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SOME STUDIES IN PATHOS GRAPHS

BY

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INTRODUCTION

This report concentrates entirely on the concept of pathos graphs of a graph. The emphasis of the work has been on the graph throertic properties of these special kinds of graphs. Various characterizations and other results discussed here pervade a large part of the graph theory. In most of the cases we havd restricted our discussion to pathos graphs of a tree.

The work has been divided into six papers of which the first three papers deal with pathos graphs (in particular pathos graphs of a tree) and the last three deal with point pathos graph (in particular point pathos graphs of a tree) the relation between pathos graphs and point pathos graphs has been also established.

ACKNOWLEDGEMENT

I am extremely grateful to Professor Alladi Ramakrishnan, the Director of MATSCIENCE for his encouragement in bringing out this piece of research work in the form of 'MATSCIENCE REPORT'.

It is my pleasure to record my sincere thanks and deep sense of gratitude to Dr.E.Sampathkumar, Department of Mathematics, Karnatak University, Dharwar, for his valuable discussions, advice and encouragement, in bringing out this work.

C O N T E N T S

ON PATHOS GRAPHS OF A GRAPH

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The concept of a pathos of a graph G was introduced by Harary(1), as a collection of minimum number of line disjoint open paths whose union is G. The path number pn(G) of a graph G is the number of paths in a pathos. Later R.G.Stanton, D.D.Cowan, and L.O.James(6) and Harary(2) have calculated the path number for certain classes of graphs like trees, cubic graphs, complete graphs, complete bipartite graphs etc.

Here we shall define pathos graph of a graph G, and study some general properties of pathos graphs of a tree and obtain a characterization of pathos graph of a tree.

1.1. DEFINITION. Let P be a pathos of a graph G. The intersection graph P(G) of P is called a pathos graph of G. Thus points in P(G) correspond to the paths in P, and two points in P(G) are joined if and only if the corresponding paths in P have a point in common.

Unless otherwise stated, by a graph we mean a finite, undirected, connected graph without loops or multiple lines.

We observe that P(G) is connected if and only if G is connected. Also it follows that a cycle of length n is covered by two line disjoint paths. Clearly, there may exist different sets of pathos for a graph G. Corresponding to these different sets of pathos, we may have different pathos graphs of G. For example see figure 1.

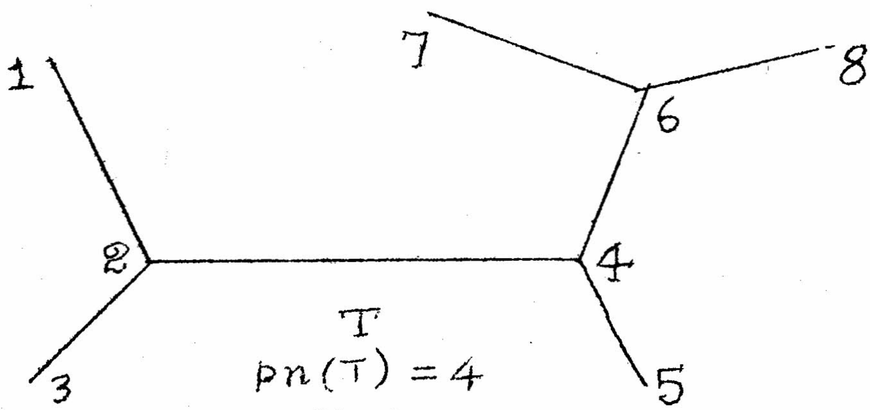


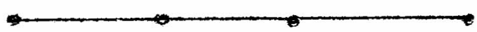
Fig.1

The possible set of pathos P of the above tree T are, the following.

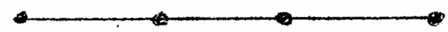
P_1	:	123 ,	5467,	86,	24
P_2	:	867,	645,	324,	12
P_3	:	1245,	32,	467,	86
P_4	:	12467,	68,	45,	32
P_5	:	32468,	67,	45,	12
P_6	:	3245,	12,	468,	67

The different pathos graphs of the tree T corresponding to the pathos P_1, P_2, P_3 are the trees $P_1(T)$, $P_2(T)$ and $P_3(T)$

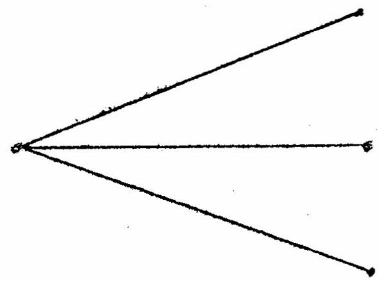
respectively. See figure 2.



$P_1(T)$



$P_2(T)$



$P_3(T)$

Figure 2

Here we note that $P_1(T) = P_2(T) \neq P_3(T)$.

1.2. SOME GENERAL PROPERTIES OF PATHOS GRAPH OF A TREE

A graph G is said to be homeomorphic to H if G can be obtained by introducing some points of degree two in some lines of H.

We observe that the path number $pn(T)$ of a tree T is equal to the path number of any tree homeomorphic with T. Further, it is easy to see that,

THEOREM 1. All trees homeomorphic to a given tree T, have the same collection of pathos graphs as that of T.

Illustration: See figure 3.

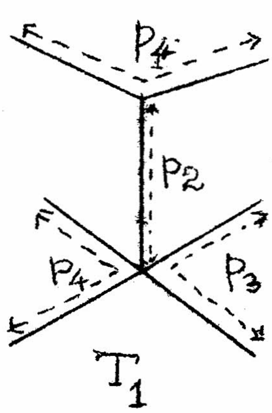
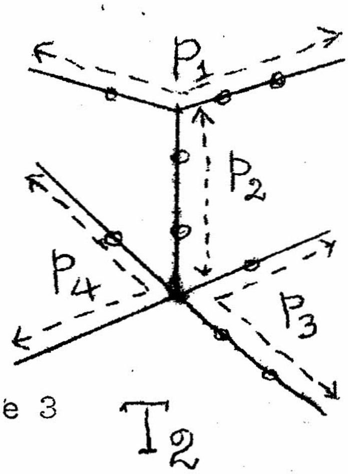


Figure 3



It can be easily seen that these two trees T_1 and T_2 have the following pathos graphs, which are one and the same, see figure 4.

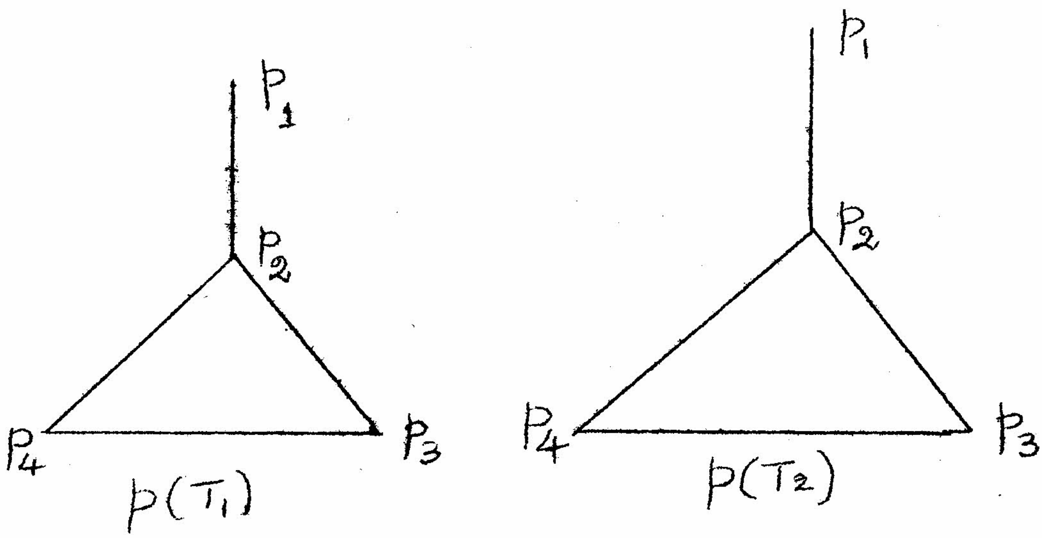


Figure 4

Now we state an important theorem without proof which follows easily. As usual let $\{r\}$ denote the smallest integer greater than or equal to r.

<u>THEOREM 2</u>. Let T be a tree and P be a pathos of T. Then the pathos graph $P(T)$ corresponding to P is the complete graph on $\left\{\frac{n}{2}\right\}$ points if and only if $T = K_{1,n}$ or T is homeomorphic to $K_{1,n}$.

Given a tree we now give an algorithm to determine a pathos P of T.

<u>ALGORITHM</u>. Choose an odd point v_1 and proceed along a path until we reach another odd point say v_2. Let P' be the $v_1 - v_2$ path thus obtained. Let v_3 be another odd point ...

different from v_1 and v_2. Then again start from v_3 and proceed along a line not previously traversed, until we obtain a new odd point v_4. Let P'' be the $v_3 - v_4$ path thus obtained. Continuing this process we get a pathos P consisting of paths

$$P', P'', P'''$$

which cover the given tree T.

REMARK 1. The end points of each path in any pathos of a tree are odd points.

REMARK 2. No two paths in a pathos of a tree, have a common end point.

REMARK 3. The path number $pn(T)$ of a tree T is equal to k, where $2k$ is the number of odd vertices in T.

Before going to the next theorem we shall make it a point that the path number of a starpoint v of T is the path number of the star-subgraph of T at v.

THEOREM 4. Let $v_1, v_2, v_3, \ldots\ldots, v_k$ be the set of all star-points of a tree T, and n_i be the path-number of v_i. Then the number of lines in any pathos graph of T is

$$\frac{1}{2} \sum_{1}^{k} n_i(n_i - 1).$$

PROOF. We observe that the path number of v_i is $\left\{\frac{d_i}{2}\right\}$ where d_i is the degree of the star-point v_i.

Corresponding to each star-subgraph at v_i in T, we have a complete subgraph on n_i points in $P(T)$. (Theorem 2). And clearly the number of lines in such a subgraph is

$$\frac{1}{2} n_i(n_i - 1).$$

Hence the total number of lines in $P(T)$ is

$$\frac{1}{2} \sum_i n_i(n_i - 1)$$

This proves the theorem.

THEOREM 5. Let T be a tree. Then for any pathos P of $T, P(T)$ is a tree if and only if $\Delta(T) \leqslant 4$, where $\Delta(T)$ is the maximum degree of any point in T.

PROOF. Let $P(T)$ of a tree T be again a tree. Suppose T contains a point of degree $n \geqslant 5$. Then corresponding to the star-subgraph $K_{1,n}$ in T, we have a complete subgraph on $\left\{\frac{n}{2}\right\}$ points in $P(T)$. Since $\left\{\frac{n}{2}\right\}$ is $\geqslant 3$, it follows that $P(T)$ contains a cycle and hence is not a tree, a contradiction. Converse part is obvious.

COROLLARY. If one of the pathos graph of a tree T is a tree, then all the other pathos graphs of T are also trees.

1.3. CHARACTERIZATION OF PATHOS GRAPH OF A TREE.

THEOREM 6. Let G be a connected graph. Then $G = P(H)$ for some tree H if and only if every block in G is complete.

PROOF. (NECESSITY. Let $G = P(H)$ for some tree H and v_1 be a star-point of H. Then corresponding to the paths in P containing v_1, we have in $P(H)$ a complete subgraph say G_1. Also, if v_2 is another star-point adjacent to v_1, then corresponding to the paths in P containing v_2, we have in $P(H)$

another complete subgraph G_2. It is easy to see that there is exactly one path belonging to the pathos P which contains v_1 and v_2. The point corresponding to this path in $P(H)$ will be the cut point common to G_1 and G_2. This implies that every block in G is complete.

