MATSCIENCE REPORT 83

NEW CONCEPTS IN ARITHMETIC FUNCTIONS

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ON ARITHMETIC FUNCTIONS AND DIVISORS OF HIGHER ORDER

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ON ARITHMETIC FUNCTIONS AND DIVISORS OF HIGHER ORDER

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It is well known that the fundamental concept of a 'divisor' leads to remarkable Arithmetic functions. In this paper we discuss properties of arithmetic functions '<u>of higher order</u>' defined through the introduction of a new concept of a '<u>divisor of higher order</u>'. We shall construct an infinite sequence of Euler like functions and the well known Euler function shall be the first member of this sequence. Particular care has been given to the construction of such divisors so that the exact formulae for these functions can be got once the canonical representation of the integer concerned is known. Asymptotic estimates of such functions are given and a study of error functions associated with the Euler like sequence is made. We would like to mention that the familiar number theoretic functions become only the first members of an infinite sequence of functions of similar behaviour.

If 'd' and 'n' are two positive integers and if $d \mid n$. We say d is a <u>first order</u> divisor of n and change the notation to $d \mid_1 n$. When 'a' and 'b' are two positive integers (a,b) rewritten as $(a,b)_1$ shall denote the largest divisor of 'a' dividing b. When $(a,b)_1 = 1$ we say 'a' is prime to 'b' <u>order</u> 1.

If 'd' and 'n' are two integers, then d is said to be a divisor of n of second order, denoted by $d \mid_2 n$ if

$$\left(\frac{n}{d}, d\right)_1 = 1.$$

(This is the definition of unitary divisor). The symbol $(a,b)_2$ represents the largest divisor 'c' of a, satisfying $c \mid_2 b$. If $(a,b)_2 = 1$ we say 'a' is prime to order 2. (a' is semiprime to 'b' in standard usage). Here comes the departure.

A divisor d of n is a divisor of third order (notation: $d \mid_3 n$) if

$$\left(\frac{n}{d}, d\right)_2 = 1$$

The symbol $(a,b)_3$ stands for the largest divisor 'c' of 'a' that satisfies $c \mid_3 b$. If $(a,b)_3 = 1$ we say 'a' is prime to b <u>order</u> 3. We generalise by saying that $d \mid_r n$ if

$$\left(\frac{n}{d}, d\right)_{r-1} = 1$$

and

$$(a,b)_r = \max \left\{ c \mid_1 a \ . \ c \mid_r b \right\}.$$

If $(a,b)_r = 1$, then 'a' is prime to 'b' <u>order</u> r.

NOTE. The definition of $d \mid_3 n$ given by us differs from the two well known extensions of the concept of a unitary divisor given by Chidambaraswamy [2] and Suryanarayana [5] respectively. The former defines 'd' to be a semi-unitary divisor of n if $(d, \frac{n}{d})_2 = 1$, as opposed to our $d \mid_3 n$ where $(\frac{n}{d}, d)_2 = 1$. The latter defines d to be a bi-unitary divisor of n if $(d, \frac{n}{d})^{**} = 1$ where $(a,b)^{**}$ represents the largest common unitary divisor of 'a' and 'b'. However in both the papers [2], and [5], the concept of a unitary divisor is just extended one step beyond.

Our definition of higher order divisor is given in such a way that the higher order divisors share many several properties in common so that it is possible to discuss together the properties of arithmetic functions of r^{th} order, as we shall see in the theorems that follow. Moreover some of the familiar number theoretic results follow as corollaries if we set r = 1, and some of the results of Cohen can be deduced it we set $n = 2$.[3]

We now define r^{th} order analogues to some well known arithmetic functions. However as $(a,b)_r \neq (b,a)_r$ in general these functions have interesting dual functions. Denote by

$$\varphi_r(n,x) = \sum_{\substack{0<a\leq x \\ (a,n)_r=1}}' 1 \quad ; \quad \varphi_r(n,n) = \varphi_r(n)$$

and its dual

$$\varphi_r^*(n,x) = \sum_{\substack{0<a\leq x \\ (n,a)_r=1}}' 1 \quad ; \quad \varphi_r^*(n,n) = \varphi_r^*(n)$$

for $r \geq 1$. We define $\varphi_0(n,x) = \varphi_0^*(n,x) = [x]$, where $[x]$ denotes the largest integer $\leq x$. Note that $\varphi_1 = \varphi_1^* = \varphi$ (Euler). We define the divisor functions

$$\sigma_{r,k}(n) = \sum_{d|_r n}' d^k \quad and \quad \sigma_{r,k}^*(n) = \sum_{d|_r n}' \left(\frac{n}{d}\right)^k$$

Before we take up the study of these functions we need to define some more functions. Let $\{F_r\}_{r=0}^{\infty}$ denote the sequence given by

$$F_0 = 0, \; F_1 = 1 \quad F_n = F_{n-1} + F_{n-2} \quad n \geq 2$$

Let $\ell(y)$ and $\ell^*(y)$ denote respectively the least integers $>$ and $\geq$ y. Further define

$$f_r(x) = \ell\left(\frac{F_{r-1}}{F_r}x\right) \text{ when } r \equiv 1 \pmod 2$$

$$f_r(x) = \ell^*\left(\frac{F_{r-1}}{F_r}x\right) \text{ when } r \equiv 0 \pmod 2$$

Let $f_r^{-1}(x)$ denote the largest integer y with $f_r(y) = x$. And if $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ be the canonical decomposition of n, then let

$$\beta_1(n) = n \quad and \quad \beta_r(n) = \prod_{i=1}^{s} p_i^{f_r^{-1}(\alpha_i)+1}$$

We will now show

LEMMA 1. If $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ be the canonical decomposition of n as a product of distinct primes, and if $d \mid_1 n$, then $d \mid_r n$ if and only if $d = \prod_{i=1}^{s} p_i^{\beta_i}$ where

$$\beta_i = 0 \quad or \quad f_r(\alpha_i) \le \beta_i \le \alpha_i$$

Proof. For $r = 1$, $f_r(\alpha_i) = 1$ and so the lemma holds trivially. For $r = 2$, $f_r(\alpha_i) = \alpha_i$ and $\beta_i = 0$ or $\beta_i = \alpha_i$ for a unitary divisor and the lemma is true.

Let $r = 3$ and $d = \prod_{i=1}^{s} p_i^{\beta_i}$ satisfy $d \mid_3 n$. Clearly $d \mid_1 n$ and so $\alpha_i \geq \beta_i$ trivially holds. Now

$$\frac{n}{d} = \prod_{i=1}^{s} p_i^{\alpha_i - \beta_i}$$

If $d \mid_3 n$ then $(\frac{n}{d}, d)_2 = 1$. Thus there is no divisor except 1 of n/d which is a divisor of d of second order. This is possible if and only if

$$\alpha_i - \beta_i < \beta_i \quad or \quad \beta_i = 0$$

For if $\alpha_i - \beta_i \geq \beta_i$ then $p_i^{\beta_i} \mid_1 (\frac{n}{d})$ and $p_i^{\beta_i} \mid_2 d$ contradiction. Thus $\alpha_i - \beta_i < \beta_i$. If $\alpha_i - \beta_i < \beta_i$ and $p_i^{\gamma_i} \mid (\frac{n}{d})$ that $0 < \gamma_i \le \alpha_i - \beta_i < \beta_i$ and so $p_i^{\gamma_i} \nmid_2 d$. Hence $(\frac{n}{d}, d)_2 = 1$.

Thus

$$\alpha_i - \beta_i < \beta_i \iff \beta_i > \frac{\alpha_i}{2} = \frac{F_2}{F_3}\alpha_i$$

Moreover β_i is an integer and so $\beta_i \geq f_3(\alpha_i)$ proving lemma for r = 3.

In general let the lemma hold for 1, 2, ... r, r even. Now $d \,|_{r+1}\, n$ if and only if $(\frac{n}{d}, d)_r = 1$ where

$$d = \prod_{i=1}^{s} p_i^{\beta_i} \qquad \frac{n}{d} = \prod_{i=1}^{s} p_i^{\alpha_i - \beta_i}$$

Now $(\frac{n}{d}, d)_r = 1$ says that there is no divisor of $\frac{n}{d}$ save 1, that is a divisor of d order r. This is possible if and only if $\alpha_i - \beta_i < \frac{F_{r-1}}{F_r}\beta_i$ or $\beta_i = 0$. For otherwise if $\alpha_i - \beta_i \geq \frac{F_{r-1}}{F_r}\beta_i$ then one can find a γ_i satisfying

$$\alpha_i - \beta_i \geq \gamma_i \geq \frac{F_{r-1}}{F_r}\beta_i$$

so that $p_i^{\gamma_i} \,|_1\, (\frac{n}{d})$ and $p_i^{\gamma_i} \,|_r\, d$ a contradiction. Thus we have

$$\alpha_i - \beta_i < \frac{F_{r-1}}{F_r}\beta_i$$

or $\qquad \beta_i > \frac{F_r}{F_{r+1}}\alpha_i \qquad$ and β_i is an integer.

Thus $\beta_i \geq f_{r+1}(\alpha_i)$ proving the lemma for r+1 odd. The proof for the case r+1 even in similar.

The higher order divisors share in common the property.

LEMMA 2. (a) If a, and n are integers then for any nonnegative integer λ

$$(a, n)_r = (\lambda n + a, n)_r = (\lambda n - a, n)_r$$

(b) We have $(n,a)_r = 1$ if and only if

$$(n,a)_r = (n, \lambda\beta_r(n) + a)_r = (n, \lambda\beta_r(n) - a)_r = 1.$$

We omit the details of the proof of (a) and (b) as they are direct consequences of the definitions. We shall need Lemma 2 in the discussion of the error functions.

THEOREM 1. If $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ as in Lemma 1, then

$$\varphi_r(n) = n \prod_{i=1}^{s} \left(1 - \frac{1}{p_i^{f_r(\alpha_i)}}\right)$$

<u>Proof.</u> We know that

$$\varphi_r(n,x) = \sum_{\substack{0<a\leq x \\ (a,n)_r=1}}' 1 = [x] - \sum_{\substack{0<a\leq x \\ (a,n)_r>1}}' 1$$

Now $(a,n)_r > 1$, if there exists a $d|_r n$, $d > 1$ with $d|_1 a$. We know from Lemma 1 that $d|_r n$ if and only if $\beta_i = 0$ $f_r(\alpha_i) \leq \beta_i \leq \alpha_i$ where $d = \prod_{i=1}^{s} p_i^{\beta_i}$. This implies that if $p_i \nmid_1 a$ and $p_i |_1 (a,n)_r$ then $p_i^{f_r(\alpha_i)} |_1 a$. Thus the combinatorial expansion leads to

$$\varphi_r(n,x) = [x] - \sum_{0<i\leq s}' \left[\frac{x}{p_i^{f_r(\alpha_i)}}\right] + \sum_{0<i<j\leq s}' \left[\frac{x}{p_i^{f_r(\alpha_i)} p_j^{f_r(\alpha_j)}}\right]$$

$$+ \cdots \quad (-1)^r \sum' \left[\frac{x}{p_1^{f_r(\alpha_1)} \cdots p_s^{f_r(\alpha_s)}}\right] \qquad (1)$$

If we pur $x = n$ in the (1) we get Theorem 1. Now (1) also indicates that

LEMMA 3. If $e_r(n,x) = \frac{x}{n}\varphi_\gamma(n) - \varphi_\gamma(n,x)$ then

$$e_\gamma(n,x) = O(n^\epsilon) \ \forall \epsilon > 0 .$$

<u>Proof.</u> We can rewrite (1) as

$$\varphi_\gamma(n,x) = x - \sum_{0<i\leq s}' \frac{x}{p_i^{f_\gamma(\alpha_i)}} + \sum_{0<i<j\leq s}' \frac{x}{p_i^{f_\gamma(\alpha_i)} p_j^{f_\gamma(\alpha_j)}}$$

$$+ O\left(1 + \sum_{p_i|n}' 1 + \sum_{p_i p_j|n}' 1 + \cdots\right)$$

$$= \frac{x}{n}\varphi_\gamma(n) + O(\psi(n))$$

where $\psi(n) = 2^s$ when $n = \prod_{i=1}^{s} p_i^{\alpha_i}$. Thus we have

$$e_\gamma(n,x) = \frac{x}{n}\varphi_\gamma(n) - \varphi_\gamma(n,x) = O(\psi(n)) = O(\tau(n)) = O(n^\epsilon) \ \forall \epsilon > 0$$

as

$$\psi(n) = 2^s \leq \prod_{i=1}^{s}(\alpha_i + 1) = O(n^\epsilon) \quad \text{see [4]}$$

This establishes the lemma.

THEOREM 2. If $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ then

$$\varphi_\gamma^*(n, \beta_\gamma(n)) = \varphi(n, \beta_\gamma(n)) \prod_{i=1}^{s}\left(1 + \frac{1}{p_i^{f_\gamma^{-1}(\alpha_i)+1}\left(1 - \frac{1}{p_i}\right)}\right)$$

<u>Proof.</u> We defined $\varphi_\gamma(n,x) = \sum_{0<a\leq x, (n,a)_\gamma = 1}' 1$

Now $(n,a)_r = 1$ can arise out of two cases. If $(n,a)_1 = 1$ then $(n,a)_r = (a,n)_r = 1$. Or $(n,a)_1 > 1$ in which case there is a $p_i \mid_1 n$ and $p_i \mid_1 a$. As $(n,a)_r = 1$ even if $d \mid_1 a$, $d \nmid_r a$ for all $d \mid_1 n$. Thus $p_i^{f_\gamma^{-1}(\alpha_i)+1} \mid_1 a$. Thus from the combinatorial expansion we have

$$\varphi_\gamma^*(n,x) = \varphi_1(n,x) + \sum_{0<i\le s}^{8}{}' \varphi_1\left(\frac{n}{p_i^{\alpha_i}}, \frac{x}{p_i^{f_\gamma^{-1}(\alpha_i)+1}}\right) +$$

$$\sum_{0<i<j\le s}{}' \varphi_1\left(\frac{n}{p_i^{\alpha_i} p_j^{\alpha_j}}, \frac{x}{p_i^{f_\gamma^{-1}(\alpha_i)+1}\, p_j^{f_\gamma^{-1}(\alpha_j)+1}}\right) + \cdots \quad (2)$$

(2)

If we put $x = \beta_\gamma(n)$ in (2) and use Lemma 2 which for $r = 1$ gives $\varphi_1(n, \lambda n+\mu) = \lambda\varphi_1(n) + \varphi_1(n,\mu)$ we get theorem 2 immediately. In fact one has from (2) the following result.

LEMMA 4. If $\mathcal{E}_\gamma^*(n_1 x) = \frac{x}{\beta_\gamma(n)} \varphi_\gamma^*(n, \beta_\gamma(n) - -\varphi_\gamma^*(n_1 x)$

then $\mathcal{E}_\gamma^*(n_1 x) = O(n^\epsilon)\ \forall\ \epsilon > 0$

We omit the details of the proof.

We are now in a position to prove

THEOREM 3. For any pair of integers n and k we have

a) $\varphi_1(n) \le \varphi_3(n) \le \varphi_5(n) \le \cdots \le \varphi_6(n) \le \varphi_4(n) \le \varphi_3(n)$

b) $\sigma_{2,k}(n) \le \sigma_{4,k}(n) \le \sigma_{6,k}(n) \le \cdots \le \sigma_{5,k}(n) \le \sigma_{3,k}(n) \le \sigma_{1,k}(n)$

c) $\sigma^*_{2,k}(n) \le \sigma^*_{4,k}(n) \le \sigma^*_{6,k}(n) \le \cdots \le \sigma^*_{5,k}(n) \le \sigma^*_{3,k}(n) \le \sigma^*_{1,k}(n)$

d) $\frac{\varphi_1^*(n,\beta_1(n))}{\beta_1(n)} \le \frac{\varphi_3^*(n,\beta_3(n))}{\beta_3(n)} \le \frac{\varphi_5^*(n,\beta_5(n))}{\beta_5(n)} \le \cdots$

$\le \frac{\varphi_6^*(n,\beta_6(n))}{\beta_6(n)} \le \frac{\varphi_4^*(n,\beta_4(n))}{\beta_4(n)} \le \frac{\varphi_2^*(n,\beta_2(n))}{\beta_2(n)}$

Proof. We shall prove (a) and (b). Theorem proofs of (c) and (d) are similar. First we observe that $\frac{F_{2k}}{F_{2k-1}}$ from an increasing sequenee and $\frac{F_{2k-1}}{F_{2k}}$ form a decreasing sequence, both converging to $\frac{\sqrt{5}-1}{2}$. Further we note that if $x < y$ then

$$\ell(x) \leq \ell(y), \quad \ell^*(x) \leq \ell^*(y) \quad \text{and } \ell(x) \leq \ell^*(y) \tag{5}$$

the first two inequalities being trivial, but the last not so. These follow from the definition of ℓ, and ℓ^* (Page 2).

