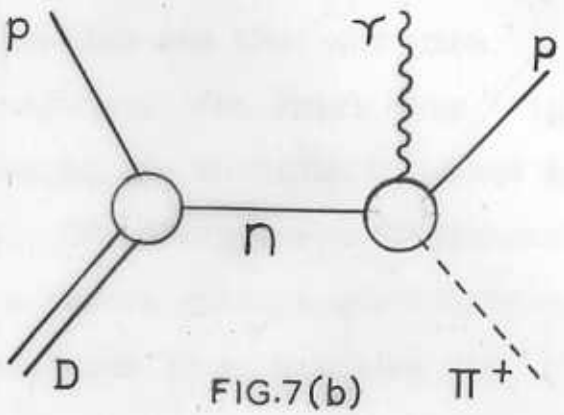

FIG.7(b)

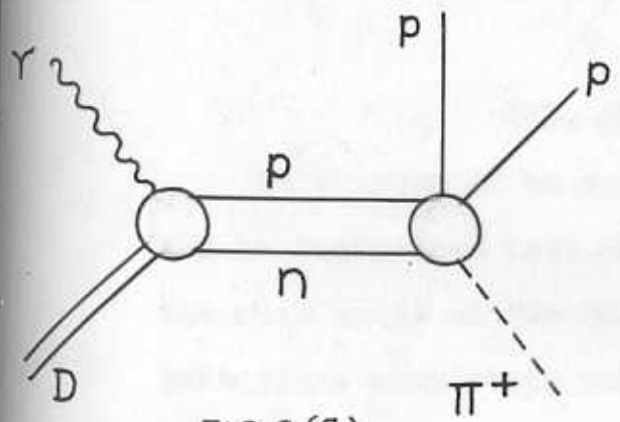

FIG.8 (a)

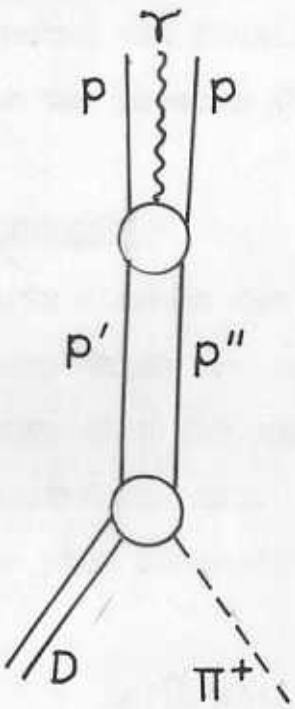

FIG.8(b)

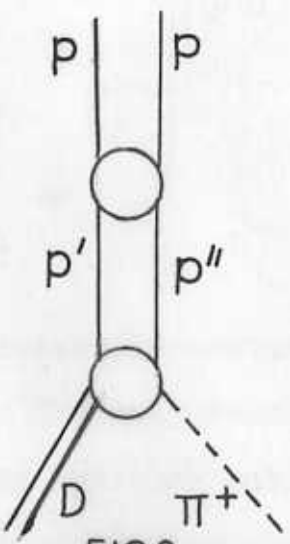

FIG.9

impulse approximation. The second term (Fig. 8a) involves the matrix element for photo-disintegration and that for pion production in nucleon-nucleon collision. The third term (Fig. 8b) represents the meso_disintegration of the deuteron followed by radiative scattering of nucleons. The three particle intermediate state cntaining two-nucleon and a photon gives a contribution to the two particle intermediate states if we consider the intermediate photon to go off as the final photon without interaction. Then this term will represent the final state interaction (between the nucleons) for the process (Fig. 9).

3. ANGULAR DISTRIBUTIONS

In chapter II we saw how the matrix element for any process can be decomposed into angular operators which are matrices in the spin space of the nucleons and which give the angular distributions completely and an energy dependent part. More speifically, the angular operators for pion absorption in deuterium can be written as

$$L_{Js'l'sl}(\hat{k}'\beta';k\beta)=\sum_{\mu'\mu} L^{\mu'\mu}_{Js'l'sl}(\hat{k}'\cdot\hat{k})\,Q_{s'\mu'}(\beta')\,Q^{*}_{s\mu}(\beta)$$

$$L^{\mu'\mu}_{Js'l'sl}(\hat{k}\cdot k)=\sum_{M} C^{l'M-\mu',s'\mu'}_{JM}\,C^{lM-\mu,s\mu}_{JM}$$

$$\times\, Y_{l'M-\mu'}(\hat{k}')\;Y^{*}_{lM-\mu}(\hat{k}) \qquad (7)$$

In (7) JSl represent respectively the total angular momentum, the total spin S (=1 for triplé-triplet and 0 for singlet→triplet transitions) and the relative orbital angular momentum of the pion. The prime denotes similar quantities in the final state.

k and k' are unit vectors denoting the directions of the initial and final set of particles in the centre of mass. M and μ are the z -components of the total angular momatum and spin. The summation over the z - components of the initial and final spins S and S' respectively can be conveniently represented by the insertion of the spin projection operators, $\sigma_{1\pm} \pm \sigma_{2\pm}$) and $\sigma_{1z} \pm \sigma_{2z}$. Substituting the values for the Clebsch-Gordans and spherical harmonics, the angular operators for various partial transitions for a given total angular momentum J can be derived and have been given by Ritus[1)]

The angular operators for photo-(and electro) disintegration of the deuteron can be derived from those for pion absorption (process I) and nucleon-nucleon scattering by operating by $\frac{1}{\sqrt{j(j+1)}} (\vec{\epsilon} \cdot \frac{\partial}{\partial \hat{k}})$ for the electric and by $\frac{i}{\sqrt{j(j+1)}} [(\hat{k} \times \vec{\epsilon}) \cdot \frac{\partial}{\partial \hat{k}}]$ for the magnetic multipole transitions (as explained in chapter II). These are given in Tables I and II respectively. In the case of electro-disintegration we have a further contribution from the logitudnal multipole transitions and these are obtained by multiplying by $\hat{k} \cdot \vec{\epsilon}$ the angular operators for those transitions and which have the same parity as in the case of electric multipole transitions. A further pointto be noted is that here a $J = 0 \longrightarrow J = 0$ transitions is allowed unlikethe other cases. The angular operators for this case are given in Table III. In these tables, we use the following notation.

$$\vec{S} = \frac{1}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \; ; \; \vec{T} = \frac{1}{2} (\vec{\sigma}_1 - \vec{\sigma}_2)$$

$$\vec{n} = (\hat{k}' \times \hat{k}) \quad ; \quad \vec{W} = (\hat{k} \times \vec{\epsilon}) \qquad (8)$$

$\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the spin operators (matrices) for nucleons 1 and 2 respectively.

1) V. I. Ritus, loc. cit

Now we consider the angular distribution for the various processes, using the notation $a_{J\ell\ell'}$ and $a_{J\ell\ell'}$ for the energy dependent parts of the matrix elements for pion absorption and photo-(or electro-) disintegration respectively.

(a) Pion absorption:- If we assume that the pion is absobbed only in an s or p state which will be the case for low energes, the angular distribution for process I is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left[3|a_{101}|^2 + 5(1+3\cos^2\theta)|a_{212}|^2 + 2|a_{010}|^2 + 2\sqrt{10}(1-3\cos^2\theta)\,\mathrm{Re}(a^*_{212}a_{010})\right] \tag{9}$$

Now, as mentioned in the introduction, the experimental differential cross-section can be represented by

$$\frac{4\pi}{\sigma}\frac{d\sigma}{d\Omega} = \frac{A+\cos^2\theta}{A+1/3} \tag{10}$$

where the quantity A has been observed to increase with increasing pion energies. We notice that for $\cos\theta = 1/\sqrt{3}$, the differential (and hence the total) cross-sections are independent of A. Using this fact we get the following relations between the energy dependent coefficients

$$3|a_{101}|^2 + 10|a_{212}|^2 + |a_{010}|^2 = \frac{\sigma}{4\pi} \tag{11}$$

Further, by equating the angle dependent and independent parts in (9) with the corresponding quantities in (10), we obtain

$$3|a_{101}|^2 + 5(1-3A)|a_{212}|^2 + 2|a_{010}|^2 + 2\sqrt{10}(1+3A)\,\mathrm{Re}(a^*_{212}a_{010}) = 0 \tag{12}$$

From the expression (11) and (12), the various energy dependent coefficients can be evaluated.

(b) Photo-disintegration:-

At low energies only an electric dipole transition is likely to occur. The corresponding differential cross-section is given by.

$$\frac{d\sigma_e \cdot d}{d\Omega} = \frac{1}{64}\left[49\,|a_{211}|^2 - 30\,|a_{111}|^2 + 36\,\mathrm{Re}\,(a^*_{111}\,a_{211})\right]\sin^2\theta + \frac{1}{16} \times \left(|a_{211}|^2 - \frac{3}{2}\,\mathrm{Re}\,(a^*_{111}\,a_{211})\right)\cdot\cos\theta + \frac{3}{32}\left[3\,|a_{211}|^2 + 13\,|a_{111}|^2 - \mathrm{Re}\,(a^*_{111}\,a_{211})\right] \tag{13}$$

The experimental cross-section is of the form $b + c\ \sin^2\theta$ [1] which gives us on comparison with (13) the relations,

$$\begin{aligned} |a_{211}|^2 &= \tfrac{3}{2}\,\mathrm{Re}\,(a^*_{111}\,a_{211}) \\ 7\,|a_{211}|^2 + 39\,|a_{111}|^2 &= 32\,b \\ 73\,|a_{211}|^2 - 30\,|a_{111}|^2 &= 64\,c \end{aligned} \tag{14}$$

from which we obtain

$$\begin{aligned} |a_{111}|^2 &= \frac{32}{3057}\,(73\,b - 14\,c) \\ |a_{211}|^2 &= \frac{64}{3057}\,(15\,b + 39\,c) \end{aligned} \tag{15}$$

Retaining only the square terms, the contribution from the

magnetic dipole transition to the differential cross-section is given by

$$\frac{d\sigma_{m.d}}{d\Omega} = \frac{1}{32}\left[\{96\cos^2\theta - 111\cos^2\theta\sin^2\theta + 36\sin\theta\cos\theta\,(1+\cos\theta+\cos^2\theta) + 15\}|a_{2121}|^2 + 12|a_{1101}|^2 + \frac{3}{2}(13 - 9\sin^2\theta\cos^2\theta)|a_{1121}|^2\right] \quad (16)$$

(e) Electro-disintegration:- In this case we have in addition to the electric and magnetic transitions the logitudinal multipole contributions also. Retaining the electric dipole and longitudinal dipole only we have

$$\frac{d\sigma}{d\Omega} = A + B \quad (17)$$

where

$$A = \frac{1}{128}\{[99(\vec{k'}\cdot\vec{\epsilon})^2 + 9(\vec{k'}\times\vec{\epsilon})^2 + \vec{\epsilon}^2 + 8(\vec{k'}\times\vec{\epsilon})\cdot(\vec{k}\times\vec{\epsilon})]|a_{2111}|^2 + 9[3(\vec{k'}\cdot\vec{\epsilon})^2 + 17(\vec{k'}\times\vec{\epsilon})^2 + \vec{\epsilon}^2]|a_{1111}|^2 + 6[14(\vec{k'}\cdot\vec{\epsilon})^2 - (\vec{k'}\times\vec{\epsilon})^2 - \vec{\epsilon}^2 - 4(\vec{k'}\times\vec{\epsilon})\cdot(\vec{k}\times\vec{\epsilon})]\,\mathrm{Re}(a^*_{1111}a_{2111})\} \quad (18)$$

and

$$B = \frac{(\vec{k}\cdot\vec{\epsilon})^2}{64}\Big[\{99\cos^2\theta' + 9(\vec{k}'\times\vec{k})^2 + \vec{k}^2\}|a_{211}|^2$$
$$+ 4\{3\cos^2\theta' + (\vec{k}'\times\vec{k})^2 + \vec{k}^2\}|a_{011}|^2$$
$$+ 18\{\cos^2\theta' + 4(\vec{k}'\times\vec{k})^2 + \vec{k}^2\}|a_{111}|^2$$
$$+ 4\{14\cos^2\theta' - 5(\vec{k}'\times\vec{k})^2 + \vec{k}^2\}\mathrm{Re}(a^*_{211}a_{011})$$
$$+ 6\{14\cos^2\theta' - 5(\vec{k}'\times\vec{k})^2 - \vec{k}^2\}\mathrm{Re}(a^*_{211}a_{111})$$
$$+ 12\{2\cos^2\theta' - (\vec{k}'\times\vec{k})^2 + \vec{k}^2\}\mathrm{Re}(a^*_{011}a_{111})\Big] \qquad (19)$$

In the above $\vec{k}$ represents the direction of $(\vec{s}_1 - \vec{s}_2)$ and $\vec{\epsilon} = ie\,\dfrac{\bar{u}(s_2)\,\vec{\gamma}\,u(s_1)}{(s_1 - s_2)^2}$

θ' is the angle of one of the outgoing nucleons in the centre of mass system of the virtual photon and the initial nucleon. A and B correspond respectively to the contributions from the electric dipole and the longitudinal dipole transitions. The energy dependent coefficients can again be determined by comparison with experimental angular distributions.

Table I.

Angular Operators for electric multipole transitions

Triplet ⟶ Triplet

$$J = j+1 = l'+1$$

$$\frac{1}{(j+1)[j(j+1)]^{1/2}}\Big[\Big\{(j+3)(\vec{k}'\cdot\vec{\epsilon})-(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{\epsilon})-i(j+1)\vec{s}\cdot(\vec{k}'\times\vec{\epsilon})\Big\}P_j'$$

$$+\Big\{\big[2\vec{k}'\cdot\vec{k}-(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{k})-i(j+1)\vec{s}\cdot\vec{n}\big]\vec{k}'\cdot\vec{\epsilon}$$

$$-2(\vec{s}\cdot\vec{n})(\vec{s}\cdot(\vec{k}'\times\vec{\epsilon}))\Big\}P_j''-(\vec{s}\cdot\vec{n})^2(\vec{k}'\cdot\vec{\epsilon})P_j'''\Big]$$

$$J = j+1 = l'-1$$

$$\frac{1}{(j+1)[j(j+2)]^{1/2}}\Big[\Big\{(2j+3)(j+2)(\vec{s}\cdot\vec{k}')^2(\vec{k}'\cdot\vec{\epsilon})$$

$$-2(j+2)(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{\epsilon})+i(j+2)(\vec{s}\cdot(\vec{k}'\times\vec{\epsilon}))$$

$$-2j(j+2)(\vec{k}'\cdot\vec{\epsilon})\Big\}P_j'+\Big(\Big\{-2(j+2)(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{k})$$

$$+(2j+3)(\vec{s}\cdot\vec{k}')^2(\vec{k}'\cdot\vec{k})+i(j+2)(\vec{s}\cdot\vec{n})$$

$$+2(\vec{k}'\cdot\vec{k})\Big\}(\vec{k}'\cdot\vec{\epsilon})-2(\vec{s}\cdot\vec{n})(\vec{s}\cdot(\vec{k}'\times\vec{\epsilon}))P_j''$$

$$-(\vec{s}\cdot\vec{n})^2(\vec{k}'\cdot\vec{\epsilon})P_j'''\Big]$$

$$J = j-1 = \ell'+1$$

$$\frac{1}{j(j^2-1)^{1/2}}\Big[\Big\{(2j-1)(j-1)(\vec{s}\cdot\vec{k}')^2(\vec{k}'\cdot\vec{\epsilon})+2(j-1)(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{\epsilon})$$

$$-i(j-1)\vec{s}\cdot(\vec{k}'\times\vec{\epsilon})+2(1-j^2)\vec{k}'\cdot\vec{\epsilon}\Big\}P_j'$$

$$+\Big(\Big\{2(j-1)(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{k})-(2j-1)(\vec{s}\cdot\vec{k}')^2(\vec{k}'\cdot\vec{k})-i(j-1)(\vec{s}\cdot\vec{n})$$

$$2\vec{k}'\cdot\vec{k}\Big\}\vec{k}'\cdot\vec{\epsilon}-2(\vec{s}\cdot\vec{n})\Big(\vec{s}\cdot(\vec{k}'\times\vec{\epsilon})\Big)P_j''-(\vec{s}\cdot\vec{n})^2(\vec{k}'\cdot\vec{\epsilon})P_j'''\Big]$$

$$J = j-1 = \ell'-1$$

$$\frac{1}{j(j(j+1))^{1/2}}\Big[\Big\{(2-j)\vec{k}'\cdot\vec{\epsilon}-(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{\epsilon})+ij(\vec{s}\cdot(\vec{k}'\times\vec{\epsilon})\Big\}P_j'$$

$$+\Big(\Big\{2\vec{k}'\cdot\vec{k}-(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{k})+ij(\vec{s}\cdot\vec{n})\Big\}\vec{k}'\cdot\vec{\epsilon}$$

$$-2(\vec{s}\cdot\vec{n})\Big(\vec{s}\cdot(\vec{k}'\times\vec{\epsilon})\Big)P_j''-(\vec{s}\cdot\vec{n})^2(\vec{k}'\cdot\vec{\epsilon})P_j'''\Big]$$

$$J = j = \ell'$$

$$\frac{2j+1}{[j(j+1)]^{3/2}}\Big[\Big\{(j-1)(j+2)\vec{k}'\cdot\vec{\epsilon}+(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{\epsilon})\Big\}P_j'+\Big(\Big\{(\vec{s}\cdot\vec{k}')(\vec{s}\cdot\vec{k})$$

$$-2\vec{k}'\cdot\vec{k}\Big\}\vec{k}'\cdot\vec{\epsilon}+2(\vec{s}\cdot\vec{n})(\vec{s}\cdot(\vec{k}\times\vec{\epsilon}))P_j''+(\vec{s}\cdot\vec{n})^2P_j'''\Big]$$

Triplet $\longrightarrow$ Singlet

$$J = j = \ell'$$

$$\frac{i(2j+1)}{j(j+1)}\Big[\vec{T}\cdot(\vec{\epsilon}\times\vec{k}')P_j'+\Big(T\cdot(\vec{k}\times\vec{k}')\Big)(\vec{k}'\cdot\vec{\epsilon})P_j''\Big]$$

Table II

Angular operators for magnetic multipole transitions

Triplet → Triplet

$$J = j+1 = \ell'$$

$$\frac{-i(2j+3)^{\frac{1}{2}}}{(j+1)[j(j+2)]^{\frac{1}{2}}}\left[\left\{-(j+2)(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{w})+i(j+2)(\vec{S}\cdot\vec{k}')\left(\vec{S}\cdot(\vec{k}'\times\vec{w})\right)\right\}P_j'\right.$$
$$-\left(\left\{(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{k})-i(j+2)(\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{n})\right\}\times(\vec{k}'\cdot\vec{w})-i\left\{\left[\vec{S}\cdot\vec{w}-(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{w})\right]\right.$$
$$\left.\left.\times(\vec{S}\cdot\vec{n})+\left(\vec{S}\cdot(\vec{n}\times\vec{k}')\right)\left(\vec{S}\cdot(\vec{k}'\times\vec{w})\right)\right\}\right)P_j''-i\left\{\left(\vec{S}\cdot(\vec{n}\times\vec{k}')\right)(\vec{S}\cdot\vec{n})\right\}(\vec{k}'\cdot\vec{w})P_j'''\Big]$$

$$J = j = \ell'+1$$

$$\frac{-i(2j+1)^{\frac{1}{2}}}{(j+1)^{\frac{3}{2}}}\left[\left\{-(\vec{S}\cdot\vec{w})+i(j-1)(\vec{S}\cdot\vec{k}')\times\left(\vec{S}\cdot(k'\times\vec{w})\right)\right\}P_j'+\left(\left\{-(\vec{S}\cdot\vec{k})\right.\right.$$
$$+i(j-1)(\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{n})\Big\}\vec{k}'\cdot\vec{w}+i\left\{\left[(\vec{S}\cdot\vec{w})-(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{w})\right]\vec{S}\cdot\vec{n}\right.$$
$$\left.\left.+\left(\vec{S}\cdot(\vec{n}\times\vec{k}')\right)\left(\vec{S}\cdot(\vec{k}'\times\vec{w})\right)\right\}\right)P_j''+i(\vec{S}\cdot\vec{n})\left(\vec{S}\cdot(\vec{n}\times\vec{k}')\right)(\vec{k}'\cdot\vec{w})P_j'''\Big]$$

$$J = j = \ell'-1$$

$$\frac{-i(2j+1)^{\frac{1}{2}}}{j(j+1)^{\frac{3}{2}}}\left[\left\{-(\vec{S}\cdot\vec{w})-i(j+2)(\vec{S}\cdot(\vec{k}'\times\vec{w}))(\vec{S}\cdot\vec{k})\right\}\times P_j'-\left(\left\{(\vec{S}\cdot\vec{k})+i(j+2)\right.\right.$$
$$(\vec{S}\cdot\vec{n})(\vec{S}\cdot\vec{k}')\Big\}\vec{k}'\cdot\vec{w}-i\left\{\left[(\vec{S}\cdot\vec{w})-(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{w}\right](\vec{S}\cdot\vec{n})\right.$$
$$\left.\left.+\left(\vec{S}\cdot(\vec{n}\times\vec{k}')\right)\left(\vec{S}\cdot(\vec{k}'\times\vec{w})\right)\right\}P_j''+i(\vec{S}\cdot\vec{n})\left(\vec{S}\cdot(\vec{n}\times\vec{k}')\right)(\vec{k}'\cdot\vec{w})P_j'''\Big]$$

$$J = j-1 = \ell'$$

$$\frac{-i(2j-1)^{1/2}}{j^{3/2}\left[j^2-1\right]^{1/2}}\left[\left\{(j-1)(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{W})-i(j-1)(\vec{S}\cdot\vec{k}')(\vec{S}\cdot(\vec{k}'\times\vec{W}))\right\}P_j'\right.$$

$$-\left(\left\{(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{k})+i(j-1)(\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{n})\right\}\vec{k}'\cdot\vec{W}+i\left\{(\vec{S}\cdot\vec{n})\left[(\vec{S}\cdot\vec{W})-(\vec{S}\cdot\vec{k}')(\vec{k}'\cdot\vec{W})\right]\right.\right.$$

$$\left.\left.+(\vec{S}\cdot(\vec{n}\times\vec{k}'))(\vec{S}\cdot(\vec{k}'\times\vec{W}))\right\}\right)P_j''-i(\vec{S}\cdot(\vec{n}\times\vec{k}'))(\vec{S}\cdot\vec{n})(\vec{k}'\cdot\vec{W})P_j'''$$

Triplet $\longrightarrow$ Singlet

$$J = j+1 = \ell'$$

$$\frac{-i(2j+3)^{1/2}}{\left[j(j^2+1)\right]^{1/2}}\left[\left\{j(\vec{T}\cdot\vec{k})(\vec{k}'\cdot\vec{W})+(\vec{T}\cdot\vec{k}')\right.\right.$$

$$\left.-(\vec{T}\cdot\vec{W})(\vec{k}'\cdot\vec{k})\right\}P_{j+1}'+(j+1)(\vec{T}\cdot\vec{W})P_{j+1}$$

$$\left.+\left\{(\vec{T}\cdot\vec{k}')-(\vec{T}\cdot\vec{k})(\vec{k}'\cdot\vec{k})\right\}(\vec{k}'\cdot\vec{W})P_{j+1}''\right]$$

$$J = j-1 = \ell'$$

$$\frac{-i(2j-1)^{1/2}}{\left[j(j^2-1)\right]^{1/2}}\left[\left\{(\vec{T}\cdot\vec{k}')-(j+1)(\vec{T}\cdot\vec{k})(\vec{k}'\cdot\vec{W})-(\vec{T}\cdot\vec{W})(\vec{k}'\cdot\vec{k})\right\}P_{j-1}'\right.$$

$$\left.-j(\vec{T}\cdot\vec{k})P_{j-1}+\left\{(\vec{T}\cdot\vec{k})-(\vec{T}\cdot\vec{k})(\vec{k}'\cdot\vec{k})\right\}P_{j-1}''\right]$$

Table III

Angular operators for longitudinal multipole transitions

Triplet → Triplet

$$J = j+1 = \ell'+1$$

$$\frac{1}{j+1}\Big[(j+1)P_j + \Big\{2(\vec{k}'\cdot\vec{k}) - (\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{k}) - i(j+1)(\vec{S}\cdot\vec{n})\Big\}P_j' - (\vec{S}\cdot\vec{n})^2 P_j''\Big](\vec{k}\cdot\vec{\epsilon})$$

$$J = j+1 = \ell'-1$$

$$\frac{1}{[(j+1)(j+2)]^{\frac{1}{2}}}\Big[\Big\{-2(j+1)^2 + (2j+3)(j+1)(\vec{S}\cdot\vec{k}')^2\Big\}P_j + \Big\{-2(j+2)(\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{k}) + (2j+3)(\vec{S}\cdot\vec{k}')^2(\vec{k}'\cdot\vec{k}) + i(j+2)(\vec{S}\cdot\vec{n}) + 2(\vec{k}'\cdot\vec{k})\Big\}P_j' - (\vec{S}\cdot\vec{n})^2 P_j''\Big](\vec{k}\cdot\vec{\epsilon})$$

$$J = j-1 = \ell'+1$$

$$\frac{1}{[j(j-1)]^{\frac{1}{2}}}\Big[\Big\{-2j^2 + (2j-1)j(\vec{S}\cdot\vec{k}')^2\Big\}P_j + \Big\{2(j-1)(\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{k}) - (2j-1)(\vec{S}\cdot\vec{k}')^2(\vec{k}'\cdot\vec{k})$$

$$-i(j-1)(\vec{S}\cdot\vec{n})+2(\vec{k}'\cdot\vec{k})\Big\}\,P_j'-(\vec{S}\cdot\vec{n})^2\,P_j''\Big]\,(\vec{k}\cdot\vec{\epsilon})$$

$$\underline{J=j-1=l'-1}$$

$$\frac{1}{j}\Big[-j\,P_j+\Big\{2(\vec{k}'\cdot\vec{k})-(\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{k})$$

$$+ij(\vec{S}\cdot\vec{n})\Big\}P_j'-(\vec{S}\cdot\vec{n})^2P_j''\Big]\,(\vec{k}\cdot\vec{\epsilon})$$

$$\underline{J=j=l'}$$

$$\frac{2j+1}{j(j+1)}\Big[j(j+1)P_j+\Big\{(\vec{S}\cdot\vec{k}')(\vec{S}\cdot\vec{k})$$

$$-2(\vec{k}'\cdot\vec{k})\Big\}P_j'+(\vec{S}\cdot\vec{n})^2P_j''\Big]\,(\vec{k}\cdot\vec{\epsilon})$$

Triplet $\longrightarrow$ singlet

$$\underline{J=j=l'}$$

$$\frac{i(2j+1)}{[j(j+1)]^{\frac{1}{2}}}\Big(\vec{T}\cdot(\vec{k}\times\vec{k}')\Big)\,(\vec{k}\cdot\vec{\epsilon})\,P_j'$$

CHAPTER VII.