SUFFICIENCY. Let G be a connected graph in which every block is complete. We now construct a tree, T, whose one of the pathos graphs is G, as follows.

Let G_i denote the blocks of G. For each G_i we take a point v_i. Suppose G_1 is an end block (i.e. the block having one and only one cut point) and $G_1, G_2, G_3, \ldots\ldots, G_r$ have a common out point. Then corresponding to this configuration we draw a path $v_1, v_2, v_3, \ldots\ldots, v_r$. Now let $G_s (s > r)$ be a block having a common cut point with some $G_i (1 \leq i \leq r)$. Then we join v_s and v_i. Repeating this process for all the blocks of G, it is easy to see that the graph thus obtained on the points v_i is a tree T'. See the figure 5.

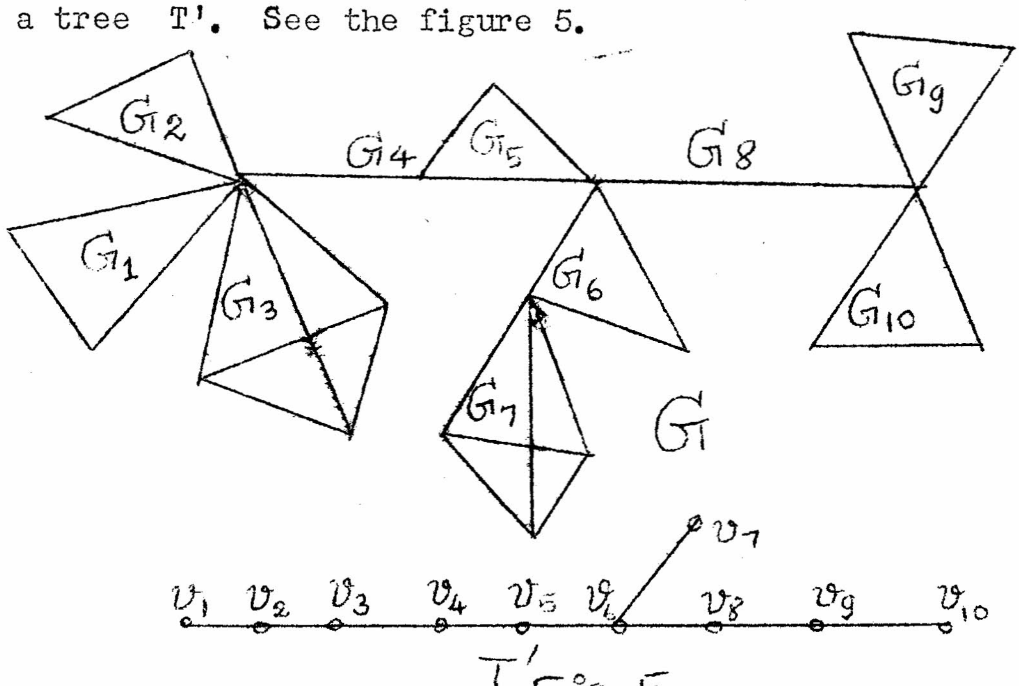


Fig. 5.

Further at each point v_i of T' we introduce $2n_i - d_i$ end lines, where n_i is the number of points in the complete block G_i and d_i is the degree of v_i in T'. Let H be the tree thus obtained. See the figure 6.

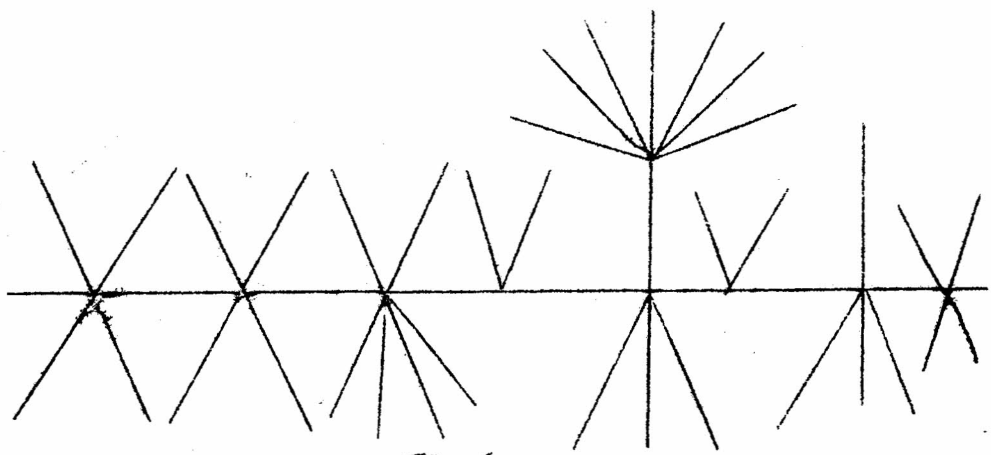


Fig 6.

We now indicate the method of selection of a pathos P of H, such that $P(H) = G$.

Corresponding to each cut-point C_i of G, let P_i be the path in H defined as follows. Let $G_1, G_2, \ldots\ldots, G_k$ be the complete blocks incident at C_i in G. Corresponding to this configuration consider the path P' in H consisting $v_1, v_2, \ldots, v_k$ in some order, say

$$P' \cdot v_1, v_2, v_3, \ldots\ldots, v_k.$$

If the degree of v_1 in H is even then we add an end line to P' at v_1. And similarly, if deg v_k is even, we add an end line at v_k also. If the degree v_1 is odd we may or may not add an end line to P' at v_1, and likewise at v_k. Let P_i be the path thus obtained. We now choose a pathos P of H such that for each cut-point C_i of G, P contains a path P_i defined as above. The other paths in P can be chosen arbitrarily. Now it is not hard to verify that $P(H) = G$.

The above construction may be seen in the following figure 7

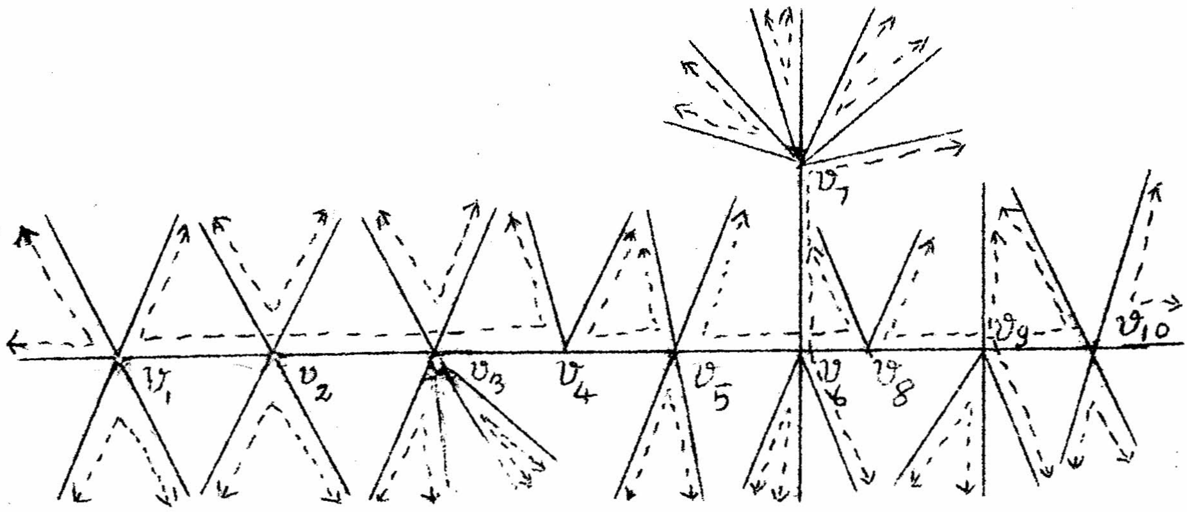


Fig 7.

As in Harary[3], the graph $L_2(G)$ of a graph is defined as $L(S(G))$. When G is a tree, it is easy to see that every block of $L_2(G)$ is complete. Hence we have

<u>COROLLARY 6.1.</u> If T is a tree, then $L_2(T)$ is a pathos graph of some tree.

Since every block in a pathos graph $P(T)$ of a tree is a clique and conversely, we have

<u>COROLLARY 6.2.</u> For a tree T,

$$B(P(T)) = Q(P(T)),$$

that is the block graph[4] of $P(T)$ is the same as the clique graph[5] of $P(T)$.

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ON PATHOS GRAPHS OF A GRAPH-II

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This is a continuation of Paper 1.

Here we study the following:

(1) Planarity of pathos graphs

(2) Traversibility of pathos graphs

(3) Some miscellaneous results

2.1. PLANARITY OF PATHOS GRAPHS

THEOREM 1. Let T be a tree and P be a pathos of T. Then the pathos graph $P(T)$ is planar if and only if $\deg v \leq 8$, for every v in T.

PROOF. NECESSITY. A graph is planar if and only if every block in it is planar. But in $P(T)$ every block is complete. And a complete graph is planar whenever its order is at most 4. But the order of any complete block in $P(T)$ can be 4, only when $\deg v \leq 8$ in T, for every v in T. This proves that the condition is necessary.

SUFFICIENCY. Let T be a tree in which $\deg v \leq 8$, for all points v. Then in $P(T)$ every block is complete of order at most 4. This implies that $P(T)$ is planar. This proves the theorem

THEOREM 2. Let T be a tree and P be a pathos of T. Then $P(T)$ is outer planar if and only if $\deg v \leq 6$, for every v in T.

PROOF. NECESSITY. Suppose T contains a point v, such that deg $v \geq 7$. Then [theorem 2, Paper 1] P(T) contains a complete subgraph corresponding to v of order ≥ 4, which is non-outer planar, a contradiction.

SUFFICIENCY. If deg $v \leq 6$, for every v in T, then order of every complete block in P(T) is ≤ 3, and hence every block is outer planar. This proves the theorem.

2.2. TRAVERSIBILITY.

THEOREM 3. Let T be a tree and P be a pathos of T. Then P(T) is eulerian if and only if every star-point of T is of degree $n \geq 5$, such that $\left\{\frac{n}{2}\right\}$ is not even.

PROOF. NECESSITY. Let P(T) be eulerian. Then every point of P(T) is of even local degree[3].

Now suppose T contains a point of degree $n \leq 4$. Then P(T) contains a block which is complete on $\left\{\frac{n}{2}\right\} \leq 2$ points. (Theorem 2, Paper 1). This impliesthat P(T) is not eulerian.

If T contains a point of degree n, such that $\left\{\frac{n}{2}\right\}$ is even then P(T) contains a complete block, which is regular of degree $\left(\left\{\frac{n}{2}\right\} - 1\right)$ and is odd. But that is not true.

SUFFICIENCY. Let G = P(T), for some tree T and G satisfy the given condition of the theorem. Then it follows that every point in P(T) belongs to a complete block on an odd number of points. This implies that every point in P(Y) is of even degree. Hence P(T) is eulerian.

THEOREM 4. $P(T)$ of a tree T is hamiltonian if and only if T is homeomorphic to $K_{1,n}$, for $n \geq 5$.

PROOF: NECESSITY. If T contains more than one star-point, then $P(T)$ contains more than one complete block and hence it contains a cut point. This implies that $P(T)$ is not hamiltonian And hence T should be homeomorphic to $K_{1,n}$.

If n is ≤ 4, then $P(T)$ is either K_2 or K_1, and hence it is not hamiltonian. This proves that n is ≥ 5.

SUFFICIENCY. Let T be homeomrophic to $K_{1,n}$, for $n \geq 5$. Then $P(T)$ will be complete on $\left\{\frac{n}{2}\right\} \geq 3$ points and thus it is hamiltonian. This proves the theorem.