Now (3) implies that for any integer 'm' we have

$$f_1(m) \leq f_3(m) \leq f_5(m) \leq \ldots \leq f_6(m) \leq f_4(m) \leq f_2(m) \tag{6}$$

We now assume $n = \prod_{i=1}^{s} p_i^{\alpha_i}$. Then if we use (4) and Theorem 1 we get (a). Now (4) and theorem(2) will give (d) on similar lines of reasoning for $f_r^{-1}(m)$

To prove (b) it is enough to observe that

$$d \mid {}_{2m}n \Rightarrow d \mid {}_{2m+2}n, \quad d \mid {}_{2m+1}n \Rightarrow d \mid {}_{2m-1}n, \quad d \mid {}_{2m}n \Rightarrow d \mid {}_{2m'+1}n$$

for any pair of integers m and m'. This follows from lemma 1. Thus the set of inequalities (b) and (c) are true. This proves the theorem. We now take up the asymptotic estimates of $\sigma_{r,k}$ and $\sigma^*_{r,k}$. Let us define two constants for $k > 0$

$$\alpha_{r,k} = \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{\varphi_{r-1}(n)}{n^{k+2}} \tag{5}$$

$$\alpha^*_{r,k} = \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{\varphi^*_{r-1}(n, \beta_{r-1}(n))}{n^{k+1}\beta_{r-1}(n)} \tag{6}$$

Our main theorem is

THEOREM 4. a) $$\sum_{n=1}^{m}{}' \sigma_{r,k}(n) = \alpha^*_{r,k} m^{k+1} + O(m^{k+\frac{1}{2}})$$

b) $$\sum_{n=1}^{m}{}' \sigma^*_{r,k}(n) = \alpha_{r,k} m^{k+1} + O(m^{k+\frac{1}{2}})$$

Proof. We shall prove the second part of Theorem 4. Part a will follow on similar reasoning. We shall first need an estimate of

$$\sum_{0 < a \leq x, \ (a,n)_r = 1}{}' a^k \tag{7}$$

Let $A(n,r,s)$ denote the s^{th} number 'a' such that $(a,n)_r = 1$. It is obvious that

$$\varphi_r(n, A(n,r,s)) = s$$

We know from Lemma 3 that

$$\varphi_r(n, A(n,r,s)) = \frac{A(n,r,s)}{n} \varphi_r(n) + o(n^{\epsilon}) = s \ \forall \epsilon > 0$$

so that

$$A(n,r,s) = \frac{ns}{\varphi_r(n)} + \frac{n}{\varphi_r(n)} o(n^{\epsilon}) \ \forall \epsilon > 0. \tag{8}$$

We deduce from theorem 3 that for $r \geq 0$ $\varphi_r(n) \geq \varphi_1(n) = \varphi(n)$ (as $\varphi_0(n) = n$). As it is known that $n/\varphi(n) = 0$ (log log n/ log n) see [4] we infer

$$\frac{n}{\varphi_r(n)} = 0 \ (\mathrm{Log\ Log}\ n)$$

so that (8) is rewritten as

$$A(n,r,s) = \frac{ns}{\varphi_r(n)} + o(n^{\epsilon}) \quad \forall \epsilon > 0 \tag{9}$$

Thus

$$\sum_{\substack{0<a\le x\\ (a,n)_\gamma=*}}' a^K = \sum_{0<s\le\varphi_\gamma(n,x)}' A(n,r,s)^K = \sum_{0<s\le\varphi_\gamma(n,x)}' \left(\frac{ns}{\varphi_\gamma(n)} + O(n^\epsilon)\right)^K \quad \forall\,\epsilon>0$$

$$= \frac{n^K}{\varphi_\gamma(n)^K}\sum_{0<s\le\varphi_\gamma(n,x)}' s^K + \frac{n^{K-1}}{\varphi_\gamma(n)^{K-1}}\sum_{0<s\le\varphi_\gamma(n,x)}' O(s^{K-1}n^\epsilon) \quad \forall\,\epsilon>0$$

$$= \frac{n^K}{\varphi_\gamma(n)^K}\left(\frac{\varphi_\gamma(n,x)^{K+1}}{K+1} + O(\varphi_\gamma(n,x)^K)\right) + O\left(\frac{n^{K-1+\epsilon}}{\varphi_\gamma(n)^{K-1}}\varphi_\gamma(n,x)^K\right)$$

$$= \frac{n^K}{\varphi_\gamma(n)^K}\left(\frac{x^{K+1}\varphi_\gamma(n)^{K+1}}{(K+1)n^{K+1}} + O(x^{K+\epsilon})\right) + O\left(n^\epsilon\varphi_\gamma(n,x)^K\right)$$

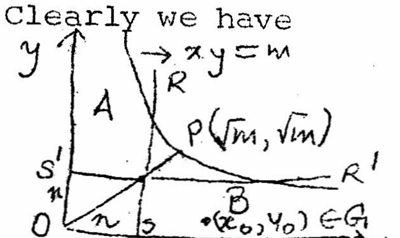
$$\forall\,\epsilon>0$$

$$= \frac{x^{K+1}\varphi_\gamma(n)}{(K+1)\,n} + O(x^{K+\epsilon}) \quad \forall\,\epsilon>0 \tag{10}$$

by (8) where x is taken as $\geq n$.

We shall return to (10) after making a geometric interpretation of $\sigma^*_{r,K}$ Consider the hyperbola $xy = m$ above the x-axis. Call a lattice point (x_0, y_0) good if $0 < x_0 \cdot y_0 \leq m$ (y with $(y_0, x_0)_{r-1} = 1$.

Let G denote the set of good lattice points. Divide the region under the curve into three nob-intersecting regions A, OP and B. Clearly we have



$$\sum_{n=1}^{m}\sigma^*_{r,K}(n) = \sum_{(x_0,y_0)\in G}' y_0^K$$

which can be split up as

$$\sum_{n=1}^{m}{}' \sigma^*_{r,K}(n) = \sum_{(x_0,y_0)\in G\cap A}{}' \sigma_{r,K}\, y_0^K + \sum_{(x_0,y_0)\in G\cap B}{}' y_0^K + \sum_{(x_0,y_0)\in G\cap (OP)}{}' y_0^K$$

$$= S_1 + S_2 + S_3 \text{ say}$$

Clearly

$$S_3 = O(m^{(K+1)/2})$$

To estimate S_2 pick a point S' on OY at a distance n from O with $n \leq \sqrt{m}$. The sum of $y_o^k = n^k$ over $R'S'$ through S' is

$$\sum_{\substack{n < x_0 \leq \frac{m}{n} \\ (n,x_0)_{r-1}=1}}{}' n^K = n^K \varphi^*_{r-1}(n, n, \tfrac{m}{n})$$

(where $\varphi^*_{r-1}(n,c,d) = \sum\limits_{c < a \leq d,\ (n,a)_{r-1}=1}' 1$). Thus we have

$$S_2 = \sum_{n=1}^{[\sqrt{m}]}{}' n^K \varphi^*_{r-1}(n,n,\tfrac{m}{n}) = \sum_{n=1}^{[\sqrt{m}]}{}' n^K O(\tfrac{m}{n}) = m \sum_{n=1}^{[\sqrt{m}]}{}' n^{K-1}$$

$$= O(m^{(K+2)/2}) = O(m^{K+\frac{1}{2}}) \text{ for } K \geq 1.$$

To estimate S_1 pick an S on OX at a distance n from O with $n \leq \sqrt{m}$. Draw RS through it. The sum of y_o^k over y_o lying on RS is

$$\sum_{\substack{n < y_0 \leq m/n \\ (y_0,n)_{r-1}=1}}{}' y_0^K = \frac{m^{K+1}\varphi_{r-1}(n)}{(K+1)n^{K+2}} + O\left(\frac{m^{K+\epsilon}}{n^{K+\epsilon}}\right) - \frac{n^K \varphi_{r-1}(n)}{K+1} + O(n^{K+\epsilon})\ \forall \epsilon > 0$$

Using (10), where x takes values n, and $\frac{m}{n}$. If we sum (12) from 1 to $[\sqrt{m}]$ we get S_1 which is

$$S_1 = \frac{m^{K+1}}{K+1}\sum_{n=1}^{[\sqrt{m}]}{}' \frac{\varphi_{r-1}(n)}{n^{K+2}} + O(m^{K+\epsilon}) + O(m^{(K+2)/2}) \quad \forall \epsilon > 0$$

so that

$$S_1 = \frac{m^{K+1}}{K+1} \sum_{n=1}^{[\sqrt{m}]} \frac{\varphi_{r-1}(n)}{n^{K+2}} + O(m^{K+\frac{1}{2}})$$

$$= \frac{m^{K+1}}{K+1} \left(\sum_{n=1}^{\infty} \frac{\varphi_{r-1}(n)}{n^{K+2}} - \sum_{n=[\sqrt{m}]+1}^{\infty} \frac{\varphi_{r-1}(n)}{n^{K+2}} \right) + O(m^{K+\frac{1}{2}})$$

$$= \alpha_{r,K} m^{K+1} + m^{K+1} O\left(\sum_{n=[\sqrt{m}]+1}^{\infty} \frac{1}{n^{K+1}} \right) + O(m^{K+\frac{1}{2}})$$

$$= \alpha_{r,K} m^{K+1} + m^{(K+1)/2} O\left(m^{\frac{(K+1)}{2}} \sum_{n=[\sqrt{m}]+1}^{\infty} \frac{1}{n^{K+1}} \right) + O(m^{K+\frac{1}{2}})$$

$$= \alpha_{r,K} m^{K+1} + O(m^{(K+2)/2}) + O(m^{K+\frac{1}{2}})$$

$$= \alpha_{r,K} m^{K+1} + O(m^{K+\frac{1}{2}}) \quad \text{for } K \geq 1 .$$

If we substitute these estimates of S_1, S_2 and S_3 in (11) we get

$$\sum_{n=1}^{m} \sigma^*_{r,K}(n) = \alpha_{r,K} m^{K+1} + O(m^{K+\frac{1}{2}})$$

proving part (b). The proof of part (a) is similar with the following changes. We have to replace $\varphi_{r-1}(n)/n$ by $\varphi_{r-1}(n, \beta_{r-1}(n)/\beta_{r-1}(n)$ and use Lemma 4 instead of Lemma 3 to get a estimate similar to (10). The proof is complete.

We deduce a few corollaries to our theorem.

COROLLARY 1. If $\sigma(n)$ denotes the sum of the divisors of n then

$$\sum_{n=1}^{m}{}' \sigma(n) \sim \frac{\pi^2}{12} m^2$$

COROLLARY 2. If $\sigma_{1,k}(n)$ denotes the sum of the k^{th} powers of the divisors of n then

$$\sum_{n=1}^{m}{}' \sigma_{1,k}(n) = \frac{\zeta(k+1)}{k+1} m^{k+1} + O(m^{k+\frac{1}{2}})$$

COROLLARY 3. If $\sigma_{2,1}(n)$ denotes the sum of the unitary divisors of n then

$$\sum_{n=1}^{m}{}' \sigma_{2,1}(n) = \frac{\pi^2 m^2}{12\zeta(3)} + O(m^{3/2})$$

Proof. Corollary 1 follows from theorem 4 if we estimate $\alpha_{1,1}$. Clearly

$$\alpha_{11} = \alpha_{11}^{*} = \frac{1}{2}\sum_{n=1}^{\infty}{}' \frac{1}{n^2} = \frac{\pi^2}{12}.$$

Corollary 2 follows if we find $\alpha_{1,k}$ which is $\zeta(k+1)/k+1$.

Corollary 3 follows from an estimate of $\sigma_{2,k}$ which is

$$\sigma_{2,1} = \sigma_{2,1}^{*} = \frac{1}{2}\sum_{n=1}^{\infty}{}' \frac{\varphi_1(n)}{n^3} = \frac{\pi^2}{12\zeta(3)}.$$

which is the result due to Cohen [3].

COROLLARY 4. For $k > 1$ we have

$$\alpha_{2,k} \leq \alpha_{4,k} \leq \cdots\cdots \leq \alpha_{5,k} \leq \alpha_{3,k} \leq \alpha_{1,k}.$$

This follows directly from theorem 3. We raise the following question. (which we do not at the moment answer). What is $\lim_{r \to \infty} \alpha_{r,k}$? Finally we take up the discussion of error functions associated with the Euler functions. (A similar discussion for r = 1 is made in [1]).

We first calculate the average value of $e_r(n, x)$ and $\mathcal{E}^*_\gamma(n, x)$ for fixed n where x is discrete.

THEOREM 5.
$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m}{}' e_\gamma(n,i) = -\frac{\varphi_\gamma(n)}{2n}$$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m}{}' \mathcal{E}^*_\gamma(n,i) = -\frac{\varphi^*_\gamma(n,\beta_\gamma(n))}{\beta_\gamma(n)}$$

<u>Proof.</u> From Lemma 2 we deduce that

$$\begin{aligned} e_r(n,i) + e_r(n,n-i) &= 0 \text{ if } (i,n) \neq 1 \\ &= -1 \text{ if } (i,n) = 1. \end{aligned}$$

so that we get

$$\sum_{i=1}^{n}{}' e_\gamma(n,i) = -\varphi_\gamma(n)/2$$

Now Lemma 1 says

$$e_\gamma(n,\lambda n+i) = \frac{\lambda n+i}{n}\varphi_\gamma(n) - \varphi_\gamma(n,\lambda n+1) = \frac{\lambda n+i}{n}\varphi_\gamma(n) - \lambda\varphi_\gamma(n) - \varphi_\gamma(n,i)$$
$$= e_\gamma(n,i)$$

Let m = $\lambda n + \mu$ for some non-negative integer λ where $0 \leq$ $0 \leq \mu < \lambda$.

Clearly
$$\frac{1}{m}\sum_{i=1}^{m}{}' e_\gamma(n,i) = \frac{1}{m}\sum_{i=1}^{n}{}' e_\gamma(n,i) + \frac{1}{m}\sum_{i=n+1}^{2n}{}' e_\gamma(n,i) + \dots \frac{1}{m}\sum_{i=(\lambda-1)n+1}^{\lambda n}{}' e_\gamma(n,i)$$
$$+ \frac{1}{m}\sum_{i=\lambda n+1}^{\lambda n+\mu}{}' e_\gamma(n,i)$$

$$= \frac{-\lambda \varphi_\gamma(n)}{2m} + \frac{1}{m} \sum_{i=1}^{\mu} O(n^{\epsilon})$$

$$= \frac{-\varphi_\gamma(n)}{2n} + O\left(\frac{1}{m}\right)$$

so that proceeding to the limit as $m \longrightarrow \infty$, we get the first part of the theorem. The second part follow on similar reasoning. However the mean over the continuous variable vanishes. To be more precise

THEOREM 6.

$$\int_0^{n} e_\gamma(n,x)\,dx = 0 \;; \quad \int_0^{\beta_\gamma(n)} \mathcal{E}_\gamma^*(n,x)\,dx = 0.$$

Proof. The above theorem is an immediate consequence of the following Lemma.

LEMMA If f is Riemann integrable in $[0,m]$ and $f(x)+f(m-x) = 0$ for all but a finite x in $[0,m]$, then $\int_0^m f(x)\,dx = 0$

Clearly

$$\int_0^m f(x)dx = \int_0^m f(m-x)sx = \frac{1}{2}\int_0^m f(x) + f(m-x)\,dx = 0$$

Note that $e_r(n_1x) + e_r(n,n-x) = 0$ for all x except when $(x,n)_r = 1$ similarly $\mathcal{E}_\gamma^*(n_1x) + \mathcal{E}_\gamma^*(n, \beta_r(n)-x) = 0$ for all x except when $(n_1x)_r = 1$. Thus Theorem 6 is true.

