MATSCIENCE REPORT 81

APPLICATIONS OF
THE THEORY OF STOCHASTIC PROCESSES
TO PHYSICAL PROBLEMS

ALLADI RAMAKRISHNAN
(Matscience, Madras-600020.)

A Reprint of the thesis Submitted to
The University of Manchester in 1951
for the Degree of Doctor of Philosophy

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-600020 (INDIA)

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A JUSTIFICATION

Twentyfive years is a short period in human history but a very long one in a particular human life. The completion of such a period in my research career is considered sufficient justification to make available as a Matscience Report my Ph.D. thesis on stochastic processes comprising the work done under my great teachers Bhabha and Bartlett with the cooperation of D.G.Kendall, Moyal and Janossy.

The I.B.M. conference on Point Processes in 1971 convinced me that the theory of product densities has not only survived but will endure. At that time when I coined the word I did not realize that it would be so well accepted by the scientific community. It is hoped that those who wish to be initiated into probability theory will treat this as an encouraging example of attempting new work from first principles.

To enable a reader to view the thesis in the light of the work that followed, I am including a note entitled 'Silver Thoughts' the contents of which are self-explanatory.

Alladi Ramakrishnan

SILVER THOUGHTS

What a fortunate visitation of God's grace it is to greet the President of India at a time when our nation is celebrating the silver jubilee of its independence! What a happy thought it is for me that I am completing twentyfive years of research in mathematical sciences and receiving in person his blessings for the book which represents the essence of my efforts during this period.

I remember vividly the day when I met Professor Bhabha during my stay in Delhi in 1947 when I was acting as a personal assistant to my great father who was engaged in the high task of drafting the Indian Constitution. At that period the atmosphere in India was charged with triumph and hope. After years of unrelenting struggle against the proud British empire the prospect of a prósperous future seemed to unfold before a redeemed people. Fascinated by the reputation of Bhabha and enchanted by his personal charm I leapt into the scientific world leaving the ready and rightful inheritance of a legal career. My gracious mother endorsed my choice and my generous father blessed me in the new endeavour.

The great excitement in science is the pursuit of the first problem and this came as a gift from Bhabha of the famous unsolved fluctuation problem of cosmic radiation. Untrammelled by conventional training I plunged into speculation and surmise which resulted in my first work on product densities in stochastic processes. Since its scope stretched beyond the immediate interests of Professor Bhabha I had to seek the guidance of Professors Bartlett and Kendall, the leaders of stochastic theory in England. Returning to India in 1951 I was given the privilege of initiating the physics department of the Madras University in

the modest position of a reader in theoretical physics. Till 1956 I worked on the theory of stochastic processes and indulged in an impertinent but enjoyable intrusion into the hallowed domain of astrophysics. Chance and circumstance then took me to the Yukawa Hall in Kyoto and to the famous Institute for Advanced Study at Princeton. Close contact with Professor Oppenheimer drew me into the domain of elementary particle physics and the four years after my return from Princeton were spent in groping and searching for new areas of research in high energy physics.

Meanwhile by a miraculous turn of fortune, through the grace of Professor Bohr, the support of our Prime Minister Nehru and the efforts of Mr.C. Subramaniam, I was invested with opportunities to serve an institute of advanced learning which corresponded in every aspect to my own vision of a haven of creative science. My own prediliction towards mathematical methods asserted itself in preference to phenomenological physics and the main contributions we made were the uncovering of new topological and combinatorial features of Feynman graphs. This encouraged me to tackle an old, neglected but unsolved problem of the transition from Pauli to Dirac matrices. The answer came out as a simple revelation rather than as a profound deduction and once it was obtained it was possible to weave systematically the logical pattern into which the Pauli and Dirac matrices could be imbedded. It yielded a new approach to matrix theory

which enabled me to understand in a deeper way the creators of matrix analysis like Sylvester, Clifford, Cayley and Hamilton on the one hand and the makers of modern physics like Pauli, Dirac, Feynman and Gell-Mann on the other.

The generous hospitality of the American institutions provided me opportunities to undertake several round the world missions accompanied by my wife and son to propagate the new methods of matrix theory in over a hundred international centres of higher learning and research. I was encouraged in my efforts by the thoughtful opinion of our sympathetic Chairman, Mr. Nedunchezhiyan that mathematics is a discipline which develops the ability of logical reasoning and rational approach. The experience of twentyfive years of research makes me yield to the exalted thought that:

'The physical universe is a revelation of mathematical logic'

The astronauts may have brought home the same message after their lunar Odyssey.

Alladi Ramakrishnan

PREFATORY NOTE

This dissertation embodies the work done by the author in Manchester during the academic years 1949-51 on the applications of the theory of stochastic processes, to physical problems. The work has been stimulated mainly by the so-called fluctuation problem relating to the cascade theory of cosmic ray showers. The author started this work in collaboration with Professor H.J. Bhabha at the Tata Institute of Fundamental Research, Bombay, and a short account of their joint work has been given in Appendices I and II not only for the sake of completeness but also to illustrate the scope of the problems that arise in a comprehensive statistical treatment of phenomena in physics.

I am deeply indebted to Professor M. S. Bartlett and Mr. J. E. Moyal who guided me throughout in the preparation of this thesis and to Mr. D. G. Kendall for his constant encouragement and advice. My thanks are also due to Professor L. Janossy with whom I had very valuable discussions on various aspects of the stochastic problems in cosmic radiation.

Alladi Ramakrishnan

University of Manchester
1st March 1951

CONTENTS

CHAPTER I

Introduction

The theory of stochastic processes has developed in recent times into one of the most important branches of science. While the theoretical ranges of the problems connected with it have given the mathematician a fertile field of investigation, their results have found numerous applications, particularly in different questions of physical and technical statistics. The object of the present work is to apply the theory of stochastic processes to physical problems, in particular those involving the notions of the change of the statistical distribution in time or space and of the correlation between statistical variables.

Essentially there are two methods to deal with such 'dynamic' statistical problems. The first method, hereinafter referred to as Method I, is especially useful in the case of Markoff processes. It is the characteristic feature of a Markoff process that for each instant, the future development of the process depends only on its present state but not on its prehistory. Method I makes direct use of this property; the procedure consists in expressing the state of the system at $t + \Delta t$ in terms of the state at t (where t is the parameter with respect to which the process develops) assuming t is continuous and passing to the limit as Δt tends to zero. If t were discrete assuming values $t_1, t_2, \ldots, t_K, \ldots$ we express the state of the system at t_K in terms of the state of the system at t_{K-1}.

The second method, hereinafter referred to as Method II, is based upon an entirely different mode of approach. In its original form, for continuous t, it yielded integral equations (integral with respect to t) and this has been described in detail in Chapter VII. This method, in the case of processes which are Markovian with respect to t, is equivalent to dealing with the stochastic process from the following point of view:.

At the outset it must be stated that the method is too general (in fact it is applicable ever to some non-Markovian processes, an elementary example of which is given in Chapter IX) to admit of a very simple complete formulation but to illustrate its use we proceed thus. Giving the most general interpretation to the state of a system, but assuming for the present that the system can occupy a discrete number of states S_1, S_2,...in any stochastic process we are interested in calculating the probability that the system is in state S_J at t given that it was in S_i at $t=0$. Instead of considering what happens in the interval between t and $t+\Delta t$ (which is the procedure adopted in Method I) we ask: what changes do occur in the interval between $t=0$ and $t=\Delta t$? Giving an equally general interpretation to the transition probability per unit t we define $R_{K\ell}$ as the probability per unit t that the system in a state S_K jumps to the state S_ℓ. Let us assume for the sake of simplicity, that $R_{K\ell}$ is independent of t and the stochastic process is homogeneous in and Markovian with respect to t. We define $P(J|i;t)$ as the required

probability. In the first instant of 'time' Δt (t is a general parameter and the word 'time' is used merely for the sake of illustration) the system jumps from the state S_i to state S_K with probability $R_{iK}\Delta t$ (Δt is an infinitesimal quantity). It is still in the state S_i with probability $1-\sum_{K,K\neq i} R_{iK}\Delta t$. Thus we can write

$$P(J|i;t) = \Big[1-\sum_{K,K\neq i} R_{iK}\Delta t\Big] P(J|i;t-\Delta t) + \sum_{K,K\neq i} P(J|K;t-\Delta t) R_{iK}\Delta t$$

yielding

$$\frac{\partial P(J|i;t)}{\partial t} = -P(J|i;t)\sum_{K,K\neq i} R_{iK} + \sum_{K,K\neq i} P(J|K;t)R_{iK} \qquad (1.1)$$

If the states do not form a discrete system but are continuous infinite, we get integrals instead of sums. In contrast to the above, Method I yields

$$P(J|i;t+\Delta t) = \Big[1-\sum_{K,K\neq J} R_{JK}\Delta t\Big] P(J|i;t) + \sum_{K,K\neq J} P(K|i;t)R_{KJ}\Delta t$$

Making Δt tend to zero we have

$$\frac{\partial P(J|i;t)}{\partial t} = -P(J|i;t)\sum_{K,K\neq J} R_{JK} + \sum_{K,K\neq J} P(K|i;t)R_{KJ} \qquad (1.2)$$

The striking contrast in approach is obvious on a mere examination of the equations (1.1) and (1.2)*.

In the case when the transition probabilities are t dependent, by the application of Method I we obtain the same equation (1.2) but we must apply Method II in its original form yielding an integral equation with respect to t and this has been described later in Chapter VII.

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* In the simple Markoff chain case, Method II is equivalent to the 'backward differential equation' of Kolmogoroff in contrast with the 'forward differential equation' obtained by Method I (cf. Arley, N. 'Stochastic processes and cosmic radiation', p.31 (1943) (Copenhagen) and W.Feller's 'Introduction to the theory of probability', p.390 (Wiley,1950)

CHAPTER II

Energy distribution of neutrons slowed down by elastic impacts

The problem of the energy distribution of neutrons cascading in an infinite homogeneous moderator which started with the paper of Condon and Breit has been considered by many authors and the integral equations of the problem have been solved by the application of the Laplace or Fourier transforms. Doubts were expressed regarding the applicability of the Laplace transformation to Adler's problem in view of the introduction of the Dirichlet discontinuous factor which was necessitated by considering the case of slowing down in a non-hydrogeneous medium. Here though no new result has been obtained (except possibly in the case of a non-constant source) the results of the various authors have been obtained in a single uniform scheme free from mathematical objections by using the Mellin's transformation and a simple device of changing the order of integration.*

Problem I.

Assuming the neutrons to be fast, the probability that a neutron of energy E_1 after an elastic collision with a proton has an energy between E_2 and $E_2 + dE_2$ is given by

$$P(E_1, E_2)\, dE_2 = dE_2 / E_1$$

* At the time when the author derived these results he was unaware that this problem had been discussed in great detail by Ivan Waller.

This is a direct consequence of the assumption that the scattering is elastic and also isotopic in the centre of gravity system of the neutron and proton, (the masses of the neutron and proton are equal). We shall also assume that the moderator is homogeneous isotropic and infinite. The problem is to find the probability $\pi(n,E)dE$ that after n collisions the neutron has energy between E and $E+dE$ given that its initial energy is E_o .

This problem can immediately be solved by the direct application of the Markoff method. The Markoff chain equation is obviously

$$\pi(n,E) = \int_{E}^{E_o} \pi(n-1,E')\, dE'/E' \quad *$$

This difference-integral equation can be solved by using the Mellin's transformation. Defining

$$P(n,S) = \int_{0}^{\infty} E^{S-1} \pi(n,E)\, dE$$

$$\pi(n,E) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} E^{-S} P(n,S)\, dS$$

$$P(n,S) = \int_{0}^{E_o} E^{S-1} dE \int_{E}^{E_o} \pi(n-1,E')\, dE'/E'$$

---

* It is immaterial whether we use E_o or ∞ as the upper limit of integration in the transformation since $\pi(n,E)$ vanishes for $E > E_o$.

Changing the order of integration

$$P(n, S) = \frac{1}{S} P(n-1, S), \quad P(1, S) = \frac{1}{S} E_o^{S-1}$$

$$\Pi(n, E) = \frac{1}{2\pi i E_o} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_o}{E}\right)^S \frac{1}{S^n} \, dS$$

The above contour integral is equal to

$$\frac{1}{E_o} \frac{t^n}{(n-1)!} H(t)$$

where

$$H(t) = 1 \quad \text{if} \quad t > 0$$

$$H(t) = 0 \quad \text{if} \quad t < 0 \qquad t = \log(E_o/E)$$

Note that

$$\int_0^{E_o} \Pi(n, E) \, dE = \int_0^{E_o} \frac{1}{E_o} \frac{(\log E_o/E)^{n-1}}{(n-1)!} \, dE$$

$$= \frac{1}{(n-1)!} \int_0^{\infty} e^{-t} t^{n-1} \, dt = 1$$

Problem II.

We shall now consider the same problem in the case of non-hydrogeneous media. Then the probability that a neutron of energy E after one collision has energy between E' and $E' + dE'$ is given by

$$\frac{1}{(1-\alpha^2)} \frac{dE'}{E}$$

where α^2 represents the fraction which gives the lowest limit to which the neutron energy is reduced.

$$\alpha = \left\{ \frac{M - m}{M + m} \right\}$$

M = Mass of the particle with which the neutron collides.

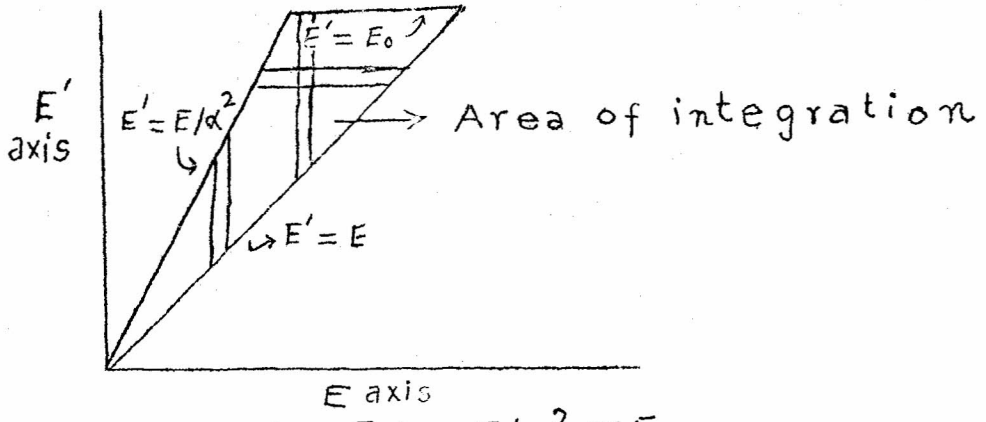
m = mass of the neutron.* Thus we have to modify our difference-integral equation as

$$\pi(n,E) = \int_{E}^{E/\alpha^2} \frac{1}{(1-\alpha^2)} \pi(n-1,E') \frac{dE'}{E'} \quad \text{if } \frac{E}{\alpha^2} < E_0$$

or

$$\pi(n,E) = \int_{E}^{E_0} \frac{1}{(1-\alpha^2)} \pi(n-1,E') \frac{dE'}{E'} \quad \text{if } \frac{E}{\alpha^2} > E_0$$

Without introducing any discontinuous factor, we use the Mellin's transformation as in the last case but it must be remembered that the double integration is performed within the shaded area shown in fig. This explains the choice of the upper limit.



E' varies from E to E/α^2 or E_0

E varies from 0 to E_0 . Changing the order

E varies from $\alpha^2 E'$ to E'

E' varies from 0 to E_0 .

*This results from the fact that masses of the colliding atoms are [illegible]. The maximum transfer of energy occurs in a head-on collisic

This yields

$$P(n,S) = \frac{P(n-1,S)}{(1-\alpha^2)}\left[\frac{x^S}{S}\right]_{\alpha^2}^{1}$$

putting $\log \frac{E_0}{E} = x$ and $\log \alpha^2 = -a$ we have since

$$\frac{P(n,S)}{P(n-1,S)} = \frac{1-\alpha^{2S}}{S(1-\alpha^2)}, \quad P(0,S) = E_0^{S-1}$$

$$\pi(n,E) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{xS}(1-e^{-aS})^n}{S^n(1-\alpha^2)^n}\,dS$$

$$= \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma_0+i\infty} \left\{1-\binom{n}{1}e^{-aS}\cdots\right\}\frac{e^{xS}\,dS}{S^n(1-\alpha^2)^n}$$

$$= \frac{1}{E_0(1-\alpha^2)^n(n-1)!}\left\{\begin{array}{l} x^{n-1} - \binom{n}{1}(x-a)^{n-1}\cdots \\ \cdots (-1)^n\binom{n}{n-1}(x-\overline{n-1}\,a)\end{array}\right\}$$

which was the Laplace solution to the problem.

Problem III.

In 1941, Adler derived the energy spectrum of neutrons in a homogeneous medium in which Q neutrons of energy E_0 are produced per second. Let v be the velocity of the neutrons, m the mass and $\lambda(v)$ the free path of scattering M

the mass of the nuclei and $P(E',E)$ the probability that a neutron of initial energy E' is scattered into an energy interval E and $E+dE$. He obtained the solution of the integral equation by the application of the Laplace transformation and by assuming a Dirichlet discontinuity factor. Due to the peculiar nature of the function $P(E',E)\,dE$ (it being expressible in the form $P(E/E')\,dE/E'$ the problem can be solved more easily by the Mellin's transformation.

The equation obtained by Adler is

$$M(E) = Q.P(E_0,E) + \int_E^{E_0} M(E')\,P(E,E')\,dE'$$

where

$$M(E) = N(E)\,\upsilon/\lambda$$

and $N(E)$ represents the energy spectrum of the neutrons. Multiplying by $E^{S-1}\,dE$ and integrating with respect to E under the condition

$$P(E,E') = \frac{1}{1-\alpha^2}\,\frac{1}{E'} \qquad \text{for } E < \alpha^2 E'$$

$$P(E,E') = 0 \qquad \text{for } E > \alpha^2 E'$$

If $m(S)$ is the Mellin transform of $M(E)$

$$m(S) = \frac{Q}{(1-\alpha^2)}\,\frac{1}{E_0}\int_{\alpha^2 E_0}^{E_0} E^{S-1}\,dE + \int_E^{E_0} M(E')\,P(E,E')\,dE$$

The above integral is evaluated in exactly the same manner as in the case of the previous problem by changing the order of integration .

$$m(s) = \left\{ \frac{Q E_0^{s-1}(1-\alpha^{2s})}{(1-\alpha^2)^s} + m(s)\frac{(1-\alpha^{2s})}{(1-\alpha^2)^s} \right\}$$

$$m(s) = \frac{Q E_0^{s-1}(1-\alpha^{2s})}{s(1-\alpha^2)-(1-\alpha^{2s})}$$

Thus we have

$$M(E) = \left(\frac{Q}{E_0}\right)\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E}\right)^s \frac{(1-\alpha^{2s})\,ds}{s(1-\alpha^2)-(1-\alpha^{2s})}$$

$$= \frac{Q}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{xs}(1-e^{-as})}{s(1-\alpha^2)-(1-e^{-as})}\,ds$$

by putting $x = \log E_0/E$ and $a = \log 1/\alpha^2$ and taking σ large we can write the above as

$$\frac{1}{2\pi i E_0}\int_B \frac{1}{s(1-\alpha^2)} e^{xs}(1-e^{-as})\left\{1-\frac{1-e^{-as}}{s(1-\alpha^2)}\right\}^{-1} ds$$

$$= \frac{1}{2\pi i E_0}\int_B \sum_{n=1}^{n=\infty}\left[\frac{e^{xs}(1-e^{-as})^n}{s^n(1-\alpha^2)^n}\right] ds$$

We immediately recognise the above expression to be

$$Q \cdot \sum_{n=1}^{n=\infty} \pi(n, E)$$

$$M(E) = Q\left\{\pi(1,E) + \pi(2,E) + \ldots + \pi(n,E) + \ldots\right\}$$

where

$$\pi(n, E) = \frac{1}{2\pi i E_0} \int_B \frac{e^{xs}(1-e^{-as})^n}{s^n (1-\alpha^2)^n} \, ds$$

Problem IV

In the case of hydrogeneous material

$$\pi(n, E) = \frac{1}{E_0} \frac{x^{n-1}}{(n-1)!}$$

Therefore

$$M(E) = \frac{Q}{E_0}\left\{1 + x + \frac{x^2}{2} + \ldots\right\} = \frac{Qe^x}{E_0} = \frac{Q}{E}$$

$$P(E) = \frac{\lambda}{v}\frac{Q}{E} = \frac{\lambda' Q}{E^{3/2}} \quad \text{where } \lambda = \sqrt{\frac{2}{m}}\,\lambda'$$

This is the formula obtained by Groenwold and Groendfik using a different method.

If Q were not a constant but time dependent we must have instead of Q the following expression, the integral equation

$$\int_{-\infty}^{t} a\,Q(t)\,e^{a(t-t')}\,dt'$$

Integrating partially we have the above equal to

$$\left\{Q - \frac{Q'}{a} + \frac{Q''}{a^2} - \cdots\right\}_{\text{at } t}$$

Thus the number of neutrons with energy E is given by

$$N(E) = \frac{\lambda}{v}\left\{Q - \frac{Q'}{a} + \frac{Q''}{a^2} - \cdots\right\}\left\{\pi(1,E) + \pi(2,E) \cdots\right\}$$

If $P(n,E) = \frac{\lambda}{v}\left(Q - \frac{Q'}{a} + \frac{Q''}{a^2} - \cdots\right)$, $P(n,E)$ can be interpreted as the number of particles which have suffered n collisions <u>and</u> have energy E. In the stationary case Waller obtained the Adler formula by first calculating $P(n,E)$ and then summing over all values of n .

In the above treatment we have not considered scattering and absorption of neutrons. The formulae in these cases have been obtained by Waller.

It is to be noted that while $\Pi(n, E)$ is a probability density, $P(n, E)$ represents the <u>average</u> density of particles that have undergone n collisions <u>and</u> have an energy E. The significance of this statement will be clear later, after the theory of product densities is explained.

CHAPTER III

One dimensional stochastic processes

Let a system be capable of occupying a set of mutually exclusive states, $S_1, S_2, \ldots S_{\eta}$ where η may be finite or enumerably infinite. For example, by the state of such a stochastic system we may mean a state represented by the value a stochastic variable X assumes, where t is the parameter with respect to which the process develops. In such a case we call the stochastic process one-dimensional. To study such a stochastic process we ask the question: what is the probability $\pi(S_J|S_i;t)$ that the system is in the state S_J at t given that it was in S_i at $t=0$? If R_{Ji} is defined as the probability per unit t that system in state S_i will jump to S_J, assuming the process to be homogeneous and Markovian with respect to t we have according to Method I (see 1.1)

$$\partial\pi(S_J|S_i;t)/\partial t = \begin{cases} -\pi(S_J|S_i;t)\sum_{K,K\neq J} R_{KJ} \\ +\sum_{K,K\neq J}\pi(S_K|S_i;t)R_{JK} \end{cases} \quad (3.1)$$

and by Method II (see 1.2)

$$\partial\pi(S_J|S_i;t)/\partial t = \begin{cases} -\pi(S_J|S_i;t)\sum_{K,K\neq i} R_{Ki} \\ +\sum_{K,K\neq i}\pi(S_J|S_K;t)R_{Ki} \end{cases} \quad (3.2)$$

The system of equations (3.1) and (3.2) are equivalent for both yield the solution (written in matrix form) in the case of

finite n ,

$$\frac{d[\pi]}{dt} = [R][\pi]$$

$$[\pi(t)] = [\pi(0)]\, e^{[R]t} \tag{3.3}$$

where

$$[R] = \begin{bmatrix} R_{11} & R_{21} & \cdot & \cdot & \cdot & R_{n1} \\ R_{12} & R_{22} & \cdot & \cdot & \cdot & R_{n2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{1n} & R_{2n} & \cdot & \cdot & \cdot & R_{nn} \end{bmatrix} \tag{3.4}$$

$$\text{where } R_{ii} = -\sum_{K(\neq i)} R_{iK}$$

It is quite clear that (3.1) deals with the rows and (3.2) with the columns of the matrix $e^{[R]t}$.

Arley and many other authors have considered these systems in great detail and so we shall not concern ourselves with such systems here.

We shall now consider the case when $X(t)$ is a continuous variable. In such cases we define a probability density $\pi(x|x_0;t)$, where $\pi(x|x_0;t)dt$ represents the probability that the stochastic variable assumes a value between

X and $X + dX$

$$\int_X \pi(X|X_0;t)\,dX = 1 \qquad *$$

If we assume that the transition probability per unit t that the stochastic variable which has a value X_1 at t 'jumps' to X_2 is given by the function $R(X_1|X_2)$ (for simplicity we shall assume that R is independent of t) according to Method I we have

$$\partial\pi(X|X_0;t)/\partial t = -\pi(X|X_0;t)\int_{X'} R(X|X')\,dX' + \int_{X'} \pi(X'|X_0;t)\,R(X'|X)\,dX' \qquad (3.5)$$

This is the fundamental integro-differential equation of a one-dimensional (the dimension referring to the stochastic variable) process** If the stochastic variable were discrete

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* In this chapter the notation $\int_X$ indicates that the integration is performed over the whole range of X. But $\int_a^b$ as usual indicates integration between the limits a and b.

** By Method II we have

$$\partial\pi(X|X_0;t)/\partial t = -\pi(X|X_0;t)\int_{X'} R(X_0|X')\,dX' + \int_{X'} \pi(X|X';t)\,R(X_0|X')\,dX' \qquad (3.5)^*$$

It is not possible to prove the equivalence of this equation and (3.5), as easily as in the case of (3.1) and (3.2). In this chapter we deal only with (3.5) in detail.

we obtain the system of simultaneous differential equations. In fact the transition from the discrete to the continuous case can be made by a limiting process and the integral equation can be described to be a brief and compact summary of the entire information contained in an infinite system of differential equations.