2.3. SOME MISCELLANEOUS RESULTS.

THEOREM 5. $P(T)$ of a tree T is bipartite if and only if $\Delta(T) \leq 4$, Where $\Delta(T)$ is the maximum degree of any point in T.

PROOF. NECESSITY. Let $P(T)$ of a tree T be bipartite. If T contains a point v of degree $n \geq 5$, then $P(T)$ contains a complete subgraph on $\left\{\frac{n}{2}\right\} \geq 3$ points and hence $P(T)$ is not bipartite, a contradiction.

SUFFICIENCY. Suppose $\Delta(T) \leq 4$. Then (Theorem 5, Paper 1) $P(T)$ is again a tree for every set of pathos of T and hence it is bipartite[2]. This proves the theorem.

Now to discuss the next theorem we make use of the following theorem due to J.Krausz[2], the English version of which is found in[1].

THEOREM A. A graph G is the line graph of some graph if and only if its lines can be partitioned in to complete subgraphs in such a way that no point lies in more than two of the subgraphs.

THEOREM 6. Let T be a tree and P be a pathos of T. Then the pathos graph $P(T)$ is a line graph if and only if every path in P passes through at most two star-points.

PROOF. NECESSITY. Let $P(T)$ be a line graph. If P contains a path which passes through more than two star-points, then it is easy to see that in $P(T)$ we have a cut point belonging to more than two complete blocks. This implies that $P(T)$ is not a line graph, a contradiction.

SUFFICIENCY. Let P be a set of pathos of a tree T satisfying the given condition of the theorem. Then every block in $P(T)$ is complete and at each out cut point there are exactly two blocks. Keeping in view, the above theorem A, it follows that $P(T)$ is a line graph.

THEOREM 7. Let T be a tree and P be a pathos of T. Then chromatic number of $P(T)$ is $\left\{\frac{\Delta(T)}{2}\right\}$, where $\Delta(T)$ is the maximum degree of a point in T.

PROOF. We observe that every block in $P(T)$ is complete and the maximum order of a complete block in $P(T)$ is $\left\{\frac{\Delta(T)}{2}\right\}$. Also it is well known that the chromatic number of a connected graph G is equal to max $\chi(B_i)$, where B_i is a block of G. Hence the result follows.

THEOREM 8. The pathos of a complete graph K_n, on n points is again complete on $\left\{\frac{n}{2}\right\}$ points, where as usual $\left\{\frac{n}{2}\right\}$ is the least integer greater than or equal to $\frac{n}{2}$.

PROOF. Let n be an even integer, say $n = 2k$. Then by section 5 in(4), the number of paths in K_n is k. Now by the help of the lemma 5.1 in(4), we shall label the pathos of K_n as follows:

a_1	a_2	a_{2k}	a_3	a_{2k-1}	.	.	.	a_k	a_{k+1}
a_2	a_3	a_1	a_4	a_{2k}	.	.	.	a_{k+1}	a_{k+2}
a_3	a_4	a_2	a_5	a_1	.	.	.	a_{k+2}	a_{k+3}
a_4	a_5	a_3	a_6	a_2	.	.	.	a_{k+3}	a_{k+4}
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
a_k	a_{k+1}	a_{k-1}	a_{k+2}	a_{k-2}	.	.	.	a_1	a_{2k}.

Case 2. Let n be an odd integer, say $n = 2k+1$. Again with the help of the lemma 5.2 in(4), the pathos of K_n may be written in the following way.

a_1	a_2	a_{2k+1}	a_3	a_{2k}	.	.	.	a_{k+1}	a_{k+2}
a_2	a_3	a_1	a_4	a_{2k+1}	.	.	.	a_{k+2}	a_{k+3}
a_3	a_4	a_2	a_5	a_1	.	.	.	a_{k+3}	a_{k+4}
a_4	a_5	a_3	a_6	a_2	.	.	.	a_{k+4}	a_{k+5}
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
a_k	a_{k+1}	a_{k-1}	a_{k+2}	a_{k-2}	.	.	.	a_{2k}	a_{2k+1}

In this case we observe that there will be some lines missing, which may be of the type,

$(a_1\ a_{2k+1}),\ (a_2\ a_{(2k+1)-1}),\ (a_3\ a_{(2k+1)-2}),$ etc..

And clearly these lines may be covered by drawing a path of the type,

$a_{2k+1}\ a_1\ a_2\ a_{(2k+1)-1}\ a_{(2k+1)-2}\ a_3\ \ldots\ldots$ and so on,

deleting some points of the type

a_1, a_{k-1}, a_{2k} etc., there by shortening the related paths.

Now we observe that from the above labelling, each of the $\left\{\frac{n}{2}\right\}$ paths of the pathos has at least one point in common with the remaining paths. This clearly shows that every point in corresponding pathos graph has adjacency with every other point in it. Hence it follows that the pathos graph of a complete graph on n points is again complete on $\left\{\frac{n}{2}\right\}$ points. This proves the theorem.

The following two theorems are the consequences of some theorems already proved. We state them without proof.

THEOREM 9. The connectivity of a pathos graph of a tree is either one or $p-1$, where $p = \left\{\frac{n}{2}\right\}$ such that $K_{1,n}$ is the star.

THEOREM 10. The line connectivity of $P(T)$ of a tree T is

$$\min\left(\left\{\frac{n_i}{2}\right\} - 1\right),$$

where n_i is the degree of the point in T.

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ON UNIQUENESS OF THE PATHOS GRAPH OF A TREE

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In general the pathos graph P(T) (Paper 1) of a tree T depends on the pathos P of T, and two pathos graphs of T may not be isomorphic. We now investigate trees for which all pathos graphs are isomorphic.

As observed earlier, all trees homeomorphic to a given tree T, have the same collection of pathos graphs as that of T. Hence without loss of generality, in this section we consider trees which do not have points of degree 2.

It is noted earlier (Theorem 2, Paper 1) that all pathos graphs of the star $K_{1,n}$ are isomorphic to the complete graph on $\left\{\frac{n}{2}\right\}$ points where $\left\{\frac{n}{2}\right\}$ denotes the least integer greater than or equal to $\frac{n}{2}$. Hence the pathos graph of $K_{1,n}$ is unique.

By a <u>two star</u> we mean the graph obtained by joining the central points of two stars. For example, the graph G of figure 1 is a two star.

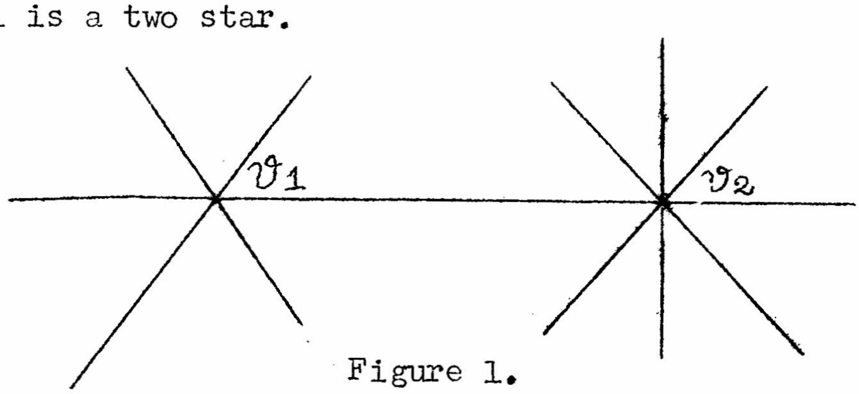


Figure 1.

It is easy to see that all pathos graphs of G are isomorphic to the graph in figure 2, which is $K_3 \cdot K_4$, that is the graph obtained by identifying a point of K_3 with a point of K_4.

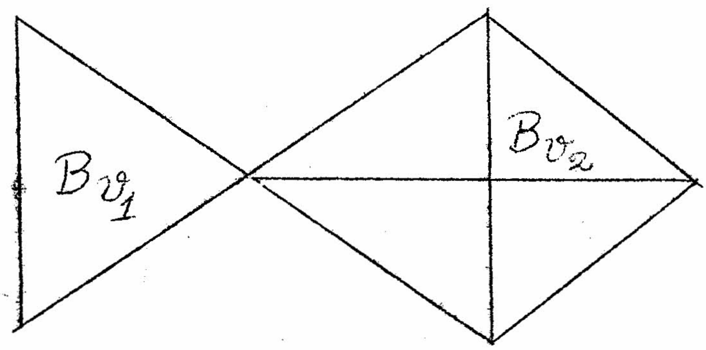


Figure 2

In general, if G is a two star, with star-points of degree m and n respectively, then all pathos graphs of G are isomorphic to $K_{\left\{\frac{m}{2}\right\}} K_{\left\{\frac{n}{2}\right\}}$ and hence the pathos graph of G is unique.

A <u>super star</u> is a tree T obtained by joining central points $v_1, v_2, v_3, \ldots\ldots, v_k$ of two or more stars to the same point v_o, called the <u>central point</u> of T. The points $v_1, v_2, v_3, \ldots\ldots, v_k$ are called <u>semi-central</u> points of T. For example, the trees T_1 T_2 in fig.3 are super stars.

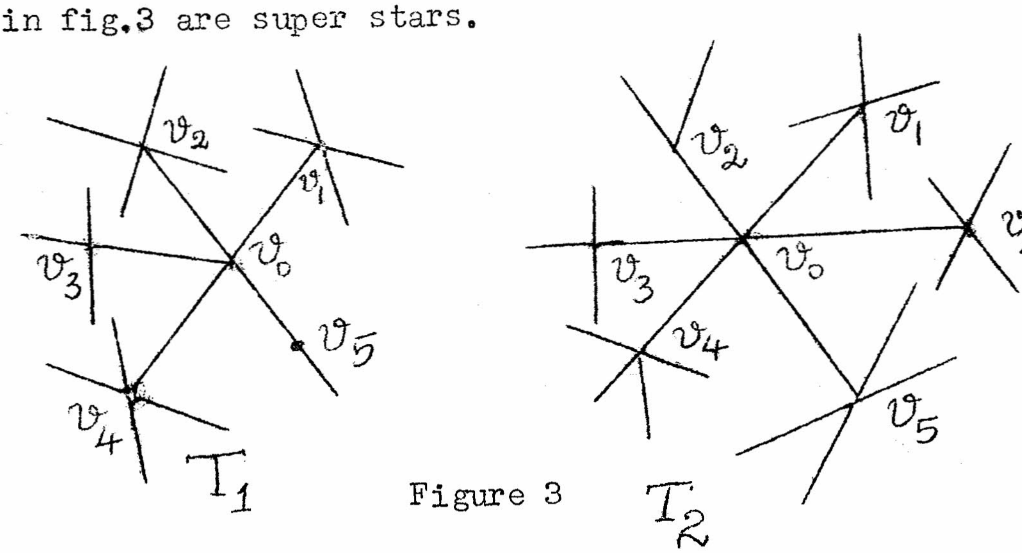


Figure 3

Let T be a super star and v_1 $v_2, \ldots\ldots, v_k$ be the semi-central points of T and v_o be the central point of T. Let n_i be the path number of v_i for $i = 0,1,2,\ldots\ldots,k$. Then the pathos graph of T consists of the complete graph K_{n_o}, and at each point of K_{n_o}, there are at most two complete and blocks. Each such end block corresponds to a unique semi-centre v_i for some $i = 1,2,\ldots\ldots,k$.

We illustrate this by means of two examples.

Let the degree of v_o be even, say 10. Then $k = 10$, and the path number of v_o is 5. Corresponding to the star $K_{1,10}$ at v_o in T we have the complete subgraph K_5 in a pathos graph of T, and a pathos graph of T is of the form as shown in Fig.4.