We now study the properties of additive error functions associated with φ_γ and φ_γ^*. Define for $s \geq 2$

$$e_\gamma(n,\alpha_1,\alpha_2,\ldots\alpha_s) = \varphi_\gamma(n, \sum_{i=1}^{s}\alpha_i) - \sum_{i=1}^{s}\varphi_\gamma(n,\alpha_i)$$

and

$$e^{*}_{\gamma}(n,\alpha_1,\alpha_2,\ldots\alpha_s) = \varphi^{*}_{\gamma}\Big(n,\sum_{i=1}^{s}{}'\alpha_i\Big) - \sum_{i=1}^{s}{}'\varphi^{*}_{\gamma}(n,\alpha_i)$$

We begin by showing

<u>THEOREM 7.</u>

a) $$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}{}' e_{\gamma}(n,\alpha_1,\alpha_2,\ldots\alpha_s) = \sum_{n=1}^{\alpha_1+\alpha_2+\ldots\alpha_s}{}'\frac{\varphi^{*}_{\gamma}(n,\beta_{\gamma}(n))}{\beta_{\gamma}(n)} - \sum_{i=1}^{s}{}'\sum_{n=1}^{\alpha_i}{}'\frac{\varphi^{*}_{\gamma}(n,\beta_{\gamma}(n))}{\beta_{\gamma}(n)}$$

and the much similar

b) $$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}{}' e^{*}_{\gamma}(n,\alpha_1,\alpha_2,\ldots\alpha_s) = \sum_{n=1}^{\alpha_1+\ldots\alpha_s}{}'\frac{\varphi_{\gamma}(n)}{n} - \sum_{i=1}^{s}\sum_{n=1}^{\alpha_i}{}'\frac{\varphi_{\gamma}(n)}{n}$$

<u>Proof:</u> We only prove the ~~first~~ part. The second equation follows on similar lines. We know

$$\frac{1}{m}\sum_{n=1}^{m}{}' e_{\gamma}(n,\alpha_1,\alpha_2,\ldots\alpha_s) = \frac{1}{m}\sum_{n=1}^{m}{}'\varphi_{\gamma}\Big(n,\sum_{i=1}^{s}\alpha_i\Big) - \frac{1}{m}\sum_{n=1}^{m}{}'\sum_{i=1}^{s}{}'\varphi_{\gamma}(n,\alpha_i) \qquad (13)$$

For any integer j we have

$$\sum_{n=1}^{m}{}'\varphi_{\gamma}(n,j) = \sum_{n=1}^{m}{}'\sum_{\substack{i=1\\(i,n)_{\gamma}=1}}^{j}{}' 1 = \sum_{n=1}^{j}{}'\sum_{\substack{i=1\\(n,i)_{\gamma}=1}}^{m}{}' = \sum_{n=1}^{j}{}'\varphi^{*}_{\gamma}(n,m)$$

$$= m\sum_{n=1}^{j}{}'\frac{\varphi^{*}_{\gamma}(n,\beta_{\gamma}(n))}{\beta_{\gamma}(n)} + O(1) \qquad (14)$$

This implies that

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}{}'\varphi_{\gamma}(n,j) = \sum_{n=1}^{j}{}'\frac{\varphi^{*}_{\gamma}(n,\beta_{\gamma}(n))}{\beta_{\gamma}(n)}$$

If in (13) we set $\sum_{i=1}^{s}{}'\alpha_i$ and α_i as j, and then use (14) and proceed to the limit $m \to \infty$ we get Theorem 7 Part a. Part b follows by observing that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m}{}' \varphi_r^*(n,j) = \sum_{n=1}^{j}{}' \frac{\varphi_r(n)}{n} .$$

This completes the proof.

Note that the right hand side of (a) and (b) are of the form

$$g_r\left(n, \sum_{i=1}^{s}{}' \alpha_i\right) - \sum_{i=1}^{s}{}' g_r(n,\alpha_i)$$

and
$$g_r^*\left(n, \sum_{i=1}^{s}{}' \alpha_i\right) - \sum_{i=1}^{s}{}' g_r^*(n,\alpha_i)$$

which resembles remarkably the forms of $e_r(n, \alpha_1, \alpha_2, \ldots\ldots \alpha_s)$ and $e_r^*(n, \alpha_1, \; _2, \ldots\ldots \alpha_s)$.

We conclude by proving a necessary and sufficient condition for a number n to be a power of a prime using $e_r(n, \alpha_1, \alpha_2)$.

THEOREM 8. A necessary and sufficient condition for n to be a power of a prime is that

$$e_r(n, \alpha_1, \alpha_2) \leq 0 \quad \circ \; \forall\, \alpha_1, \alpha_2 \in Z^+ = \{1,2,3,\ldots\}. \qquad (15)$$

Proof. The necessity part is easy to establish. We know that

$$\varphi_r(n, \alpha_1+\alpha_2) = \alpha_1+\alpha_2 - \left[\frac{\alpha_1+\alpha_2}{p^{f_r(m)}}\right]$$

$$\varphi_r(n,\alpha_1) = \alpha_1 - \left[\frac{\alpha_1}{p^{f_r(m)}}\right] ; \quad \varphi_r(n,\alpha_2) = \alpha_2 - \left[\frac{\alpha_2}{p^{f_r(m)}}\right]$$

where $n = p^m$ and $[x]$ represents the largest integer $\leq$ x.

Now as $[x+y] \geqslant [x] + [y]$, the necessity part follows directly.

To prove sufficiency let (15) hold and let $n = \prod_{i=1}^{s} p_i^{\beta_i}$, $s > 1$. We shall get a contradiction. Consider the two numbers $p_i^{f_r(\beta_i)}$, $p_j^{f_r(\beta_j)}$ for any two distinct i, j with $1 \leqslant i < j \leqslant s$. As these numbers are relatively prime there exist positive integral solutions to

$$\left| x p_i^{f_r(\beta_i)} - y p_j^{f_r(\beta_j)} \right| = 1$$

Without loss of generality let $y p_j^{f_r(\beta_j)} > x p_i^{f_r(\beta_i)}$

Consider now an integer m satisfying

$$m \equiv 0 \left(\mathrm{mod}\ p_1^{f_r(\beta_1)} \cdots p_s^{f_r(\beta_s)}\right) \tag{16}$$

and let

$$m' = \prod_{i=1}^{s} p_i^{f_r(\beta_i)}$$

One can show that $(a,n)_r = 1$ if and only if

$$(\lambda m' + a, n)_r = (\lambda m' - a, n)_r = 1 \tag{17}$$

Now consider the intervals $\left(0,\ y p_j^{f_r(\beta_j)}\right]$ and $\left(m-2,\ m + y p_j^{f_r(\beta_j)} - 2\right]$

It is evident from (16) and (17) that for every 'a' with $0 < a \leqslant y p_j^{f_r(\beta_j)} - 2$ and $(a,n)_r = 1$ there is an 'm+a' with $m < m + a \leqslant m + y p_j^{f_r(\beta_j)} - 2$ and $(m+a,n)_r = 1$. But neither $x p_i^{f_r(\ \ i)}$ are prime to n order r (we use Lemma 1 here). Yet as $(1,n)_r = 1$ we have $(m-1),n)_r = 1$.

Thus

$$\varphi_\gamma(n, m-2, \quad m + y \quad p_j^{f_r(\beta_j)} - 2) = \varphi_\gamma(n, \; y \quad p_j^{f_r(\beta_j)}) + 1 \qquad (18)$$

which is the same as saying

$$e_r(n, \alpha_1, \alpha_2) = 1 > 0$$

if we set $\alpha_1 = m-2$, $\alpha_2 = p_j^{f_r(\beta_j)}$ in (18), a contradiction

to our assumption (15) for some $\alpha_1, \alpha_2 \in Z^+$ (actually for

infinitely many as the solutions to (16) are infinite). Thus

s = 1 which establishes sufficiency. The proof is complete.

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REFERENCES

1) K. Alladi, On a Generalization of the Euler function, Professor S. Ramanujan Commemmoration Volume, Oxford Press, Madras, India to appear December 1974.

2) Chidambaraswamy

3) E. Cohen, On Arithmetic functions associated with the unitary divisors of an integer, Math. Z. 1960, 74

4) W. J. LeVeque, Topics in number theory, Vol. 1, Addison Wesley, Reading, Mass, 1960, pp. 120-140.

5) D. Suryanarayana, On the number of bi-unitary divisors of an integer, Proc. Conf. on Arithmetic Functions, Springer-Verlag 19, Berlin, Heidelberg, New York.

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ON THE ASYMPTOTIC DISTRIBUTION OF FUNCTIONS MODULO AN INTEGER **

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ON THE ASYMPTOTIC DISTRIBUTION OF FUNCTIONS MODULO AN INTEGER

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Introduction:-

One of the features of number theoretic functions ia that though the rate of growth is not predictable, yet the summatory values of the function behave very well. A typical example is the function $\tau(n)$ which represents the number of divisors of n, a positive integer. The equation $\tau(x)=2$ has infinitely many solutions in x (integers) namely the primes, and $\tau(x)=m$ also has a solution however large m be. And it is known [1]

$$\sum_{n=1}^{m}{}' \tau(n) = m\log m + (2\gamma-1)m + O(\sqrt{m}) \quad (1)$$

where γ is the Euler's constant. In this paper we shall deal with the asymptotic behaviour when functions are summed over a set of integers with positive natural density. Here our interest will centre around functions "uniformly asymptotic" (see. Definition 1) and functions which can be expressed in terms of these uniformly asymptotic functions.

UNIFORMLY ASYMPTOTIC FUNCTIONS Here and in what follows by an integer we refer to an integer ≥ 0 and $Z^+=\{1,2,3,\cdots\}$ Whenever we speak of a function we mean a real valued function with $f(x)\geq 0$ Now let $A\subset Z^+$. For real x denote by

$$A(x) = \sum_{a\in A,\ a\leq x}{}' 1$$

and by $\delta(A)$ the limit of (if it exists)

$$\lim_{x\to\infty} \frac{A(x)}{x} = \delta(A)$$

$\delta(A)$ is called the natural density of A.

Let f be a function and let $\sum_{0<n\leq x}{}' f(n) = F(x)$ diverge to infinity monotonically. In this paper we shall be interested in limits of the form

$$\lim_{x\to\infty} \sum_{\substack{0<n\leq x\\ n\in A}}{}' f(n) \Big/ \sum_{0<n\leq x}{}' f(n) = \ell$$

It is quite natural to expect $\ell=\delta(A)$. However it will be convenient to know what exactly the members of A are besides knowing $\delta(A)>0$.

This leads to the definition of a uniformly asymptotic function.

Definition 1: Let f be a function and λ an integer. If we have

$$\lim_{x\to\infty} \frac{\sum'_{0<n\le x,\ n\equiv\mu(\mathrm{mod}\,\lambda)} f(n)}{\sum'_{0<n\le x} f(n)} = \frac{1}{\lambda} \qquad (3)$$

for all residue classes $\mu\,(\mathrm{mod}\ \lambda)$ then we say 'f' is uniformly asymptotic modulo λ. By $U(\lambda)$ is meant the set of all 'f' uniformly asymptotic modulo λ. If $f \in U(\lambda)\ \forall \lambda \in Z^+$ we say 'f' is a uniformly asymptotic function.

Lemma 1: If $f_1, f_2 \ldots\ldots f_n \in U(\lambda)$ and $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ are real constants >0, then $f \in U(\lambda)$ where

$$f = \sum_{i=1}^{n}{}' \alpha_i f_i$$

Proof: It is evident from Definition 1 that $g \in U(\lambda)$ implies $\alpha g \in U(\lambda)$ where α is a constant >0. Now let $g, h \in U(\lambda)$. We will show that $(g+h) \in U(\lambda)$. Denote by

$$u_m = \frac{\sum'^{m}_{n=1,\ n\equiv\mu(\mathrm{mod}\,\lambda)} (g(n)+h(n))}{\sum'^{m}_{n=1} (g(n)+h(n))}$$

where μ is some residue class modulo λ. Clearly if $a_m, b_m, c_m \,\&\, d_m$ denote

$$a_m = \sum'^{m}_{\substack{n=1\\ n\equiv\mu(\mathrm{mod}\,\lambda)}} g(n) \ ; \quad b_m = \sum'^{m}_{\substack{n=1\\ n\equiv\mu(\mathrm{mod}\,\lambda)}} h(n) \ ; \quad c_m = \sum'^{m}_{n=1} g(n) \ ; \quad d_m = \sum'^{m}_{n=1} h(n)$$

then

$$u_m = \frac{a_m + b_m}{c_m + d_m} \qquad (4)$$

Now let $\varepsilon_m = \left|\frac{1}{\lambda} - \frac{a_m}{c_m}\right|$ and $\varepsilon'_m = \left|\frac{1}{\lambda} - \frac{b_m}{d_m}\right|$. Clearly (4) indicates that u_m lies in between $\frac{a_m}{c_m}$ and $\frac{b_m}{d_m}$ so that

$$\left|u_m - \frac{1}{\lambda}\right| \le \max(\varepsilon_m, \varepsilon'_m)$$

Now as $g, h \in U(\lambda)$, ε_m and $\varepsilon'_m \to 0$ as $m \to \infty$ so that $u_m \to \frac{1}{\lambda}$ as $m \to \infty$. This proves $g + h \in U(\lambda)$

It is now a straightforward deduction that $f \in U(\lambda)$ proving the lemma.

It is an easy exercise to verify that if $f(n) = n^K$ with $K \in Z^+$ then f is uniformly asymptotic. Now Lemma 1 actually tells us that

Lemma 2: If $f(x)$ is a polynomial then 'f' is uniformly asymptotic.

Actually Lemma 2 becomes a particular case of a more general theorem we shall prove presently.

<u>Theorem:-</u> Let f be a function and let $\sum\limits_{n\leq x}' f(n)$ diverge. If

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1 \qquad (5)$$

then f is uniformly asymptotic. Conversely if f is uniformly asymptotic and if the following limit exists

$$\lim_{n\to\infty} \frac{f(n)}{f(n-1)} = \ell \qquad (6)$$

then $\ell = 1$.

<u>Proof:-</u> Consider a $\lambda \in Z^+$ and a residue μ of λ with $0 \leq \mu < \lambda$. Let x be a real number and partition $[0, x]$ as $[0, \mu]$, $[\mu, \mu+\lambda]$ ..... $[\lambda n' + \mu, x]$. Now as (5) holds we have for any $K \in Z^+$

$$f(n+k) = f(n) + o(f(n))$$

Choose $k \leq \lambda$. Now every integer n between $\lambda i + \mu$ and $\lambda(i+1)+\mu$ so that

$$\sum_{n=\lambda i+\mu+1}^{\lambda(i+1)+\mu}{}' f(n) = \sum_{n=\lambda i+\mu}^{\lambda(i+1)+\mu}{}' f(\lambda i+\mu) + o(f(\lambda i+\mu))$$

$$= \lambda f(\lambda i+\mu) + o(f(\lambda i+\mu)) \qquad (7)$$

clearly

$$\sum_{0<n\leq x}' f(n) = \lambda \sum_{\substack{n\leq x \\ n\equiv\mu(\mathrm{mod}\,\lambda)}}' f(n) + \sum_{\substack{0<n\leq x \\ n\equiv\mu(\mathrm{mod}\,\lambda)}}' o(f(n)) = \lambda \sum_{\substack{0<n\leq x \\ n\equiv\mu(\mathrm{mod}\,\lambda)}}' f(n) + o\Big(\sum_{0<n\leq x}' f(n)\Big)$$

as $\sum\limits_{0<n\leq x}' f(n)$ diverges. This implies that

$$\lim_{x\to\infty} \sum_{\substack{0<n\leq x \\ n\equiv\mu(\mathrm{mod}\,\lambda)}}' f(n) \Big/ \sum_{0<n\leq x}' f(n) = \frac{1}{\lambda}$$

or $f \in U(\lambda)$ since μ was arbitrary. As $\lambda \in Z^+$ is arbitrary, f is uniformly asymptotic.

Conversely let $f \in U(\lambda)\ \forall \lambda \in Z^+$. Now let $\ell < 1$. Then $\sum\limits_{n=1}^{\infty}' f(n) < \infty$ so that $f \notin U(\lambda)$ a contradiction. Thus $\ell \geq 1$. Let $\ell > 1$ so that (6) gives $f(n+1) = \ell f(n) + o(f(n))$

Now using arguments similar to (7) we have for $x = \lambda n + \mu - 1$ for some residue class mod λ

$$\sum_{0<n\leq x}' f(n) = (1+\ell+\ell^2+\cdots \ell^{\lambda-1}) \sum_{\substack{0<n\leq x \\ n\equiv\mu(\mathrm{mod}\,\lambda)}}' f(n) + o\Big(\sum_{0<n\leq x}' f(n)\Big)$$

which gives

$$\lim_{x\to\infty} \sum_{\substack{0<n\leq x \\ n\equiv\mu(\text{mod}\,\lambda)}}' f(n) \Big/ \sum_{\alpha n\leq x}' f(n) = \frac{1}{1+\ell+\ell^2+\dots\ell^{\lambda-1}} \neq \frac{1}{\lambda}$$

a contradiction to $f\in U(\lambda)$. Thus $\ell=1$ proving the theorem.

It is however not necessary for the limit ℓ to exist in (6) if $f\in U(\lambda)\ \forall \lambda\in Z^+$. We now give an example of a function which is uniformly asymptotic without limit $f(x+1)/f(x)$ existing. For real x let $[x]$ denote the largest integer $\leq x$. Let the fractional part of x denoted by $\{x\}$ stand for $x-[x]$. Let $\theta>0$ be an irrational and define a function f by

$$f(n)=\{n\theta\} \quad n\in Z^+ \qquad (8)$$

<u>Theorem 2:</u> The function f in (8) is uniformly asymptotic. We need two lemmas to prove our theorem.

