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LECTURE NOTES ON

GRAVITATIONAL COLLAPSE AND GRAVITATIONAL RADIATION

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P R E F A C E

This report consists of the material presented in a course of lectures on advanced topics in the theory of gravitation, at the Institute of Mathematical Sciences, Madras, during the year 1971. The prerequisites for following this course have been covered in an earlier report on the basic course given by the same author (Mat. Report 71). Most of the material in this is based on the original research papers by pioneers working in the fields of gravitational collapse and gravitational radiation.

Starting with a study of Schwarzschild singularity which today is the most controvercial, I have studied the gravitational implosion problem through various solutions as those of Oppenheimer-Snyder, Vaidya and others. In this context I have referred to the problems associated with boundary condition in general relativity and with singularities.

While studying gravitational radiation we have first established the nature of radiation as being Quadrapole. After going through in detail regarding the classification of fields we have finally presented some exact solutions representing gravitational radiation fields.

Even with this coverage I still feel that the present report would serve only as a basic study in understanding there advanced topics, gravitational collapse and gravitational radiations. Any incompleteness or omission of some other work is only because of the intended nature of the present coverage. The list of references presents only those works on which I have based my lectures and those from which the materials have been taken.

I wish to thank Professor Alladi Ramakrishnan for his interest and encouragement.

A.R. Prasanna

CONTENT

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GRAVITATIONAL COLLAPSE

The study of Radio Sources in the universe presents one of the most fascinating challenges of modern Astronomy and Cosmology. On the one hand the distribution of the sources tend to indicate clues to the possible cosmological models, while on the other the mechanism of their energy production (which is enormous $\sim O\ 10^{60}$ ergs) is the foremost problem for Astrophysicists. During the early part of 1950's Astronomers tend to beleive that the source of such high energies could be in collision of galaxies. Though in the beginning some spectroscopic analysis inclined to support this view, very soon it was found to be an inadequate explanation.

In 1963, Hoyle and Fowler[1] put forward the view that the energy associated with radio objects could come from the gravitational collapse of super-massive bodies. Hoyle and Fowler suggested that the gravitational potential energy of a highly compressed mass could account for the amount of energy in question. From the simple Newtonian values for gravitational potential energy, we have its numerical value $\delta\ GM^2/R$ for a spherical body of mass M and radius R, with $\delta = 3(5 - n)$ for a configuration of polytropic index n. As an example it can be seen that if $\delta = 1$, for a mass $= 10^6$ M , a gravitational potential energy of 10^{60} erg is obtained when the mass is compressed to a sphere of radius 3.8 R . The same could be achieved for a smaller aver ge density if we consider masses of the order of 10^8 M , wherein the required energy may be obtained, for a radius of 3.8×10^4 R . This idea of gravitational collapse soon gained currency and many started investigation on this problem from the general relativity point of view. It happened that there were already some solutions in general relativity wherein non-static interior fields (allowing for the motion of fluid particles) were studied. We shall take up these and other solutions obtained more recently in the course of these lectures.

Analysing the problem of gravitational collapse we find that we are concerned with general relativistic interior solutions studied by external observers. Now since the solutions to be considered are spherically symmetric (no rotation is considered personally) the exterior field is described by the well-knwon Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 + \left(1 - \frac{2m}{r}\right) dt^2 \quad (1.1)$$

where $m = GM/C^2$, M being the mass of the source, G the gravitational constant and C, the velocity of light in vacuum.

For the external observer, it is apparent that as the body is collapsing the radius shrinks continuously and a stage will reach when $r = 2m$ and the singularity approached. Now the following questions arise

1) What happens to the body when it reaches the stage $r = 2m$?

2) Will it continue to collapse and reach the other singularity $r = 0$?

3) If so how to prevent such a catastrophe?

These questions can be clearly understood and answered if we know in detail the nature of Schwarzschild singularity at $r = 2m$. Hence before proceeding further with the collapse problem we now consider the Schwarzschild singularity.

Schwarzschild Singularity

The question with any singularity is whether it is a coordinate singularity or a physical singularity. By coordinate singularity we mean that which can be removed by a transformation of coordinates. It is generally believed that Schwarzschild singularity is not a space-time singularity and can be removed by a coordinate transformation. The arguments in favour of this run as follows:

Any space-time singularity must be characterised by the fact that the associated scalars of the field should become infinite at that point or along that region. In this case if, we consider the determinant $g_{ij} = g = -r^4 \sin^2\theta$, or the curvature invariant $R_{hijk} R^{hijk} = 48\, m^2/r^6$, both are regular at r = 2m. Hence one thinks that the singularity may be a removable one. Eddington[2] (1924) and later Finkelstein[3] (1958) have shown that the transformation

$$t = t' \pm 2m \ln\left(\frac{r'}{2m} - 1\right), \quad x^\alpha = x'^\alpha, \qquad (1.2)$$

takes the metric (1.1) into the form

$$ds^2 = -\left(1-\frac{2m}{r}\right)dr^2 \pm \frac{4m}{r}\, dr\, dt - r^2 d\Omega^2 + \left(1-\frac{2m}{r}\right)dt^2 \qquad (1.3)$$

which can further be written as,

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - \frac{2m}{r}(dt \pm dr)^2 \qquad (1.4)$$

This metric is apparently free from the singularity at r=2m. But we notice that essentially the transformation (1.2) itself is singular at r=2m and hence not a justifiable one. Further the Finkelstein form of the metric is not static (as $g_{44} \neq 0$) and not invariant under the transformation t $\to$ -t. This Schwarzschild surface r=2m acts as a unidirectional membrane. If we take the positive sign in the last term of (1.4), light signals that propogate along null geodesis into the future (increasing t) can leave r=0 but cannot reach there. If we use the negative sign the situation will be reversed. Signals propogated into the future can get inside the Schwarzschild surface but cannot get out.

Kruskal[4)] reviewing the problem purely from the analytical extension point of view showed that the Schwarzschild manifold can be extended for the region $r > 2m$ through the transfomation

$$u = (r/2m - 1)^{1/2} e^{r/4m} \cosh(t/4m)$$
$$v = (r/2m - 1)^{1/2} e^{r/4m} \sinh(t/4m) \tag{1.5}$$

to obtain the metric

$$ds^2 = f^2(-du^2 + dv^2) - r^2 d\Omega^2 \tag{1.6}$$

with $f^2 = (32m^3/r)\, e^{-(r/2m)}$, and the radial coordinat r is defined through

$$(r/2m - 1)\, e^{r/2m} = u^2 - v^2 \tag{1.7}$$

Though now in the form (1.6) there is no singularity at r=2m, the new coordinate system corresponds to an accelerated frame and hence the metric is not again static.

Restricting to the transformation which yield only static form Rosen[5] has shown that by considering the transformation

$$r - 2m = u^2/8m \tag{1.8}$$

one can get

$$ds^2 = -\left[1 + u^2/16m^2\right] du^2 - \left(1/64m^2\right)\left(u^2+16m^2\right)^2 d\Omega^2 + \left[u^2/(u^2+16m^2)\right] dt^2 \tag{1.9}$$

wherein the infinity at g_{11} is removed. But still we have $g_{44} = 0$ at r = 2m, and this makes g^{44} infinite, for $u = 0$. As Rosen further points out the form (1.9) though helpful to study the immediate neighbourhood of r=2m, does not satisfy the boundary condition at infinity, in the sense that as $u \to \infty$, (1.9) does not go over to the flat metric. But this seems to have been remedied by taking

$$r - 2m = w^2/(w + m/2) \tag{1.10}$$

so that as $w \to \infty$, r = W, while for $|w| \ll m$, $w = u/4$. This transformation yields the metric

$$ds^2 = -\frac{(w+m)^4}{(w+m/2)^4}\left[dw^2 + (w+m/2)^2 d\Omega^2\right] + \frac{w^2}{(w+m)^2} dt^2 \tag{1.11}$$

which is actually the familiar form of Schwarzschild solution in isotropic coordinates for $w = r' - m/2$, given as

$$ds^2 = -\left(1+ m/2r'\right)^4 \left(dr'^2 + r'^2 d\Omega^2\right) + \left[\frac{1-m/2r'}{1+m/2r'}\right]^2 dt^2 \quad (1.12)$$

Thus we see that whatever the transformation one considers, it is impossible to avoid the singularity at r=2m completely. One may go into spurious forms like that of Finkelstein or Kruskal through singular transformations, but the surface r=2m is such that the region $r < 2m$ is always unphysical. In fact r=2m forms an impenetrable boundary for light rays and particles. This can further be seen through the following considerations.

For an external observer who is in a Schwarzschild field his coordinate time 't' is related to the proper time 's' through

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 .$$

Since ds is always finite as $r \to 2m$, $dt \to \infty$. Hence from the point of view of the external observer it takes infinite time for the object to collapse into Schwarzschild radius. Since the surface $r < 2m$ is unphysical, no light ray (signal) can penetrate this surface and reach the observer. That is if there are objects which have already reached r=2m then they can never be observed and recently such objects have been named as black holes.

Having thus seen the barrier property of r=2m, it is essential to understand, whether any collapse is possible at all above this region. For if a given spherical configuration continues to be stable for all radius above r=2m, no collapse would be possible. Chandrasekhar (1964) for the first time considered this problem from the general relativity point of view. He studied the dynamical instability of gaseous masses approaching Schwarzschild limit as follows. Starting from a spherically symmetric configuration in general relativity he showed that under purely radial perturbations the instability of the equilibrium configuration occurs at a radius

$$R = \frac{K}{\gamma - 4/3} \; \frac{2GM}{c^2}$$

where K is a constant depending on the density distribution in the configuration. In the case of a homogeneous sphere he obtained K = 19/42. Thus Chandrasekhar has shown that if there can arise a small perturbation for an equilibrium configuration with radius of the order given above, the entire distribution can collapse continuously. It is to be noted that this Radius is greater than the Schwarzschild radius and hence the collapse is observable. It is of interest to note that Chandrasekhar has found for a polytrope of index 3 the dynamical instability will occur at a radius of the order of 0.5 light years
If we consider the approximate relation for a quasar as given by Hoyle and Fowler,

$$R = \frac{5.8 \times 10^{18}}{T_0} \left(M/M_\odot \right)^{1/2}$$

and note the observation of Michel (1964) that for a body of $M = 10^8$ the energy release would be maximum when the central temperature is of the order of 3×10^5 K we get $R \simeq 0.49$ light year.