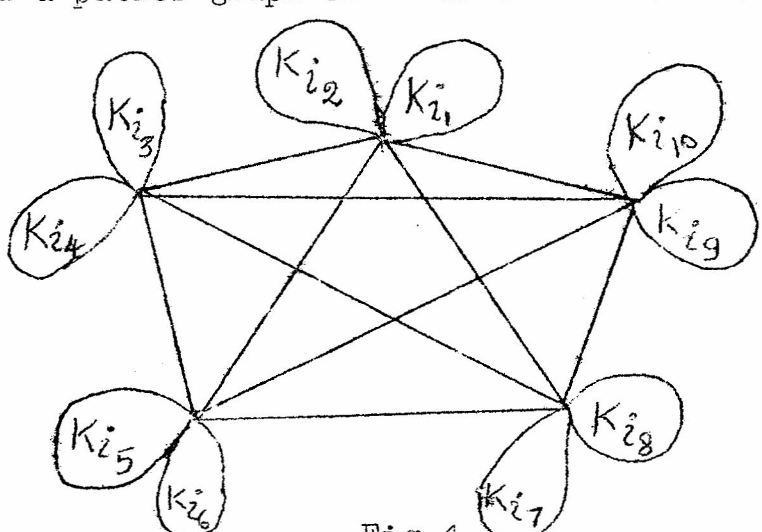


Fig.4

where K_{i_r} is the complete graph on i_r points and

$i_r \in \{n_1, n_2, \ldots\ldots, n_{10}\}$.

We observe that corresponding to the star subgraph at v_i for $i = 1,2,3,\ldots,k$, we have a complete subgraph K_{n_i} in a pathos graph of T and this is an end block attached to K_5, as shown in the figure 4.

If deg v_o is odd say 9, then $k = 9$, and the path number of v_o is again 5. Corresponding to the star-subgraph $K_{1,9}$ at v_o, we have again the complete subgraph K_5 in any pathos graph of T. In this case the pathos graph of T is of the form as shown in Figure 5.

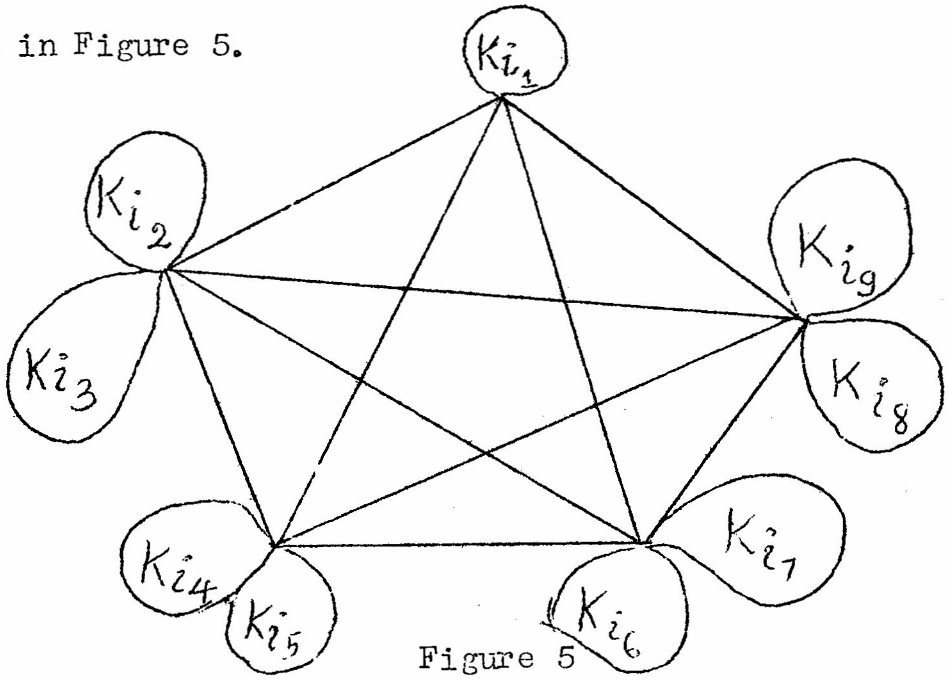


Figure 5

Thus we see that

(1) If deg v_o is even, then there are exactly two complete subgraphs K_{n_i} for $i \geq 1$, at each point of K_{n_o}.

(2) If deg v_o is odd, then there is exactly one complete subgraph K_{n_i} for $i \geq 1$, at one point and two complete subgraphs at all other points of K_{n_o}.

For different such combinations of complete end blocks as shown in figure 4 or 5, we may have different pathos graphs of T.

We now examine when all pathos graphs of T are isomorphic.

Case 1. Suppose deg v_o is even. In this case as observed above there are exactly two complete subgraphs K_{n_i} for $i \geq 1$, at each point of K_{n_o}. Hence it is easy to see that all pathos graphs of T are isomorphic if and only if all the complete graphs K_{n_i} for $i \geq 1$, except possibly one, are isomorphic. In otherwords all the pathos graphs of T are isomorphic if and only if the path numbers of all points v_i for $i = 1,2,\ldots\ldots,k$, except possibly one, are equal.

Case 2. Suppose deg v_o is odd. As observed earlier, in a pathos graphs of T, there is one complete subgraph K_{n_i} for $i \geq 1$ at one point of K_{n_o}, and two complete subgraphs at all other points of K_{n_o}. In this case also, clearly all pathos graphs are isomorphic if and only if all the complete subgraphs K_{n_i} for $i \geq 1$ in a pathos graphs of T are isomorphic. In otherwords, all the pathos graphs of T are isomorphic if and only if all the path numbers of v_i for $i = 1,2,\ldots\ldots,k$ are equal. We now prove,

THEOREM 1. The pathos graph of a tree T is unique if and only if it is in one of the following forms,

(1) T is homeomorphic to a star

(2) T is homeomorphic to a two star

(3) T is homeomorphic to a super star T_1

Satisfying the following conditions:

Let v_o be the centre of T_1 and $v_1\ v_2, \ldots\ldots, v_k$ be the semi-centres of T_1.

(a) If deg v_o is even, then the path numbers of all points v_1, for $i = 1,2,\ldots,k$ except possibly one, are all equal.

(b) If deg v_o is odd, then the path numbers of all points v_i for $i = 1,2,\ldots\ldots,k$ are all equal.

PROOF. As we have discussed already, a star, a two star, a super star satisfying the above conditions have unique pathos graphs. So any tree homeomorphic to any one of these graphs also has a unique pathos graph. This proves sufficiency.

We now indicate the necessity by an example. For this we take a tree (Fig. 6) which is not homeomorphic to any of the trees mentioned in the theorem.

Let $P = \{P_1, P_2, \ldots\ldots, P_{15}\}$ be the set of pathos as shown in figure 6. The pathos graph $P(T)$ of the tree T, corresponding to P is the graph in figure 7. Let $P' = \{P_1', P_2', \ldots, P_{15}'\}$ be the pathos of T as shown in figure 8. Then the pathos graph $P'(T)$ of T corresponding to P' is given in figure 9. We observe that $P(T) \neq P'(T)$. This proves the necessity and hence the theorem.