<u>Lemma 3:</u> If $\alpha_1,\alpha_2\dots\alpha_n,\dots$ is a sequence with $\alpha_n\in[0,1]$ and uniformly distributed then

$$\lim_{m\to\infty}\frac{\alpha_1+\alpha_2+\dots\alpha_m}{m}=\frac{1}{2}.$$

(Note: By uniform distribution is meant the following. Let $0\leq\alpha<\beta\leq 1$ and $\varphi_n(\alpha,\beta)=\sum_{i\leq n,\ \alpha_i\in[\alpha,\beta]}' 1$ and let

$$D_n(\alpha,\beta)=\left|\frac{\varphi_n(\alpha,\beta)}{n}-(\beta-\alpha)\right|$$

If $D_n(\alpha,\beta)\to 0$ as $n\to\infty\ \forall\ 0\leq\alpha<\beta\leq 1$ then the sequence $(\alpha_n)_{n=1}^{\infty}$ is uniformly distributed or u.d. in $[0,1]$ (sec [2] for details). If for any $(\alpha_n)_{n=1}^{\infty}$ the sequence of fractional parts of α_n ie $\{\alpha_n\}_{n=1}^{\infty}$ is u.d in $[0,1]$ then $(\alpha_n)_{n=1}^{\infty}$ is u.d. mod 1)

To prove the lemma let us partition $[0,1]$ into $\left[0,\frac{1}{2N}\right],\left[\frac{1}{2N},\frac{2}{2N}\right]$ .... $\left[\frac{2N-1}{2N},\frac{1}{1}\right]$ and let $\beta_r=\frac{r}{2N}$ $r=0,1,2,3,\dots.2N$. Clearly as (α_n) is u.d. in $0,1$ we have

$$\varphi_m(\beta_{r-1},\beta_r)=\frac{m}{2N}+o(m).$$

Now

$$\frac{\alpha_1+\alpha_2+\dots\alpha_m}{m}\leq\frac{\sum_{r=1}^{2N}\beta_r(\varphi_m(\beta_{r-1},\beta_r))}{m}=\frac{\sum_{r=1}^{2N}\beta_r\left(\frac{m}{2N}+o(m)\right)}{m}$$

$$=\frac{1}{2}+\frac{1}{4N}+o(1)$$

so that

$$\limsup_{m\to\infty} \frac{\alpha_1+\alpha_2+\cdots \alpha_m}{m} \le \frac{1}{2}+\frac{1}{4N}.$$

Similarly

$$\frac{\alpha_1+\alpha_2+\cdots \alpha_m}{m} \ge \frac{\sum_{r=1}^{2N} \beta_{r-1} \varphi_m(\beta_{r-1},\beta_r)}{m} = \frac{\sum_{r=1}^{2N} \beta_{r-1}\left(\frac{m}{2N}+o(m)\right)}{m} = \frac{1}{2}-\frac{1}{4N}+o(1)$$

or $\liminf_{m\to\infty} \frac{\alpha_1+\alpha_2+\cdots\alpha_m}{m} \ge \frac{1}{2}-\frac{1}{4N}$.

Now as the choice of N is arbitrary $\limsup = \liminf$ ~~lim~~ proving lemma 3.

<u>Lemma 4:</u> If $(\alpha_n)_{n=1}^{\infty}$ is u.d. mod 1. and 'c' a constant then so is $(\beta_n)_{n=1}^{\infty}$ where $\beta_n = c+\alpha_n$.

<u>Proof:-</u> Pick α and β such that $0\le\alpha<\beta\le 1$ Find $\alpha-c$, and $\beta-c$ mod 1 and let these be γ and δ respectively. If $\gamma<\delta$ then $\delta-\gamma=\beta-\alpha$. Now $\{\beta_n\}\in[0,1]$ and $\{\alpha_n\}\in[0,1]$ with the condition that $\{\alpha_n\}\in[\gamma,\delta]$ if and only if $\{\beta_n\}\in[\alpha,\beta]$. If $\delta<\gamma$ then denote by $I_1=[0,\delta]$ and $I_2=[\gamma,1]$. We have then $\{\beta_n\}\in[\alpha,\beta]$ if and only if $\{\alpha_n\}\in I_1\cup I_2$. Now $I_1\cap I_2=\phi$ and $|I_1\cup I_2| = |I_1|+|I_2|$ (where $|I|$ denotes the length of an interval I). Clearly as $\{\alpha_n\}$ is u.d. mod 1 we have $(\beta_n)_{n=1}^{\infty}$ is u.d. mod 1 also, proving the lemma.

<u>Proof of theorem 2:</u> It is known that $\{n\theta\}$ is u.d. in $[0,1]$ (see [2] so that lemma 3 gives

$$\sum_{0<n\le x}{}' f(n) = \frac{x}{2}+o(x) \tag{9}$$

Now pick any $\lambda\in Z^+$ and let μ be a residue of λ with $0<\mu\le\lambda$

Clearly

$$f((\lambda n+\mu)\theta) = \{(\lambda n+\mu)\theta\} = \{n\lambda\theta+\mu\theta\}$$

Now $\lambda\theta$ is irrational so that $(n\lambda\theta)$ is u.d. mod 1. Use $\alpha_n = n\lambda\theta$ in Lemma 3 and $c=\mu\theta$. $\{\beta_n\} = f((\lambda n+\mu)\theta)$ is u.d. in $[0,1]$ Applying Lemma 3 we get

$$\sum_{\substack{0<n\le x\\ n\equiv\mu \pmod{\lambda}}}{}' f(n) = \frac{x}{2\lambda}+o\left(\frac{x}{\ }\right. \tag{10}$$

Now (9) and (10) together give that $f \in U(\lambda)$ as the choice of was arbitrary. As λ itself is arbitrary, f is a uniformly asymptotic function proving theorem 2.

We now study the behaviour of functions over the set of integers relatively prime to an integer. Here our interest shall be on function which can be expressed in terms of uniformly asymptotic functions. Here we come across an interesting analogue of the Riemann Zeta function. Define for $s > 1$

$$\zeta_d(s,N) = \sum_{\substack{n=1 \\ (n,N)=d}}^{\infty}{}' \frac{1}{n^s}, \quad \zeta_1(s,N) = \zeta(s,N), \quad \zeta(s,1) = \zeta(s) \qquad (11)$$

As $(n, N) = 1 \iff (d,N) = \left(\frac{n}{d}, N\right) = 1 \;\forall d \mid n$ the following are immediate deductions.

$$\sum_{\substack{n=1 \\ (n,N)=1}}^{\infty}{}' \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s,N)}; \quad \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty}{}' \frac{\tau(n)}{n^s} = \zeta(s,N)^2; \quad \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty}{}' \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1,N)}{\zeta(s,N)} \qquad (12)$$

under suitable domains of convergence, where φ denotes the Euler function $\sum_{\alpha a \leq n,\, (a,n)=1}' 1$, μ the Moebius function (see [1]) and τ the function mentioned in (1). Let us obtain the value of $\zeta(s,N)$ in terms of $\zeta(s)$. Now (11) implies that

$$\zeta(s) = \sum_{d \mid N}{}' \zeta_d(s,N) = \sum_{d \mid N}{}' \frac{1}{d^s} \zeta(s,N) = \zeta(s,N)\,\sigma_s(N)/N^s$$

which gives $\zeta(s,N) = \zeta(s)\, N^s/\sigma_s(N)$ where $\sigma_s(N)$ denotes the sum of the s^{th} powers of the divisors of N.

We give two more definitions before going to prove the theorems. In (2) if $\ell > \delta(A)$ we say f is strongly asymptotic over A. If $\ell < \delta(A)$ then f is weakly asymptotic over A. Further let R(N) denote the set of all integers relatively prime to N so that

$$\delta(R(N)) = \varphi(N)/N \quad \text{as} \quad R(N) = \bigcup_{j=1}^{\varphi(N)} A_j$$

where $A_j = \{n + \mu_j\}$ with $0 < \mu_j < N$, μ_j being the j^{th} number relatively prime to N. Note also that if $f \in U(N)$ then

$$\sum_{\substack{0<n\leq x,\ n \in R(N)}}{}' f(n) \sim \frac{\varphi(N)}{N} \sum_{0<n\leq x}{}' f(n)$$

<u>Theorem 3:</u> Let $f \in U(N)$ and $\sum\limits'_{0<n\leq x,(n,N)=1} f(n) = c_N x^s + O(x^{s-\epsilon})$ where $c_N, \epsilon > 0$ and $s > 1$. If $F(n) = \sum\limits'_{d|n} f(d)$ then F is weakly asymptotic over R(N) and

$$\lim_{m\to\infty} \frac{\sum\limits'^{m}_{n=1,(n,N)=1} F(n)}{\sum\limits'^{m}_{n=1} F(n)} = \frac{\varphi(N)\,\zeta(s,N)}{N\,\zeta(s)} \tag{13}$$

<u>Proof:</u> From the definition of F we have

$$\sum'^{m}_{n=1} F(n) = \sum'^{m}_{n=1}\sum'_{d|n} f(d) = \sum'^{m}_{d=1}\sum'^{[m/d]}_{d'=1} f(d') = \sum'^{m}_{d=1} c\left[\frac{m}{d}\right]^s + O\left(\left[\frac{m}{d}\right]^{s-\epsilon}\right)$$

$$= \sum'^{m}_{d=1}\left[c\left(\frac{m}{d}\right)^s + O\left(\left(\frac{m}{d}\right)^{s-1}\right) + O\left(\left(\frac{m}{d}\right)^{s-\epsilon}\right)\right]$$

$$= cm^s \sum'^{m}_{d=1} \frac{1}{d^s} + o(m^s)$$

(as $\sum\limits'^{m}_{d=1}\left(\frac{m}{d}\right)^{s-\epsilon} = o(m^s)$ where $s-\epsilon$ is $<, =$ or > 1 as $s>1$).

$$= cm^s\left(\sum'^{\infty}_{d=1}\frac{1}{d^s} - \sum'^{\infty}_{d=m+1}\frac{1}{d^s}\right) + o(m^s)$$

$$= cm^s\zeta(s) + o(m^s) + O(m)$$

$$= cm^s\zeta(s) + o(m^s) \tag{14}$$

Now

$$\sum'^{m}_{\substack{n=1\\(n,N)=1}} F(n) = \sum'^{m}_{\substack{n=1\\(n,N)=1}}\sum'_{d|n} f(d) = \sum'^{m}_{\substack{d=1\\(d,N)=1}}\sum'^{[m/d]}_{\substack{d'=1\\(d',N)=1}} f(d')$$

$$= \sum'^{m}_{\substack{d=1\\(d,N)=1}} \frac{\varphi(N)}{N} c\left[\frac{m}{d}\right]^s + O\left(\left(\frac{m}{d}\right)^{s-\epsilon}\right) \quad \text{as } f \in U(N)$$

$$= c\frac{\varphi(N)}{N} m^s \sum'^{\infty}_{\substack{d=1\\(d,N)=1}} \frac{1}{d^s} + o(m^s) \tag{15}$$

(for reasons similar to (14).

$$= \frac{\varphi(N)}{N} c\, m^s \zeta(s,N) + o(m^s)$$

Now (15) and (14) together give (13). And as $\zeta(s,N) < \zeta(s)$ we have the limit $< \delta(R(N))$ or F is weakly asymptotic over R(N).

Theorem 4:- Let f be a function and $F(n)=\sum_{d|n}' f(d)$. If $F \in U(N)\ \forall N \in Z^+$ and $\sum'_{0<n\le x,\,(n,N)=1} F(n) = K_N x^s + O(x^{s-\epsilon}),\ K_N, \epsilon > 0,\ s > 1$ then f is strongly asymptotic over R(N) and

$$\lim_{m\to\infty} \frac{\sum\limits_{\substack{n=1\\(n,N)=1}}^{m}{}' f(n)}{\sum\limits_{n=1}^{m}{}' f(n)} = \frac{\varphi(N)\,\zeta(s)}{N\ \zeta(s,N)} \tag{16}$$

Proof:- As $F(N)=\sum_{d|n}' f(d)$, we have from the Moebius inversion formula (see [1]) $f(n)=\sum_{d|n}' \mu(d) F(\frac{n}{d})$

Now

$$\sum_{n=1}^{m}{}' f(n) = \sum_{n=1}^{m}{}' \sum_{d|n}{}' \mu(d) F(\tfrac{n}{d}) = \sum_{d=1}^{m}{}' \mu(d) \sum_{d'=1}^{[\frac{m}{d}]}{}' F(d')$$

$$= \left(\sum_{d=1}^{m}{}' \mu(d)\left(\frac{cm^s}{d^s}\right)\right) + o(m^s) \quad \text{(for reasons similar to (14))}$$

$$= cm^s \sum_{d=1}^{\infty}{}' \frac{\mu(d)}{d^s} + cm^s \sum_{d=m+1}^{\infty}{}' O\left(\frac{1}{d^s}\right) + o(m^s)$$

$$= \frac{cm^s}{\zeta(s)} + o(m^s) \tag{17}$$

from (12). Again

$$\sum_{\substack{n=1\\(n,N)=1}}^{m}{}' f(n) = \sum_{\substack{n=1\\(n,N)=1}}^{m}{}' \sum_{d|n}{}' \mu(d) F(\tfrac{n}{d}) = \sum_{\substack{d=1\\(d,N)=1}}^{m}{}' \mu(d) \sum_{\substack{d'=1\\(d',N)=1}}^{[\frac{m}{d}]}{}' F(d')$$

$$= \left(\sum_{\substack{d=1\\(d,N)=1}}^{m}{}' \mu(d)\left(c\frac{m^s}{d^s}\frac{\varphi(N)}{N}\right)\right) + o(m^s) \quad \text{as } F \in U(N)$$

$$= cm^s \sum_{\substack{d=1\\(d,N)=1}}^{\infty}{}' \frac{\mu(d)}{d^s}\frac{\varphi(N)}{N} + O\left(cm^s \sum_{d=m+1}^{\infty}{}' \frac{1}{d^s}\right) + o(m^s).$$

$$= \frac{cm^s}{\zeta(s,N)}\frac{\varphi(N)}{N} + o(m^s)$$

Now (17) and (18) together imply (16). Now as $\zeta(s,N) \le \zeta(s)$ the limit is $> \varphi(N)/N = \delta(R(N))$ which gives that F is strongly asymptotic over R(N). The proof is complete.

We now deduce two interesting results from theorems 3 and 4.

Theorem 5:- a) $\sigma_K(n) = \sum'_{d|n} d^K$ then for $K>0$

$$\sum_{\substack{n=1 \\ (n,N)=1}}^{m}{}' \sigma_K(n) \sim \frac{m^{K+1}\varphi(N)}{(K+1)\ N}\, \mathcal{G}(K+1,N)$$

b) If $\varphi(n)$ denotes the Euler function then

$$\sum_{\substack{n=1 \\ (n,N)=1}}^{m}{}' \varphi(n) = \frac{m^2}{2\,\mathcal{G}(2,N)}$$

Proof: Set $f(x) = x^K$ in Theorem 3. Then $F(n) = \sigma_K(n)$ Here $C = \frac{1}{K+1}$ and $S = K+1 > 1$. Part (a) follows from (15). Note that $f \in U(N)$ as f is uniformly asymptotic (Lemma 2).

Set $F(x) = x$ in Theorem 4. Then $f(n) = \varphi(n)$. Here $C = 1/2$ and $S = 2$. Part b follows as $F(n) = U(N)$ for Lemma 2 gives that F is uniformly asymptotic.

Our final theorem deals with the case $\sum'_{\substack{0<n\leq x \\ (n,N)=1}} f(n) \sim C_N x^S + O(\ \ )$ where $C_N > 0$ and $S = 1$

Theorem 6:- Let $f \in U(N)_K$ $\forall N \in Z^+$ and $\sum'_{0<n\leq x, (n,N)=1} f(n) = C_N x + O(1)$, $C_N > 0$.

Let $F(n) = \sum'_{d|n} f(d)$. Then F is weakly asymptotic over R(N) and

$$\lim_{m\to\infty} \frac{\sum'^{m}_{n=1, (n,N)=1} F(n)}{\sum'_{0<n\leq m} F(n)} = \left(\frac{\varphi(N)}{N}\right)^2$$

Proof: Very much similar to the proof of Theorem 3. We omit the details but give the sketch of it.

The only change comes in (15) where $S = 1$. So $\frac{1}{d} = g(d)$ is uniformly asymptotic, (we deduce this from Theorem 1) and so an extra $\varphi(N)/N$ appears in the limit.

Corollary If $\tau(n)$ represents the function given in (1) Then

$$\sum_{\substack{n=1 \\ (n,N)=1}}^{m} \tau(n) = \left(\frac{\varphi(N)}{N}\right)^2 m \log m + O(m)$$

(a result known to Cordon and Rogers [3])

Corollary follows if we set $k = 0$ and use (1) to estimate $\sum_{n=1}^{m}{}' \tau(n)$

References: 1) W. J. LeVeque, Topics in Number Theory, Vol. 1, Addison Wesley, Reading, Mass, p.p. 100-140

2) Ivan Niven, Introduction to Diophantine Approximations, Interscience Publishers, John Wiley and Sons, New York, London 1963 p.p. 10-40

3) B. Gordon and K. Rogers, Sums of the divisor function, Canad. J. of Math., 16, 1964, p.p. 151-158

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FUNCTIONAL ANALOGUES TO DISTRIBUTION AND DENSITY+

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FUNCTIONAL ANALOGUES TO DISTRIBUTION AND DENSITY

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In this paper we discuss functional analogues to the concepts of density of integer sequences and uniform distribution in the sense of Weyl [6] and Niven [5]

Part 1:

Throughout this section whenever we refer to a function 'f' we mean a function on $[0,1]$ which has atmost a finite number of discontinuities and $f(x) > 0$, $0 \leq x \leq 1$. Let $A = \{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[0, 1)$ and 'f' a function. Let α, β be real numbers such that $0 \leq \alpha < \beta \leq 1$. We denote by

$$F_n(\alpha,\beta) = \sum_{\substack{\alpha_i \in [\alpha,\beta) \\ i \leq n}} f(\alpha_i) \tag{1.1}$$

and by

$$\mathcal{D}_n(\alpha,\beta) = \left| \frac{F_n(\alpha,\beta)}{F_n(0,1)} - (\beta-\alpha) \right| \tag{1.2}$$

If $D_n(\alpha,\beta) \to 0$ as $n \to \infty$ for all $0 \leq \alpha < \beta \leq 1$ we say that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is uniformly f-distributed in $[0,1)$ and denote it in short by A is u.d(f) in $[0,1)$. This is the central idea of this section.