Direct applications of this equation to physical problems can be found easily. Taking for instance the ionisation or the radiation loss of energy of an atom or particle in its passage through matter. The problem can be formulated as follows:

Given the probability that a particle of energy E jumps down to an energy lying between E' and $E'+dE'$ per unit thickness as $R(E,E')\,dE'$, we have to calculate the probability $\pi(E,t)dE$ that the particle has energy between E and $E+dE$ at t, under given initial conditions. Obviously $\pi(E,t)$ satisfies

$$\partial\pi(E,t)/\partial t = -\pi(E,t)\int_0^E R(E,E')\,dE + \int_E^\infty \pi(E',t)\,R(E',E)\,dE' \qquad (3.6)$$

In the case of ionisation, the form of $R(E,E')$ was given by Bloch, and in the case of radiation loss it has been given by Bethe and Heitler. If $R(E,E')dE'$ can be put in the form $R(E'/E)\,dE'/E$ then the above integral equation can be solved by the use of the Mellin's transformation. For on applying the Mellin's transformation defined by

$$P(S,t) = \int_0^\infty \pi(E,t)\,E^{S-1}\,dE \qquad (3.7)$$

the integral equation reduces to

$$\frac{\partial P(S,t)}{\partial t} = -A(S)P(S,t) \qquad (3.8)$$

$$P(S,t) = P(S,0)e^{-A(S)t}$$

And by the inversion formula

$$\pi(E,t) = \frac{1}{2\pi i E_0}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E}\right)^S e^{-A(s)t}\,dS. \qquad (3.9)$$

if

$$P(S,0) = E_0^{S-1} \qquad \text{i.e } \pi(E,0) = \delta(E_0-E)$$

In the usual deterministic approach to collision loss, we deal with the following problem.

Let a particle of energy E enter matter. Due to collision processes it loses energy. The loss in energy per unit thickness is a function of E the energy of the particle, say $F(E)$ i.e.

$$F(E) = -\frac{\partial E}{\partial t} \qquad (3.10)$$

From this it is possible to calculate the loss in energy in passing through matter of thickness t; for

$$\int_0^t dt = \int_{E_0}^{E} -\frac{dE}{F(E)} \quad \text{or} \quad \int_{E}^{E_0}\frac{dE}{F(E)} = t \qquad (3.11)$$

The solution of this equation yields $E = \overline{E(t)}$ It must be remembered that $\overline{E}(t)$ is the <u>average</u> energy of the particle and $F(E)$ represents the average loss of energy (per unit length) of the particle. In all treatments it has been implicitly assumed that the above equation gives the average energy but no formal proof has been given.

Defining $\pi(E,t)$ as before we obtain

$$\frac{\partial \pi(E,t)}{\partial t} = -\pi(E,t)\int_0^E R(E,E')\,dE' + \int_E^\infty \pi(E',t)R(E',E)\,dE' \qquad (3.12)$$

By the deterministic approach

$$\frac{d\bar{E}}{dt} = F(\bar{E}) \qquad (3.13)$$

Now we shall establish a connection between $F(\bar{E})$ and $R(E,E')$ If we define $F(E)$ as the average loss per unit distance of a particle of energy E

$$F(E) = \int_{E'} R(E,E')(E-E')\,dE' \qquad (3.14)$$

$$F(\bar{E}) = \int_E \pi(E,t)\,dE \int_{E'} R(E,E')(E-E')\,dE' \qquad (3.15)$$

$$= \int_{\bar{E}} R(\bar{E},E')(\bar{E}-E')\,dE'$$

Now consider the equation

$$\frac{\partial \pi(E,t)}{\partial t} = -\pi(E,t)\int_0^E R(E,E')\,dE' + \int_E^\infty \pi(E',t)\,R(E',E)\,dE \qquad (3.16)$$

Multiply by E and integrate. We have

$$\frac{\partial \bar{E}}{\partial t} = -\bar{E}\int_0^E R(E,E')\,dE' + \int_E E\,dE \int_E^\infty \pi(E',t)\,R(E',E)\,dE' \qquad (3.17)$$

Changing the order of integration of the second term we have for the second term

$$\pi(E',t)\,dE' \int_E R(E',E)\,E\,dE \qquad (3.18)$$

Put $E = (E-E') + E'$ and the entire equation reduces to

$$\frac{\partial \bar{E}}{\partial t} = \int_0^{E_0} \pi(E,t)\,dE \int_0^E R(E,E')(E'-E)\,dE' = F(\bar{E}) \qquad (3.19)$$

Now we shall deal with an approximation when $R(E,E')$ is such a function that $R(E,E') \Rightarrow 0$ as E' deviates from E. In such a case defining

$$\int_{E'} E' R(E,E')\,dE' = \beta$$

the average loss in energy per unit thickness of a particle of energy E

$$\int_{E'} (E')^2 R(E,E')\,dE' = \delta$$

the mean square energy loss per unit thickness of a particle of energy E the fundamental integro-differential equation

reduces to

$$\frac{\partial \pi(E,t)}{\partial t} = \frac{\partial}{\partial E}(\beta\pi) + \frac{1}{2}\frac{\partial^2}{\partial E^2}(\delta\pi) \qquad (3.20)$$

in the second approximation. In the first approximation we have

$$\frac{\partial \pi(E,t)}{\partial t} = \frac{\partial}{\partial E}(\beta\pi) \qquad (3.21)$$

If we assume β is independent of E and $\pi(E,0) = \delta(E_0 - E)$ then

$$\pi(E,t) = \delta(E_0 - \beta t - E) \qquad (3.22)$$

i.e. there is a deterministic loss of βt. But the loss is not deterministic if we go to a second approximation which yields the differential equation (3.20) derived by Feller.

In the case of Brownian motion, the stochastic variable is the displacement x instead of the energy E and accordingly we have the equation

$$\frac{\partial \pi(x,t)}{\partial t} = \frac{\partial}{\partial x}(\beta\pi) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\delta\pi) \qquad (3.23)$$

if we assume $\beta = 0$ i.e. where $R(x,x') = R(x-x') = R(x'-x)$ we get the Einstein's diffusion equation. If the total number of particles is N, and they are identical then we define

$$\phi(x,t) = N\pi(x,t)$$

mean density of particles. (The significance of 'mean density' will be explained in the chapter on product density). ϕ satisfies the Einstein equation

$$\partial\phi(x,t)/\partial t = (\delta/2)\,\partial^2\phi(x,t)/\partial x^2$$

CHAPTER IV

Multi-dimensional stochastic processes

Until now we have been dealing with Markoff stochastic processes defined by a function $\pi(\alpha|\beta ; t_1, t_2)$ which is the probability that a system is in a state S_α at $t = t_1$, given that it was in the state S_β at $t = t_2$ where t is the parameter with respect to which the stochastic process progresses. If the process is homogeneous in t_1 and t_2

$$\pi(\alpha|\beta ; t_1, t_2) = \pi(\alpha|\beta ; t_1 - t_2)$$

or

$$\pi(\alpha|\beta ; t)$$

writing t for $t_1 - t_2$. For example, if x is a stochastic variable the state of the system may be represented by the value x assumes. If x were discrete, π represents a probability magnitude; if x were continuous it represents a probability density.

To determine π we should know the transition probability per unit time defined by

$$\text{limit}\left[\pi(\alpha|\beta ; t)/t\right] \text{ as } t \Rightarrow 0 = R(\alpha|\beta) \quad (\alpha \neq \beta)$$

Such a type of stochastic process we called a one-dimensional stochastic process. The concept of a stochastic process was

extended by Cramer to an n-dimensional process, i.e. we have random variables or in other words an n-dimensional random variable

$$Z(t) = \{ X_1(t), X_2(t), \ldots, X_n(t) \}$$

chosen as the basic variable of the process. Then we have to define a function

$$\pi(X_1, X_2, \ldots X_n \mid \xi_1, \xi_2, \ldots, \xi_n ; t)$$

which is the probability that the random variables have the values $X_1, X_2, \ldots X_n$ at t given that they had the values $\xi_1, \xi_2, \ldots \xi_n$ at $t = 0$. This leads us to the concept of 'a branching stochastic process' and the transition probabilities per unit t in such a case will be no longer simple as explained below.

The term 'branching stochastic process' was applied by Kolmogoroff to a certain type of stochastic process described by him in the following manner*.

We shall consider an aggregate of objects (for instance molecules, electrons, photons or neutrons etc.) of n different

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* The notation used is similar to that of Kolmogoroff. This chapter on branching stochastic processes has been introduced for the sake of completeness. It forms a link between Chapters III and IV.

types $T_1, T_2, \ldots, T_n$, and suppose that in the interval* $(t = t_\tau, t = t_{\tau+1})$ one object of the type T_K, with probability $P_K^\alpha(t_\tau, t_{\tau+1})$ is transformed into the aggregate $S_\alpha = \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_n T_n$ consisting of α_1 objects of type T_1, α_2 objects of type T_2, ..., α_n objects of type T_n. The object is to find the probability that there are

η_1 objects of type T_1

η_2 objects of type T_2

. . . . . .

η_n objects of type T_n

at t_J, given that there were $\xi_1, \xi_2, \ldots, \xi_n$ objects of type $T_1, T_2, \ldots, T_n$ respectively at t_i. Let us define the above probability by

$$\pi(\eta_1, \eta_2, \ldots, \eta_n \mid \xi_1, \xi_2, \ldots, \xi_n ; t_J, t_i)$$

It is quite clear that π is uniquely determined if P_K^α is known. We shall not here write down the equation which π satisfies. The process is better understood by citing an actual physical example. Before doing so let us remind ourselves that in all such processes we assume, according to Kolmogoroff, that the probabilities are independent of 1) how and when the original object of the type T_K came into existence

* For the moment, with Kolmogoroff, we assume t to be discrete represented by the sequence $t_1, t_2, \ldots, t_K, \ldots$.

it being only assumed that it existed at the moment say $t = t_1$.
2) What may happen to other possible objects under consideration of type $T_1, T_2, \ldots, T_n$ distinct from T_K at $t = t_1$ or from objects that may arise from the given object at $t = t_1$.

This stochastic scheme has various applications in biology, chemistry and the physics of elementary particles. In chemical and physical problems the scheme must naturally be applied for continuous t (t say, representing time) and instead of $P_K^\alpha (t_r, t_{r+1})$ we have P_K^α defined per unit t. A simple example can be cited to illustrate such a process.

Black body radiation

In his theory of radiation Einstein assumes

1) A_{nm} is the probability per unit time that an atom jumps from the energy state E_n to the state E_m spontaneously $(E_n > E_m)$

2) B_{nm} is the probability per unit time that an atom jumps from the energy state E_n to E_m in the presence of radiation of frequency ν_{nm}

$$\nu_{nm} = \{(E_n - E_m)/h\}$$

3) C_{nm} is the probability per unit time that an atom jumps from the energy state E_m to the state E_n $(E_n > E_m)$ in the presence of radiation of frequency ν_{nm}.

The most detailed formulation of the process is obtained by defining the probability that at t there are

α_1 atoms in the E_1 state

α_2 atoms in the E_2 state

. . . . . .

β_{nm} photons in the ν_{nm} state

(n and m running through all the positive integers), as $\Pi(\alpha_1, \alpha_2, .., \beta_{nm}, .., t)$. Using method I we proceed to find the change in the value of Π after a time Δt .

Moyal has discussed this problem in detail using method I. The physically important case arises when t tends to infinity and the photons and atoms are in a state of 'dynamic' equilibrium.

In Appendix I, the stochastic problem of cosmic radiation is discussed using such a Π function.

CHAPTER V

The Theory of Product Densities

Many stochastic problems arise in physics where we have to deal with a stochastic variable representing the number of particles distributed in a continuous infinity of states characterised by a parameter E, and this distribution varies with another parameter t (which may be continuous or discrete; if t represents time or thickness it is of course continuous). This variation occurs due to transitions characteristic of the stochastic process under consideration. If the E-space were discrete and the states represented by $E_1, E_2, \ldots$ then it would be possible to define a function

$$\pi(\nu_1, E_1; \nu_2, E_2; \ldots; t)$$

representing the probability that there are ν_1 particles in E_1, ν_2 particles in E_2 $\overset{etc.}{\ldots}$ at t. The variation of π with t is governed by the transitions defined for the process; $\nu_1, \nu_2, \ldots$ are thus stochastic variables, and it is possible to study the moments or the distribution function of the sum of such stochastic variables

$$N = \nu_1 + \nu_2 + \ldots$$

with the help of the π function which yields also the correlation between the stochastic variables ν_i .

But if the E-space is a continuum no such π function can be defined, for we have a continuous infinity of stochastic variables representing the numbers in the elementary ranges dE. The concept of correlation has to be generalized and a consistent formulation is necessary before we deal with such a system.

In quantum mechanics such processes arise since the transitions may occur between continuous sets of variables. Examples of such transitions are collision loss, radiation loss by fast particles or pair creation by high-energy photons.

The method described in this paper is quite general, and the word 'particle' is used to facilitate understanding of the problem from a physical point of view. It is also to be noted that the continuous parameter referred to in the title is E and not t. There is no restriction on t. Whether t is continuous or discrete depends upon the definition of transition probabilities.

2. Description of the method

Let $M(E;t)$ represent the stochastic variable denoting the number of particles with parametric values less than E. Then $dM(E;t)$ represents the stochastic variable denoting the number of particles in the elementary range dE. We shall assume that the probability that there occurs one particle in dE is proportional to dE, while the probability that there occurs more than one particle, say n, is of order $(dE)^n$ and hence is vanishingly small compared to the probability for the occurrence of one particle. Thus it is possible to define a function

$f_1(E;t)$ such that

$$f_1(E;t)\,dE = \mathcal{E}\{dM(E;t)\} \tag{5.1}$$

where $\mathcal{E}\{dM(E;t)\}$ represents the average number of particles in dE (the $\mathcal{E}$ denoting expectation value). If we say that the probability that n particles occur in dE is $P(n)$ where n is zero or a positive integer, then

$$\begin{aligned} P(1) &= f_1(E;t)\,dE + O(dE)^2 \\ &= \mathcal{E}\{dM(E;t)\} + O(dE)^2 \\ P(0) &= 1 - f_1(E;t)\,dE - O(dE)^2 \\ P(n) &= O(dE)^n \quad (n>1) \end{aligned} \tag{5.2}$$

If we define

$$\sum_n n^r P(n) = \mathcal{E}\{n^r\} \tag{5.3}$$

then we have

$$\mathcal{E}\{n^r\} = \mathcal{E}\{[dM(E;t)]^r\} = \mathcal{E}(n) = \mathcal{E}\{dM(E;t)\} \tag{5.4}$$

Thus all the moments of the stochastic variable $dM(E;t)$ are equal to the probability that the stochastic variable assumes the value 1. But it must be noted that while $f_1(E;t)\,dE$ is a probability magnitude, its integral over E for fixed t yields only the mean number of particles in the range of integration, as the integration does not correspond to the addition of

infinitesimal probabilities relating to mutually exclusive events. Thus

$$\mathcal{E}\left\{M(E_u)-M(E_\ell)\right\}=\int_{E_\ell}^{E_u} f_1(E)\,dE=\int_{E_\ell}^{E_u}\mathcal{E}\{dM(E)\} \qquad (E_u>E_\ell) \tag{5.5}$$

We drop the variable t, as for the present purpose the dependence on t is irrelevant.

We now introduce the concept of product density. We form the product of the stochastic variables $dM(E_1)$, $dM(E_2)$ and accordingly define

$$f_2(E_1,E_2)\,dE_1\,dE_2=\mathcal{E}\left\{dM(E_1)\,dM(E_2)\right\} \tag{5.6}$$

which is also equal to the joint probability that a particle lies in dE_1 and a particle lies in dE_2 when dE_1, dE_2 do not overlap. When they do, a degeneracy occurs, and

$$\mathcal{E}\left\{\left[dM(E_1)\right]^2\right\}=\mathcal{E}\left\{\left[dM(E_1)\right]\right\}=f_1(E_1)\,dE \tag{5.7}$$

We shall call f_2 a product density of degree 2. In view of the degeneracy it directly follows that

$$\begin{aligned}\mathcal{E}\left\{\left[M(E_u)-M(E_\ell)\right]^2\right\}&=\int_{E_\ell}^{E_u}\int_{E_\ell}^{E_u}\mathcal{E}\left\{dM(E_1)\,dM(E_2)\right\}\\&=\int_{E_\ell}^{E_u} f_1(E)\,dE+\int_{E_\ell}^{E_u}\int_{E_\ell}^{E_u} f_2(E_1,E_2)\,dE_1\,dE_2\end{aligned} \tag{5.8}$$

Similarly, we can define product densities of any higher order, say n , where

$$f_n(E_1, E_2, \ldots, E_n)\, dE_1\, dE_2 \cdots dE_n = \mathcal{E}\{dM(E_1)\, dM(E_2) \ldots dM(E_n)\} \tag{5.9}$$

also represents the joint probability that there lies one particle in dE_1, one in dE_2, one in dE_3 ... one in dE_n provided the intervals $dE_1, dE_2, \ldots, dE_n$ do not overlap. We shall call f_n a product density of degree n . It should be noted that in the above expression no two of the dE_i's should overlap, for if $dE_n = dE_{n-1}$ then

$$\mathcal{E}\{dM(E_1)\, dM(E_2) \ldots dM(E_{n-2})\, dM(E_{n-1})^2\}$$

$$= \mathcal{E}\{dM(E_1)\, dM(E_2) \ldots dM(E_{n-2})\, dM(E_{n-1})\} \tag{5.10}$$

$$= f_{n-1}(E_1, E_2, \ldots, E_{n-1})\, dE_1\, dE_2 \cdots dE_{n-1}$$

This consideration is of importance when we are concerned with the n-th order moments of $[M(E_u) - M(E_l)]$. Before proceeding to this problem we shall consider special cases of f_n which yield results useful for the problem of determining the higher moments of $[M(E_u) - M(E_l)]$.

Consider the case of N particles distributed in a continuum represented by the E space.. N is a constant and not a stochastic variable. But the N particles are distributed in the E space according to probability laws. We can then write in this case

$$f_1(E)\, dE = N f_1^0(E)\, dE \,, \quad \left(\int_{\text{whole range}} f_1^0(E)\, dE = 1 \right) \tag{5.11}$$

where $f_1^0(E)\,dE$ represents the probability that any one of the N particles selected at random lies in dE . By a simple argument it follows that

$$f_2(E_1, E_2)\,dE_1\,dE_2 = N(N-1)\,f_1^0(E_1)\,f_1^0(E_2)\,dE_1\,dE_2 \tag{5.12}$$

$$f_N(E_1, E_2, \ldots, E_N)\,dE_1\,dE_2 \cdots dE_N = N!\,f_1^0(E_1)\,f_1^0(E_2)\cdots f_1^0(E_N)\,dE_1\,dE_2\cdots dE_N$$

If we take any finite fragment of the E space ΔE and write

$$\int_{\Delta E} f_1^0(E)\,dE = P_{\Delta E} \tag{5.13}$$

we get the mean number of particles in ΔE as $NP_{\Delta E}$, and the mean square deviation of the number of particles in ΔE as $NP_{\Delta E}(1-P_{\Delta E})$ in accordance with the well-known binomial distribution law. It is quite clear from our definitions that in this case product densities of order greater than N vanish, and when $N=1$ the density of degree 1 coincides with the ordinary concept of probability density.

Now we are in a position to calculate the general (say τ th) moment of the number of particles in any finite range $\Delta E = E_u - E_\ell$ represented by

$$\mathcal{E}\left\{\left[M(E_u) - M(E_\ell)\right]^\tau\right\} = \mathcal{E}\left\{M_{\Delta E}^\tau\right\} \tag{5.14}$$

In view of the degeneracies mentioned we can write the τ th

moment as

$$\sum_{S=1}^{S=\gamma} C_S^{\gamma} \int_{E_\ell}^{E_u} \int_{E_\ell}^{E_u} \cdots \int_{E_\ell}^{E_u} f_S(E_1, E_2, \ldots, E_S)\, dE_1\, dE_2 \cdots dE_S \qquad (5.15)$$

The coefficients C_S^{γ} are functions of γ and S and in no way dependent on the function f. To calculate C_S^{γ} we take the particular case of the distribution of N particles in the E space. Substitute the values for the various product densities in terms of $f_1^0(E)$ and integrate over the whole E-space. Then we get

$$N^{\gamma} = \sum_{S=1}^{S=\gamma} C_S^{\gamma}\, N(N-1)\cdots(N-S+1) \qquad (5.16)$$

These coefficients are now obtained by taking $N = 1, 2, \ldots$. This yields*

$$C_1^{\gamma} = C_{\gamma}^{\gamma} = 1, \quad C_2^{\gamma} = \tfrac{1}{2}(2^{\gamma} - 2) \quad \ldots \qquad (5.17)$$

Thus we have obtained the γth moment of the stochastic variable as the sum of the integrals of the product densities of order equal to and less than γ. Now if we consider the case

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* In this discussion we shall use the bar notation to denote the average operation to avoid any confusion between the expectation symbol $\mathcal{E}$ used before and $\mathcal{E}_{\Delta E}$ denoting the energy of the system.

where

$$f_r(E_1, E_2, \ldots, E_r) = f_1(E_1) f(E_2) \cdots f_1(E_r), \quad (5.18)$$

i.e. when there is no correlation between particles in two different energy ranges the rth moment of the number of particles in a finite range ΔE is

$$\mathcal{E}\{N^r_{\Delta E}\} = C^r_1 \mathcal{E}\{N_{\Delta E}\} + C^r_2 [\mathcal{E}\{N_{\Delta E}\}]^2 + \cdots \quad (5.19)$$
$$\cdots + C^r_r [\mathcal{E}\{N_{\Delta E}\}]^r$$

which can be shown to coincide with the moments of a variable with a Poisson distribution.

Let $\pi(n, \Delta E)$ be the distribution function of the number of particles in ΔE.

$$\mathcal{E}\{N^r_{\Delta E}\} = \sum_n n^r \pi(n, \Delta E) \quad (5.20)$$

If $G(u, \Delta E)$ be the generating function corresponding to $\pi(n, \Delta E)$ by definition $G(u, \Delta E) = \mathcal{E}(u^n \pi(n, \Delta E))$

$$\mathcal{E}(N^r_{\Delta E}) = \left[\left(u \frac{d}{du}\right)^r G(u, \Delta E)\right]_{u=1} \quad (5.21)$$

Let this be equal to

$$b^r_1 [G'(u)]_{u=1} + b^r_2 [G''(u)]_{u=1} + \cdots + b^r_r \left[G^{(r)}(u)\right]_{u=1} \quad (5.22)$$

The coefficients b_i^r are independent of $G(u)$: But in a Poisson distribution we know that $G(u, \Delta E) = e^{(u-1)\xi(N_{\Delta E})}$

$$\xi(N_{\Delta r}^r) = \sum_{i=1}^{i=r} b_i^r \left[\xi(N_{\Delta E})\right]^i \qquad (5.23)$$

Compare (5.23) with (5.19). Equating the coefficients we have

$$C_i^r \equiv b_i^r \qquad (5.24)$$

Hence

$$\int_{\Delta E} \cdots \int_{\Delta E} f_r(E_1, E_2, \ldots, E_r) = \left[\frac{\partial^r G_1(u, \Delta E)}{\partial u^r}\right]_{u=1}$$

Janossy in his treatment of the statistical problem of cosmic radiation dealt with the function $\pi(n, \Delta E)$ and hence $G(u, \Delta E)$, while the author deals with the function f_r. The above relation proves the equivalence of the two methods once the appeal to the generating function is made.

We shall now extend the theory of product densities to a slightly different type of stochastic problem. In analogy with the notation of statistical mechanics we shall call the aggregate of particles a 'system'. The energy of a system is the sum of the energies of its members. In particular we shall call $\xi_{\Delta E}$ the energy of the partial system of particles whose individual energies lie in the range ΔE. $\xi_{\Delta E}$ is thus a stochastic variable and it immediately follows from the

interpretation of product densities that

$$\overline{\xi_{\Delta E}} = \int_{\Delta E} E f_1(E)\, dE$$

$$\overline{\xi^2_{\Delta E}} = \int_{\Delta E}\int_{\Delta E} E_1 E_2 f_2(E_1, E_2)\, dE_1\, dE_2 + \int_{\Delta E} E_1^2 f_1(E_1)\, dE_1 \qquad (5.25)$$

The derivation of the higher moments of $\xi_{\Delta E}$ is much more complicated than that of the higher moments of $N_{\Delta E}$. For the sake of completeness we shall here give the rule for the derivation of the γth moment of $\xi_{\Delta E}$. It is quite clear that it will be a sum of integrals involving product densities of order $\gamma, \gamma-1, \gamma-2, \ldots, 1$ in view of the degeneracies mentioned before. To find what the integrals involving the product densities of order $p<\gamma$ are, we proceed as follows.

In the previous problem the number of ways in which this $(\gamma-p)$-fold degeneracy can occur was found to be C_p^γ. In other words, it is the number of different ways in which γ distinguishable balls can be distributed in p indistinguishable boxes. In the above case the problem is much more complicated and we break up the C_p^γ number of ways into 'complexions'. A 'complexion' is characterised by a set of numbers $l_1, l_2, \ldots, l_p$ such that $l_1 + l_2 + \ldots + l_p = \gamma$. Let us call a typical complexion 'belonging to p' as C_{p_i}. Thus

$$\sum_i C_{p_i} = C_p^\gamma$$

The summation over i indicates the summation over 'complexion'.