We have seen that for a spherically symmetric collapsing object the radius $r = 2m$ forms a natural barrier, reading which the object ceases to communicate with an external observer. But from the point of view of a comoving observer a continuous collapse is possible and as I mentioned last time this problem of implosion was studied by Datt , 1938, Oppenheimer and Snyder 1939 and more recently by Hoyle-Narlikar, (1964) and Vaidya (1968).

Let us now consider the implosion problem as studied by Oppenheimer and Snyder. The basic idea here again is to construct non-static solution, for the interior of a spherically symmetric matter distribution, of the Einstein's field equations and then match it with a proper exterior solution and thus make the solution complete. To begin with we have the metric form

$$ds^2 = e^{\lambda} dt^2 - e^{\nu}(dr^2 + r^2 d\Omega^2) \qquad (2.1)$$

where λ and ν are functions of r and t. Considering a comoving observer the velocity four-vector will be given by $(0, 0, 0, V^4)$. If we now recollect the solution obtained in the case of uniform cosmological model we had the physical situation analogously and hence we can start from the metric

$$ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2 \right] \qquad (2.2)$$

where we have taken $k = 1$. (positive curvature). For the field represented by this metric we had the pressure and density as given by

$$\left.\begin{aligned} \frac{8\pi G}{c^2} p &= -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{c^2}{R^2}, \\ 8\pi G \rho &= \frac{3\dot{R}^2}{R^2} + \frac{3c^2}{R^2} \end{aligned}\right\} \qquad (2.3)$$

At this stage we make an important assumption that the pressure is zero for the imploding object. This gives us from 2.3

$$\left.\begin{aligned} \dot{R}^2 &= \left(\frac{A}{R} - c^2\right) \\ \text{and} \quad \frac{\dot{R}^2}{R^2} &= -\frac{c^2}{R^2} + \frac{8\pi G \rho}{3} \end{aligned}\right\} \qquad (2.4)$$

where A is a constant. From (2.4) it can be seen that the density ρ satisfies the relation

$$\rho R^3 = \text{Constant} \qquad (2.5)$$

From the second of (2.4) it can be seen that in equilibrium i.e. when $\dot{R} = 0$, $R = R_0$ and $\rho = \rho_0$ are connected through

$$R_0 = \left(\frac{8\pi G \rho_0}{3c^2}\right)^{-1/2} \qquad (2.6)$$

Redefining the r coordinate such that $R = 1$ at maximum phase we get the metric (2.2) to be

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-\alpha r^2} + r^2 d\Omega^2\right] \tag{2.7}$$

with $\alpha = \dfrac{8\pi G \rho_0}{3c^2}$.

(2.8)

and the equation (2.4) now turns out to be

$$\frac{\dot{R}^2}{R^2} = \frac{\alpha(1-R)}{R^3} \tag{2.9}$$

which can be integrated to get

$$\sqrt{\alpha}\, t = \pi/2 - \sin^{-1}\sqrt{R} + \sqrt{R}\sqrt{1-R} \tag{2.10}$$

It is to be noted that unlike in the case of R-W line-element where $r = 0$ could be any assigned observer (particle) here $r = 0$ uniquely defines the centre of the imploding object. Having thus obtained the metric describing the interior field it is now important to obtain the metric for the exterior field of this body.

In order to get this let us start with the spherically symmetric form

$$ds^2 = dt^2 - e^{\mu} dr^2 - r^2 e^{\nu} d\Omega^2 \tag{2.11}$$

where μ and ν are functions of say $\tau = t/T$, $\xi = r_b/r$, T a fixed time interval and r_b some fined length. Since we want the field exterior we have to solve the set of equations $R_{ij} = 0$. While solving this set we notice that the equations would no longer be partial in r and t, but ordinary with $r^{-3/2}t$ as a variable. The solution is given by

$$ds^2 = dt^2 - R^2(\chi)\left\{\frac{K^2 dr^2}{1 - \frac{\alpha r_b^3}{r}} + r^2 d\Omega^2\right\} \tag{2.12}$$

where

$$\chi = \frac{t r_b^{3/2}}{r^{3/2}}, \qquad K = \frac{1}{R}\frac{\partial(rR)}{\partial r} \tag{2.13}$$

and R satisfies the relation

$$R_\chi^2 / R^2 = \alpha\,(1-R)/R^3 \tag{2.14}$$

We notice that since the metric (2.12) describes a region where $R_{ij} = 0$ (empty), by Birkhoff's theorem it must be just a trans form of Schwarzschild's exterior solution. Hence it is important to express this in the Schwarzschild form and further adjust the constants so that the solution matches with the interior solution across the boundary.

Starting with the Schwarzschild form as given by

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2 d\Omega^2 + \left(1 - \frac{2m}{\bar{r}}\right) dT^2 \tag{2.15}$$

we first express the interior solution (2.7) in Schwarzschild coordinates through the following transformation

$$\bar{r} = r R \quad , \quad T = \phi\left(\int \frac{r\,dr}{1-\alpha r^2} + \int \frac{dt}{R\dot{R}}\right) \tag{2.16}$$

where ϕ is any differentiable function. This takes the line-element (2.7) into

$$ds^2 = e^{\nu} dT^2 - e^{\lambda} d\bar{r}^2 - \bar{r}^2 d\Omega^2 \tag{2.17}$$

with

$$e^{\lambda} = \frac{1}{1-\alpha r^2 - r^2\dot{R}^2} \,, \quad e^{\nu} = \frac{R^2\dot{R}^2(1-\alpha r^2)}{(1-\alpha r^2 - r^2\dot{R}^2)\,\phi_1^2} \tag{2.18}$$

where ϕ_1 is the derivative of ϕ with respect to its argument. Using (2.9) we get

$$e^{-\lambda} = 1 - \frac{\alpha r^2}{R} = 1 - \frac{\alpha r^3}{\bar{r}} \tag{2.19}$$

Let us say that $r = r_b$ denotes the boundary of the distribution. Then using the definition of α from (2.8), we get at the boundary $r = r_b$

$$e^{-\lambda} = 1 - \frac{2m}{\bar{r}} \,, \quad m = \frac{MG}{c^2} \,, \quad M = \frac{4}{3}\pi r_b^3 \rho_0 \tag{2.20}$$

giving the continuity of g_{11} .

Thus we have described through the metrics (2.7) and (2.12) the field (interior and exterior) of an imploding object. In order to calculate the time required for the object to contract from $\bar{r} = r_b$ to $\bar{r} = 2m$,

$$\int_{R=1}^{R=\alpha r_b^2} dt \tag{2.21}$$

for $\bar{r} = r_b R$ and

$$\left(\bar{r} = 2m \quad \text{means} \quad R = \frac{2m}{r_b} = \alpha r_b^2 \, , \quad \alpha = \frac{8\pi G \rho_0}{3c^2} \right)$$

$$\int_1^{\alpha r_b^2} \frac{dR}{\dot{R}} = \int_1^{\alpha r_b^2} \alpha^{-1/2} \left(\frac{1-R}{R} \right)^{1/2} dR \tag{2.22}$$

$$= \frac{1}{\alpha^{1/2}} \left[\frac{\pi}{2} - \sin^{-1} \alpha^{1/2} r_b + \alpha^{1/2} r_b \sqrt{1 - \alpha r_b^2} \right] \tag{2.23}$$

which is finite.

Considering the exterior field as given by (2.12) along with the relation

$$\left(\frac{dR}{d\chi} \right)^2 = \frac{\alpha (1-R)}{R} \tag{2.24}$$

We observe first that for an imploding star $dR/d\chi$ should be negative. Now for a radial light pulse $ds = 0$ and $d\Omega = 0$. Hence we get for the external observer

$$\left(1 - \frac{\alpha r_b^3}{r}\right)^{1/2} dt = \pm R(\chi)\, K\, dr$$

The positive and negative signs characterise the outward moving and inward moving light rays respectively. Also we have $\bar{r} = Rr$

$$\begin{aligned} d\bar{r} &= r \frac{\partial R}{\partial t} dt + \frac{\partial}{\partial r}(rR)\, dr \\ &= r\left(\frac{dR}{d\chi}\right) \frac{r_b^{3/2}}{r^{3/2}} dt + R K\, dr \end{aligned} \qquad (2.26)$$

$$d\bar{r} = dt \left[-\frac{r_b^{3/2}}{r^{1/2}} \frac{(1-R)^{1/2}}{R^{1/2}} \alpha^{1/2} \pm \left(1 - \frac{\alpha r_b^3}{r}\right)^{1/2} \right] \qquad (2.27)$$

It can be easily seen that when the positive sign is used $d\bar{r}$ can vanish for $\bar{r} = rR = \alpha r_b^3 = 2m$.

Hence it verifies the fact already mentioned that in an imploding object, for an outward moving light ray $\bar{r} = \alpha r_b^3$ forms an impenetrable barrier. i.e. light rays can move in to the object, but when the object reaches Schwarzschild radius no light ray can escape out. By a symmetric argument it may be verified from above that for an expanding object again $\bar{r} = \alpha r_b^3$ forms a barrier such that light rays can move out of it but cannot enter in.

Lecture 3

In the last lecture we have discussed the collapse of a cold body (as it is normally referred to) through Oppenhimer-Snyder solution. Further collapse considered was with zero pressure and hence there was no force of resistence. But we know that such a situation can hardly be realistic (1) due to the fact that the release of energy from the collapsing body must be accompanied by a flow of radiation (2) and as McVittie points out the process of contraction is non-adiabatic.

This leads us to consider more realistic pictures wherein we shall have non-zero internal pressure and a radiation belt surrounding the collapsing object. The moment we say that there is a belt of radiation outside the contracting star, the Schwarzschild exterior solution becomes no more feasible for an external observer as it deals only with a region devoid of any form of distribution. Hence we must search for new solutions both interior and exterior such that;

(1) In the interior of the star $p \neq 0$

(2) In the exterior $R_{ij} \neq 0$, but $R_{ij} = \rho V_i V_j$

where ρ is the radiation density and V^i is a null vector of radiation propagation.

In this lecture we will concentrate on the solution for the radiation region alone. After the advent of Schwarzschild's solutions in 1916, no discussion was made of any pertinent local solution in general relativity for non-static manner. In 1939, for the first time prof Narlikar raised the question of the field of a radiating star in general relativity. Narlikar remarks that

"If the principle of energy is to hold good, that is if the combined energy of the matter and field is to be conserved, the system must be an isolated system surrounded by flat space-time. A spherical radiating mass would probably be surrounded by a finite and non-static envelope of radiation with radial symmetry. This would be surrounded by a radial field of gravitational energy becoming weaker and weaker as it runs away from the central body until at least the field is flat at infinity. It has yet to be seen whether and how this view of the distribution of energy is substantiated by the field equations of relativity."