If $A = \{\alpha_n\}_{n=1}^{\infty}$ denotes a sequence in $[0,1)$ we say two functions f and g are A - equivalent, (notation: $f \overset{A}{\sim} g$) if A is u.d.(f) and u.d ' in $[0,1)$. Clearly this is an equivalence relation. We shall characterise in Theorem 1.6 the relation $f \overset{A}{\sim} g$.

Note that if $f(x) = 1$ and A is u.d (f) in $[0,1)$ then A is uniformly distributed in the sense of Weyl [6]. We also apply the concept u.d. f' to numerical integration.

Let $I = [\alpha,\beta) \subset [0,1)$. Let α_{n_1} be the first number of $A = \{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ that lies in I. Let α_{n_2} be the next member of A in I,... If $\beta_i = \alpha_{n_i}$ for i = 1,2,3,... then $B = \{\beta_i\}$ is called the restriction of A to I. We begin by proving

<u>LEMMA 1:1:</u> If $A = \{\alpha_n\}_{n=1}^{\infty}$ is uniformly distributed in $[0,1)$ and $I \subset [0,1)$ then $\{\beta_i\}_{i=1}^{\infty}$ the restriction of A to I is uniformly distributed in I.

<u>Proof:</u> Let $[\alpha,\beta) \subset [0,1)$. Let α', β' be real numbers with $\alpha \leq \alpha' < \beta' \leq \beta$. Denote by

$$\varphi_m(\alpha',\beta') = \sum_{\alpha' \leq \beta_i \leq \beta',\ i \leq m}' 1 \quad ; \ \varphi_m(\alpha,\beta) = m$$

Now let

$$\mathcal{D}_m(\alpha',\beta') = \left| \frac{\varphi_m(\alpha',\beta')}{m} - \frac{\beta'-\alpha'}{\beta-\alpha} \right| = \left| \frac{\varphi_m(\alpha',\beta')}{n_m} \cdot \frac{n_m}{m} - \frac{\beta'-\alpha'}{\beta-\alpha} \right| \qquad (1.3)$$

Now as $\beta_i = \alpha_{n_i}$ we deduce that

$$\varphi_m(\alpha',\beta') = \sum_{\alpha_i \in [\alpha',\beta'],\ i \leq n_m}' 1 \qquad (1.4)$$

Now as A is u.d. in $[0,1)$, (1. indicates that $\varphi_m(\alpha',\beta')/n_m \to \beta'-\alpha'$ as $m \to \infty$. Moreover $n_m/n \to 1/(\beta-\alpha)$ so that from (3) we infer that $D_m(\alpha',\beta') \to 0$ for all $\alpha \leq \alpha' < \beta' \leq \beta$ which establishes Lemma 1.1.

For a more quantitative estimate of $D_m(\alpha',\beta')$ one can show using that

$$0 \leq \mathcal{D}_m(\alpha', \beta') \leq \sup_{(\alpha',\beta') \subset [\alpha,\beta]} \mathcal{D}_m(\alpha',\beta') = \mathcal{D}_m \leq 2 D_{n_m} \quad (1.5)$$

where D_N denotes as usual the discrepancy of the first N terms of A.

By a rational step function f on $[0,1]$ we mean a step function which has $f(x)$ rational, $0 \leq x \leq 1$, and its points of discontinuity $y_o, y_1, \ldots y_h$ are all rational.

<u>THEOREM 1.2:</u> If 'f' is a rational step function then there exist sequences which are uniformly f-distributed in $[0,1]$.

<u>Proof:</u> Let the points of discontinuity of f, $y_1 \ldots y_h$ be rational. As the y_i are rational it is possible to subdivide $[0,1]$ into intervals $I_1, I_2 \ldots I_k$ defined by points $0 = x_o < x_1 < x_2 < \ldots < x_{k-1} < x_k = 1$ where $I_r = [x_{r-1}, x_r)$, such that $|I_r| = \frac{1}{k}$, $r = 1,2,\ldots,K$, and the y_i s form a subset of the x_js. We are now sure that f is continuous in each I_k and is also constant. For $x \in I_r$ let $f(x) = q_{r/s_r}$, $r = 1,\ldots K$. Consider the rationals s_{r/q_r}, $r=1,2,\ldots K$. If $q = [q_n \ldots, q_k]$ denotes the l.c.m of q_i s them rewrite s_{r/q_r} as $p_{r/q}$ $r=1,2,\ldots K$.

Consider any sequence $A = \{\alpha_n\}_{n=1}^{\infty}$ that is u.d. in $[0,1)$. Let A_r denote the restriction of A to I_r. Clearly by Lemma 1.1 each A_r is u.d. in I_r $r=1,2,\ldots K$.

<u>Construction:-</u> Pick the first p_1 members from A_1, the first p_2 members from $A_2, \ldots$ and put them side by side with members of A_i

preceding those of A_j if $i < j$. Clearly we have $p_1 + p_2 + \dots p_k = P$ members. Repeat this performance with the members of A_i, without its first p_i members and lay these next the P members formed. Continue the process to get a sequence $\{\beta_n\}_{n=1}^{\infty}$ which is a rearrangement of A.

<u>Claim:-</u> $B = \{\beta_n\}_{n=1}^{\infty}$ is u.d. (f) in $[0,1)$

Let $0 \leq \alpha < \beta \leq 1$ and n an arbitrary integer with $n = \lambda P + \mu$, $0 \leq \mu < P$. Let $[\alpha,\beta)$ be split using the x_js as $[\alpha, x_\ell), [x_\ell, x_{\ell+1}) \dots [x_{m-1}, \beta)$. Clearly we have split up $[\alpha,\beta)$ as

$$[\alpha,\beta) = \bigcup_{r=1}^{K} \{[\alpha,\beta) \cap I_r\} = \bigcup_{r=\ell}^{m} \{[\alpha,\beta) \cap I_r\}$$

so that

$$F_n(\alpha,\beta) = \sum_{\beta_i \in [\alpha,\beta),\, i \leq n}' f(\beta_i) = \sum_{r=\ell}^{m}{}' \sum_{\substack{\beta_i \in [\alpha,\beta) \cap I_r \\ i \leq n}}' f(\beta_i)$$

which reduces to

$$F_n(\alpha,\beta) = \sum_{r=\ell}^{m}{}' \frac{q_r}{s_r} \sum_{\substack{\beta_i \in [\alpha,\beta) \cap I_r \\ i \leq n}}' 1 = \sum_{r=\ell}^{m}{}' \frac{q_r}{s_r} \varphi_n(I'_r) \qquad (1.6)$$

where $I'_r = [\alpha,\beta) \cap I_r$ and $\varphi_n(I) = \sum_{\alpha_i \in I,\, i \leq n}' 1$. Clearly for $1 < r < m$ we have

$$\varphi_n(I'_r) = \varphi_{\lambda P}(I'_r) + O(P) = \lambda p_r + O(P)$$

For $r = 1$ we note that the restriction of A to I_1 is u.d. so that

$$\varphi_n(I'_\ell) = \varphi_{\lambda P}(I'_\ell) + O(P) = \frac{x_\ell - \alpha}{|I_\ell|} \lambda p_\ell + O(P) + o(\lambda)$$

and similarly

$$\varphi_n(I'_m) = \varphi_{\lambda P}(I'_m) + O(P) = \frac{\beta - x_{m-1}}{|I_m|} \lambda p_m + O(P) + o(\lambda)$$

so (6) reduces to

$$F_n(\alpha,\beta) = \left(\sum_{r=l+1}^{m-1} \frac{q_r}{s_r}\lambda p_r\right) + \lambda p_l \frac{q_l}{s_l}\, \frac{x_l-\alpha}{|I_l|} + \lambda p_m \frac{q_m}{s_m}\, \frac{\beta - x_{m-1}}{|I_m|} + O(P) + o(\lambda)$$

$$= q\lambda(m-l+1) + \lambda q \frac{x_l-\alpha}{|I_l|} + \lambda q \frac{\beta - x_{m-1}}{|I_m|} + O(P) + o(\lambda)$$

$$= K\lambda q\,(\beta-\alpha) + O(P) + o(\lambda) \qquad (1.7)$$

where $K \doteq 1/|I_1|$. Clearly we have

$$F_n\,(0,1) = K\lambda q + O(P) + o(\lambda). \qquad (1.8)$$

Now (7) and (8) together imply thus as $n \to \infty$, $(\lambda \to \infty)$ (2) holds with α_i replaced $\{\beta_i\}$ so that $\{\beta_n\}_{n=1}^{\infty}$ is u.d. (f) in $[0,1)$ as claimed.

We now apply the concept of u.d.f to Numerical Integration. For the step function f discussed above let f^* denote

$$f(x) = \frac{p_r}{p\,|I_r|} \quad \text{when} \quad f(x) = \frac{q_r}{s_r} \qquad (1.9)$$

We call f^* as the normaliser of f.

Let $R\,[0,1]$ denote all Riemann Integrable functions in $[0,\ 1]$

We are now in a position to prove our main theorem which is

<u>THEOREM 1.3.</u> If f is a rational step function and $A = \{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ is u.d. (f), and $\phi \in R\,[0,1]$ then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \phi(\alpha_i) = \int_0^1 \phi(x)\, f^*(x)\, dx$$

<u>Proof:</u> As in Theorem 1.2. we divide $[0,1) = \bigcup_{r=1}^{k} I_r$, and $f(x) = q_r/s_r$, $x \in I_r$. The sequence β_n is renamed as α_n here. We can straightaway write

$$\frac{1}{n}\sum_{i=1}^{n}{}' \phi(\alpha_i) = \frac{1}{n}\sum_{r=1}^{K}\sum_{\alpha_i\in I_r,\, i\le n}{}' \phi(\alpha_i) \tag{1.10}$$

Clearly we have

$$\frac{\varphi_n(I_r)}{n} = \frac{p_r}{p} + o(1)$$

so we rewrite (10) as

$$\frac{1}{n}\sum_{i=1}^{n}{}' \phi(\alpha_i) = \sum_{r=1}^{K}\frac{\varphi_n(I_r)}{n}\cdot\frac{1}{\varphi_n(I_r)}\sum_{\substack{\alpha_i\in I_r\\ i\le n}}{}' \phi(\alpha_i) \tag{1.11}$$

As the restriction of A in I_j is uniformly distributed we deduce from Weyl's criterion [6] that

$$\lim_{n\to\infty}\frac{1}{\varphi_n(I_r)}\sum_{\substack{\alpha_i\in I_r\\ i\le n}}{}' \phi(\alpha_i) = \frac{1}{|I_r|}\int_{x_{r-1}}^{x_r}\phi(x)\,dx \tag{1.12}$$

On applying (12) to (11) we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}{}' \phi(\alpha_i) = \sum_{r=1}^{K}\frac{p_r}{p|I_r|}\int_{x_{r-1}}^{x_r}\phi(x)\,dx = \int_0^1 \phi(x)f^*(x)\,dx$$

proving the theorem as claimed.

<u>Corollary:</u> $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}{}' \frac{\phi(\alpha_i)}{f^*(\alpha_i)} = \int_0^1 \phi(x)\,dx$.

Now if ϕ, g^* R 0,1 (with $\int_0^1 g^*(x)dx = 1$) then one has

<u>THEOREM 1.4:</u> There exists a sequence u.d. (f) in $[0,1)$, where f is a rational step function such that

$$\left|\frac{1}{n}\sum_{i=1}^{n}{}' \phi(\alpha_i) - \int_0^1 \phi(x)g^*(x)\,dx\right| < \varepsilon$$

This is a straightforward deduction of

<u>LEMMA 1.5:</u> If $g \in R\,[0,1]$, then there is a rational step function f such that

$$\left|\int_0^1 g(x) - f(x)\,dx\right| < \delta \quad \forall\ \delta > 0$$

We omit the details of the proof.

Our final theorem of this section characterises $f \overset{A}{\sim} g$.

THEOREM 1.6: If $\{\alpha_n\}_{n=1}^{\infty} = A$ is u.d. (f) in $[0,1]$, then a necessary and sufficient condition that $f \overset{A}{\sim} g$ is that there exists a positive constant K such that $f(x) = Kg(x)$ holds for all but a finite number of x $[0,1]$.

Proof: The sufficiency is easy to establish. As $f \in R[0,1]$ and $f > 0$ we have

$$F_n(0,1) = \sum_{\alpha_i, i \leq n}{}' f(\alpha_i)$$

as a monotonic increasing sequence diverging to infinity. Thus

$$\frac{\sum_{i=1, \alpha_i \in I}^{n}{}' g(\alpha_i)}{\sum_{i=1}^{n}{}' g(\alpha_i)} = \frac{\sum_{i=1, \alpha_i \in I}^{n}{}' Kf(\alpha_i) + \varepsilon(\alpha_i)}{\sum_{i=1}^{n}{}' f(\alpha_i) + \varepsilon(\alpha_i)} \tag{13}$$

$$= \frac{\left(K \sum_{i=1, \alpha_i \in I}^{n}{}' f(\alpha_i)\right) + O(1)}{\left(K \sum_{i=1}^{n}{}' f(\alpha_i)\right) + O(1)}$$

As $\sum' f$ diverges if we proceed to the limit as $n \rightarrow$, we observe that (13) gives $f \overset{A}{\sim} g$.

Now let there be no constant K such that $f(x) = K g(x)$ for all but a finite number of x. Thus there exists a constant C such that $f(x) > Cg(x)$ or $f(x) < eg(x)$ has infinitely many solutions. For otherwise $f(x) = Cg(x)$ for all but a finite number of x which gives a contradiction. Now If both inequalities have infinitely many solutions then there are constants C', C" with C' < C" such that $f(x') = C'y(x")$ and $f(x") = C"g(x")$ at points x', x" which are points of continuity of f. Otherwise the infinity of x which have $f(x) > Cg(x)$ must have two points of continuity where $\frac{f(x)}{g(x)}$ is distinct. Denote by K the maximum of $\frac{f(x)}{g(x)}$, where x is continuous. say x_o and there is a point of continuity of f and g where $\left(f(x)/g(x)\right) = K' < K$.

Thus consider an interval I with $x_o \in I$ where $K' < \frac{f(x)}{g(x)} < K$.

For this I we have

$$\frac{\sum\limits_{i \le n,\ \alpha_i \in I}' g(\alpha_i)}{\sum\limits_{i \le n}' g(\alpha_i)} > \frac{\sum\limits_{\alpha_i \in I,\ i \le n}' (K'+\varepsilon) f(\alpha_i) + O(1)}{\sum\limits_{i \le n}' (K'+\varepsilon) f(\alpha_i) + O(1)}$$

so that

$$\liminf_{n \to \infty} f \frac{\sum\limits_{\alpha_i \in I,\ i \le n}' g(\alpha_i)}{\sum\limits_{i \le n}' g(\alpha_i)} > |I|$$

a contradiction to $\{\alpha_n\}_{n=1}^{\infty} = A$ is u.d. (g). Thus $f(x) = K\,g(x)$ for all but a finite number of x $[0,1]$ proving the theorem.

<u>Part 2:</u>

Now we take up the discussion of functional analogues to the concepts of density and distribution modulo an integer, in the sense of Niven [5]. Whenever we refer to a function 'f' in this section we mean $f(n) > 0$ $n \in Z^+ = \{1,2,3,\ldots\}$ and $\sum\limits_{0<n\le x}' f(n)$ diverges to infinity with x monotonically.

Let $A \subset Z^+$. Denote by $A_f(x)$ and $Z_f(x)$ the following

$$A_f(x) = \sum_{\substack{0<n\le x \\ n\in A}}' f(n) \quad ; \quad Z_f(x) = \sum_{\substack{0<n\le x \\ n\in Z^+}}' f(n).$$

we denote by $\delta_f(A)$ the limit of the following (if it exists)

$$\lim_{x\to\infty} \frac{A_f(x)}{Z_f(x)} = \delta_f(A) \qquad (2.1)$$

and call $\delta_f(A)$ as the f- density of A. When $f(x) = K$ then $\delta_f(A) = \delta(A)$ the natural density of A.

The members of A shall be represented by a_n, $n=1,2,3,\ldots.$ where $a_i < a_j$ if $i < j$. We only discuss sets which have infinitely many members for trivially $\delta_f(A) = 0$ when A is finite as $Z_f(x) \to \infty$ as $x \to \infty$

Now we go to the generalisation of Niven's concept of uniform distribution modulo an integer. Denote by

$$a_f(x,\mu,\lambda) = \sum_{\substack{0<n\le x,\, n\in A \\ n\equiv\mu \pmod{\lambda}}}' f(n)$$

If

$$\lim_{x\to\infty} \frac{a_f(x,\mu,\lambda)}{A_f(x)} = \frac{1}{\lambda} \qquad (2.2)$$

for all $0 \le \mu < \lambda$, $\mu \in Z^+$ we say A is uniformly f- distributed modulo λ and denote it by A is u.d. f(mod λ). Note that $\lambda \ne 1$ for $\lambda = 1$ is trivial and moreover uniform f-distribution modulo 1 (in $[0,1)$) has been introduced in § 1.

