ON WEIGHTED EIGENVALUE PROBLEMS AND APPLICATIONS

By

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Declaration

I, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

Anoop T.V.

Dedicated to my parents.

Abstract

The main objective of this thesis is to find a large class of weight functions that admits a positive principal eigenvalue for the weighted eigenvalue problems for the Laplacian and the p-Laplacian. More specifically, for a connected domain Ω in $\mathbb{R}^N$ with $N \geq 2$, we study sufficient conditions for a function $g \in L^1_{loc}(\Omega)$ to admit a pair (λ, u), with $\lambda \in \mathbb{R}^+$ and $u > 0$ a.e. such that u is a weak solution of the following problem:

$$-\Delta_p u \quad = \quad \lambda g|u|^{p-2}u, \qquad \text{in } \Omega, \tag{1}$$

where $1 < p < N$ and $\Delta_p u := \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})$ is the p-Laplace operator. Such a λ, if exists, is called a principal eigenvalue of (1). In the literature, most of the sufficient conditions for the existence of a positive principal eigenvalue demand that the weight function g or its positive part g^+ to be in $L^{\frac{N}{p}}(\Omega)$. However, in the field of applications one may need to consider weights that are not belonging to any of the Lebesgue spaces.

We look for a weak solution of (1) in $\mathcal{D}^{1,p}_0(\Omega)$, where

$$\mathcal{D}^{1,p}_0(\Omega) := \text{completion of } \mathcal{C}^\infty_c(\Omega) \text{with respect to } \left\|\nabla \cdot\right\|_p \text{ norm .}$$

Now the existence of a positive principal eigenvalue for (1) is closely related with the existence of a minimizer for the functional $J(u) = \displaystyle\int_\Omega |\nabla u|^p$ on the level set $\mathcal{M}_p = \left\{ u \in \mathcal{D}^{1,2}_0(\Omega) \colon \int_\Omega g|u|^p = 1 \right\}$. If the map G, $G(u) = \int_\Omega g^+|u|^p$, is compact, then a direct variational method ensures the existence of a min-

imizer for J on $\mathcal{M}_p$. If g^+ is in $L^{\frac{N}{p}}(\Omega)$, the dual of $L^{\frac{p^*}{p}}(\Omega)$, then the map G is compact. This is mainly a consequence of three facts (i) the continuous embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ (ii) the compactness of the embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into $L^p_{loc}(\Omega)$ (iii) the density of $\mathcal{C}_c^\infty(\Omega)$ in $L^{\frac{N}{p}}(\Omega)$.

The main novelty of our results is that we allow weights that are not in any of the Lebesgue spaces, but only in certain weak Lebesgue spaces. For this we make use of the finest embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into the Lorentz space

$$L(p^*,p) = \left\{ u \text{ measurable } : \int_0^{|\Omega|} [t^{\frac{1}{p^*}} u^*(t)]^p \, \frac{dt}{t} < \infty \right\},$$

where u^* denotes the one dimensional decreasing rearrangement of u. The Lorentz space $L(p^*,p)$ is a Banach space with a suitable norm and it is a proper subspace of the Lebesgue space $L^{p^*}(\Omega)$. However, when g^+ is in the Lorentz space $L(\frac{N}{p},\infty)$ (the dual space of $L(\frac{p^*}{p},1)$), the map G is not necessarily compact. For example, when $g(x) = \frac{1}{|x|^p}$ and Ω contains the origin, the map G is not compact, indeed $\frac{1}{|x|^p}$ is in $L(\frac{N}{p},\infty)$. In this thesis, we find a large class of admissible weights in $L(\frac{N}{p},\infty)$ that admits a minimizer for J on $\mathcal{M}_p$. Namely, the closure of $\mathcal{C}_c^\infty(\Omega)$ in $L(\frac{N}{p},\infty)$ that will henceforth be denoted by $\mathcal{F}_{\frac{N}{p}}$:

$$\mathcal{F}_{\frac{N}{p}} := \overline{C_c^\infty(\Omega)}^{\|\cdot\|(\frac{N}{p},\infty)} \subset L\left(\frac{N}{p},\infty\right).$$

We prove that J admits a minimizer on $\mathcal{M}_p$, when g^+ is in $\mathcal{F}_{\frac{N}{p}}$.