THE EQUAL TIME COMMUTATOR IN ELECTRODYNAMICS

1. Introduction

In this chapter, we shall apply Low's procedure to some electrodynamic problems with a view to understanding the role of the equal time commutator in electrodynamics. The equal time commutator occurs both in Low's theory and in the dispersion theory, the latter being model independent and hence more general. The difference between the two approaches is that we are interested in the time-ordered product in the former case and the retarded product involving a commutator in the latter. Although the name "equal time commutator" has come into vogue

in dispersion theory, we shall use the same nomenclature for the corresponding term in Low's approach.

We can see how the equal time commutator arises indispersion theory by considering for simplicity the scattering of two identical neutral scalar bosons interacting through their self-field. If p and k represent the moments of the two incoming particles and p' and k' the moments of the two outgoing particles, the S-matrix element for the process can be written down using the "asymptotic condition"[1] to reduce the particles in the state vector into current operators. Thus

$$\langle p'k'|S|pk\rangle = \frac{i}{\sqrt{2\omega_k}} \int_{-\infty}^{\infty} d^4x\, e^{ikx} K_x \langle p'k'|A(x)|p\rangle + \langle p'k'|I|pk\rangle$$

$$= i\int_{-\infty}^{\infty} \frac{d^4x}{\sqrt{2\omega_k}} e^{ikx} \langle p'k'|j(x)|p\rangle + \langle p'k'|I|pk\rangle \qquad (1)$$

Here K_x is the operator $(\Box_x - m^2)$ which operating on the field variable $A(x)$ gives the current operator $j(x)$. The second term indicates propagation without scattering of the particles. Using translational invariance this could be written as

$$\langle p'k'|S|pk\rangle = \frac{i(2\pi)^4}{\sqrt{2\omega_k}} \delta^4(p'+k'-p-k)\langle p'k'|j(0)|p\rangle \qquad (2)$$

Now

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1) H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cim. 1 , 205 (1955).

$$\langle p'k'|j(0)|p\rangle = \frac{i}{\sqrt{2\omega_k}}\int d^4x e^{-ik'x} K_x \langle p'|T(A(x)j(0))|p\rangle \quad (3)$$

$$T(A(x)j(0)) = \theta(x)[A(x), j(0)] + j(0)A(x) \quad (4)$$

$$K_x T(A(x)j(0)) = \theta(x)[j(x), j(0)] - j(0)j(x)$$
$$-\delta(x_0)[A(x), j(0)] - \frac{d}{dt}\{\delta(x)[A(x), j(0)]\} \quad (5)$$

Here $\theta(x)$ is the step function which is equal to 1 when $x_0 > 0$ and 0 if $x_0 < 0$. If we assume that $j(x)$ contains no time derivatives of $A(x)$ the fourth term on the right hand side is zero. The third term is the equal time commutator term for an obvious reason and its contribution to the matrix element is given by

$$\frac{i}{\sqrt{4\omega_k\omega_{k'}}}\int e^{-ik'x}\{\delta(x_0)\langle p'|[A(x), j(0)]|p\rangle\, d^4x$$

$$= \frac{-i}{\sqrt{4\omega_k\omega_{k'}}}\int e^{-ik'x + i(p-p')\frac{x}{2}} \cdot \delta(x_0)\langle p'|[A(\tfrac{x}{2}), j(-\tfrac{x}{2})]|p\rangle d^4x$$

$$= -i\int e^{-i(k+k')\frac{x}{2}} \cdot G(p, p', x)\, d^4x \quad (6)$$

where causality implies that $G(p, p', x) = 0$ unless $x = 0$. Hence we can express G as a sum of finite derivatives of $\delta(x)$ i.e.

$$G(p, p', x) = \sum_{n=1}^{N} g_n(p, p')\,\delta^n(x) \quad (7)$$

so that the matrix element will have the contribution

$$\sum_{n=1}^{N} g_n (p, p') (k+k')^n$$

which is a polynomial.

The term corresponding to the equal time commutator in the Lagrangian theory always arises when we are converting into current operator a particle in the state vector, the field variable of which occurs either in bilinear or multilinear combination in the interaction Hamiltonian. This term brings, as we shall see, unrenormalised quantities in its wake.

Now as already mentioned the equal time commutator which term contributes a polynomial to the matrix element in the dispersion theoretic approach. But for its explicit evaluation one has to assume an interaction and the canonical commutation relations. For example in the pseudoscalar meson theory with pseudoscalar coupling we find with unrenormalised operators

$$[\varphi(x), \dot{\varphi}(y)]\, \delta(x_0 - y_0) = 0$$

$$\left[\frac{\partial \varphi(x)}{\partial x_0}, \dot{\varphi}(y)\right] \delta(x_0 - y_0) = 3i\lambda \varphi^2(x)\, \delta(x-y) \qquad (8)$$

Now the statement is usually made that these equal time commutator terms need not be evaluated explicitly[1] as they can be subtracted out since they do not depend on energy[2]. It is therefore interesting to study the role of the equal time commutator in the conventional Lagrangian approach for as we shall show in section 3 , by examples

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1) S. Gasiorowicz, Fortschritte der Physik, 8, 665 (1960)

2) M.L.Goldberger, Phys. Rev., 99 , 979 (1955); Also "Lectures on Dispersion Relations" by J.C.Taylor, Rochester University.

involving the electromagnetic interaction, this term in fact forms part and in some cases even the whole of the matrix element which would mean that it is dependent both on energy and momentum transfer.

We should expect that any method such as Low's procedure which enables one to write the matrix element of a process in the Heisenberg representation and which is non-perturbative in the sense mentioned earlier should when applied to problems in electrodynamics reproduce the well-established results of perturbation theory under suitable approximations. In the course of the above investigation, we show that this is the case in the various problems in quantum electrodynamics viz., the Compton ~~xxxx~~ effect, electron-electron scattering, electron-positron scattering and the bremsstrahlung. Explicit expressions for the lowest order matrix elements are derived and they have been shown to be identical with those obtained by the Feynman method. All the Feynman graphs have been obtained as a consequence without the need of separate enumeration. The relative signs of the various terms in the matrix elements are also obtained without recourse to ad hoc principles; i.e. the matrix elements obtained are properly symmetrised if the final state involves identical particles.

3. The role of the equal time commutator in

(a) The Compton effect :

First we shall consider the Compton effect, viz., the scattering of a photon by an electron. Using Low's procedure, as described in Chapter II and 'contracting' on the photons the matrix element for the process can be written as

$$\langle p'q'|S|pq\rangle = (-i)^2 \int\int \frac{dx\,dy}{\sqrt{4q_0 q_0'}}\, e^{iqx}\, e^{-iq'y}$$

$$\cdot \langle p'|P(j_\mu(y)\, j_\nu(x))|p\rangle \qquad (9)$$

where p and p' represent the initial and final four momenta of the electron and q and q' the initial and final four momenta of the photon. $j_\nu(x)$ and $j_\mu(y)$ are the Heisenberg photon current operators which satisfy the equation of motion

$$\Box A_\nu(x) = -j_\nu(x) = -ie\,\bar{\psi}(x)\gamma_\nu\psi(x) \qquad (10)$$

and similarly for $j_\mu(y)$. Using the translational invariance of the Heisenberg operators, the x and y dependence of the operators can be removed. The integration of the four-dimensional y yields the energy-momentum conservation and we have

$$\langle p'q'|S|pq\rangle = -i\int \frac{dx'}{\sqrt{4q_0'q_0}}\, e^{iqx'}$$

$$\cdot\,\delta(p'+q'-p-q)\langle p'|P(j_\mu(0)\,j_\nu(x'))|p\rangle \qquad (11)$$

where $x' = x - y$. To remove the x' dependence a complete set of intermediate states has to be interposed and the integration of the variable x' gives the momentum of the intermediate state and the energy denominators. Finally, we obtain

$$\langle p'q'|S|pq\rangle = \frac{-i}{\sqrt{4q_0'q_0}}\,\delta(p'+q'-p-q)$$

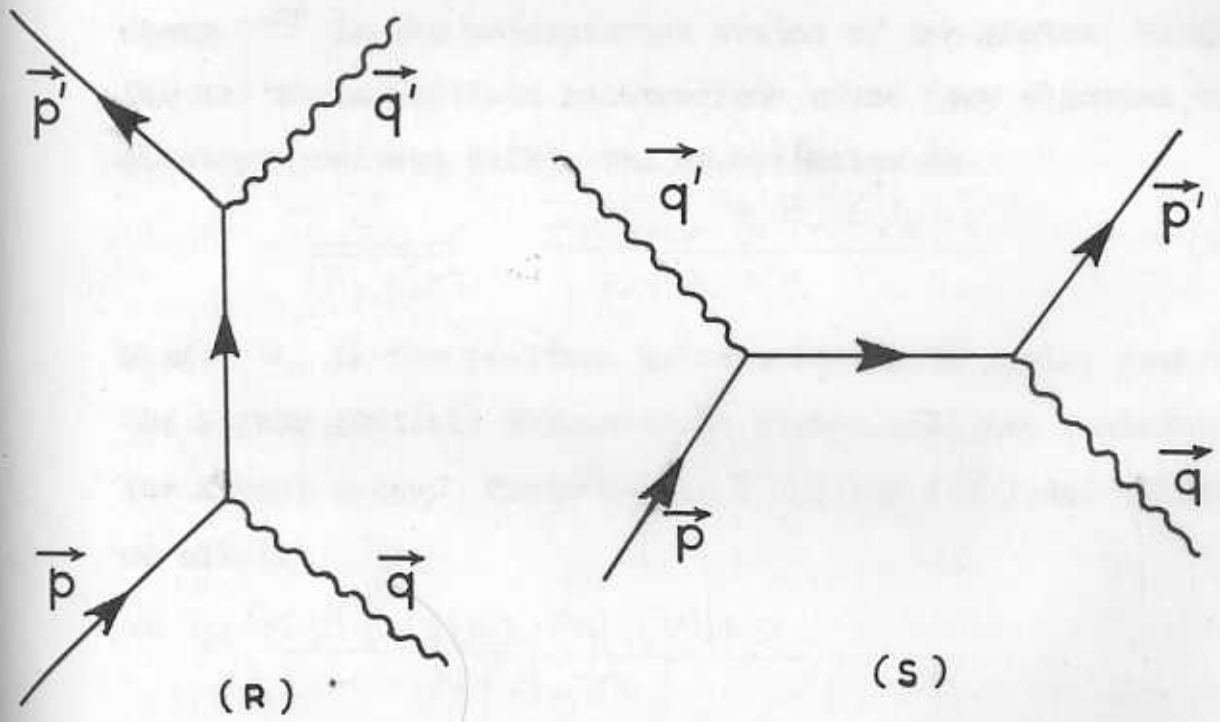
$$\cdot\Bigg[\sum_{\vec{n}=\vec{p}+\vec{q}} \frac{\langle p'|j_\mu(0)|n\rangle\langle n|j_\nu(0)|p\rangle}{p_0+q_0-n_0+i\varepsilon}$$

$$+\sum_{\vec{n}=\vec{p}'-\vec{q}} \frac{\langle p'|j_\nu(0)|n\rangle\langle n|j_\mu(0)|p\rangle}{p_0'-q_0-n_0+i\varepsilon}\Bigg] \qquad (12)$$

The above expression is exact and if we switch over to the interaction representation, we will get the perturbation expansion. We shall deduce the lowest order matrix element combining the energy denominators suitably to obtain the usual Feynman matrix elements for the process.

First we shall evaluate the first term in the square bracket in (12) in the lowest order. For the one-electron intermediate state, a replacement of the state vectors and current operators in the matrix elements in the numerator by the corresponding quantities in the interaction representation and use of the usual commutation relations between creation and annihilation operators enables us to simplify

FIG. 1 .



this term to

$$\frac{-e^2}{\sqrt{16 p_0 p_0' n^2}} \quad \frac{\bar{u}_{p'} \epsilon_\mu \gamma_\mu u_n \bar{u}_n \epsilon_\nu \gamma_\nu u_p}{p_0 + q_0 - n_0} \qquad (13)$$

where ϵ is the polarisation vector of the photon. Similarly for the three particle intermediate state (one electron plus an electron-positron pair), the contribution is

$$-\frac{e^2}{\sqrt{16 p_0 p_0' n^2}} \quad \frac{\bar{u}_{p'} \epsilon_\mu \gamma_\mu v_n \bar{v}_n \epsilon_\nu \gamma_\nu u_p}{p_0 + q_0 + n_0} \qquad (14)$$

where v_n is the positron spinor. It can be easily seen that the higher particle intermediate states will not contribute in the lowest order. Hence adding (13) and (14) and simplifying we obtain

$$\frac{\langle p' | j_\mu(0) | n \rangle \langle n | j_\nu(0) | p \rangle}{p_0 + q_0 - n_0} = \frac{-e^2}{\sqrt{4 p_0 p_0'}} \left[\bar{u}_{p'} \epsilon_\mu \gamma_\mu \frac{1}{i(\not{p} + \not{q}) + m} \epsilon_\nu \gamma_\nu u_p \right] \qquad (15)$$

Expression (15) is precisely the matrix element corresponding to one of the Feynman diagrams[1] corresponding to the Feynman diagram (Diagram R in fig. 1). Ramakrishnan et al[2] have shown from different considerations that the Feynman

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1) R.P. Feynman, Quantum Electrodynamics Lecture Notes, Caltech, (1953).

2) A. Ramakrishnan, T.K.Radha and R. Thunga, Proc. Ind. Acad. Sci. 52 , 228 (1960).

propagator in momentum representation can be conveniently split up into two parts -- one arising from the positive energy part of the intermediate state and the other from the negative energy part. They show that

$$\frac{1}{i(\not{p}+\not{q})+m} = \frac{1}{2}\left[\frac{i\overset{+}{\not{P}}_n - m}{n_0(p_0+q_0-n_0)} - \frac{i\overline{\not{P}}_n - m}{n_0(p_0+q_0+n_0)}\right] \qquad (32a)^{16}$$

where $\overset{+}{P}$ is a four-vector with energy component $n_0 = +\sqrt{(\vec{p}+\vec{q})^2+m^2}$ and $\overline{P}_n$ the four-vector with energy component $-n_0$. We obtainthe same result, but we use a different language - the one and three particle intermediate states. The three particle intermediate state here corresponds to Feynman's negative energy propagation.

In a similar way, the second term in (18)[12] can be evaluated. As shown earlier, the one-particle and three-particle intermediate states alone contribute and adding them we obtain

$$\frac{\langle p'|j_\nu(0)|n\rangle\langle n|j_\mu(0)|p\rangle}{p_0'-q_0-n_0+i\varepsilon} = \frac{-e^2}{\sqrt{4p_0p_0'}}\left[\bar{u}_{p'}\epsilon_\mu\gamma_\mu\,\frac{1}{i(\not{p}-\not{q}')+m}\,\epsilon_\nu\gamma_\nu u_p\right] \qquad (32b)^{17}$$

Expression (32b) is identical with the Feynman matrix element corresponding to the other diagram.

(Diagram 5 in ref. 1 , p. 195)[169]

Adding (※15) and (※17) we obtain the complete matrix element for the Compton effect in the lowest non-vanishing order.

Now one of the principal features of Low's method or that of Lehmann, Symanzik and Zimmermann is the transcription of some or all of the particles in the initial or final state into current operators and there is a variety of choice in doing this. In the above, we have converted the initial and final photons into current operators. If, instead, we convert one of the photons and one of the electrons into current operators, we will obtain the equal time commutator term, as shown below.

$$\langle p'q'|S|pq\rangle = -\frac{1}{\sqrt{4p_0' q_0}}\,\overline{u}_{p'} \cdot \iint dx\,dy\; e^{-iqx}\, e^{-ip'y} \langle q'|[P(J(y) j_\mu(x)) + i\delta(x-y) e\epsilon_\mu \gamma_\mu \psi(x)]|p\rangle \qquad (18)$$

Here we have converted the initial photon and final electron into current operators. $J(y)$ is the electron current operator given by

$$J(y) = \left(\gamma \frac{\partial}{\partial y} + m\right)\psi(y) = ie\gamma A(y)\psi(y) + \delta m\, \psi(y) \qquad (19)$$

Removing the x and y dependence of the operators and integrating them out, we obtain

$$\langle p'q'|S|pq\rangle = -i\,\frac{\delta(p'+q'-p-q)}{\sqrt{4p_0'q_0}}\,\bar{u}(p')\cdot\Bigg[\sum_{\vec{n}=\vec{p}'+\vec{q}'}\frac{\langle q'|J(0)|n\rangle\langle n|j_\mu(0)|p\rangle}{p_0'+q_0'-n_0+i\varepsilon}$$

$$-\sum_{\vec{n}=\vec{p}'-\vec{p}}\frac{\langle q'|j_\mu(0)|n\rangle\langle n|J(0)|p\rangle}{p_0'+n_0-p_0-i\varepsilon}$$

$$+ie\epsilon_\mu\gamma_\mu\,\frac{\langle q'|J(0)|p\rangle}{i(\not{p}-\not{q}')+m}\Bigg] \qquad (2\Theta)$$

The equation (2Θ) is exact and the operators and state vectors are in the Heisenberg representation. The third term in (2Θ) arises from the equal time commutator and when we rewrite it as

$$-i\,\frac{\delta(p'+q'-p-q)}{\sqrt{4p_0'q_0}}\,\bar{u}p'\,ie\epsilon_\mu\gamma_\mu u_m(p-q')\cdot\frac{1}{(p-q')^2+m^2}\,\bar{u}_m(p-q')\langle q'|J(0)|p\rangle \qquad (2\#)$$

We find that the first half of the matrix element represents the bare electron photon vertex and the second half, the physical electron-photon vertex. Thus when we write the former in terms of renormalised quantities, we will be having the renormalisation constants as factors[1] and they have to be

1) This point has been noticed by Goldberger, Oehme and Nambu while studying the single variable dispersion relations for nucleon-nucleon scattering. * M.L.Goldberger et. al. (Ann. Phys., 2, 226, (1957)).

evaluated suitably. In the lowest order of course the matrix element is well-defined.

Reverting to the interaction representation and thereby to the perturbation expansion, the first term in (20) can be evaluated in the lowest order. The contribution from the one-particle intermediate state is the same as (∃) . However for the three-particle intermediate state, this term vanishes in the second order if we make a direct evaluation. This is apparent if we rewrite the numerator of the first term in terms of creation and annihilation operators. The first factor in the numerator viz., $<q'|J(0)|n>$ would be

$$= {}_0\!<d_{q'}|ie\gamma_\mu A_\mu(0)\psi(0)|\, a^\dagger_{p_1}\, a^\dagger_{p_2}\, b^\dagger(p_3)>_0$$

$$=<d_{q'}|ie\gamma_\mu\left[(\hat{a}_p+\hat{d}^\dagger_p)(\hat{a}_p+b^\dagger_p)\right]|a^\dagger_{p_1}\, a^\dagger_{p_2}\, b^\dagger_{p_3}>_0 \qquad (22)$$

$$= 0$$

since we cannot match all the creation and annihilation operators in the state vectors and the current operator.

This result, of course, is not correct and the three-particle intermediate state does contribute as we have seen earlier. But no ambiguity arises if we interpret the numerator of the first term in (20) in which all the quantities and in the Heisenberg representation as the product of the matrix elements for three-particles going to two particles and two

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†) a, b and d represent the annihilation operators for the electron, positron and photon repsectively and the symbol, Λ_0 above the operators indicates that they carry the wave function associated with the operators.

particles going to three particles and then replace it by the product of similar matrix elements in the interaction representation. The result is that we obtainthe complementary term (14). Thus we see that we have to exercise some caution when we are pulling out a fermion as a current operator.

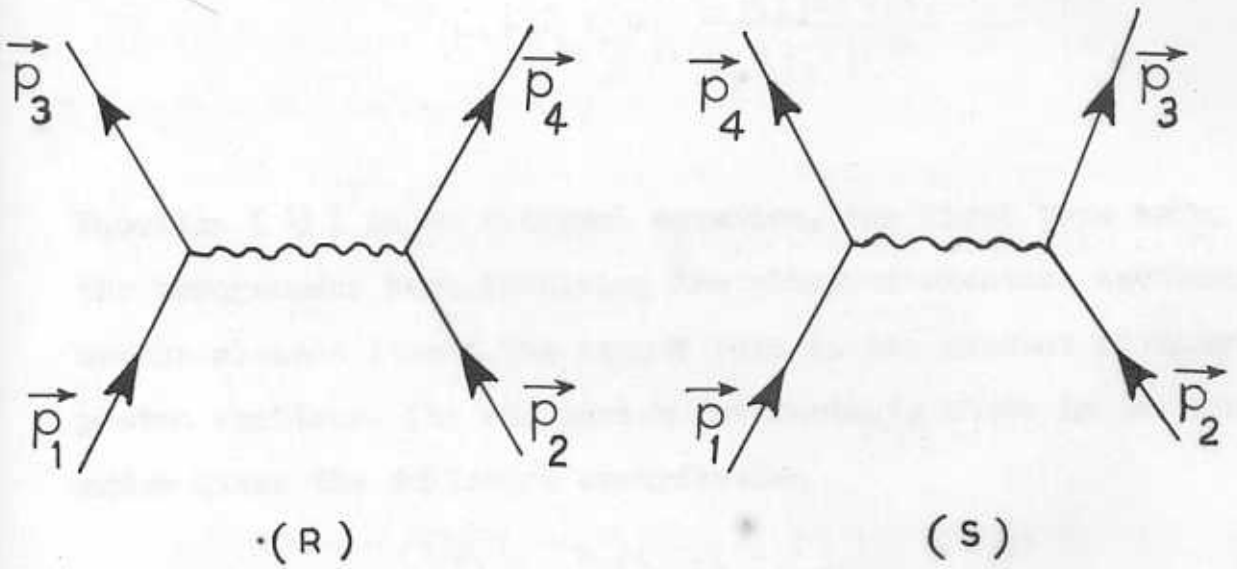
The second term in (20 24) vanishes identically. The third term directly gives the propagator for the other Feynman diagram and in the lowest order corresponds to the diagram of reference 1, page .

As a third alternative, we can pull out both the electrons from the state vector and convert them into current operators. But as pointed out, one has to be careful in such cases and it is convenient to keep the fermions in the state vector andconvert only the bosons into current operators.

(b) Electron-electron and electron-positron scattering.

The matrix element for Moller scattering can be written as products of the ~~xxxxx~~ electron-photon x vertices by converting two of the electrons, one from the initial and the other from the final state into current operators. In this case, the equal time commutator is unavoidable and in fact it represents the matrix element corresponding to one of the Feynman diagrams (Diagram R in ~~ref.~~ fig. 2, page 202). The complete matrix element can be written as

FIG. 2.



(R) (S)

$$\langle p_3 p_4 | S | p_1 p_2 \rangle = -i \frac{\delta(p_1 + p_2 - p_3 - p_4)}{\sqrt{4 E_1 E_3}}$$

$$\cdot \Big[\bar{u}_{p_3} \Big\{ \sum_{\substack{n \\ \vec{n} = \vec{p}_1 + \vec{p}_2}} \frac{\langle p_4 | J^+(0) | n \rangle \langle n | J(0) | p_2 \rangle}{p_{10} + p_{20} - n_0 + i\varepsilon}$$

$$+ \sum_{\substack{n \\ \vec{n} = \vec{p}_4 - \vec{p}_1}} \frac{\langle p_4 | J(0) | n \rangle \langle n | J^+(0) | p_2 \rangle}{p_{40} - p_{10} - n_0 + i\varepsilon} \Big\} u_{p_1}$$

$$+ ie \bar{u}_{p_3} \gamma_\mu u_{p_1} \frac{\langle p_4 | j_\mu(0) | p_2 \rangle}{(p_4 - p_2)^2} \Big] \qquad (23)$$

Equation (23) is an integral equation, the first term being the homogeneous term involving the electron-electron scattering matrix element itself. The second term is the product of electron-photon vertices. The one photon intermediate state in the lowest order gives the following contribution

$$\frac{1}{2\omega_n} \frac{(\bar{u}_{p_3} \gamma_\mu u_{p_2})(\bar{u}_{p_4} \gamma_\mu u_{p_1})}{(E_4 - E_1) - \omega_n} \qquad (24)$$

where ω_n represents the energy of the photon. The above corresponds to the emission of a photon by the electron of momentum p_2 and its absorption by the electron of momentum p_1. We must add to this the complementary term representing the emission of the photon by the electron of momentum p_1 and its subsequent absorption by the electron of momentum p_2. This yields the complete matrix element corresponding to one of the Feynman diagrams, (Diagram 8 in fig ref. 2, p. 202)

$$\frac{(\bar{u}_{p_3}\gamma_\mu u_{p_2})(\bar{u}_{p_4}\gamma_\mu u_{p_1})}{(p_4 - p_1)^2}$$

since

$$\frac{1}{2\omega_n}\left[\frac{1}{(E_4 - E_1) - \omega_n} - \frac{1}{(E_4 - E_1) + \omega_n}\right] = -\frac{1}{(p_4 - p_1)^2} \qquad (28)^{25}$$

The next higher particle intermediate state viz., one-photon plus electron-positron pair gives rise to disconnected Feynman diagrams in the second order.

The third term in (24)[20] is the equal time commutator term and in the lowest order it represents the matrix element corresponding to the diagram R of ref. 1, p. viz.,

$$\frac{(\bar{u}_{p_3}\gamma_\mu u_{p_1})(\bar{u}_{p_4}\gamma_\mu u_{p_2})}{(p_4 - p_2)^2}$$

We notice that the relative sign between this "compounded" second term of (23) and the third term of the same, which is usually attributed to the Pauli exclusion principle, is obtained naturally in this method. This means that we need not symmetrize the matrix element with respect to the interchange of the two final electrons. This confirms the point made in Chapter II . We also notice that the relative sign of the two terms in Compton scattering has come out correctly and even with energy denominators (equation (27)) we notice that "crossing symmetry", which in this context would mean

that for every diagram with the initial photon absorbed first and the final photon emitted later there is also another diagram with the final photon emitted first and the initial photon absorbed later, is satisfied.