The most fundamental functions for uniform distribution happen to be functions uniformly <u>asymptotic</u> modulo λ introduced for the first time in [1] by the author. We describe them briefly. If we have

$$\lim_{x \to \infty} \frac{\displaystyle\sum_{\substack{0<n\leq x \\ n \equiv \mu \bmod \lambda}}' f(n)}{\displaystyle\sum_{0<n\leq x}' f(n)} = \frac{1}{\lambda} = \lim_{x\to\infty} \frac{Z_f(x,\mu,\lambda)}{Z_f(x)} \qquad (2.3)$$

for all $0 \leq \mu < \lambda$, $\mu \in Z^+$ then f is uniformly asymptotic modulo λ (or u.a mod in short.) By $U(\lambda)$ is meant the set of all f, u.a mod λ It was for example shown in [1] that if

$$\lim_{n \to \infty} \frac{f(n-1)}{f(n)} = 1 \qquad (2.4)$$

then $f \in U(\lambda)\ \forall \lambda \in Z^+$. However (2.4) is not a necessary condition as is demonstrated by the following example.

If θ is irrational and $f(n) = n\theta - [n\theta] = (n\theta)$, then $f \in U(\lambda)$ for all $\lambda \in Z^+$

We begin by proving

<u>THEOREM 2.1:</u> If $f \in U(\lambda)$ and $\delta_f(A) < 1$ 1, then if A is u.d. $f \pmod{\lambda}$ so is $\bar{A} = Z^+ - A$.

<u>THEOREM 2.2:</u> If $f \in U(\lambda)$ and $\delta_f(A) = 1$ then A is u.d. $f \pmod{\lambda}$.

<u>Proof:-</u> Denote by $\bar{a}_f(x,\mu,\lambda)$ and $\bar{A}_f(x)$ the following:

$$\bar{a}_f(x,\mu,\lambda) = \sum_{\substack{0<n\leq x,\, n\in\bar{A} \\ n\equiv\mu(\mathrm{mod}\,\lambda)}}' f(n) \ ; \quad \bar{A}_f(x) = \sum_{\substack{0<n\leq x \\ n\in\bar{A}}}' f(n)$$

with the above notation we deduce that

$$a_f(x,\mu,\lambda) + \bar{a}_f(x,\mu,\lambda) = Z_f(x,\mu,\lambda) \qquad (2.5)$$

Now (2.5) reduces to

$$\frac{a_f(x,\mu,\lambda)}{A_f(x)}\circ\frac{A_f(x)}{Z_f(x)}+\frac{\bar{a}_f(x,\mu,\lambda)}{\bar{A}_f(x)}\circ\frac{\bar{A}_f(x)}{Z_f(x)}=\frac{Z_f(x,\mu,\lambda)}{Z_f(x)} \tag{2.6}$$

Now as $A_f(x)/Z_f(x) = 1 - \left(\bar{A}_f(x)/Z_f(x)\right)$ we infer from (2.6) that

$$\frac{\bar{A}_f(x)}{Z_f(x)}\left(\frac{\bar{a}_f(x,\mu,\lambda)}{\bar{A}_f(x)}-\frac{a_f(x,\mu,\lambda)}{A_f(x)}\right)=\frac{Z_f(x,\mu,\lambda)}{Z_f(x)}-\frac{a_f(x,\mu,\lambda)}{A_f(x)} \tag{2.7}$$

Now if we proceed to the limit $x \to \infty$ then $\bar{A}_f(x)/Z_f(x) \Rightarrow \delta_f(\bar{A})$

Further as $f \in U(\lambda)$, $Z_f(x,\mu,\lambda)/Z_f(x) \to \frac{1}{\lambda}$ by (2.3).

If we assume A to be u.d.f(mod λ), the right side of (2.7) vanishes because of (2.2) and (2.3). But as $\delta_f(\bar{A})<1$, $\delta_f(\bar{A}) \neq 0$ as $\delta_f(\bar{A}) = 1-\delta_f(A)$. Thus $\bar{a}_f(x,\mu,\lambda)/\bar{A}_f(x) \to 1/\lambda$ as $x \to \infty$ which means Theorem 2.1 is established.

If $\delta_f(A) = 1$ then $\delta_f(\bar{A}) = 0$ so that the left side of (2.7) vanishes. Thus as $f \in U(\lambda)$ we see (2.3) holds and so (2.2) holds which means Theorem 2.2 is true.

Examples:- 1) If $A = F$ denotes the Fibonacci Sequence given by

$$F_n = F_{n-1} + F_{n-2} \qquad n \geqslant 2 \qquad F_0 = 0 \quad F_1 = 1$$

and if

$$f(F_n) = 2 \quad \text{when} \quad n \equiv 0 \pmod 3$$

$$f(F_n) = 1 \quad \text{when} \quad n \not\equiv 0 \pmod 3$$

then $A = F$ is u.d. f modulo 2.

2) If $A = \mathfrak{S}$ denotes the set of square free integers.

and if

$$f(S) = 1 \quad S \in \mathfrak{S} \qquad S \equiv 1 \pmod 2$$

$$f(S) = 2 \quad S \in \mathfrak{S} \qquad S \equiv 2 \pmod 2$$

then $\mathfrak{S}$ is u.d. f modulo 2.

<u>THEOREM 2.3</u> If $A \subset Z^+$ and 'f' a function such that $x \sim y \Rightarrow f(x) \sim f(y)$ x, y Z^+ then $\delta_f(A)$ exists and is equal to $\delta(A)$, *if $\delta(A) \neq 0$ exists*

<u>Proof:</u> It is obvious that

$$\lim_{n \to \infty} \frac{f(n+1)}{f(n)} = 1 \tag{2.8}$$

We shall make 'f' a continuous function by the following process. For $n < x < n+1$, $n \in Z^+$ define $f(x)$ as satisfying

$$\frac{f(x) - f(n)}{x - n} = \frac{f(n+1) - f(n)}{1} \tag{2.9}$$

Clearly from the definition of f in (2.8) we have either

$$f(n) \leq \int_n^{n+1} f(x)\,dx \leq f(n+1)$$

or

$$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)$$

Now (2.8) implies that we can write

$$f(n) = \int_{n-1}^{n} f(x)\,dx + o\,(f(n)).$$

which gives

$$\sum_{0<n\leq x}{}' f(n) = \int_0^x f(x)\,dx + \sum_{0<n\leq x}{}' o(f(n))$$

$$= \int_0^x f(x)\,dx + o\left(\sum_{0<n\leq x}{}' f(n)\right) = Z_f(x) \tag{2.10}$$

as $Z_f(x) \to \infty$ as $x \to \infty$. If $\delta(A) = \delta \neq 0$ and a_n the

n^{th} member of A then

$$a_n = n\bar{\delta} + o(n) \qquad (2.11)$$

where $\bar{\delta} = 1/\delta$. Clearly from (2.11) we have

$$\sum_{\substack{0<n\le x \\ n\in A}}' f(n) = \frac{1}{\quad} \sum_{\substack{0<n\le A(x) \\ 0<n\bar{\delta}\le x}}' f(n\bar{\delta}) + o\Big(\sum_{0<n\bar{\delta}\le x}' f(n\bar{\delta})\Big) \qquad (2.12)$$

Now one can show that

$$f(n\bar{\delta}) = \frac{1}{\bar{\delta}} \int_{(n-1)\bar{\delta}}^{n\bar{\delta}} f(x)\,dx + o(f(n\bar{\delta})) \qquad (2.13)$$

so that arguments similar to those of (2.10) gives on putting together (2.11) and (2.12)

$$A_f(x) = \sum_{\substack{0<n\le x \\ n\in A}}' f(n) = \delta(A)\int_0^{A(x)\bar{\delta}} f(x)\,dx + o\Big(\sum_{0<n\bar{\delta}\le x}' f(n\bar{\delta})\Big) \qquad (2.14)$$

Now as $x\sim y \Rightarrow f(x)\sim f(y)$ and as $Z_f(x)\to\infty$ as $x\to\infty$ one can show that

$$\lim_{t\to\infty}\frac{f(t)}{Z_f(t)} = \lim_{t\to\infty}\frac{f(t)}{\int_0^t f(t)dt} = 0 \qquad (2.15)$$

which is the same as saying $t\sim t'$ gives

$$Z_f(t) \sim Z_f(t')$$

and
$$\int_0^t f(x)\,dx \sim \int_0^{t'} f(x)dx$$

For a proof of (2.15) see [1]. Thus (2.14) by virtue of (2.15) reduces to

$$A_f(x) = \delta(A)\int_0^x f(x)\,dx + o\Big(\sum_{0<n\le x}' f(n)\Big) \qquad (2.16)$$

Clearly from (2.10) and (2.16) we infer

$$\lim_{x\to\infty} \frac{A_f(x)}{A_f(x)} = \delta_f(A) = \delta(A)$$

which establishes theorem 2.3.

~~One can also show on similar lines of reasoning the converse of theorem 2.3.~~

~~THEOREM 2.4:- Let $A \subset Z^+$ and 'f' a function with x~~

~~$x \sim y \Rightarrow f(x) \sim f(y)$, $x, y \in Z^+$. If $\delta_f(A)$ exists and is non-zero, then so does $\delta(A)$ and $\delta(A) = \delta_f(A)$.~~

~~Actually theorems 2.3 and 2.4 imply~~

THEOREM 2.5:- If $x \sim y \Rightarrow f(x) \sim f(y)$ and $A \subset Z^+$ is u.d. mod λ, with $\delta(A) \neq 0$ then A is u.d.f mod λ. ~~Conversely if A is u.d.f mod λ, then A is u.d.(mod λ)~~

PART 3 :-

We now go back to sequences $A = \{\alpha_n\}_{n=1}^{\infty} \in [0,1]$ that are u.d. (f) (where by f we mean a function with $f(x) > 0$, and at most a finite number of discontinuities, in the sense of § 1. We discuss analogues to (2.1) and (2.2) in the present section.

If λ be any modulus and $\mu \in Z^+$ with $0 \leq \mu < \lambda$, denote by $A_\mu = \{a_{\lambda n+\mu}\}_{n=1}^{\infty}$. If each A_μ, $\mu = 0, 1, \ldots \lambda-1$ is u.d. (f) in $[0,1]$ we say that $A = \{\alpha_n\}$ is uniformly distributed in $[0,1]$ strongly mod λ (notation: A is u.d. (f) in $[0,1]$ S. (mod λ).)

For any sequence $B = \{\alpha_n \in A \mid n \in A' \subset Z^+\}$, and $\bar{B} = \{\alpha_n \in A \mid n \in Z^+ - A\}$ denote by $\delta_f(B)$ and $\delta_f(\bar{B})$ the following limits if they exist

$$\delta_f(B) = \lim_{n\to\infty} \frac{\varphi_n(0,1)}{F_n(0,1)} \quad ; \quad \delta_f(B') = \lim_{n\to\infty} \frac{\overline{\varphi_n}(0,1)}{F_n(0,1)} \qquad (3.1)$$

where

$$F_n(0,1) = \sum_{i \leq n} f(\alpha_i) \quad ; \quad \varphi_n(0,1) = \sum_{\substack{i \leq n \\ \alpha_i \in B}} f(\alpha_i), \quad \overline{\varphi}_n(0,1) = \sum_{\substack{i \leq n \\ \alpha_i \in \overline{B}}} f(\alpha_i)$$

Cle-rly as $\overline{B} = A - B$, $\delta_f(B) + \delta_f(\overline{B}) = 1$. Let A be u.d. (f) in $[0,1]$

THEOREM 3.1: If $\delta_f(B) < 1$ and B is u.d. f in $[0,1]$ then So is $\overline{B}$.

THEOREM 3.2: If $\delta_f(B) = 1$ ~~and~~ then B is u.d. (f) in $0,1$ ...

Proof: With the usual notation we have

$$\varphi_n(\alpha,\beta) + \overline{\varphi}_n(\alpha,\beta) = F_n(\alpha,\beta)$$

so that we deduce

$$\frac{\varphi_n(\alpha,\beta)}{\varphi_n(0,1)} \cdot \frac{\varphi_n(0,1)}{F_n(0,1)} + \frac{\overline{\varphi}_n(\alpha,\beta)}{\overline{\varphi}_n(0,1)} \cdot \frac{\overline{\varphi}_n(0,1)}{F_n(0,1)} = \frac{F_n(\alpha,\beta)}{F_n(0,1)} \tag{3.2}$$

Now as (3.1) indicates that

$$\frac{\varphi_n(0,1)}{F_n(0,1)} = 1 - \frac{\overline{\varphi}_n(0,1)}{F_n(0,1)}$$

we rewrite (3.2) as

$$\frac{\overline{\varphi}_n(0,1)}{F_n(0,1)} \left[\frac{\overline{\varphi}_n(\alpha,\beta)}{\overline{\varphi}_n(0,1)} - \frac{\varphi_n(\alpha,\beta)}{\varphi_n(0,1)} \right] = \frac{F_n(\alpha,\beta)}{F_n(0,1)} - \frac{\varphi_n(\alpha,\beta)}{\varphi_n(0,1)} \tag{3.3}$$

Clearly as $n \to \infty$ $F_n(\alpha,\beta)/F_n(0,1) \to \beta - \alpha$. Moreover $\overline{\varphi}_n(0,1)/F_n(0,1) \to \delta_f(\overline{B})$. If $\delta_f(B) < 1$ then and B is u.d. f, $[0,1]$ then as $\delta_f(\overline{B}) \neq 0$ we infer theorem 3.1. If $\delta_f(B) = 1$ then $\delta_f(\overline{B}) = 0$ so that theorem 3.2 is true.

We return to the above theorems after proving

THEOREM 3.3: If $A = \{\alpha_n\}_{n=1}^{\infty}$ is u.d. (f) in $[0,1]$ S (mod λ)

and $\lambda' \in Z^+$ divides λ , then A is u.d. (f) in $[0,1]$ S. mod λ' .

Proof: Theorem 3.3 is a direct consequence of a concept we call blending of sequences. If $S_1, S_2, \ldots S_k$ are K sequences whose n^{th} terms are represented by $S_{r,n}$, $n = 1,2, \ldots \infty$ $r = 1,2,\ldots k$. Define a sequence

$$\mathcal{S} = s_{\lambda K+\mu} = S_{\mu,\lambda}, \quad 0 \le \mu < K, \lambda = 1,2,\ldots\infty$$

$\mathcal{S}$ is called a 'blending' of S_1, S_2, ...S_k . Clearly if each S_i, $i = 1,2,\ldots k$ is u.d. (f) in $[0,1]$, S is also u.d. (f) in $[0,1]$. In the above theorem have A_μ, $\mu = 0,1,2\ldots\lambda-1$ as u.d. (f) in $[0,1]$. Now there are exactly λ/λ' classes mod λ that leave a remainder μ' (mod λ'). These classes determine sets $A'_{\mu_1}, A_{\mu_2}\ldots A_{\mu(\lambda/\lambda')}$ which when blended give $A'_{\mu'} = \{\alpha_{\lambda' n+\mu'}\}_{n=1}^{\infty}$

Thus $A'_{\mu'}$ is u.d. (f) in $[0,1]$ for $\mu' = 0,1,2,\ldots\lambda'-1$, proving the theorem.

We have as a corollary

COROLLARY:- If $A = \{\alpha_n\}_{n=1}^{\infty}$ is u.d. (f) in $[0,1]$. Strongly (mod λ) then $\{\alpha_n\} = A$ is u.d. (f)

Now the above corollary together with theorem 3.1 gives

THEOREM 3.4:- If B denotes the union of some of the A_μ, and $\bar{B} = A - B$, then both B, and $\bar{B}$ are u.d. (f) in $[0,1]$.

We omit the details of the proof.

Our next question is obvious. Are there sequences (given an f) that are u.d. f S(mod λ) for some λ. Consider the step function f in $[0,1]$ and the number P we defined. Define a cyclic operation

$$C(\vec{x}) = C(x_1, x_2, \ldots x_p) = (x_p, x_1 \ldots x_{p-1})$$

$$\text{and} \quad C^\lambda (x_1, x_2, \ldots x_p) = C\ (C \ldots \ (\vec{x}) \ldots \text{imes.}$$

Rearrange the $_i$ so constructed cyclically mod as follows. Define for $\lambda \in Z^+$

$$(\gamma_{(\lambda-1)P+1}, \ldots \gamma_{\lambda P}) = C^\lambda(\beta_{(\lambda-1)P+1}, \ldots \beta_{\lambda P})$$

One can show on rather straightforward but laborious computation that if $\{\alpha_n\}$ were u.d. in $[0,1]$ S(mod P) then γ_n is u.d. (f) in $[0,1]$ s. mod p. (Note. Whenever f(x)=1 we omit mentioning f). Here one has only to show that if $A = \{\alpha_n\}$ is u.d. in $[0,1]$ S. mod P, and $I \subset 0,1$ then the restriction of A to I is also u.d. in I, S. (mod P). Thus we have

THEOREM 3.5 If $A = \{\alpha_n\}_{n=1}^\infty$ is u.d. in $[0,1]$ s mod P and f a rational step function (with P as defined in Theorem 1.1) then A can be rearranged so as to be u.d. f in $[0,1]$ od P. We conclude by producing sequences in 0,1 that are u.d. S (mod)

We observe that the two most common sequences possess this property.