Thus we write

$$\overline{\xi^{r}_{\Delta E}} = C^{r}_{r}\int_{\Delta E}\cdots\int_{\Delta E} E_1 E_2 \cdots E_r f_r(E_1, E_2, \ldots, E_r)\, dE_1 dE_2 \cdots dE_r$$
$$+ C^{r}_{r-1}\int_{\Delta E}\cdots\int_{\Delta E} E_1^2 E_2 \cdots E_{r-1} f_{r-1}(E_1, E_2, \ldots, E_{r-1})\, dE_1 dE_2 \cdots dE_{r-1}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \sum_i C_{p_i}\int_{\Delta E}\cdots\int_{\Delta E} E_1^{l_1} E_2^{l_2}\cdots E_p^{l_p} f_p(E_1, E_2, \ldots, E_p)\, dE_1 dE_2 \cdots dE_p$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

as p takes values r to 1. Of course $C^{r}_{1} = C^{r}_{r} = 1$. The coefficients C^{r}_{p} break up into 'complexions' only when $p > 1$ or $p < r$. To illustrate this, suppose we take the product density of degree r and ask for a two fold degeneracy. If the variables are $E_1, E_2, \ldots, E_r$ the two fold degeneracy occurs in two different complexions i.e. when, for example,

$$E_1 = E_r = E_{r-1} \qquad \text{or} \qquad E_1 = E_r,\ E_2 = E_{r-2}$$

In one case we have the term

$$\int_{\Delta E}\cdots\int_{\Delta E} C_{p_i} E_1^3 E_2 E_3 \cdots E_{r-2} f_{r-2}(E_1, \ldots, E_{r-2})\, dE_1 \cdots dE_{r-2}$$

In the other case we have

$$\int_{\Delta E}\cdots\int_{\Delta E} C_{p_J} E_1^2 E_2^2 E_3 \cdots E_{r-2} f_{r-2}(E_1, \ldots E_{r-2})\, dE_1 \cdots dE_{r-2}$$

In this case

$$C_{p_i} = \binom{r}{3} \cdot \quad C_{p_J} + C_{p_i} = C^{r}_{p} \cdot$$

CHAPTER VI

Applications of the Method of Product Densities to Physical Problems

The theory of product densities besides being applicable to physical problems which we shall consider presently, also assists in the interpretation of certain functions in physics whose mathematical significance in probability theory has not been clarified.

1. Born and Green Functions in the Kinetic Theory of Liquids.

Born and Green in their formulation of a general kinetic theory of liquids introduced what they called a 'set of multiform distribution functions'. It is to be mentioned that these functions have not distribution functions in the language of probability theory but they are product densities. To assist in the interpretation of those functions the following properties (not explicitly mentioned in the previous chapter) must be noted.

1) Product densities are defined with respect to a continuum which in general can be n-dimensional. If a product density is integrated with respect to τ dimensions it is still a product-density with respect to the continum of $n-\tau$ dimensions. Thus the Born and Green functions n_k, f_k, g_k are connected. They are all product-densities.

2) Taking in general a product density of degree k as $f_k(x_1, x_2, .., x_k)$ let us ask the question:
What is the effect of integrating f_k with respect to only one variable ?

Writing

$$\int_{X_h} f_h(X_1, \cdots, X_h)\, dX_1 \cdots dX_h = \phi_{h-1}(X_1, \cdots, X_{h-1})\, dX_1 \cdots dX_{h-1} \qquad (6.1)$$

This represents the expectation value of the product of the number of particles in the range of integration of X_h and the number of particles in $dX_1, dX_2, \cdots, dX_{h-1}$. ϕ_{h-1} is no longer a product-density and $\phi_{h-1}\, dX_1 \cdots dX_{h-1}$ no longer a probability magnitude.

In particular when we have a total number of N and only N particles in the entire range

$$\phi_{h-1} = (N-h+1) f_{h-1} \quad \text{if} \quad f_h\, dX_h \qquad (6.2)$$

is integrated over the whole range of X_h since the number of particles outside the infinitesimal ranges $dX_1 \cdots dX_{h-1}$ is $N-h+1$ <u>conditional</u> upon the existence of particles in $dX_1 \cdots dX_{h-1}$ and so it is not a random variable. Born and Green state that this result is obvious, but it is not clear why it is so unless we interpret their functions as product densities.

The above result can also be derived by the following trivial reasoning. If we confine our attention to h labelled particles $p_1, p_2, \cdots, p_h$ and define a function

$$f_h^1(X_1, \cdots, X_h)$$

such that

$$f_h^1(X_1, \cdots, X_h)\, dX_1 \cdots dX_h$$

represents the probability that p_1 lies in dX_1 , p_2 lies in dX_2 , then f_h^1 is a <u>probability density</u> and is subject to the normal rules of normalization giving

$$\int_{X_h} f_h^1\, dX_h = f_{h-1}^1$$

$$\int f_1^1\, dX_1 = \int_{X_1}\int_{X_2}\cdots\int_{X_h} f_h^1\, dX_1 \cdots dX_h = 1 \qquad (6.3)$$

If we now assume the particles are identical by a simple argument we have

$$f_h = N(N-1)\cdots(N-h+1)\, f_h^1 \qquad (6.4)$$

Thus

$$\int_{X_h} f_h\, dX_h = (N-h+1)\, f_{h-1}$$

It is to be noted that even when a 'particular' particle is distributed in the continuum independent of another 'particular' particle i.e. when

$$f_h^1 = f_1^1(X_1)\, f_1^1(X_2) \cdots f_1^1(X_h) \qquad (6.5)$$

then

$$f_k = 1\left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\cdots\left(1 - \frac{k-1}{N}\right) f_1(x_1) f_1(x_2)\cdots f_1(x_k)$$

But of course in the liquid state

$$f_k^1 \neq f_1^1(x_1)\cdots f_1^1(x_k)$$

Finally, a general remark can be made that the effect of integrating a product density can always be found by going back to its definition and noting the degeneracies mentioned in the previous chapter.

2. Distribution functions of many electron systems.

Similar considerations apply to the ''distribution functions' of many electron systems. Using the notation of Mott and Sneddon where the distribution functions of a many electron system of Z electrons are defined as $\rho_Z, \rho_{Z-1}, \ldots, \rho_1$ (defined with respect to a continuum q) we identify them as product densities of degree $Z, Z-1, \ldots, 1$ respectively (product densities of order greater than z vanish since there are only z electrons). Hence all the results stated for Born and Green functions apply to these ρ's as well. Mott and Sneddon stated that to obtain ρ_m from ρ_Z $(m<Z)$ we integrated ρ_Z with respect to $dq_{m+1}\cdots dq_Z$ and the 'integrations must be carried out in such a way that each configuration must appear only once ;

that is $dq_{m+1} \cdots dq_z$ must be replaced by $\frac{dq_{m+1} \cdots dq_z}{(z-m)!}$. This argument is not correct and is based upon a mixing up of the concepts of probability and product densities. Their statement must be replaced by the mathematical arguments outlined above using the concept of product densities which yields the required result

$$\int_{v_{m+1}} \int_{v_{m+2}} \cdots \int_{v_z} \rho_z \, dq_{m+1} \cdots dq_z = (z-m)! \, \rho_m \qquad (6.6)$$

and thus no appeal need be made to fictitious rules of integration.

3. Structure functions of Khintchine

Let us now go back to the statistical 'system' of particles distributed in an energy continuum. If the entire system consists of a definite number of particles say N, the problem of the energy distribution of a partial system can be considered from a different viewpoint. For convenience we shall label the particles as $P_1, P_2, \ldots, P_N$. We should of course remember that these particles are indistinguishable and this labelling is used only for the purpose of mathematical computation. Let the energy of each of these be a stochastic variable represented by $\epsilon_1, \epsilon_2, \cdots, \epsilon_N$ respectively. Let us assume that these stochastic variable are independent and that the distribution function of each of them is identical and given by $\Omega_1(E)$.

Let us define a stochastic variable

$$X_n = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$$

Let the distribution function of X_n be $\Omega_n(X_n)$

$$\Omega_n(X_n)' = \iint\cdots\int \Omega_1(\epsilon_1)\Omega_1(\epsilon_2)\cdots\Omega_1(\epsilon_{n-1})\,\Omega_1\Big(X_n - \sum_{i=1}^{n-1}\epsilon_i\Big)\, d\epsilon_1 d\epsilon_2 \cdots d\epsilon_n$$

If the characteristic function of $\Omega_n(X_n)$ be $\phi_n(u)$ then

$$\phi_n(u) = \left[\phi_1(u)\right]^n \tag{6.8}$$

It is quite clear that $\phi_n(u)$ and $\Omega_n(X_n)$ are the c.f. (characteristic function) and distribution function of the sum of any n of the stochastic variable $\epsilon_1, \epsilon_2, \cdots, \epsilon_N$. We can now speak of the conditional probability density $\pi_n(E|E_0)$ where $\pi_n(E|E_0)dE$ is the probability that X_n assumes a value between E and $E+dE$ given that $X_N = E_0$. By the definition of the conditional probability it follows that

$$\pi_n(E|E_0) = \Omega_n(E)\,\Omega_{N-n}(E_0 - E) \Big/ \Omega_N(E_0) \tag{6.9}$$

This is the basic formula obtained by Khintchine using abstract mathematical arguments. Khintchine calls the Ω's the structure functions. It must be noted that Ω's have no physical meaning when taken in isolation. For if there were only one particle there is no reason why its energy should have a probability distribution defined by a particular function. The central

problem of statistical mechanics arises from the fact that N particles assume energies in a random manner subject to one restirction that their total energy is constant. There would be of course be no 'randomness' if there were only one particle. But to arrive at the formula for $\Pi_n(E|E_0)$ we conceive of the fictitious Ω's.

Now we shall derive the product densities in this case on the basis of the above formula. From the definition of product densities it follows

$$f_1(E) = N\Pi(E|E_0) = N\Omega_1(E)\Omega_{N-1}(E_0-E)\Big/\Omega_N(E_0)$$

$$f_2(E_1,E_2) = \frac{N(N-1)\Omega_1(E_1)\Omega_1(E_2)\Omega_{N-2}(E_0-E_1-E_2)}{\Omega_N(E_0)} \qquad (6.10)$$

. . . . . .

$$f_n(E_1,\ldots E_n) = \frac{N(N-1)\cdots(N-n+1)\,\Omega_1(E_1)\cdots\Omega_{N-n}(E_0-E_1-E_{n-1})}{\Omega_1(E_0)}$$

The terms $N, N(N-1),\cdots$ occur due to the indistinguishable nature of the particles. Confining ourselves to n 'particular' particles say $P_1,\ldots,P_n$ the probability that P_1 has energy E_1, P_2 has energy $E_2,\cdots$ is given by

$$\frac{\Omega_1(E_1)\,\Omega_1(E_2)\cdots\Omega_1(E_{n-1})\,\Omega_{N-n}(E_0-E_1-\cdots-E_n)}{\Omega_N(E_0)} \qquad (6.11)$$

Hence the above formula for f_n follows by a simple arguement. Note that the probability that a particular particle P_i has

an energy E_i is not independent of another particle P_J having an energy E_J. For the joint probability of the even defined by $\epsilon_i = E_i$, $\epsilon_J = E_J$ is

$$\left[\Omega_1(E_i)\,\Omega_1(E_J)\,\Omega_{N-2}(E_0 - E_i - E_J) \Big/ \Omega_N(E_0)\right] \qquad (16.12)$$

$$\neq \left(\frac{\Omega_1(E_i)\,\Omega_{N-1}(E_0 - E_i)}{\Omega_N(E_0)}\right)\left(\frac{\Omega_1(E_J)\,\Omega_{N-1}(E_0 - E_J)}{\Omega_N(E_0)}\right)$$

Khintchine derives the asymptotic formula for large N for $\Pi(E|E_0)$ using the central limit theorem which is applicable even when the actual form of Ω is not known.

4. The Statistical Problem of Cosmic Radiation.

It is a well-known phenomenon of cosmic radiation that when a fast electron passes near a nucleus it may create a photon by Bremsstrahlung, which in its turn may create a pair, and this 'cascade' process leads to the formation of a shower of electrons and photons (the term electron includes both positive and negative electrons). The cascade theory first put forward by Bhabha and Heitler and independently by Carlson and Oppenheimer in 1937 and subsequently developed by many authors predicts only the average number of photons and electrons in a shower. But the processes of radiation and pair creation are governed by probability laws, and hence cascade multiplication is essentially a statistical process. Since the Bethe-

Heitler cross-sections for radiation and pair creation are continuous in the energy variable the problem of cascade multiplication presents the mathematical difficulties outlined in the introduction. We shall assume in what follows that the problem as regards thickness of matter through which the particles pass is one-dimensional, i.e. depends on only one parameter t which is the thickness of matter traversed, the lateral spread of showers being neglected. This assumption is made in all treatments of the cascade theory.

Statement of the problem: Given the initial energy spectrum of photons and electrons at $t=0$ and that

(a) an electron of energy E radiates a quantum and has residual energy lying between E' and $E'+dE'$ with a probability per unit thickness $R(E,E')dE'$;

*(b) x x x x x x

(c) an electron of energy E loses energy by collision (or ionization) and has residual energy between E' and $E'+dE'$ with a probability per unit thickness $\rho(E,E')dE'$, we must calculate the fluctuation of the total number of particles in a shower about the mean. The actual form of the functions in (a), (b), (c) is known, but the actual functions need not be specified for the purpose of this paper. The integro-differential equations of the problem (equations (6.15) A to D below) have been completely solved by Bhabha and Ramakrishnan.

* (b) a photon of energy E creates a pair of electrons, one of which has an energy lying between E' and $E'+dE'$ with a probability per unit thickness $R'(E,E')dE'$;

It will be seen from the foregoing treatment that the fluctuation (second-order moments) of the total number of particles in the entire energy range can be calculated through the determination of (1) the density of particles in any particular energy range, (2) the correlation between particles in two different energy ranges. In this respect the work of Bhabha and Ramakrishnan differs from all previous treatments except that of Scott and Uhlenbeck. In fact Arley had considered this difficulty insoluble and constructed idealised models to correspond to the process of cascade multiplication. To avoid the well-known (first mentioned) complications which stand in the way of a rigorous mathematical treatment of probability theory in a continuum Bhabha and Ramakrishnan first treated the total energy range as being made up of discrete energy states and pased to the continuum as a limit of the discrete case. The method of probability-generating functions was used, but once the formulation of the continuum is assumed the equations of the problem of density 2 follow in as easy a manner as the equations for density of degree 1.

Density of degree 1. Let $f_1(E)$ and $g_1(E)$ represent densities of degree 1 with respect to the number of electrons and photons respectively. From the processes (a), (b), (c) the integral equations involving f_1 and g_1 follow immediately:

$$\frac{\partial f_1(E,t)}{\partial t} = -f_1(E,t)\Big[\int_0^E R(E,E')\,dE' + \int_0^E \rho(E,E')\,dE' + \int_E^\infty f_1(E',t)[R(E',E)+\rho(E',E)]\,dE' + 2\int_E^\infty g_1(E',t)R(E',E)\,dE' \tag{6.13A}$$

$$\frac{\partial g_1(E,t)}{\partial t} = -g_1(E,t)\int_0^E R'(E,E')\,dE' + \int_E^\infty f_1(E',t)R(E',E'-E)\,dE' \tag{6.13B}$$

The equations are self-explanatory. These equations were obtained by Landau and Rumer (neglecting collision loss) and later Bhabha and Chakrabarthi introduced the term $\beta\,\partial f_1/\partial E$ for collision loss. This approximation can be obtained by assuming that the function $\rho(E,E')$ is such that it becomes vanishingly small as differs from E appreciably. If we write

$$\int_0^E (E-E')\,\rho(E,E')\,dE' = \beta \tag{6.14}$$

to a first approximation, we can remove the terms involving collision loss and replace then by $\beta\,\dfrac{\partial f_1}{\partial E}$.

Density of degree 2. (1) Let $f_2(E_1,E_2)\,dE_1\,dE_2$ represent the probability of joint occurrence of an electron in dE_1 and an electron in dE_2 .

(2) Let $fg_{1,1}(E_1,E_2)\,dE_1\,dE_2$ represent the joint probability of the occurrence of an electron in dE_1 and a photon in dE_2 .

(3) Let $g_2(E_1, E_2)\,dE_1\,dE_2$ represent the joint probability of the occurrence of a photon in dE_1 and a photon in dE_2.

Then f_2, $fg_{1,1}$, g_2 represent the product densities. From the definitions it follows that

$$f_2(E_1, E_2) = f_2(E_2, E_1), \quad g_2(E_1, E_2) = g_2(E_2, E_1)$$

$$fg_{1,1}(E_1, E_2) \neq fg_{1,1}(E_2, E_1)$$

(It should be noted that $fg_{1,1}$ is a single symbol). Let us compute the variation of the above product densities.

Variation of $f_2(E_1, E_2)$. We vary the parameter t by dt.

(1) Either the electron in E_1 or the electro in E_2 state may move out of the state by radiation or collision loss and hence there is a depletion in the probability

$$f_2(E_1, E_2)\,dE_1\,dE_2$$

the measure of which is given by

$$f_2(E_1, E_2)\Big[\int_0^{E_1} R(E_1, E) + P(E_1, E)\,dE + \int_0^{E_2} \{R(E_2, E) + P(E_2, E)\}\,dE\Big]dE_1\,dE_2\,dt$$

(2) Consider the probability $f_2(E, E_1)\,dE\,dE_1$. The electron in the E-state may radiate or lose energy by collision loss and convert $f_2(E, E_1)$ to $f_2(E_2, E_1)$. The measure of

this contribution to the probability is given by

$$\left[\int_{E_2}^{\infty}\left\{f_2(E,E_1)\left[R(E,E_2)+\rho(E,E_2)\right]dE\right\}\right]dE_1\,dE_2\,dt$$

(3) Consider the probability $f_2(E,E_2)\,dE\,dE_2$. The electron in the E-state may radiate or lose energy by collision loss and convert $f_2(E,E_2)$ to $f_2(E_1,E_2)$ state, and the measure of this contribution to the probability is given by

$$\left[\int_{E_1}^{\infty}\left\{f_2(E,E_2)\left[R(E,E_1)+\rho(E,E_1)\right]dE\right\}\right]dE_1\,dE_2\,dt$$

(4) Consider the probability $fg_{1,1}(E_1,E)\,dE\,dE_1$. The photon of energy E may create a pair one of which has an energy E_2 or a photon of energy E belonging to $fg_{1,1}(E_2,E)$ may create a pair one of which has the energy E_1, this giving a contribution to f_2 the measure of which is given by

$$\left[2\int_{E_2}^{\infty}fg_{1,1}(E_1,E)\,R'(E,E_2)\,dE+2\int_{E_1}^{\infty}fg_{1,1}(E_2,E)\,R'(E,E_1)\,dE\right]\times$$
$$\times\,dE_1\,dE_2\,dt$$

The factor 2 enters because the function

$$R'(E,E') = R'(E,E-E')$$

is symmetrical as regards the creation of positive and negative electrons.

(5) Lastly, a photon of energy E_1+E_2 may create a pair of energy E_1 and E_2 and the measure of this contribution is given by

$$2g_1(E_1+E_2)R'(E_1+E_2, E_1)\,dE_1\,dE_2\,dt$$

It is to be noted that we are computing the variation of $f_2(E_1, E_2)\,dE_1\,dE_2$ with t and the contribution due to the process (5) is of the same order as the contribution from the processes (1)-(4). It was to obtain this extra term that Bhabha and Ramakrishnan originally adopted the discrete model and passed to the limiting case of a continuum. But once the concept of product densities is understood then this extra term follows as a necessary consequence. It only means that the creation of a particle of energy E_1 by a photon of energy E_1+E_2 inevitably implies the creation of another particle of energy E_2.

Similarly, we compute the variation of $g_2(E_1, E_2)$, $fg_{1,1}(E_1, E_2)$ and $fg_{1,1}(E_2, E_1)$ and thus obtain the following integro-differential equations:

$$\begin{aligned}\frac{\partial f_2(E_1,E_2)}{\partial t} = f_2(E_1,E_2)\Big\{&\int_0^{E_1}\big[R(E_1,E)+P(E_1,E)\big]\,dE \\ &+\int_0^{E_2}\big[R(E_2,E)+P(E_2,E)\big]\,dE \\ &+\int_0^{\infty} f(E,E_1)\big[R(E,E_2)+P(E,E_2)\big]\,dE \\ &+\int_{E_2}^{\infty} f_2(E,E_2)\big[R(E,E_1)+P(E,E_1)\big]\,dE \\ &+2\int_{E_1}^{\infty} fg_{1,1}(E_1,E)R'(E,E_2)\,dE+2\int_{E_2}^{\infty} fg_{1,1}(E_2,E)R'(E,E_1)\,dE \\ &+2g_1(E_1+E_2)R'(E_1+E_2,E_1)\Big\}\end{aligned} \qquad (6.15A)$$

$$\frac{\partial f g_{1,1}(E_1,E_2)}{\partial t} = -f g_{1,1}(E_1,E_2)\left\{\int_0^{E_1}\left[R(E_1,E)+P(E_1,E)\right]dE + \int_0^{E_2} R'(E_2,E)\,dE + \int_{E_1}^{\infty} f g_{1,1}(E,E_2)\left[R(E,E_1) + P(E,E_1)\right]dE + \int_{E_1}^{\infty} 2g(E,E_2)\,R'(E,E_1)\,dE + \int_{E_2}^{\infty} f_2(E_1,E)\,R(E,E-E_2)\,dE + f_1(E_1+E_2)\,R(E_1+E_2,E_1)\right\} \tag{6.15B}$$

$$\frac{\partial f g_{1,1}(E_2,E_1)}{\partial t} = -f g_{1,1}(E_2,E_1)\left\{\int_0^{E_2}\left[R(E_2,E)+P(E_2,E)\right]dE + \int_0^{E_1} R'(E_1,E)\,dE + \int_{E_2}^{\infty} f g_{1,1}(E,E_1)\left[R(E,E_2) + P(E,E_2)\right]dE + \int_{E_2}^{\infty} 2g_2(E,E_1)\,R'(E,E_2)\,dE + \int_{E_1}^{\infty} f_2(E_2,E)\,R(E,E-E_1)\,dE + f_1(E_1+E_2)\,R(E_1+E_2,E_2)\right\} \tag{6.15C}$$

$$\frac{\partial g_2(E_1,E_2)}{\partial t} = -g_2(E_1,E_2) + \left\{\int_0^{E_1} R'(E_1,E)\,dE + \int_0^{E_2} R'(E_2,E)\,dE\right\} + \int_{E_2}^{\infty} f g_{1,1}(E,E_1)\,R(E,E-E_2)\,dE + \int_{E_1}^{\infty} f g_{1,1}(E,E_2)\,R(E,E-E_1)\,dE \tag{6.15D}$$

As regards the initial conditions of the problem, if at $t = 0$ there is only one particle (electron or photon) it is clear that all the product densities (order greater than 1) vanish at $t = 0$ as a necessary consequence of our interpretations.

The above equations were obtained by Bhabha and Ramakrishnan as a result of a long and tedious limiting process.* Assuming the energy states to be discrete they obtained the number n_K in any particular state E_K with a probability distribution $\pi(n_K)$ such that

$$\sum_{n_K=1}^{n_K=\infty} n_K^2 \pi(n_K) = \mathcal{E}(n_K^2) = \mathcal{E}(n_K) = \sum_{n_K=0}^{\infty} n_K \pi(n_K)$$

where n_K takes integral values $0, 1, 2, \ldots$. It is clear from the above equations that since $\pi(n_K)$ is non-negative the equation implies

$$\mathcal{E}(n_K^2) = \mathcal{E}(n_K) = \pi(1), \quad \pi(n_K) = 0 \quad n_K > 1$$

and

$$\pi(0) = -\pi(1)$$

This result was not noted by Bhabha and Ramakrishnan (in their paper already referred to). They passed to the continuum by assuming that there are ρ states per unit energy range, whence ρdE_K states in the range dE_K (a process quite familiar in quantum mechanics), or

$$\mathcal{E}(n_K)\rho\, dE_K = \mathcal{E}\{n(E_K)\, dE_K\}$$

All these results are implied in our formulation and hence the limiting process can be avoided.

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*See Appendix I.

D.G.Kendall has independently arrived at similar methods regarding the formulation of product densities in dealing with a problem on population, the continuous infinity of states in his case referring to the ages of the individuals in the population. He has also discussed in this paper the alternative method of characteristic functional (which comprises theoretically all moment and probability relations).

Recently, Janossy has approached the fluctuation problem by a novel method*. He has introduced the function $\Pi(n, E, t)$ representing the probability distribution of the number of particles with energy greater than E and dealt with this function $\Pi(n, E, t)$ directly thus avoiding the mathematical difficulties outlined in the introduction. But has has had to use the method of generating functions, leading to integral equations for the moments of the stochastic variable representing the number of particles of energy greater than E; these equations are just as complicated as those obtained by Bhabha and Ramakrishnan. But for the calculation of the probability $\Pi(n, E, t)$ for small values of n by numerical integration Janossy's method is perhaps the most suitable one available. The variation of higher product densities can be written down in the form of integral equations but would serve no practical purpose in view of their increasing complexity.

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apter VII.

Bartlett and Kendall have extended Janossy's approach to provide integral equations for the characteristic functional and discuss their solution in the population growth problem.

5. Penetrating Showers in the Theory of Cosmic Radiation.

The Theory of penetrating showers is still in a fluid state. We shall here discuss the mathematical theory of two statistical models suggested to explain the phenomenon.

Model I: Nucleon Cascades.

Statistically, the problem can be stated as follows in the most general mathematical terms.

It is given that a particle of energy E creates a particle of energy E-E' and has a residual energy lying between E' and E' + dE' with a probability per unit thickness equal to $R(E,E')\,dE'$. Since we do not distinguish between the two particles $R(E, E') = R(E, E-E')$.

Our problem is to calculate the mean and mean square deviation of the number of particles above a certain energy at thickness t given that at $t = 0$ there is one particle of energy E_0. We shall use the product-density technique to deal with this problem. We accordingly define

1) $f_1(E,t)\,dE$ as the probability of a particle lying in dE. $f_1(E,t)$ is thus a product-density of degree 1. The mean number in a finite range ΔE is given by

$$\bar{n}_{\Delta E} = \int_{\Delta E} f_1(E,t)\,dE \qquad (6.16)$$

2) $f_2(E_1, E_2, t)\,dE_1 dE_2$ as the joint probability that a particle lies in dE_1 and a particle in dE_2 .