Vaidya (1951) solved this outstanding problem completely and the solution is today known as Vaidya's metric for a radiating star. I will now describe Vaidya's solution starting from first principles. Let a star of mass M and radius r_0 start radiating at time t_0 and at time t_1 let the outer surface of the radiation zone be at r_1. Hence we have a belt of radiation for $r_0 < r < r_1$ in $t_0 < t < t_1$. Let the radiation density be σ and V^i the 4-vector along which the radiation propagates and hence a null vector. The energy-momentum tensor for the region under consideration may be obtained analogous to the case of perfect fluid distribution and hence here p being zero we have

$$T^{ij} = \sigma V^i V^j , \qquad V^i V_i = 0 \tag{3.1}$$

Thus we have the system of equations

$$R_{ij} = -8\pi\sigma V_i V_j \tag{3.2}$$

$R = 0$, because trace of T^{ij}, $T = 0$. Since we are concerned with a spherical mass, we assume the field to be spherically symmetric and hence start with the metric

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\Omega^2 + e^{\nu} dt^2 \tag{3.3}$$

where λ and ν are functions of r and t . We consider the radiation to flow only along the radial direction and thus have $V^i = (V^1, 0, 0, V^4)$. Using this along with (3.3) in (3.2) we get the three equations,

$$e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} + \frac{\dot{\lambda}}{r} e^{-(\lambda+\nu)/2} = 0 \tag{3.4}$$

$$e^{-\lambda}\left[\frac{\lambda'-\nu'}{r} - \frac{2}{r^2}\right] + \frac{2}{r^2} = 0 \tag{3.5}$$

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu'-\lambda'}{2r}\right] + e^{-\nu}\left[\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right] = 0 \tag{3.6}$$

and from $V_i V^i = 0$, we get

$$-e^{\lambda}(V^1)^2 + e^{\nu}(V^4)^2 = 0 \tag{3.7}$$

In solving these equations we first note that the solution must reduce to Schwarzschild's static metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 + \left(1-\frac{2M}{r}\right) dt^2 \tag{3.8}$$

at $r = r_0$, $t = t_0$ and for $r \geq r_1$ at $t = t_1$.
Having this in mind let us take

$$e^{-\lambda} = \left(1 - \frac{2m}{r}\right) \tag{3.9}$$

where $m = m(r,t)$ and feeding it in equation (3.4) we get

$$e^{-\lambda/2}\frac{\partial m}{\partial r} + e^{-\nu/2}\frac{\partial m}{\partial t} = 0 \tag{3.10}$$

or

$$\frac{dm}{d\tau} = 0$$

where

$$\frac{d}{d\tau} = e^{-\lambda/2}\frac{\partial}{\partial r} + e^{-\nu/2}\frac{\partial}{\partial t} \tag{3.12}$$

The operator $\frac{d}{d\tau}$ is nothing but differential operator following the fluid motion. Far from (3.7) it can be seen that $V^1 = e^{-\lambda/2}$, $V^4 = e^{-\nu/2}$ From (3.10) we get

$$e^{\nu/2} = -\frac{\dot{m}}{m'}\left(1-\frac{2m}{r}\right)^{-1/2} \tag{3.13}$$

On substituting in (3.5) for ν and λ, we get

$$\left(\frac{\dot{m}'}{\dot{m}} - \frac{m''}{m'}\right)\left(1-\frac{2m}{r}\right) = \frac{2m}{r^2} \tag{3.14}$$

whose first integral is

$$m'\left(1-\frac{2m}{r}\right) = f(m) \tag{3.15}$$

where $f(m)$ is any arbitrary function of m . In order to solve the third of field equations viz. (3.6) let us consider the conservation law

$$\left(T_1^{\,j}\right)_{;j} \equiv 0 \tag{3.16}$$

which on expansion gives

$$\frac{\partial}{\partial r}(T_1^{\,1}) + \frac{\partial}{\partial t}(T_1^{\,4}) - \frac{\nu'}{2}\left(T_4^{\,4}-T_1^{\,1}\right) + \frac{2}{r}\left(T_1^{\,1}-T_2^{\,2}\right) + T_1^{\,4}\left(\frac{\dot{\lambda}+\dot{\nu}}{2}\right) = 0 \tag{3.17}$$

Using in this the fact that

$$T_1^4 e^{(\nu-\lambda)/2} + T_4^4 = 0 \quad \text{and} \quad T_1' + T_4^4 = 0 \tag{3.18}$$

(3.6) reduces to

$$\frac{d}{d\tau}\left(r^2 e^{-\lambda} T_4^4 \right) = 0 \tag{3.19}$$

i.e.

$$\frac{d}{d\tau}\left\{ m'\left(1 - \frac{2m}{r}\right)\right\} = 0$$

which means $\frac{dm}{d\tau} = 0$. This we know is true from (3.1‥ .
Thus we find that all the equations are satisfied, provided,

$$e^{-\lambda} = \left(1 - \frac{2m}{r}\right), \quad e^{\nu} = \left(\frac{\dot{m}}{m'}\right)^2 \left(1 - \frac{2m}{r}\right)^{-1} \tag{3.20}$$

where $m = m(r, t)$ and $m'\left(1 - \frac{2m}{r}\right) = f(m)$

Thus we get the solution for the radiation zone as obtained by Vaidya,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 + \frac{\dot{m}^2}{f^2}\left(1 - \frac{2m}{r}\right) dt^2 \tag{3.21}$$

with $m'\left(1 - \frac{2m}{r}\right) = f(m)$

This radiation zone is on one side attached to the boundary of the radiating source and on the other empty region represented by $R_{ij} = 0$. Hence it is necessary to study what happens across the boundaries, Since we have not yet obtained the solution for the interior of the source we will not consider the boundary on this side. For the exterior boundary we have at any time $t = t_1$ the radial coordinate $r = r_1$ and for $r > r_1$ the region is represented by the Schwarzschild solution

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 + \left(1-\frac{2M}{r}\right) dt^2 \qquad (3.22)$$

where M is the total mass of the source and radiation. By comparing (3.21) with (3.22) for $m = M$, we need for the continuity of g_{ij},

$$m = M \;, \qquad \dot{m} = -f(m) \qquad (3.23)$$

Here $r = R(t)$ an every expanding function of time. We take $\dot{m}$ to be negative for the reason that as the time passes m decreases. Using the equation satisfied by m viz.,

$$m'\left(1-\frac{2m}{r}\right) = f(m) \qquad (3.24)$$

along with (3.23) we get after simple calculation (cf. Vaidya 1951) the relation connecting r_1 and t_1,

$$r_1 + 2M\log(r_1 - 2M) - t_1 = \text{a constant} \qquad (3.25)$$

However, we find that the condition at the boundary $r = R(t)$ still kept $f(m)$ undetermined. From the definition of $f(m)$ it is clear that it is governed by the conditions in the interior of the radiating source.

The exterior field of a radiating star as given by (3.) now must be fitted on to a proper interior solution. We have to first obtain the solutions for the interior of the star wherein both matter and radiation are present. Before going to solve the field equations for the interior we shall now introduce Vaidya's radiation coordinates and express the solution obtained above in these coordinates.

The radiation coordinate 'u' as introduced by Vaidya is a solution of the differentiation along the line of flow

$$\frac{du}{d\tau} = 0 \; ; \; or \qquad e^{-\lambda/2} u' + e^{-\nu/2} \dot{u} = 0 \tag{4.1}$$

which here satisfies

$$du = \frac{\dot{m}}{f} dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \tag{4.2}$$

After performing this transformation the line-element (3.) takes the more simpler form

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2\,du\,dr - r^2 d\Omega^2 \tag{4.3}$$

where m = m(u)., u being a function of (r, t). Lindquist et al (1965) have studied this solution (4.3) in detail, and shown that the radiation density σ is given by

$$\sigma = -\frac{1}{4\pi r^2} \frac{1}{(r+u)^2} \frac{dm}{du} \tag{4.4}$$

where $\gamma = (1 + v^2 - 2m/r)^{1/2}$, $v = V^1$ the radial component of velocity of the observer. Since the energy density should always be pasitive we get $\frac{dm}{du} < 0$.

We will now consider the interior solution as given by Vaidya (1966). As in the other cases we start with a general spherically symmetric form

$$ds^2 = -e^{\alpha} dr^2 - r^2 e^{\beta} d\Omega^2 + e^{2\phi} dt^2 \tag{4.5}$$

where α, β, ϕ are functions of r and t. The interior of the star und r consideration is supposed to be filled with a perfect fluid along with radiation. For such a system the energy-momentum tensor T_i^j is given by

$$T_i^j = M_i^j + E_i^j \quad , \qquad \text{with} \tag{4.6}$$

$$M_i^j = (\rho + p) V_i V^j - p\delta_i^j , \tag{4.7}$$

$$E_i^j = \sigma q_i q^j \tag{4.8}$$

where ρ and p are the density and pressure of the matter distribution and σ the radiation density. The four velocities V^i and q^i are such that $V^i = (0, 0, 0, V^4)$ and $q^i = (q^1, 0, 0, q^4)$ and $V^i V_i = 1$, $q_i q^i = 0$.