A similar consideration shows that in the case of electron positron scattering also the equal time commutator term forms part of the matrix element. The complete matrix element for this case is given below

$$\langle p_{e'}\, p_{p'} | S | p_e p_p \rangle = -i\,\delta(p_e + p_p - p_{e'} - p_{p'})$$

$$\cdot \Bigg[\bar{u}_{p_{e'}} \sum_{\substack{n \\ \vec{n} = \vec{p}_p + \vec{p}_e}} \left\{ \frac{\langle p_{p'} | J^{+}_{(0)} | n \rangle \langle n | J_{(0)} | p_p \rangle}{p_{p_0} + p_{e_0} - n_0 + i\varepsilon} \right\} u_{p_e}$$

$$- \sum_{\substack{n \\ \vec{n} = \vec{p}_{p'} - \vec{p}_e}} \bar{u}_{p_{e'}} \left\{ \frac{\langle p_{p'} | J_{(0)} | n \rangle \langle n | J^{+}_{(0)} | p_p \rangle}{p_{e_0} + n_0 - p_{p'_0} - i\varepsilon} \right\} u_{p_e}$$

$$-ie\, \bar{u}_{p_{e'}} \gamma_\mu\, u_{p_e}\, \frac{\langle p_{p'} | j_\mu(0) | p_p \rangle}{(p_{p'} - p_p)^2}$$

The first term represents the Feynman diagram corresponding to electron-positron annihilation and their subsequent creation (Bhabha scattering . As shown earlier, the complementary term has to be included to get the ~~as~~ complete propagator for the virtual photon. The second term vanishes identically. The third term is the equal time commutator and it represents the matrix element corresponding to electron-positron scattering with exchange of a single photon.

(c) Other processes.

Some further examples where the equal time commutator term gives the whole mx matrix element or part thereof are the following:

(i) Electron-proton scattering where in the lowest order of the electromagnetic coupling constant, the complete matrix element which is the product of the electro-magnetic vertices of the proton and electron is given by the equal time commutator.

(ii) Bremsstrahlung which may be considered to be the limiting case[1] of the radiative scattering of an electron by a proton as the proton mass is made infinite.

(iii) Electro-pion production wherein the lowest order, the equal time commutator term gives the completematrix element for the process, making it equivalent to pion production by a virtual photon, - a result which was derived inChapter Sec. .

(iv) Electron-disintegration of the deuteron which again can be equated to disintegration by a virtual photon, as mentioned in Chapter . The contribution to the matrix element for the photo-production of pions from nucleons arising from the photon-pion interaction term

1) A. Ramakrishnan et. al., Proc. Ind. Acad. Sci. 53 , 306 (1961).

~~i.e.~~

$$ie\left(\varphi^* \frac{\partial \varphi}{\partial x_\mu} - \varphi \frac{\partial \varphi^*}{\partial x_\mu}\right) A_\mu(x)$$

which is bilinear in the pionfield variable φ and hence can give rise to an equal time commutator term which is the "pion current" term .

In all these cases, we see that the equal time commutator term forms the complete matrix element (processes i, ii, iii and iv of C) or a part thereof in the other cases considered. Thus we are forced to the conclusion that in the conventional approach using an interaction Hamiltonian we have to include the contribution from the equal time commutator in the calculation. Another notable feature that emerges from our considerations is that there is no need for separate symmetrization or antisymmetrization of the matrix element as the matrix elements exhibit crossing symmetry and the effect of Pauli principle in the derivation itself.

PART II

CHAPTER VIII

PHOTO PRODUCTION OF SINGLE PIONS FROM NUCLEONS IN THE STRIP APPROXIMATION TO THE MANDELSTAM REPRESENTATION[1)]

1. Introduction

The Mandelstam representation[2)] is a prescription for the location of the singularities of the S-matrix elements of a process involving two incoming and two outgoing particles. Hence in the beginning there appeared to be no way of including the contributions to the unitarity condition from intermediate states with more than two particles in a consistent way within the framework of the representation. Thus in the earlier applications of the representation there was a feeling that the representation might be valid in a restricted domain, namely, the low energy region[3)] . The approach of Chew and Mandelstam[4)] to the problem of pion-pion scattering using the representation which might be regarded as a generalisation of the effective range analysis familar in the low energy nucleon-nucleon problem rested on the plausible assumption that the behaviour of an analytic function in a small region of the complex plane is

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2) S. Mandelstam, Phys. Rev., 112, 1344 (1958).
3) S. Gasiorowicz, Fort-Schritte der Phys., 8 , 665 (1960).
4) G.F.Chew and S.Mandelstam, Phys. Rev., 119 , 467 (1960).

dominated by nearby singularities. The domain where most of the experiments are done and where the singularity spectrum is simplest is the low-energy region or more specifically the region below the threshold for the production of an additional particle. High mass singularities were neglected or approximately represented by empirical constants. These assumptions and restrictions led to the reduction of the double-variable dispersion relations to a single variable one, the resulting relations being the partial wave amplitudes for the lowest angular momenta.

Cini and Fubini[1] also made an equivalent approach to low energy problems. They showed that a consistent procedure based on the neglect of inelastic processes in the unitarity condition leads also to the reduction of the two-dimensional representation to a one-dimensional one which has the correct analyticity properties in all variables, fulfills the crossing relations and is approximately unitary.

This reduction is enabled by the fact that for some processes, the lower limits of integration of the two variables in each of the terms of the Mandelstam representation are never reached at the same time; that is, while the lower limit of the one starts from the normal threshold value, the lower limit of the other variable may start at the value of the inelastic threshold and since for the low-energies considered the contribution from this region

1) M. Cini and S. Fubini, Ann. Phys., 10 , 352 (1960).

will be in any case small we can expand the denominator containing the corresponding variable in a powerseries retaining only the first few terms. But this procedure may not always work. Consider the process

$$\gamma + \pi \rightarrow N + \bar{N}$$

which forms the third channel for the process of photoproduction of a pion from a nucleon. Now depending on whether we are considering the isoscalar or isovector part of the photon interaction, we have two or three pion intermediate states respectively as given by the left hand members of figures (1a) and (1b). Expanding the vertices in these graphs, we obtain the graphs on the right hand side. We easily see that while for the isoscalar part both the variables s and t do not touch their respective thresholds viz., $(m+1)^2$ and 4 at the same time so that the Cini-Fubini approximation holds; the same is not true for the isovector part with three pions in the intermediate states since both the variables s and t touch their threshold values viz., $(m+1)^2$ and 9 simultaneously.

These approximations have, as mentioned, only a limited range of validity and are therefore reliable only for those values of the kinematic variables for which the singularities coming from the inelastic processes are sufficiently far off. But the "strip" approximation recently suggested by Chew and Frautschi[1] and also independently

1) G.F.Chew and S. Frautschix, Phys. Rev., 123, 1478 (1960).

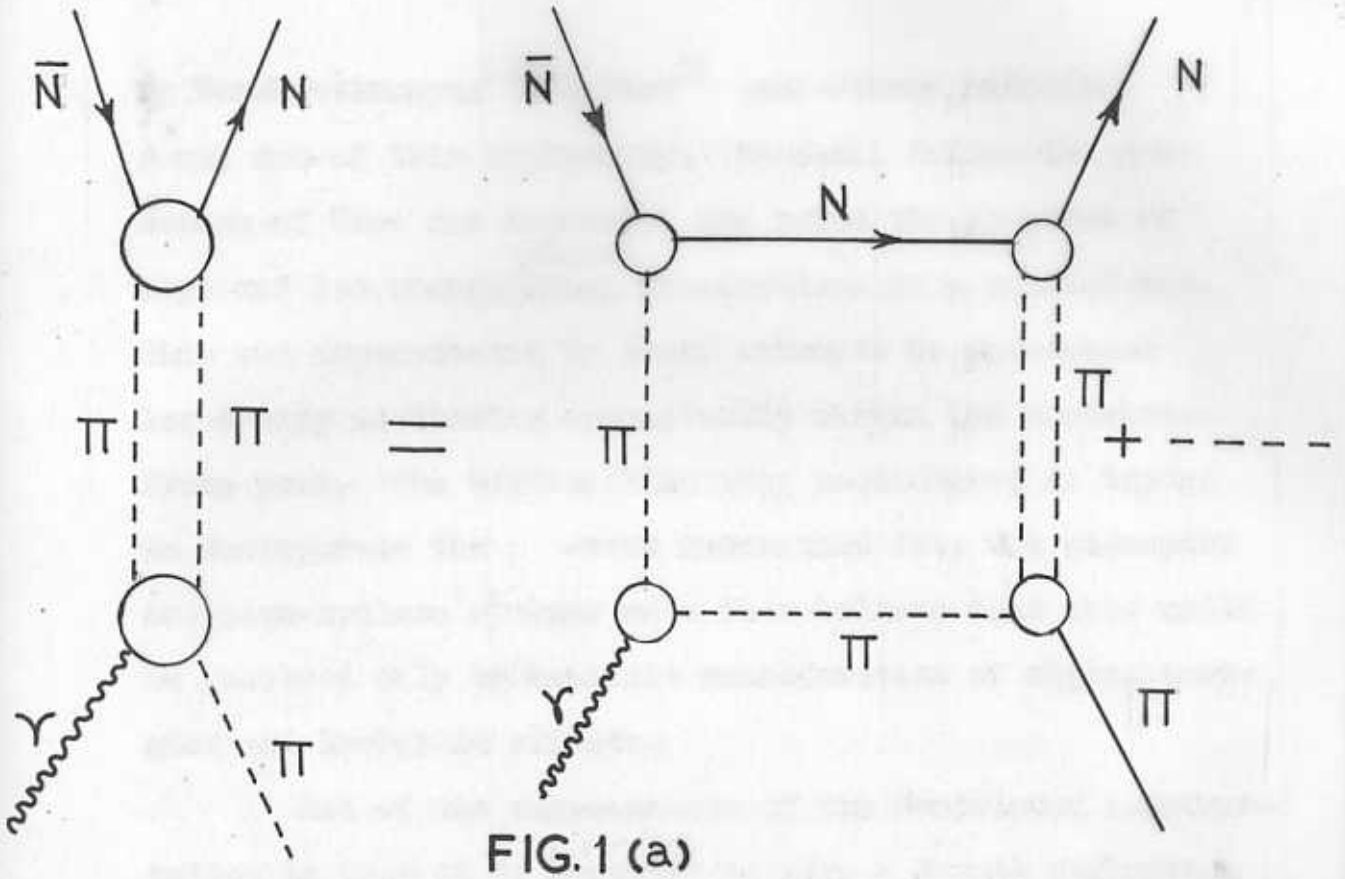


FIG. 1 (a)

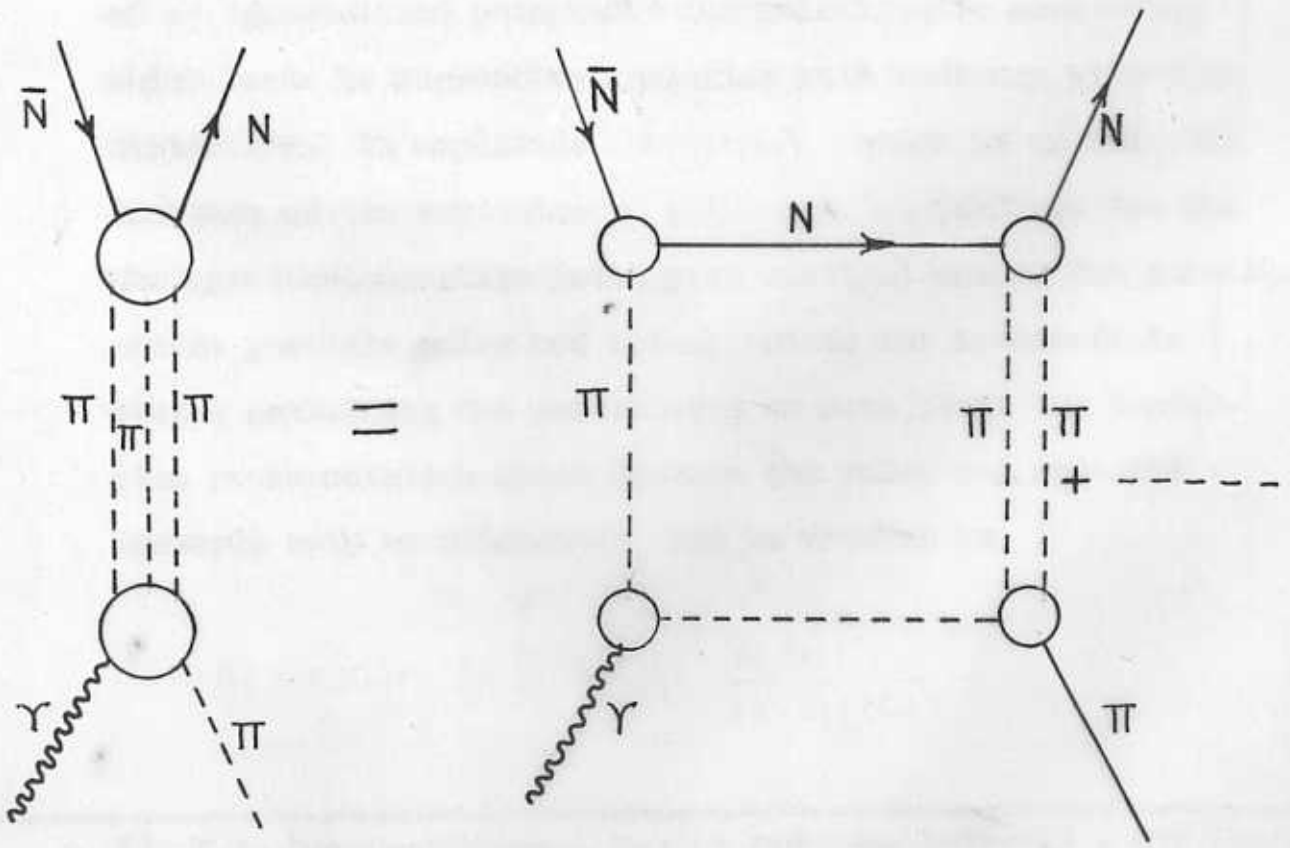


FIG. 1(b)

by Ter-Martirosyan[1], Gribov[2] and others indicates a way out of this difficulty. We shall follow the procedure of Chew and Frautschi who treat the dynamics of high and low energy strong interactions in a unified way. This was necessitated by their attempts to understand low energy scattering consistently within the Mandelstam frame work. The difficulties they encountered in trying to incorporate the p -wave resonances into the pion-pion and pion-nucleon systems made them believe that they could be resolved only by explicit consideration of higher energies and inelastic effects.

One of the consequences of the Mandelstam representation is that it is possible to give a formal definition of a "generalized potential" for relativistic scattering which leads to suggestive analogies with ordinary potential scattering. An amplitude $A(s,t,u)$ which is an analytic function of the variables s, t and u (defined for the photoproductionproblem inthe next section) except for possible single particle poles and branch points for intermediate states containing two particles or more, obeys the Mandelstam representation which, ignoring the poles and spin and isotopic spin complications, can be written as

$$A(s,t,u) = \frac{1}{\pi^2}\iint ds'dt' \frac{A_{13}(s',t')}{(s'-s)(t'-t)} +$$

1) K.A. Ter-Martirosyan, Soviet Physics- JETP, 12 , 575 (1961)
2) V.N.Gribov, Nucl. Phys., 22 , 249 (1961)

$$+\frac{1}{\pi^2}\iint ds'du'\ \frac{A_{12}(s',u')}{(s'-s)(u'-u)}$$

$$+\frac{1}{\pi^2}\iint dt'du'\ \frac{A_{23}(t',u')}{(t'-t)(u'-u)} \qquad (1)$$

Fixing the energy variable, s this could be written as

$$A(s,t,u) = \frac{1}{\pi}\int dt'\ \frac{A_3(t',s)}{t'-t} + \frac{1}{\pi}\int \frac{A_2(u',s)}{u'-u}\ . \qquad (2)$$

A_3 , the discontinuity in channel III can be split up into

$$A_3(t,s) = V_3(t,s) + \frac{1}{\pi}\int ds'\ \frac{A_{13}^{I(el)}(s',t)}{s'-s}$$

where $A_{13}^{I(el)}$ contains the scattering into elastic (two-particle) intermediate states in channel I and V_3 ~~III~~ , which serves as a generalized direct potential for channel I , contains everything else. The generalized exchange potential can be defined in terms of Channel II with u used in place of t. If the generalized potentials are given the elastic double spectral functions for channel I can be computed from the expressions for the double spectral functions in terms of the absorptive parts

$$A_{13}^{I(el)}(s,t) = \frac{1}{\pi q\sqrt{s}}\left[\iint \frac{dt_2\,dt_3\,A_3^*(t_2,s)A_3(t_3,s)}{K_1^{\frac{1}{2}}(q^2;t,t_2,t_3)}\right.$$

$$\left.+\iint \frac{du_2\,du_3\,A_2^*(u_2,s)A_2(u_3,s)}{K_1^{\frac{1}{2}}(q^2;t,u_2,u_3)}\right] \qquad (3)$$

$$A_{12}^{I(el)}(s,u) = \frac{1}{\pi q_{\sqrt{s}} \sqrt{s}} \left[\iint dt_2 \, du_3 \right.$$

$$\left. \cdot \frac{(A_3^*(s,t_2) A_2(s,u_3) + A_2^*(s,u_3) A_3(s,t_2))}{K_2^{\frac{1}{2}}(q_{\sqrt{s}}^2; u, t_2, u_3)} \right. \qquad (4)$$

$$\frac{1}{K_1^{\frac{1}{2}}(q_{\sqrt{s}}^2; t, t_2, t_3)} = 0 \text{ unless } t^{\frac{1}{2}} > t_2^{\frac{1}{2}} + t_3^{\frac{1}{2}}$$

$$\frac{1}{K_1^{\frac{1}{2}}(q_{\sqrt{s}}^2; t, u_2, u_3)} = 0 \text{ unless } t^{\frac{1}{2}} > u_2^{\frac{1}{2}} + u_3^{\frac{1}{2}}$$

$$\frac{1}{K_2^{\frac{1}{2}}(q_{\sqrt{s}}^2; u, t_2, u_3)} = 0 \text{ unless } u^{\frac{1}{2}} > t_2^{\frac{1}{2}} + u_3^{\frac{1}{2}} \qquad (5)$$

Equations (2) and (3) along with the expression for the double spectral functions bear a close resemblance to the set of equations determining non-relativistic potential scattering (of the type whichm obeys a Mandelstam representation) the difference being that the generalized potential is energy dependent, becomes complex above the inelastic threshold and has relativistic kinematic factors.

From equations (2), (3) and (4) , the complete matrix element for the process for fixed energy, $A(s,t,u)$ can be evaluated as follows. For convenience let us take the case of pion-pion scattering. Let us assume that $A_2 = 0$ which means A_{12} is also zero. From the inequalities (5) we see that $A_{13}(s,t) = 0$ if $t^{\frac{1}{2}} < t_2^{\frac{1}{2}} + t_3^{\frac{1}{2}}$.

But we know that $A_3(s,t_i) = 0$ if $t_i < 4$ (i.e. below the threshold for the scattering). Hence $A_{13}(s,t) = 0$ if $t < 16$. Thus

$$A_3(s,t) = V_3(s,t) \quad \text{for} \quad t < 16 \qquad (6)$$

Now if V_3 (the direct potential) is given, then $A_3(s,t)$ is known for $t < 16$. Again applying the inequality in t we find that $A_{13}(s,t)$ can be calculated for $t < 36$ since $A_3(s,t)$ is known for $t < 16$. From this again we can calculate $A_3(s,t)$ for $t < 36$ from equation (3).[1]

The above procedure assumes that the potential V_3 is known which is not the case for the relativistic case. This is where the strip approximation comes in and is useful.

---

1) The statement implied in the above that the matrix element $A(s,t,u)$ can be determined completely once we know the single (A_2 and A_3) and the double (A_{12} and A_{13}) spectral functions is true only if there is no subtractions in the matrix element, i.e. if all the functions are well-behaved at infinity. If however there are subtractions, the subtraction constants are not determined by the spectral functions and thus the matrix element $A(s,t,u)$ is not determined by the spectral functions alone. The introduction of Regge poles however removes this difficulty. (See, G.F. Chew, S.C. Frautschi and S. Mandelstam, Phys. Rev. ~~126~~, 1202 (1962)).

In terms of the double spectral functions the strip approximation can be stated as follows. The scattering amplitudes in the physical regions are dominated by the strips of the double spectral functions lying between each of the thresholds for the three reactions involved in a Mandelstam graph (see Fig. 2) and the corresponding thresholds for inelastic processes (i.e. production of one more particle). Therefore the double spectral (or strip) functions can be calculated using the elastic unitarity conditions along . More specitifically, the strip approximation is the assumption that

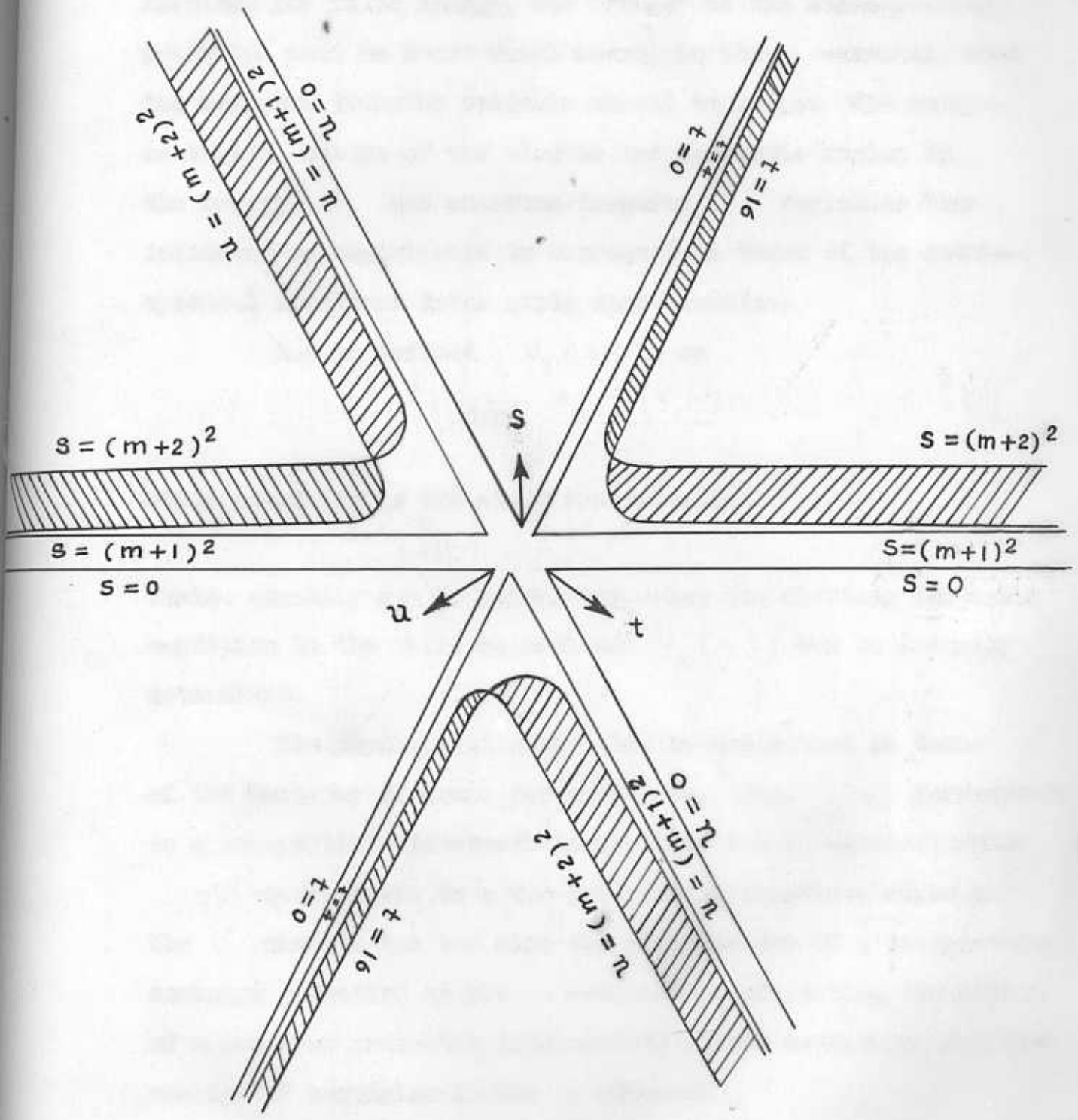
$$A_{13}^{inel}(s) = A_{13}^{el}(t)$$
$$A^{el}(s) = A_{13}^{inel}(t) \qquad (7)$$

and similar relations for the other double spectral functions, "elastic" and "inelastic" refer to the elastic and inelastic parts. The above approximation is based on the following experimental facts:

(i) The elastic scattering cross-sections tend to be large at low energies, with resonances inthe low-angular momentum states,

(ii) At highenergies all elastic cross-sections show a characteristic diffraction pattern consisting of a peak in the forward direction. In the case of pion-nucleon and nucleon-nucleon collisions, the scattering takes place mainly in the range $-20 \leq t \leq 0$ for t the square of the four-momentum transfer. In order that (i) may be

FIG.2. The Strip Regions



feasible for fixed energy, the "range" of the corresponding potential must be short which means, in the s -channel, the momentum transfer variable should be large. The complementarity nature of the elastic and inelastic region in the energy s and momentum transfer t variables thus indicated by experiments is expressed in terms of the double-spectral functions inthe strip approximation.

Now we defined $V_3(s,t)$ as

$$V_3(s,t) = \int ds' \frac{A_{13}^{inel}(s',t)}{s'-s} \quad . \tag{8}$$

Since according to the strip approximation

$$A_{13}^{inel}(s,t) = A_{13}^{el}(t,s)$$

thmis quantity can be calculated using the elastics unitarity condition in the third channel and $V_3(s,t)$ can be formally determined.