Defining

$$\overline{n n}_{\Delta E} = \int_{\Delta E}\int_{\Delta E} f_2(E_1, E_2)\,dE_1 dE_2 \qquad (6.17)$$

according to the theory of product densities if $\overline{n^2}_{\Delta E}$ is the mean square number of particles in ΔE

$$\overline{n^2}_{\Delta E} = \overline{n}_{\Delta E} + \overline{n n}_{\Delta E} \qquad (6.18)$$

The diffusion equations for f_1 and f_2 are

$$\frac{\partial f_1(E,t)}{\partial t} = -f_1(E,t)\int_0^E R(E, E')\,dE' + 2\int_E^{E_0} f_1(E', t)\,R(E', E)\,dE' \qquad (6.19)$$

(We assume that at $t = 0$, there is a primary of energy E_0 .

The factor 2 results since $R(E,E') = R(E, E-E')$

$$\begin{aligned}\frac{\partial f_2(E_1, E_2, t)}{\partial t} = {} & -f_2(E_1, E_2, t)\left[\int_0^{E_1} R(E_1, E)\,dE + \int_0^{E_2} R(E_2, E)\,dE\right] + 2\int_{E_1}^{E_0} f_2(E', E_2, t)\\ & R(E', E_1)\,dE' \\ & + 2\int_{E_2}^{E_0} f_2(E_1, E', t) R(E', E_2)\,dE' \qquad (6.20)\\ & + 2 f_1(E_1 + E_2, E_1)\end{aligned}$$

These equations are quite general. We can solve them completely if we assume that $R(E,E')\,dE'$ can be expressed as $R(\epsilon)\,d\epsilon$, $\epsilon = E'/E$ by applying the Mellin's transformation.

$$P_1(s,t) = \int_0^\infty E^{s-1} f_1(E,t)\,dE$$

$$P_1(s,q,t) = \int_0^\infty \int_0^\infty f_2(E_1,E_2) E_1^{s-1} E_2^{q-1}\,dE_1\,dE_2 \tag{6.21}$$

Then we have

$$\frac{\partial P_1(s,t)}{\partial t} = -A_s P_1(s,t)$$

$$\frac{\partial P_2(s,q,t)}{\partial t} = -P_2(s,q,t)(A_s + A_q) + P_1(s+q-1)\,\alpha(s,q) \tag{6.22}$$

where

$$A_s = \phi - 2\phi_s, \quad \phi = \int_0^1 R(\epsilon)\,d\epsilon,$$

$$\phi_s = \int_0^1 R(\epsilon)\,\epsilon^{s-1}\,d\epsilon \qquad \alpha(s,q) = 2\int_0^1 R(\epsilon)\,\epsilon^{s-1}(1-\epsilon)^{q-1}\,d\epsilon$$

The solutions are

$$P(s,t) = E_0^{s-1}\, e^{-A_s t},$$

$$P_2(s,q,t) = E_0^{s+q-2}\,\alpha(s,q)\,x$$

$$x\left(\frac{e^{-A_{s+q-1}t} - e^{-(A_s+A_q)t}}{A_s + A_q - A_{s+q-1}}\right) \tag{6.23}$$

f_1 and f_2 are obtained by inverting the $P_1(s,t)$ and $P_2(s,q,t)$. The mean and mean square numbers of particles with energy greater than E_c are given by

$$\bar{n}(>E_c) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E_c}\right)^{s-1}\frac{e^{-A_s t}}{(s-1)}\,ds \qquad (6.24)$$

$$\overline{n^2}(>E_c) = \frac{1}{(2\pi i)^2}\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{1}{E_c}\right)^{s+q-2}\frac{P_2(s,q,t)}{(s-1)(q-1)}\,ds\,dq$$

A very simple result can be obtained if we integrate the integro-differential equations over the whole range 0_- to E_0. Defining

$$\bar{n} = \int_0^{E_0} f_1(E)\,dE, \quad \overline{nn} = \int_0^{E_0}\int_0^{E_0} f_2(E_1,E_2)\,dE_1\,dE_2,$$

$$\overline{n^2} = \bar{n} + \overline{nn}$$

we have

$$\frac{\partial \bar{n}}{\partial t} = \phi\,\bar{n}, \quad \frac{\partial \overline{nn}}{\partial t} = -2\overline{nn}\,\phi + 4\overline{nn}\,\phi + 2\frac{d\bar{n}}{dt} \qquad (6.25)$$

i.e.

$$\bar{n} = e^{\phi t}, \quad \overline{nn} = 2e^{2\phi t} - 2e^{\phi t}, \quad \overline{n^2} = 2\bar{n}^2 - \bar{n}$$

which is the familar result for a Furry distribution. The above simplification results in view of the homogeneity assumption for R which according to physical fact is not valid for low energies. That is why we ask the question: What is the mean and mean square number of particles above a certain energy E_c ?

Model II of Janossy and Heitler*

Recently Janossy and Heitler have worked out the probabilities for the production of n mesons by a nucleon in its passage through matter of thickness t on the basis of two assumptions

1) A low energy cut off at E_c.

2) An energy spectrum for the initial primary:(See also A. Ramakrishnan, Proc.Phys.Soc. A 63 (1950), 861). We shall now deal with the more complicated problem of the calculation of the probabilities for the emission of n mesons above the critical energy. Stating in mathematical terms the most general formulation is as follows.

Given that the probability per unit thickness that a nucleon of energy E produces a meson of energy between E' and E' + dE' is R(E,E') dE' (the mesons have no capacity to multiply) and that a nucleon is incident at t = 0 our aim is to calculate the mean and mean square number of mesons

*Janossy and Heitler suggested this model in their paper Proc. Phys. Soc. A. 62, (1949), 669 later Janossy in Proc. Phys. Soc. A. 63 (1950), 241 used Model I.

produced above the critical energy E_c represented by $\overline{m}\,(>E_c)$ and $\overline{m^2}\,(>E_c)$ respectively.

Using the method of product densities and assuming that at $t = 0$ the nucleon has energy E_o we define

1) $f_1(E,t;E_o)dE$ is the probability that the initial nucleon has an energy between E and $E + dE$ at thickness t. In this case there is only one nucleon and so the ordinary concept of probability density coincide with the density of degree 1. Higher product densities of nucleons vanish.

2) $g_1(E,t;E_o)$ and $g_2(E_1,E_2,t;E_o)$ as densities of degree one and two of mesons.

3) $fg_{1,1}(E_1,E,t;E_o)dE_1dE_2$ as the joint probability that the nucleon lies in dE_1 and a meson in dE_2. Then f_1, g_1, g_2 and $fg_{1,1}$ satisfy

$$\frac{\partial f_1(E,t;E_o)}{\partial t} = -f_1(E,t;E_o)\int_0^E R(E,E')\,dE' + \int_E^{E_o} f_1(E',t;E_o)\,R(E',E'-E)\,dE' \tag{6.26}$$

$$\frac{\partial g_1(E,t;E_o)}{\partial t} = \int_E^{E_o} f_1(E',t;E_o)\,R(E',E)\,dE \tag{6.27}$$

$$\frac{\partial fg_{1,1}(E_1,E_2,t;E_o)}{\partial t} = -fg_{1,1}(E_1,E_2,t;E_o)\int_0^{E_1} R(E_1,E)\,dE + \int_{E_1}^{E_o} fg_{1,1}(E',E_2)\,R(E',E'-E_1)\,dE' + f_1(E_1+E_2)\,R(E_1+E_2,E_2) \tag{6.28}$$

$$\frac{\partial g_2(E_1, E_2, t; E_0)}{\partial t} = \int_{E_1}^{E_0} f g_{1,1}(E', E_2; E_0) R(E', E_1) dE' + \int_{E_2}^{E_0} f g_{1,1}(E', E_1; E_0) R(E', E_2) dE' \quad (6.29)$$

If we now assume $R(E, E')\,dE' = R(\epsilon)\,d\epsilon$, $\epsilon = \frac{E'}{E}$, then applying the Mellin's transformation

$$\nu(s,t) = \int_0^{E_0 \text{ or } \infty} f_1(E,t)\, E^{s-1} dE, \quad \mu(s,t) = \int_0^{\infty} g_1(E.t) E^{s-1} dE$$

$$\nu\mu(s,q,t) = \int_0^{\infty}\int_0^{\infty} f g_{1,1}(E_1, E_2, t)\, E_1^{s-1} E_2^{q-1} dE_1 dE_2$$

$$\mu\mu(s,q,t) = \int_0^{\infty}\int_0^{\infty} g_2(E_1, E_2, t)\, E_1^{s-1} E_2^{q-1} dE_1 dE_2 \quad (6.30)$$

We obtain the equations

$$\frac{\partial \nu(s,t)}{\partial t} = -A_s \nu(s,t), \quad \nu(s,t) = E_0^{s-1} e^{-A_s t}$$

$$\frac{\partial \mu(s,t)}{\partial t} = \nu_s \beta_s, \quad \mu(s,t) = \beta_s \int_0^t \nu(s,t)\, dt$$

$$\frac{\partial \nu\mu(s,q,t)}{\partial t} = -\nu\mu(s,q,t)\beta_s + \nu\mu(q,s)\beta_q$$

$$\frac{\partial \mu\mu(s,q,t)}{\partial t} = \nu\mu(s,q,t)\beta_s + \nu\mu(q,s)\beta_q$$

$$A_s = \phi - \phi_s, \quad \phi = \int_0^1 R(\epsilon)\, d\epsilon$$

$$\phi_s = \int_0^1 R(1-\epsilon)\, \epsilon^{s-1} d\epsilon, \quad \beta_s = \int_0^1 R(\epsilon)\, \epsilon^{s-1} d\epsilon \quad (6.31)$$

yielding the solutions

$$\nu\mu(s,q,t) = e^{-Ast}\int_0^t e^{Ast}\,\nu(s+q-1,t)\,\alpha(s,q)\,dt$$

$$\alpha(s,q) = \int_0^1 R(\epsilon)\,\epsilon^{s-1}(1-\epsilon)^{q-1}\,d\epsilon$$

$$\mu\mu(s,q,t) = \int_0^t \{\nu\mu(s,q,t)\beta_s + \nu\mu(q,s,t)\beta_q\}\,dt \quad (6.32)$$

Thus

$$\overline{m}(>E_c) = \int_{E_c}^{E_0} g_1(E,t)\,dE = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{1}{E_c}\right)^{s-1}\frac{\mu(s,t)}{(s-1)}\,ds \quad (6.33)$$

and

$$\overline{m^2}(>E_c) = \overline{m}(>E_c) + \overline{mm}(>E_c)$$

where

$$\overline{mm}(>E_c) = \left(\frac{1}{2\pi i}\right)^2\int_{\sigma-i\infty}^{\sigma+i\infty}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{1}{E_c}\right)^{s+q-2}\frac{\mu\mu(s,q,t)}{(s-1)(q-1)}\,ds\,dq \quad (6.24)$$

A simple result can be obtained if we require the mean and means square number of mesons ($\overline{m}$ and $\overline{m^2}$) in the whole, energy range 0 to E_0. For then we have, defining

$$\overline{n} = \int_0^{E_0} f_1(E,t)\,dE, \quad \int_0^{E_0}\int_0^{E_0} f g_{1,1}(E_1,E_2)\,dE_1\,dE_2 = \overline{nm}$$

and $\overline{mm}$ as $\int_0^{E_0}\int_0^{E_0} g_2(E_1,E_2,t)\,dE_1\,dE_2$

$$\frac{d\bar{n}}{dt} = 0 \quad \bar{n} = 1 \qquad \text{i.e. there is only one nucleon}$$

$$\frac{d\bar{m}}{dt} = \phi\bar{n} \;, \quad \bar{m} = \phi t$$

$$\frac{d\overline{nm}}{dt} = -\overline{nm}\,\phi + \overline{nm}\,\phi + \frac{d\bar{m}}{dt} \qquad \text{i.e. } \overline{nm} = \phi t$$

$$\frac{d\overline{nlm}}{dt} = 2\phi^2 t \;, \quad \overline{mm} = \bar{m}^2 \tag{6.35}$$

i.e.

$$\overline{m^2} = \bar{m} + \bar{m}^2$$

the familiar result for a Poisson distribution.

CHAPTER VII

Method II or the method of Regeneration Points

Until now, we have been dealing with the applications of Method I to physical problems. It must however be mentioned that the product density technique can be applied in conjunction with Method I or Method II provided that the densities are defined with respect to a variable, say, E different from t the variable with respect to which the stochastic process develops*. We shall in this chapter outline Janossy's application of Method II to the statistical problem of cosmic radiation which yielded integral equations with respect to t. We shall also deal with the application of this method to some simple multiplicative processes. This method had been used earlier by Palm, Bellman and Harris to processes where the difficulty relating to a continuous infinity of stochastic variable did not arise. The historical development of this method has been described in detail by Bartlett and Kendall who called it the method of regeneration points.

1. Outline of Janossy's treatment

Janossy, in order to avoid the complication of dealing with a continuous infinity of stochastic variables defined a function $\Pi^{(i)}(n, E, E_0, t)$ representing the probability that

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* In the next chapter we deal with a class of stochastic processes where the product densities have been defined with respect to t.

there are n electrons above the energy E at t, given that at $t=0$ there was a primary of energy E_0. If the primary is an electron we put $i=1$, if it is a photon $i=2$. He visualised the stochastic process from the following point of view. Somewhere between 0 and t, say between o and t the primary, if it is an electron may create a photon and the initial electron and the newly created photon will then become independent primaries of stochastic processes of 'duration' $t-\tau$. If at $t=0$ we have a photon, it may create a pair of electrons between τ and $\tau+d\tau$ and these electrons become independent primaries of stochastic processes of duration $t-\tau$. Viewing the development of the process thus, we find that. $\pi^{i}(n,E,E_0,t)$ satisfies the integral equations (for i=1, i=2)

$$\pi^{(1)}(n,E,E_0,t) = \left[\int_0^t e^{-\alpha_1\tau}\,d\tau\right]\times \tag{7.3}$$
$$\times\int_E^{E_0}\left\{\sum_{n_1+n_2=n}\pi^{(1)}(n_1,E,E',t-\tau)\,\pi^{(2)}(n_2,E,E_0-E',t-\tau)\right\}\times$$
$$\times R(E_0,E')\,dE' + \delta(n-1)\,e^{-\alpha_1 t}$$

$$\pi^{(2)}(n,E,E_0,t) = \left[\int_0^t e^{-\alpha_2\tau}\,d\tau\right]\times \tag{7.4}$$
$$\times\int_E^{E_0}\left\{\sum_{n_1+n_2=n}\pi^{(1)}(n_1,E,E',t-\tau)\,\pi^{(1)}(n_2,E,E_0-E',t-\tau)\right\}R'(E_0,E')\,dE'$$
$$+\delta(n)\,e^{-\alpha_2 t}$$

where $\alpha_1 = \int_0^{E_0} R(E_0,E')\,dE'$, $\alpha_2 = \int_0^{E} R'(E_0,E')\,dE'$ (7.2)

$R(E_0,E')\,dE'$ represents the probability per unit t that an electron of energy E_0 drops to the interval lying between E' and $E'+dE'$ radiating a photon of energy E_0-E'.

represents the probability per unit t that a photon of energy splits up into a pair of electrons one of which has an energy lying between E' and $E'+dE'$. The equations are self-explanatory. For example in equation (7.1) $e^{-\alpha_1 \tau}$ is the probability that no event occurs between 0 and τ and $e^{-\alpha_1 \tau} R(E_0, E') dE' d\tau$ is the probability that the first event defined by a transition of electron from E_0 to E' occurs. The first event may occur between 0 and t, hence the integration over τ, or it may not occur at all with a probability $e^{-\alpha_1 \tau}$ hence the term $\delta(n-1)\, e^{-\alpha_1 t}$ where $\delta(X)=1$ for $X=0$ and zero otherwise. The energy transitions may occur over the whole range 0 to E_0, hence the integration over E'. The summation $\sum_{n_1+n_2=n}$ indicates that the new primaries at τ must create n_1 and n_2 electrons respectively such that $n_1+n_2=n$. Similar arguments apply to the equation for $\pi^{(2)}$. Differentiating the equations with respect to t we have

$$\frac{\partial \pi^{(1)}(n, E, E_0, t)}{\partial t} = -\alpha_1 \pi^{(1)}(n, E, E_0, t) + \int_E^{E_0} \Big\{ \sum_{n_1+n_2=n} \pi^{(1)}(n_1, E, E', t)\, \pi^{(2)}(n_2, E, E_0-E', t) \Big\} \times R(E_0, E')\, dE' \tag{7.3}$$

$$\frac{\partial \pi^{(2)}(n, E, E_0, t)}{\partial t} = -\alpha_2 \pi^{(2)}(n, E, E_0, t) + \int_E^{E_0} \Big\{ \sum_{n_1+n_2=n} \pi^{(1)}(n_1, E, E', t)\, \pi^{(1)}(n_2, E, E_0-E', t) \Big\} \times R'(E_0, E')\, dE' \tag{7.4}$$

This was the procedure adopted by Janossy. But if we follow the view point of the author, (outlined in the introduction

Chapter I) the above integro-differential equations could have been obtained directly. To proceed to solve these equations we use the method of generating functions. We also assume that $R(E, E')dE'$ and $R'(E, E')\,dE'$ can be expressed as functions of E'/E i.e. say $R(\epsilon)\,d\epsilon$, and $R'(\epsilon)\,d\epsilon$ respectively.

Defining

$$G_1(u, \epsilon, t) = \sum_n \pi^{(1)}(n, \epsilon, t)\, u^n, \quad \epsilon = E/E_0$$

$$F(u, \epsilon, t) = \sum_n \pi^{(2)}(n, \epsilon, t)\, u^n$$

we obtain

$$\frac{\partial G(u, \epsilon, t)}{\partial t} = -\alpha_1 G(u, \epsilon, t) + \int_\epsilon^1 G(u, \epsilon/\epsilon', t) F(u, \epsilon/1-\epsilon, t) R(\epsilon')d\epsilon' \tag{7.5}$$

$$\frac{\partial F(u, \epsilon, t)}{\partial t} = -\alpha_2 F(u, \epsilon, t) + \int_\epsilon^1 G(u, \frac{\epsilon}{\epsilon'}, t)\, G_1(u, \frac{\epsilon}{1-\epsilon'}, t) R'(\epsilon')d\epsilon' \tag{7.6}$$

These were the equations obtained by Janossy.

2. Simplified problem of Cosmic Radiation

In the Janossy equations we shall make ϵ tend to zero. Writing $\pi^i(n, \epsilon, t)$ for $\epsilon = 0$ as $\pi^i(n, t)$ and defining $G_1(u, t)$ and $F(u, t)$ accordingly

$$\frac{\partial \pi^{(1)}(n, t)}{\partial t} = \left\{ -\pi^{(1)}(n, t) + \sum_{n_1 + n_2 = n} \pi^{(1)}(n_1, t)\, \pi^{(2)}(n_2, t) \right\} \alpha' \tag{7.7}$$

$$\frac{\partial \pi^{(2)}(n,t)}{\partial t} = \left\{ -\pi^{(2)}(n,t) + \sum_{n_1+n_2=n} \pi^{(1)}(n_1,t)\,\pi^{(1)}(n_2,t) \right\} \alpha_2 \tag{7.8}$$

$$\frac{\partial G}{\partial t} = (GF - G)\,\alpha_1 \tag{7.9}$$

$$\frac{\partial F}{\partial t} = (G^2 - F)\,\alpha_2 \tag{7.10}$$

The initial conditions are: At t=0

$$\pi^{(1)}(n,0) = 0 \quad \text{for} \quad n \neq 1 \quad \text{i.e.} \quad n = 0, 2, 3, 4, \ldots$$

$$\pi^{(1)}(1,0) = 1$$

and

$$\pi^{(2)}(n,0) = 0 \quad \text{for} \quad n \neq 0 \quad \text{i.e.} \quad n = 1, 2, 3, 4, \ldots$$

$$\pi^{(2)}(0,0) = 1$$

But for all t, $\pi^{(1)}(0,t) = 0$. This means that for an electron initiated shower there is at least one electron at all thicknesses since we do not have an absorption probability for electrons. We apply these conditions to equations (7.7) and (7.8). Since

$$\frac{\partial \pi^{(2)}(1,t)}{\partial t} = \left\{ \pi^{(1)}(1,t)\,\pi^{(1)}(0,t) + \pi^{(1)}(0,t)\,\pi^{(1)}(1,t) - \pi^{(2)}(1,t) \right\} \alpha_2$$

and $\pi(0,t) = 0$ for all t, we have $\pi^{(2)}(1,t) = A e^{-\alpha_2 t}$. But at $t = 0$, $\pi^{(2)}(1,t) = 0$ whence $A = 0$ and $\pi^{(2)}(1,t) = 0$ for all t. Now applying this result to equation (7.7) we have

$$\frac{d\pi^{(1)}(2,t)}{dt} = -\alpha_1 (1 - e^{-\alpha_2 t})\,\pi^{(1)}(2,t)$$

$$= f(t)\,\pi^{(1)}(2,t)$$

where $f(t) = -\alpha_1 (1 - e^{-\alpha_2 t})$

$$\pi^{(1)}(2,t) = A e^{\int_0^t f(t)\,dt}$$

But $\pi^{(1)}(2,t)=0$ at $t = 0$ whence $A = 0$ and $\pi^{(1)}(2,t) = 0$ for all t. By successive application of thdse conditions we get $\pi^{(2)}(3,t) = 0$ for all t, $\pi^{(1)}(4,t) = 0$ for all t and in general

$$\pi^{(1)}(n,t) = 0 \text{ for all t if n is even}$$
$$\pi^{(2)}(n,t) = 0 \text{ for all t if n is odd.}$$

This corresponds to the physical situation; since electrons are produced in pairs an electron initiated shower should have an odd number of electrons and a photon initiated shower should have an even number.

It should be noted that the same equations represent the photon distribution but the equations will then satisfy different initial conditions. Thus if we are concerned with the number of photons at t, interpreting $\pi^{(1)}$, $\pi^{(2)}$, G and F accordingly we write the initial conditions as

$$\pi^{(1)}(n,0) = 1 \text{ if } n = 0 \quad \pi^{(1)}(n,0) = 0 \text{ if } n \neq 0$$

$$\pi^{(2)}(n,0) = 1 \text{ if } n = 1 \quad \pi^{(2)}(n,0) = 0 \text{ if } n \neq 1$$

Under these conditions we find $\pi^{(1)}(n,t)$ and $\pi^{(2)}(n,t)$ exist for all values of t, that is, the number of photons can be even or odd.

It is a special feature of Janossy's approach that the simultaneous equations connect a photon initiated shower of electrons with an electron initiated shower of electrons while the well-known equations of the cascade theory* connect the number of photons of a shower with the number of electrons.

Operating on G and F and writing

$$\left(\frac{\partial G_1}{\partial u}\right)_{u=1} = \bar{n} \quad , \quad \left(\frac{\partial F}{\partial u}\right)_{u=1} = \bar{m}$$

$$\left(\frac{\partial^2 G_1}{\partial u^2}\right)_{u=1} = \nu - \bar{n} \quad , \quad \left(\frac{\partial^2 F}{\partial u^2}\right)_{u=1} = \mu - \bar{m}$$

we find that $\bar{n}$, $\bar{m}$, ν, μ satisfy the equations

$$\frac{d\bar{n}}{dt} = \bar{m}\alpha, \quad \frac{d^2\bar{m}}{dt^2} + \beta\frac{d\bar{m}}{dt} - 2\alpha\beta\bar{m} = 0 \qquad (7.11)$$

$$\frac{d^2\mu}{dt^2} + \beta\frac{d\mu}{dt} - 2\alpha\beta\mu = 8\bar{n}\bar{m}\alpha\beta \qquad (7.12)$$

$$\nu - \bar{n}^2 = \int_0^t \mu\alpha\, dt$$

According to the quantum-mechanical cross-sections now available α is infinite, contrary to physical facts. So, to build up our statistical model, we shall take $\alpha = \beta$ **. By choosing a suitable unit of thickness we can without loss of generality put $\alpha t = \tau$.

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* For literature on cascade theory see Janossy (1948)

**For $\alpha \neq \beta$ the author has not been able to find explicit solutions for the equations.

$$\frac{\partial G}{\partial \tau} = GF - G \ , \quad \frac{\partial F}{\partial \tau} = G^2 - F \tag{7.13}$$

At $\tau = 0$, $G = u$, $F = 1$ for electron shower, .

$G = 1$, $F = u$ for photon shower.

To solve the equation we put

$$g(\tau) = G(\tau) e^{\tau}, \quad f(\tau) = F(\tau) e^{\tau} \text{ and } y = e^{-\tau}$$

This yields

$$\frac{\partial g}{\partial y} = -gf, \qquad \frac{\partial f}{\partial y} = -g^2$$

The solution of these equations is readily found to be

Electron Distribution

$$G = \sqrt{1-u^2}\, e^{-\tau} \operatorname{Cosech} \left\{ (e^{-\tau} - 1)\sqrt{1-u^2} + \coth^{-1} \frac{1}{\sqrt{1-u^2}} \right\}$$

$$F = \sqrt{1-u^2}\, e^{-\tau} \coth \left\{ (e^{-\tau} - 1)\sqrt{1-u^2} + \coth^{-1} \frac{1}{\sqrt{1-u^2}} \right\} \tag{7.14}$$

Photon Distribution

$$G = \sqrt{u^2-1}\, e^{-\tau} \operatorname{Cosech} \left\{ (e^{-\tau} - 1)\sqrt{u^2-1} + \coth^{-1} \frac{u}{\sqrt{u^2-1}} \right\}$$

$$F = \sqrt{u^2-1}\, e^{-\tau} \coth \left\{ (e^{-\tau} - 1)\sqrt{u^2-1} + \coth^{-1} \frac{u}{\sqrt{u^2-1}} \right\} \tag{7.15}$$

In his symposium paper Bartlett (1949) obtained the differential equation for the generating function $\Pi(Z, W, t)$

for the joint distribution of electrons and photons, z and w being the corresponding variables. He obtained the equation (his equations (32) and (33) with $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = 0$)

$$\frac{\partial \pi}{\partial t} = (ZW - Z)\frac{\partial \pi}{\partial Z} + (Z^2 - W)\frac{\partial \pi}{\partial W} \qquad (7.16)$$

It is quite clear that the equations (3) and (4) are the equations for the marginal distributions. In a private communication he has pointed out that the above equation can be completely solved when the equations for the marginal distribution can be solved and these are mathematically the same as (3) and (4) with Z and W in place of n_1 and n_2. We give here the result,

Electron initiated shower

$$\pi(Z,W) = e^{-t\sqrt{W^2-Z^2}}\,\mathrm{cosech}\left\{(e^{-t}-1)\sqrt{W^2-Z^2}\right. \qquad (7.17)$$
$$\left. + \coth^{-1}\frac{W}{\sqrt{W^2-Z^2}}\right\}$$

Photon initiated shower

$$\pi(Z,W) = e^{-t\sqrt{W^2-Z^2}}\coth\left\{(e^{-t}-1)\sqrt{W^2-Z^2}\right. \qquad (7.18)$$
$$\left. + \coth^{-1}\frac{W}{\sqrt{W^2-Z^2}}\right\}$$

Putting Z=1 or W=1 we get the generating functions of the marginal distributions.