The field equations are given by

$$R_i^j - \frac{1}{2} R g_i^j = -8\pi T_i^j \tag{4.9}$$

Using (4.5) - (4.8) in (4.9) we get the set of equations in the unknowns $\rho, p, \sigma, v^i, q^i$ and the potentials α, β and ϕ. By properly inspecting this set we find the relations,

$$T_1^1 - T_2^2 = T_1^4 e^{\phi - \alpha/2} \tag{4.10}$$

$$T_2^2 = T_3^3 = -p \;, \quad T_1^1 + T_4^4 - T_2^2 = \rho \,, \tag{4.11}$$

and
$$T_4^1 = \sigma e^{\phi - \alpha/2} \tag{4.12}$$

Essentially we find that if we solve for the metric potentials we will immediately get the values for ρ, p and σ. From the field equations we substitute for T_1^1, T_2^2 and T_1^4 in (4.10) and get the differential equation,

$$\frac{\beta''}{2} + \phi'' + \phi'^2 - \frac{\alpha'\beta'}{4} - \frac{\beta'\phi'}{2} - \frac{\alpha'\phi'}{2} - \frac{1}{r}\left(\frac{\alpha'}{2} + \phi'\right) - \frac{1}{r^2}$$
$$+ \frac{e^{\alpha-\beta}}{r^2} + e^{(\alpha - 2\phi)}\left[\frac{\ddot{\beta}}{2} - \frac{\ddot{\alpha}}{2} - \frac{\dot{\alpha}^2}{4} + \frac{\dot{\beta}^2}{2} - \frac{\dot{\alpha}\dot{\beta}}{4} - \frac{\beta\dot{\phi}}{2} + \frac{\dot{\alpha}\dot{\phi}}{2}\right]$$
$$+ e^{(\alpha/2 - \phi)}\left[\dot{\beta}' + \phi'\dot{\beta} + \left(\frac{\beta'}{2} + \frac{1}{r}\right)(\dot{\beta} - \dot{\alpha})\right] = 0 \tag{4.13}$$

where an overhead dash and dot denotes partial differentiation with respect to r and t respectively. This equation is more or less a consequence of the nature of the comoving coordinates used and hence one could call it a coordinate condition. However, since we have three unknowns in α, β and ϕ, we need two more equations for a complete solution. One can obtain the other two conditions from the physical nature of the distribution as for example the equation of state or thermodynamic and energy transfer properties of the fluid distribution. Vaidya solves this system by adopting conditions such that the solution obtained in this case should be a generalisation of Oppenheimer-Snyder solution obtained earlier. In order to accomplish this he chooses,

$$e^{\alpha} = S^2 e^{\lambda}, \quad e^{\beta} = S^2 e^{\mu} \tag{4.14}$$

where $S = S(t)$, and λ, μ as functions of r and u the retarded time defined through the relation $du/d\tau = 0$. Substituting for α, β from (4.14) in (4.13) and collecting the coefficients of S^2 and $\dot{S}^2$ seperately and equating them to zero we get

$$2\phi' = u'(\lambda_u - \mu_u), \tag{4.15}$$

$$\frac{\mu_{rr}}{2} + \frac{u'}{2}(\lambda_{ur} - \mu_{ur}) - \frac{1}{2}u'^2\mu_{uu} - \frac{1}{2}u'\mu_u u'_u - \frac{1}{4}u'\mu_r\mu_u - \frac{1}{2r}u'\mu_u - \frac{1}{2r}\lambda_r + \frac{1}{r^2}e^{\lambda-\mu} = 0 \tag{4.16}$$

Here (4.15) is one of the assumed relations referred to earlier. Further in order to make (4.16) amenable for integration we assume a relation between λ and μ as given by

$$u'\mu_u = \lambda_r - 2\mu_r \tag{4.17}$$

From the defining equation for u viz.,

$$u'e^{-\alpha/2} + \mathring{u}e^{-\phi} = 0 \tag{4.18}$$

We get

$$u'_r = \frac{1}{2} u' (\lambda_r + u' \mu_u) \tag{4.19}$$

Using (4.17) and (4.19) in (4.16) we get a simple second order partial differential equation

$$\lambda_{rr} - \mu_{rr} - (\lambda_r - \mu_r)^2 - \frac{2}{r}(\lambda_r - \mu_r) - \frac{2}{r^2} + \frac{2}{r^2} e^{\lambda-\mu} = 0 \tag{4.20}$$

which on integration gives,

$$e^{\mu-\lambda} = r^2 a(u) + r\, b(u) + 1 \tag{4.21}$$

(4.17) and (4.19) will now give,

$$u'(ar^2 + br + 1) = f, \quad a = a(u), \quad b = b(u) \quad f = f(u) \tag{4.22}$$

With this the form of the metric now reduces to

$$ds^2 = -S^2 e^{-2\phi}\left[\frac{dr^2}{(ar^2+br+1)^2} + \frac{r^2 d\Omega^2}{ar^2 + br + 1}\right] + e^{2\phi} dt^2 \tag{4.23}$$

along with the relations

$$2\phi'(ar^2 + br + 1) = -u'(a_1 r^2 + b_1 r) \tag{4.24}$$

$$u'(ar^2 + br + 1) = f(u) \tag{4.25}$$

$a_1 = da/du$, $b_1 = db/du$. The expressions for ρ, p and σ are now given by

$$8\pi\rho = \frac{3}{4}(D_t\beta)^2 + 3\left(a - \frac{1}{4}b^2\right)\frac{e^{2\phi}}{S^2} \tag{4.26}$$

$$8\pi p = -\left[D_{tt}\beta + \frac{3}{4}(D_t\beta)^2 + \left(a - \frac{1}{4}b^2\right)\frac{e^{2\phi}}{S^2} + D_{rt}\beta\right] \tag{4.27}$$

(

$$8\pi\sigma = D_{rt}\beta + \frac{rb+2}{r\times} D_r \phi \qquad (4.28)$$

where $D_t = e^{-\phi}\frac{\partial}{\partial t}$, $D_r = e^{-\alpha/2}\frac{\partial}{\partial r}$ and (4.29)

$$e^{\beta} = S^2 e^{-2\phi}(ar^2+br+1)^{-1} \qquad (4.30)$$

As we have already mentioned the solution sought by Vaidya was a generalisation of Openheimer-Snyder solution. Hence we now find that if we choose 'a' and 'b' as constants in the solution (4.23), we get $\phi' = 0$ so that by a proper choice of t it can be put equal to zero. Then the metric (4.23) reduces to

$$ds^2 = dt^2 - S^2\left(\frac{d\bar{r}^2}{1-\alpha\bar{r}^2} + \bar{r}^2 d\Omega^2\right), \quad \bar{r}^2 = r^2 \times (ar^2+br+1)^{-1}$$

$$\alpha = a - \frac{1}{4}b^2 \qquad (4.31)$$

which is the well-known Oppenheimer-Snyder solution for a collapsing star.

Having thus obtained the interior solution we have to now fit it on to the exterior solution obtained earlier. In order to accomplish this we will express the exterior solution in a suitable form as follows. We start from the solution

$$ds^2 = \left(1-\frac{2m}{R}\right)du^2 - R^2 d\Omega^2 + 2du\,dR, \quad m = m(u) \qquad (4.32)$$

and change $R \rightarrow r$ by putting $R = R(r, u)$ This transforms the metric (4.32) to

$$ds^2 = \left(1 - \frac{2m(u)}{R} + 2R_u\right)du^2 - R^2 d\Omega^2 + 2R_r\,du\,dr$$

Since the interior solution does not have the term in $du\,dr$ we will transform this further by changing u to t such that in the resulting metric term is absent. This gives us,

$$ds^2 = -\left(1-\frac{2m}{R}+2R_u\right)\left(u'^2 dr^2 - u_t^2 dt^2\right) - R^2 d\Omega^2 \qquad (4.34)$$

with

$$u'\left(1-\frac{2m}{R}+2R_u\right) = -R_r \qquad (4.36)$$

But $R = R(r,u), \simeq R(r,t)$ gives

$$R' = R_r + R_u u', \quad R_t = R_u u_t \qquad (4.36)$$

Hence we have the external field of a radiating star described by the metric

$$ds^2 = -\left[1-\frac{2m(u)}{R}+\frac{2R_t}{u_t}\right]\left(u' dr^2 - u_t^2 dt^2\right) - R^2 d\Omega^2 \qquad (4.37)$$

with $$u'\left[1-\frac{2m(u)}{R}+\frac{R_t}{u_t}\right] = -R', \qquad (4.38)$$

and the density of radiation σ is given by

$$\sigma = -\left(\frac{dm}{du}\right)\frac{1}{4\pi R^2}\left[1-\frac{2m(u)}{R}+\frac{2R_t}{u_t}\right]^{-1} \qquad (4.39)$$

As Vaidya points out this ρ solution when m is a constant = M, gives a very general form of Schwarzschild's exterior solution.

Thus we find that the complete solution for a radiating star can be obtained if the interior solution (4.23) is fitted on to the exterior solution (4.37) across a boundary $r = r_0$. In order to get this fitting we have at our disposal the functions $a(u)$, $b(u)$, $f(u)$, $m(u)$ and $R(r, t)$. We will first fix up the function R such that at $r = r_0$, the coefficients of $d\Omega^2$ in the two metrics and their first derivatives are continuous giving $R_0(t)$ undetermined. The defining equations for u in the interior and exterior as given by (4.25) and (4.38) have two undetermined functions of t which on proper fixing yield the continuity of u and u_t across $r = r_0$. The requirements for the continuity of u', g_{11}, p and ρ yields four equations (Vaidya 1966) of which only three are independent. Regarding $f(u)$ one can fix it by assuming the total luminosity $-\dfrac{dm}{du}$ for an observer at a large distance to be zero as $R \to 2m(u)$ the schwarzschild surface. Thus we will have three equations to determine four unknowns a, b, $R_0(t)$ and m . choosing different b's one can get different relations among these three equations. One can obtain the two relations

$$(D_t R_0)^2 = \frac{2m_0}{R_0} - \frac{r_0^2\left(a_0 - \frac{1}{4}b_0^2\right)}{a_0 r_0^2 + b_0 r_0 + 1}$$

and

$$\frac{1}{R_0}\left(\frac{dR_0}{du}\right) = \frac{1}{f_0}\left(\frac{df_0}{du}\right) - \frac{m_0}{R_0^2}$$

Using these with the two of the old set one can determine the unknown functions. The material and radiation densities at $r = r_0$ are given by

$$\rho_0 = \left(\frac{3m_0}{4\pi R_0^3}\right), \quad \sigma_0 = \frac{-\beta_0^2\, dm/du}{(4\pi R_0^2 r_0^2 S^2)}$$

Having studied in particular two interesting solutions one representing the adiabatic and the other non-adiabatic cases we shall now discuss the merits and demerits of these situations and briefly go over some other solutions of field equations attempting to represent a collapsing body.

To begin with we had the Oppenheimer-Snyder solution which though mathematically simple be hardly realistic. The assumption $p = 0$ is very idealistic from the point of view of a compressing body. Particularly if one has in mind the cosmological bodies such as quasars then the mass considered to account for the heavy energy output gives a density quite high and at such densities one can hardly expect the matter to be incoherent.

On the other hand Vaidya's solution could be little more realistic from the point of view of a co-moving observer. For except for observers in the immediate vicinity of the object the field in the radiation belt could hardly be of any effect. Some of the physical properties of Vaidya's solution are as follows.

By choosing properly the functions $a(t)$ and $b(t)$ such that $a - b^2/4 > 0$ we can make ρ the density of the collapsing object everywhere positive.

p and σ will always be non-negative.

The rate of contraction $D_t R_0$ of the boundary of the sphere is similar to that in the adiabatic case of Oppenheimer and Snyder.