The approximation can also be understood in terms of the Curkosky diagrams for a process. Fig. (3a) corresponds to a two-particle intermediate state in the s -channel while (3b) corresponds to a two-particle intermediate state in the t channel but has also the significance of a two-particle exchange potential in the s -channel representing the effect of a sum over inelastic intermediate states containing definite numbers of particles in the s -channel.

We shall study in this chapter the problem of the photo-production of a single pion by a nucleon in the above

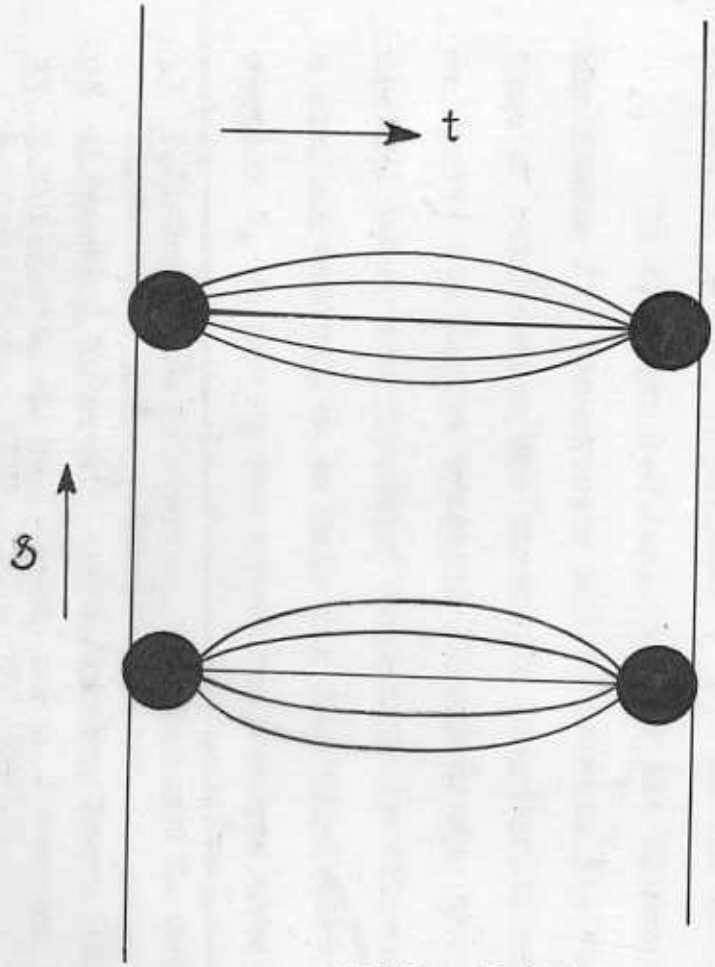


FIG. 3(a)

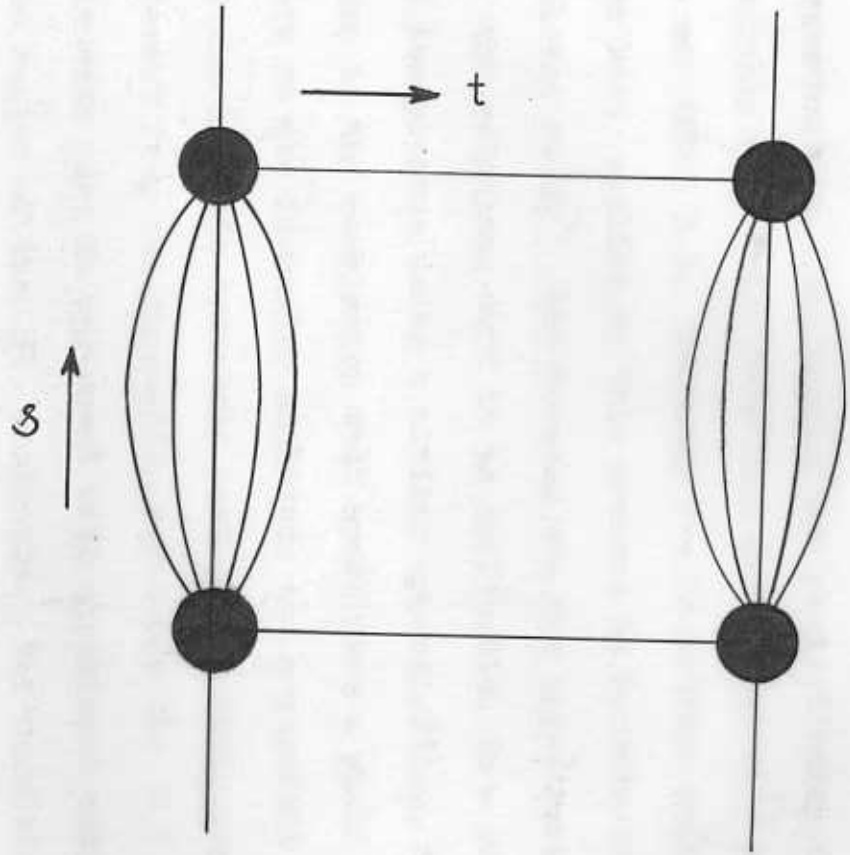


FIG. 3(b)

approximation. This problem was studied using the single variable dispersion relations by Chew et al[1] hereafter referred to as (CG L N). Recently the Mandelstam representation has been applied to this process by Gourdin et al[2] and Solovev et al[3] who however use the one-dimensional form of the relations when it is applicable. In a detailed study of the process using a similar approximation, Ball[4] has come to the conclusion that apart from a small additive term to the isoscalar amplitude and a possible contribution to the isotopic symmetric part of the photo production matrix element from the three-pion resonance the C G L N matrix elements are in agreement with experiment upto at least the region of the 33 resonance. For energies beyond this where the threshold for extra particle productions sets in, we have to use an approximation such as the strip approximation.

In the next section, we give the kinematics and invariant amplitudes for the process and in section 3 , we study the question of subtraction and kinematic singularities. In section 4, we derive the double spectral functions for the photo production channel using the "elastic" unitarity condition. In Section 5, a similar evaluation is made for the third channel. Finally in section 6, we set up the equations for the absorptive parts

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1) G.F.Chew, M.L.Goldberger, F.E.Low and Y. Nambu, Phys. Rev., 106 , 1345 (1957).

2) M.Gourdin, D.Lurie* and A.Martin, Nuovo Cim., 18, 933 (1960)

3) L.D.Solov'v, G. Bialkowski and A. Jurewicz, Sovietx Phys. JETP , 13 , 589 (1961).

4) J.S.Ball, Phys. Rev., 124 , 2014 (1961)

in terms of the double spectral functions. The solution of these equations should yield us a matrix element valid for arbitrary energies (with the momentum transfer restricted).

2. Kinematics and the invariant amplitudes

We define the variables

$$s = -(p_1+k)^2 = -(p_2+q_\pi)^2$$

$$u = -(p_1-q_\pi)^2 = -(p_2-k)^2$$

$$t = -(p_2-p_1)^2 = -(k-q_\pi)^2 \qquad (9)$$

with

$$s+u+t = 2m^2+1 \qquad (10)$$

Here p_1 and p_2 are the four-momenta of the initial and final nucleons, q_π and k those of the pion and photon respectively and m is the mass of the nucleon. The three variables s, t and u are respectively the squares of the total energy in their respective centres of mass systems, for the three connected processes represented by the same diagram (fig. 4) viz.,

$$\gamma + N \rightarrow N + \pi \qquad (I)$$

$$\gamma + \bar{N} \rightarrow \bar{N} + \pi \qquad (II)$$

and

$$\gamma + \pi \rightarrow N + \bar{N} \qquad (III)$$

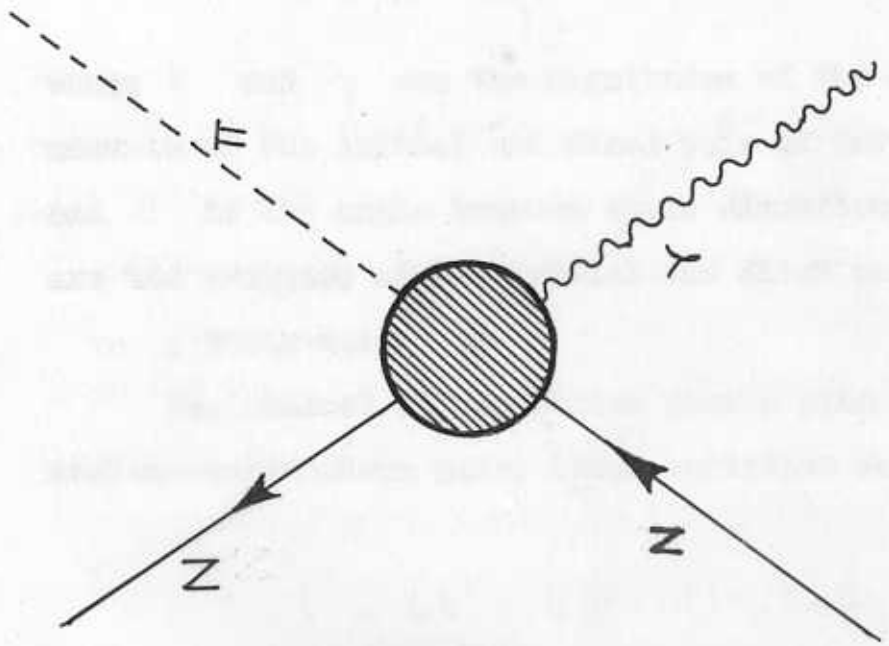

FIG. 4

For channel I (photoproduction of a pion from a nucleon), the variables are given in the centre of mass of the photon-nucleon (or the pion nucleon) system by

$$s = (E_1 + k)^2 = (E_2 + \omega)^2$$

$$t = 1 - 2k\omega + 2kq_1 \cos\theta$$

$$u = m^2 - 2E_2 k - 2kq_1 \cos\theta \qquad (12)$$

where k and q_1 are the magnitudes of the centre of mass momenta of the initial and final pair of particles respectively and θ is the angle between their directions. E_1 and E_2 are the energies of the initial and final nucleon — and m, their mass.

For channel III (a photon plus a pion going into a nucleon-antinucleon pair) these variables represent

$$t = 4E^2 = 4(p^2 + m^2) = (k + \omega_{q_1})^2$$

$$u = m^2 + 1 - 2pq_1 \cos\varphi - 2\omega_{q_1} E$$

$$= m^2 - 2pq_1 \cos\varphi - 2q_1 E$$

$$s = m^2 + 2pq_1 \cos\varphi - 2q_1 E$$

$$= m^2 + 1 + 2pq_1 \cos\varphi - 2\omega_{q_1} E \qquad (13)$$

Here q_1 is the magnitude of the centre of mass momentum of the pion (or photon) and $\vec{p}$ that of the final nucleon

(or antinucleon) . φ is the angle between the directions of the initial and final system of particles and E is the energy of the nucleon (antinucleon).

We shall now write down the invariant amplitudes for the photo production of a pion from a nucleon as given by C & L N . The relation between the S and the T matrix elements is

$$S_{fi} = i(2\pi)^4 \delta(k + p_1 - p_2 - q_f) \frac{1}{(2\pi)^6} \cdot \frac{m}{2(k\omega_{q_f} p_{10} p_{20})^{\frac{1}{2}}} T_{fi} \tag{13a}$$

Because of the energy-momentum conservation, only three of the four vectors p_1 , p_2 , k and q_f are independent and we can choose the independent vectors to be k , q_f and $P = \frac{p_1 + p_2}{2}$. Forming all possible independent scalars from the three vectors above, the γ -matrices and the polarization vector ϵ , of the photon, we obtain the following four independent forms, on using the Dirac equation and gauge-invariance which is imposed by requiring that the matrix element vanishes if we replace in it ϵ by k in

$$M_A = i\gamma_5 (\gamma.\epsilon)(\gamma.k)$$

$$M_B = 2i\gamma_5 \left[(\epsilon.P)(q_f.k) - (\epsilon.q_f)(P.k) \right]$$

$$M_C = \gamma_5 \left[(\gamma.\epsilon)(q_f.k) - (\gamma.k)(q_f.\epsilon) \right]$$

$$M_D = 2\gamma_5 \left[(\gamma.\epsilon)(P.k) - (\gamma.k)(\epsilon.P) - im(\gamma.\epsilon)(\gamma.k) \right] \tag{14}$$

Here the γ_5 matrix appears in each of the fundamental forms because of the pseudoscalar nature of the pion.

In terms of these invariant forms, the photoproduction matrix element can be written as

$$T = M_A A + M_B B + M_C C + M_D D \qquad (15)$$

Because of the fact that in an electromagnetic interaction isotopic spin can be conserved or can change by ± 1, we must introduce three (in contrast to two for the case of pion nucleon scattering) invariant quantities, namely, an isotopic symmetric, an isotopic antisymmetric and an isoscalar part. In terms of these ~~xxx~~ each ~~xxx~~ of the invariant coefficients A, B, C, D can be written in the form

$$A_\alpha = \frac{1}{2}\left\{\tau_\alpha, \tau_3\right\} A^{(+)} + \frac{1}{2}\left[\tau_\alpha, \tau_3\right] A^{(-)} + \tau_\alpha A^{(0)} \qquad (16)$$

Here α refers to the isotopic spin index of the pion.

We thus have twelve invariant amplitudes for the problem, viz., $A^{(\pm,0)}$, $B^{(\pm,0)}$, $C^{(\pm,0)}$ and $D^{(\pm,0)}$.

The amplitudes for channel II are obtained from (14) and (15) by noticing the crossing symmetry of the amplitudes A, B, C and D under the interchange of s and u. From fig. (4) we see that going over from process I to Process II amounts to changing p_1 to $-p_2$ and p_2 to $-p_1$, i.e. P to $-P$, leaving the pion and photon variables unaffected. From (9) it follows that this is equivalent to the exchange $s \rightarrow u$, $t \leftrightarrow t$. Under the exchange of the τ matrices in (13) only the antisymmetric combination will change sign. The crossing relations are given by

$$A^{(+,0)}(s,u,t) \longrightarrow A^{(+,0)}(u,s,t)$$

$$B^{(+,0)}(s,u,t) \longrightarrow B^{(+,0)}(u,s,t)$$

$$C^{(-)}(s,u,t) \longrightarrow C^{(-)}(u,s,t)$$

$$D^{(+,0)}(s,u,t) \longrightarrow D^{(+,0)}(u,s,t)$$

$$A^{(-)}(s,u,t) \longrightarrow -A^{(-)}(u,s,t)$$

$$B^{(-)}(s,u,t) \longrightarrow -B^{(-)}(u,s,t)$$

$$C^{(+,0)}(s,u,t) \longrightarrow -C^{(+,0)}(u,s,t)$$

$$D^{(-)}(s,u,t) \longrightarrow -D^{(-)}(u,s,t) \qquad (17)$$

The amplitudes for process II are thus obtained from those for process I in a very simple way with utmost a change of sign.

3. Kinematic singularities and subtractions

We have next to examine whether the C G L N amplitudes defined by (14) and (15) are free from kinematic singularities, i.e. singularities other than those postulated by Mandelstam. Ball[1] has examined this question and we will present his arguments here. By taking appropriate traces

1) J.S.Ball, loc. cit.

over the nucleon and photon indices, we can form scalars which are analytic functions of the momenta and hence by the Hall-Wightman theorem[12] are analytic functions of the scalar products of the momenta which will be assumed to satisfy the Mandelstam representation. Now the scalars A, B, C, D can be written as linear combinations of these analytic scalars. If the transformation between these two do not introduce kinematic singularities, both sets will satisfy the Mandelstam representation. As the basic set, Ball chooses the non-gauge invariant set of amplitudes $B_i\ (i = 1, 2, \cdots 8)$ by writing the T -matrix as

$$T = \sum_{i=1}^{8} B_i(s,t,u)\, N_i(p_1, p_2, k, \epsilon, \gamma) \qquad (18)$$

where the N_i's are independent Lorentz-invariant matrices that can be formed out of γ , ϵ and the independent momentum four-vectors and are given by

$$N_1 = i\gamma_5 \gamma.\epsilon \gamma.k \ , \quad N_2 = i\gamma_5 (p_1+p_2).\epsilon$$

$$N_3 = 2i\gamma_5 q_\gamma.\epsilon \ , \quad N_4 = 2i\gamma_5 k.\epsilon$$

$$N_5 = \gamma_5 \gamma.\epsilon \ , \quad N_6 = \tfrac{1}{2}\gamma_5 \gamma.k (p_1+p_2).\epsilon$$

$$N_7 = \gamma_5 \gamma.k\, k.\epsilon \ , \quad N_8 = \gamma_5 \gamma.k\, q_\gamma.\epsilon \qquad (19)$$

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1) D. Hall and A.S.Wightman, Kg. Danske Videnskab, Selskab, Mat.-Fys. Medd. 31, No. 5 (1957).

We now obtain constraints on the B_i's by imposing gauge invariance inthe form mentioned earlier on 2.8(18). The result is

$$(s-u)B_2 = 2(t-1)B_3$$

$$B_5 + \frac{1}{4}(u-s)B_6 + \frac{1}{2}(t-1)B_8 = 0 \qquad (20)$$

Imposing these conditions on (T), we obtain the relations between the set A, B, C, D and the B_i's

$$A = B_1 - mB_6 \;, \quad B = \frac{2B_2}{t-1}$$

$$C = -B_8 \;, \quad D = -\frac{1}{2}B_6 \qquad (21)$$

Ball has shown that the amplitudes B_i are free from kinematic singularities so that the amplitude B will have an extra pole in the t variable.

The fixed momentum transfer relations for photo production given by C G L N were written without subtractions.[1] Solovev et al [2] argue that if we can assume that the photoproduction for differential cross-section behaves at infinity like that of the forward scattering of pions on nucleons, then there is no need for subtraction in the photo production problem since there is an energy dependent factor in the relation between the invariant amplitudes and the physical

1) G.F.Chew, Report on the theory of strong coupling of ordinary ~~mm~~ particles, Kiev Conference (1960) and discussion thereon.

2) L.D.Solovev et al , Soviet Physics - JETP, loc. cit.

photo production, matrix element due to gauge invariance whereas no such factor appears in the scattering problem. However Ball argues that the C G L N formula are probably valid without subtraction for the (-) amplitudes while a three-pion resonance would require a subtraction in the (+) amplitudes. The two-pion resonance will necessitate a subtraction in the isoscalar (0) amplitude. Writing the representation for the B_i's in the most general form (including single spectral functions,)we observe that since the amplitudes A, C and D have no poles in the t variable, they do not also have a one-dimensional spectral function in that variable so that they could be written as

$$A = R_1(s;u)\left(\frac{1}{m^2-s} \pm \frac{1}{m^2-u}\right)$$
$$+ \frac{1}{\pi}\left\{\int_{(m+1)^2}^{\infty} ds' \frac{\rho_s(s')}{s'-s} \pm \int_{(m+1)^2}^{\infty} du' \frac{\rho_u(u')}{u'-u}\right\}$$
$$+ \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_4^{\infty} dt' \frac{A_{13}(s',t')}{(s'-s)(t'-t)}$$
$$+ \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{(m+1)^2}^{\infty} du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)}$$
$$+ \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} du' \int_4^{\infty} dt' \frac{A_{23}(u',t')}{(u'-u)(t'-t)} \qquad (22)$$

and similarly for C and D . $R_1(s;u)$ is the residue.

Because of the relation $B = \frac{2B_2}{t-1}$, the Mandelstam representation for the B amplitude is

$$B = \frac{R_2(s;u)}{t-1}\left(\frac{1}{s-m^2} \pm \frac{1}{u-m^2}\right) + \frac{R_2(t)}{t-1}$$

$$+ \frac{1}{(t-1)} \cdot \frac{1}{\pi}\left\{\int_{(m+1)^2}^{\infty} \frac{\rho_s(s')\,ds'}{s'-s} + \int_{(m+1)^2}^{\infty} \frac{\rho_u(u')\,du'}{u'-u}\right.$$

$$\left. + \int_{4}^{\infty} \frac{\rho_t(t')\,dt'}{t'-t}\right\}$$

$$+ \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{4}^{\infty} dt' \frac{A_{13}(s',t')}{(s'-s)(t'-t)}$$

$$+ \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} ds' \int_{(m+1)^2}^{\infty} du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)}$$

$$+ \frac{1}{\pi^2}\int_{(m+1)^2}^{\infty} du' \int_{4}^{\infty} dt' \frac{A_{23}(u',t')}{(u'-u)(t'-t)} \qquad (23)$$

4. The strip functions for channel I .

In this section we shall evaluate the strip (or double spectral) functions for the photo production problem. The procedure is as given by Mandelstam[1]. To start with, we go over from the inveriant set of amplitudes (14) and (15) to the corresponding centre of mass amplitudes (as calculations will be easier in terms of the Pauli spin matrix σ appearing there then in terms of the γ matrices appearing in the.

1) S. Mandelstam, loc. cit.

invariant amplitudes). Denoting by F_I the centre of mass amplitude we have

$$F_I = i\vec{\sigma}.\vec{\epsilon}\, \mathcal{J}_1 + \frac{\vec{\sigma}.\vec{q}\ \vec{\sigma}.(\vec{k}\times\vec{\epsilon})}{q_1 k}\mathcal{J}_2$$

$$+ \frac{i\vec{\sigma}.\vec{k}\,\vec{q}.\vec{\epsilon}}{k q_1}\mathcal{J}_3 + \frac{i\vec{\sigma}.\vec{q}\,\vec{q}.\vec{\epsilon}}{q^2}\mathcal{J}_4 \qquad (24)$$

$\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ and $\mathcal{J}_4$ are functions of the three variables s, u and t. The $\mathcal{J}_i$'s are related to the invariant amplitudes by the relations

$$\alpha = e(W-m)\mathcal{J}_1 = (W-m)A + (W-m)^2 D - \frac{(t-1)}{2}(C-D)$$

$$\beta = \frac{e(W+m)(m+E_2)}{q_1}\mathcal{J}_2 = -(W+m)A + (W+m)^2 D - \frac{(t-1)}{2}(C-D)$$

$$\gamma = \frac{e\mathcal{J}_3}{q_1} = (W-m)B + (C-D)$$

$$\delta = \frac{e(m+E_2)}{q^2}\mathcal{J}_4 = -(W+m)B + (C-D) \qquad (25)$$

Here

$$e = \frac{8\pi W (E_1+m)^{-\frac{1}{2}}(E_2+m)^{-\frac{1}{2}}}{(W-m)} \qquad (26)$$

$W = s^{\frac{1}{2}}$ is the total energy in the centre of mass frame. Our amplitudes α, β, γ and δ are connected with the amplitudes F_1, F_2, F_3, F_4 of G G L N by the relations

$$\alpha = (W - m) F_1$$
$$\beta = (W + m) F_2$$
$$\gamma = F_3$$
$$\delta = F_4 \qquad (27)$$

The differential cross section in the centre of mass frame is given in terms of the amplitude F_I by

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} |F_I|^2 \qquad (28)$$ (1)

We have chosen the set of functions α , β , γ , δ as basis in view of the fact that the transformation from A, B, C, D to α , β , γ , δ do not introduce any further singularities.

We shall now proceed to evaluate the double spectral functions. Define the direction of the photon $\vec{k}$ as the z direction. $\vec{k}$ and $\vec{q}_1$ define the z-x plane. The polarization vector $\vec{\epsilon}$ of the photon lies in the x-y plane and we shall assume it to form an angle ψ with the x -axis. We shall now use the "elastic" unitarity condition written in terms of the absorptive part, F_I' of the amplitude F_I

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1) Actually the differential cross-section in the centre-of-mass , on restoring the normalisation factors and using the two-particle density of states, is $\frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{q}{k} \cdot |$ matrix element $|^2$, but a factor $\frac{1}{8\pi W}$ has been included in F_I so that the differential cross section is as given by (28).

i.e. we shall retain in the complete set of intermediate states in F_I' , only the state containing the pion and the nucleon so that the absorptive part can be written as the product of the pion-nucleon scattering amplitude and the photo-production matrix element itself. Writing the pion-nucleon scattering matrix element in the centre of mass frame as

$$I = a + \frac{\vec{\sigma}.\vec{q}_{\nu_2}\,\vec{\sigma}.\vec{q}_{\nu_1}}{q_{\nu_2} q_{\nu_1}} b \qquad (29)$$

where q_{ν_1} and q_{ν_2} are the momenta of the initial and final pions, respectively, we can write the unitarity condition as[1)]

$$F_I'(s,z) = \eta \int_{-1}^{1} dz_2 \int_0^{2\pi} d\varphi \,.$$
$$\times I^*(s,z_3) F_I(s,z_2) \qquad (30)$$

where η is a normalization factor, $z_1 = \cos\theta_1$, $z_2 = \cos\theta_2$, and (θ_2, φ) represents the angle between the initial photon-nucleon direction and the intermediate pion-nucleon direction, and z_3 the cosine of the angle between the intermediate pion-nucleon direction and that of the final pion or nucleon. Now the unitarity condition only says that equation (30) is valid in the physical region. F_I' must then be obtained in the unphysical region by analytical continuation. I and F_I can be expressed as analytic functions of t or

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1) We shall introduce the isotopic spin labels of the amplitudes later.

equivalently of z by means of the fixed energy variable dispersion relations similar to equation (2)

$$I^{*}(\nu, z_3) = \frac{1}{\pi}\int dz'_3 \; \frac{I_2^{*}(\nu, z'_3) + I_3^{*}(\nu, z'_3)}{z'_3 - z_3}$$

$$F_I(\nu, z_2) = \frac{1}{\pi}\int dz'_2 \; \frac{F_I^{2}(\nu, z'_2) + F_I^{3}(\nu, z'_2)}{z'_2 - z_2} \qquad (31)\ 1)$$

In (31) I_2 and I_3 refer to the absorptive parts of the second (pion-nucleon scattering) and third $(\pi + \overline{\pi} \to N + \overline{N})$ channels in the scattering problem and F_I^2 and F_I^3 are the absorptive parts of the second/third channels in the photo production problem. For simplicity the absorptive parts I_2 and I_3 (as also F_I^2 and F_I^3) have been written under the same integral though they contribute to different regions of the variables of integration. Thus $I_2(\nu, z)$ will be non-zero only if $z < 1 - \frac{\{\nu - (m-1)^2\}}{2q^2}$ apart from a δ -function at $z = 1 - \frac{(\nu - m^2 - 2)}{2q^2}$ and $I_3(\nu, z)$ will be non-zero only if $z > 1 + \frac{t}{2q^2}$. Similarly F_I^2 will be non-zero only if $z < \frac{k\omega - \frac{1}{2}(m^2 - 2m - 1 - \nu)}{kq}$ again apart from a δ -function corresponding to the nucleon pole and F_I^3 will be non-zero only if $z > \frac{k\omega + \frac{3}{2}}{kq}$.