I.J.Good (1949), in his contribution to the Symposium discoussion has derived equations (3) and (4) as special case of a more general equation. This he obtained by a different method, by first considering the time (or thickness) parameter t as a discrete and passing to continuous t as a limiting case.

The mean number of electrons and photons for electron and photon initiated showers are given below.

Electron initiated shower

$$\text{Mean number of electrons} = \frac{2}{3}\left(e^{\tau} + \frac{1}{2}e^{-2\tau}\right)$$
$$\text{Mean number of photons} = \frac{1}{3}\left(e^{\tau} - e^{-2\tau}\right) \qquad (7.19)$$

Photon initiated shower

$$\text{Mean number of electrons} = \frac{2}{3}\left(e^{\tau} - e^{-2\tau}\right)$$
$$\text{Mean number of photons} = \frac{1}{3}\left(e^{\tau} + 2e^{-2\tau}\right) \qquad (7.20)$$

It is to be noted that if we do not distinguish between the photons and electrons the total number of particles is e^{τ} which is as it should be. Bartlett considered also an approximate 'randomised' solution for the simplified problem of cosmic radiation but he has asked me to point out that the above solution for the mean number indicates that the right value for the constant occurring in his solution is $\frac{1}{3}$, and not $\frac{1}{4}$ as first suggested by him.

Calculating the mean and mean square deviation of the number of electrons and photons under the assumption $e^{t} \gg e^{-t}$

we find

Mean square deviation of electrons $\sim$ (mean number of electrons)^2

Mean square deviation of photons $\sim$ (mean number of photons)^2

in the case of both photon and electron initiated showers.

If we have two stochastic variables X and Y and we assume that

$$E(X^2) - \{E(X)\}^2 = \sigma^2(X) = \{E(X)\}^2$$

$$E(Y^2) - \{E(Y)\}^2 = \sigma^2(Y) = \{E(Y)\}^2$$

and writing $Z = X+Y$, if we have

$$E(Z^2) - \{E(Z)\}^2 = \sigma^2(z) = \{E(z)\}^2 \qquad (7.21)$$

it follows that $E(XY) - E(X)E(Y) = E(X)E(Y)$. Since we know that the total number of particles obeys a Furry distribution* and that the number of electrons as well as the number of photons obey a Furry law approximately (and that $E(X)$ and $E(Y)$ are of the same order of magnitude) it follows that the correlation coefficient between electrons and photons is approximately 1.

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*By a Furry process we mean a simple multiplicative process of only one type of particle. The generating function of the process is given as the solution of $\partial G/\partial t = G^2 - G$, $G = u$ at $t=0$ the 'birth' probability per unit t being arranged to be unity. $E(n) = e^t$, $\sigma^2(n) = E(n^2) - [E(n)]^2 = [E(n)]^2 - E(n)$ If $e^t \gg 1$, $\sigma^2(n) \simeq [E(n)]^2$

In view of the peculiar nature of the generating functions obtained it is relevant to state a simple result of general validity regarding generating functions.

Assuming that a generating function $G_1(u)$ is regular at $u=0$ it can be expanded in the form of a Taylor series about $u=0$

$$G_1(u) = G_1(0) + G_1'(0)u + G_1''(0)\frac{u^2}{2!} + \ldots\ldots$$

But by definition

$$G_1(u) = P(0) + P(1)u + P(2)u^2 + \ldots\ldots$$

where $P(n)$ is the corresponding probability for a discrete distribution.

Hence

$$P(n) = \left\{ G_1^{(n)}(0) \,/\, n! \right\} \qquad (7.22)$$

It immediately follows that if $G_1(u) = G_1(-u)$. Then $P(n)=0$ when n is odd and that if $G_1(u) = -G_1(-u)$ then $P(n)=0$ when n is even.

In the case of electron initiated shower of electrons to recognise that G is odd and F is even we expand

$$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$$

Here

$$B = \coth^{-1}\frac{1}{\sqrt{1-u^2}} = \cosh^{-1}\frac{1}{u} = \sinh^{-1}\frac{\sqrt{1-u^2}}{u}$$

Hence for the electron distribution of an electron initiated shower*

$$G = u\sqrt{1-u^2}\, e^{-\tau} \Big/ \{ \sinh A + \sqrt{1-u^2} \cosh A \} \qquad (7.23)$$

where

$$A = (e^{-\tau} - 1)\sqrt{1-u^2}$$

By similar considerations we recognise F to be an even function of u when it represents the generating function for the distribution of photons.

3. The negative binomial distribution

We shall construct a statistical model which is slightly different from the Furry model. We shall assume that whenever the initial particle (ancestor) gives 'birth' to a new particle, these newly created or secondary particles and their descendents have a probability to break up (into two) equal to β times the corresponding probability for the initial 'ancestor'. If G is the generating function of the distribution of the total number of particles (omitting the ancestor) and F that arising from each secondary particle (including that particle itself) then

$$\frac{\partial G}{\partial t} = GF - G \,, \quad (G = 1 \text{ at } t = 0) \qquad (7.24)$$

* In a private communication D.G.Kendall has pointed out that these results were obtained by Consael using Method I.

where

$$\frac{\partial F}{\partial t} = (F^2 - G)\beta \qquad (F = u \text{ at } t = 0) \tag{7.25}$$

If $\beta = 1$, F represents a Furry distribution, and $G = F/u$

If $\beta = 0$, then $F = u$ and

$$\frac{\partial G}{\partial t} = Gu - G, \quad G = e^{(u-1)t} \tag{7.26}$$

giving the generating function of a Poisson distribution. For general β

$$G = \left[u - (u-1)e^{\beta t}\right]^{-1} / \beta \tag{7.27}$$

For $\beta \to 0$, $G = e^{(u-1)t}$ and for $\beta \to 1$, $G \to 1/[u-(u-1)e]$

and for general the mean number of particles omitting the ancestor is

$$\pi = \left[e^{\beta t} - 1\right] / \beta$$

The mean square deviation is $\beta \bar{n}^2 + \bar{n}$, giving the deviation for this distribution which is variously known as the Polya or negative binomial or Pascal distribution.

We could have obtained the following difference-differential equation for $\pi(n,t)$, the probability that there are n secondaries produced by the primary,

$$\frac{\partial \pi(n,t)}{\partial t} = -[1+\beta n]\pi(n,t) + [1+\beta(n-1)]\pi(n-1,t) \tag{7.28}$$

$$\pi(n,0) = 0 \quad \text{if} \quad n \neq 0, \quad \pi(n,0) = 1, \text{ if } n = 0$$

By successive operations we obtain

$$\pi(n,t) = \left(\frac{1-e^{-\beta t}}{\beta}\right)^n \frac{1\cdot(1+\beta)(1+2\beta)\cdots(1+\overline{n-1}\beta)e^{-t}}{n!} \quad (7.29)$$

$$\pi(0,t) = e^{-t}$$

Arley considered the distribution

(7.30)

$$\pi(n,x) = \left(\frac{\lambda x}{1+\beta\lambda x}\right)^n \frac{1(1+\beta)(1+2\beta)\cdots(1+\overline{n-1}\beta)\ (1+\beta\lambda x)^{-1/\beta}}{n!}$$

Putting $\beta = 1$

$$\pi(n,x) = \frac{1}{(1+\lambda x)}\left\{1-\frac{1}{1+\lambda x}\right\}^n \quad (7.31)$$

To interpret this as a Furry distribution he used the transformation

$$e^{\lambda t} = (1+\lambda x) \quad (7.32)$$

This artificial transformation is not necessary if we take the model suggested above. The variables t and x are connected by the relation

$$e^{\beta t} = (1+\beta\lambda x) \quad (7.33)$$

To obtain the Polya distribution Arley assumed the probability that a particle is born when there are n particles in

existence at χ to be

$$\frac{\lambda(1+\beta n)}{(1+\beta\lambda\chi)}\,d\chi$$

This quantity is clearly equal to $(1+\beta n)\,dt$ and hence we obtain the difference equation given above.

Recently Anscombe (1950) has recalled a number of ways in which the negative binomial distribution can arise in biology. One of them bears a close relation to the above derivation. 'If colonies or groups of individuals are randomly distributed over an area in a Poisson distribution we obtain a negative binomial distribution for the total count if the numbers of individuals in the colonies are distributed independently in a logarithmic series distribution'. The problem considered above corresponds to the case of a Poisson distribution of colonies arising in time, each colony, once formed, growing as a Furry distribution; this case is also briefly mentioned by Anscombe.

It should be pointed out that the process discussed above is essentially identical with the simple birth and immigration process of Kendall by regarding the initial ancestor as a source of 'immigrants'. The main object of the above discussion has been to compare it with Arley's treatment.

4. The simple 'birth, death, and immigration' process of Kendall

If we assume the probability of the 'death' of a particle (per unit range of t) to be equal to μ, if we let the

corresponding rate of 'immigration' into the population be equal to η, and if we take the probability per unit t of 'birth' to be unity, then by simple arguments we are led to the following equations for G, the generating function for the number of individuals in a population generated by 'immigration' and the function F which denotes the generating function arising from each immigrant.

$$\frac{\partial G}{\partial t} = (GF - G)\eta \qquad \frac{\partial F}{\partial t} = (F-1)(F-\mu) \tag{7.34}$$

$$G = 1 \text{ at } t=0 \qquad F = u \text{ at } t=0$$

when $\mu \neq 1$

$$F = 1 + \frac{m a e^{mt}}{1 - a e^{mt}} \qquad G = \left(\frac{1 - a e^{mt}}{1-a}\right)^{-\eta}$$

when $\mu = 1$

$$F = \frac{u - (u-1)t}{1 - (u-1)t} \qquad G = (1 - (u-1)t)^{-\eta}$$

If at $t=0$, we have n_0 members of the population then the generating function for the distribution of the population at t is obviously GF^{n_0}.

5. Multiple production

We shall now introduce a new feature into the type of stochastic process until now discussed - that of multiple

production. We assume that a particle is replaced by a statistical distribution of particles represented by $\phi(n)$ with a probability α per unit t. (Without loss of generality we can put $\alpha=1$ by suitably choosing a unit of t) $\phi(0)$represent the probability per unit t of absorbtion. Also $\phi(1)=0$ since 'replacement' and the existence of the particle are mutually exclusive. If G is the generating function of the total number of particles at t then G obviously satisfies

$$\frac{\partial G}{\partial t} = \phi(G) - G \tag{7.35}$$

If we know that if a replacement occurs, exactly N particles replace the existing particle (or in other words the existing particle produces N-1 particles then $\phi(u)=u^N$ and

$$\frac{\partial G}{\partial t} = G^N - G$$

If $n=2$ we get a Furry process. For a general ϕ it may be difficult to solve for G completely but it is quite easy to get the first and second moments from G. Putting

$$\left(\frac{\partial G}{\partial u}\right)_{u=1} = \bar{n}\,,\quad \left(\frac{\partial^2 G}{\partial u^2}\right)_{u=1} = \overline{n^2} - \bar{n},$$

$$[\partial\phi(u)/\partial u]_{u=1} = \bar{a}\,,\quad [\partial^2\phi(u)/\partial u^2]_{u=1} = \overline{a^2} - \bar{a}$$

we have

$$\bar{n} = e^{(\bar{a}-1)t},\quad \overline{n^2} - \bar{n}^2 = \frac{\overline{a^2} - 2\bar{a} + 1}{\bar{a}-1}\bar{n}(\bar{n}-1) \tag{7.36}$$

If we assume that at t=0 there is a primary and the primary

produce n secondaries with a probability per unit t equal to α, and that the secondaries do not have the capacity to multiply then if G be the generating function of the distribution of the total number of particles then

$$\frac{\partial G}{\partial t} = (G u^n - G)\alpha, \quad G = e^{(u^n - 1)\alpha t} \tag{7.37}$$

Le Couteur's model for nuclear evaporation

There are many physical experiments, such as nuclear disintegration, in which a single primary particle gives rise to several secondary particles by a process which is itself unobservable. A possible, much simplified mathematical model is a population in which each member has probabilities $\lambda(t)dt$ of giving birth, $(\mu-\gamma)(t)dt$ of death and $\gamma(t)dt$ of emigration in the time interval t to $t+dt$. Only the emigrants are supposed to be observable and one has to calculate the total number of emigrants out of a population generated by a single ancestor.

If G be the generating function of the distribution of emigrants then G satisfies the equation

$$\frac{\partial G}{\partial t} = G^2\lambda + (\mu-\gamma) + u\gamma - G(\mu+\lambda) \tag{7.38}$$

If λ, μ, γ are independent of t, we can solve the equation completely.

$$\frac{\partial G_1}{\partial t} = \lambda (G_1 - \alpha)(G_1 - \beta)$$

where

$$\alpha, \beta = \left(\frac{\mu+\lambda}{\lambda}\right) \pm \sqrt{\left(\frac{\mu+\lambda}{\lambda}\right)^2 - 4\left(\frac{\mu - \gamma + u\gamma}{\lambda}\right)}$$

$$G_1 = \frac{\beta - \alpha}{\left(\frac{1-\alpha}{1-\beta}\right) e^{(\alpha-\beta)\lambda t} - 1} + \beta \qquad (7.39)$$

at $t \to \infty$, $e^{(\alpha-\beta)\lambda t} \to \infty$ and $G_1 \to \beta$. Thus

$$G \to \left(\frac{\mu+\lambda}{\lambda}\right) - \sqrt{\left(\frac{\mu+\lambda}{\lambda}\right)^2 - 4\left(\frac{\mu - \gamma + u\gamma}{\lambda}\right)}$$

which is Moyal's solution of Le Couteur's problem.

6. A problem on counters

We shall now consider a simple stochastic problem relating to electron counters which are supposed to register random events such that the probability of an event in any small interval of time of length Δt (Δt is an infinitesimal quantity) independently of previous events is $\alpha \Delta t + O\Delta t$ or simply Δt (provided we suitably choose the unit of t). The probability that n events occur during a time interval of length t is given by the Poisson law

$$\frac{e^{-t}\,t^{n}}{n!}$$

We shall assume that every event is followed by a dead time a during which no event is registered. Our problem is to calculate $\pi(n,t)$ that n events are recorded in time t. It is quite clear that $\pi(n,t)$ satisfies the integral equation

$$\pi(n,t) = \int_0^a e^{-\tau}\,\pi(n,t-\tau)\,d\tau + \int_a^t e^{-\tau}\,\pi(n-1,t-\tau)\,d\tau \quad (7.41)$$
$$\chi\delta(n)\,e^{-t}$$

The equation is self-explanatory. If $G_1(u,t)$ is the generating function of $\pi(n,t)$

$$G_1(u,t) = \int_0^a e^{-\tau}\,G_1(u,t-\tau)\,d\tau + \int_a^t e^{-\tau}\,u\,G(u,t-\tau)\,d\tau \quad (7.42)$$
$$+ e^{-t}$$

Differentiating with respect to t we have

$$\frac{\partial G(u,t)}{\partial t} = (u-1)\,e^{-a}\,G(u,t-a) \quad (7.43)$$

We know that $G_1(u,\tau) = 1$, for $\tau < a$. So we choose an N such that $t - Na < a$. By iteration we obtain

$$G(u,t) = 1 + \frac{(u-1)\,e^{-a}\,(t-a)}{1!} + \cdots$$
$$\cdots + \frac{(u-1)^N\,e^{-Na}\,(t-Na)^N}{N!} \quad (7.44)$$

$$\pi(n,t) = \sum_{m=n}^{m=N} (-1)^{m-n} \binom{m}{n} e^{-ma}\,(t-ma)^m \quad (7.45)$$

CHAPTER VIII

A Comparison of Methods I and II

In this chapter we shall discuss a stochastic process from the point of view of methods I and II. The comparison illustrates in a striking manner how these two entirely different methods of approach lead to identical results.

Mathematical formulation of the problem:

Given that at t=0 (t is the parameter with respect to which the stochastic process 'develops') there is a 'parent' particle of energy E_0 and that the probability per unit t that the 'parent' of energy E produces a secondary of energy between $E-E'$ and $E-(E'+dE')$ consequently dropping to an energy state between E' and $E'+dE'$ is $R(E,E')\,dE'$, our problem is to find the probability distribution of the number of 'secondaries' produced. It is assumed that secondaries do not in their turn produce further particles or lose energy in any other manner. One aspect of the problem, the distribution in energy of the secondaries has been discussed before, using the method of product densities.

We shall first discuss the mathematical theory of such a stochastic process and later cite some examples in physics.

Application of Method I

Defining $\pi(n, E_0, E, t)\,dE$ as the joint probability that the parent lies between E and $E+dE$, and has

produced n secondaries in the range t. Using the Markoff nature of the process we at once write the diffusion equation of the process

$$\frac{\partial \pi(n, E_0, E, t)}{\partial t} = -\pi(n, E_0, E, t)\int_0^E R(E, E')\,dE' + \int_E^{E_0} \pi(n-1, E_0, E', t)\,R(E', E)\,dE' \qquad (8.1)$$

A complete Mellin's transformation solution of this equation can be obtained provided $R(E, E')\,dE'$ can be expressed in the form $R(\epsilon)d\epsilon$ where $\epsilon = E'/E$. In such a case define

$$P(n, E_0, S, t) = \int_0^{\infty \text{ or } E_0} \pi(n, E_0, E, t)\,E^{S-1}\,dE \qquad (8.2)$$

It is immaterial whether the upper limit of integration is E_0 or ∞ since the function π is zero for $E > E_0$. Also define

$$\int_0^1 R(\epsilon)\,d\epsilon = \phi = \text{constant}, \quad \omega(\epsilon) = \frac{R(\epsilon)}{\phi} \qquad (8.3)$$

The above integrodifferential equation is transformed thus

$$\frac{\partial P(n, E_0, S, t)}{\partial t} = -P(n, E_0, S, t) + P(n-1, E_0, S, t)\,\phi\,\omega_S \qquad (8.4)$$

where

$$\omega_S = \int_0^1 \omega(\epsilon)\,\epsilon^{S-1}\,d\epsilon \qquad (8.5)$$

The initial condition is given by $P(1, E_0, S, 0) = E_0^{S-1}$

$P(n, E_0, S, 0) = 0$, for $n \neq 0$.

The solution of the above equation is

$$P(n, E_0, S, t) = E_0^{S-1} \omega_s^n \frac{(\phi t)^n}{n!} e^{-\phi t} \tag{8.6}$$

$$\pi(n, E_0, E, t) = \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E}\right)^S \omega_s^n \frac{(\phi t)^n}{n!} e^{-\phi t} \, ds \tag{8.7}$$

In actual physical problems, however, it is unreal to assume that the 'parent' particle produces secondaries whatever be its energy. Usually we define a critical energy E_c such that

$$R(E, E') = 0 \qquad \text{for } E < E_c$$

If we make the assumption of a cut-off for the cross-section function equation (8.1) breaks up into two parts; writing

$$\pi^{(1)}(n, E_0, E, t) \equiv \pi(n, E_0, E, t) \quad (E < E_c)$$
$$\pi^{(2)}(n, E_0, E, t) \equiv \pi(n, E_0, E, t) \quad (E > E_c)$$

$$\frac{\partial \pi^{(2)}(n, E_0, E, t)}{\partial t} = -\pi^{(2)}(n, E_0, E, t) \int_0^{E_0} R(E, E') dE' + \int_E^{E_0} \pi^{(2)}(n-1, E_0, E', t) R(E', E) dE' \tag{8.8}$$

$$\frac{\partial \pi^{(1)}(n, E_0, E, t)}{\partial t} = \int_{E_c}^{E_0} \pi^{(2)}(n-1, E_0, E', t)\, R(E', E)\, dE' \qquad (8.9)$$

It is to be noted that the solution of equation (8.1) assuming no cut-off is identical with $\pi^{(2)}(n, E_0, E, t)$ for $E > E_c$. Thus

$$\pi^{(2)}(n, E_0, E, t) \text{ for } E > E_c = \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-\phi t} \frac{(\phi t)^n}{n!} \omega_s^n \left(\frac{E_0}{E}\right)^s ds \qquad (8.10)$$

To reduce equation (8.9) we first integrate $\pi^{(1)}(n, E_0, E, t)$ with respect to E from 0 to E_c and defining

$$P^{(1)}(n, E_0, E_c, t) = \int_0^{E_c} \pi^{(1)}(n, E_0, E, t)\, dE \qquad (8.11)$$

apply the Mellin's transformation with respect to the variable E_c. Define

$$P^{(1)}(n, E_0, E_c, t) = \int_0^{E_0 \text{ or } \infty} P^{(1)}(n, E_0, E_c, t)\, E_c^{s-1}\, dE_c \qquad (8.12)$$

The upper limit of integration is E_0 or ∞ since $P^{(1)}(n, E_0, E_c, t)$ vanishes for $E_0 < E_c$.

$$\begin{aligned} &\int_0^{E_c} dE \int_{E_c}^{E_0} \pi^{(2)}(n-1, E_0, E', t)\, R(E', E)\, dE' \\ &= \int_{E_c}^{E_0} \pi^{(2)}(n-1, E_0, E', t) \int_0^{E_c} R(E', E)\, dE \\ &= \int_{E_c}^{E_0} \pi^{(2)}(n-1, E_0, E', t)\, \psi(E', E_c)\, dE' \end{aligned} \qquad (8.13)$$

Since $R(E, E')\,dE' = R(\epsilon)\,d\epsilon$ where $\epsilon = E'/E$

$$\psi(E', E_c) = \psi(\epsilon) \qquad \text{where } \epsilon = E_c/E' \tag{8.14}$$

We note that, since in equation (8.13) $E' > E_c$

$$\int_{E_c}^{E_0} \pi^{(2)}(n-1, E_0, E', t)\,\psi(E', E_c)\,dE' = \int_{E_c}^{E_0} \pi((n-1), E_0, E', t)\,\psi(E', E_c)\,dE' \tag{8.15}$$

where $\pi(n-1, E_0, E', t)^{E_c}$ is the solution of equation (8.1) assuming no cut-off.

$$P^{(1)}(n, E_0, s, t) = \int_0^{E_0} P^{(1)}(n, E_0, E_c, t)\,E_c^{s-1}\,dE_c \tag{8.16}$$

We have

$$\frac{\partial P^{(1)}(n, E_0, s, t)}{\partial t} = \int_0^{E_0} E_c^{s-1}\,dE_c \int_{E_c}^{E_0} \pi(n-1, E_0, E', t)\,\psi(E', E_c)\,dE'$$

$$= \int_0^{E_0} \pi(n-1, E_0, E', t) \int_0^{E'} \psi(E', E_c) \times \left(\frac{E_c}{E'}\right)^{s-1} \frac{dE_c}{E'}$$

$$= e^{-\phi t} \frac{(\phi t)^{n-1}}{(n-1)!} E_0^{s}\,\omega_{s+1}\,\eta_s \tag{8.17}$$

where $\eta_s = \frac{1}{\phi} \int_0^1 \psi(\epsilon)\,\epsilon^{s-1}\,d\epsilon \qquad \epsilon = E_c/E'$ (8.18)

$$\frac{1}{\phi}\,\psi(t) = \int_0^{E_c} \omega(E', E)\,dE, \qquad \epsilon = E_c/E' \tag{8.19}$$

$$P^{(1)}(n, E_0, S, t) = \prod_{n}\phi t \; E_0^{S} \omega_{S+1}^{n-i} \eta_S \qquad (8.20)$$

Thus

$$P^{(1)}(n, E_0, E_c, t) = \frac{1}{2\pi i}\prod_{n}\phi t \int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E}\right)^{S} \omega_{S+1}^{n-1} \eta_S \, dS \qquad (8.21)$$

In theory we can have a limiting distribution as t tends to ∞.

$$\Pi^{(2)} \to 0 \qquad \text{as} \quad t \to 0$$

$$P^{(1)}(n, E_0, E_c, t) \to \frac{i}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E_c}\right)^{S} \omega_{S+1}^{n-1} \eta_S \, dS \qquad (8.22)$$

since $\prod_{n}\phi t \to 1$

In particular choose $\omega(\epsilon) = \beta \epsilon^{\beta-1}$

This yields

$$P^{(1)}(E_0, E, n) = \left(\frac{E_c}{E_0}\right)^{\beta}\left(\beta \log \frac{E_0}{E_c}\right)^{n-1} \Big/ (n-1)! \qquad (8.23)$$

This is a Poisson distribution (shifted by 1) with mean

$$1 + \beta \log(E_0/E_c).$$

A simple solution can be obtained if the initial energy E_0 has a probability Spectrum given by

$$F(E_0)\,dE_0 = \gamma\left(\frac{E_c}{E_0}\right)^{\gamma+1}\frac{dE_0}{E_c} \qquad (8.24)$$

Notice that $\int_{E_c}^{\infty} F(E_0)\, dE_0 = 1$

Considering equations (8.8) and (8.9) we write

$$P_n^{(1)} = \int_{E_c}^{\infty} F(E_0)\, dE_0 \int_0^{E_c} \pi^{(1)}(n, E_0, E, t)\, dE \qquad (8.25)$$

$$P_n^{(2)} = \int_{E_c}^{\infty} F(E_0)\, dE_0 \int_{E_c}^{E_0} \pi^{(2)}(n, E_0, E, t)\, dE \qquad (8.26)$$

$P_n^{(1)}$ thus represents the joint probability that n secondaries are produced and the primary or parent has an energy less than E_c.