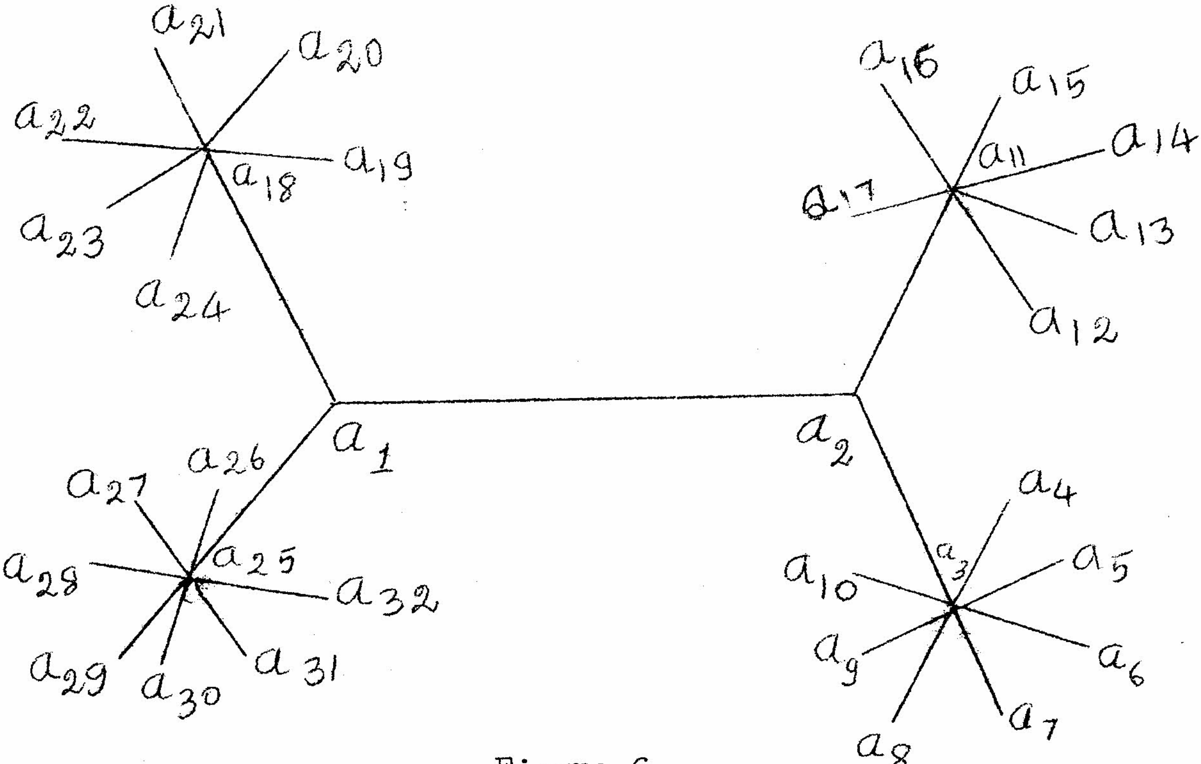


Figure 6

The set of pathos $P = \{P_1\ P_2\ \ldots\ldots\ P_{15}\}$

P_1 : $a_1\ a_2$

P_2 : $a_{26}\ a_{25}\ a_1\ a_{18}\ a_{24}$

P_3 : $a_{19}\ a_{18}\ a_{20}$

P_4 : $a_{21}\ a_{18}\ a_{22}$

P_5 : $a_{23}\ a_{18}$

P_6 : $a_{32}\ a_{25}\ a_{31}$

P_7 : $a_{30}\ a_{25}\ a_{29}$

P_8 : $a_{28}\ a_{25}\ a_{27}$

P_9 : $a_{10}\ a_3\ a_9$

P_{10} : $a_8\ a_3\ a_9$

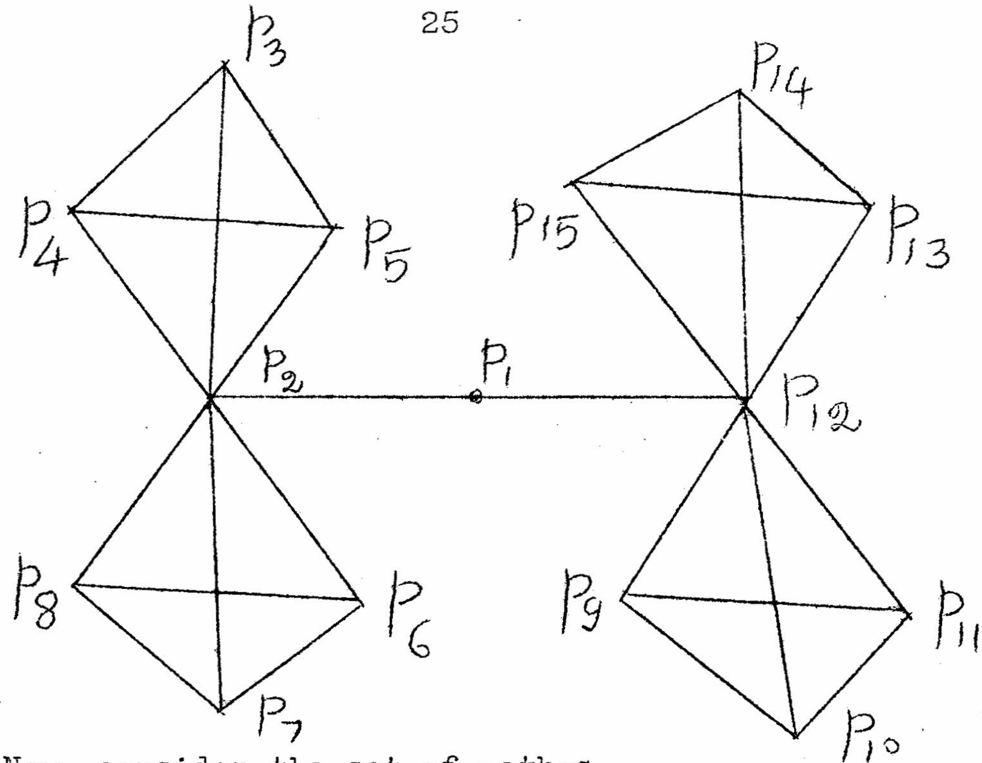
P_{11} : $a_6\ a_3\ a_5$

P_{12} : $a_4\ a_3\ a_2\ a_{11}\ a_{12}$

P_{13} : $a_{13}\ a_{11}$

P_{14} : $a_{14}\ a_{11}\ a_{15}$

P_{15} : $a_{16}\ a_{11}\ a_{17}$



Now, consider the set of pathos

$$P' = \left\{ P'_1, P'_2, P'_3 \ \ldots\ldots\ldots\ldots\ P'_{15} \right\} \text{ in } T$$

where

$P'_1 \cdot b_1\ b_2\ b_3\ b_4\ b_5\ b_6$

$P'_2 \cdot b_{21}\ b_{20}\ b_3$

$P'_3 \cdot b_{22}\ b_{20}\ b_{23}$

$P'_4 \cdot b_{24}\ b_{20}\ b_{25}$

$P'_5 \cdot b_{25}\ b_{20}\ b_{27}$

$P'_6 \cdot b_{28}\ b_2$

$P'_7 \cdot b_{29}\ b_2\ b_{30}$

$P'_8 \cdot b_{31}\ b_2\ b_{32}$

$P'_9 \cdot b_4\ b_{13}\ b_{12}$

$P'_{10} \cdot b_{19}\ b_{13}\ b_{18}$

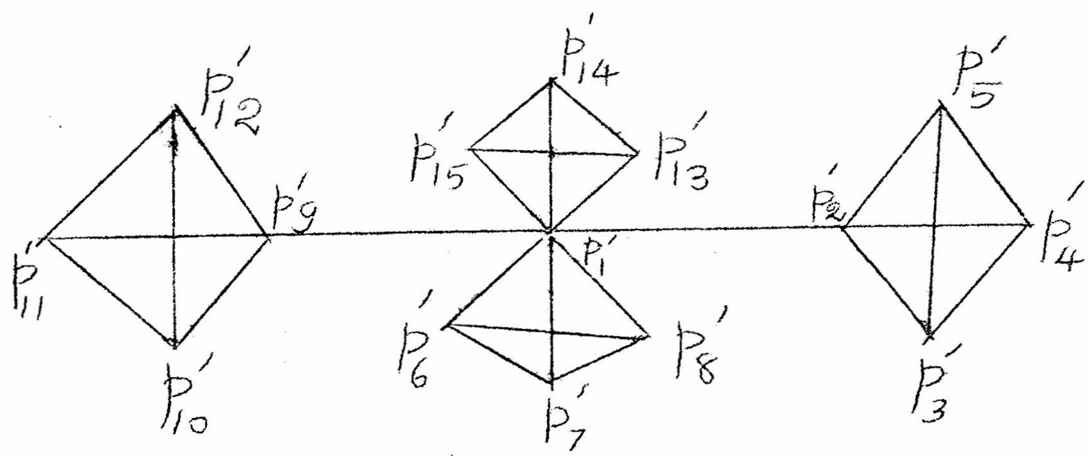
$P'_{11} \cdot b_{17}\ b_{13}\ b_{16}$

$P'_{12} \cdot b_{15}\ b_{13}\ b_{14}$

$P'_{13} \cdot b_{11}\ b_5\ b_{10}$

$P'_{14} \cdot b_9\ b_5\ b_8$

$P'_{15} \cdot b_7\ b_5$



Similarly, one can observe that any tree which is not homeomorphic to the trees mentioned in the theorem will have at least two non isomorphic pathos graphs. This proves the theorem.

ON POINT PATHOS GRAPHS OF A GRAPH

(Graphs defined on (0,1) matrix)

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INTRODUCTION:

Let $A = [a_{ij}]$ be a (0,1) matrix. The graph $G(A)$ of the matrix A is defined as follows. The point set of $G(A)$ consists of all a_{ij}'s with $a_{ij}=1$. Two such points are adjacent if and only if they appear either in the same row or in the same column of A.

The idea of the matrix graphs was provided by the well known theorem of Konig-Egervary[2] on (0,1) matrices.

THEOREM: (Konig-Egervary). Let A be a (0,1) matrix of size $m \times n$. The minimum number of lines that cover all the 1's in A is equal to the maximum number of 1's in A with no two of the 1's on a line.

In ((3) p.2) Hedetniemi indicated that the matrix graph thus constructed has its point independence number equal to the maximum number of 1's in the matrix with no two of the 1's on a line.

In [1] C.R.Cook has defined the clique-vertex graph as one of the special case of the matrix graphs and discussed some of its properties.

The concept of a pathos of a graph G was introduced by F.Harary as a collection of minimum number of line disjoint open paths whose union is G. The path number $P_n(G)$ of a graph G is the number of paths in a pathos. R.G.Stasston, D.D.Cowan and L.O.James have calculated the path number for certain classes of graphs like trees, cubic graphs complete graphs etc. L.Lovasz[6) has also produced some good bounds for $P_n(G)$.

The point-pathos matrix A of G is a pxk matrix $\{a_{ij}\}$ where $a_{ij} = 1$ if the point v_i belongs to the path p_i in P where P is the set of pathos in G and $a_{ij} = o$ otherwise.

Let P be a pathos of a graph G. The point pathos graph $_pP(G)$ of G is the matrix graph of point-pathos matrix of G. That is the point set of $_pP(G)$ is the set of ordered pairs (p_i, v_j) where v_j is a point of G on the path p_i in P and two ordered pairs (p_i, v_j) and (p_m, v_n) are adjacent if and only if either $p_i = p_m$ or $v_j = v_n$.

Recall that in general path number of any graph is not determined. However, path number is determined for tree and some special classes of graphs.

In this paper we mainly deal with point pathos graph $_pP(T)$ of a tree T and obtain its characterisation. We also show that the row graph of the point pathos matrix of a graph G is isomorphic to the pathos graph P(G)[7). And the clique graph (or block graph) of a column graph of the point pathos matrix is also isomorphic to the pathos graph P(G).

Let T be a tree and P be a set of pathos of T. The rows of the point pathos matrix of T correspond to the paths in P of T and the columns correspond to the points of T. The column sums of this matrix correspond to the number of pathos in P, in which a point appears.

In figure 1 a tree T and the set of paths in P are given. In table 1, the point pathos matrix corresponding to the tree T and paths in it are given. In figure 2, the point pathos graph $_pP(T)$ of a tree (T in figure 1) is given.

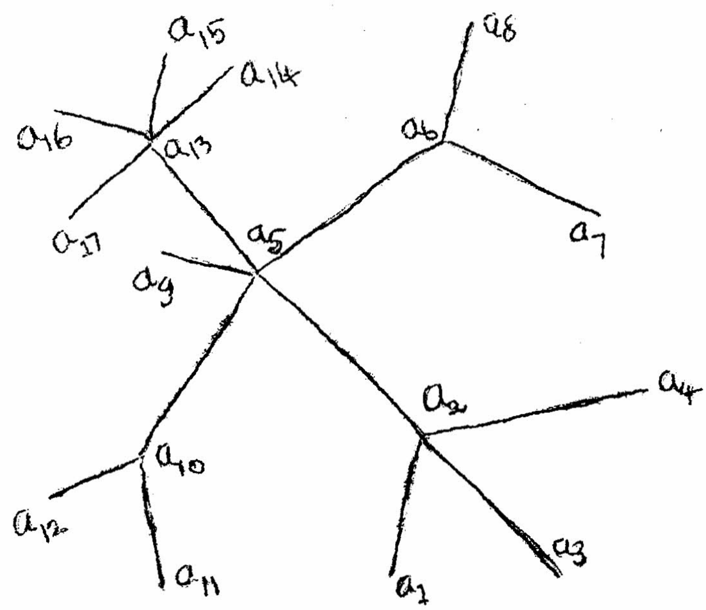

Figure 1

p_1	:	$a_1\ a_2\ a_3$	p_5	:	$a_{12}\ a_{10}\ a_5\ a_9$
p_2	:	$a_4\ a_2\ a_5\ a_6\ a_7$	p_6	:	$a_{10}\ a_{11}$
p_3	:	$a_6\ a_8$	p_7	:	$a_{16}\ a_{13}\ a_{15}$
p_4	:	$a_5\ a_{13}\ a_{14}$	p_8	:	$a_{17}\ a_{13}$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}
p_1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
p_2	0	1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
p_3	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0
p_4	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0
p_5	0	0	0	0	1	0	0	0	1	1	0	1	0	0	0	0	0
p_6	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
p_7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
p_8	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1

Table 1

Point pathos matrix of T in figure 1

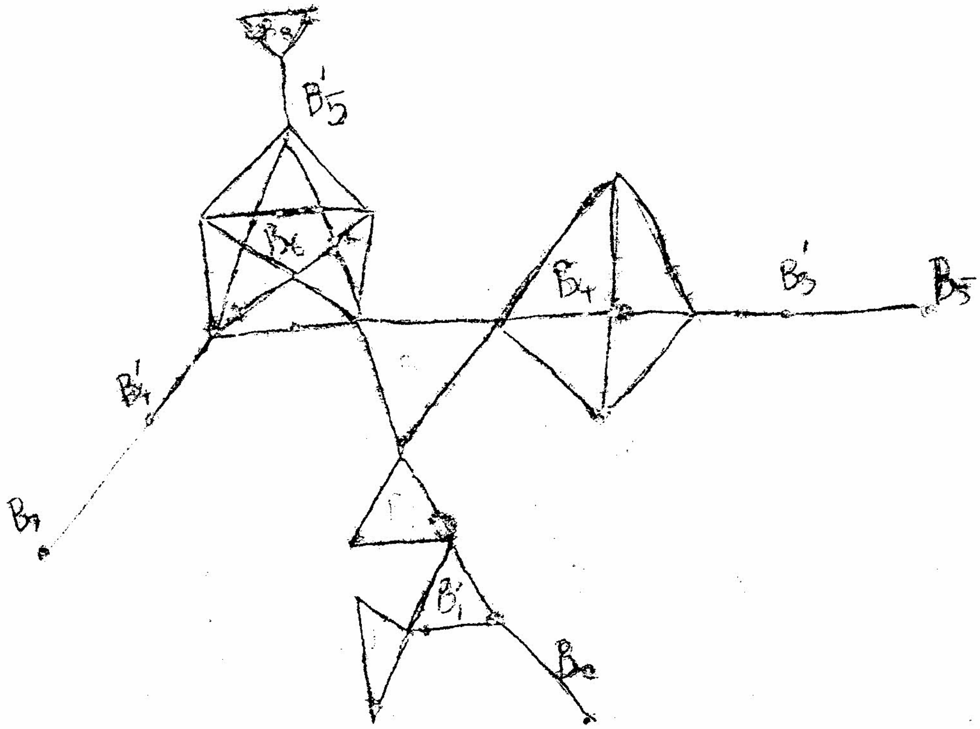

Figure 2

Point pathos graph $_pP(T)$ of a tree T in figure 1.