THEOREM 3.6:- If θ is irrational and $\alpha_n = n\theta - [n\theta] = (n\theta)$ then $\{\alpha_n\}$ is u.d. in $[0,1]$ S mod λ $\forall \lambda \in Z^+$.

Proof: We observe that

$$(\{\lambda n + \mu\}\theta) = (n\{\lambda\theta\} + \mu\theta)$$

Now $\lambda\theta$ is irrational and $\mu\theta$ a constant. Thus $(\{\lambda n + \mu\}\theta)$

is u.d. in $[0,1]$ or $\{\alpha'_n\}$ is u.d. in $[0,1]$ s. mod λ $\forall \lambda \in Z^+$.

The final theorem is more complicated to prove. Write the sequence of rationals in $[0,1]$ in natural order. To be more precise the rationals in the Farey Sequence of order n (see [3]) are written in ascending order and precede those of the Farey Sequence of order n+1. This set is denoted by $\{r_n\}_{n=1}^{\infty}$. We now show

THEOREM 3.7:- The sequence $\{r_n\}_{n=1}^{\infty}$ is u.d. in $[0,1]$ s. (mod λ) $\forall \lambda \in Z^+$.

Proof:- Let $\varphi(n)$ denote the Euler φ function. We know 3

$$\Phi(m) = \sum_{n=1}^{m} \varphi(n) = \frac{3m^2}{\pi^2} + O(m \log m) \tag{3.4}$$

Thus $\Phi(m+1)/\Phi(m) \to 1$ as $m \to \infty$. Let λ, μ be given integers with $0 \leq \mu < \lambda$. Denote for $[\alpha, \beta] \subset [0,1]$

$$\varphi_n(\alpha, \beta) = \sum_{\substack{r_i \in [\alpha, \beta),\ i \leq n \\ i = \mu (\mathrm{mod}\,\lambda)}}' 1 \tag{3.5}$$

and

$$\psi_n(\alpha, \beta) = \sum_{r_i \in [\alpha, \beta),\ i \leq n}' 1 . \tag{3.6}$$

One has from (3.5) and (3.6)

$$\psi_n(0,1) = n, \text{ and } \varphi_n(0,1) = [(n-\mu)/\lambda]$$

Now consider a rational j/m' with fixed denominator m'. Clearly the number of such m' such that $j/m' \in [\alpha, \beta)$ is $\varphi(m', \beta m') - \varphi(m', \alpha m')$

where

$$\varphi(n, x) = \sum_{\substack{0 < a \leq x \\ (a,n)=1}} 1$$

The number of these j/m' that are of the form r_i, $i \equiv \mu \pmod{\lambda}$ is

$$\frac{\varphi(m', \beta m') - \varphi(m', \alpha m')}{\lambda} + O(1) \tag{3.7}$$

Now summing (3.7) with m' from 1 to m we get

$$\varphi_{\Phi(m)}(\alpha, \beta) = \frac{\psi_{\Phi(m)}(\alpha,\beta)}{\lambda} + O(m) \tag{3.8}$$

Thus

$$\mathcal{D}_{\Phi(m)}(\alpha,\beta) = \left| \frac{\varphi_{\Phi(m)}(\alpha,\beta)}{\varphi_{\Phi(m)}(0,1)} - (\beta-\alpha) \right| = \left| \frac{\frac{\psi_{\Phi(m)}(\alpha,\beta)}{\lambda} + O(m)}{(\Phi(m)-\mu)/\lambda} - (\beta-\alpha) \right|$$

$$= \left| O\left(\frac{1}{m}\right) + \frac{\psi_{\Phi(m)}(\alpha,\beta)}{\Phi(m)} - (\beta-\alpha) \right| \leq O\left(\frac{1}{m}\right) + D_{\Phi(m)}(\alpha,\beta) \tag{3.9}$$

where $D_N(\alpha,\beta)$ represents the discrepancy in (α,β) of the first N terms of r_i. Clearly

$$D_{\Phi(m)}(\alpha,\beta) \leq 2D_{\Phi(m)} = O\left(\frac{1}{m}\right) \tag{3.10}$$

(see Niederreiter [4] for details) so that (3.9) reduces to

$$\mathcal{D}_{\Phi(m)}(\alpha,\beta) = O\left(\frac{1}{m}\right) \tag{3.11}$$

thus

$$\mathcal{D}_{\Phi(m)} = \sup_{0\leq\beta\leq 1} \mathcal{D}_{\Phi(m)}(0,\beta) = O\left(\frac{1}{m}\right)$$

For any integer n, there exists m such that $\Phi(m) \leq n < \Phi(m+1)$

Now (3.4) indicates that $n - \Phi(m) = O(m)$ so that (3.8) takes a more general form

φ

$$\varphi_n(\alpha,\beta) = \frac{\psi_n(\alpha,\beta)}{\lambda} + O(m) \tag{3.12}$$

so compatation similar to (3.10) yields

$$\mathcal{D}_n(\alpha,\beta) = O\left(\frac{1}{m}\right) \tag{3.13}$$

or

$$\mathcal{D}_n = \sup_{0\leq\beta\leq 1} \mathcal{D}_n(\alpha,\beta) = O\left(\frac{1}{m}\right)$$

Now (3.13) and (3.12) together give that $\mathcal{D}_n(\alpha,\beta) \to 0$ as $n \to \infty$ or r_i, $i \equiv \mu \pmod{\lambda}$ is u.d. in $[0,1]$ proving theorem.

Moreover we have on observing (3.13), (3.11) and (3.4)

$$\mathcal{D}_n = O\left(\frac{1}{\sqrt{n}}\right)$$

This completes the proof.

REFERENCES

1) K. Alladi, On the asymptotic distribution of functions modulo on integer, submitted for publication

2) K. Alladi, On a generalisation of the Euler function, Prof. S. Ramanujan Com. Vol, Madras, India 1974 p.p.

3) W. J. LeVeque, Topics in Number Theory, Vol. 1, Addison Wesley, Reading Mass. 1960

4) H. Niederreiter, The Distribution of Farey Points, Math. Ann. 201, 1973, pp. 341-345

5) I. Niven, Uniform distribution of sequences of integers, Trans. Amer. Math. Soc., 1961, Vol. 98, No. 1, pp. 52-61

6) H. Weyle, Über die Gleichverteilung von Zahlen modulo eins, Math. Ann, Vol. 77 (1916) pp. 313-352

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A FUNCTIONAL ANALOGUE TO KOKSMA'S INEQUALITY

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This is an addenda to "Functional Analogues to Distribution and Density" [1] by the same author, and so we refer to this paper for necessary background. We actually be concerned with §1 of [1].

Let $R[0,1]$ denote the set of Riemann Integrable functions in $[0,1]$, and let $\varphi \in R[0,1]$. Let f be a rational step function with normaliser f^*, and $A = \{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ is u.d. (f).

It was shown in [1] that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1} \varphi(\alpha_i) = \int_0^1 \varphi(x) f^*(x)\, dx \qquad (1)$$

Now (1) can be rewritten as

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{\varphi(\alpha_i)}{f^*(\alpha_i)} = \int_0^1 \varphi(x)\, dx \qquad (2)$$

For a subinterval $[\alpha, \beta)$ of $[0,1)$ if φ represents the characteristic function of $[\alpha, \beta)$ then (2) implies that A is u.d$\left(\frac{1}{f^*}\right)$ in $[0,1]$. This also follows from theorem 1.6 as

$$\frac{1}{f^*} = K f \quad \text{and so} \quad f \overset{A}{\sim} \frac{1}{f^*}$$

Let now $A^*(t,N) = \sum_{\substack{i=1 \\ \alpha_i \le t}}^{N} \frac{1}{f^*(\alpha_i)}$, $F^*(N) = \sum_{i=1}^{N} \frac{1}{f^*(\alpha_i)}$

and

$$R_N^*(t) = \frac{A^*(t,N)}{F^*(N)} - t \quad \text{and} \quad P_N^*(t) = \frac{A^*(t,N)}{N} - t$$

Denote by

$$D_N^* = \sup_{0\le t<1} |R_N^*(t)| \quad \text{and by} \quad \delta_N^* = \sup_{0\le t<1} |\rho_N^*(t)|$$

Clearly we have $\lim_{N\to\infty} D_N^* = \lim_{N\to\infty} \delta_N^* = \lim_{N\to\infty} D_N = 0,$

when A is u.d. (f), where D_N stands for

$$D_N = \sup_{0\le t<1} D_N(t)$$

where $D_n(t) = D_n(0,t)$ as in [1]. With the above notation

it is clear that (2) can be restated in an equivalent from

as.
$$\lim_{nN\to\infty} \frac{1}{F^*(N)} \sum_{i=1}^{n} \frac{\varphi(\alpha_i)}{f^*(\alpha_i)} = \int_0^1 \varphi(x)\,dx \qquad (2^*)$$

We now show that for $\varphi \in R\,[0,1]$ with bounded variation $V(\varphi)$

<u>THEOREM 1:</u>
$$\left| \int_0^1 \varphi(t)\,dt - \frac{1}{F^*(N)} \sum_{i=1}^{N} \frac{\varphi(\alpha_i)}{f^*(\alpha_i)} \right| \le V(\varphi) D_N^*$$

<u>Proof:</u> As $\varphi \in R[0,1]$ we see that

$$\int_0^1 R_N(t)\,d\varphi(t) = \int_0^1 \frac{A^*(N,t)}{F^*(N)}\,d\varphi(t) - \int_0^1 t\,d\varphi(t) = I_1 - I_2.$$

Plainly using integration by parts.

$$I_2 = \varphi(1) - \int_0^1 \varphi(t)\,dt \qquad (3)$$

Define a function $c^*(t,x)$ as

$$c^*(t,x) = \frac{1}{f^*(x)} \qquad x < t$$

$$= 0 \quad \text{Otherwise}$$

so that

$$I_1 = \frac{1}{F^*(N)} \int_0^1 \sum_{i=1}^{N} C^*(t,\alpha_i)\, d\varphi(t) = \frac{1}{F^*(N)} \sum_{i=1}^{N} \int_0^1 C^*(t,\alpha_i)\, d\varphi(t)$$

$$= \frac{1}{F^*(N)} \sum_{i=1}^{N} \int_{\alpha_i}^{1} \frac{1}{f^*(\alpha_i)}\, d\varphi(t) = \frac{1}{F^*(N)} \sum_{i=1}^{N} \left\{ \frac{\varphi(1)}{f^*(\alpha_i)} - \frac{\varphi(\alpha_i)}{f^*(\alpha_i)} \right\}$$

$$= \varphi(1) - \frac{1}{F^*(N)} \sum_{i=1}^{N} \frac{\varphi(\alpha_i)}{f^*(\alpha_i)} \qquad (4)$$

Now (4) and (3) together imply

$$\left| \int_0^1 R_N^*(t)\, d\varphi(t) \right| = \left| \frac{1}{F^*(N)} \sum_{i=1}^{N} \frac{\varphi(\alpha_i)}{f^*(\alpha_i)} - \int_0^1 \varphi(t)\, dt \right| \qquad (5)$$

so that Theorem 1 follows from (5) by the definition of $V(\varphi)$ and D_N^* .

If we had instead considered

$$\int_0^1 \rho_N^*(t)\, d\varphi(t)$$

then computation similar to (3), (4) and (5) would yield

<u>THEOREM 2:</u>

$$\left| \frac{1}{N} \sum_{i=1}^{N} \frac{\varphi(\alpha_i)}{f^*(\alpha_i)} - \int_0^1 \varphi(t)\, dt \right| \leq V(\phi)\delta_N^* + \varphi(1)\,\delta_N^*(1)$$
$$\leq \{V(\varphi) + \varphi(1)\}\,\delta_N^*$$

We omit the details of the proof. Actually theorem 2 is a quantitative estimate of (2).

If we set $f(x) = K$ so that $f^*(x) = 1$ $0 \leq x \leq 1$, then Koksma's inequality follows from Theorem 1. The case $f(x) = K$ corresponds to uniform distribution in the sense of Weyl.

REFERENCE:

1) K. Alladi, Functional Analogues to distribution and density.

This note is actually part of [1], and will be incorporated when [1] is written in revised form.

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10.1.75

ON GENERALIZED EULER FUNCTIONS AND RELATED TOTIENTS

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ON GENERALIZED EULER FUNCTIONS AND RELATED TOTIENTS

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In this paper we discuss two generalizations of the Euler function $\varphi(n)$ and use these functions to make estimates of the averages connected with the greatest common divison (a,b) and the least common multiple $[a,b]$ of two integers 'a' and 'b'

§ 1

Define for real $r \geqslant 1$ a function φ_r by

$$\sum_{d|n} \varphi_r(d) = n^r \tag{1.1}$$

Clearly from (1.1) we infer by Moebius Inversion [4]

$$\varphi_r(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^r \tag{1.2}$$

where μ is the Mobius function. For integral values of r, $\varphi_r(n)$ is Jordan's function $J_r(n)$ (see [2]) which can be written in more general form

$$\varphi_r(n, x_1, x_2, \dots x_r) = \sum_{\substack{(a_1, a_2, \dots a_r, n) = 1 \\ a_1 \le x_1,\ a_2 \le x_2, \dots a_r \le x_r}}{}' 1 \qquad 0 < a_i,\ i = 1, 2, \dots r \tag{1.3}$$

with the notation

$$\varphi_r(n, x) = \varphi_r(n, x_1, x_2, \dots x_r),\quad x_i = x,\ i = 1, \dots, r$$
$$\varphi_r(n, n) = \varphi_r(n) \tag{1.4}$$

If $n = \prod_{i=1}^{s} p_i^{\alpha_i}$ one can deduce from (1.3) the following

$$\varphi_r(n, x_1, x_2, \dots x_r) = [x_1][x_2] \dots [x_r] - \sum_{0 < i \le s}{}' \left[\frac{x_1}{p_i}\right]\left[\frac{x_2}{p_i}\right] \dots \left[\frac{x_r}{p_i}\right]$$
$$+ \sum_{0 < i < j \le s}{}' \left[\frac{x_1}{p_i p_j}\right]\left[\frac{x_2}{p_i p_j}\right] \dots \left[\frac{x_r}{p_i p_j}\right] - \dots \tag{1.5}$$

where $[x]$ represents for real x the largest integer $\leq x$. Now (1.4) and (1.5) together imply that for integral r

$$\varphi_r(n,x) = \sum_{d|n} \mu(d) \left[\frac{x}{d}\right]^r \quad (1.6)$$

Then we can define $\varphi_r(n,x)$ for all real r using (1.6) so that Moebius inversion for two variables again indicates that

$$\sum_{d|n} \varphi_r\left(\frac{n}{d}, \frac{x}{d}\right) = [x]^r \quad (1.7)$$

One can show (1.7) and (1.6) to be equivalent from Moebius Inversion given below. If

$$F(n,x) = \sum_{d|n} f\left(\frac{n}{d}, \frac{x}{d}\right)$$

then

$$f(n,x) = \sum_{d|n} \mu(d) \; F\left(\frac{n}{d}, \frac{x}{d}\right)$$

Actually (1.7) indicates that

$$\varphi_r(n,x) = \frac{x^r}{n^r} \varphi_r(n) + O(x^{r-1} \tau(n)) \quad (1.8)$$

and

$$\varphi_r(n,x_1,x_2,\ldots x_r) = \frac{x_r}{n} \varphi_r(n,x_1,x_2,\ldots x_{r-1},n) + O(\varphi_r(r,x_1,x_2,\ldots x_{r-1})) \quad (1.9)$$

where $\tau(n)$ represents the number of divisors of n. We begin by making an asymptotic estimate of $\varphi_r(n)$.

THEOREM 1.

$$\sum_{0<n\leq x} \varphi_r(n) = \frac{x^{r+1}}{(r+1)\,\zeta(r+1)} + O(x^r \log x)$$

Proof. We have by (1.2)

$$\sum_{0<n\leq x}{}' \varphi_r(n) = \sum_{0<n\leq x}{}' \sum_{d|n}{}' \mu(d)\left(\frac{n}{d}\right)^r = \sum_{0<d\leq x} \mu(d) \sum_{0<d'\leq x/d}{}' d'^r$$

$$= \sum_{0<d\leq x}{}' \mu(d)\left\{\left(\frac{x}{d}\right)^{r+1}\Big/(r+1) + O\left(\left(\frac{x}{d}\right)^r\right)\right\}$$

$$= \frac{x^{r+1}}{r+1} \sum_{0<d\leq x}{}' \frac{\mu(d)}{d^{r+1}} + O\left(x^r \sum_{0<d\leq x}{}' \frac{\mu(d)}{d^r}\right)$$

$$= \frac{x^{r+1}}{r+1}\left\{\sum_{d=1}^{\infty}{}' \frac{\mu(d)}{d^{r+1}} - \sum_{d>x}{}' \frac{\mu(d)}{d^{r+1}}\right\} + O(x^r \log x)$$

$$= \frac{x^{r+1}}{(r+1)\,\zeta(r+1)} + O\left(x^{r+1} \sum_{d>x}{}' \frac{1}{d^{r+1}}\right) + O(x^r \log x)$$

$$= \frac{x^{r+1}}{(r+1)\,\zeta(r+1)} + O(x^r \log x)$$

Theorem 1 will enable us to make an estimate of the average of $\varphi_r(n)/n^r$ once we use Abels Summation formula given below.