$P_n^{(2)}$ represents the joint probability that n secondaries are produced and the primary or parent has an energy greater than E_c.

Writing $P_n = P_n^{(1)} + P_n^{(2)}$ (8.27)

P_n represents the probability that n secondaries are produced irrespective of the energy state of the primary.

$$\sum_n P_n = 1$$

$$\frac{\partial P_n^{(2)}}{\partial t} = -P_n^{(2)} \phi + P_{n-1}^{(2)} \phi\, \omega_{r+1} \qquad (8.28)$$

$$\frac{\partial P_n^{(1)}}{\partial t} = P^{(2)} \phi (1 - \omega_{r+1}) \qquad (8.29)$$

where

$$\omega_\gamma = \int_0^{E_0} \omega \left(\frac{E'}{E}\right)\left(\frac{E'}{E}\right)^{\gamma+1} dE'/E' \tag{8.30}$$

We obtain the solution of the above equations as

$$P_n^{(2)} = \frac{(\phi t)^n}{n!} e^{-\phi t} \omega_{\gamma+1}^n \tag{8.31}$$

$$P_n^{(1)} = \overline{|}_n \phi t \; \omega_{\gamma+1}^{n-1} (1 - \omega_{\gamma+1}) \tag{8.32}$$

where $\overline{|}_n a$ is the incomplete Gamma function defined by

$$\overline{|}_n a = \int_0^a \frac{z^{n-1}}{(n-1)!} e^{-z} dz \tag{8.33}$$

As $t \to \infty$, $\overline{|}_n \phi t \to 1$

$$P_n^{(2)} \to 0 \tag{8.34}$$

$$P_n^{(1)} \to \omega_{\gamma+1}^{n-1} (1 - \omega_{\gamma+1}) \tag{8.35}$$

It should be noted that these simple solutions are possible only because we have averaged over the initial energy spectrum. If we are dealing with a primary of definite energy $E_0 > E_c$ we must use the Mellin's transform solution given before. But even in this case a drastic simplification results

if we do not assume a cut-off, i.e. if we take $E_c = 0$ (an unreal assumption). We do not then have a splitting up of the fundamental equation (8.1). If we define $S(n, E_0, t)$ as the probability that n secondaries have been produced given that the initial energy is E_0 we obtain the equation

$$\frac{\partial S(n, E_0, t)}{\partial t} = \phi \left\{ -S(n, E_0, t) + S(n-1, E_0, t) \right\} \quad (8.36)$$

Compare this equation with (8.28). The solution of (8.24) is

$$S(n, E_0, t) = S(n, t) = \frac{(\phi t)^n}{n!} e^{-\phi t} \quad (8.37)$$

$S(n, E_0, t)$ is independent of E_0 . It is a poisson distribution. If the initial energy E_0 is far greater than E_c then for small n where the average number of secondaries produced is not so great as to reduce the energy of the primary below the critical energy the Poisson formula must be a good approximation.

It is to be noted, for $E_c = 0$, there can exist no limiting distribution as $t \to \infty$ since the mean $\phi t \to \infty$.

Approach by Method II

We shall approach the above problem from the point of view of Method II. We define the function $S(n, E, t)$ which represents the probability that n secondaries are produced by a primary of energy E. It is clear that

$$S(n, E_0, t) = \int_0^{E_0} \pi(n, E_0, E, t)\, dE \quad (8.38)$$

The method of regeneration points enables us to deal with this function directly. In the first instant of time dt the initial primary would create a secondary of energy lying between $E_0 - E$ and $E_0 - (E+dE)$ with probability $R(E_0, E)\,dE\,dt$ and would not have created a secondary at all with probability $1 - \int_0^{E_0} R(E_0, E)\,dE\,dt$. This argument immediately leads us to the equation

$$\frac{\partial S(n, E_0, t)}{\partial t} = -S(n, E_0, t)\int_0^{E_0} R(E_0, E)\,dE + \int_0^{E} S(n-1, E, t)\,R(E_0, E)\,dE \tag{8.39}$$

This equation is the counterpart of (8.1) obtained by Method I. But method II possesses one advantage over the other method. If a limiting distribution $S(n, E_0) = S(n, E_0, t)$ as $t \to \infty$ exists then we can deal with $S(n, E_0)$ directly

$$\frac{\partial S(n, E_0, t)}{\partial t} = 0 \qquad \text{at } t = \infty \tag{8.40}$$

$$S(n, E_0)\,\phi(E_0) = \int_0^{E_0} S(n-1, E)\,R(E_0, E)\,dE \tag{8.41}$$

where

$$\phi(E_0) = \int_0^{E_0} R(E_0, E)\,dE \tag{8.42}$$

or

$$S(n, E_0) = \int_0^{E_0} S(n-1, E)\, \omega(E_0, E)\, dE \qquad (8.43)$$

since

$$\omega(E_0, E) = R(E_0, E) / \phi(E_0)$$

,

If we assume the cut-off such that for $E < E_c$, $S(n, E) = 0$ if $n > 1$, $S(0, E) = 1$, then we have for $n > 1$

$$S(n, E_0, E_c) = \int_{E_c}^{E_0} S(n-1, E, E_0)\, \omega(E_0, E)\, dE \qquad (8.44)$$

$$S(1, E_0, E_c) = \int_0^{E_c} \omega(E_0, E)\, dE$$

This equation was obtained by Bartlett* through 'static' considerations using 'partition' arguments. The above equation can be completely solved by the Mellin's transformation if we assume as before

$$W(E, E')\, dE' = \omega(\epsilon)\, d\epsilon \qquad \text{where } \epsilon = E'/E$$

If this were so we can write

$$S(n, E_0, E_c) \quad \text{as} \quad S(n, E_c/E_0)$$

$$S(n, E_c/E_0) = \int_{E_c}^{E_0} S(n-1, E_c/E')\, \omega(E'/E_0)\, dE'/E_0$$

$$S(1, E_c/E_0) = \int_0^{E_c} \omega(E'/E_0)\, dE'/E_0 = \psi(E_c/E_0) \qquad (8.45)$$

*In a private communication.

Now write

$$E_c/E_0 = \epsilon\,, \quad E'/E_0 = \epsilon'\,, \quad E_c/E' = \epsilon/\epsilon'$$

This yields

$$S(n,\epsilon) = \int_{\epsilon}^{1} S(n-1,\ \epsilon/\epsilon')\,\omega(\epsilon')\,d\epsilon' \quad (n>1)$$

$$S(1,\epsilon) = \psi(\epsilon) \tag{8.46}$$

Now applying the Mellin's transformation with respect to and defining

$$q(n,s) = \int_{0}^{1\text{ or }\infty} \epsilon^{s-1} d\epsilon \int_{\epsilon}^{1} S(n-1,\ \epsilon/\epsilon')\,\omega(\epsilon')\,d\epsilon' \tag{8.47}$$

It is immaterial whether the upper limit of integration is 1 or ∞ since $S(n,\epsilon) = 0$ for $\epsilon > 1$. Changing the order of integration we have

$$\begin{aligned}
&\int_{0}^{1} \omega(\epsilon')\,d\epsilon' \int_{0}^{\epsilon'} S(n-1,\ \epsilon/\epsilon')\,\epsilon^{s-1}\,d\epsilon \\
&= \int_{0}^{1} \omega(\epsilon')\,\epsilon'^{s}\,d\epsilon' \int_{0}^{\epsilon'} S(n-1,\ \epsilon/\epsilon') \left(\frac{\epsilon}{\epsilon'}\right)^{s-1} \frac{d\epsilon}{\epsilon'} \\
&= \omega_{s+1}\, q(n-1,s)
\end{aligned} \tag{8.48}$$

Thus

$$q(n,s) = \omega_{s+1}\, q(n-1,s)$$

$$q(1,s) = \int_{0}^{1} \psi(\epsilon)\,\epsilon^{s-1}\,d\epsilon = \eta_s \tag{8.49}$$

(8.50)

$$q(n,S) = \omega_{S+1}^{n-1}\,\eta_S \tag{8.51}$$

Thus

$$S(n, E_0, E_c) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{E_0}{E_c}\right)^S \omega_{S+1}^{n-1}\,\eta_S\,dS \tag{8.52}$$

since $\epsilon = E_c/E_0$

This result beautifully confirms our earlier result obtained from method I.

Now we shall cite a few physical applications of our results.

Janossy and Heitler have used the above model to explain the production of mesons in penetrating showers. Such a model is also useful to calculate the number of knock-on electrons produced by fast mesons.

Le Couter in the evaporation theory of nuclear disintegration used an equation which is equivalent to Bartlett's. But the main difficulty in that case is that the cross-section $\omega(E, E')\,dE'$ is not homogeneous in E and E'.

CHAPTER IX

On a Class of Stochastic Processes

Let us consider events occurring along the time axis. We can define product densities along the time axis (till now we considered product densities with respect to another continuous variable E and these changed with respect to t). Let the product density of degrée n of events be

$$f_n(t_1, t_2, \ldots, t_n)$$

Let us assume that we deal with a class of product densities defined by

$$f_n(t_1, t_2, \ldots, t_n) = f_1(t_1) f_1(t_2 - t_1) \cdots f_1(t_n - t_{n-1}) \quad (9.1)$$

provided $t_1, t_2, \ldots, t_n$ are ordered ($t_1 < t_2 < \cdots < t_n$) * where f_1 is the product density of degree one. This represents a class of stochastic processes examples of which are easily found. For instance

1. Poisson events (a trivial example)
2. Electron counter problems
3. 'Renewal' problems

can be cited as illustrative of such processes.

If $\pi(n, t)$ represents the probability that n events have occurred in time t, what happens between t and $t + dt$

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* If $t_1, t_2, \ldots, t_n$ are not ordered f_n still exists. f_n is a product density and hence is symmetrical in the variable $t_1, t_2, \ldots, t_n$. We can therefore order them and then apply (9.1)

is not only dependent on the fact that n events have occurred in time t but upon the 'prehistory' - i.e. when the last event happened. Hence the usual method of expressing $\Pi(n+t+dt)$ in terms of $\Pi(n,t)$ is unsuitable for such problems. We shall show that method II and the product density technique yield identical results.

Product density approach

Let $G(u,t)$ be the generating function corresponding to $\Pi(n,t)$ and by a result proved in Chapter V

$$\left[\frac{\partial^n G(u,t)}{\partial u^n}\right] = \int_0^t dt_n \int_0^t dt_{n-1} \cdots \int_0^t f_n(t_1, t_2, \ldots, t_n)\, dt_1$$

Since f_n is symmetrical in $t_1, t_2, \ldots, t_n$ we can write the above as

$$n! \int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} f_n(t_1, t_2, \ldots, t_n)\, dt_1$$

$$= n! \int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} f_1(t_1) f_1(t_2 - t_1) \cdots$$

$$\cdots f_1(t_n - t_{n-1})\, dt_1 \qquad (9.2)$$

Let the Laplace transform* (L.T.) with respect to the variable t of the function $f_1(t)$ be $\psi(s) = \int_0^\infty e^{-st} f_1(t)\, dt$

* In this chapter L.T. means Laplace transform.

Then since

$$\left[\frac{\partial G(u,t)}{\partial u}\right]_{u=1} = \int_0^t f_1(t)\,dt$$

the L.T. of

$$\left[\frac{\partial G(u,t)}{\partial u}\right]_{u=1} = \frac{\psi(s)}{s} \tag{9.3}$$

The L.T. of

$$\left[\frac{\partial^n G(u,t)}{\partial u^n}\right]_{u=1} = n!\frac{[\psi(s)]^n}{s} \tag{9.4}$$

If $g(u,s)$ be the L.T. of $G(u,t)$

$$g(u,s) = \frac{1}{s}\left\{\frac{1}{1-(u-1)\psi(s)}\right\} \tag{9.5}$$

If $p(n,s)$ is the L.T. of $\pi(n,t)$

$$P(n,s) = \frac{1}{s}\frac{[\psi(s)]^n}{[1+\psi(s)]^{n+1}} \tag{9.6}$$

In any stochastic problem of this type we must determine either $\psi(s)$ or $f_1(t)$.

Approach by Method II

Let us consider the above class of stochastic processes from the point of view of the method of regeneration points (Method II). The relation (9.1) between the product densities

of degree n and 1 is equivalent to stating that if we are given that an event occurred at t = 0, the probability that an event occurs between t and $t+dt$ (whether or not an event occurred between 0 and t) is determined by the function $f_1(t)dt$ and is independent of what happened before t = 0. It also follows that, given an event happened at t = 0, the probability that the next event happens between t and $t+dt$ is also independent of what happened before t = 0. This probability can be expressed by the function $-\frac{d\phi(t)}{dt}.dt$ where $\phi(t)$ represents the probability that no event occurs between 0 and t given that an event occurred at t = 0. Usually $\phi(t)$ can be expressed as $e^{-\int_0^t \alpha(\tau)d\tau}$ where $\alpha(t)\,dt$ has the obvious meaning of the probability that an event occurs between t and $t+dt$ given that no event occurred between 0 and t and that an event occurred at t = 0. According to the method of regeneration points we at once write the integral equation

$$\pi(n,t) = -\int_0^t \pi(n-1, t-\tau)\phi'(\tau)d\tau + \delta(n)\phi(t) \quad (2.7)$$

where $\phi'(\tau) = \frac{d\phi(\tau)}{d\tau}$ and $\delta(n) = 1$ for n = 0 or zero otherwise

$$G(u,t) = -\int_0^t u\,G(u, t-\tau)\phi'(\tau)d\tau + \phi(t) \quad (2.8)$$

$$g(u,s) = -g(u,s)(s\,\sigma(s) - 1)u + \sigma(s) \quad (2.9)$$

where $\sigma(s)$ is in the L.T. of $\phi(t)$

$$g(u,s) = \sigma(s)/[1 + u(s\,\sigma(s) - 1)] \quad (2.10)$$

Also

$$\left[\frac{\partial g(u,s)}{\partial u}\right]_{u=1} = \frac{\psi(s)}{s} = \frac{1}{s}\,\frac{1-s\sigma(s)}{s\sigma(s)} \tag{9.11}$$

or $$\sigma(s) = \frac{1}{s}\,\frac{1}{1+\psi(s)} \tag{9.12}$$

Thus $$g(u,s) = \frac{1}{1-(u-1)(1+\psi(s))} \tag{9.13}$$

a result derived by the product density method.

Thus $P(n,s)$ the L.T. of $\pi(n,t)$ is

$$\frac{1}{s}\,\frac{\psi(s)^n}{[1+\psi(s)]^{n+1}} = \sigma(s)(1-s\sigma(s))^n \tag{9.14}$$

In any stochastic problem we must determine $\sigma(s)$ or $\psi(s)$ i.e. $\phi(t)$ or $f_1(t)$ in order to obtain $P(n,s)$ i.e. $\pi(n,t)$. We shall now apply the above results to some probability problems relating to counters.

A. Counter problems

A counter is supposed to register random events such that the probability of an event in any small interval of time of length Δt (Δt is an infinitesimal quantity) independently of previous events is $\propto \Delta t + o\,\Delta t$ or simply Δt (provided we choose the unit of time suitably). The probability that exactly n events occur during a time interval of length t is given by the

Poisson law

$$e^{-t} t^n / n!$$

Due to resolving time, the counter is unable to register all events. It is customary to treat two cases:

Type I:- After each registration the counter is locked for a constant time a. An event is registered if and only if no registration has taken place during a time a preceding it.

Type II:- The counter registers an event if and only if no event has occurred during the preceding time interval of length a. Here an event occurring at a moment when the counter is locked prolongs the inoperative period. In theory the counter can remain locked indefinitely.

Type II has been discussed in Chapter VII, by directly solving the integral equation. We shall here obtain the solution for the probability distribution of the number of registered events in time t as particular cases of the results derived above.

In the following discussion we assume

(1) that the initial condition is specified as follows:

At $t = 0$ an event is registered. We require the probability $\pi(n,t)$ that n events are registered in time t. Omitting the initial event

$$\pi(n,t) = 0 \text{ at } t = 0 \text{ for } n \neq 0$$

$$\pi(0,0) = 1$$

(2) that $\phi(t)$ represents the probability that no event is registered in time t.

(3) that $f_1(t)$ represents the product density of degree one of registered events.

For type I counter it is more easy to compute $\phi(t)$ rather than $f_1(t)$

$$\phi(t) = e^{-(t-a)} \quad \text{for } t > a$$

$$\phi(t) = 1 \quad \text{for } t < a$$

L.T. of $\phi(t) = \sigma(s) = \dfrac{e^{-as}}{s+1} + \dfrac{1-e^{-as}}{s}$

$$= \frac{e^{-as}}{s(s+1)} + \frac{1}{s} \qquad (2.15)$$

L.T. of $\pi(n,t)$, the probability that n registrations occur in time t, is

$$\sigma(s)(1-s\sigma(s))^n = \frac{e^{-nas}}{(s+1)^n}\left[\frac{1}{s} - \frac{e^{-as}}{s(s+1)}\right]$$

$$= \frac{1}{s}\left[\frac{e^{-nas}}{(s+1)^n} - \frac{e^{-(n+1)as}}{(s+1)^{n+1}}\right]$$

Thus

$$\frac{\partial \pi(n,t)}{\partial t} = H(t-na)\, e^{-t}\frac{t^{n-1}}{(n-1)!} - H(t-\overline{n+1}a)\, e^{-t}\frac{t^n}{n!}$$

$$\pi(n,t) = \int_0^t H(t-na)\, e^{-t}\frac{t^{n-1}}{(n-1)!}\, dt - \int_0^t H(t-\overline{n+1}a)\, e^{-t}\frac{t^n}{n!}\, dt$$

$$= \int_{na}^t e^{-t}\frac{t^{n-1}}{(n-1)!}\, dt - \int_{(n+1)a}^t e^{-t}\frac{t^n}{n!}\, dt \qquad (2.16)$$

For the counter of type II it is easy to calculate $f_1(t)$ the product density of the first degree relating to registered events.

$$f_1(t)\,dt = H(t-a)\,e^{-a}\,dt \qquad (9.17)$$

where H is the Heaviside Unit function i.e. $H(x) = 0$ for $x<0$, $H(x)=1$ for $x>0$. The above expression results from arguing that an event is recorded when it is not preceded by an event in time interval a and is not also preceded by the event occurring at $t = 0$ within a time interval a . Thus

$$\psi(s) = \text{L.T. of } f_1(t) = \frac{1}{se^{a(s+1)}} \qquad (9.18)$$

And this yields

$$G(u,t) = 1+\frac{(u-1)\,e^{-a}(t-a)}{1!} + \cdots + \frac{(u-1)^N e^{-Na}(t-Na)^N}{N!} \qquad (9.19)$$

where N is a number such that $(N+1)a > t > Na$

$$\Pi(n,t) = \sum_{m=n}^{m=N} (-1)^{m-n} \binom{m}{n} (t-ma)^m \qquad (9.20)$$

a result derived in Chapter VII.

We can now derive the L.T. of $\pi(n,t)$ when it represents the probability of n registrations occurring in time t for a more general type of counter in which the registered and unregistered events are followed by two different 'dead times' a and b respectively. We proceed as follows:-

At $t = 0$ an event is registered. We ask: What is the probability $\phi(t)$ that no event is registered up to time t. Let us assume $a > b$.

We can write $\phi(t)$ as

$$\phi(t) = \left\{ H(t) - H(t - \overline{a-b}) \right\} + \phi^{\overline{II}}(t - \overline{a-b})\, H(t - \overline{a-b}) \qquad (9.21)$$

The first part expresses the condition that no event is registered for $t < a - b$. $\phi(t) = 1$ for $t < a - b$. $\phi^{\overline{II}}(t - \overline{a-b})$ represents the probability that no event is registered in time $(t - \overline{a-b})$ on the hypothetical assumption that we are dealing with a counter of type II with dead time b. As has been previously mentioned, for type II counter we do not usually compute $\phi^{\overline{II}}(t)$ but only $f_1^{\overline{II}}(t)$, the product density of registered events of a type II counter with dead time b. But the L.T.'s of $\phi^{\overline{II}}(t)$ and $f_1^{\overline{II}}(t)$ are connected by the relation (9.12). Thus

$$\text{L.T of } \phi(t) = \text{L.T of } \left\{ H(t) - H(t - \overline{a-b}) \right\} + \text{L.T of } \left\{ \phi^{\overline{II}}(t - \overline{a-b}) \right\} \qquad (9.22)$$

We know that the L.T. of

$$f_1^{\overline{II}}(t) = \frac{1}{s e^{b(s+1)}}$$

Thus we have L.T. of $f_1(t)$ the product density of registered events of the generalized type of counter given by

$$\psi(s) = \left\{ \frac{1}{s e^{as+b} + e^{(a-b)s} - 1} \right\} \qquad (9.23)$$

Put b = 0 we get counter type I.

Put a = b we get counter type II.

Put a = b = 0 we get a poisson distribution for $\Pi(n,t)$ characteristic of an ideal counter.

The same result is obtained for $a < b$ following exactly the same arguments.

B. Random points on a line *

We shall now deal with a slightly more complicated problem which is analogous to the class of stochastic processes considered in this chapter.

Suppose we are concerned with a random distribution of N points in a line of length t. We define a random point as a 'right neighbour' if it is at a distance less than a

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* This problem considered by Lovera was reviewed by Feller W. in Mathematical Reviews (1949) pp.121, vol.10, no.2. From Feller's review it was clear that Lovera had considered only the first and second moments and not the whole distribution.

from the preceding point. We ask: What is the probability of getting m 'non-right neighbours'? We shall assume that the point which has no points on its left but is at a distance less than a from t = 0 is a 'right neighbour'.

Let us take q as the probability per unit length that a point lies on the line. It is quite clear that $q = \frac{1}{t}$. The probability that any one of the random points lies between t and $t+dt$ is $\frac{N}{t}.dt$ and it is this quantity $\frac{N}{t}$ that corresponds to the 'probability per unit distance that an event occurs' in the previous problem (which we normalized to 1). We shall now form the product densities $f_1(t_1), f_2(t_1, t_2)$ ..... of 'non-neighbours'. They are

$$f_1(t_1) = H(t_1 - a)\, Nq\,(1 - qa)^{N-1}$$

$$f_2(t_1, t_2) = H(t_1 - a)\, H(t_2 - t_1 - a)\, N(N-1)q^2 \times (1 - 2qa)^{N-2}$$

. . . . . .

$$f_n(t_1, t_2, \ldots, t_n) = H(t_1 - a)\, H(t_2 - t_1 - a) \ldots$$

$$\ldots H(t_n - t_{n-1} - a)\, N(N-1)\ldots(N-n+1)\, q^n (1 - nqa)^{N-n} \qquad (9.24)$$

The factors, N, N(N-1),.....are obtained due to the fact that there are a finite number of points N and there is consequently a diminution by one in the 'available' number as we 'fill' each infinitesimal range $dt_1, dt_2, \ldots, dt_n$ to define the product densities. qa is the probability that a point occurs in the length a We impose the condition that in the case of the product density of degree n none of these n points should be preceded by a point within a distance a of any one of them. So we get the term $(1 - nqa)^{N-n}$.

Thus if $G_1(u)$ is the generating function of the distribution function of 'non-neighbours'

$$\left[\frac{\partial^n G_1(u)}{\partial u^n}\right]_{u=1} = \int_0^t dt_n \int_0^t dt_{n-1} \cdots \int_0^t f_n(t_1, t_2, \ldots, t_n)\, dt_1$$

$$= N(N-1)\cdots(N-n+1) \times q^n (1-nqa)^{N-n} (t-na)^n \quad (9.25)$$

Note that if $f_n \neq 0$ $t_1 > a, t_2 \geq 2a, \ldots, t_n > na$

So choose an M such that $(M+1)a > t > Ma$

All f_n with $n > M$ vanish.

So

$$G_1(u) = \left[1 + (u-1)Nq(1-qa)^{N-1}(t-a) \ldots . \right.$$
$$\left. + (u-1)^M N(N-1)\cdots(N-M+1) q^M \right. \quad (9.26)$$
$$\left. (1-qMa)^{N-M} (t-Ma)^N \right] (M!)^{-1}$$

Noting that $q = \frac{1}{t}$ we have

$$G_1(u) = 1 + \frac{(u-1)Nq^N(t-a)^N}{1!} + \ldots$$
$$\ldots + \frac{(u-1)^M N(N-1)\cdots(N-M+1) q^N (t-Ma)^N}{M!} \quad (9.27)$$

Putting $aq = a/t = y$

$$G_1(u) = 1 + \binom{N}{1}(u-1)(1-y)^N + \ldots$$
$$\ldots + \binom{N}{M}(u-1)^M (1-My)^N \quad (9.28)$$

If $M > N$ we stop with the term involving $(u-1)^N$.

APPENDIX I.

The Limiting Process

We shall here outline the limiting process adopted by Bhabha and Ramakrishnan to avoid the well-known complications which stand in the way of a rigorous mathematical treatment of probability theory of a continuum. They made the following assumptions:

A) Electrons and photons exist only in discrete states each with a definite energy.

B) A limiting process exists by which the number of such states in any unit energy interval at E, say ρ, can be made as large as we please.

C) The probabilities for the elementary processes of radiation and pair creation change from one state to a neighbouring state in such a way that the difference between them becomes smaller than any number however small in the limit in which the number of states in any energy interval tends to infinity.

These assumptions in fact correspond to the well known procedure in quantum mechanics in which one considers all the phenomena to take place in a large box of dimension ℓ. The states of the electrons and photons are then discrete and their number tends to infinity as ℓ tends to infinity. Then for the quantum mechanical transition probabilities, (C) is also satisfied.