Substituting the expressions (24) and (29) for F and I respectively, we can finally write the unitarity condition in terms of the absorptive parts of the amplitudes α , β , γ and δ as

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1) We assume the pole terms to be included under the integral sign by using δ -function factors and suitably redefining the lower limit of integration.

$$F_I'(s,z_1) = \frac{i\alpha_2'(s,z_1)}{e(W-m)}\left(\sigma_x\cos\psi+\sigma_z\sin\psi\right)$$

$$-\frac{\beta_2'(s,z_1)\cdot q}{e(W+m)(m+E_2)}\left(1-z_1^2\right)^{\frac{1}{2}}\sin\psi$$

$$+\frac{i\beta'(s,z_1)\cdot q}{e(W+m)(m+E_2)}\left(-\sigma_x z_1\cos\psi-\sigma_y z_1\sin\psi+\sigma_z\left(1-z_1^2\right)^{\frac{1}{2}}\cos\psi\right)$$

$$+i\gamma'(s,z_1)\cdot\frac{q}{e}\sigma_z\left(1-z_1^2\right)^{\frac{1}{2}}\cos\psi$$

$$+i\delta'(s,z_1)\cdot\frac{q^2}{e(m+E_2)}\left(1-z_1^2\right)^{\frac{1}{2}}\cos\psi\left(\sigma_z z_1+\sigma_x\left(1-z_1^2\right)^{\frac{1}{2}}\right)$$

$$=\eta\iiiint\frac{dz_2'\,dz_3'\,dz_2\,d\varphi}{(z_2'-z_2)(z_3'-z_3)}\cdot\left[a_2^*(s,z_3')+a_3^*(s,z_3')\right.$$

$$+\left\{b_2^*(s,z_3')+b_3^*(s,z_3')\right\}\left\{z_3-i\left(-\sigma_x z_1\left(1-z_2^2\right)^{\frac{1}{2}}\sin\varphi\right.\right.$$

$$+\sigma_y\left[z_1\left(1-z_3^2\right)^{\frac{1}{2}}\cos\varphi-\left(1-z_1^2\right)^{\frac{1}{2}}z_2\right]$$

$$\left.\left.\left.+\sigma_z\left(1-z_1^2\right)^{\frac{1}{2}}\left(1-z_2^2\right)^{\frac{1}{2}}\sin\varphi\right)\right\}\right]\times$$

$$\times\Big[\, i\big(\mathcal{F}_1^2(s,z_2')+\mathcal{F}_1^3(s,z_2')\big)(\sigma_x\cos\psi+\sigma_y\sin\psi)$$

$$+\big(\mathcal{F}_2^2(s,z_2')+\mathcal{F}_2^3(s,z_2')\big)\times\Big\{-(1-z_2^2)^{\frac{1}{2}}(\cos\varphi\sin\psi-\sin\varphi\cos\psi)$$

$$+i\,(-z_2\cos\psi\,\sigma_x-z_2\sin\psi\,\sigma_y+(1-z_2^2)^{\frac{1}{2}}(\cos\varphi\cos\psi+$$

$$+\sin\varphi\sin\psi)\,\sigma_z\Big\}$$

$$+i\,\big(\mathcal{F}_3^2(s,z_2')+\mathcal{F}_3^3(s,z_2')\big)(1-z_2^2)^{\frac{1}{2}}(\cos\varphi\cos\psi+\sin\varphi\sin\psi)\,\sigma_x$$

$$+i\,\big(\mathcal{F}_4^2(s,z_2')+\mathcal{F}_4^3(s,z_2')\big)(1-z_2^2)^{\frac{1}{2}}$$

$$\times(\cos\varphi\cos\psi+\sin\varphi\sin\psi)\Big\{(1-z_2^2)^{\frac{1}{2}}\cos\varphi\,\sigma_x$$

$$+(1-z_2^2)^{\frac{1}{2}}\sin\varphi\,\sigma_y+z_2\,\sigma_z\Big\}\Big] \qquad (32)$$

In the above, the absorptive parts of the photoproduction amplitudes are indicated by superscripts and those for the pion-nucleon scattering amplitudes by subscripts. The number in each case represents the channel concerned.

We now perform the φ and z_2 integrations. The main point to be noted here is that the integration, term by term, gives the same type of functions viz.,

$$\frac{1}{\sqrt{k}} \log \frac{z_1 - z_2' z_3' + \sqrt{k}}{z_1 - z_2' z_3' - \sqrt{k}}$$

where

$$k = z_1^2 + z_2'^2 + z_3'^2 - 1 - 2 z_1 z_2' z_3' \qquad (33)$$

It
~~If~~ as in the case of pion-nucleon scattering has both right and left-hand cuts in the z_1 variables (see Mandelstam[1]). This is a feature of all two particle interactions involving two-particle intermediate states.

The next step is to write dispersion relations for the absorptive parts α_1' , β_1' , γ_1' , δ_1'

$$\alpha_1'(s,t) = \frac{1}{\pi} \int \frac{dt'}{t'-t} \alpha_{13}(s;t') - \frac{1}{\pi} \int \frac{dt'}{t'-t} \alpha_{12}(s;t') \qquad (34)$$

and similar relations for β_1' , γ_1' and δ_1' with the right and left hand cuts displayed. Such relations are simple consequences of writing a double variable dispersion relation like (1) in the single variable form (2) . Now the other expression for F_I' viz., (32) on integration over φ and z_2 must also be an analytic function of t and hence of z_1 with discontinuities of magnitude $2\rho_{13}$ and $2\rho_{12}$ as z_1 crosses the positive and

1) S.Mandelstam, loc. cit.

negative real axes. Here ρ represents any one of the quantitites α , β , γ , δ . A comparison of the left and right hand sides of the equation (32) with equations (34) for α' , β' , γ' , δ' substituted in it should therefore give expressions for the double spectral functions on identifying the cuts. For this, it is necessary to disentangle the spin-dependent and spin independent terms in (32). A point to be noted here is that once such an identification is made the factors containing the direction of polarization of the photon (such as $\cos\psi$ or $\sin\psi$) happen to be the same on both sides so that they drop out and our choice of the polarization vector in an arbitrary direction in the $x-y$ plane has been justified.

Comparison of the spin independent terms in (32) gives rise to the following equations involving $\beta_{13}(s,z_1)$ and $\beta_{12}(s,z_1)$ only.

$$\beta_{13}(s,z_1) = \frac{e(W+m)(m+E)}{q} \iint dz_2\, dz_3 \frac{K_1(z_1,z_2,z_3)}{1-z_1^2}$$

$$\cdot\Big\{\big(a_2^*(s,z_3)\mathcal{F}_2^2(s,z_2)+a_3^*(s,z_3)\mathcal{F}_2^3(s,z_2)\big)(z_2-z_1z_2)$$

$$-\big(b_2^*(s,z_3)\mathcal{F}_2^2(s,z_2)+b_3^*(s,z_3)\mathcal{F}_2^3(s,z_2)\big)(z_2^2-z_3^2)$$

$$+\big(b_2^*(s,z_3)\mathcal{F}_1^2(s,z_2)+b_3^*(s,z_3)\mathcal{F}_1^3(s,z_2)\big)(z_2-z_1z_3)\Big\} \qquad (35)$$

$$\beta_{(12)}(s,z_1) = \frac{e(W+m)(m+E_R)}{q}\iint dz_2\,dz_3\,\frac{K_2(z_1,z_2,z_3)}{1-z_1^2}$$
$$\Big\{\Big(a_2^*(s,z_3)\,\mathcal{F}_2^2(s,z_2)+a_3^*(s,z_3)\,\mathcal{F}_2^3(s,z_2)\Big)$$
$$\times(z_3-z_1z_2)$$
$$-\Big(b_2^*(s,z_3)\,\mathcal{F}_2^2(s,z_2)+b_3^*(s,z_3)\,\mathcal{F}_2^3(s,z_2)\Big)$$
$$\times(z_2^2-z_3^2)$$
$$+\Big(b_2^*(s,z_3)\,\mathcal{F}_1^2(s,z_2)+b_3^*(s,z_3)\,\mathcal{F}_1^3(s,z_2)\Big)$$
$$\times(z_2-z_1z_3)\Big\} \qquad (36)$$

The primes over z_2 and z_3 have been dropped. It will be noticed that the combinations of absorptive parts appearing in (35) and (36) are similar to those in pion-nucleon scattering. This is because of the restrictions on the functions K_1 and K_2 which are defined by

$$K_1(z_1,z_2,z_3) = -\left[k(z_1,z_2,z_3)\right]^{-\frac{1}{2}}$$

$$\text{for } z_1 > z_2z_3+(z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}}$$

$$= 0 \text{ for } z_1 < z_2z_3+(z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}}$$

$$K_2(z_1,z_2,z_3) = \left[k(z_1,z_2,z_3)\right]^{-\frac{1}{2}}$$

$$\text{for } z_1 < z_2z_3+(z_2^2-1)(z_3^2-1)^{\frac{1}{2}}$$

$$= 0 \text{ for } z_1 > z_2 z_3 - (z_2^2-1)^{\frac{1}{2}} (z_3^2-1)^{\frac{1}{2}} \qquad (37)$$

K_1 survives only if both z_2 and z_3 are positive or negative and K_2 survives only if one of them is positive and the other negative. The points

$$z_1 = z_2 z_3 \pm (z_2^2-1)^{\frac{1}{2}} (z_3^2-1)^{\frac{1}{2}}$$

are the points at which k changes sign.

Similarly the comparison of the coefficients of $i\sigma_y$, $i\sigma_x$ and $i\sigma_z$ in (32) gives rise to the following equations for the double spectral functions

$$\begin{aligned}
&\alpha_{13(12)}(s,z_1) - z_1 \beta_{13(12)}(s,z_1) \\
&= \sum_i \iint dz_2\, dz_3\, K_{1(2)}(z_1,z_2,z_3) \\
&\times \Big[a_i^*(s,z_3)\, \mathcal{F}_1^i(s,z_2) - z_2\, a_i^*(s,z_3)\, \mathcal{F}_2^i(s,z_2) \\
&+ z_3\, b_i^*(s,z_3)\, \mathcal{F}_1^i(s,z_2) \\
&+ \Big\{ z_2(z_3 - z_1 z_2) - z_2 z_3 + \frac{z_1 (z_3 - z_1 z_2)^2}{1-z_1^2} \Big\} \\
&\times b_i^*(s,z_3)\, \mathcal{F}_2^i(s,z_2) \Big] \qquad (38)
\end{aligned}$$

$$\begin{aligned}
&\alpha_{13(12)}(s,z_1) - z_1 \beta_{13(12)}(s,z_1) + (1-z_1^2)\, \delta_{13(12)}(s,z_1) \\
&= \sum_i \iint dz_2\, dz_3\, \frac{K_{1(2)}(z_1,z_2,z_3)}{1-z_1^2}
\end{aligned}$$

$$\times\Big[\Big\{a_i^*(s,z_3)\,\mathcal{F}_1^i(s,z_2)-z_2\,a_i^*(s,z_3)\,\mathcal{F}_2^i(s,z_2)$$
$$+z_3\,b_i^*(s,z_3)\,\mathcal{F}_1^i(s,z_2)\Big\}(1-z_1^2)$$
$$+(z_3-z_1z_2)^2\,a_i^*(s,z_3)\,\mathcal{F}_4^i(s,z_2)$$
$$-\Big\{z_2(z_3-z_1z_2)+z_3(z_2-z_1z_3)\Big\}\,b_i^*(s,z_3)\,\mathcal{F}_2^i(s,z_2)$$
$$-(z_3-z_1z_2)(z_2-z_1z_3)\,b_i^*(s,z_3)\,\mathcal{F}_3^i(s,z_2)$$
$$+(z_3-z_1z_2)(z_3^2-z_2^2)\,b_i^*(s,z_3)\,\mathcal{F}_4^i(s,z_2)\Big] \qquad (39)$$

$$\gamma_{13(12)}(s,z_1)+\beta_{13(12)}(s,z_1)+z_1\,\delta_{13(12)}(s,z_1)$$
$$=\sum_i\iint dz_2\,dz_3\;\frac{K_{1(2)}(z_1,z_2,z_3)}{1-z_1^2}$$
$$\times\Big[\Big\{a_i^*(s,z_3)\,\mathcal{F}_2^i(s,z_2)+a_i^*(s,z_3)\,\mathcal{F}_3^i(s,z_2)$$
$$+z_2\,a_i^*(s,z_3)\,\mathcal{F}_4^i(s,z_2)\Big\}(z_3-z_1z_2)$$
$$+(z_2-z_1z_3)\,b_i^*(s,z_3)\,\mathcal{F}_1^i(s,z_2)$$
$$+(z_3^2-z_2^2)\,b_i^*(s,z_3)\,\mathcal{F}_2^i(s,z_2)$$
$$+z_3(z_3-z_1z_2)\,b_i^*(s,z_3)\,\mathcal{F}_3^i(s,z_2)$$
$$+z_1(z_3-z_1z_2)\,b_i^*(s,z_3)\,\mathcal{F}_4^i(s,z_2)\Big] \qquad (40)$$

In the above, the summation over i indicates that terms like $a_i^*(s,z_3)\,\rho_i(s,z_2)$ should be replaced by $a_2^*(s,z_3)\,\rho_2(s,z_2) + a_3^*(s,z_3)\,\rho_3(s,z_2)$ in the double spectral functions ρ_{13} and they should be replaced by $a_2^*(s,z_3)\rho_3(s,z_2)+a_3^*(s,z_3)\rho_2(s,z_2)$ in ρ_{12} and similarly for products containing b.

Before solving the above equations for the double spectral functions we shall introduce the isotopic spin labels of the amplitudes. The complete matrix element for photo production including isotopic spin is

$$F_I = \tfrac{1}{2}\left\{\tau_\alpha, \tau_3\right\} F_I^{(+)} + \tfrac{1}{2}\left[\tau_\alpha, \tau_3\right] F_I^{(-)} + \tau_\alpha F_I^{(0)} \qquad (41)$$

On the right hand side of the unitarity condition (30) we have the product of the pion-nucleon scattering amplitude the complete matrix element of which is

$$\tfrac{1}{2}\left\{\tau_\alpha, \tau_\beta\right\} I^{(+)} + \tfrac{1}{2}\left[\tau_\alpha, \tau_\beta\right] I^{(-)}$$

and the photo production amplitude

$$\tfrac{1}{2}\left\{\tau_\beta, \tau_3\right\} F_I^{(+)} + \tfrac{1}{2}\left[\tau_\beta, \tau_3\right] F_I^{(-)} + \tau_\beta F_I^{(0)}$$

We observe that a $\left\{\tau_\alpha, \tau_3\right\}$ can be generated both by multiplying $\left\{\tau_\alpha, \tau_\beta\right\}$ by $\left\{\tau_\beta, \tau_3\right\}$ and $\left[\tau_\beta, \tau_3\right]$ by $\left[\tau_\alpha, \tau_\beta\right]$ Similarly the antisymmetric part $\left[\tau_\alpha, \tau_3\right]$ is obtained by multiplying the symmetric part as well as the antisymmetric part in the other and the isoscalar part τ_α is obtained by the antisymmetric part in each of the matrix elements/ by multiplying both the symmetric and antisymmetric part of I with the isoscalar part of F_I. Incorporating these in the equations for

the double spectral functions viz., equations (35), (36) and (38) to (40) and solving we obtain finally

$$\alpha^{(\pm,0)}_{13(12)}(s,z_1) = e(W-m)\sum_{\tau}\sum_{i}\iint dz_2\,dz_3$$

$$\cdot\frac{K_{1(2)}(z_1,z_2,z_3)}{1-z_1^2}\Big\{a^{*}_{i\tau}(s,z_3)\,\mathcal{J}_1^{i\tau}(s,z_2)(1-z_1^2)$$

$$-a^{*}_{i\tau}(s,z_3)\,\mathcal{J}_2^{i\tau}(s,z_2)(z_2-z_1z_3)$$

$$+b^{*}_{i\tau}(s,z_3)\,\mathcal{J}_1^{i\tau}(s,z_2)\Big[z_3(1-z_1^2)+z_1(z_2-z_1z_3)\Big]$$

$$+b^{*}_{i\tau}(s,z_3)\,\mathcal{J}_2^{i\tau}(s,z_2)\,2z_3(z_1z_3-z_2)\Big\} \qquad (42)$$

$$\beta^{(\pm,0)}_{(13(12)}(s,z_1) = \frac{e(W+m)(m+E_2)}{q_v}\sum_{\tau}\sum_{i}\iint dz_2\,dz_3$$

$$\cdot\frac{K_{1(2)}(z_1,z_2,z_3)}{1-z_1^2}$$

$$\times\Big\{a^{*}_{i\tau}(s,z_3)\,\mathcal{J}_2^{i\tau}(s,z_2)(z_3-z_1z_2)$$

$$-b^{*}_{i\tau}(s,z_3)\,\mathcal{J}_2^{i\tau}(s,z_2)(z_2^2-z_3^2)$$

$$+b^{*}_{i\tau}(s,z_3)\,\mathcal{J}_1^{i\tau}(s,z_2)(z_2-z_1z_3)\Big\} \qquad (43)$$

$$\gamma^{(\pm,0)}_{13(12)}(s,z_1) = \frac{e}{q}\sum_{\tau}\sum_{i}\iint dz_2\,dz_3\;\frac{K_{1(2)}(z_1,z_2,z_3)}{1-z_1^2}$$

$$\cdot(z_3 - z_1 z_3)\Big\{a^{*}_{i\tau}(s,z_3)\,\mathcal{F}_3^{i\tau}(s,z_2)$$

$$+\Big(a^{*}_{i\tau}(s,z_3)\,\mathcal{F}_4^{i\tau}(s,z_2)$$

$$+ b^{*}_{i\tau}(s,z_3)\,\mathcal{F}_2^{i\tau}(s,z_2)\Big)\;\frac{(z_2 - z_1 z_3)}{1-z_1^2}$$

$$+ b^{*}_{i\tau}(s,z_3)\,\mathcal{F}_3^{i\tau}(s,z_2)\left(z_3 + \frac{z_1(z_2 - z_1 z_3)}{1-z_1^2}\right)$$

$$+ b^{*}_{i\tau}(s,z_3)\,\mathcal{F}_4^{i\tau}(s,z_2)\,z_1\left(1 - \frac{(z_3^2 - z_2^2)}{1-z_1^2}\right)\Big\} \qquad (44)$$

$$\delta^{(\pm,0)}_{13(12)}(s,z_1) = \frac{e(m+E_2)}{q^2}\sum_{\tau}\sum_{i}\iint dz_2\,dz_3$$

$$\cdot K_{1(2)}(z_1,z_2,z_3)\;\frac{(z_3 - z_1 z_2)}{(1-z_1^2)^2}$$

$$\cdot\Big\{a^{*}_{i\tau}(s,z_3)\,\mathcal{F}_4^{i\tau}(s,z_2)(z_3 - z_1 z_2)$$

$$-\Big[b^{*}_{i\tau}(s,z_3)\,\mathcal{F}_2^{i\tau}(s,z_2)$$

$$+ b^{*}_{i\tau}(s,z_3)\,\mathcal{F}_3^{i\tau}(s,z_2)\Big](z_2 - z_1 z_3)$$

$$+ b^{*}_{i\tau}(s,z_3)\,\mathcal{F}_4^{i\tau}(s,z_2)(z_3^2 - z_2^2)\Big\} \qquad (45)$$

In the above, we have used the notation $\sum_\tau$ to indicate that

(i) for the isotopic spin symmetric part, $\mathfrak{A}$ (the '+' amplitude) we have to replace each $a_i^* \mathcal{F}^i$ by

$$a_i^{*(+)} \mathcal{F}^{i(+)} - 2 a_i^{*(-)} \mathcal{F}^{i(-)}$$

(ii) for the isotopic spin antisymmetric part replace each $a_i^* \mathcal{F}^i$ by

$$a_i^{*(+)} \mathcal{F}^{i(-)} - a_i^{*(-)} \left[\mathcal{F}^{i(+)} + \mathcal{F}^{i(-)} \right]$$

and (iii) in the isoscalar amplitude replace $a_i^* \mathcal{F}^i$ by

$$\left[a_i^{*(+)} - 2 a_i^{*(-)} \right] \mathcal{F}^{i(0)}$$

Equations (42) through (45) have to be re-expressed in terms of the t variables with the help of the relations

$$z = \frac{k\omega_q + \frac{1}{2}(t-1)}{kq}$$

$$= \frac{(s-m^2)(s-m^2+1)+2s(t-1)}{(s-m^2)\left\{(s-m^2+1)^2-4s\right\}^{\frac{1}{2}}} \qquad (46)$$

for the photoproduction amplitude and

$$z = 1 + \frac{t}{2q^2}$$

$$= 1 + \frac{2st}{\left\{s-(m+1)^2\right\}\left\{s-(m-1)^2\right\}} \qquad (47)$$

for the pion-nucleon scattering amplitude. But as the resulting expressions are rather cumbersome, we have preferred to give the strip functions in terms of the z variables.

5. The strip functions for channel III.

The invariant matrix element for the process $\gamma+\pi\to N+\bar{N}$ is obtained from the corresponding matrix element (15) for channel I (photo production) by observing that in going over from channel I to III , the following transformations take place :

$$p_1 \to -p_1 \; ; \quad q_1 \to -q_1 \tag{48}$$

and the positive energy spinor for the antinucleon has to be used; more specifically,

$$M_A \longrightarrow i\gamma_5\, \gamma.\epsilon\, \gamma.k$$

$$M_B \longrightarrow 2i\gamma_5\left[-(p_2-p_1).\epsilon\, q_1.k + (p_2-p_1).k\, q_1.\epsilon\right]$$

$$M_C \longrightarrow -\gamma_5\,(\gamma.\epsilon\, q_1.k - \gamma.k\, q_1.\epsilon)$$

$$M_D \longrightarrow 2\gamma_5\left[\gamma.\epsilon\,(p_2-p_1).k - \gamma.k\,(p_2-p_1).\epsilon - im\,\gamma.\epsilon\,\gamma.k\right] \tag{49}$$

These have to be sandwiched between the initial antinucleon spinor v and the adjoint of the final nucleon spinor u where

$$v(p_1) = + \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} -\vec{\sigma}\cdot\vec{p}_1 \\ E+m \end{pmatrix} ; \quad u(p_2) = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m \\ \vec{\sigma}\cdot\vec{p}_2 \end{pmatrix} \tag{50}$$

Here E is the energy of the nucleon (or the antinucleon) in its centre of mass). Using the explicit representations,

$$\gamma_5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} ; \quad \gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} ; \quad j = 1, 2, 3 \tag{51}$$

for the γ -matrices, the matrix element in the centre of mass frame can be evaluated and is given by

$$F_{III} = iL \frac{\vec{p}\cdot\vec{\epsilon}}{p} + M \frac{\vec{\sigma}\cdot(\vec{p}\times\vec{\epsilon})}{p}$$

$$+ N' \frac{\vec{\sigma}\cdot\vec{p}\ \vec{p}\cdot(\vec{\epsilon}\times\vec{l})}{p^2 l} + P \frac{\vec{\sigma}\cdot(\vec{\epsilon}\times\vec{l})}{l} \tag{52}$$

where

$$L = \frac{lp}{m}(A + 4E^2 B)$$

$$M = -\frac{2lEp}{m} C$$

$$N = (E+m)N' = \frac{p^2 l}{m}(A + 2ED)$$

$$P = l\left(A - \frac{2E^2 D}{m}\right) \tag{53}$$

In the above A, B, C, D are functions of $t = 4E^2$ and of $\cos\varphi = \frac{\vec{l}\cdot\vec{p}}{lp}$ where φ, l and p are quantities defined in Sec. 2.

Starting from the centre-of-mass matrix elements and retaining in the unitarity condition the lowest mass two particle intermediate state, i.e. the two pion state, we can evaluate the double spectral functions for the III channel in the same manner as in the previous section. In this case, we require the matrix elements for the process $\pi+\pi \to N+\bar{N}$ and $\gamma+\pi \to \pi+\pi$. The centre of mass matrix element for the first of these can be obtained from the invariant matrix element for pion-nucleon scattering in the same manner as we derived the matrix element for $\gamma+\pi \to N+\bar{N}$ from the photoproduction matrix element and has been given by Frazer and Fulco[1]. It is given by

$$F = G\,\frac{\vec{\sigma}\cdot\vec{p}}{p} + H\,\frac{\vec{\sigma}\cdot\vec{\eta}}{\eta} \qquad (54)$$

where $\vec{\eta}$ and $\vec{p}$ are the centre of mass momenta of the intermediate ~~(pions)~~ and final (nucleon or antinucleon) particles respectively. The amplitudes G and H are given in terms of the invariant amplitudes A and B for pion nucleon scattering (in the notation of ref. p.) by

$$G = -\frac{1}{m}\left\{A + \frac{B}{E+m}\,\vec{p}\cdot\vec{\eta}\right\}$$

$$H = E\,\frac{B}{m} \qquad (55)$$

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1) W.R.Frazer and J.R.Fulco, 117 , 1603 (1960).