LEMMA 1. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a monotonic increasing of real numbers, $\lambda_n \to \infty$ as $n \to \infty$, $\{c_n\}_{n=1}^{\infty}$ a sequence of real or complex numbers. Let 'f' be a function with a continuous derivative in $[\lambda_1, \infty)$ and denote by

$$C(x) = \sum_{\lambda_n \leq x} c_n.$$

Then

$$\sum_{\lambda_n \leq x} c_n f(\lambda_n) = C(x) f(x) - \int_{\lambda_1}^{x} C(t) f'(t)\,dt.$$

For a proof of Lemma 1 (see [4]). If we set $\lambda_n = n$, $f(x) = 1/x^r$ and $c_n = \varphi_r(n)$ then Lemma 1 and Theorem 1 together give

THEOREM 2. $$\lim_{x\to\infty} \frac{1}{x} \sum_{0<n\le x}{}' \frac{\varphi_r(n)}{n^r} = \frac{1}{\zeta(r+1)}$$

Note that if $\sigma_r(n)$ denotes $\sum_{d|n} d^r$ then one can show (see [2])

THEOREM 2*. $$\lim_{x\to\infty} \frac{1}{x} \sum_{0<n\le x}{}' \frac{\sigma_r(n)}{n^r} = \zeta(r+1)$$

From this we infer that $\varphi_r(n)$, n^r, $\sigma_r(n)$ are roughly geometry. Actually $\varphi_r(n)$ and $\sigma_r(n)$ have lot of connections. One can show for integral r the non-trivial result

$$\varphi_r(n)\,\sigma_r(n) \leqslant n^{2r} \tag{1.10}$$

As r is an integer (1.3) reveals that for any $d\ n$

$$\varphi_r(d, dn) \geq \varphi_r(n, dn)$$

so that we have trivially

$$\sum_{d|n} \varphi_r(d, dn) \geqslant \sum_{d|n} \varphi_r(n, dn)$$

which on observing (1.2) can be written as

$$\sum_{d|n} \varphi_r(d)\, n^r \geqslant \sum_{d|n} d^r\, \varphi_r(n)$$

or

$$n^{2r} \geqslant \varphi_r(n)\,\sigma_r(n)$$

from (1.1) and so (1.10) is true. As it is known that

$$\sigma_r(n) = O(n^r \log\log n)$$

we have from (1.10) the following

THEOREM 3. For all integral r, there exists a constant c_r such that

$$\varphi_r(n) > \frac{c_r\, n^r}{(\log\log n)}$$

We now make an estimate of the average error involved in the approximation given in (1.8). Denote by $e_r(n,x)$

$$e_r(n,x) = \frac{x^r}{n^r}\,\varphi_r(n) - \varphi_r(n,x)\,.$$

THEOREM 4. For any pair of integers $r, i > 0$ we have

$$\lim_{m\to\infty} \frac{1}{m}\sum_{n=1}^{m}{}' e_r(n,i) = \frac{i^r}{\zeta(r+1)} - \sum_{j=1}^{i}{}' \frac{\varphi_r(j,j,i,i,\dots i)}{j}$$

Proof. We know that

$$\frac{1}{m}\sum_{n=1}^{m}{}' e_r(n,i) = \frac{i^r}{m}\sum_{n=1}^{m}{}' \frac{\varphi_r(n)}{n^r} - \frac{1}{m}\sum_{n=1}^{m}{}' \varphi_r(n,i) \qquad (1.11)$$

We know from Theorem 2

$$\lim_{m\to\infty} \frac{1}{m}\sum_{n=1}^{m}{}' \frac{\varphi_r(n)}{n^r} = \frac{1}{\zeta(r+1)}$$

So we only have to estimate the second summation in (1.11). We have

$$\frac{1}{m}\sum_{n=1}^{m}{}' \varphi_r(n,i) = \frac{1}{m}\sum_{n=1}^{m}{}' \sum_{\substack{(a_1,a_2,\dots,a_r,n)=1\\ 0<a_j\le i,\ j=1,2,\dots r}}{}' 1$$

$$= \frac{1}{m}\sum_{a_1=1}^{i}{}' \sum_{\substack{(n,a_2,\dots a_r,a_1)=1\\ n\le m,\ a_j\le i\\ j=2,3,\dots r}}{}' 1$$

$$= \frac{1}{m}\sum_{a_1=1}^{i}{}' \varphi_r(a_1,i,i,\dots i,m)$$

$$= \frac{1}{m} \sum_{a_1=1}^{m}{}' \frac{m}{a_1} \varphi_r(a_1, i, i, \dots i, a_1) + O\left(\varphi_r(a_1, i, i, \dots i)\right)$$

$$= \sum_{j=1}^{i}{}' \frac{\varphi_r(j, j, i, i, \dots i)}{j} + O\left(\frac{1}{m}\right)$$

If we proceed to the limit $m \to \infty$ we get theorem 4. For the case $r = 1$ theorem 4 reduces to the simple form (see [1])

$$\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m}{}' \epsilon_1(n, i) = \frac{6i}{\pi^2} - \sum_{j=1}^{i}{}' \frac{\varphi(j)}{j} = o(i)$$

We now take up an estimate of the average value of (a,b) and use $\varphi_r(n)$ to help us. But first we prove a very interesting relation connecting $\varphi_r(n)$ and (a,b). This is due to Jagannathan and Ranganathan [3] who stated it without proof in a slightly different form. We supply here a proof.

LEMMA 2. For all real $r \geqslant 1$ we have

$$n \sum_{d|n}{}' \frac{\varphi_r(d)}{d} = \sum_{l=1}^{n}{}' (l, n)^r$$

<u>Proof.</u> First we write the right side as

$$\sum_{l=1}^{n}{}' (l,n)^r = \sum_{l=1}^{n}{}' \sum_{\substack{(l,n)=d \\ d|n}}{}' (l,n)^r = \sum_{d|n}{}' d^r \varphi\left(\frac{n}{d}\right) \tag{1.12}$$

We know that

$$n\sum_{d|n}' \frac{\varphi_r(d)}{d} = \sum_{d|n}' \left(\frac{n}{d}\right)\sum_{d'|d}' \mu(d')\left(\frac{d}{d'}\right)^r = \sum_{d|n}' \left(\frac{n}{d}\right)\sum_{d'|d}' \mu\left(\frac{d}{d'}\right) d'^r$$

$$= \sum_{d|n}' d^r \sum_{e|\frac{n}{d}}' \mu(e)\left(\frac{n}{d}\right) = \sum_{d|n} d^r \sum_{e|\frac{n}{d}}' \mu\left(\frac{n}{de}\right) e$$

$$= \sum_{d|n} d^r \varphi\left(\frac{n}{d}\right) = \sum_{l=1}^{n}{}' (l,n)^r$$

using (1.12). This establishes the lemma.

Define for real $r \geq 1$, $P_r(n)$ a generalisation of Pillai's function

$$P_r(n) = \sum_{l=1}^{n}{}' (l,n)^r = \sum_{d|n}' \varphi_r(d)\left(\frac{n}{d}\right) = \sum_{d|n}' d^r \varphi\left(\frac{n}{d}\right) \tag{1.13}$$

Estimates for $r = 1$ can be made and one can show that

$$\sum_{0<n\leq x}' P_1(n) = \frac{3}{\pi^2} x^2 \log x + O(x^2) \tag{1.14}$$

and equivalently using Abel's summation formula

$$\sum_{0<n\leq x}' \frac{P_1(n)}{n} = \frac{6}{\pi^2} x \log x + O(x) \tag{1.15}$$

Put crudely (1.15) implies that $P_1(n)/n$ behaves like $6 \log n/\pi^2$ or the average value of (a,n) is $6 \log n/\pi^2$. We make asymptotic estimates of $P_r(n)$ $r > 1$ using (1.13) in two ways

THEOREM 5.

$$\sum_{0<n\leq x}' P_r(n) = \frac{x^{r+1}\zeta(r)}{(r+1)\zeta(r+1)} + O(x^{r+1-\varepsilon}) \qquad 0<\varepsilon<1, \quad r-\varepsilon>1$$

Proof. Method 1

We have

$$\sum_{0<n\le x}{}' P_r(n) = \sum_{0<n\le x}{}' \sum_{d|n}{}' d^r \varphi\left(\frac{n}{d}\right) = \sum_{0<d\le x}{}' \varphi(d) \sum_{0<d'\le \frac{x}{d}}{}' d'^r$$

$$= \sum_{0<d\le x}{}' \varphi(d) \left\{ \left(\frac{x}{d}\right)^{r+1} \Big/ r+1 + O\left(\left(\frac{x}{d}\right)^r\right) \right\}$$

$$= \frac{x^{r+1}}{r+1} \sum_{0<d\le x}{}' \frac{\varphi(d)}{d^{r+1}} + O\left(x^r \sum_{0<d\le x}{}' \frac{\varphi(d)}{d^r} \right)$$

$$= \frac{x^{r+1}}{r+1} \left\{ \sum_{d=1}^{\infty}{}' \frac{\varphi(d)}{d^{r+1}} - \sum_{d>x}{}' \frac{\varphi(d)}{d^{r+1}} \right\} + O(x^{r+1-\varepsilon})$$

$$= \frac{x^{r+1}\zeta(r)}{(r+1)\zeta(r+1)} + O(x^{r+1-\varepsilon})$$

Method 2

We also know

$$\sum_{0<n\le x}{}' P_r(n) = \sum_{0<n\le x}{}' \sum_{d|n}{}' \varphi_r(d)\left(\frac{n}{d}\right) = \sum_{\substack{d|n \\ 0<d\le x}}{}' d \sum_{0<d'\le x/d}{}' \varphi_r(d')$$

$$= \sum_{0<d\le x}{}' d \left\{ \frac{\left(\frac{x}{d}\right)^{r+1}}{(r+1)\zeta(r+1)} + O\left(\left(\frac{x}{d}\right)^r \log\left(\frac{x}{d}\right)\right) \right\} \quad \text{using Theorem 1}$$

$$= \frac{x^{r+1}}{(r+1)\zeta(r+1)} \sum_{0<d\le x}{}' \frac{1}{d^r} + O\left(x^r \sum_{0<d\le x}{}' \frac{1}{d^{r-1}} \log\frac{x}{d} \right)$$

$$= \frac{x^{r+1}\zeta(r)}{(r+1)\zeta(r+1)} + O(x^{r+1-\varepsilon})$$

We infer from Theorem 5 by Abel's summation formula

COROLLARY.
$$\lim_{x \to \infty} \frac{1}{x} \sum_{0<n\leq x}' \frac{P_r(n)}{n^r} = \frac{\zeta(r)}{\zeta(r+1)}$$

§ 2.

We turn our attention to an analogue of $P_r(n)$ which is

$$A_r(n) = \sum_{a=1}^{n} [a,n]^r \tag{2.1}$$

where by $[a,n]$ is meant $an/(a,n)$. It is interesting to observe that

$$A_r(n) = \sum_{a=1}^{n}{}' [a,n]^r = \sum_{a=1}^{n}{}' \frac{a^r n^r}{(a,n)^r} = n^r \sum_{d|n}' \varphi^{(r)}\left(\frac{n}{d}\right) \tag{2.2}$$

where by $\varphi^{(r)}(n)$ is meant

$$\varphi^{(r)}(n) = \sum_{0<l\leq n,\ (l,n)=1}' l^r \tag{2.3}$$

a generalization of Euler's $\varphi(n)$ attributed to Thacker (see [2]). We make use of (2.2) to make an asymptotic estimate of $A_r(n)$.

THEOREM 6.
$$\sum_{0<n\leq x}' A_r(n) = \frac{x^{2r+2}\,\zeta(r+2)}{2(r+1)^2\,\zeta(2)} + O\left(x^{2r+1+\epsilon} \cancel{\log x}\right)$$

Proof. We have by (2.2)

$$\sum_{0<n\leq x}' A_r(n) = \sum_{0<n\leq x}' n^r \sum_{d|n}' \varphi^{(r)}\left(\frac{n}{d}\right) \tag{2.4}$$

If $\varphi(n,x) = \sum_{0<a\leq x,\ (a,n)=1}' 1$, then $\varphi(n,x) = \frac{x}{n}\varphi(n) + O(n^{\epsilon})\ \forall \epsilon > 0$

so that we infer

$$\varphi^r(n) = \frac{n^r \varphi(n)}{(r+1)} + O(n^{r+\varepsilon}) \ \forall \varepsilon > 0 \tag{2.5}$$

Thus (2.5) and (2.4) together imply that

$$\sum_{0<n\leq x}{}' A_r(n) = \sum_{0<n\leq x}{}' n^r \sum_{d|n}{}' \left\{ \frac{\left(\frac{n}{d}\right)^r \varphi\left(\frac{n}{d}\right)}{(r+1)} + O\left(\left(\frac{n}{d}\right)^{r+\varepsilon}\right) \right\} \quad \forall \varepsilon > 0$$

$$= \frac{1}{r+1} \sum_{0<n\leq x}{}' n^{2r} \sum_{d|n}{}' \frac{\varphi(n/d)}{d^r} + O\left(\sum_{0<n\leq x}{}' n^{2r+\varepsilon} \sum_{d|n}{}' 1 \right) \ \forall \varepsilon > 0$$

$$= \frac{1}{r+1} \sum_{0<d\leq x}{}' \frac{1}{d^r} \sum_{0<d'\leq \frac{x}{d}}{}' \varphi(d') d'^{2r} d^{2r} + O(x^{2r+1+\varepsilon}) \ \forall \varepsilon > 0$$

$$= \frac{1}{r+1} \sum_{0<d\leq x}{}' d^r \sum_{0<d'\leq \frac{x}{d}}{}' \varphi(d') d'^{2r} + O(x^{2r+1+\varepsilon}) \ \forall \varepsilon > 0 \tag{2.6}$$

If we use Abel's summation formula we get

$$\sum_{0<n\leq x}{}' \varphi(n) n^{2r} = \left\{ \frac{3x^2}{\pi^2} + O(x \log x) \right\} x^{2r} - \int_1^x \left\{ \frac{3t^2}{\pi^2} + O(t \log t) \right\} 2r t^{2r-1} dt$$

$$= \frac{3x^{2r+2}}{\pi^2} + O(x^{2r+1} \log x) - \frac{2r}{2r+2} \cdot \frac{3}{\pi^2} x^{2r+2} + O(x^{2r+1} \log x)$$

$$= \frac{6x^{2r+2}}{\pi^2 (2r+2)} + O(x^{2r+1} \log x) \tag{2.7}$$

substituting estimate (2.7) in (2.6) we get

$$\sum_{0<n\leq x}{}' A_r(n) = \frac{1}{r+1}\sum_{0<d\leq x}{}' d^r\left\{\frac{6x^{2r+2}}{\pi^2(2r+2)d^{2r+2}} + O\left(\frac{x^{2r+1}}{d^{2r+1}}\log\left(\frac{x}{d}\right)\right)\right\} + O(x^{2r+1+\varepsilon})$$

$$= \frac{x^{2r+2}\,6}{2(r+1)^2\pi^2}\sum_{0<d\leq x}{}' \frac{1}{d^{r+2}} + O(x^{2r+1}\log x) + O(x^{2r+1+\varepsilon})\ \forall\varepsilon>0$$

$$= \frac{x^{2r+2}\,\zeta(r+2)}{2(r+1)^2\,\zeta(2)} + O(x^{2r+1+\varepsilon})\ \forall\varepsilon>0.$$

and the proof is complete.

If we set $r = 1$ in Theorem 6 we get

$$\sum_{0<n\leq x}{}' A_1(n) = \frac{x^4\,\zeta(3)}{8\,\zeta(2)} + O(x^{3+\varepsilon})\ \forall\varepsilon>0.$$

Now Abel's summation formula implies that

$$\sum_{0<n\leq x}{}' \frac{A_1(n)}{n} = \frac{x^3\,\zeta(3)}{6\,\zeta(2)} + O(x^{2+\varepsilon})\ \forall\varepsilon>0.$$

which means $A_1(n)/n$ behaves like $n^2\zeta(3)/2\zeta(2)$. Thus the average value of $[a,n]$ is $n^2\zeta(3)/2\zeta(2)$.

REFERENCES

1) K.Alladi, On a generalisation of the Euler function, Prof. S. Ramanujan Comm. Vol., Madras, Dec. 1974, pp.114-124.

2) L.E.Dickson, Theory of Numbers, Vol.1, Chelsea, New York, 1962 pp.110-140.

3) N.Jagannathan and N.R.Ranganathan, 'On generalized Clifford groups II', Reports on Mathematical Physics (to appear)

4) W.J.LeVeque, Topics in Number Theory, Vol.1, Addison Wesley, Reading, Mass, 1960, pp.120-140.