The definition of the Bethe-Heitler and Bloch cross-sections undergo the following change in notation in the discrete case. In the discrete case we define

1) $R_{\alpha i}$ as the probability per unit thickness that an electron of energy E_α passes into the state E_i after radiation of a photon of energy $E_\alpha - E_i$. $\sum_i R_{\alpha i}$ is the total probability per unit thickness that an electron of energy E_α will radiate a photon of any energy whatsoever. The cross-section in the continuous distribution case is $\int_0^{E_\alpha} R(E_\alpha, E_i)\, dE_i$ and so when we pass to the limit we must replace $\sum_i R_{\alpha i}$ by $\int_0^{E_\alpha} R(E_\alpha, E_i)\, dE_i$. If ρ_i is the number of states per unit energy interval at E, as ρ_i tends to infinity we must replace,

$R_{\alpha i}\rho_i\, dE_i$ by $R(E_\alpha, E_i)\, dE_i$

Note that $R_{\alpha i} \to O\left(\frac{1}{\rho}\right) \to 0$ as ρ tends to infinity.

2) Similarly $R'_{\alpha i}$ is the probability per unit thickness that a photon of energy E_α will create a pair of electrons one of which has energy E_i.

3) and $\rho_{\alpha i}$ as the probability per unit thickness that an electron of energy E_α drops to E_i through collision loss of energy $E_\alpha - E_i$.

Our method consists in breaking up the phenomena which can be observed at thickness t into elementary events each described in the maximum possible detail consistent with the laws of quantum mechanics. We first replace the continuous energy range from 0 to E_0 (E represents the upper limit of the energy of a photon or electron and is equal to the

energy of the initial particle at t = 0) by a very large number of discrete states of energies

$$E_1, E_2, \ldots, E_N$$

An elementary event at thickness t is one in which n_1 electrons (positive or negative) and m_1 photons occur in the state E_1 n_2 electrons and m_2 photons occur in the state E_2..... etc. For such an elementary event we assign the probability

$$\Pi(n_1, n_2, \ldots; m_1, m_2, \ldots; t)$$

The sum of all the probabilities of all possible mutually exclusive events must be unity. We now define*

$\sum n_i \Pi(n_1, n_2 \ldots; m_1, m_2, \ldots; t) = \overline{n_i}$ = average number of electrons in the state E_i.

$\sum n_i^2 \Pi(n_1, n_2, \ldots; m_1, m_2, \ldots; t) = \overline{(n_i)^2}$ mean square number of electrons in E_i.

Similarly $\overline{m_i}$ and $\overline{m_i^2}$ are defined as the mean and mean square number of photons in E_i. Also

$$\overline{n_i n_j} = \sum n_i n_j \Pi(n_1, n_2, \ldots; m_1, m_2 \ldots; t)$$

$$\overline{n_i m_j} = \sum n_i m_j \Pi(n_1, n_2, \ldots; m_1, m_2, \ldots; t)$$

$$\overline{m_i m_j} = \sum m_i m_j \Pi(n_1, n_2, \ldots; m_1, m_2, \ldots; t)$$

etc

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* In this discussion $\sum$ indicates summation over all possible mutually exclusive events i.e. over all values of n_i, and n_i and m_i take all possible values.

For practical purposes it is not necessary to know Π since one is seldom in a position to measure experimentally the probabilities Π . For practical purposes it is sufficient to know the mean number of particles in any energy interval and the mean-square deviation of that number. For this purpose we must know $\bar{n}$, $\overline{n^2}$ and $\bar{m}$, $\overline{m^2}$ etc.

We shall compute the change in Π neglecting collision loss. By a process quite familiar in the theory of stochastic processes we obtain the following differential equation.*

$$\frac{d\Pi(n_1,\ldots;m_1,\ldots;t)}{dt} = -\Big(\sum_{i,r<i}\sum n_i R_{ir} + \sum_{i,r<i}\sum m_i R'_{ir}\Big)\times \Pi(n_1,\ldots;m_1,\ldots;t)$$

$$+ \sum_{i,r<i}\sum (n_i+1) R_{ir}\, \Pi(n_1,\ldots n_r-1;n_i+1,\ldots;m_1,\ldots \check{n}_{i-r-1}\ldots m_{i-r-1},\ldots;t) \quad \text{(a.1)}$$

$$+ \sum_{i,r<i}\sum (m_i+1) R'_{ir}\, \Pi(n_1,\ldots n_r-1;\ldots m_i,\ldots m_i+1,\ldots;t)$$

We now use the method of the probability generating function defined by

$$G(u_1\ldots u_N, v_1\ldots v_N;t) = \sum \prod_{i,j} \Pi(n_1\ldots n_N;m_1\ldots m_N;t) \times u_i^{n_i} v_j^{n_j}$$

where $\sum$ as stated before indicates summation over all values of n_i and m_j

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* Here $\sum$ denotes summation over all the indices

This yields the master equation

$$\frac{dG}{dt} = -\left\{\sum_{i}\sum_{r<i} U_i \frac{\partial G}{\partial u_i} R_{ir} + \sum_{i}\sum_{r<i} U_i \frac{\partial G}{\partial v_i} R'_{ir}\right\}$$
$$+\sum_{i}\sum_{r<i} u_r v_{i-r} \frac{\partial G}{\partial u_i} R_{ir} + \sum_{i}\sum_{r<i} u_r u_{i-r} \frac{\partial G}{\partial v_i} R'_{ir} \qquad (a.2)$$

from which, for example, equations for $\bar{n}_i$, $\overline{n_i^2}$, ---

etc. can be derived.

For example

$$\frac{d}{dt}\bar{n}_i = \sum_{j>i} \bar{n}_j R_{ji} - \bar{n}_i\left(\sum_{j<i} R_{ij}\right) + 2\sum_{j>i} \bar{n}_j R'_{ji} \qquad (a.3)$$

To pass to the continuous case we multiply by both sides by

ρ and make ρ tend to infinity and write

$$\bar{n}(E_i) = \lim_{\rho\to\infty} \bar{n}_i \rho \qquad \overline{n^2}(E_i) = \lim_{\rho\to\infty} \overline{n_i^2}\rho$$

Thus $\bar{n}(E)$ is $O\left(\frac{1}{\rho}\right)$. But note that

$\sum_{j<i} R_{ij}\bar{n}_j$ is replaced by $\int_{E_i}^{\infty} \bar{n}(E_j,t)\, R(E_j,E_i)\, dE_j$

$-\bar{n}_i \sum_{j<i} R_{ij}$ is replaced by $-\bar{n}(E_i,t)\int_{E_i}^{0} R(E_i,E_j)\, dE_j$

and $\sum_{j>i} \bar{n}_j R'_{ji}$ by $\int_{E_i}^{\infty} \bar{m}(E_j,t)\, R'(E_j,E_i)\, dE_j$

where $\bar{n}(E,t)$ and $\bar{m}(E,t)$ are densities of degree one of electrons and photons respectively. Thus we obtain the cascade equations for density of degree one.

The equation for $\overline{n_i^2} - \bar{n}_i$ is as follows

$$\frac{d}{dt}\left(\overline{n_i^2} - n_i\right) = -2\left(\overline{n_i^2} - \bar{n}_i\right)\left(\sum_{j<i} R_{ij}\right)$$
$$+\left[2\left\{\sum_{j>i} \overline{n_i n_j} R_{ji}\right\} + 4\sum_{j>i} \overline{n_i m_j} R'_{ji}\right.$$
$$\left. + \left[2\,\overline{m}_{2i} R'_{2i,i}\right]\right] \qquad (a.4)$$

As ρ tends to infinity all the terms on the right hand side are of $O(\frac{1}{\rho})$ except the term $\left[2\sum_{j>i}\overline{n_i n_j}K_{ji}+4\sum_{j>i}\overline{n_i m_j}K_{ji}\right]$ which is of $O(\frac{1}{\rho^2})$ and so becomes vanishingly small in comparison with the others. The solution of the above equation then becomes (for actual boundary conditions of the problem corresponding to a shower started by a single electron or photon)

$$\overline{n^2(E)} = \overline{n(E)}$$

As regards the equations for $\overline{n_i n_j}$, $\overline{m_i m_j}$, $\overline{n_i m_j}$ we remember that as ρ tends to infinity each of them is $O(\frac{1}{\rho^2})$ and we write

$$\overline{n_2(E_i,E_j)} = \lim_{\rho\to\infty} \overline{n_i n_j}\rho^2$$

$$\overline{m_2(E_i,E_j)} = \lim_{\rho\to\infty} \overline{m_i m_j}\rho^2$$

and

$$\overline{nm(E_i,E_j)} = \lim_{\rho\to\infty} \overline{n_i m_j}\rho^2$$

The equations for $\overline{n_2(E_i,E_j)}$, $\overline{m_2(E_i,E_j)}$, $\overline{nm(E_i,E_j)}$ obtained from the 'master' equation then go over into the equations for the product densities of degree two given in Chapter VI.

Note: In the notation of Chapter VI

$$\overline{n}(E_i,t) = f_1(E_i,t)$$
$$\overline{m}(E_i,t) = g_1(E_i,t)$$
$$\overline{n_2}(E_i,E_j;t) = f_2(E_i,E_j;t)$$
$$\overline{m_2}(E_i,E_j;t) = g_2(E_i,E_j;t)$$
$$\overline{nm}(E_i,E_j;t) = fg_{11}(E_i,E_j;t)$$

APPENDIX II

Complete solution of the Cascade equations of Bhabha and Ramakrishnan

We can completely solve the cascade equations of Bhabha and Ramakrishnan given in Chapter VI* if we assume that R and R' have their respective forms for complete screening, namely

$$R(E,E') = \left\{ \frac{E'}{E} - \left(\frac{4}{3}+\alpha\right)\left(1-\frac{E}{E'}\right)\right\}\frac{1}{E} \qquad (b.1)$$

$$R'(E,E') = \left\{ 1 - \left(\frac{4}{3}+\alpha\right)\left(\frac{E'}{E} - \frac{E'^2}{E^2}\right)\right\}\frac{1}{E} \qquad (b.2)$$

α being a small constant (see for example Bhabha and Chakrabarty 1943). In this case it is convenient to take the Mellin transforms of $n(E;t)$ and $m(E;t)$. thus

$$\nu(\gamma;t) = \int_0^\infty E^{\gamma-1} n(E;t)\, dE \qquad (b.3)$$

$$\gamma(\gamma;t) = \int_0^\infty E^{\gamma-1} m(E;t)\, dE \qquad (b.4)$$

Then the functions ν and γ satisfy the equations obtained from (6.13) by applying the same transformation. Introducing the vector $\phi(\gamma;t)$ with two components defined by

$$\phi(\gamma;t) = \begin{pmatrix} \nu(\gamma;t) \\ \gamma(\gamma;t) \end{pmatrix} \qquad (b.5)$$

it follows from (6.13) that ϕ satisfies the matrix equation

$$\frac{d\phi(\gamma;t)}{dt} = \tau(\gamma)\,\phi(\gamma;t) \qquad (b.6)$$

where

$$\tau(\gamma) = \begin{pmatrix} -A_\gamma & B_\gamma \\ C_\gamma & -D \end{pmatrix} \qquad (b.7)$$

with the functions A_γ to D given by (see for example Bhabha and Chakrabarty 1943)

$$A_\gamma = \left(\frac{4}{3}+\alpha\right)\left\{\frac{d}{d\gamma}\log\Gamma(\gamma) + \gamma - 1 + \frac{1}{\gamma}\right\} + \frac{1}{2} - \frac{1}{\gamma(\gamma+1)}$$

$$B_\gamma = 2\left\{\frac{1}{\gamma} - \left(\frac{4}{3}+\alpha\right)\frac{1}{(\gamma+1)(\gamma+2)}\right\} \qquad (b.8)$$

$$C_\gamma = \frac{1}{\gamma+1} + \left(\frac{4}{3}+\alpha\right)\frac{1}{\gamma(\gamma-1)} \qquad (b.9)$$

$$D = \frac{7}{9} - \frac{1}{6}\alpha$$

γ in (b.8) being the Euler-Mascheroni constant.

The matrix τ has the two eigenvalues $-\lambda_\gamma$ and $-\mu_\gamma$ given by

$$\begin{pmatrix}\lambda_\gamma \\ \mu_\gamma\end{pmatrix} = \frac{1}{2}(A_\gamma + D) \mp \frac{1}{2}\left\{(A_\gamma - D)^2 + 4B_\gamma C_\gamma\right\}^{\frac{1}{2}} \qquad (b.10)$$

τ can be brought to the diagonal form by the matrix

$$\sigma(\gamma) = \begin{pmatrix} D - \lambda_\gamma & \mu_\gamma - D \\ C_\gamma & -C_\gamma \end{pmatrix} \qquad (b.11)$$

so that defining a matrix

$$\Lambda(\gamma; t) = \begin{pmatrix} e^{-\lambda_\gamma t} & 0 \\ 0 & e^{-\mu_\gamma t} \end{pmatrix} \qquad (b.12)$$

one sees at once that the matrix $\sum(\gamma; t) = \sigma(\gamma)\Lambda(\gamma; t)$ satisfies

$$\frac{d}{dt}\left\{\sigma(\gamma)\Lambda(\gamma; t)\right\} = \tau(\gamma)\left\{\sigma(\gamma)\Lambda(\gamma; t)\right\} \qquad (b.13)$$

note: See page 128A for a note on * in the previous page (2nd line)

Any solution of (b.6) is simply obtained by multiplying the matrix $\sigma\Lambda$ from the right by a suitable vector. For a shower excited by a single electron of energy E_0, with the boundary condition $n(E;0) = \delta(E-E_0)$, or equivalently, $\nu(\gamma;0) = E_0^{\gamma-1}$, the solution of (b.6) is the well known one

$$\phi(\gamma;t) = \sigma(\gamma)\Lambda(\gamma;t)\begin{pmatrix}1\\1\end{pmatrix}\frac{E_0^{\gamma-1}}{\mu_\gamma - \lambda_\gamma} \qquad (b.14)$$

It follows from (b.13) that the matrix $\Sigma^{-1}(\gamma;t) = \Lambda^{-1}(\gamma;t)\sigma^{-1}(\gamma)$ satisfies the equation

$$\frac{d}{dt}\Sigma^{-1}(\gamma;t) = -\Sigma^{-1}(\gamma;t)\,\tau(\gamma) \qquad (b.15)$$

where

$$\Lambda^{-1}(\gamma;t) = \begin{pmatrix} e^{\lambda_\gamma t} & 0 \\ 0 & e^{\mu_\gamma t}\end{pmatrix}$$
$$\sigma^{-1}(\gamma) = \frac{1}{(\mu_\gamma-\lambda_\gamma)C_\gamma}\begin{pmatrix} C_\gamma & \mu_\gamma - D \\ C_\gamma & -(D-\lambda_\gamma)\end{pmatrix} \qquad (b.16)$$

To solve the equations for the density of degree two we now make a double Mellin Transformation, thus

$$\nu_2(\gamma,s;t) = \int_0^\infty dE \int_0^\infty dE' E^{\gamma-1}E'^{s-1}n_2(E,E';t) \qquad (b.17)$$

$$\nu_\gamma(\gamma,s;t) = \int_0^\infty dE \int_0^\infty dE' E^{\gamma-1}E'^{s-1}n_m(E,E';t) \qquad (b.18)$$

$$\gamma_2(\gamma,s;t) = \int_0^\infty dE \int_0^\infty dE' E^{\gamma-1}E'^{s-1}m_2(E,E';t) \qquad (b.19)$$

Denoting by $\phi^{(2)}(r,s;t)$ the vector

$$\phi^{(2)}(r,s;t) = \begin{pmatrix} \nu_2(r,s;t) \\ \nu_r(r,s;t) \\ \nu_r(s,r;t) \\ \gamma_2(r,s;t) \end{pmatrix} \tag{b.20}$$

and performing a double Mellin transform on the equations for density of degree two one finds without any difficulty that $\phi^{(2)}$ satisfies the equation

$$\frac{d}{dt}\phi^{(2)}(r,s;t) = \{\tau(r)\otimes 1 + 1\otimes\tau(s)\}\phi^{(2)}(r,s;t) + \psi(r,s;t) \tag{b.21}$$

where 1 is the 2×2 unit matrix and ψ is given by

$$\psi(r,s;t) = \begin{pmatrix} a_1(r,s)\,\gamma(r+s-1;t) \\ a_2(r,s)\,\nu(r+s-1;t) \\ a_2(s,r)\,\nu(r+s-1;t) \\ 0 \end{pmatrix} \tag{b.22}$$

and *

$$a_1(\gamma,s)=\frac{2\Gamma(\gamma)\Gamma(s)}{\Gamma(\gamma+s)}\left\{1-\left(\frac{4}{3}+\alpha\right)\frac{\gamma s}{(\gamma+s+1)(\gamma+s)}\right\} \quad \text{(b.23)}$$

$$a_2(\gamma,s)=\frac{\Gamma(\gamma)\Gamma(s)}{\Gamma(\gamma+s)}\left\{\frac{s}{\gamma+s}+\left(\frac{4}{3}+\alpha\right)\frac{\gamma}{s-1}\right\} \quad \text{(b.24)}$$

The multiplication sign $\otimes$ in (b.21) denotes the direct product of two matrices. It follows immediately that in the absence of the ψ on the right the solution of (b.21) would be a direct product of the solutions of (b.6). Let $\zeta(\gamma,s;t)$ be a 4×4 matrix satisfying

$$\frac{d}{dt}\zeta(\gamma,s;t)=\{\tau(\gamma)\otimes 1+1\otimes\tau(s)\}\zeta(\gamma,s;t) \quad \text{(b.25)}$$

so that $\zeta^{-1}(\gamma,s;t)$ satisfies

$$\frac{d}{dt}\zeta^{-1}(\gamma,s;t)=-\zeta^{-1}(\gamma,s;t)\{\tau(\gamma)\otimes 1+1\otimes\tau(s)\} \quad \text{(b.26)}$$

Then a solution of (b.21) is

$$\phi^{(2)}(\gamma,s;t)=\zeta(\gamma,s;t)\int_0^t\zeta^{-1}(\gamma,s;t)\,\psi(\gamma,s;t)\,dt \quad \text{(b.27)}$$

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* $a_1(\gamma,s)$ is the same as $S(\gamma+1,s-1)$ defined by formula (41a) of the paper by Scott and Uhlenbeck. Their (41a) is however in error by having $\Gamma(\gamma+s+1)$ in place of $\Gamma(\gamma+s+2)$

This solution satisfies the boundary condition $\phi^{(2)} = 0$ at $t = 0$, corresponding to the fact that at $t = 0$ the shower is initiated by a single electron of energy E_0 only.

It follows from (b.13) and (b.25) that

$$\zeta(r,s;t) = \{\sigma(r) \otimes \sigma(s)\}\{\Lambda(r;t) \otimes \Lambda(s;t)\}$$
$$= \zeta_0(r,s)\{\Lambda(r;t) \otimes \Lambda(s,t)\} \quad \text{(b.28)}$$

where $\zeta_0(r,s) = \sigma(r) \otimes \sigma(s)$ is given explicitly by (b.11) and is independent of t. It follows from (b.11), (b.12) and (b.14) that

$$\psi(r,s;t) = \psi_1(r,s) E_0^{r+s-2} e^{-\lambda_{r+s-1} t} + \psi_2(r,s) E_0^{r+s-2} e^{-\mu_{r+s-1} t}$$

where

$$\psi_1(r,s) = \frac{1}{\mu_{r+s-1} - \lambda_{r+s-1}} \begin{pmatrix} a_1(r,s) C_{r+s-1} \\ a_2(r,s)(D - \lambda_{r+s-1}) \\ a_2(s,r)(D - \lambda_{r+s-1}) \\ 0 \end{pmatrix} \quad \text{(b.30)}$$

$$\psi_2(r,s) = \frac{1}{\mu_{r+s-1} - \lambda_{r+s-1}} \begin{pmatrix} -a_1(r,s) C_{r+s-1} \\ a_2(r,s)(\mu_{r+s-1} - D) \\ a_2(s,r)(\mu_{r+s-1} - D) \\ 0 \end{pmatrix} \quad \text{(b.31)}$$

Corresponding to the two parts ψ_1 and ψ_2 of ψ, one now finds that $\phi^{(2)}$ itself splits into two parts $\phi_1^{(2)} + \phi_2^{(2)}$. Since the only t dependent factors in the integral in (b.27) are the exponentials, one finds easily

$$\phi_1^{(2)}(r,s;t) = E_0^{r+s-2} \zeta_0(r,s) \eta(\lambda_{r+s-1}) \times^* \zeta_0^{-1}(r,s) \psi_1(r,s) \quad \text{(b.32)}$$

and

$$\phi_2^{(2)}(r,s;t) = E_0^{r+s-2} \zeta_0(r,s) \eta(\mu_{r+s-1}) \times \zeta_0^{-1}(r,s) \psi_2(r,s) \quad \text{(b.33)}$$

* $\times$ denotes ordinary multiplication

where $\eta(x)$ is the diagonal matrix with the four elements

$$\frac{e^{-xt} - e^{-(\lambda_r+\lambda_s)t}}{\lambda_r+\lambda_s-x}, \quad \frac{e^{-xt} - e^{-(\lambda_r+\mu_s)t}}{\lambda_r+\mu_s-x},$$

$$\frac{e^{-xt} - e^{-(\mu_r+\lambda_s)t}}{\mu_r+\lambda_s-x} \quad \text{and} \quad \frac{e^{-xt} - e^{-(\mu_r+\mu_s)t}}{\mu_r+\mu_s-x}$$

(b.34)

Terms of the type $\exp(-(\lambda_r+\lambda_s)t)$ in (b.32) can be combined with the corresponding term in (b.33). We then obtain after some calculation

$$\phi^{(2)}(r,s;t) = \phi'(r,s;t)+\phi_\lambda(r,s:t)+\phi_\mu(r,s;t) \quad \text{(b.35)}$$

where

$$\phi'(r,s;t) = E_0^{r+s-2}\zeta_0(r,s)\{\Lambda(r;t)\otimes\Lambda(s;t)\}\psi'(r,s) \quad \text{(b.36)}$$

$$\phi_\lambda(r,s;t) = E_0^{r+s-2} e^{-\lambda_{r+s-1}t} \times \quad \text{(b.37)}$$
$$\zeta_0(r,s)\eta_0(\lambda_{r+s-1})\zeta_0^{-1}(r,s)\psi_1(r,s)$$

$$\phi_\mu(r,s;t) = E_0^{r+s-2} e^{-\mu_{r+s-1}t} \times \quad \text{(b.38)}$$
$$\zeta_0(r,s)\eta_0(\mu_{r+s-1})\zeta_0^{-1}(r,s)\psi_2(r,s)$$

the vector v' being given by

$$v'(r,s)=\begin{cases}\dfrac{1}{(\lambda_r+\lambda_s-\lambda_{r+s-1})(\lambda_r+\lambda_s-\mu_{r+s-1})}\Big\{a_1(r,s)\,C_{r+s-1}\\ +a_2(r,s)\dfrac{\mu_s-D}{C_s}(D-\lambda_r-\lambda_s)+a_2(s,r)\times\\ \dfrac{\mu_r-D}{C_r}(D-\lambda_r-\lambda_s)\Big\}\\ \dfrac{1}{(\lambda_r+\mu_s-\lambda_{r+s-1})(\lambda_r+\mu_s-\mu_{r+s-1})}\Big\{a_1(r,s)\\ \times C_{r+s-1}\\ -a_2(r,s)\dfrac{D-\lambda_s}{C_s}(D-\lambda_r-\mu_s)+a_2(s,r)\times\\ \dfrac{\mu_r-D}{C_r}(D-\lambda_r-\mu_s)\Big\}\\ \dfrac{1}{(\mu_r+\lambda_s-\lambda_{r+s-1})(\mu_r+\lambda_s-\mu_{r+s-1})}\Big\{a_1(r,s)\\ \times C_{r+s-1}\\ +a_2(r,s)\dfrac{\mu_s-D}{C_s}(D-\mu_r-\lambda_s)-a_2(r,s)\times\\ \dfrac{D-\lambda_r}{C_r}(D-\mu_r-\lambda_s)\Big\}\\ \dfrac{1}{(\mu_r+\mu_s-\lambda_{r+s-1})(\mu_r+\mu_s-\mu_{r+s-1})}\times\\ \Big\{a_1(r,s)\,C_{r+s-1}-a_2(r,s)\dfrac{D-\lambda_s}{C_s}(D-\mu_r-\mu_s)\\ -a_2(s,r)\dfrac{D-\lambda_r}{C_r}(D-\mu_r-\mu_s)\Big\}\end{cases}\qquad\text{(b.39)}$$

$\eta_0(x)$ is just the diagonal matrix obtained from $\eta(x)$ by replacing the numerator of every element in the latter by $\zeta_0^{-1}(r,s)\psi_1(r,s)$.

Denoting the vector in (b.37) namely $\xi_0(r,s)\eta_0(\lambda_{r+s-1})\lambda$ by $v_\lambda(r,s)$ one finds after a somewhat tedious calculation that its first component is

$$\Big[\{D(D-x)(x_r+x_s)+\lambda_r\mu_r x_s+\lambda_s\mu_s x_r-x x_r x_s\}a_1(r,s)+C_{r+s-1}+(D-\lambda_s)(\mu_s-D)C_s^{-1}\{D(x_r+x_s)-\lambda_r\mu_r+\qquad\text{(b.40)}$$

$$(\lambda_s-\chi)(\mu_s-\chi)\}\, a_2(r,s)(D-\lambda_{r+s-1})+(D-\lambda_r)(\mu_r-D)$$
$$C_r^{-1}\{D(\chi_r+\chi_s)-\lambda_s\mu_s+(\lambda_r-\chi)(\mu_r-\chi)\}a_2(s,r)$$
$$(D-\lambda_{r+s-1})]\{(\mu_{r+s-1}-\lambda_{r+s-1})(\lambda_r+\lambda_s-\chi)(\lambda_r+\mu_s-\chi)$$
$$(\mu_r+\lambda_s-\chi)(\mu_r+\mu_s-\chi)\}^{-1}$$

where $\chi=\lambda_{r+s-1}$ and $\chi_r=\lambda_r+\mu_r-\chi$ (b.41)

The last component of $v_\lambda(r,s)$ is

$$[(C_r C_s(\chi_r+\chi_s)a_1(r,s)C_{r+s-1}+\{A_s(\chi_r+\chi_s)$$
$$-\mu_s\lambda_s+(\lambda_r-\chi)(\mu_r-\chi)\}a_2(r,s)C_r(D-\lambda_{r+s-1})$$
$$+\{A_r(\chi_r+\chi_s)-\mu_r\lambda_r+(\lambda_s-\chi)(\mu_s-\chi)\}$$
$$a_2(s,r)C_s(D-\lambda_{r+s-1})]\{(\mu_{r+s-1}-\lambda_{r+s-1})$$
$$(\lambda_r+\lambda_s-\chi)(\lambda_r+\mu_s-\chi)(\mu_r+\lambda_s-\chi)$$
$$(\mu_r+\mu_s-\chi)\}^{-1} \quad \text{(b.42)}$$

In the paper of Scott and Uhlenbeck the approximation is made of putting all the elements of $\eta_0(\chi)$ equal to zero except the first element on the diagonal. This is erroneous, since the other elements on the diagonal of $\eta_0(\chi)$ make a contribution to (b.40) and (b.42) which is comparable with that of the first diagonal element.