There is only one amplitude for the process $\gamma + \pi \longrightarrow \pi + \pi$ [1] which we shall denote by Q. Then proceeding as in the last section, we obtain the double spectral functions as

$$M^{(0)}_{13(23)}(t,\xi_1) = \gamma \sum_j \iint d\xi_2\, d\xi_3 \cdot \frac{K_{1(2)}(\xi_1,\xi_2,\xi_3)}{(1-\xi_1^2)^2}(\xi_2-\xi_1\xi_3)(\xi_3-\xi_1\xi_2) \cdot \left[H_j^{*(+)}(t,\xi_3) - 2H_j^{*(-)}(t,\xi_3)\right] \cdot Q_j^{(0)}(t,\xi_2) \tag{56}$$

$$N^{(0)}_{13(23)}(t,\xi_1) = \gamma \sum_j \iint d\xi_2\, d\xi_3 \cdot \frac{K_{1(2)}(\xi_1,\xi_2,\xi_3)}{1-\xi_1^2}(\xi_3-\xi_1\xi_2) \cdot \left\{\left[G_j^{*(+)}(t,\xi_3) - 2G_j^{*(-)}(t,\xi_3)\right] \times Q_j^{(0)}(t,\xi_2) + \frac{(\xi_3-\xi_1\xi_2)}{1-\xi_1^2}\left[H_j^{*(+)}(t,\xi_3) - 2H_j^{*(-)}(t,\xi_3)\right] \times Q_j^{(0)}(t,\xi_2)\right\} \tag{57}$$

1) M. Gourdin and A. Martin: Nuovo Cimento, 16, 78 (1960).

$$P^{(o)}_{13(23)}(t,\xi_1) = \zeta \iint d\xi_2\, d\xi_3$$

$$\cdot \frac{K_{1(2)}(\xi_1,\xi_2,\xi_3)}{(1-\xi_1^2)^2} \cdot \xi_1(\xi_3-\xi_1\xi_2)(\xi_2-\xi_1\xi_3)$$

$$\cdot\left[H_j^{*(+)}(t,\xi_3) - 2H_j^{*(-)}(t,\xi_3)\right]$$

$$\cdot Q_j^{(o)}(t,\xi_2) \qquad (58)$$

We observe that only the isotopic scalar parts of the strip functions survive since for the reaction $\gamma+\pi\to\pi+\pi$ only the isoscalar part of the photon contributes to the matrix element. Also there is no spectral function corresponding to $L_{13(23)}$. The ξ's which are the cosines of the angles in this case are given in terms of the s, u and t variables by the relations

$$\xi_1 = \cos\varphi = \frac{s-m^2+2lE}{2pl}$$

$$= -\frac{u-m^2+2lE}{2pl} \qquad (59)$$

$$\xi = \frac{t^{\frac{1}{2}}(t+2s-3)}{(t-4)^{\frac{1}{2}}(t-1)} \qquad (60)$$

for the reaction $\gamma+\pi\to\pi+\pi$ and

$$\xi = \frac{2(s-1-m^2)+t}{(t-4)^{\frac{1}{2}}(t-4m^2)^{\frac{1}{2}}} \qquad (61)$$

for the reaction $\gamma+\pi\to N+\bar{N}$. ζ is a normalisation factor.

6. Equations for the absorptive parts.

We can now obtain equation connecting the absorptive parts of the photo production amplitudes and the double spectral functions by using the dispersion relations for absorptive parts. The "elastic" part of each of the absorptive parts of $\alpha^{(\pm,0)}$, $\beta^{(\pm,0)}$, $\gamma^{(\pm,0)}$ and $\delta^{(\pm,0)}$ obeys in dispersion relations, given by equation (34) in which the double spectral functions are the ones calculated in sec. 4, namely, the ones for the photoproduction channel. We recall that inthe evaluation of these spectral functions we fixed the energy in the "elastic" region for the process. To obtain the "inelastic" parts of the absorptive parts $\alpha_1^{(\pm,0)}$ etc. we use the strip approximation

$$\rho_{st}^{inel}(s) \doteq \rho_{st}^{el}(t)$$

$$\rho_{su}^{inel}(s) = \rho_{su}^{el}(u) \qquad (62)$$

Now $\rho_{st}^{el}(t)$ is given by the double : spectral functions for the third channel if we make use of the equations. (25) and (53) to express the amplitudes α , β , γ , δ in terms of the amplitudes L , M , N and P . Explicitly these are given by

$$\alpha = \frac{m}{(W+m)(W+m+2E)}\left[\left\{\frac{mW(t-1)}{E^2} + \frac{m(t-1)(W-m-2E)(W-m)}{4E(E+m)}\right.\right.$$
$$\left.\left. +2(W^2-m^2)\left[W + \frac{(W-m-2E)(Wm+m^2-2E^2)}{4E(E+m)}\right]\right\} N\right.$$
$$- \frac{(t-1)}{4E^2\ell}\left\{4W(E+m) - E(W-m-2E)(W-m)\right\} M$$
$$- \left\{\frac{(t-1)}{E}\left[\frac{(W-m-2E)(W-m)}{4(E+m)} + \frac{W}{E}\right]\right.$$
$$\left. + \frac{[W^2-(m+2E)^2](W^2-m^2)}{2E(E+m)}\right\} P\Bigg]$$

$$\beta = m\left[\left\{\frac{m(t-1)}{4E(E+m)} + \frac{(W+m)(Wm+m^2-2E^2}{2E(E+m)}\right\} N\right.$$
$$\left. - \frac{(t-1)}{4E\ell} M - \frac{1}{4E(E+m)}\left\{(t-1)+2(W+m)(W+m+2E)\right\} P\right]$$

$$\gamma = \frac{m}{4E^2(E+m)}\left[\frac{(E+m)}{\ell}\left\{2M-(W-m)L\right\}\right.$$
$$\left. - \left\{2m+(W-m)E\right\} N + (2-W+m)P\right]$$

$$\delta = \frac{m}{4E^2(E+m)}\left[\frac{(E+m)}{\ell}\left\{2M+(W+m)L\right\}\right.$$
$$\left. +(W+m)(E-2m)N + (W+m+2)P\right] \quad (63)$$

$\rho^{el}_{s,u,}(u,s)$ is obtained on using the crossing relations (17) along with the equation (25).

Thus the final equation for the absorptive part, valid for all energies is given by

$$\rho_1^{(\pm,0)}(s,u,t)=\frac{1}{\pi}\int_4^\infty \frac{dt'}{t'-t}\left[\rho_{13}^{(\pm,0)}(s,t')+\rho_{13}^{(\pm,0)}(t',s)\right]$$
$$+\frac{1}{\pi}\int_{(m+1)^2}^\infty \frac{du'}{u'-u}\left[\rho_{12}^{(\pm,0)}(s,u')+\rho_{12}^{(\pm,0)}(u',s)\right] \quad (64)$$

The corresponding equation for the absorptive part of the III channel $(\gamma+\pi\to N+\bar{N})$ is

$$\rho_3^{(0)}(t,u,s)=\frac{1}{\pi}\int_{(m+1)^2}^\infty \frac{ds'}{s'-s}\left[\rho_{13}^{(0)}(t,s')+\rho_{13}^{(0)}(s',t)\right]$$
$$+\frac{1}{\pi}\int_{(m+1)^2}^\infty \frac{du'}{u'-u}\left[\rho_{23}^{(0)}(t,u')+\rho_{23}^{(0)}(u',t)\right] \quad (65)$$

In (64) and (65) we have not displayed the pole terms and possible subtraction terms. As already mentioned, a two pion-resonance will necessitate a subtraction in the isoscalar amplitude for photo-production and a three pion resonance would require a subtraction in the iso-symmetric part of the amplitude. Recently there has been evidence for both these resonances.[1] The pole as well as the subtraction terms can be easily incorporated in the equations.

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1) For two pion resonance : G.Goldhaber et al , Phys. Rev. Letters, 3 , 181 (1959); A. Erwin et al , Phys. Rev. Letters, 6 , 628 (1961); E.Pickup et al , Phys. Rev. Letters, 7 , 192 (1961). For the 3-pion resonances : B. Maglic et al , Phys. Rev. Letters, 7 , 178 (1961); M.L.Stevenson et al , Phys. Rev., 125 , 687 (1962); P.L.Bastien et al , Phys. Rev. Letters, 8 , 114 (1962); D.D.Carmody et al , Phys. Rev. Letters, 8 , 117 (1962).

7. Conclusion

Equationsx (64) taken in conjunction with the crossing relations (17) and the expressions for the double spectral functions viz., equations (42–45) and (56–58) form the basis for a study of the photoproduction of pions from nucleons in the strip approximation. As is apparent, a knowledge of the absorptive parts of the matrix elements for the processes:

$$\pi + N \longrightarrow \pi + N$$

$$\pi + \pi \longrightarrow N + \bar{N}$$

and

$$\gamma + \pi \longrightarrow \pi + \pi$$

is necessary for this purpose. When these are available with sufficient reliability, they can be fed into the above equations and the resulting expression for the absorptive parts for the photoproduction matrix element can be expected to be correct for arbitrary energies. But as the number of amplitudes involved in the problem makes a numerical evaluation a formidable task, we have made a study in the next chapter, of the meaning of the strip approximation in the case of a much simpler problem, namely, the photoproduction of pions on pions.

CHAPTER IX

THE STRIP APPROXIMATION AND PHOTO PRODUCTION OF PIONS ON PIONS

1. INTRODUCTION

We apply, in this chapter the strip approximation to the photo production of pion on pions.

A study of the process of photo production of pions ons pions has a special role in the present framework of the theory of elementary particles. A knowledge of this process leads to a better understanding of the phenomenon of photo production of pions from nucleon and in this respect the role of the process is similar to that of pion-pion scattering in pion-nucleon interactions. The process is the simplest that could be thought of for studying the consequences of any particular theory since the matrix element does not have in its structure complications due to spin or isospin. Thus the algebra involved in the evaluation of the matrix element is not much compared to more complicated processes like pion nucleon scattering or nucleon-nucleon scattering.

The process

$$\gamma + \pi \longrightarrow \pi + \pi \qquad (1)$$

has been studied in the Cini Fubini approximation to the Mandelstam

p representation by Gourdin and Martin, Solve'v and Wong[1]. However, as mentioned in the previous chapter, the very nature of the one dimensional approximation restricts our considerations to only low energies. The strip approximation enables us to include both high and low energy phenomena in a single scheme. Ball and Frazer[2] applied this approximation to calculate the inelastic effects on pion-nucleon scattering for partial waves of angular momentum $l \geq 2$ (for which only the approximation is likely to work) in the energy range of agreement with the observed position of the peaks, but the higher resonances and also for $K^- - p$ scattering. They found qualitative agreement in the assignment of quantum numbers and rough agreement with the observed position of the peaks, but the elastic effects thus calculated were found to exceed the unitarity limit. On the other hand Cassandro[3] who has made a similar calculation finds that the "strip" integrals reproduce only about 30% of the inelastic effect. Chew et al[4] argue that the inclusion of more than two particles in the intermediate state in the unitarity condition might remove the difficulty regarding violation of

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1) M.Gourdin and A. Martin, Nuo.Cim. 16, 78 (1960); L.D.Solove'v, Soviet Physics - JETP, 13, 418 (1961); H.S.Wong, Phys.Rev.Lett. 5, 70 (1960)
2) J.S. Ball and W.R.Frazer, Phys. Rev.Letters 7, 204 (1961)
3) M.Cassandro, Nuovo Cimento 26, 196 (1962)
4) G.F. Chew, S.C.Frautschi and S. Mandelstam, Phys. Rev. 126, 1202 (1962)

unitarity mentioned above. We show in this chapter that for the simple case of process (1), the use of the strip approximation along with the one-dimensional solution for the elastic absorptive part leads to the result that the contribution from the inelastic part is of order 1/s for large s where s is the square of the total centre of mass energy for the process. This would mean that the inelastic cross-section goes down like $1/s^2$)

2. Kinematics.

As usual we introduce the three Mandelstam variables s, u, and t defined by

$$s = -(k + q_1)^2 = -(q_2 + q_3)^2$$

$$u = -(k - q_2)^2 = -(q_3 - q_1)^2$$

$$t = -(k - q_3)^2 = -(q_2 - q_1)^2 \qquad (2)$$

where

$$s + t + u = 3 \qquad (3)$$

k and q_{v_1} are the moments of the initial photon and pion and q_{v_2} and q_{v_3} are those of the final pions. By the symmetry of the process all the three variables represent the square of the centre of mass energies of the same process, different regions of the variables contributing to the matrix element. In the centre of mass of the corresponding processes, the variables are given by

$$\begin{aligned} s &= (\omega_k + k)^2 = 4\omega_q^2 \\ u &= 1 - 2k\omega_q + 2kq\cos\theta \\ t &= 1 - 2k\omega_q - 2kq\cos\theta \end{aligned} \qquad (4)$$

where θ is the angle between the directions of the initial and final pairs of particles.

In the present problem, there is only one Lorentz and gauge-invariant combination of the available vectors for the process. The invariant matrix element can be written as

$$\langle \pi\pi | T | \gamma\pi \rangle = \frac{\epsilon_{mnrs}\, q_{v_1}^m q_{v_2}^n q_{v_3}^r e^s \epsilon_{\alpha\beta\gamma} F(s,u,t)}{(16\, k\omega_k \omega_{q_{v_2}} \omega_{q_{v_3}})^{1/2}} \qquad (5)$$

Here E_{mnrs} and $\epsilon_{\alpha\beta\gamma}$ are the completely antisymmetric tensors of ranks four and three respectively; α, β and γ refer to the isotopic spin labels of the pion.

In the centre of mass system, the matrix element (5) can be written as

$$\langle \pi\pi | T | \gamma\pi \rangle = \frac{e\cdot(\underline{k}\times\underline{q})\,\epsilon_{\alpha\beta\gamma}\,F(s,u,t)}{4\,\omega_q\sqrt{\omega_k k}}$$

The differential cross-section when the initial photon is unpolarized is given in terms of the amplitude $F(s,u,t)$ by

$$\frac{d\sigma}{d\Omega} = \frac{1}{128\pi^2}\,k q^3 \sin^2\theta\,\left|F(s,u,t)\right|^2 \qquad (7)$$

3. The strip approximation

To study the process (1) under the strip approximation, we have to evaluate the double-spectral functions. Since we have only one amplitude to deal with, theevaluation is particularly simple and in fact has been given by Mandelstam.[1] Taking only the lowest two particle intermediate state as the two pion state in the unitarity condition, the double spectral functions in the elastic region of the s -variable are given by

$$F_{13}(s,t) = 4\iint dz_2\,dz_3\,K_1(z_1,z_2,z_3)\cdot\left[I_3^*(s,z_3)\,F_3(s,z_2)+I_2^*(s,z_3)\,F_2(s,z_2)\right]$$

$$F_{12}(s,u) = 4\iint dz_2\,dz_3\,K_2(z_1,z_2,z_3)\times\left[I_3^*(s,z_3)\,F_2(s,z_2) + I_2^*(s,z_3)\,F_3(s,z_2)\right] \qquad (8)$$

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1) S.Mandelstam, Phys.Rev., 112 , 1344 (1958).

where $z_1 = \cos\theta$. z_2 and z_3 are respectively the cosines of the angles between the initial and intermediate set of particles and the intermediate and final set of particles. I represents the amplitude for pion-pion scattering through $I = 1$ state.

K_1 and K_2 are given by the same expressions as in Chapter

We can introduce the isotopic spin labels now. The unitarity condition, if we retain only the two-pion intermediate state, gives the absorptive part of the photo pion production amplitude as a product of the matrix element for pion-pion scattering and photo-production of pions on pions (process (1)). The complete matrix element for the former as given by Chew and Mandelstam[1] is

$$A\delta_{\beta\gamma}\delta_{\xi\eta} + B\delta_{\beta\xi}\delta_{\gamma\eta} + C\delta_{\beta\eta}\delta_{\gamma\xi}$$

where ξ and η are the isotopic spin labels of the intermediate state pions and β and γ those of the final pions. When this is combined with the photopion production matrix element which contains the isotopic spin operator $\epsilon_{\alpha\eta\xi}$ we find that only the combination $B - C$ of the amplitudes for pion-pion interaction survives. But $B - C = A^{(1)}$ where $A^{(1)}$ is the pion-pion scattering matrix element in the isotopic state, $T = 1$ which is precisely the state in which there is a resonance

1) G.F.Chew and S. Mandelstam, 119 , 467 (1960).

Since we will be making use of the dominance of the resonance later, it is a fortunate circumstance that only the amplitude corresponding to the isotopic spin state with $T=1$ is non-vanishing

The quantities in (8) have to be re-expressed in terms of the t variable using the relations

$$z_1 = \frac{s^{1/2}(2t+s-3)}{(s-4)^{1/2}(s-1)}$$

$$z_3 = \frac{2t_3+s-4}{s-4}$$

$$z_2 = \frac{s^{1/2}(s+2t_2-3)}{(s-4)^{1/2}(s-1)} \qquad (9)$$

Further we need to require the absorptive parts I_2, I_3, F_2 and F_3 of the pion-pion scattering, we can use the resonance model with a finite width and write the matrix element for pion-pion and photo pion interaction. For

absorptive part as

$$I_i(s,z) = \frac{3\Gamma(2t+s-4)/(s-4)}{s_n - s - i\Gamma\left(\frac{s}{4}-1\right)^{3/2}}$$

where s_n denotes the position of the resonance in the (11) state.

The amplitude for process (1) has been given in the one-dimensional approximation by Solov'ev[1] . He points out that while the four-pion vertex with four internal nucleon lines diverges so that it becomes necessary to introduce into the Lagrangian (in the field-theoretic approach) a corresponding counter-term,

1) L.D.Solove'v ; loc. cit.

which introduces the pion-pion coupling constant, a similar vertex with one photon and three pion external lines converges so that there is no need for new counter-terms and anew coupling constant as used; for instance, by Gourdin and Martin[1]. He gives two solutions of which we take the one given below. The complete matrix element neglecting F and higher partial waves is given by

$$|f(s)| = |f(\nu)| e^{i\delta(\nu)} \tag{11}$$

where $\delta(\nu) = \delta_{11}$ the pion-pion scatteringphase shift in the resonant state and

$$|f(\nu) = \frac{\Lambda \Phi(\nu)}{s-1} \tag{12}$$

Here $\nu = q^2$, Λ is the pole contribution from the processe $\gamma + \pi \longrightarrow N + \bar{N}$ and Φ is given by

$$\Phi = \left| \frac{(\nu - \nu_k)^2}{(\nu + b - \nu_k)(\nu_k + b - \nu)} \right|^{\frac{(\nu - \nu_k)}{2b}} \left| \frac{(\nu_k + b - \nu)}{\nu + b - \nu_k} \right|^{1/2} \cdot \left\{ \nu \longrightarrow -\nu - 9/8 \right\} \tag{13}$$

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1) M.Gourdin and A. Martin; loc. cit.

The solution is valid for the region where

$$\delta = \begin{cases} 0 & \nu < \nu_k - b,\ \nu > \nu_k + b \\ (\pi/2b)(\nu + b - \nu_k) & \nu_k - b < \nu < \nu_k \\ (\pi/2b)(\nu_k + b - \nu) & \nu_k < \nu < \nu_k + b \end{cases} \tag{14}$$

Here $\nu_k = 1{\cdot}5$ and $b = 0{\cdot}4$.

Thus the absorptive part for the photo-pion production channel is given by

$$F_i = \frac{\Lambda}{s_i - 1} \Phi(\nu_i) \sin\delta(\nu_i) \tag{15}$$

Using relations (10) and (15) for the absorptive parts of the pion-pion scattering and photopion production matrix elements respectively and changing from the angle to the t variables using relations (9) we can rewrite the double spectral functions (8) as

$$F_{13}(s,t) = -6 \iint \frac{dt_2\, dt_3\, K_1(s,t,t_2,t_3)\, s^{1/2}}{(s-4)^{3/2}(s-1)} \Gamma^2 \Lambda \cdot \left[\frac{(t_3-4)^{1/2}}{(t_2-1)} \frac{(2s+t_3-4)\sin\delta\left(\frac{t_2}{4}-1\right)\Phi\left(\frac{t_2}{4}-1\right)}{(s_r-t_3)^2+\Gamma^2\left(\frac{t_3}{4}-1\right)^3} \right]$$

$$-6 \iint \frac{du_2\, du_3\, K_1'(s,t,u_2,u_3)\, s^{1/2}\, \Gamma^2 \Lambda}{(s-4)^{3/2}(s-1)} \cdot \left[\frac{(u_3-4)^{1/2}}{(u_2-1)} \frac{(4-2s-u_3)\sin\delta\left(\frac{u_2}{4}-1\right)\Phi\left(\frac{u_2}{4}-1\right)}{(s_r-u_3)^2+\Gamma^2\left(\frac{u_3}{4}-1\right)^3} \right] \tag{16}$$

where

$$K_1(s,t,t_2,t_3) = -\left[\frac{s}{(s-1)^2(s-u)}\left\{(s+2t-3)^2+(s+2t_2-3)^2\right\}\right.$$
$$+\frac{(2t_3+s-4)^2}{(s-4)^2}-1$$
$$\left.-\frac{2s(s+2t-3)(s+2t_3-4)(s+2t_2-3)}{(s-4)^2(s-1)^2}\right]^{-\frac{1}{2}} \qquad (17)$$

and

$$K_1'(s,t,u_2,u_3) = -\left[\frac{s}{(s-1)^2(s-u)}\left\{(s+2t-3)^2\right.\right.$$
$$\left.+(3-s-2u_2)^2\right\}+\frac{(4-s-2u_3)^2}{(s-4)^2}-1$$
$$\left.-\frac{2s(s+2t-3)(3-s-2u_2)(4-s-2u_3)}{(s-4)^2(s-1)^2}\right]^{-\frac{1}{2}} \qquad (18)$$

$$F_{12}(s,u) = +b\iint dt_2\,du_3\,\frac{K_2(s,u,t_3,u_2)\,s^{1/2}}{(s-4)^{3/2}(s-1)}$$

$$\times \Gamma^2 \Lambda \left[\frac{(t_3-4)^{1/2}}{(u_2-1)} \frac{(2s+t_3-4)\sin\delta\left(\frac{u_2}{4}-1\right)\Phi\left(\frac{u_2}{4}-1\right)}{(s_r-t_2)^2+\Gamma^2\left(\frac{t_2}{4}-1\right)^3} \right]$$

$$-6\iint dt_2\, du_3 \frac{K_2'(s,u,u_2,t_2)\, s^{1/2}}{(s-4)^{3/2}(s-1)} \Gamma^2 \Lambda$$

$$\times \left[\frac{(u_3-4)^{1/2}}{(t_2-1)} \frac{(4-2s-u_3)\sin\delta\left(\frac{t_2}{4}-1\right)\Phi\left(\frac{t_2}{4}-1\right)}{(s_r-u_3)^2+\Gamma^2\left(\frac{u_3}{4}-1\right)^3} \right] \tag{19}$$

with

$$K_2(s,u,t_3,u_2) = \left[\frac{s}{(s-1)(s-4)} \left\{ (3-s-2u)^2 + (3-s-2u_2)^2 \right\} + \frac{(2t_3+s-4)^2}{(s-4)^2} - 1 - \frac{2s(2t_3+s-4)(3-s-2u)(3-s-2u_2)}{(s-4)^2(s-1)} \right]^{-1/2}$$

and (20)

$$K_2'(s,u,t_2,u_3) = \left[\frac{s}{(s-1)^2(s-4)} \left\{ (3-s-2u)^2 + (s+2t_2-3)^2 \right\} + \frac{(4-s-2u_3)^2}{(s-4)^2} - 1 - \frac{2s(3-s-2u)(s+2t_2-3)(4-s-2u_3)}{(s-4)^2(s-1)} \right] \tag{21}$$

To obtain the inelastic parts of the double spectrum functions we use the strip approximation which prescribes that

$$F_{13}^{inel}(s,t) = F_{13}^{el}(t,s)\,; F_{12}^{inel}(s,u) = F_{12}^{el}(u,s) \tag{22}$$

In view of the symmetry of the problem, we need not calculate the double spectral functions for the third channel but can obtain them

immediately from (16) and (19) by the change of variables indicated in (22) .

The dispersion relation for the absorptive part of channel I is

$$F_1(s,t) = \frac{1}{\pi}\int_4^\infty dt' \frac{F_{13}(s,t')}{t'-t} dt' + \frac{1}{\pi}\int_4^\infty du' \frac{F_{12}(s,u')}{u'-u} + \frac{1}{\pi}\int_4^{16} dt' \frac{F_{13}(t',s)}{t'-t} + \frac{1}{\pi}\int_4^{16} du' \frac{F_{12}(u',s)}{u'-u} \tag{23}$$

We should expect that if the one-dimensional solution (15) is good, the solution of equation (23) retaining only the first two terms on the right hand side should not differ appreciably from (15) . We shall therefore be content with estimating the contribution from the inelastic part given by the last two terms, $\mathbb{K}$ for large values of s .

For very large, s , $K_1(t, s, t_2, t_3)$ can be approximated by

$$\frac{(t-1)(t-4)}{t^{1/2}} \left\{ 4s^2(t-4) + 4s(t-3)(t-4) - 4s(t+2t_3-4)(t+2t_2-3) \right\}^{-1/2}$$

$$= \frac{(t-1)(t-4)}{t^{1/2}} \left\{ 4s^2(t-4) - 16st_2t_3 - 8st_3(t-3) - 8st_2(t-4) \right\}^{-1/2}$$

Substituting this and similar expressions for K_1', K_2 and K_2', we find on majorising the integrals over t' that the contribution from the first term of $F_{13}(t, s)$ is a convergent integral and is of the order of $1/s$ as is also the case for the contribution from the second term of $F_{13}(t, s)$. We can neglect the last term of (23) since for large positive values of s and fixed values of t, u will be large and negative so that the denominator of the integral occurring in it will never vanish and will be large for large values of u.

Conclusion

That the inelastic contributions are of the order $1/s$ for every large s seems to be a significant feature. It is interesting to note that a similar result follows for the process of pion-pion scattering which is a little more complicated in that the matrix element here involves three amplitudes corresponding to the three values, 0, 1 and 2 for the total isotopic spin of the two pions. But if we assume a resonance in the $T = 1$ state we need deal with only one amplitude. Both the absorptive parts appearing in the double spectral functions I_{13} and I_{12} (where I represents the pion-pion scattering amplitude in the $T = 1$, $J = 1$ state) will correspond to those of pion-pion scattering in the second and third channels. Using the expression (10) for the absorptive parts, we have

$$I_{13}(s,t_1) = -\frac{9}{4}\iint \frac{dt_2\,dt_3}{(s-4)}\,\Gamma^4 K_1(s,t_1,t_2,t_3)$$

$$\times\ \frac{(2t_3+s-4)(2t_2+s-4)}{\left[(s_\pi-s)^2+\Gamma^2\left(\frac{s}{4}-1\right)^3\right]^2}$$

$$-\frac{9}{4}\iint \frac{du_2\,du_3}{(s-4)}\,\Gamma^4 K_1'(s,t_1,u_2,u_3)$$

$$\times\ \frac{(4-s-2u_3)(4-s-2u_2)}{\left[(s_\pi-s)^2+\Gamma^2\left(\frac{s}{4}-1\right)^3\right]^2} \qquad (25)$$

where all the variables correspond to those for pion-pion scattering. As in the case of photoproduction of pions on pions we can write the expression for $I_{13}(t_1,s)$ which represents the inelastic contribution by the simple change of variables $t_1 \longleftrightarrow s$ and introducing a numerical factor given by the crossing matrix.