In both the above cases R=2M gives an event horizon. Further the conclusions drawn in either cases depend mostly on the initial assumptions put in, namely, uniform density, perfect spherical symmetry etc. From the point of view of a comoving observer there is nothing that prevents the collapse into the singularity R=0. This is a very important question and should be considered in detail.

Hoyle and Narlikar (1964) attempted to solve this catastrophe of collapsing into a singularity through the introduction of C field. Their model of collapse is as follows:

In the case of the interior of the collapsing object they consider along with the gravitational field, a creation field through the field equations

$$R_i^k - \tfrac{1}{2} R \delta_i^k = -8\pi G \left[T_i^k - f \left(C_i C^k - \tfrac{1}{2} \delta_i^k C_l C^l \right) \right] \qquad (5.1)$$

where f is a coupling constant and the C-field satisfies the source equation

$$f C^i{}_{;i} = j^i{}_{;i} \quad , \quad j^i = \rho \frac{dx^i}{ds} , \qquad (5.2)$$

The interior solution is now similar to that of elliptic cosmology ($k = 1$) as given by

$$ds^2 = dt^2 - S^2(t) \left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2 \right] . \qquad (5.3)$$

The field equations now give

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2}{S^2} + \frac{\alpha}{S^2} = 4\pi G f \dot{C}^2, \qquad \rho S^3 = \text{constant} \quad (5.4)$$

$\alpha = 8\pi G \rho_0 / 3$, ρ_0 being the density at maximum. The conservation equation $C^i_{;i} = 0$ gives for $\dot{C}$

$$\dot{C} = A / S^3 , \qquad \text{A being a constant} \qquad (5.5)$$

Using this in (5.4) one can integrate it to get

$$S\dot{S}^2 = \frac{\alpha}{S} - \frac{4\pi G f A^2}{3 S^3} + \text{constant.}$$

Taking $\dot{S} = 0$ at $S = 1$, the constant can be adjusted. By assuming the initial effects of C-field to be very small Hoyle-Narlikar, concludes that

$$\dot{S}^2 = \frac{\alpha(1-S)}{S} - \frac{4\pi G f A^2}{3 S^4} \qquad (5.6)$$

As S decreases S^{-4} gains importance and there are these two zeros for $\dot{S}$. The object thus oscillates between two limits for S , say S_1 (maximum), S_2 (min.). The oscillations are gradually damped as S_1 decreases. Hoyle-Narlikar feel that neutrino emission could be a main source of damping. However the object never attains a static state. For the exterior region again Schwarzschild solution is chosen and by expressing the interior solution in a non-co-moving coordinates the fit seems to be obtained at the boundary. In fact with the inclusion of C-field the fit could be obtained only after considering C to be a function of both r and t. As they point out no proper fitting could be obtained in this case except for the asymptotic case $S \rightarrow 0$. This is a very severe restriction and may not be applicable to the proper problem in question.

Hence the inclusion of C-field (the so called negative energy field) to prevent the catastrophic collapse into a singularity of a positive energy source (gravitational field) does not seem to be satisfactory in view of the serious difficulties that arise with respect to the boundary of the imploding object.

Nariai and Tomita (1965) writing on the problem of gravitational collapse point out that both Oppenheimer-Snyder and Hoyle-Narlikar solutions for the collapse problem are unsatisfactory from the point of view of fitting of exterior and interior solutions. They seem to obtain a new solution wherein they adopt the O'Brien-Synge's boundary conditions. I should like to refer the interested reader to the original paper as there is nothing special to mention about. It is to be remembered that the singularity with respect to the Schwarzschild solution ($r = 2m$) is intrinsic as it does form a barrier and only by singular coordinate transformation one may achieve solution that appear to be free from singularity.

However Nariai and Tomita seem to obtain a new exterior solution which is an extension of Schwarzichild's solution. But their entire proceedure depends on the admissibility of O'Brien-Synge's boundary conditions. Their approach does give a new light on the mathematics of the problem of collapse. However the physics of the collapse problem is still to be understood.

Nariai (1967) has given another model of collapse with a pressure gradient which of course is one more in the series of solutions given by Vaidya, Mcvittie and others. Since the procedure behind the construction of these solutions is almost the same, we may not go into the details of these solutions. The special feature of Nariai's solution is that it is a non-static generalisation of Buchdal's work for an Emden polytrope of index 5. Further in this model there occurs a region of maximum collapse after which the object tends to explode. Nariai (1967) has further given an oscillating model for collapse wherein the radius S lies between the limits $0 < S_1 < S < S_2$ for a positive definite density in the region $0 < r < r_b$.

It is to be noticed that in all the above models an important assumption of perfect spherical symmetry has been made. This basically rules out the possibility of considering rotation with collapse. When the question of catastrophic collapse came up the first thought that occurred was to prevent the catastrophe by introducing rotation. The idea being that as the object collapses to a smaller radius conservation of angular momentum would lead to ejection of mass from the parent body and this along with centrifugal acceleration could prevent further collapse. If this is to be considered the first sacrifice should be the perfect spherical symmetry of the collapsing object as there would be a preferred direction along the axis of rotation.

It is of importance to consider collapse of a body with deviations from spherical symmetry. Doroschkevich et al (1966) have studied this aspect of the collapse problem as follows. Since very high rotation would lead to a complete breakdown of spherical symmetry and since no equilibrium configuration is known for it, they consider rotation as a small perturbation (1st order) and superpose this on the original symmetric solution. Thus they still maintain the characteristic Schwarzschild radius $r=2m$ in the solution and their claim is that even with the deviation the essential feature of $r = 2m$ which forms a self locking for the body, remains unchanged. As the compression goes on, perturbation grows up, but because of the gravitational self-locking these perturbation will not affect the region outside $r=2m$. The first order time-independent deviations from the spherical Schwarzschild field may be divided into 2-types:

1) 'multipole' perturbations, which change purely the spatial components of the metric tensor as $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3$) and pure temporal part g_{44}

2) rotational perturbations comes in through the appearance of terms like $g_{\alpha 4}$.

Regge and Wheeler (1957) have pointed out that "multipole" time-independent perturbations have to increase infinitely on approaching the Schwarzschild surface.

Kerr has obtained a solution for the Vacuwn field equation which some have interpreted as the exterior solution for a rotating body. But this is not generally accepted as the solution has not been derived from first principles including rotation.

However, the collapse of a rotating body can be better understood only when we accomplish an exact solution for a rotating configuration in general relativity. Instead of trying to satisfy ourselves regarding the smallness of the quantitative corrections due to rotation, when considered as a perturbation, one should understand the qualitative effects of rotation which could change the entire configuration and the field associated with it.

The problems viewed from this-point essentially brings in a study of axially symmetric solutions in general relativity and their stability. The type of singularities that one encounters then may be essentially different and thus there is a vast scope for theoretical research in this field of study.

GRAVITATIONAL RADIATION

Having known too well about the electromagnetic radiation, the prediction of the existence of gravitational wave is just but a step. But the theory to explain the existence and propogation of gravitational waves is quite complicated. General theory of Relativity was the first successful theory of gravitation which precicted the existence of gravitational waves.

In analogy with electromagnetic waves, Einstein in 1916, tried to obtain solution representing gravitational waves by approximate linear solutions of the field equations (This is because, basically the field equations being non-linear, superposition of solutions would not be possible for general cases). Starting from the field equations

$$R_i^{\ j} - \frac{1}{2} R \delta_i^{\ j} = -\frac{8\pi G}{c^4} T_i^{\ j}, \qquad (1.1)$$

Einstein studied the solutions

$$g_{ij} = \delta_{ij} + h_{ij} \qquad (1.2)$$

where δ_{ij} is the Lorentz metric and h_{ij} first order quantities. Substituting (1.2) in (1.1) we get

$$\Box \phi_i^{\ j} = \frac{16\pi G}{c^4} T_i^{\ j} \qquad (1.3)$$

where $\phi_i^{\ j} = h_i^{\ j} - \frac{1}{2} \delta_i^{\ j} h$, satisfies the coordinate condition

$$\phi_{i\ ,j}^{\ j} = 0 \qquad (1.5)$$

Solution of (6.3) is given by

$$\phi_i^{\,j}(r,t) = \frac{-4G}{c^4}\int \frac{\left(T_i^{\,j}\right)_{\text{retarded}}\, d^3x'}{|r-r'|} \tag{1.6}$$

Thus given a certain source distribution T_i^j. One could know the approximate wave solution associated with it. Trautman (1965) has criticized this approach of linear approxmation as follows. If one obtains on approximate solution as Synage has pointed out it can be thought of as an exact solution for a different distribution obtained directly from Einstein's field equations. Now if this is done for a radiative weak field solution one finds that the corresponding flux of matter counter-balances the out flow of gravitational radiation as computed from the pseudo-tensor.

As a matter of fact the linearized solution cited by Einstein can be obtained directly from generalised Newton's law of gravitation for 4-space. We have in the four-space the D'alembertian equation

$$\Box\phi = 4\pi k\rho \tag{1.7}$$

whose solution is given by the retarded potential

$$\phi(\vec{R_0},t) = -k\int \frac{\rho(\vec{r},t-R/c)}{R}\, dV \tag{1.8}$$

where $R = \vec{R_0} - \vec{r}$, $R = |\bar{R}|$ and $\vec{r}$ the variable of integration, and the integral is over the 3-space, t = const. The expansion into multipole of this solution of (6.7) yields

$\phi = \frac{-KM}{R_0} - \frac{k \pm n.P}{CR_0^2}$ + Quadrapole and higher multipoles, Where M is the total mass and P the total momentum of the isolated system. In order to get the total power E radiated, the analogous electromagnetic expression is used, viz.,

$$E = \frac{c}{k} \oint_S (\nabla\phi)^2 ds$$

where S is a surface at sufficiently large distance from the source. Hence it is apparent that the terms which can survive as $R \to \infty$ are only those of the order 1/R in $\nabla\phi$. From the expression obtained it can be seen that as M and P are constants for a given isolated system, the terms that could contribute are only the Quadrapole and higher order terms. Thus one finds that the gravitational radiation is essentially of the Quadra-pole type. As Trautman points out it is essential in this argument to have the gravitational mass identified to be same as inertial mass. Since this has been assured by the principle of equivalence we can arrive at the above conclusion. Some of these points may be taken up for further discussion when we come to the question of energy associated with these waves.

Having thus seen that the gravitational radiation is essentially of Quadrapole type let us now consider how we can represent it in a Riemannian manifold, and how it propogates.