THEOREM:1 Let T be a tree. Then the number of points in $_pP(T)$ is $\sum_i n_i$, where n_i is the path number of a point v_i in T.

PROOF: If n_i is the path number of a point v_i in T then v_i lies on n_i paths in P. Thus the total number of 1's appearing in the i-th column of the point pathos matrix of G is n_i. And hence total number of 1's appearing in that matrix is $\sum_i n_i$. This is the number of points in $_pP(G)$. This proves the theorem.

THEOREM:2. Let T be a tree with $pn(v_i)$ as the path number of a non pendant point v_i in T and n_i as the length of a path p_i of P in T. Then the total number of lines in $_pP(T)$ is given by

$$\sum_i n_i \frac{(n_i+1)}{2} + \sum_i \frac{pn(v_i)(pn(v_i)-1)}{2}$$

PROOF: In a point pathos matrix of a tree, the number of 1's is every row corresponds to the number of points in the corresponding path p_i of P in T. And by definition of point pathos graph, points corresponding to 1's in a row induce a complete block in ${}_pP(T)$. If n_i for $i = 1, 2, \ldots\ldots\ldots, n$ are the lengths of the paths p_i in P of T, then the number of lines in the corresponding complete blocks in ${}_pP(T)$ will be clearly

$$\sum_i \frac{(n_i + 1)\left((n_i + 1) - 1\right)}{2}$$

$$\text{i.e.} \qquad \frac{n_i(n_i + 1)}{2} \qquad \text{(A)}$$

Similarly, the points corresponding to the number of 1's in each column form a complete block in ${}_pP(T)$, which corresponds to the number of paths p_i in P having v_i as a common point. This is clearly path number $pn(v_i)$ of the point v_i. So again the number of lines corresponding to such complete blocks will be

$$\sum_i \frac{pn(v_i)\left(pn(v_i) - 1\right)}{2} \qquad \ldots\ldots\ldots\ldots \qquad \text{(B)}$$

From (A) and (B) it follows that the total number of lines in ${}_pP(T)$ is

$$\sum_i \frac{n_i(n_i + 1)}{2} + \sum_i \frac{pn(v_i)\left(pn(v_i) - 1\right)}{2}$$

This proves the theorem.

Let $A = \{a_{ij}\}$ be a (o,1) point pathos matrix with m rows. Then the points of the <u>row graph</u> $R(A)$, $\{u_1, u_2, \ldots\ldots\ldots, u_m\}$, correspond to the rows of A and two points of R(A) are adjacent if and only if there is a column of A with 1's in the two rows corresponding to the points.

<u>THEOREM: 3</u>. The row graph R(A) of a point pathos matrix of a tree T is isomorphic to P(T), the pathos graph of the tree T.

<u>PROOF</u>: Clearly, there is a one-one correspondence between the points of the row graph of the point pathos matrix and the points of the pathos graph P(T) of the three T. Two points in the row graph of the point-pathos matrix of the tree T, are adjacent if and only if the ~~tow~~ rows have a 1 in the same column. This is equivalent to saying that the corresponding paths of the pathos have a point in common. This implies that the row graph R(A) is the interaction graph of the pathos P of the tree T. This proves that the row graph of the point pathos matrix of the graph T is isomorphic to its pathos graph P(T).

<u>Example</u>. The following graph in figure 3 is the row graph of a tree T shown in figure 1,

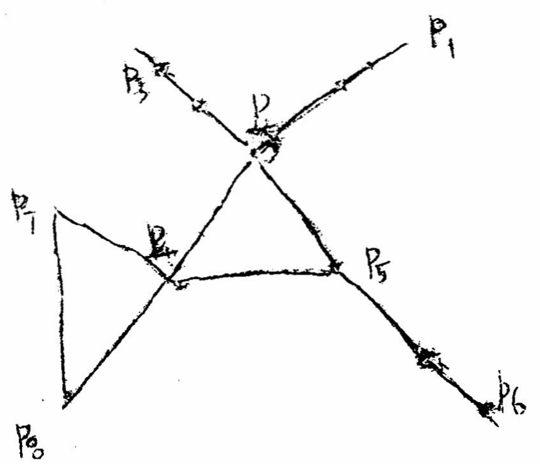

Figure 3

Let $A = [a_{ij}]$ be a (o,1) point pathos matrix with n columns. Then the points of the column graph C(A), $\{v_i, v_j, \ldots, v_n\}$ correspond to the columns of A and two points of C(A) are adjacent if and only if there is a row of A with 1's in two columns corresponding to the points.

THEOREM:4 Let T be a tree and P be the set of pathos in T. Then the clique graph (or block graph) of the column graph of the point-pathos matrix is again isomorphic to the pathos graph P(T) of the tree T.

PROOF: By the definition of the pathos graph P(G) of a graph G, there is one-one correspondence between the paths in P of G and thd points in P(G).

By the definition of a column graph there is a one-one correspondence between the complete blocks of C(A), the column graph of point-pathos matrix A, and the paths p_i in pathos P of tree T.

Also, by the definition of the clique graph, there exists one-one correspondence between the cliques (or blocks) of G and the points of clique graph (or blocks) of G and the points of clique graph (or block graph).

Now, since every block in the column graph C(A) of a tree T is complete, it follows that there exists one-one correspondence between the paths p_i in P of T, and hence the number of points in the corresponding pathos graph P(T) of a tree T and the points of the clique graph (or block graph) of the column graph of the point pathos matrix of the tree T.

On the other hand any two points are adjacent in P(T) implies that the corresponding paths p_i, p_j in P of T have a point in common. Consequently this implies that the complete blocks (or cliques) Q_i, Q_j in C(A) corresponding to p_i, p_j in P of T, have a point in common. This in turn implies that the two points in the clique graph (or block graph) of a column graph of a point pathos matrix, corresponding to Q_i, Q_j are adjacent. Thus it follows that clique graph (or block graph) of the column graph C(A) of the point pathos matrix of a tree T is isomorphic to the pathos graph P(T). This proves the theorem.

<u>Example</u>.

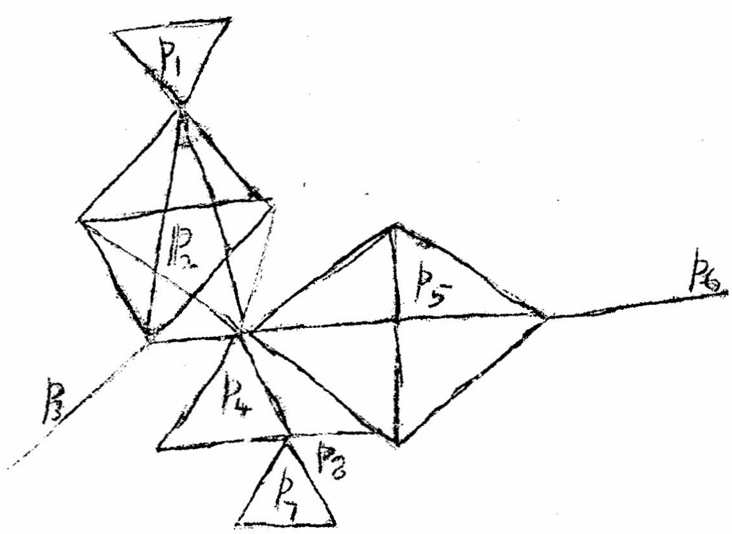

Figure 4

The graph in the above example (see figure 4) is the column graph C(A) of a point pathos matrix of a tree T in figure 1. It is easy to observe that clique graph (or block graph) of this graph C(A) is isomorphic to the pathos graph P(T) of the tree T.

The following Corollary follows from the above theorems.

Corollary 4.1: Let T be a tree and A be its point pathos matrix. Then the row graph R(A) of A is isomorphic to the clique graph (block graph) of its column graph C(A).

CHARACTERIZATION OF $_pP(T)$ OF A TREE T

The following two theorems characterize the point pathos graphs of a tree T.

THEOREM 5: A graph G is point pathos graph $_pP(T)$ of some path T if and only if G is complete.

PROOF: NECESSITY. Any path T of length n contains (n+1) points and has pn(T) = 1. So the point pathos matrix contains one and only one row and has as many number of columns as there are points in T. Naturally this matrix has all 1's in the same row. This implies clearly that pP(T) is complete on (n+1) points.

SUFFICIENCY: Suppose G be a complete graph, say K_n. Then it is easy to see that the graph obtained by removing all chords and a line of K_n, is a path T of length n-1, which has a path number unity and n points. Clearly K_n is $_pP(T)$ of T thus obtained. This proves the theorem.

THEOREM 6: A graph G is the point pathos graph of a tree T(Which is not a path), i.e. G = $_pP(T)$ if and only if

(a) Every block of G is complete.

(b) There exist two subsets Π_1 and Π_2 of the set Π of blocks of G, such that

i) $\pi = \pi_1 \cup \pi_2$

ii) No two blocks belonging to the same set have a point in common.

iii) At each point of any block belonging to one of the subsets say π_2, there is a block belonging to π_1.

PROOF: NECESSITY. Let G be a point pathos graph of a tree T. Consider the point pathos matrix A of T. By the definition of G, the 1's in A correspond to the points of A and every block in G is complete. Also, each column (or row) in A corresponds to a complete block in G, and conversely. The order of a complete block in G is equal to the number of 1's appearing in the corresponding column (or row) of A.

Let A be a mxp matrix and

$$\pi_1 = \{B_1, B_2, B_3, \ldots\ldots\ldots, B_m\}$$

be the set of blocks of G corresponding to the rows of A. Also let $\pi_2 = \{B_1', B_2', B_3', \ldots\ldots\ldots, B_p'\}$ be the set of blocks of G corresponding to the columns of G. It is easy to see that no two blocks in π_1 or in π_2 have a common point.

Now, consider B_1' of the set π_2, and let its order be r. This block corresponds to the first column in A and there are "r" 1's in the first column. Let

$$i_1, i_2, i_3, \ldots\ldots, i_r$$

be the rows of A, which have 1's common with the first column of A. Then each of the blocks

$$B_{i_1}, B_{i_2}, B_{i_3}, \ldots\ldots, B_{i_r}$$

of G have a common point with B_1'. Likewise we can show that every block in π satisfies the condition (ii) of (b). This proves that the conditions in the theorem are necessary.

<u>SUFFICIENCY</u>: Let G be a connected graph satisfying the given conditions in the theorem..

Let π be the set of blocks of G and π_1, π_2 be two subsets of π satisfying the given conditions of the theorem. Thus at each point for every block B' in there is a block belonging to π_1.

Let G' be the graph obtained from G by contracting every block belonging to π_1 into a single point. Clearly G' is also a graph in which every block is complete. Hence, by an earlier theorem (Theorem 6, Chapter I) G' is a pathos graph of some tree T. Now the construction of the tree T can be undertaken in two ways.