It is clear from (b.14) and (b.23) that if v' in (b.36) were replaced by the vector having all its components equal to 1 then the right hand side of (b.36) would simply become $\phi(r;t)\otimes\phi(s;t)$. It is convenient to separate out this part and replace $v'(r,s)$ by

$$v''(r,s)=v'(r,s)-\begin{pmatrix}1\\1\\1\\1\\\vdots\end{pmatrix} \quad \text{(b.43)}$$

Denoting a vector with the four components $n_2(E,E';t)$ $n_m(E,E';t)$, $n_m(E',E;t)$ and $n_2(E,E';t)$ by $f(E,E';t)$ one obtains through a reversal of the transformations (17 to 19)

$$f(E,E';t) = \frac{1}{(2\pi i)^2}\int_{c-i\alpha}^{c+i\alpha} d\gamma \int_{c-i\alpha}^{c+i\alpha} ds\, E^{-\gamma} E'^{-s} \phi^{(2)}(\gamma,s;t) \qquad (b.44)$$

$$= f_0(E,E';t) + f''(E,E'';t) + f_\mu(E,E';t) + f_\lambda(E,E';t)$$

c being some positive real number. $f_0(E,E';t)$ is the vector with the components

$n(E;t)n(E';t)$, $n(E;t)m(E';t)$, $n(E';t)m(E;t)$ and $m(E;t)m'(E';t)$

while

$$f''(E,E';t) = \frac{1}{(2\pi i)^2}\int_{c-i\alpha}^{c+i\alpha}\int \frac{1}{E_0^2}\left(\frac{E_0}{E}\right)^{\gamma}\left(\frac{E_0}{E'}\right)^{s} \times \qquad (b.45)$$

$$\zeta_0(\gamma,s)\{\Lambda(\gamma;t)\times\Lambda(s;t)\}v''(\gamma,s)\,d\gamma\,ds$$

$$f_\lambda(E,E';t) = \frac{1}{(2\pi i)^2}\int\int \frac{1}{E_0^2}\left(\frac{E_0}{E}\right)^{\gamma}\left(\frac{E_0}{E'}\right)^{s} \times \qquad (b.46)$$

$$e^{-\lambda_{\gamma+s-1}t}\zeta_0(\gamma,s)\,\eta_0(\lambda_{\gamma+s-1})\,\zeta_0^{-1}(\gamma,s)\,\psi_1(\gamma,s)\,d\gamma\,ds$$

$$f_\mu(E,E';t) = \frac{1}{(2\pi i)^2}\int\int \frac{1}{E_0^2}\left(\frac{E_0}{E}\right)^{\gamma}\left(\frac{E_0}{E'}\right)^{s} \qquad (b.47)$$

$$e^{-\mu_{\gamma+s-1}t}\zeta_0(\gamma,s)\,\eta_0(\mu_{\gamma+s-1})\,\zeta_0^{-1}(\gamma,s)\,\psi_2(\gamma,s)\,d\gamma\,ds$$

To get the mean of the square of the number of particles or photons in the energy interval from E_1 to E_2 one has to integrate the f's with respect to E and E' from E_1 to E_2. This simply changes any particular f in (b.45, b.46, b.47) to $F(E_1,E_1)+F(E_2,E_2)-2F(E_1,E_2)$ where $F(E_1,E_2)$ is obtained from the corresponding f

b 45-47

in (λ) by replacing $E_0^{-2}(E_0/E)^{\gamma}(E_0/E')^{s}$ in the integrand by

$$\frac{1}{(\gamma-1)(s-1)}\left(\frac{E_0}{E}\right)^{\gamma-1}\left(\frac{E_0}{E'}\right)^{s-1}$$

* (See page 128)

Thus it follows (this

$$\overline{N(E_1,E_2)^2}-\{\overline{N(E_1,E_2)}\}^2=\overline{N(E_1,E_2)}+[\{F''(E_1,E_1)+F''(E_2,E_2)-2F''(E_1,E_2)\}+\{F\lambda(E_1,E_1)+F\lambda(E_2,E_2)-2F\lambda(E_1,E_2)\}+\{F\mu(E_1,E_1)+F\mu(E_2,E_2)-2F\mu(E_1,E_2)\}]_1 \qquad (b.48)$$

when the 1 at the end of the square bracket indicates that the first component alone of the vector in the brackets is to be taken. To obtain the mean of the squares of the number of photons $M(E_1,E_2)$ in the energy interval from E_1 to E_2 one has a corresponding expression with M replacing N in (b.48), and a suffix 4 at the end of the square brackets. By letting $E_2\longrightarrow\infty$ one gets $N(E_1)$ the total number of particles of energy greater than E_1. In that case, provided $E_1\ll E_0$, the contribution of the last two F's in each curly bracket in (b.48) is negligible compared with the first, and one obtains to a good approximation

$$\overline{N(E_1)^2}-\{\overline{N(E_1)}\}^2=\overline{N(E_1)}+\{F''(E_1,E_1)+F\lambda(E_1,E_1)+F\mu(E_1,E_1)\}_1 \qquad (b.49)$$

The double integrals for the f's given by (b.45, b.46, b.47) or the corresponding ones for the F's can be evaluated by the saddle point method. It is true as usual that but for small t the contribution of f_μ is negligible compared with that

of f'' or f_λ while even in f'' given by (b.45), the contribution comes mainly from the first element in the matrix $\Lambda(r;t)\otimes\Lambda(s;t)$ namely $\exp\{-(\lambda_r+\lambda_s)t\}$ which exceeds the others by an order of magnitude for small t. Thus, one essentially requires to know only the first component of the vector $v''(r,s)$ given by (b.39).

The circumstance that there is a non-vanishing fourth component to the matrix in (b.45) and likewise a non-vanishing fourth component of the matrix v_λ given by (b.42) results in the mean square deviation of the number of quanta in any energy interval differing from its Poissonion value. This means that there is a correlation between the numbers of quanta in two different energy intervals despite the fact that in the equation for m_2, there is no correlation term on the right.

SUMMARY: On the basis of the quantum mechanical cross-sections for radiation emission by electrons and pair creation by quanta when screening is complete, the fluctuation in the mean number of electrons and photons in any given energy interval in a cascade is calculated. The expression for the mean square deviation of this number is given explicitly in the form of a double integral which can be evaluated by the saddle point method

* (page 127) : The mean of the number $N(E_1,E_2)$ of particles whose energies lie between E_1 and E_2 is $\overline{N(E_1,E_2)} = \int_{E_1}^{E_2} n(E,t)\,dE$ and Bhabha (1950) has shown that $\overline{N(E_1,E_2)^2} = \overline{N(E_1,E_2)} + \int_{E_1}^{E_2}\int_{E_1}^{E_2} n_2(E,E';t)\,dE\,dE'$ where the continuous function $n_2(E,E';t)$ together with the two other functions $n_m(E,E';t)$ and $m_2(E,E';t)$ is determined by equations III to V in page 128A.

* Equations referred to here are (6·13) and (6·15) They appear in the paper of Bhabha and Ramakrishnan with some changes in notation. We give them as they appear in that paper for a clear understanding of what follows

$$\frac{d}{dt}n(E;t) = \int_E^\infty dE' R(E',E)\, n(E';t) - \int_0^E dE' R(E,E')\, n(E;t) + 2\int_0^\infty dE' R'(E,E')\, m(E';t)$$

$$= T_E^{11}\{n(E;t)\} + T_E^{12}\{m(E;t)\} \qquad \text{I (6·13A)}$$

$$\frac{d}{dt}m(E;t) = \int_E^\infty dE' R(E',E'-E)\, m(E';t) - \int_0^\infty dE' R'(E,E')\, m(E;t)$$

$$= T_E^{21}\{n(E;t)\} + T_E^{22}\{m(E;t)\} \qquad \text{II (6·13B)}$$

T_E^{kl} here stand as abbreviations for the four operations which occur in these equations

(6·15) →
$$\frac{d}{dt}n_2(E,E';t) = \int_E^\infty d\epsilon \cdot R(\epsilon,E)\, n_2(\epsilon,E') + \int_{E'}^\infty d\epsilon\, R(\epsilon,E')\, n_2(E,\epsilon) - \left\{\int_0^E d\epsilon\, R(E,\epsilon) + \int_0^{E'} d\epsilon \cdot R(E',\epsilon)\right\} n_2(E,E') + 2\int_{E'}^\infty d\epsilon\, R'(\epsilon,E')\, nm(E,\epsilon) + 2\int_E^\infty d\epsilon \cdot R'(\epsilon,E)\, nm(E',\epsilon) + 2m(E+E')\, R'(E+E',E)$$

$$= T_E^{11}\{n_2(E,E')\} + T_{E'}^{11}\{n_2(E,E')\} + T_E^{12}\{nm(E',E)\} + T_{E'}^{12}\{nm(E,E')\} + 2R'(E+E',E)\, m(E+E') \qquad \text{III}$$

$$\frac{d}{dt}nm(E,E';t) = T_E^{11}\{nm(E,E')\} + T_E^{12}\{m_2(E,E')\} + T_{E'}^{21}\{n_2(E,E')\} + T_{E'}^{22}\{nm(E,E')\} + R(E+E',E)\, n(E+E') \qquad \text{IV}$$

$$\frac{d}{dt}m_2(E,E';t) = T_E^{21}\{nm(E,E')\} + T_{E'}^{21}\{nm(E',E)\} + T_E^{22}\{m_2(E,E')\} + T_{E'}^{22}\{m_2(E,E')\} \qquad \text{V}$$

Here $n(E;t)$, $m(E;t)$ $n_2(E,E';t)$, $nm(E,E';t)$ and $m_2(E,E';t)$ have replaced $f_1(E,t)$, $g_1(E,t)$ $f_2(E_1,E_2;t)$, $fg_{1,1}(E_1,E_2;t)$ and $g_2(E_1,E_2;t)$ respectively in the equations (6.13) and (6.15) and ρ has been put equal to 0.

APPENDIX III

A discrete model having the properties of a continuum of stochastic variables

We can build up a system of particles occupying a discrete number of states (finite or enumerably infinite) which presents features analogous to the continuous system provided that we assume that in every state there can be 1 or 0 particles. In other words we have a sequence of stochastic variables

$$n_1, n_2, \ldots, n_i, \ldots .$$

each of which can assume the values of 0 or 1. In such a case we can define the probability that n_i assumes the value 1 as ${}_{(1)}P_i$. We can define a joint probability of degree two that n_i assumes the value 1 and n_J assume the value 1 as ${}_{(2)}P_{iJ}$. We can also in the same way define a joint probability of degree m that m variables $n_i, n_J, \ldots$ each assume the value 1, represented by ${}_{(m)}P_{iJK\ldots}$. By simple arguments it follows that if ν is the sum of stochastic variables, i.e.

$$\nu = \sum_i n_i$$

the γth moment of ν is given by

$$\overline{\nu^\gamma} = \sum_S \left\{ C^\gamma_S \sum_i \sum_J \ldots \sum_{(S)} P_{iJK\ldots} \right\} \qquad (c.1)$$

In particular

$$\overline{\nu} = \sum_i {}_{(1)}P_i \quad , \quad \overline{\nu^2} = \sum_i {}_{(1)}P_i + \sum_i \sum_J {}_{(2)}P_{iJ} \quad i \neq J \qquad (c.2)$$

It is quite clear that if these stochastic variables (n_i) are independent and the number of such variables tends to infinity keeping $\bar{\nu}$ finite, then ν approaches a Poisson distribution.

Recently Koopman [illegible] considered such a sequence if it possesses Markovian properties. We shall call the sequence Markovian if the probability that n_J assumes the value 0 or 1, given that n_i has assumed the value 0 or 1 $(i<J)$ does not depend upon the additional information about what value n_K has assumed. $(K<i)$. Accordingly we define

$$a_J = p(n_J = 1 \mid n_{J-1} = 1)$$

$$b_J = p(n_J = 1 \mid n_J = 0)$$

where a_J and b_J are the probabilities that n_J assumes the value 1 given that n_{J-1} has assumed the value 1 or 0, respectively. We further define P_J^K as the probability that $n_J = 1$ given that $n_K = 1 \ (K<J)$, and Π_J^K as the probability that $n_J = 1$ given that $n_K = 0 \ (K<J)$. Following Koopman, P_J^K and Π_J^K satisfy the classical probability equations

$$P_J^K = P_{J-1}^K a_J + (1 - P_{J-1}^K) b_J, \quad P_{K+1}^K = a_{K+1} \quad (c.3)$$

$$\Pi_J^K = \Pi_{J-1}^K a_J + (1 - \Pi_{J-1}^K) b_J, \quad \Pi_{K+1}^K = b_{K+1} \quad (c.4)$$

It is quite easy to prove that the joint probability of degree n that n members of the sequence $n_J, n_\ell \ldots$ are

each equal to 1 given that $n_K = 1$ $(K < J < \ell \ldots)$ defined by

$$(n) P^{K}_{J\ell}$$

can be uniquely determined in terms of the joint probability of degree one

$$(n) P^{K}_{J\ell \ldots} = P^{K}_{J} P^{J}_{\ell} \cdots \quad \text{(a product of } n \text{ such terms)} \tag{c.5}$$

Note that here we have denoted P^{K}_{J} as probability of degree one which we originally called $_{(1)}P_{J}$.

Now let us assume that the a_i's and b_i's are independent of i . This immediately yields the result

$$P^{1}_{K} = C^{K-1} + b\left(\frac{1 - C^{K-1}}{1 - C}\right), \quad C = a - b \tag{c.6}$$

Also

$$P^{J}_{K} = P^{1}_{K-J+1} \tag{c.7}$$

Using the above result we obtain

$$\overline{\nu}_{\infty} = \sum_{i=2}^{i=\infty} P^{1}_{i} = \left\{\frac{\ell + a}{\ell - a}\right\} \tag{c.8}$$

the condition for the convergence of the sum being $nb \to \ell$ as $n \to \infty$, $b \to 0$. Note $C \to a$.

Also

$$\overline{\nu^{2}_{\infty}} = \frac{\ell a + \ell + a}{\ell^{2}} + \overline{\nu}^{2}_{\infty} \tag{c.9}$$

This result agrees with that of Koopman who by a different method obtained the generating function ν as

$$\phi(u) = \left[1 - \frac{1-u}{1-\gamma u}\right] \exp\left\{\frac{-\bar{m}(1-u)(1-\gamma) + (1-u)}{1-\gamma u}\right\} \quad \text{(c.10)}$$

Koopman's $\bar{m} = \bar{\nu}_{\infty} + 1$, $c = \gamma$ i.e. $\bar{m}$ includes the value of the initial stochastic variable.

APPENDIX IV

On an integral equation of Chandrasekhar and Munch

Part I

Recently, Chandrasekhar and Munch (hereafter referred to as C.M.) obtained an integro-differential equation in dealing with the fluctuations in brightness in the Milky Way. In this note we shall obtain their equation more directly by using the device of a simple reformulation of their problem. This was formulated by them as follows:*

The problem being considered as a one-dimensional one, our object is to calculate the probability distribution $q(u, \epsilon)\, du$ of radiation of intensity u at $t = \epsilon$ where t is the linear distance along the line of sight. It is given 1) that there is a deterministic contribution of radiation of magnitude $\beta\, d\tau$ from the element of length $d\tau$ at $t = \tau$ (this is due to the stars occurring in a uniform distribution along t) to the intensity u at $t = \epsilon$ 2) that clouds occur in a Poisson distribution in any element of length t given by

$$\frac{e^{-\alpha t}(\alpha t)^n}{n!}$$

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* The language used in the formulation is that of the author. It is intended to facilitate the reformulation.

A given cloud has a transparency factor q (i.e. it reduces the intensity of radiation of the light of the stars immediately behind it by a factor q) with a probability density $\psi(q)$. In other words, α is the probability per unit t that radiation of given intensity u 'jumps' to an interval between uq and $u(q+dq)$.

If we formulate the problem in a different manner, the integral equation of C.M. can be derived very easily and no recourse need be had to the rather long procedure they have adopted. We re-formulate the problem as follows:

Given that 1) there is a deterministic contribution of magnitude $\beta d\tau$ from $d\tau$ at $t=\tau$ to the intensity u at $t=\epsilon$ 2) the probability per unit t (or distance) that an intensity of magnitude u drops to an interval lying between uq and $u(q+dq)$ is $\alpha\cdot\psi(q)\,dq$. (This of course in the physical problem is due to the intercention of clouds, but for mathematical purposes, only the transition probability per unit t is required) our purpose is to calculate the probability distribution of intensity u at $t=\epsilon$ defined by the function $g(u,\epsilon)$.

The two formulations are <u>equivalent</u>. (This fact may not have been noticed by C.M.) as a mere examination will reveal. (The equivalence is discussed in detail presently). But the second formulation yields the integral equation immediately if we use the Markovian property of the stochastic process defined by $g(u,\epsilon)$.

Let us increase ϵ by $d\epsilon$ and calculate $g(u, \epsilon + d\epsilon)$ *

1) There is a contribution to the probability $g(u, \epsilon + d\epsilon)du$ if the state (u', ϵ) existing with probability $g(u', \epsilon)\, du'$ drops to the state u and the measure of this contribution is

$$\alpha \cdot g(u', \epsilon)\, \psi(u/u')(du/u')\, du'\, d\epsilon \quad , \quad \binom{u}{u'} = q$$

and so the total measure of the contribution is

$$\alpha\, du\, d\epsilon \int_u^\infty \psi(u/u')\, du'/u'\, g(u', \epsilon)$$

2) There is a depletion in the probability of the state $(u, \epsilon + d\epsilon)$ if the radiation of intensity u at ϵ drops to u' at $\epsilon + d\epsilon$ and the measure of this depletion is

$$g(u, \epsilon)\, \alpha \cdot d\epsilon\, du \int_0^u \psi(u'/u)\, \frac{du'}{u} = \alpha \cdot g(u, \epsilon)\, du\, d\epsilon$$

since

$$\int_u^\infty \psi(u'/u)\, du'/u = 1$$

3) Due to the deterministic contribution of magnitude $\beta d\epsilon$, the state $(u - \beta d\epsilon, \epsilon)$ 'moves' to $(u, \epsilon + d\epsilon)$

Thus

$$g(u, \epsilon + d\epsilon)\, du = g(u - \beta d\epsilon, \epsilon)\, du + \alpha g(u, \epsilon)\, du\, d\epsilon + du\, d\epsilon \int_u^\infty g(u', \epsilon)\, \alpha\, \psi(u/u')\, \frac{du'}{u'}$$

---

* This is a standard procedure to solve stochastic problems of this type. See for example A. Ramakrishnan, Proc. Camb. Phil. Soc. 46 (1950) p. 593.

Making $d\epsilon$ tend to zero

$$\frac{\partial g(u,\epsilon)}{\partial \epsilon} = -g(u,\epsilon)\alpha - \beta \frac{\partial g(u,\epsilon)}{\partial u} + \alpha \int_u^\infty g(u',\epsilon)\,\psi(u/u')\,du'/u'$$

This is the integro-differential equation obtained by C.M. putting $\alpha = 1$, $\beta = 1$. The above equation is more general since α and β can be functions of ϵ. Note that

$$\int_u^\infty g(u',\epsilon)\,\psi(u/u')\,du'/u' = \int_0^1 g(u/q,\epsilon)\,\frac{\psi(q)}{q}\,dq$$

where the transition probability $u' \to u$ is homogeneous in u' and u. In view of this the integral equation is reducible by using the Mellin's transformation

$$P(S,\epsilon) = \int_0^\infty u^{S-1} g(u,\epsilon)$$

where S is a complex variable. This transformation reduces the equation to

$$\frac{\partial P(s,\epsilon)}{\partial \epsilon} = -\alpha p(s,\epsilon) + \eta(s)\alpha p(s,\epsilon) + \beta(s-1)p(s-1,\epsilon)$$

where

$$\eta(s) = \int_0^\infty q^{s-1}\psi(q)\,dq$$

$$g(u,\epsilon) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} p(s,\epsilon)\,\epsilon^{-s}\,ds$$

When $S = n+1$, where n is a non-negative integer gives the n-1th moment of u. In such a case the difference - differential equation for $p(n,t)$ can be solved by iteration as shown by C.M.

We shall now examine in detail the duality between the two formulations.

The duality consists in the realisation of the fact that the probability of a transition from u occurring in an interval dt is equal to the probability that an interception to u occurs. If $R(u,u',t)du'$ is the transition probability per unit t that a transition $u \to u'$ takes place then the total probability of a transition from u is $\int_{u'} R(u,u',t)du' = \alpha(u,t)$ If the probability of an interception to u is independent of u, $\alpha(u,t)$ will be independent of u. The conditional probability that given a transition from u has occurred, the transition is of the (u,u') type, is given by

$$\psi(u,u',t) = R(u,u',t) / \alpha(u,t)$$

Note that $\int_{u'} \psi(u,u',t)\,du' = 1$ for all u, t,

of course, the total probability that any transition takes place is given by $\int_u g(u,t)\,\alpha(u,t)\,du$

This is of no importance since, if α is a function of u, to evaluate the above we must know $q(u,t)$ which is the object of the problem. If α is independent of u it reduces to α since $\int_u q(u,t)\,du = 1$

In this case $R(u,u')\,du'$ is expressible as a function of q where $q = u'/u$ Writing it as $R(q)\,dq$ α is independent of u

$$R(q,t) = \alpha(t)\psi(q,t) \quad \text{or}$$

$$R(q,t) = \alpha(t)\,\psi(q) \quad \text{if } \psi \text{ is}$$

independent of t.

If α is independent of t it can be normalized to 1 without loss of generality but suitably choosing a unit of t. It is to be noted that even if α were a function of t the number of transitions (which is equal to the number of interceptions) obeys a Poisson distribution with mean $\bar{n} = \int_0^t \alpha\,dt$. The distribution $\pi(n,t)$ is Poisson if the probability that an interception occurs in an interval dt is <u>independent</u> of the occurrence of an interception in any other interval. If this were not so, $\pi(n,t)$ will not be Poissonian and for example we may have to define $\alpha(n,t)\,dt$ as the total probability that interception occurs in dt given that n interceptions have occurred previously. Then we must write

$$R(n,q,t) = \alpha(n,t)\,\psi(n,q,t)$$

where $R(n, q, t)$ is [illegible] taking place in dt given that n transitions have already taken place. In such a case the stochastic process defined by $g(u,t)$ is no longer Markovian, but the Markovian property can be restored by defining a new function $g(n, u, t)$ where $g(n, u, t)\, du$ is the joint probability that the intensity at u lies between u and $u+du$ and that n collisions have occurred previously. $g(n, u, t)$ satisfies the integral equation

$$\frac{\partial g(n,u,t)}{\partial t} = -g(n,u,t)\,\alpha(n,t) - \beta \frac{\partial g(n,u,t)}{\partial u} + \int_0^1 g(n-1, u/q, t)\, R(n-1, q, t)\, \frac{dq}{q}$$

Part II

In atomic physics the same type of equation is obtained by considering the following stochastic process.

An electron in its passage through matter loses energy by radiating a photon. The probability per unit distance that electron of energy u drops to a state between u and $u+du$ radiating a photon of energy $u-u'$ is $R(u, u')\, du'$. The electron through ionisation loses energy deterministically at the rate β per unit distance. In such a case if $g(u,t)\, du$ is the probability that the energy lies between u and $u+du$ the $g(u,t)$ satisfies

$$\frac{\partial g(u,t)}{\partial t} = -g(u,t)\,\alpha + \beta \frac{\partial g(u,t)}{\partial u} + \int_u g(u',t)\, R(u', u)\, du'$$

Here we have $+\beta$ instead of $-\beta$ since we deal with collision loss as contrasted with intensity gain. The number of photons which is equal to the number of transitions correspond to the number of clouds in the Astro-physical problem. Phenomenologically, of course, the two processes are different. In the astro-physical case it is the cloud which causes the transitions (and as such α is not a function of u but atmost only of t). In the electron case it is the transitions that create the photons and so α is only a function of u and not of t. In fact only in the case of 'complete screening' (screening refers to the nuclei of matter through which the electron passes) is the assumption of homogeneity for $R(u,u')\,du'$ and hence the independence of α on u justified.

To calculate $\pi(n,t)$ the distribution of photons we have to introduce a function $P(n,u,t)$ such that $P(n,u,t)\,du$ represents the joint probability that n photons have been emitted and the energy of the electron lies between u and $u+du$

$$\pi(n,t) = \int_u P(n,u,t)\,du$$

$$\frac{\partial P(n,u,t)}{\partial u} = -P(n,u,t)\,\alpha(u) + \beta\,\frac{\partial P(n,u,t)}{\partial u} + \int P(n-1,u',t)\,R(u',u)\,du'$$

It must incidentally be mentioned that the above process is not realistic since at such energies when radiation loss takes place, pair creation by photons occurs. If this were taken into account we are led to the cascade equations with collision loss and the distribution of the photons and electrons becomes a very complicated problem. (See for example: A.Ramakrishnan, Proc. Camb. Phil. Soc. 46 (1950) p.598;

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