Lichnerowicz has given a very nice treatment in connection with gravitational radiation, wherein he shows that gravitational stock waves travel as null surfaces of discontinuity. Since by definition of admissible coordinates one has to have g_{ij} and $g_{ij,k}$ continuous across any 3-space, we can expect to have discontinuity only in the second order derivatives of g_{ij}.

A gravitational shock wave is defined as a 3-space Σ across which there are irremovable discontinuities in some of the second order derivatives of g_{ij}.

In general a surface Σ is a shock wave if the connected set of differential equations do not offer a unique solution for the quantity which is expected to be discontinuous. In order to study gravitational shock wave we use the vacuum field equations $R_{ij} = 0$. Consider the 3-surface Σ defined by $f(x^i) = 0$. Transforming to Gaussian coordinates $\bar{x}^i$ we will have Σ defined by $\bar{x}^4 = 0$ and across this we have except for $\bar{g}_{ij,44}$ all the other quantities continuous. From the components of $\bar{R}_{ij}$ it can be seen that of the ten components $\bar{g}_{ij,44}$ only 6 viz., $\bar{g}_{\alpha\beta,44}$ occur in them. Hence the condition for Σ to be a gravitational shock wave implies that

$$\bar{R}_{ij} = 0,$$

should not determine $\bar{g}_{\alpha\beta,44}$ uniquely. This is possible only if we have $\bar{g}^{44} = 0$. Now going back to the original system of coordinates (x^i) in which Σ is $f(x^i) = 0$, we find that

$$\bar{g}^{44} = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} = 0 \qquad 1.10$$

Hence we find that the 3-space $f(x^i) = 0$ can be a gravitational wave, provided it satisfies the condition

$$g^{ij} f_{,i} f_{,j} = 0 \qquad 1.11$$

i.e. $f(x^i) = 0$ is a null-surface. This establishes the fact that gravitational shock waves propagate with velocity c.

At this stage it is interesting to compare this with the result of Pauli and Fierz. They, while trying to investigate what relativistic wave equation is satisfied by particles of spin 2 and zero rest mass argued that since for a particle of spin S, 2(2S+1) field components are needed, here one needs a second rank symmetric tensor and the wave equation they obtained is

$$\Box \psi_{\mu}^{\nu} = 0,$$

with the supplementary conditions

$$\psi_{\mu}{}^{\nu}{}_{,\nu} = 0.$$

These we find are same as the wave equation obtained by Einstein in his linearized theory (1.3) along with the coordinate conditions (1.5). Since we know from electromagnetic theory that the waves associated with a particle of zero rest mass (photon) travel along null geodesics we have here analogous result that in the linearised theory of gravitation we can associate gravitation (spin 2) with gravitational waves.

Since the essence of general theory of Relativity is the associated set of non-linear equations the interpretations of gravitations would not be proper when we study exact solutions for gravitational waves.

2. Classification of Fields

Having seen that the gravitational radiation is of Quadrapole moment and that the waves propagate as null surfaces, we shall now come to the role of the curvature tensors in the study of radiation theory. Gravitation being characterised by the curvature of the space-time, it can be seen that in a given gravitational field, the acceleration between the neighbouring particles is expressed through the Riemann-Christoffel tensor, through a study of geodeosic deviation. Thus the Riemann tensor (called as Field variation tensor by Pirani) characterises the variations in the gravitational field which further is expected to be associated with the propagation of waves. Also while discussing the gravitational shock waves we considered them as discontinuities in second order derivatives of the metric potential and from the definition of the Riemann-tensor we know that the second order derivatives directly enters in it and hence the discontinuities appear in the components of the tensor itself.

Petrov (1954) classified the vacuum field solutionsinto three basic classes by using the Riemann-tensor. Synge (1964) generalised the same idea into general solutions (when the matter tensor is non-zero) by classifying the Weyl tensor. Since the symmetry properties of the Weyl tensor are same as those of the Riemann-tensor, the classification of Weyl tensor is just similar to that of Riemann tensor.

Before going into classification let us understand what classification means. Synge defines a class as follows. Given two events E and $\bar{E}$ when are they identical ? If we examine the neighbourhoods of events, then for comparison we need not only the metric tensors g_{ab} and $\bar{g}_{ab}$, but also their derivatives. The second order derivatives do not uphold the identity of events. Synge classifies events themselves in terms of the Weyl tensor. Suppose W_{abcd} is the Weyl tensor at E and $\bar{W}_{pqrs}$ is at $\bar{E}$ then through tensor covariance we have

$$\bar{W}_{pqrs} = W_{abcd} \, X^{a}_{\bar{p}} \, X^{b}_{\bar{q}} \, X^{c}_{\bar{r}} \, X^{d}_{\bar{s}} \quad . \tag{2.1}$$

Then the question is, given W and $\bar{W}$ can we find the sixteen numbers X's such that the above equality holds. It would be possible only if W and $\bar{W}$ belongs to the same class. After thus defining the class, we now come to Petrov's eigenvalue problem as follows.

Consider the equation

$$W_{abcd} \, F^{cd} = \lambda \, g_{abcd} \, F^{cd}, \tag{2.2}$$

where, $g_{abcd} = g_{ac} \, g_{bd} - g_{ad} \, g_{bc}$ (2.3)

F^{cd} an antisymmetric tensor, and λ a scalar. In order to solve the set of 6 equations (2.2), it is well known that the consistency condition requires satisfaction of a 6×6 determinantal equation. Our entire method of classification depends on the 6 eiganvalues to be obtained which may or may not be distinct.

Further if these eigenvalues are complex then they occur in conjugate pairs. Synge points out in order that two Weyl tensors be of the same class, it is necessary (but not sufficient) that the eigenvalues of W_{abcd} with respect to g_{abcd} should be same as those of W_{pqrs} with respect to g_{pqrs}. A visual representation of the Weyl tensor which takes full advantage of its symmetry properties and gives some insight into its local character is provided by Petrov's six dimensional notation as follows.

Correlate the number pairs in the range 1,2,3,4 to single numbers in the range 1,... 6 as given by

$$\begin{array}{cccccc} 23, & 31, & 12, & 14, & 24, & 34 \\ 1 & 2 & 3 & 4 & 5 & 6, \end{array} \qquad (2.4)$$

such that the Weyl tensor now becomes a symmetric 6-tensor W_{AB} which may be represented as a matrix. In general W_{AB} will have 21 elements but in view of the cyclic condition

$$W_{abcd} + W_{acdb} + W_{adbc} = 0, \qquad (2.5)$$

we get

$$W_{14} + W_{25} + W_{36} = 0 \qquad (2.6)$$

Further writing now the antisymmetric tensor F^{ab} as E^{A} in the same notation we find that (2.2) reduces to the form

$$W_{AB}\, F^{B} = \lambda\, g_{AB}\, F^{B} \qquad (2.7)$$

and the corresponding characteristic equation is

$$\det\ (W_{AB} - \lambda\, g_{AB}) = 0 \qquad (2.8)$$

Since the computation is to be done at an event, by principle of equivalence we can take $g_{ab} = \delta_{ab}$ at a point. Further choosing coordinates such that $x_4 = it$, we have for g_{AB} (2.3)

$$g_{AB} = \text{diag} \left\{ 1, 1, 1, 1, 1, 1 \right\} . \tag{2.9}$$

It is known that the Weyl tensor satisfies the ten identities $g^{ad} W_{abcd} = 0$. This in the new notation along with (2.9) gives us 10 relations and thus the number of independent components of W_{AB} is now only ten.

$$W_{AB} = \begin{pmatrix} M & N \\ N^T & +M \end{pmatrix} \tag{2.10}$$

where,

$$M = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix} \qquad N = \begin{pmatrix} W_{14} & W_{15} & W_{16} \\ W_{24} & W_{25} & W_{26} \\ W_{34} & W_{35} & W_{36} \end{pmatrix}. \tag{2.11}$$

are 3 x 3 symmetric matrices. Further both M and N are traceless. M is traceless because $g^{ad} W_{ancd} = 0$, and N is traceless because $W_{a(bcd)} = 0$ M is real and N is imaginary ($x^4 = it$). Writing the column matrix E^A as

$$F = \begin{pmatrix} G \\ H \end{pmatrix} \tag{2.12}$$

with $G = \mathrm{col}\ \{F_1, F_2, F_3\}$, $H = \mathrm{col}\ \{F_4, F_5, F_6\}$ we have G real and H imaginary. Using (2.12) and (2.10) in (2.8) which by virtue of $g_{ab} = \delta_{ab}$ is $WF = \lambda F$, we get

$$\begin{pmatrix} M & N \\ N & M \end{pmatrix} \begin{pmatrix} G \\ H \end{pmatrix} = \Lambda \begin{pmatrix} G \\ H \end{pmatrix} \tag{2.13}$$

This is equivalent to the equation

$$KJ = \Lambda J, \quad \text{with } K = M + N, \quad J = G + H \tag{2.14}$$

wherein both K and J are complex matrices. The eigenvalue problem is now defined by the characteristic equation

$$\det \left| K - \lambda I \right| = 0 \tag{2.15}$$

which is a cubic with complex coefficients. The classification of the tensor W_{abcd} is now done according to the nature of the three roots of this cubic, λ_1, λ_2, λ_3. Corresponding to these three eigenvalues let J_1, J_2, J_3 be the eigenvectors.

In classifying the tensor W_{abcd} using these λ's and J's we use the following theorems (Synge (1964)) concerning null vectors and eigenvectors, which have very simple proofs.

1. If two null vectors are orthogonal, they are collinear.

2. An orthogonal triad of non-zero vectors no two of which are collinear cannot contain a null vector as a member.

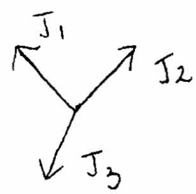
3. If J_1 and J_2 are eigenvectors corresponding to the eigenvalues λ_1 and λ_2 and if $\lambda_1 = \lambda_2$, then J_1 and J_2 are orthogonal, linearly independent, and both of them cannot be null.

With this the classification runs as follows.

1. $\lambda_1, \lambda_2, \lambda_3$ all distinct $\lambda_1 + \lambda_2 + \lambda_3 = 0.$

Then J_1, J_2, J_3 all non-null, mutually orthogonal with uniquely determined directions. Vectors all distinct

$$K = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

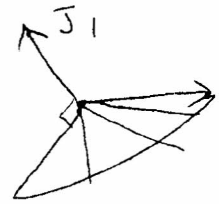


field : Petrov type I.

2a. $\lambda_1 \neq \lambda_2 = \lambda_3.$ $\lambda_1 + 2\lambda_2 = 0.$

J_1 non-null with uniquely determined direction and all vectors in a plane orthogonal to J_1. Vectors all distinct.

$$K = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$



field : Petrov type D.