<u>Case 1</u>. If the tree T does not contain points of degree 2, then T can be easily constructed from G' by using the method mentioned in theorem 6 of Chapter I.

<u>Case 2</u>. Suppose the tree contains points of degree 2. In this case we can construct the required tree T' as follows.

Let G" be the graph obtained from G by contracting the blocks belonging to π_2 into a single point. Then replace every complete block of order $n(n \geqslant 3)$ by a cycle of length n. (This can be easily done by removing all chords of every complete block of order n (n $\geq$ 3). Then further remove one line from every cycle of order n (n $\geq$ 4) which is not incident to any star-point.

In case the complete block is a triangle, then remove the line which is adjacent to an even star-point.

Also, in a case in which each of the point of the complete block is incident to a block, then remove the line that is incident to two points of even degree suitably.

Let T' be the tree thus obtained. Now one can easily verify that the point pathos graph $_pP(T')$ is the graph G. This proves the sufficiency and hence the theorem.

Now, by the help of the following example we can show that the condition (b) of the theorem is irredundant.

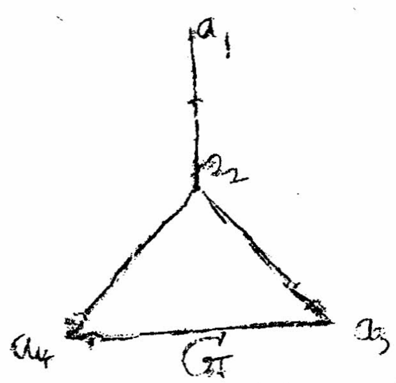

Figure 5

Here we observe that every block in G is complete but it is not a point pathos graph of any tree.

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ON POINT PATHOS GRAPHS-II

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This is a continuation of Paper .

Here we study the following

(1) Traversibility of $p^P(T)$

(2) Planarity of $p^P(R)$.

(3) Some miscellaneous results.

TRAVERSIBILITY

THEOREM 1. Let T be a tree and P be a set of pathos in T. Let d_i denote the degree of a star-point v_i in T. Then the point pathos graph $p^P(T)$ of the tree T is eulerian if and only if the length of every path p_i in P is even and for every star-point v_i in T, $\left\{\frac{d_i}{2}\right\}$ is odd.

PROOF. SUFFICIENCY. Let T be a tree satisfying the condition of the theorem. Then as observed above, the set π of complete blocks of $p^P(T)$ can be divided into two subsets π_1 and π_2, such that

(1) No two blocks in π_1 or π_2 are adjacent.

(2) At every point of each complete block in π_2, there is a complete block belonging to π_1.

Also, there is a one-one correspondence between the star-point of T and the blocks of π_2. Similarly there is a one-one correspondence between the paths in P and the complete blocks in π_1.

It can be seen clearly, that for every path p_i of length n in P of T, there is a complete block of order $(n+1)$ in $p^P(T)$ belonging to π_1, and conversely. And for every star-point v_i of degree d_i in T, there is a complete block of order $\left\{\frac{d_i}{2}\right\}$ in $p^P(T)$, and conversely.

Now, in view of the above discussion if T satisfies the condition of the theorem then it implies clearly that the degree of every point in $p^P(T)$ is even and hence $p^P(T)$ is eulerian.

<u>NECESSITY.</u> Let $p^P(T)$ be eulerian. Then the degree of every point in $p^P(T)$ is even.

Let π_i and π_2 be the two sets of blocks in $p^P(T)$ as described earlier. Then every complete block B_i in π_1 corresponds to a path p_i in P, and conversely. If n is the order of B_i then $n-1$ is the length of the corresponding path p_i. Since $p^P(T)$ is eulerian, it follows that n is odd and hence $n-1$, the length of p_i is even. This proves that the length of every path p_i in P of the tree T is even.

Further, every block B_i' in π_2 corresponds to a star-point v_i in T, and conversely. If d_i is the degree of v_i then the order of the corresponding block B_i' in $p^P(T)$ is $\left\{\frac{d_i}{2}\right\}$. Now, since $p^P(T)$ is eulerian it follows that $\left\{\frac{d_i}{2}\right\}$ is odd. This completes the necessity. Thus the theorem is proved.

THEOREM 2. Let T be a tree and P be a set of pathos. Then $p^P(T)$ is hamiltonian if and only if T is a path.

PROOF. By an earlier theorem[1] $p^P(T)$ of a tree T is complete if and only if T is a path. Also we observe that if T contains at least one star-point then the point pathos matrix contains at least two rows and hence clearly $p^P(T)$ contains a cut point. This completes the proof of the theorem.

PLANARITY

THEOREM 3. Let T be a tree P be a set of pathos in T. Then $p^P(T)$ is planar if and only if

(1) Length of every path p_i of P in T is $\leq$ 3.

(2) For every star-point v_i in T, $\left\{\frac{d_i}{2}\right\}$ is $\leq$ 4.

PROOF. We observe that in $p^P(T)$ every block is complete. Also it is known[2] that a graph G is planar if and only if every block in it is planar.

As we discussed earlier, the set of complete blocks π in $p^P(T)$ can be divided into two subsets π_1 and π_2, such that every block of order $(n+1)$ corresponds to a unique path p_i of length n in P, and conversely. Also every star-point v_i of degree d_i corresponds to a complete block B_i' in π_2 of order $\left\{\frac{d_i}{2}\right\}$, and conversely.

Now, since every block in $p^P(T)$ is complete, it follows clearly that every block in it is planar if and only if the order of the block is 4. This in turn implies that $p^P(T)$ is planar if and only if

(1) Length of every path p_i in P of T is $\leq$ 3.

(2) For every star-point v_i in T, $\left\{\frac{d_i}{2}\right\}$ is $\leq$ 4.

This proves the theorem.

We now state the following theorem for outerplanarity of a point pathos graph $p^P(T)$ of a tree T without proof. For the proof is analogous to the proof of the above theorem.

THEOREM 4. Let T be a tree and P be a set of pathos. Then $p^P(T)$ is outerplanar if and only if

(1) Length of every path p_i in P of T is $\leq$ 2.

(2) For every star-point v_i in T, $\left\{\frac{d_i}{2}\right\}$ is $\leq$ 3.

MISCELLANEOUS RESULTS

In theorem 5 chromatic number of a point pathos graph $p^P(T)$ and in theorem 6 its point point covering number are discussed.

THEOREM 5. Let T be a tree and P be a set of pathos in T. Then chromatic number of $p^P(T)$ is given by

$$\chi(p^P(T)) = \max\left(\max\left[n(p_i) + 1\right], \max\left\{\frac{d_i}{2}\right\}\right)$$

where $n(p_i)$ denotes the length of a path p_i in P of T and d_i is the degree of a star-point v_i in T. $\{r\}$ has the usual meaning.

PROOF. We know that every block in $p^P(T)$ is complete. Let π_1 and π_2 be two subsets of the set π of blocks of $p^P(T)$. If $n(p_i)$ is the length of a path p_i in P of T, then clearly

$$\max \lceil n(p_i) + 1 \rceil \quad \ldots\ldots\ldots\ldots \quad (A)$$

indicates the maximum order of a corresponding complete block in π_1.

Similarly, if d_i is the degree of a star-point v_i in T, then it follows clearly

$$\max \left\{ \frac{d_i}{2} \right\} \quad \ldots\ldots\ldots\ldots \quad (B)$$

indicates the maximum order of a corresponding complete block in π_2.

Thus, we observe that from (A) and (B)

$$\max \left(\max \lceil n(p_i)+1 \rceil, \max \left\{ \frac{d_i}{2} \right\} \right)$$

gives the maximum order of any complete block in $p^P(T)$.

But, it is known that chromatic number of a complete graph K_n is n and also the chromatic number of a graph is equal to the maximum of the chromatic number of its blocks.

Thus, it follows that the chromatic number of the point pathos graph $p^P(T)$ of a tree T is given by

$$\max \left(\max \lceil n(p_i)+1 \rceil, \max \left\{ \frac{d_i}{2} \right\} \right)$$

where $\{r\}$ has the usual meaning. This proves the theorem.

Recall that for a graph G, the clique number $w(G)$ of a graph G is the number of points in the largest complete subgraph of G.

<u>COROLLARY 5.1.</u> For a tree T

$$\chi(p^P(T) = \omega(p^P(T)).$$

Corollary follows easily from the above theorem and the definition.

THEOREM 6. For a tree T,

$$\alpha_{oo}(p^P(T)) = pn(T),$$

where α_{oo} denotes the point-point covering number of G and $pn(T)$ represents the path number of a tree T.

PROOF Let π be the set of complete blocks of the graph $p^P(T)$ of the tree T and π_1 and π_2 be the two subsets of π as defined earlier. Then every point of each block in π_2 is incident to a complete block in π_1. And no two blocks of the same subset have a point in common. And hence the complete blocks of the same set are independent.

Thus, it follows that the complete blocks of π_1 cover all the points of π_1 together with all the points of the blocks of π_2 and hence the points of $p^P(T)$.

Now, observe that all the points of each complete block in π_1 are covered by a single point. That means the point-point covering number of $p^P(T)$ is same as the number of complete blocks in π_1. But there is one-one correspondence between the complete blocks in π_1 and the paths in P of T. Thus, it follows that

$$\alpha_{oo}(p^P(T)) = pn(T).$$

This proves the theorem.

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ON UNIQUENESS OF THE POINT PATHOS GRAPHS

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As noted earlier pathos graph of a graph need not be unique. Hence it follows that point pathos graph of a graph need not be unique.

In this Paper we investigate graphs for which point pathos graph is unique. As in the case of uniqueness of pathos graph of a graph, we show here that the point pathos graph of a tree T is unique if and only if T is one of the following, satisfying some additional conditions to be stated.

(1) T is homeomorphic to a star,

(2) T is homeomorphic to a two star,

(3) T is homeomorphic to a super star.

Let T be a tree homeomorphic to a star $K_{1,n}$, and v be the central point of T. Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant points of T and r_i, $i = 1,2,3,\ldots,n$ be the number of points in the v-v_i path, $i = 1,2,3,\ldots,n$. Let P be a pathos of T. Now we have the following Cases.

Case 1. Let the degree n of v be even. Clearly the end points of every path p_i in P are pendant points. So it follows that all paths in P have exactly one point in common, and hence the point pathos matrix corresponding to P of T contains $\left\{\frac{n}{2}\right\}$ rows such that there will be one and only one column common to all rows which contain 1's in each of its rows. Thus the point pathos graph consists of a central complete block B_v corresponding to v in T of order $\left\{\frac{n}{2}\right\}$. Also at each point of B_v there is a complete block of order (r_i+r_j), where (r_i+r_j) is the number of points in the corresponding path in P, and conversely. This can be seen in the following illustration, taking $n = 10$.

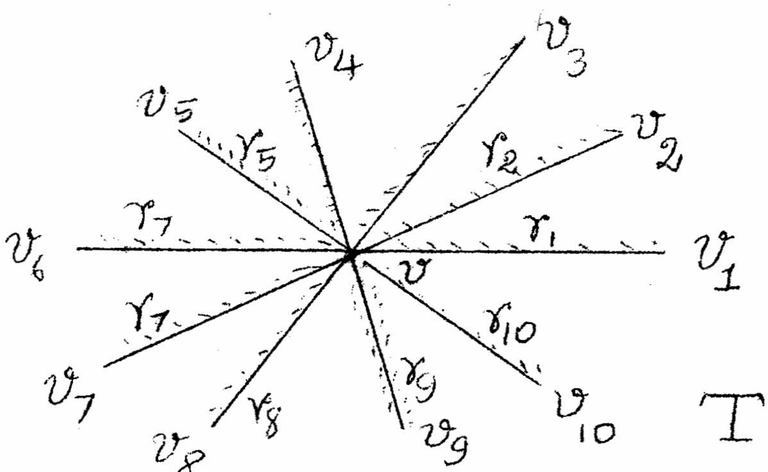

Figure 1.