In this case again the only factor which contains the variable s is the function K_1 (and K_1'). In the limit of large s this behaves like $1/s$ again. Of course this result as well as the similar one for the photoproduction of pions on pions depends on the form of the absorptive parts assumed. Still the result gives us hope that the strip approximation may yield results in agreement with experiment at high energies.

CHAPTER X

THE PIONIC DECAY OF THE Ξ PARTICLES[1)]

Introduction

In this chapter we use the "pole" approximation to study the pionic decay mode of the cascade particle. The "pole" term in the matrix element for a process in dispersion theory corresponds to the renormalized Born approximation of the conventional Lagrangian theory. Recently the "pole" approximation has been used in various problems in weak interactions of elementary particles. Bernstein et al[2)] utilized the dominance of the pion pole, in the matrix element $\langle p\bar{n}|\partial_\alpha P_\alpha|0\rangle$ (where $\partial_\alpha P_\alpha$ behaves like a pseudoscalar and $|p\bar{n}\rangle$ is the proton-antineutron state) in order to explain the rather involved calculation of Goldberger and Treiman[3)] for the rate of decay of the charged pion. Feldman et al[4)] applied the pole approximation to a study of the non-leptonic decay modes of Σ and Λ^0 which possess a number of curious features. The decay modes $\Sigma^+ \to n + \pi^+$

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1) A.P.Balachandran and K. Venkatesan, Progr. Theor. Phys. 26, 792 (1961)

2) J. Bernstein et al , Nuovo Cimento, 17, 757 (1960). Also "Lectures on the weak interactions of the strongly interacting particles" by M. Gell-Mann, The Summer School in Theoretical Physics, Bangalore,(1961).

3) M.L.Goldberger and S.B.Treiman, Phys. Rev., 110, 1478 (1958)

4) G. Feldman, P.T.Matthews and A. Salam, Phys. Rev., 121, 302 (1961).

and $\Sigma^- \to n + \pi^-$ show no up-down asymmetry while the mode $\Sigma^+ \to p + \pi^0$ seems to have an asymmetry parameter very near unity. Assuming global symmetry Feldman et al were able to show that the Σ decay asymmetries could be explained and assuming essentially one more parameter the Λ and Σ decay times and asymmetries could be correlated.

The success of the pole approximation in the problem of pion decay has encouraged us to use it to study the pionic decay modes of the Ξ particle. Recently Fowler et al [1] have presented data on the magnitude of the asymmetry parameter α_Ξ and the lifetime of the decay of Ξ^- into a $\Lambda^0 + \pi^-$. We have utilised these values to obtain the possible values for the renormalized $\Xi - \Xi - \pi$ coupling constant G_1. Fowler et al have also noted that the decay asymmetry α_Ξ and the asymmetry parameter in $\Lambda^0 \to p + \pi^-$ or $n + \pi^0$ have opposite signs. We have used this result to suggest a possible mechanism for the Σ decay which can explain the curious pattern of the asymmetries. Finally we again suggest a pole approximation calculation for determining the mass difference of K_1^0 and K_2^0 and the self-mass of Σ^0.

Calculations

(i) Let us first consider the decay of Ξ^- into a Λ^0 and a π^-. The results for the mode $\Xi^0 \to \Lambda^0 + \pi^0$ will follow from this if we use the $|\Delta \vec{T}| = \frac{1}{2}$ rule. The diagram of interest

1) Fowler et al, Phys. Rev. Letters, 6, 134 (1961)

for the Ξ^- decay is the pole diagram shown in Fig. 1 a. The diagram with the K -meson pole (Figure 1 b) makes a comparatively small contribution as we shall see. G_1 denotes the renormalized $\Xi-\Xi-\pi$ coupling constant; G_2 the coupling constant for $\Xi-K-\Lambda$ vertex; $a+b\gamma_5$, the $\Xi-\Lambda$ vertex and c is the coupling constant for the $K-\pi$ weak vertex. The matrix element corresponding to Fig. 1 (a) is given by

$$M_1 = \frac{iG_1}{\sqrt{2E_\pi}} \cdot \frac{1}{p_\pi^2 - 2p\Xi p_\pi} \bar{u}_\Xi \gamma_5 [A + B\gamma_5] u_\Lambda . \qquad (1)$$

where p denotes the four-momentum of the particle represented by the suffix and A and B are given by

$$A = -(m_\Xi + m_\Lambda)a \quad ; \quad B = (m_\Lambda - m_\Xi)b, \qquad (2)$$

m denoting the mass of a particle. The matrix element corresponding to Fig. 1 (b) is

$$M_2 = \frac{iG_2}{\sqrt{2E_\pi}} \cdot \frac{1}{p_\pi^2 + m_K^2} c\bar{u}_\Xi \gamma_5 u_\Lambda \qquad (3)$$

where E_π is the energy of the pion. Therefore the total matrix element M is given by

$$M = M_1 + M_2 = \frac{iG_1}{\sqrt{2E_\pi}} \cdot \frac{1}{p_\pi^2 - 2p_\Xi - p_\pi} \cdot \left\{ \bar{u}_\Xi [B + \beta\gamma_5] u_\Lambda \right\} \qquad (4)$$

where

$$\begin{aligned} \beta &= A + \alpha \\ \alpha &= \frac{G_2 c}{G_1} \cdot \frac{p_\pi^2 - 2p_\Xi p_\pi}{p_\pi^2 + m_K^2} \end{aligned} \qquad (5)$$

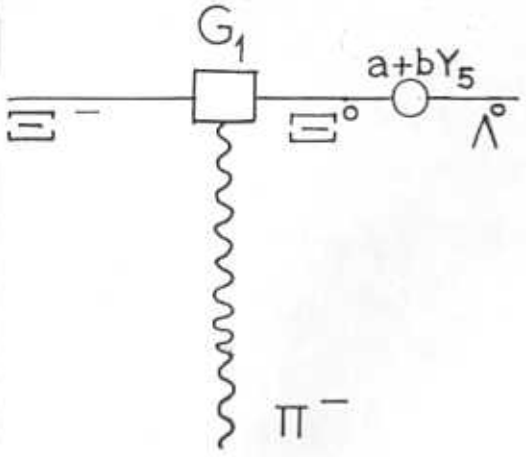


FIG. 1(a)

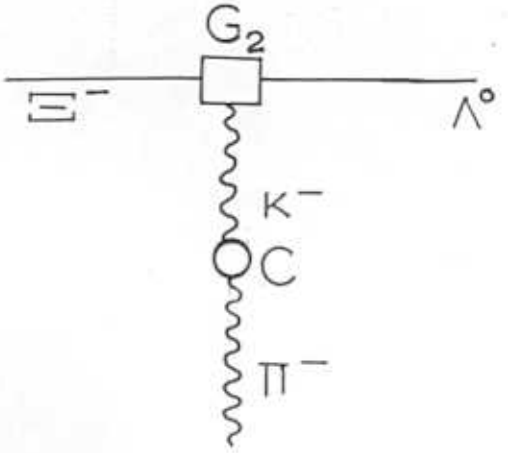
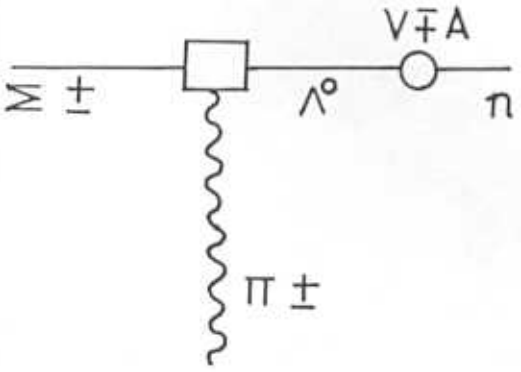
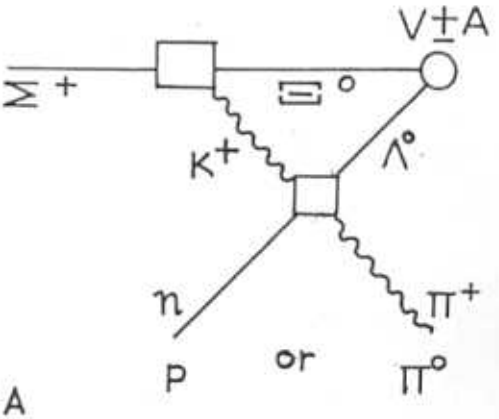


FIG. 1(b)





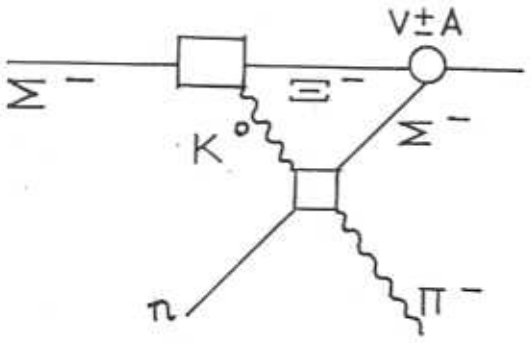
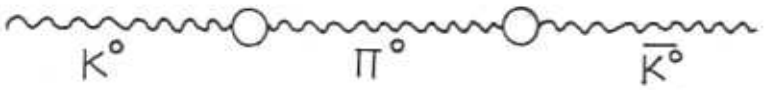


FIG. 2



□ Denotes the strong and

○ The weak vertex figure

FIG. 3

Squaring M and fixing the initial spin of the Ξ^- in the z - direction, we find that

$$|M|^2 = 4m_{\Xi}\left[E_{\Lambda}\left(|B|^2+|\beta|^2+m_{\Lambda}\left(B^2-\beta^2\right)\right.\right.$$
$$\left.\left.-2|\vec{p}_{\Lambda}|\,\mathrm{Re}\,(B\beta^*)\cos\theta\right]\right. \qquad (6)$$

E_{Λ} is the energy of Λ^0 . (6) can be written as $A'+B'\cos\theta$ where

$$A' = 4m_{\Xi}\left[E_{\Lambda}\left(|B|^2+|\beta|^2+m_{\Lambda}\left(B^2-\beta^2\right)\right)\right]$$

$$B' = -8\,m_{\Xi}\,|\vec{p}_{\Lambda}|\,\mathrm{Re}\,(B\beta^*) \qquad (7)$$

In the above, θ is the angle between the z - direction and the direction of the decay particles (Λ or π).

Now the asymmetry parameter α_{Ξ} which can be measured experimentally is defined by

$$\alpha_{\Xi} = \frac{N_{up}-N_{down}}{\frac{1}{2}\left(N_{up}+N_{down}\right)} \qquad (8)$$

where N denotes the number of particles. Now

$$N_{up} = K\cdot\int_0^{\pi/2}\left(A'+B'\cos\theta\right)\sin\theta\,d\theta$$

$$N_{down} = K\cdot\int_{\pi/2}^{\pi}\left(A'+B'\cos\theta\right)\sin\theta\,d\theta \qquad (9)$$

where K is a constant.

Hence

$$\alpha_{\Xi} = \frac{B'}{A'} = -\frac{2|\vec{p}_\Lambda| \, \mathrm{Re}(B\beta^*)}{B^2(E_\Lambda + m_\Lambda) + \beta^2(E_\Lambda - m_\Lambda)} \qquad (10)$$

If the Ξ particle is not completely polarized and its degree of polarization is P then $\alpha_{\Xi} = \frac{B'}{A'} \cdot P$

Using (6) , we can calculate the life-time τ of the particle. Assuming it to be unpolarized, τ is given by

$$\frac{1}{\tau} = \frac{G_1^2 E_\Lambda |\vec{p}_\Lambda|}{2\pi m_\Xi m_\Lambda} \cdot \left[\frac{1}{2m_\Xi - 1}\right]^2 \cdot \left[B^2(E_\Lambda + m_\Lambda) + \beta^2(E_\Lambda - m_\Lambda)\right] \qquad (11)$$

From kinematics, we have $|\vec{p}_\Lambda| = 124 \cdot 5 \ \mathrm{MeV}/c$;

$E_\Lambda = 1121 \ \mathrm{MeV}$; $E_\pi = 196 \ \mathrm{MeV}$

From (5) we then find that

$$\beta = \left\{ -2432\, a + 2 \cdot 213 \frac{G_2}{G_1} c \right\} \qquad (12)$$

If we assume that $\frac{G_2}{G_1} \sim \frac{1}{10}$ and $c \sim a \sim b$, then the second term on the right hand side of (12) is $\cdot 2213\, a$ which is small (in absolute value) compared to the first term. Therefore we shall omit it from our consideration which means that we shall be setting $\beta = A$. Fowler et al give a value $\alpha_{\Xi} = \pm 0 \cdot 69$ for the asymmetry parameter, the sign being unspecified. Using this α in (10) we obtain a quadratic equation for (a/b) with the solutions

$(a/b) = \mp\ 0.58$ or $\mp\ 4.42$. Using these values in (11) we arrive at the following values for the dimensionless parameter $\frac{G_1^2 b^2}{4\pi}$,

$$\frac{G_1^2 b^2}{4\pi} = 0.3296 \times 10^{-11} \qquad \left|\frac{a}{b}\right| = 0.58$$

and

$$\frac{G_1^2 b^2}{4\pi} = 0.4298 \times 10^{-12} \qquad \left|\frac{a}{b}\right| = 4.42 \qquad (13)$$

τ is independent of the sign of (a/b) as can be seen from (11) . If we assume $b \sim c$ and take the value of 0.3×10^{-11} for c^2 as calculated in the next subsection we get for the first of these possibilities $(G_1^2/4\pi) \sim 1$.

(ii) The experiments of Fowler et al indicate that the asymmetry parameters α_{Ξ} and α_{Λ} have opposite signs, the absolute sign of each being undetermined. This suggests that the observed pattern of the Σ -decay asymmetries might be ascribed to a parity clash between diagrams involving Λ and those involving Ξ . Figs. 2 (a), 2 (b) and 2 (c) are the relevant diagrams. Fig. 2 (a) with the Λ pole is assumed to be dominant. For Ξ intermediate states pole diagrams do not exist and those in next in order of simplicity are taken into account. The $\Xi^- - \Sigma^-$ vertex can arise from $\Xi^- \to \Lambda^\circ + \pi^- \longrightarrow \Sigma^-$. It is clear that for suitable values of the parameters involved these three diagrams can yield parity-conserving $\Sigma^+ \to n + \pi^+$ and $\Sigma^- \to n + \pi^-$ decay modes and parity-violating $\Sigma^+ \to p + \pi^\circ$ decay mode. The proton from Σ^+ decay and the Λ from

Ξ decay will have helicities of the same sign. Such a procedure would however imply that relations like $R(\Sigma^+ \to p + \pi^0) \approx 2R(\Lambda^0 \to p + \pi^-)$ [1] are in the nature of dynamical accidents.

(iii) Finally we wish to remark that the pole approximation can be employed to calculate the coupling constant for the weak $K-\pi$ vertex, c. The fact that the K_0 to $\bar{K}_0$ matrix element is connected with the $K_1^0 - K_2^0$ mass difference can be seen as follows: The mass matrix is diagonal in the $K_1^0 - K_2^0$ representation, i.e.

$$\begin{array}{c} \\ K_1^0 \\ K_2^0 \end{array} \begin{array}{c} \begin{array}{cc} K_1^0 & K_2^0 \end{array} \\ \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \end{array} = \frac{m_1 + m_2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{m_1 - m_2}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (14)$$

In the K_0, $\bar{K}_0$ representation this matrix becomes

$$\begin{array}{c} \\ K_0 \\ \bar{K}_0 \end{array} \begin{array}{c} \begin{array}{cc} K_0 & \bar{K}_0 \end{array} \\ \begin{pmatrix} & \\ & \end{pmatrix} \end{array} = \frac{m_1 + m_2}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{m_1 - m_2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (15)$$

From (15) we see that the transition K_0 to $\bar{K}_0$ involves the mass-difference δm between K_1^0 and K_2^0, the exact relation being

$$\delta m = 2 \langle K_0 | \bar{K}_0 \rangle$$

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1) S.B.Treiman, Nuovo Cimento, 15, 916 (1960).

From Fig. 3 we g find that

$$\frac{\delta m}{2} = \langle K_0 | \bar{K}_0 \rangle = \frac{c^2}{2m_K} \cdot \frac{1}{1 - m_K^2}$$

Taking the $K_1^\circ - K_2^\circ$ mass difference δm to be $\sim 10^{-5}$ we obtain $c^2 \sim 0.3 \times 10^{-11}$. A precise value for δm would allow us to determine t c more accurately. Similarly the pole approximation could be used to find the self energy of Σ° or Λ° due to transitions of the type $\Sigma^\circ \to \Lambda^\circ \to \Sigma^\circ$.

12)

1) Muller et al , Phys. Rev. Letters, 4 , 418 (1960).

PART III

CHAPTER XI

SOME NEW STOCHASTIC ASPECTS IN CASCADE THEORY[1)]

1. Introduction

In the earlier chapters of this thesis we have used some of the methods which are available for computing the cross-sections for various processes which involve a production of only a few additional particles in the final state and at low energies. At higher energies encountered in cosmic radiation (10^5 to 10^{18} ev), multiple-production of particles, particularly pions, is quite frequent and for such energies and processes no reliable method based on field theory exists. Statistical theories, such as that of Fermi[2)], hydrodynamic theories of Heisenberg[3)] and Landau[4)], models like the isobar and fireball[5)] models and many other theories[6)] have been suggested to explain the various aspects of multiple-pion production, but each of them has had only a partial success.

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1) A. Ramakrishnan and K. Venkatesan, Proc. Cosmic Ray Symposium, Department of Atomic Energy, Government of India (1961). See also A.Ramakrishnan,"Elementary Particles and Cosmic Rays", Pergamon Press (1962).

2) E. Fermi, Progress of Theoretical Physics, 5 , 570 (1950); Phys. Rev., 81 , 683 (1951); 92, 452 (1953), 93, 1434 (1954).

3) W. Heisenberg, Z. Phys. 113 , 61 (1939); 126, 569 (1949), 133, 65 (1952).

4) L.D.Landau, Izv. Akad. Nauk. U.S.S.R. 17 , 57 (1953).

5) Z. Koba and S.Takagi, Progr. Theoret.Phys. 7, 123 (1952); W.L.Kraushaar and L.J.Marks,Phys.Rev. 93, 326 (1954); G. Cocconi, Phys. Rev., 111, 1699 (1958).

6) see, e.g. A.Ramakrishnan, "Elementary Particles and Cosmic Rays", Pergamon Press (1962).

There is another aspect to cosmic ray phenomena where a direct application of field-theoretic techniques would again not be possible, but where however use can be made of cross-sections for various processes evaluated using such methods, and this is cascade phenomena, i.e. the cumulative effect of fundamental particle interactions when a single particle enters a finite thickness of matter. This particle collides with air nuclei in the atmosphere and in each collision a number of other particles are produced. These secondary particles in their passage through matter collide with other nuclei and produce further particles. Thus a cascade is generated and one of the main problems in cosmic rays is to study the nature of this multiplication. In any typical collision particles of different types at various energies and angles of emergence are produced and the probabilities of the individual collisions are determined by the corresponding quantum mechanical cross-sections. The successive collisions are considered independent and cascade multiplication then becomes a stochastic process which can be studied by methods of classical probability.

But further restrictions have to be imposed to make calculations tractable. Thus, for instance, one can consider the cascades to develop only longitudinally and ignore the lateral spread of particles in which case one can deal not only with the mean values of the various quantities involved but also their probability distributions or at least the fluctuations about

the mean. One canalso include lateral spread but then one has to be satisfied with the mean values of the various quantities.

It is however generally felt that the literature on the cascade theory has grown out of proportion to its utility in the understanding of cosmic ray phenomena or in the application of probabilistic methods. This is essentially due to the fact that in any discussion, only 'conventional' questions are formulated. New ways of looking at the problem may however reveal hitherto unnoticed aspects of the theory. In the subsequent sections, we present some attempts at such a departure from conventional methods.

2. An interesting relationship between the 'old' and the 'new' approaches to cascade theory

The "old" approach to cascade theory was concerned with the number of particles of a given type at a depth which is the thickness of matter traversed assuming the shower to be initiated by a particle at $t = 0$. We confine ourselves to the longitudinal development of the cascades and neglect their lateral spread. The main object is to obtain the probability distribution for the number of particles with specified energies at a given depth t defined by

$$\pi_I^i (n_1, n_2, \ldots n_m; E_1, E_2 \ldots E_m | E_0; t)$$

which denotes the probability that there exist n_j particles of type j each of which has an energy greater than E_j $(j = 1, 2, \ldots m)$ at t if the cascade is initiated at $t = 0$ by a particle of type i with an energy E_0.

This definition of the distribution function for the number of particles at a given depth was natural in the early days since cascades were experimentally observed in cloud chambers in which the number of particles at a given depth can be easily recorded.

The use of emulsion techniques in the study of cascades suggested however that it would be more convenient to estimate the energies of the particles at the point of their production rather than of all the particles at any particular thickness. Based on this, Ramakrishnan and Srinivasan[1] suggested a new approach to cascade theory. Instead of the function Π_I defined earlier, they define a new function $\Pi_{II}^{i}(n_1, n_2 \ldots n_m ; E_1, E_2, \ldots E_m | E_0, t)$ which is the probability that n_j particles of type j, each with energy greater than E_j <u>at the point of its production</u> are produced between 0 and t if the cascade is initiated at $t = 0$, by a particle of type i with energy E_0. They used the method of product densities, introduced earlier by Ramakrishnan[2]. In E-space which is a continuum, the product density of degree n, $(E_1, E_2 \ldots E_n)$ is defined such that $(E_1, E_2, \ldots E_n)\, dE_1\, dE_2 \ldots dE_n$ represents the probability that there is a particle in dE_1, a particle in $dE_2 \ldots$ and a particle in dE_n <u>irrespective</u> of the number elsewhere. The product density functions in the new approach which are over E space as well as t space are defined as follows:

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1) A. Ramakrishnan and S.K.Srinivasan, Proc. Ind. Acad. Sci., 44, 263 (1956).

2) A. Ramakrishnan, Proc. Camb. Phil.Soc., 46, 595 (1950).

$$F_k(E_1, E_2 \ldots E_k; t_1, t_2, \ldots t_k)\, dE_1 dE_2 \ldots dE_k \cdot dt_1 dt_2 \ldots dt_k$$

is the joint probability that a particle of energy between E_1 and $E_1 + dE_1$ is produced between t_1 and t_1+dt_1, a particle of energy between E_2 and $E_2 + dE_2$ is produced between t_2 and $t_2 + dt_2$ and a particle of energy between E_k and $E_k + dE_k$ is produced between t_k and $t_k + dt_k$. Using the equation for product densities of degree 1, the first moment of π_{II} has been calculated numerically by this method.[1)]

The essential difference between the old and new approaches to cascade theory, while apparent from the definition of the two types of product density functions given above, is strikingly brought out by the integral equations for the functions π_I and π_{II} derived by using the method of regeneration point which is quite useful in studying a class of non-Markovian stochastic processes which however have a simple dependence on previous history. Consider a stochastic process describing the occupation states of a system as a parameter t varies and let the occupation states be denoted by S_1, S_2, ... We shall call the process non-Markovian if given that the system is in a state S_j at t, we are not able to predict what happens between t and $t+dt$. But let us assume that there exists a

1) S.K.Srinivasan and N.R.Ranganathan, Proc. Ind. Acad. Sci., 45, 69 (1957).

set S_1' , S_2' .... among the occupation states of the system such that if the system is found in any one of these states at t , it is possible to predict what happens between t and $t+dt$. Such processes are called regenerative with respect to these states; i.e. when the system reaches the regenerative state, it 'loses' all memory and starts afresh.

For simplicity we shall consider a simple cascade consisting of only one type of particle and the development of the cascade is characterised by the cross-section $R(E'|E)\,dE'\,dt$ representing the probability that a particle of energy E drops to an energy interval lying between E' and $E'+dE'$ producing another particle of energy $E-E'$ in traversing a matter of thickness dt .

As defined earlier, let $\Pi_I(n, E|E_0; t)$ represent the probability that n particles are at t with the energy of each particle greater than E and $\Pi_{II}(n, E|E_0; t)$, the probability that there are produced n particles between 0 and t , each having energy greater than E at the point of production, the energy of the primary in both cases being E_0 . This difference in reckoning the energies of the particles gives rise to a corresponding difference in the summation in the equations satisfied by them as shown below.

Consider the first regeneration point τ where the primary particle of energy E_0 is replaced by two particles of energy E' and E_0-E' which become in turn independent

primaries for the subsequent development of cascades in the interval $t-\tau$. For $\Pi_I(n, E \mid E_0; t)$, these two primaries together produce n particles with energy greater than E in a distance $t-\tau$, i.e. there exist at t , n particles with energy greater than E . However in the new approach, the counting of particles is different as it depends on the energy of the particle at the point of production. Since even the primary particle with an energy E_0 at $t=0$ forms part of the system of n particles with primitive energy greater than E that are produced between 0 and t , it is clear that the two independent primaries at the first regeneration point τ need produce together only $n-1$ particles, the two primaries being included in the counting. It is to be noted that in the new approach it is quite possible that the same particle will be counted many times at various "production" points provided at each such point, the particle has an energy greater than the specified value.

We can now write down the integral equation satisfied by Π_I and Π_{II} using elementary arguments.

$$\Pi_I(n, E \mid E_0; t) = \int_0^t d\tau e^{-R(E_0)\tau} \int_E^{E_0} R(E' \mid E_0)\, dE'$$

$$\times \sum_{m+m'=n} \Pi_I(m, E \mid E'; t-\tau)$$

$$\times \Pi_I(m') E \mid E_0 - E'; t-\tau)$$

$$+ e^{-R(E_0)t}\delta(n-1)\Pi_I(n,E|E_0;t). \qquad (1)$$

$$\Pi_{II}(n,E|E_0;t) = \int_0^t d\tau e^{-R(E_0)\tau}\int_E^{E_0} R(E'|E_0)dE'$$

$$\times \sum_{m+m'=n-1} \Pi_{II}(m,E|E';t-\tau)$$

$$\times \Pi_{II}(m',E|E_0-E';t-\tau)$$

$$+ e^{-R(E_0)t}\delta(n-1)\Pi_{II}(n,E|E_0;t) \qquad (2)$$

where

$$R(E_0) = \int_0^{E_0} R(E'|E_0)dE'$$

and $\delta(n-1) = 1$ when $n=1$ and is zero otherwise.