2b. $\lambda_1 \neq \lambda_2 = \lambda_3$, $\lambda_1 + 2\lambda_2 = 0.$

J_1 non-null with uniquely determined direction J_2 null with uniquely determined direction orthogonal to J_1. Two distinct vectors.

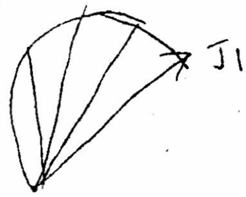
$$K = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2+1 & i \\ 0 & i & \lambda_2-1 \end{pmatrix}$$

field : Petrov type II

3a. $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

J_1 null with uniquely determined direction and all vectors orthogonal to J_1. Two distinct vectors.

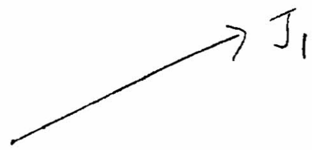
$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & i & -1 \end{pmatrix}$$



field : Petrov type N.

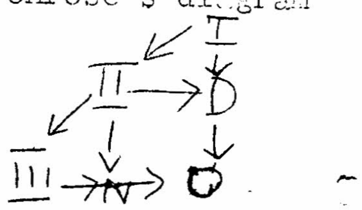
3b. $\lambda_1 = \lambda_2 = \lambda_3 = 0$. J_1 null, others non-null and only one distinct vector.

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$



field : Petrov type III.

3c. $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and No distinct vector. In this case the Weyl tensor vanishes and hence the space-time represented by such solutions will be conformally flat. This class is represented by O. The hirarchy in this method of classification is well brought out by Penrose's diagram



Pirani (1957) had concluded that the fields belonging to Petrov type II, N, and III, alone can represent gravitational radiation. However later in a paper with Bondi and Robinson (1959) he has shown that the earlier conclusion is incorrect.

Apart from the matrix method of classification given above we have tensor method of classification due to Debevér (1959) and Spinor method of classification due to Penrose (1960). I should like to present now the tensor method of Debever later developed by Sachs and the Spinor method, I shall leave for a future consideration. This method depends on a theorem due to Debever which as given by Sachs says that

In every empty-space there exist (s) atleast one and atmost four direction(s) $k^a \neq 0$, such that

$$k_{[a}R_{b]ij[c}k_{d]}k^i k^j = 0, \quad k_a k^a = 0 \tag{2.16}$$

The arrangement of the null vectors k^a (Debever vectors) determines the Petrov type.

1. If we can choose the vector k^μ such that

$$R_{abcd}\, k^d = 0 \tag{2.17}$$

then the solutions are of type N, and all the four rays will be coincident at a point.

2. If (2.17) is not satisfied, we try

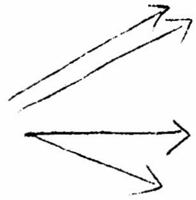
$$R_{abc[d}\, k^c k_{e]} = 0 \tag{2.18}$$

and if this is true the field is of type III, and at each point we will have 3 coincident rays and one distinct ray.

3. If (1) and (2) are not possible, we consider

$$R_{abc[d}\, k_{e]} k^d k^c = 0 \tag{2.19}$$

which if satisfied specifies the field as of type II. If there are two null vectors k^{μ} and m^{μ} such that (2.19) could be satisfied for both, then the field is of type D. In type II fields the vectors will be 3 rays with one being a double ray and all the three being distinct. On the other hand for type D there will be two distinct rays each being a double ray.



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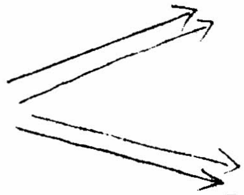

D.

4. If none of the above possibilities exist then by the theorem we must have

$$k_{[a} R_{b]ij[c} k_{d]} \; k^i k^j = 0$$

which corresponds to Petrov type I and all the four vectors will be distinct rays. Sachs has pointed out that a Riemann tensor which obeys any one of the above equations (2.17)-(2.20) will satisfy all succeeding sets.

Further he has shown that the hierarchy of types(as exhibited in Penrose diagram) has considerable physical significance, through the expansion of vacuum Riemann tensor of a retarded multipole field as given by

$$R_{abcd} = r^{-1} N_{abcd} + r^{-2} \underline{III}_{abcd} + r^{-3} \underline{II}_{abcd} + r^{-4} \underline{I}_{abcd} + r^{-5} \underline{I}'_{abcd} + O(r^{-6}) \qquad (2.21)$$

Here the tensors N_{abcd} ... are independent of r and belong to the indicated Petrov type. (2.21) suggests immediately considering the wave front discontinuity along with the monopole term of the expansion, that each of the type is basically of type N, the most degenerate case. Sachs calls the type N tensor which appears in a 'wave zone' role as 'far field' whereas the types III and II are called the 'semi-far' and 'intermediate' fields.

Before concluding this section I should like to present an interesting paper of Pirani (1959) wherein he has worked out the gravitational field of a fast-moving particle. Trautman (1958,a,b) has shown from an analysis of boundary conditions that at great distances from an isolated system the dominant terms in the Riemann tensor will have the form of type II even though the exact form may be of type I. Pirani in this work has shown that in the rest frame of an observer moving with respect to the Schwarzschild mass at high speed, the dominant terms in the Riemann tensor have the type II form which is of radiation type. The entire analysis is done through a Vierbein formalism, for the physical components of a tensor are its components in an orthogonal frame specified by a tetrad $\lambda^{\mu}_{(\alpha)}$ of unit vectors. In order to find the gravitational field of a moving particle in terms of the Riemann tensor, it is sufficient to calculate the physical components of that tensor in the frame of an observer moving relative to the Schwarzschild mass.

By constructing proper tetrads and referring the given Schwarzschild field Pirani calculates the matrix R_{AB} (A,B, ranging 1,... 6 as explained already) in the relative rest frame and shows that the structure of R_{AB} is analogous to that of a field representing plane gravitational waves as given by Bondi, Pirani, Robinson. Thus he has shown that we can associate a gravitational wave pattern with a fast moving inertial particle, as we have the association of electromagnetic waves with a moving charge.

3. Exact Solutions

We saw through linear approximation of Einstein, that Gravitational radiation is of quadrapole type and further the classification of gravitational fields according to the number of null vectors associated with the Weyl tensor. For a complete understanding of the phenomena of Gravitational Radiation it is appropriate to study exact solutions of Einstein field equations representing gravitational radiation. Again the earliest attempt to get at an exact solution representing gravitational waves is that due to Einstein and Rosen (1937).

Suppose we start from a spherically symmetrical field. If we want to solve for an empty space (matter free) then we know by Birkhoff's theorem that every solution of $R_{ij} = 0$, with spherical symmetry is just a transform of Schwarzschild's static solution and hence no radiation could be expected in that field. Having thus barred spherical symmetry from our considerations we will be left with cylindrical symmetry and plane symmetry. Einstein-Rosen solution for cylindrical waves, is expressed through the metric

$$ds^2 = e^{\gamma - 2\psi}(dt^2 - dr^2) - r^2 e^{-2\psi} d\varphi^2 - e^{2\psi} dz^2, \qquad (3.1)$$

where γ and ψ depend on 'r' and 't' alone. The field equations that are considered here are $R_{ij} = 0$, because there is no matter content. With this ψ and γ are required to satisfy the set of equations

$$\psi_{rr} + \frac{1}{r}\psi_r - \psi_{tt} = 0 \tag{3.2}$$

$$\gamma_r = r(\psi_r^2 + \psi_t^2), \quad \gamma_t = 2r\psi_r\psi_t . \tag{3.3}$$

(3.2) is nothing but the wave equation in cylindrical coordinates and a solution of it representing outward travelling waves is given by

$$\psi = A\left[\alpha_0 \, \omega r \cos \omega t + \beta_0 \, \omega r \sin \omega t\right] \tag{3.4}$$

Using this in (3.3) γ can be evaluated. This solution of Einstein and Rosen do not seem to be practicable as it represents an infinite rod to ~~be~~ the source of the field and no particular way of loss of mass can be ascertained.

Marder (1958) claims that this solution cannot be properly fitted on with any interior field in a physically plausible way. In order to overcome this Marder suggests superimposing cylindrical wave solution of the free space field equations on a static field which in turn is extended to the interior of the cylinder.

Krishna Rao (1964) has made the analysis of cylindrical waves mainly through Lichnerowicz's conditions. Lichnerowicz (1960) interprets the discontinuities of the curvature tensor R_{hijk} across a null hypersurface of a Riemannian manifold (4 dims) as a wave being propogated along the null geodesics and further at every point of the wave zone a null vector exists such that

$$k_l R_{\mu\nu,\eta\delta} + k_\mu R_{\nu l,\eta\delta} + k_\nu R_{l\mu,\eta\delta} = 0 \qquad (3.5)$$

$$k^\mu R_{\mu\nu,\eta\lambda} = 0 \qquad (3.6)$$

These two together give

$$R_i^{\,j} = \tau k_i k^j \qquad (3.7)$$

where τ is a scalar, and this is defined by Lichnerowicz as a state of "total radiation", and when $\tau = 0$ one gets pure gravitational radiation. Considering the radiation to be travelling along the radial direction alone one gets from (3.7) the set

$$R^1_1 + R^4_4 = 0, \quad R^1_1 + R^1_4 = 0, \quad R^2_2 = R^3_3 = 0. \qquad (3.8)$$

Starting from a metric similar to that of Einstein-Rosen (3.1) Krishna Rao gets for (3.8)

$$\psi'' + \frac{\psi'}{r} - \ddot{\psi} = 0, \qquad (3.9)$$

$$\gamma' + \dot{\gamma} - r(\psi' + \dot{\psi})^2 = 0 \qquad (3.10)$$

$$\gamma'' - \ddot{\gamma} + \psi'^2 - \dot{\psi}^2 = 0 \qquad (3.11)$$

In the pure gravitational waves case we have seen that satisfies

$$\gamma' - r(\psi'^2 + \dot{\psi}^2) = 0 \tag{3.12}$$

$$\dot{\gamma} - 2r\psi'\dot{\psi} = 0 \tag{3.13}$$

which is just a splitting of (3.10) above. If ψ_0 and γ_0 are the solutions of (3.9), (3.12) and (3.13) then it can be verified that the solutions of the required set is given by

$$\psi = \psi_0 \quad , \gamma = \gamma_0 + f(r - t) \tag{3.14}$$

Krishna Rao discusses in particular three cases

(1) Corresponding to $\psi_0 = 0$, $\gamma_0 = 0$ (waves on flat-space)

(2) Corresponding to Marder's solution (waves superimposed on an infinite cylinder)

(3) Corresponding to gravitational pulse waves of Weber and Wheeler.