Now, corresponding to this tree T, homeomorphic to $K_{1,n}$ $(n = 10)$ and the set of pathos

$$P = \left\{ v_1\, vv_2,\ v_3\, vv_4,\ v_5\, vv_6,\ v_7\, vv_8,\ v_9\, vv_{10} \right\}$$

we have the following point pathos graph.

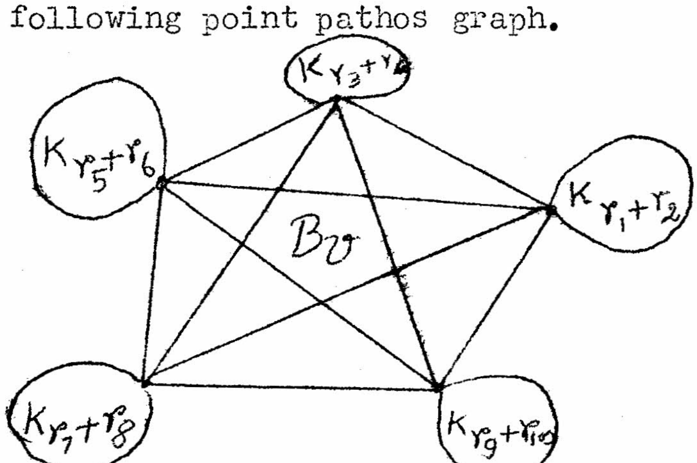

Figure 2

Here the orders of the complete blocks adjoint to B_v varies according to the different combination of (r_i+r_j). Thus we observe that corresponding to different pathos of T, we have different point pathos graphs of the type as shown in figure 2.

Now, it can be seen that all the point pathos graphs will be isomorphic if and only if all the end blocks except possibly one are isomorphic. In otherwords, all the point pathos graphs are isomorphic if and only if all the numbers r_i's except possibly one are equal.

Case 2. Suppose the degree n of v is odd, say n = 9.

Let P' be the set of pathos defined as follows.

$$P' = \left\{ v_1vv_2, \ v_3vv_4, \ v_5vv_6, \ v_7vv_8, \ vv_9 \right\} .$$

As in the above case we have in the point pathos graph $p^{P'}(T)$, a central complete block on $\left\{\frac{9}{2}\right\} = 5$ points, corresponding to the star $K_{1,9}$ at v. Further we have the end blocks

$$K_{r_1+r_2}, \ K_{r_3+r_4}, \ K_{r_5+r_6} \ \ldots\ldots\ldots , \ K_{r_9} ,$$

corresponding to the paths

$$v_1vv_2, \ v_3vv_4, \ v_5vv_6, \ \ldots\ldots\ldots, vv_9.$$

Corresponding to different pathos P' we have different point pathos graphs $p^{P'}(T)$. We observe in this case that all the point pathos graphs $p^{P*}(T)$ are isomorphic if and only if all the end blocks are isomorphic. This is possible if and only if all the r_i's, $i = 1,2,3,\ldots\ldots,n$ are equal.

In view of the above discussion we have the following theorem.

THEOREM 1. Let T be a tree, homeomorphic to $K_{1,n}$ and v be its central point. Let v_i, $i = 1,2,3,\ldots\ldots,n$ be the pendant points of T and r_i be the number of points in the $v - v_i$ path. Then the point pathos graph $p^P(T)$ of T is unique if the following conditions are true.

(1) If n is even, all r_i's are equal except possibly one .

(2) If n is odd, all r_i's are equal.

A two star is a tree containing exactly two star-points (See figure 3)

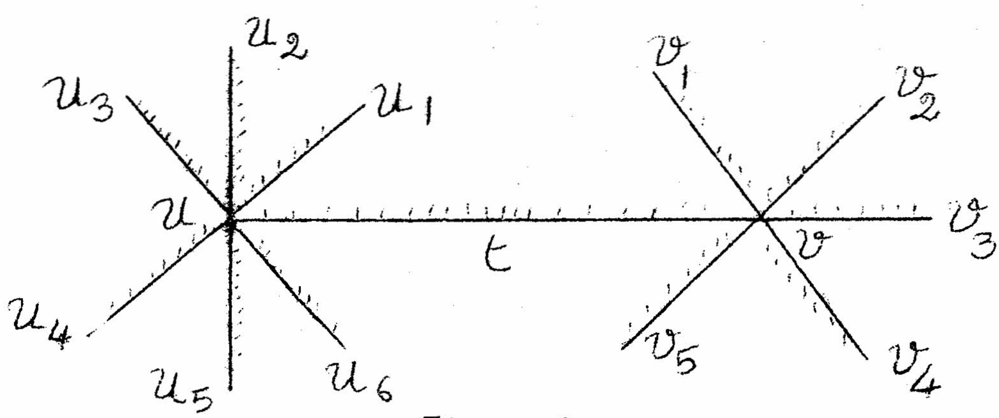

Figure 3

Let T be a tree homeomorphic to a two star. Let u and v be the two star-points such that deg $u = m$ and deg $v = n$. Let r_i, $i = 1,2,\ldots\ldots,m$ be the number of points in the $u - u_i$ path and s_j, $j = 1,2,\ldots\ldots,n$ be the number of points in the $v - v_j$ path. Let t be the number of points in the $u - v$ path.

Let us consider the two star T as made up of two trees T_1 and T_2, each homeomorphic to a star say $K_{1,m}$ and $K_{1,n}$

respectively, such that their central points u and v are joined by a u - v path. Let P be a pathos of T. Then the point pathos graph of T, the two star, will be of the form shown in Figure 4, consisting of three parts where each closed curve represents complete block.

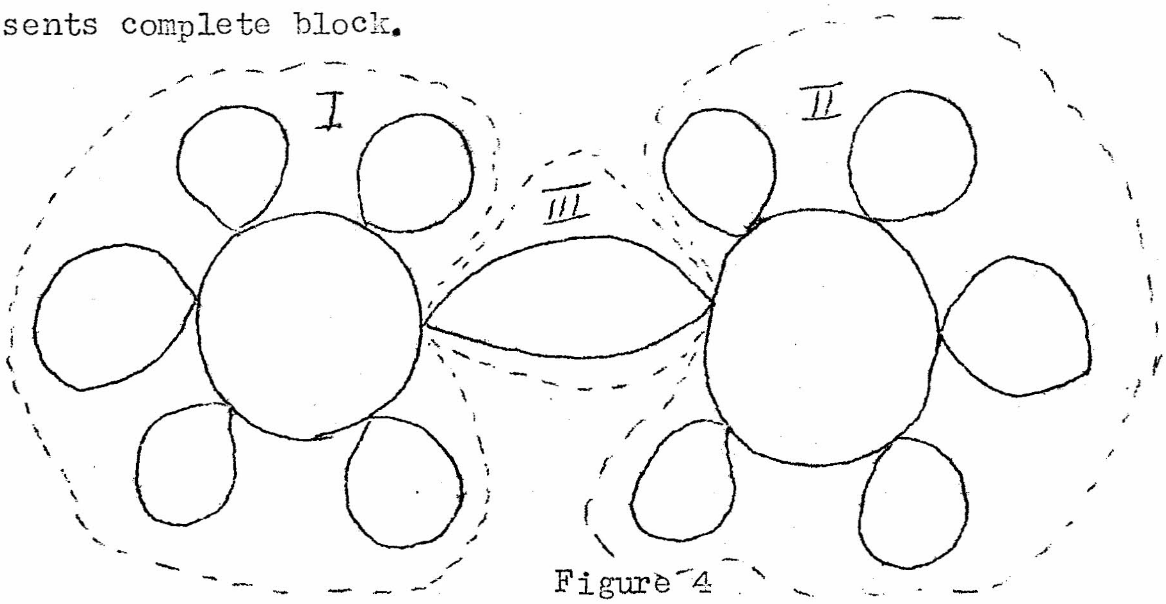

Figure 4

Part I (See figure 4) together with part III corresponds to the point pathos graph of the subtree T_1 at u, homeomorphic to a star $K_{1,m}$ and part III together with part III corresponds to the point pathos graph of the subtree T_2 at v homeomorphic to the star $K_{1,n}$. Also part III corresponds to a point pathos graph of the u-v path in T, which is incidently a complete block. Thus one can observe that the structure of part I together with part III and part II together with Part III are exactly similar to the one that we discussed in point pathos graph of a tree homeomorphic to a star. (Theorem 1).

Now, regarding the order of the complete block B, we have the following three cases.

Case 1. Suppose m and n the degrees of the star points are even. In this case u and v cannot be end points of any path p_i in P. Any path p_i in P containing u and v will be of the type

$$u_i - u \text{ path} + u - v \text{ path} + v - v_j \text{ path}.$$

corresponding to such path we have the complete block B, which is of the form $K_{r_i+t+s_j}$ for some $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n$.

Case II. Suppose both m and n are odd. In this case the complete block B may have the following possibilities.

(i) $B = K_t$. This happens when there is a path p_i in P, having u and v as end points.

(ii) $B = K_{t+r_i}$. This happens when there is a path p_i in P of the type

$$u_i - u \text{ path} + u - v \text{ path}$$

(iii) $B = K_{t+s_j}$. This is a case in which the path p_i in P will be of the form

$$u - v \text{ path} + v - v_j \text{ path}.$$

(iv) $B = K_{r_i+t+s_j}$. This is possible when the path p_i in P takes the following form

$$u_i\text{-}u \text{ path} + u\text{-}v \text{ path} + v\text{-}v_j \text{ path}.$$

Case III. Suppose one of the two numbers m and n, the degrees of the star-points, say m is odd. In this case the following possibilities may wrise for the complete block B.

(i) $B = K_{t+s_j}$. This happens when there is a path p_i in P of the form

$$u - v \text{ path} + v - v_j \text{ path.}$$

(ii) $B = K_{r_i+t+s_j}$. In this case the corresponding path p_i in P will be of the form

$$u_i - u \text{ path} + u - v \text{ path} + v - v_j \text{ path.}$$

A careful analysis of the cases I, II and III shows that uniqueness of pathos graph can occur only in case I under the conditions stated in the following theorem.

THEOREM 2. Let T be a tree homeomorphic to a two-star. Then the point pathos graph $p^P(T)$ of T is unique if

(1) The degree of the each star-point in T is even.

(2) Length of every maximal star-free path is one and the same except at most the one which joins the two star-points.

Analogously, one can prove the following theorem.

THEOREM 3. Let T be a tree homeomorphic to a super star. Then the point pathos graph $p^P(T)$ of the tree T is unique if T satisfies the following conditions.

(1) The degree of every semicentral point and the central point is even and all the semicentral points have the same degree except at most one.

(2) All maximal star-free pathos of the type $v_i - v_i'$ incident at the same semicentral point v_i, have the same length.

(3) All maximal star-free paths of the type $v_i - v_i'$ incident at all the points v_i, $i = 1, 2, \ldots\ldots\ldots, n$ have the same length except at most one.

(4) The legth of every maximal star-free path of the type $v - v_i$ is one and the same for every $i = 1, 2, \ldots\ldots, n$.

It will not be difficult to show that any tree which is different from the trees mentioned in theorems 1, 2 and 3 will have at least two different non-isomorphic pathos graphs. Hence the converse part of our main problem also follows.