We observe that the integral equations (1) and (2) satisfied by Π_I and Π_{II} are exactly the same except for the condition on summations over m and m'. It is remarkable that the change of the condition from $m+m'=n$ to $m+m'=n-1$ should give such entirely different information as are contained in the functions Π_I and Π_{II}. It may look surprising that such a trivial looking difference should give rise to two quite different approachs to cascade theory, but one has to remember that one is dealing with an integral equation for the functions Π_I and Π_{II}.

Let us define the moment generating function

$G_{I,II}(u,E|E_0;t)$ corresponding to $\Pi_I(u,E|E_0;t)$

and $\Pi_{II}(n, E|E_0; t)$ as

$$G_{I,II}(u, E|E_0; t) = \sum_{n=0}^{\infty} u^n \Pi_{I,II}(n, E|E_0; t) \qquad (3)$$

On differentiating the integral equations with respect to t, we can obtain the following differential equations for G_I and G_{II} with respect to t,

$$\frac{\partial G_I(u, E|E_0; t)}{\partial t} = \int_0^{t_0} \left\{ G_I(u, E|E'; t) \times G_I(u, E|E_0 - E'; t) - G_I(u, E|E_0; t) \right\} R(E'|E_0)\, dE' \qquad (4)$$

$$\frac{\partial G_{II}(u, E|E_0; t)}{\partial t} = \int_0^{t_0} \left\{ u G_{II}(n, E|E'; t) \times G_{II}(n, E|E_0 - E'; t) - G_{II}(u, E|E_0; t) \right\} R(E'|E_0)\, dE' \qquad (5)$$

In the more realistic models of cascade theory it will not be possible to determine the functions Π_I and Π_{II} and only the first few moments of these distribution functions can be calculated.

2. Multiple production of particles in a cascade.

Our next observation is regarding an error in the usual formulation of the problem of the multiple production of particles in a cascade. The type of theories mentioned in the introduction for multiple-pion production assume that even in a single nucleon-nucleon collision multiple production is possible. Per contra, another type of theory assumes 'plural' production, i.e. it is assumed that only one meson is produced in single nucleon-nucleon collisions but a nucleon when passing through a nucleus produces an internucleon cascade by successive collisions with nucleons in the nucleus. For simplicity the collisions inside the nucleus are considered independent and the methods of cascade theory are used, the total thickness of matter km traversed being the diameter of the nucleus. Unlike the case of a cascade in finite thicknesses of matter in the gross , it is not possible to analyse the development of the intra-nuclear cascade at various "thicknesses" but we can only make calculations on the basis that the nucleon has passed through the entire nucleus; i.e. the cascade hypothesis is used to compute the cross-sections for the production of particles in the various energy ranges in a single encounter between a nucleon and a nucleus.

We can define a cross-section

$$J_n(E_1, E_2, \cdots E_n | E_0)\, dE_1 \cdots \cdots dE_n\, dt$$

that particles are produced in the energy intervals

$$(E_1, E_1 + dE_1), (E_2, E_2 + dE_2), \cdots \cdots (E_n, E_n + dE_n)$$

and none elsewhere by a particle of energy in passing through a matter of thickness dt. The total cross-section is given by

$$J = \sum_n \frac{1}{n} \int_{E_1} \cdots \int_{E_n} J_n(E_1, E_2 \cdots E_n | E_0)\, (dE_1 \cdots dE_n)$$

The mean number of particles produced in dt in the energy interval dE_1 is given by

$$\rho(E_1 | E_0)\, dE_1\, dt = \sum_n \frac{dE_1\, dt}{(n-1)} \cdot \int_{E_2} \cdots \cdots \int_{E_n} J_n(E_1, E_2 \cdots E_n | E_0)\, dE_2 \cdots$$

and mean number of particles in dt is

$$\rho(E_0) = \int_{E_1} \rho(E_1 | E_0)\, dE_1$$

$$= \sum_n \frac{1}{n} \int_{E_1} \cdots \int_{E_n} (n J_n(E_1, E_2 \cdots E_n | E_0)$$

We observe that the expression for the cross-section is different from that for the mean density $\rho(E_0)$ except for the case $n = 1$ when the function representing the mean density is also the cross-section for the replacement of a particle of energy E_0 by one of energy E_1. (However, if we write the equations obeyed by the product densities including degree one, they will be different since the total cross-section J is different

in both cases and this occurs in the equations for product densities). Thus even for writing the equations for mean number it is not enough to know the ρ for the process, as is generally assumed, but it is also necessary to know the detailed cross-sections J_n . This fact may prove helpful in gleaning the multiplicative nature of the cross-section even from the mean number.

4. Ambigenous stochastic processes and cascade showers with lateral spread

An ultra-high energy ($> 10^{15}$ eV) primary cosmic ray particle incident on the atmosphere produces an enormous number of particles in successive interactions and these particles are scattered over a large area. Such showers developed in the atmosphere in which the lateral spread of the particles may extend asfar as one kilometre are called extensive air showers. In a study of these showers we are interested in the average number of particles in the energy range $E, (E+dE)$ at the thickness t . Let t be along the z -axis and r the radial distance from the z -axis in the x-y plane. Let the particle move at an angle lying between θ and $\theta + d\theta$ with the z-axis. θ_x and θ_y are the angles which the projection of the particle direction in the x-z and y-z planes make with the z -xn axis. Thus the number of particles can be denoted by $\pi(E, r, \theta, t)\, dE\, dr\, d\theta$

We include the distribution in r and θ by defining the cross-section $\sigma(\varphi)\,d\varphi$ to be the probability that the incident particle is deflected at an angle φ and $\varphi + d\varphi$ with its original direction in travelling a thickness t. $\sigma(\varphi)$ is an even function as the scattering should be symmetrical about the original direcion.

Using the usual probability arguments, we can write the variation in the number of particles within an interval θ and $\theta + d\theta$ caused by scattering as given by

$$-\Pi(E,r,\theta;t)\int_{\varphi}\sigma(\varphi)\,d\varphi + \int_{\varphi}\sigma(\varphi-\theta)\Pi(E,r,\varphi;t)\,d\varphi$$

There is also a deterministic lateral displacement $\theta\,dt$ due to the fact that the particles with co-ordinates $r-\theta\,dt$ at t move to r at $t+dt$; i.e.

$$\Pi(r,\theta,t+dt) = \Pi(r-\theta\,dt,\theta;t)$$

The change in the number density at r due to variation dt in t assuming only this process is

$$\Pi(r,\theta,t+dt) = \Pi(r-\theta\,dt,\theta;t)$$

$$= \Pi(r,\theta,t) - \theta\,dt\,\frac{\partial\Pi(r,\theta,t)}{\partial r}$$

Expanding $\Pi(E,r,\theta+\varphi;t)$ as a Taylor series in θ, we have

$$-\Pi(E,r,\theta;t)\int\sigma(\varphi)\,d\varphi+\Big[\int_{\varphi}\Big\{\Pi(E,r,\theta;t)$$

$$+\varphi\nabla\Pi(E,r,\theta,t)+\frac{\varphi^2}{2}\nabla^2\Pi(E,r,\theta;t)$$

$$+\cdots\Big\}\sigma(\varphi)\,d\varphi\Big]$$

The second term within the square brackets vanishes as σ is an even function. Here usually the Landau-approximation[1] is made that all moments of φ higher than φ^2 are negligible which implies that the probability for deflection $\sigma(\varphi)$ rapidly tends to zero as φ differs from zero; i.e. the scattering is confined to extremely small angles. This approximation is considered necessary since even the cumulative angular deviation θ is treated to be small enough so that $\sin\theta = 0$ i.e. the path length t traversed is identical with z -axis. If the Landau approximation is not made finite values of φ will be allowed and hence θ can take values greater than $\pi/2$. This means that the particle can be back-scattered which will disturb the Markovian nature of the process with respect to z and such a situation is supposed to lead to considerable mathematical difficulties.[2]

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1) L. Landau, J.Phys., 8 , 237 (1940)

2) Kamata and Nishimura who believe that the Landau approximation can be improved by replacing θ by $\sin\theta$ do not seem to have realised this point. J. Kamata and K. Nishimura, Progr. Theor. Phys., Suppl. 6, 135, 1958.

Recently, however, a new type of stochastic process called "ambigenous" has been studied[1] which involves back-scattering. To illustrate the process, let us consider the simple case of a particle characterized by its velocity which can take positive or negative values as the one-dimensional parameter x representing the distance travelled by the particle is varied. If the particle is assumed to move in the direction $+x$ when the velocity is positive, it is necessarily implied that it moves in the direction of $-x$ when the velocity is negative. If we choose time as the one-dimensional parameter with respect to which the process is supposed to be developing, then transition probabilities can be defined per unit time for the velocity to change sign and Markovian properties can be ascribed to be stochastic process unfolding with t. But if x is chosen as the parameter of the process, the concept of development with x breaks down since the velocity can take positive and negative values.

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1) Alladi Ramakrishnan, R. Vasudevan and P. Rajagopal, J. Math. Anal. & Applns., 1, 145 (1960)

A consequence of the above difficulty is that a differential equation cannot be written down for its motion. A way out for this problem, suggested by Bellmann[1] is to switch off the back-scattering probabilities outside a finite interval of space (0, L) considering only the probabilities, that the particle emerges at 0 or L over all time. We are not concerned with what happens between 0 and L and beyond this interval. This is precisely the idea of "Switching off". We wish to suggest that this method can be applied to multiple particle production in cascades in which m of the particles go down (into the earth) and n-m are back-scattered, the thickness of the atmosphere providing the finite interval referred to above. Further the possibility of including back-scattering may lead to a formulation of the problem of lateral spread without making the Landau approximation.

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1) R. Bellmann et al; Proc. Natal. Acad. Sci. US 43, 517 (1957); J. Math. Mech. 7, 149 (1957).

CHAPTER XII.

ON THE EQUIVALENCE OF THE FIELD THEORETIC AND FEYNMAN FORMALISMS*

1. Introduction

The chronological operator P which orders the relative time co-ordinates of the interaction Hamiltonian, products of which appear in the Dyson expansion of the S -matrix

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int\int\int \ldots \int dx_1 dx_2 dx_3 \ldots dx_n \cdot \left(\Phi_f, P\left(H_I(x_1) H_I(x_2) \ldots H_I(x_n)\right), \Phi_i \right) \quad (1)$$

does not prescribe the ordering of the operators within any single interaction Hamiltonian $H_I(x)$ which represents the vertex containing operators corresponding to the same space-time point, x . Other considerations have to be employed to re-order the creation and annihilation operators occurring in $H_I(x)$ and one such was the introduction of the normal product operator, N, which rearranges the operators such that the creation operators stand always to the left of the annihilation operators so that the vacuum expectation of any product of operators in normal product form is zero. The necessity for the introduction of such an operator which stems from its property just mentioned arose out of the following difficulty pointed out by Heisenberg [1]. In electrodynamics the "matter" current $j_\mu(x)$

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1) W. Heisenberg, Zeits. fur Phys. 90, 209 (1934), 92, 692 (1934).

* Alladi Ramakrishnan, V.Devanathan and K.Venkatesan, Journal of Mathematical Analysis and Applications (in press)

is $e\,\bar{\psi}(x)\gamma_\mu\psi(x)$ where $\psi(x)$ is the field operator for the electron and e the electromagnetic coupling constant. If one computes the vacuum expectation value of this operator assuming that the space-time label of $\bar{\psi}$; which is x is slightly different from the corresponding label of ψ which is x' and then takes the limit as $x \to x'$, then one gets an infinite result. Heisenberg attributed this infinity to the contribution from the sea of all the occupied negative energy states. He also suggested that by redefining the current density four-vector as the commutator

$$\frac{e}{2}\left[\bar{\psi}(x)\gamma_\mu\,,\,\psi(x)\right]$$

we would find that its vacuum expectation value is zero. Since the normal product possesses exactly this property we can use, instead of the above form which is obtained by symmetrizing the theory in particle and antiparticle variables, the normal product form of the current operator and since the boson and fermion operators commute, all the vertices represented by H_I 's can be supposed to be in normal product form.

We may also argue for the necessity of a normal product in the S -matrix in another way. Since the vacuum (as also the one-particle state) is an eigen-state of S , if the latter could be written in the form $1+T$ we see immediately that the vacuum expectation of the operator T should be zero which would mean that it is in a normal product form. Of course this does not

ensure that the individual H_I's should be in normal product form.

However, Ramakrishnan and collaborators demonstrated that the notion of a normal product is really not necessary if one is interested only in physical processes with prescribed initial and final states for which the reduction of the S -matrix to the Feynman form can be achieved much more simply and transparently than is the case with the complicated algebraic procedure of Wick[1]. They use some concepts of stochastic theory for this purpose and give two methods for reducing the S -matrix to the Feynman form. In the first method they[1] build up an interaction Hamiltonian in which the fermion creation and annihilation operators are not all in the same order as in the product but such that the creation operator b^+ for a positron always occurs to the left of either an a^+ which is the creation operator for an electron or b which is the positron annihilation operator. The interaction Hamiltonian then is

$$H_I \supset \sum_{p,p'} \left(\hat{a}^+_{p'} \hat{a}_p + \hat{b}_{p'} \hat{a}^+_p + \hat{b}^+_{p'} \hat{b}_p + \hat{b}_{p'} \hat{a}_p \right) \qquad (2)$$

where the symbol $\supset$ indicates that all non-essential factors

1) G.C.Wick, Phys. Rev., 80 , 268 (1950).

have been omitted and the symbol Λ is to indicate that each creation or annihilation operator is accompanied by the corresponding wave function. A unique prescription for this choice of H_I is based on the following stochastic arguments. The interaction at a vertex takes place not at a single space-time point but in a small interval of time, Δ. (This may obviate the difficulty regarding the vacuum expectation of the current operator mentioned above.)* The process of pair annihilation at t represents the transition of a positive energy electron at t to a negative energy state at $t+\Delta$, the interaction taking place in the interval Δ so that the electron destruction operators should be placed to the right of the positron destruction operator. In the case of pair creation in the interval $t-\Delta$ and t, the negative energy state of the electron at t has to be traced back to a positive energy state at t, so that in this case $b^+_{p'}$ should be placed to the left of a^+_p. For electron and positron scattering the creation operators will be to the left of the annihilation operators thus justifying the interaction Hamiltonian (2).

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* Recently Cainiello has suggested that the limiting operation of $x \to x'$ need not necessarily make the vacuum expectation value of the current operator infinite but the latter may be merely undefined under such an operation. E.R.Caianiello, Nuovo Cimento, 14, 185 (1959)

With the Hamiltonian (2), Ramakrishnan et al reduce the S-matrix element to Feynman form in a simple way in which the commutation relations of operators corresponding to different time points are ignored and the process is viewed ab initio in the Feynman sequence.

In the second method Ramakrishnan et al[1] use the conventional interaction Hamiltonian which however is not in normal product form. In the next section we will make a comparative study of this method and that of Wick which brings out the fact that if the stochastic concept of realisation of a typical sequence of events is used to write out the integrand of the S-matrix, the reduction to the Feynman form of the matrix element becomes very simple.

2. Some remarks on the use of Wick's theorem

As mentioned in the introduction, Ramakrishnan and his collaborators[2] have re-examined the connection between the field-theoretic and Feynman description of quantum mechanical collision processes and have arrived at a new and simpler

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1) A. Ramakrishnan et al, Journal of Math. Analysis and Applications,

2) A. Ramakrishnan et al, loc. cit. ibid.

proof of the equivalence of these two formalisms using stochastic arguments. In the second of these methods they[1] use the conventional interaction Hamiltonian which is not in normal product form and the stochastic concept of a realisation of a "typical sequence of events" in the integrand of the S -matrix,. We shall show in this section that if this concept of realisation is applied in to Wick's procedure, we obtain a simplfication which is similar to that of the method of Ramakrishnan and Ranganathan[2].

We shall first briefly recall Wick's procedure of reducing the S -matrix which contains the chronological product of operators into a sum of normal products. The first step is to replace the Dyson chronological operator P which occurs in the S -matrix expansion (1) by the time-ordered product T of Wick , the two being the same in the conventional theories, since the fermion operators which alone may cause a relative sign between P and T products occur in bilinear combination. The time-ordering fixes the relative positions of the interaction Hamiltonian density $H_I(x)$ in the integrand of the S -matrix , but H_I which

1) A. Ramakrishnan et al , loc cit

2) A. Ramakrishnan and N. R. Ranganathan, Journal of Math Analysis and Applications.

itself is a product of annihilation and creation operators has to be arranged inthe normal product form i.e. with the creation operators to the left of destruction operators, for reasons mentioned in the introduction. We shall now show that the requirement that $H_I(x)$ should be put in normal product form does not in any way enter the scattering process where the reduction will concern operators belonging to different times (except for potential scattering in the lowest order, which actually turns out to be a trival case) and in such a case the procedure suggested by Ramakrishnan et al using the conventional Hamiltonian is identical with that using $H_I(x)$ defined in normal product form.

The fundamental definition used by Wick is

$$T(UV) = N(UV) + \dot{U}\dot{V} \qquad (3)$$

which defines the 'contraction' of two operators U and V indicated by dots placed over them. N is the normal product operator. Writing $H_I(x)$ in the normal product form, we find the integrand of the S-matrix to involve mixed T products which can be expressed by means of Wick's theorem as a sum of normal products with all possible contractions (omitting, of course contractions between factors already in normal product form). For instance in quantum electrodynamics the integrand of the second order term in the S-matrix expansion can be written as

$$T\left(N\left(\bar{\psi}(x_1)\,\not{A}(x_1)\,\psi(x_1)\,N\left(\bar{\psi}(x_2)\,\not{A}(x_2)\,\psi(x_2)\right)\right)\right)$$

$$= N\left(\bar{\psi}(x_1)\not{A}(x_1)\psi(x_1)\bar{\psi}(x_2)\not{A}(x_2)\psi(x_2)\right)$$

$$+ N\left(\bar{\psi}(x_1)\dot{\not{A}}(x_1)\psi(x_1)\bar{\psi}(x_2)\dot{\not{A}}(x_2)\psi(x_2)\right)$$

$$+ N\left(\bar{\psi}(x_1)\not{A}(x_1)\dot{\psi}(x_1)\dot{\bar{\psi}}(x_2)\not{A}(x_2)\psi(x_2)\right)$$

$$+ N\left(\dot{\bar{\psi}}(x_1)\not{A}(x_1)\psi(x_1)\bar{\psi}(x_2)\not{A}(x_2)\dot{\psi}(x_2)\right)$$

$$+ N\left(\bar{\psi}(x_1)\dot{\not{A}}(x_1)\ddot{\psi}(x_1)\ddot{\bar{\psi}}(x_2)\dot{\not{A}}(x_2)\psi(x_2)\right)$$

$$+ N\left(\ddot{\bar{\psi}}(x_1)\dot{\not{A}}(x_1)\psi(x_1)\bar{\psi}(x_2)\dot{\not{A}}(x_2)\ddot{\psi}(x_2)\right)$$

$$+ N\left(\dot{\bar{\psi}}(x_1)\not{A}(x_1)\ddot{\psi}(x_1)\ddot{\bar{\psi}}(x_2)\not{A}(x_2)\dot{\psi}(x_2)\right)$$

$$+ N\left(\ddot{\bar{\psi}}(x_1)\dot{\not{A}}(x_1)\dddot{\psi}(x_1)\dddot{\bar{\psi}}(x_2)\dot{\not{A}}(x_2)\ddot{\psi}(x_2)\right) \qquad (10)^4$$

where $\not{A} = eA_\mu\gamma_\mu$ and the contracted pairs are the Feynman propagators in configuration space. Each normal product on the right hand side of (10)4 corresponds to a Feynman diagram in configuration space. As is evident, for a given initial and final state, there is one and only one type of normal product which has a non-vanishing matrix element between given states and this normal product reduces to a product of propagators and the normal productor of operators to be matched with the initial and final states, yielding the initial and final state wave functions.

It is to be noted that Wick's method is quite general and the reduction into normal products is made without reference to any specific process with given initial and final states. Ramakrishnan et. al. have emphasized that all the algebraic techniques involved in the method of Wick can be avoided and the procedure much simplified by specifying the initial and final states from the beginning itself. They do not work with the field operators but with their components, i.e. the annihilation and creation operators with their wave functions and consider a 'typical sequence' of events which contributes a non-vanishing term to the matrix element and then sum over all momenta to get the respective propagators corresponding to the intermediate states. Of course, the integral over space and time of the non-vanishing term corresponding to a typical sequence will vanish if energy and momenta are not conserved for the process. The procedure therefore consists in building up the propagator from the non-vanishing constituents of the integrand of the S-matrix rather than picking out the non-vanishing matrix element from a general reduction of an n th order product of field operators. No use, however, is made of the normal product form of $H_I(x)$.

We shall now give a brief description of the method of ~~making~~ matching prescribed by Ramakrishnan et. al. and then show that it is identical with the procedure of Wick provided we take only the non-vanishing term in the latter. The integrand of the n th order term of the S-matrix

taken between the initial and final states, $|i\rangle$ and $|f\rangle$ can be written as

$$\langle f|[n]\cdots[l]\cdots[k]\cdots[j]\cdots[1]|i\rangle \quad {}^{5}_{(*)}$$

for a typical realisation of the time-ordered sequence of operators with definite momentum labels. For simplicity we shall consider the one-particle initial and final states defined by

$$|i\rangle = a^{+}(p_1)|\rangle_0 \quad ; \quad |f\rangle = a^{+}(p_2)|\rangle_0$$

In (* 5) , a typical bracket contains

$$\hat{a}^{+}_{p}\,\hat{a}_{p'} + \hat{a}^{+}_{p}\,\hat{b}^{+}_{p'} + \hat{b}_{p}\,\hat{a}_{p'} + \hat{b}_{p}\,\hat{b}^{+}_{p'} \qquad (6)$$

which, leaving aside the interaction potential which is not relevant for the discussion, is the integrand of H_I with definite momentum labels, Explicitly

$$\hat{a}_{p} = \frac{1}{\sqrt{2E_p}}\, a_p u_p e^{ipx}$$

and similarly each creation and annihilation operator is accompanied by the corresponding wave function. The creation operator $\hat{a}^{+}_{p_1}$ which operating on the vacuum gives the initial state has to be matched with the corresponding annihilation operator a_{p_1} having the same momentum label which for the particular realisation may occur in the k th bracket.

Looking at the structure of the interaction (6) we find that this annihilation operator a_{p_i} may be accompanied either by an $\hat{a}^{+}_{p'}$ or a $\hat{b}_{p'}$; which in turn has to be matched with the corresponding annihilation or creation operator.

Case (i) : If it is $\hat{a}^{+}_{p'}$ then it has to be matched with the corresponding annihilation operator $a_{p'}$ in the ℓ th bracket with $t_\ell > t_k$. t denotes time.

Case (ii) : If in the k^{th} bracket $\hat{a}_{p_i}$ is accompanied by a $\hat{b}_{p'}$ which is the annihilation operator for a positron (in electrodynamics) we observe that the corresponding creation operator $b^{+}_{p'}$ should occur in an earlier bracket $[j]$ with $t_j < t_k$ since the positron which is destroyed should have been created earlier. From (6) we see that b^{+} occurs always to right of an a^{+} or a b so that in bringing $b_{p'}$ to the left of $\hat{b}^{+}_{p'}$, it is has necessarily to cross over the left-hand member of the bracket containing $b^{+}_{p'}$ which yields the negative sign, usually attributed to the positron propagator. In Wick's procedure the negative sign is already fed in by writing $H_I(x)$ in normal product form, i.e. in a typical term of $H_I(x)$, b^{+}_{p} occurs to the left but has a negative sign.

This process of matching is continued until all the annihilation and creation operators in the integrand of the S-matrix are exhausted except the creation operator

$a^+_{p_2}$ which for the particular realisation may occur in the m th' bracket. This has to be matched with the annihilation operator a_{p_2} in the final state. The matching of all intermediate operators (leaving aside the operators $a^+_{p_1}$ and a_{p_2}) corresponds in Wick's procedure to contracted pairs and the matching of $a^+_{p_1}$ and a_{p_2} actually yields the initial and final state wave functions which, in Wick's method, are obtained from the uncontracted field operators in the non-vanishing normal product when taken between the initial and final states. This can be seen as follows. In moving the creation and annihilation operators in the various brackets of (5) we have left the wave functions attached to them undisturbed in their respective brackets. Thus for case (i) , the k^{th} bracket after removing the creation and annihilation operators would be $[\bar{u}_{p'} u_{p_1}]_k \, e^{i(p_1 - p')x_k}$

and the ℓ th bracket would be $[\bar{u}_{p''} u_{p'}] e^{i(p'-p'')x_\ell}$

Thus in juxta position, with $t_\ell > t_k$, we would have

$$[\bar{u}_{p''} u_{p'}]_\ell [\bar{u}_{p'} u_{p_1}]_k e^{i(p_1-p')x_\ell} e^{i(p'-p'')x_k}$$

Now $u_{p'} \bar{u}_{p'}$ can be identified easily to be a single element of the positive energy part of the Feynman propagator in momentum space. The sum of all such terms for all possible realisations of momenta and spin will give the positive part of the Feynman propagator.

For case (ii) on the other hand, we found that there should be a $b^{+}_{p'}$ in a bracket j, with $t_j < t_k$. If we now use the prescription that the brackets, bereft of their creation and annihilation operators should be juxtaposed, starting from the right, inthe same sequence in which the matching of the operators has been done, then since each of the brackets is a scalar we should have for the spinor part of the wave functions

$$-\left[\bar{u}_{p''}\, v_{p'}\right]_j \left[\bar{v}_{p'}\, u_{p'}\right]_k$$

the negative sign arising for the reason mentioned above. Summing over all possible momenta and spins, $v_{p'}$, $\bar{v}_{p'}$ can be recognized to be the Feynman propagator for negative energies. This process of obtaining the propagators can be continued until we are left with just one spinor at the extreme right and one at the extreme left which correspond to the wave functions for the initial and final states.

Thus we see that the method of Ramakrishnan is equivalent to taking the non-vanishing terms of the normal product sum, xxxxx between the initial and final states in Wick's procedure. All we need to employ in this simplified method is the commutation relations between annihilation and creation operators without having to evaluate the commution relations of the field variables in which case one is forced to consider not the spinors themselves but their components.