In the first case Lichnerowicz's conditions are satisfied everywhere whereas in the second and third cases they are satisfied asymptotically. We next consider Plane-waves. The study of plane-waves as an exact solution of gravitational field equations was initiated by Bondi-Pirani and Robinson (1959) and they defined plane-waves as being represented by non-flat solutions of the field equations $R_{ij} = 0$. Further they assumed the symmetry of plane gravitational waves to be analogous to that of plane electromagnetic waves. Their study deals with admission of a 5 parameter group of motions by a space-time characterising plane-waves. If the waves are travelling in the z-direction then the space-time consist of 3-parametric groups of translation along Ox, Oy, and the plane t-z = const and 2-parameter groups of rotations (null) which leave the plane t-z = const, invariant. However, they presume that these plane gravitational waves exist only as asymptotic limits in nature.

Takeno (1961) has made an extensive study of various solutions representing plane-waves. His definition of plane-waves though is similar to that of B, P, R he assumes a further condition which allows one to select physical solutions from a possible innumerable solutions of $R_{ij} = 0$. His definition of plane-waves is as follows.

(1) A plane wave g_{ij} is a non-flat solution of $R_{ij} = 0$

(2) and it has the property that in some suitable coordinate system all its components are functions of a single variable $Z = Z(x^i)$ (the phase function)

$$g_{ij} = g_{ij}(Z), \text{ satisfying} \tag{3.15}$$

$$g^{ij} Z_{,i} Z_{,j} = 0, \quad Z = (z, t), \quad (Z_{,3} \neq 0 \quad Z_{,4} \neq 0) \tag{3.16}$$

One can see that in this definition the existence of a 'Cartesian-like' coordinate system is tacitly assumed and the plane-waves propogate in the z direction with fundamental velocity. By the second of (3.16) we find that at any instant all points of equal phase lie in a plane z = const, which assures the planeness of the waves. Takeno further classifies plane waves into 3 types

(1) (z-t) type where z = f(z-t)

(2) (t/z) type where z = f(t/z)

(3) and general type where no transformation can yield z a function of (z-t) or t/z.

The general form of the metrics representing plane gravitational waves are given by

$$ds^2 = -A\,dx^2 - 2D\,dx\,dy - B\,dy^2 - (C-E)\,Z^2 dz^2 - 2ZE\,dz\,dt + (C+E)\,dt^2 \qquad (3.17)$$

where A,B,C,D,E are functions of (z-t), and

$$ds^2 = -A\,dx^2 - 2D\,dx\,dy - Z^2(C-E)\,dz^2 - B\,dy^2 - 2ZE\,dz\,dt + (C+E)\,dt^2 \qquad (3.18)$$

where A,B,C,D,E,Z are functions of (t/z).

Takeno's form of the (z-t) type wave metric is similar to that of Bondi's form

$$ds^2 = e^{2\phi}(d\tau^2 - d\xi^2) - u^2\{\cosh 2\beta\,(d\eta^2 + d\zeta^2) + \sinh 2\beta \cos 2\theta\,(d\eta^2 - d\zeta^2) - 2\sinh 2\beta \sin 2\theta\, d\eta\, d\zeta\} \qquad (3.19)$$

with $u = \tau - \xi$, provided A,B,D of (3.17) satisfy,

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$$AB - D^2 = u^4 \tag{3.20}$$

However, Takeno has shown that the (z-t) type & (t/z) type are mutually transformable and thus he concludes that these two types are only different manifestations of the same field of plane gravitational waves.

Last time we had referred to an important aspect of Riemann-tensor as appears in determining the relative acceleration between test particles. The equations of geodesic deviation

$$\frac{\delta^2 \eta^\mu}{ds^2} + R^\mu{}_{\nu\rho\sigma} u^\nu \eta^\rho u^\sigma = 0 \tag{3.21}$$

When referred to a tetrad wherein U^ν is the time like vector and the other three being space-like vectors defined by parallel propagation along the chosen geodesic, will be analogous to the Newtonian equation

$$\frac{d^2 x^a}{dt^2} + k^a{}_b(t) X^b = 0, \quad k^a{}_b = \frac{\partial^2 V}{\partial x^a \partial x^b} \tag{3.22}$$

In the case of fixed plane of polarization $k^a{}_b = 0$, $a \neq b$. Further $K^1{}_1 = 0$ and $k^2{}_2 = -k^3{}_3$. Hence we find that two test particles in the way of the plane-waves suffer accelerations in opposite directions perpendicular to the direction of propagation of the waves. This establishes the transverse nature of the plane-gravitational waves.

Finally let us consider waves from bounded systems. Since spherical symmetry cannot admit wave solutions we consider the waves from axi-symmetric isolated system. This has been worked out in detail by Bondi, Metzner and Vander Berg (1962). The solution obtained by Bondi et al represents field at large spatial distance from the source with axi-symmetry and no incoming radiation.

Although in general one can use any system of coordinates to work with, it is preferable to work in certain systems wherein the boundary conditions assume simpler form. Bondi introduces a system

$$x^i = (r, \theta, \phi, u) \tag{3.23}$$

where u is the retarded time (t-r), θ and ϕ kind of polar angles with the symmetry axis as polar axis and r is a radial coordinate chosen in such a way that the 2-surface du = dr = 0 has the area $4\pi r^2$. Further the time coordinate u is defined such that $du = d\theta = d\phi = 0$ represents an outgoing light ray.

In this system the metric tensor g_{ij} takes the form

$$g_{ij} = \begin{pmatrix} 0 & 0 & 0 & -e^{2\beta} \\ 0 & r^2 e^{2\gamma} & 0 & -r^2 U e^{2\gamma} \\ 0 & 0 & r^2 \sin^2\theta \, e^{-2\gamma} & 0 \\ -e^{2\beta} & -r^2 U e^{2\gamma} & 0 & -(r^{-1} V e^{2\beta} - r^2 U^2 e^{2\gamma}) \end{pmatrix} \tag{3.24}$$

where U, V, β, γ are functions of r, θ and u. In order to ensure the regularity in the neighbourhood of the polar axis the functions $V, \beta, U/\sin\theta, \gamma/\sin\theta$ have to be regular as $\sin\theta \to 0$. It is believed that the space sufficiently far from the system is covered by one patch of coordinates of the type (3.23) (3.24). In these coordinates the absence of inward flowing radiation may be expressed by the assumption that the functions U, V, β, γ for sufficiently large r can be written as a power series in 1/r with coefficients depending on θ and u only.

The field equations $R_{ij} = 0$ now will give us 6 equations and out of these six Bondi et al consider

$$R_{11} = R_{12} = R_{22} = R_{33} = 0 \tag{3.25}$$

as main equations. R_{41} vanishes as a result of the Bianchi identities and further $R_{42} = 0$ and $R_{44} = 0$ are used as supplementary conditions. With this the solution is represented by .

$$\begin{aligned}
\gamma &= c(u,\theta)r^{-1} + O_3 \\
\beta &= -\tfrac{1}{4}c(u,\theta)^2 r^{-2} + O_3 \\
U &= -(c_2 + 2c\cot\theta)r^{-2} + [2N(u,\theta) + 3cc_2 + 4c^2\cot\theta]r^{-3} + O_4 \\
V &= r - 2M(u,\theta) - [N_2 + N\cot\theta - c_2^2 - 4cc_2\cot\theta - \tfrac{1}{2}c^2(1 + 8\cot^2\theta)]r^{-1} + O_2
\end{aligned} \tag{3.26}$$

Where O_n means a term which vanishes as r^{-n} for $r \to \infty$

The suffix 2 means partial differentiation with respect to θ for constant r, ϕ, μ and C, M, and N are functions of integrations depending on the type of matter under consideration. The supplementary conditions $R_{02} = 0$, $R_{00} = 0$, reduce the independence of M and N to give

$$M_4 = -c_4^2 + \tfrac{1}{2}(c_{22} + 3c_2 \cot\theta - 2c)_4$$
$$-3N_4 = M_2 + 3cC_{42} + 4cC_4 \cot\theta + C_0 C_2 \quad (3.27)$$

Thus it can be seen that given $C(u,\theta)$ the entire situation is fully determined. In other words the flow of information in the system is entirely controlled by the single function C. If $C_4 = 0$ for $u > 0$ the system must remain static and the moment C_0 starts deviating from zero the other functions begin to vary. Thus if anything were to happen at the source to affect the field and produce changes, it would be possible only by affecting C_0 and vice-versa. All the <u>news</u> in the field is contained in the function C_0 and hence it is called by Bondi as <u>News function</u>. In general the structure of the equations indicates that if γ, M and N are known for $u = a$ and C_0 is known for all u in $a \leq u \leq b$ then the system is fully determined in the interval $a \leq u \leq b$.

Looking back at the solution we find that

$$g_{44} = -\left(1 - \frac{2M(u,\theta)}{r} + O_2\right) \quad (3.28)$$

and thus $M(u,\theta)$ is called the <u>mass aspect</u>. (Moller 1964)

For static system M = m, the mass of the system. Bondi, defining $m(u)$ of the system as the mean value of $M(u,\theta)$ over the sphere

$$m(u) = \frac{1}{2}\int_0^{\pi} M(u,\theta)\sin\theta\, d\theta \tag{3.29}$$

gets by integrating (3.27)

$$m_4 = -\frac{1}{2}\int_0^{\pi} c_4{}^2 \sin^2\theta\, d\theta \tag{3.30}$$

For,

$$\int_0^{\pi}(c_{22} + 3c_2\cot\theta - 2c)\sin\theta\, d\theta = \left(\frac{c\sin^2\theta}{\sin\theta}\right)_2\Bigg|_0^{\pi} = 0 \tag{3.31}$$

because of the regularity conditions. Hence the main conclusion of Bondi is that

"the mass of a system is constant if and only if there is no news. If there is news, the mass decreases monotonically as long as the news continues".

In order to avoid the dependence of the result just on the assumed definition of mass, the attention is confined to systems which are initially and finally static. Since for a static system $m(u)$ is equal to the total mass or energy of the system and since the right hand side of (3.29) is always positive for $c_4 > 0$, one can see that the system must lose energy if the news function is different from zero at the intermediate stage. This loss of energy as Bondi points out could be ascribed physically only to the emission of gravitational waves from the system.

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