We consider the case $p = 2$ separately, because of the simple and rich theory available for the Laplacian. Moreover, certain results that hold for the p-Laplacian hold good for the Laplacian under weaker assumptions. For $g^+ \in \mathcal{F}_{\frac{N}{p}}$, using a variant of the strong maximum principle, we show that λ_1, the minimum of J on $\mathcal{M}_p$, is indeed a principal eigenvalue of (1). A necessary condition, namely a Pohozev type identity for the existence of a principal eigenvalue is obtained for certain class of weight functions. The radial symmetry of the eigenfunctions corresponding to λ_1 are also discussed when Ω is $\mathbb{R}^N$ or a ball centred at the origin. The existence of a nontrivial

solution branch for certain types of nonlinear equations is studied as an application of the weighted eigenvalue problem using the bifurcation theory. The existence of an infinite sequence of eigenvalues is obtained using the Ljusternik-Schirelmann theory. The weighted eigenvalue problem for the Laplacian on bounded domains in $\mathbb{R}^2$ is also studied. We obtain various sufficient conditions for the existence of a positive principal eigenvalue by making use of the optimal embeddings of $H_0^1(\Omega)$ in the classes of Orlicz and Lorentz-Zygmund spaces.

Notations

$$
\begin{aligned}
|A| &= \text{ Lebesgue measure of the set } A. \\
|x| &= \text{ Euclidean norm of the vector } x. \\
s_n \to s &= s_n \text{ converges to } s \text{ as } n \text{ goes to } \infty. \\
s_n \downarrow &= s_n \text{ is decreasing.} \\
s_n \downarrow s &= s_n \text{ is decreasing and } s_n \to s. \\
\omega_N &= \text{ Lebesgue measure of the unit ball in } \mathbb{R}^N \text{ .} \\
\mathcal{M}(\Omega) &= \{f : \text{f is real valued measurable function on } \Omega\}. \\
\mathbb{R}_+ &= \{r \in \mathbb{R} : r > 0\}. \\
\overline{\lim} &= \limsup . \\
\underline{\lim} &= \liminf .
\end{aligned}
$$

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Contents

List of Publications

- Eigenvalue problems with weights in Lorentz spaces, with Mythily Ramswamy, Marcello Lucia, Calculus of Variations and PDE , Volume 36, No: 3, 2009, 355-376.
- Positive solution branch for nonlinear eigenvalue problems in $\mathbb{R}^N$. with Jagmohan Tyagi, Nonlinear Analysis, Theory, Methods and Application, Volume 74, No. 6, 2011.
- The weighted eigenvalue problems for p-Laplacian with weights in weak Lebesgue spaces, Electronic Journal of Differential Equations, Volume 2011, No. 64, 2011, pp. 1–22.

Chapter 1

Introduction

The main object of this thesis is to obtain a large class of weight functions that admit a positive principal eigenvalue for the weighted eigenvalue problems for certain linear and quasilinear elliptic partial differential operators. More specifically, for a domain Ω in $\mathbb{R}^N$ with $N \geq 2$, we study the sufficient conditions for a function $g \in L^1_{loc}(\Omega)$ to admit a pair (λ, u), with $\lambda \in \mathbb{R}^+$ and $u > 0$ a.e. such that

$$- \Delta_p u \quad = \quad \lambda g|u|^{p-2}u, \quad \text{in } \Omega, \tag{1.1}$$

where $\Delta_p u := \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})$ is the p-Laplace operator and $1 < p < N$.

For $p = 2$, the 2-Laplacian is the usual Laplace operator, which is the prototype for the elliptic partial differential equations. For $p \neq 2$, the p-Laplacian is the prototype for more general quasilinear operators. The p-Laplace operator for $p \neq 2$ is useful in the study of non Newtonian fluids (dilatant fluids $(p > 2)$, pseudo-plastic fluid $(p < 2)$), in torsional creep problems $(p \geq 2)$ as well as in glaciology $(p \in (1, \frac{4}{3}])$. The weighted eigenvalue problem arises when one linearises certain nonlinear partial differential equation around a known solution. A principal eigenvalue of the linearised problem plays an important role in the study of bifurcation, stability and many other qualitative properties of solutions of nonlinear partial differential equations. In many applications, one may have to consider singular weight functions.

We look for a weak solution of (1.1) in a suitable Banach space, say X, of functions to be defined later. A real number λ is said to be an eigenvalue of (1.1), if there exists a non-zero u such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \lambda \int_\Omega g\, |u|^{p-2} u\, v, \quad \forall\, v \in X. \tag{1.2}$$

In this case, we say that u is an eigenfunction associated to λ. If at least one of the eigenfunctions corresponding to λ is of constant sign, then we say that λ is a *principal eigenvalue*. If all the eigenfunctions corresponding to λ are constant multiples of each other, then we say that λ is *simple*.

In the literature, most of the sufficient conditions for the existence of a positive principal eigenvalue demand that either the weight function g or its positive part g^+ to be in an appropriate Lebesgue space. However, in the fields of applications the nonlinearity may be discontinuous with indefinite sign in order to model strong attraction or repulsion reactions in the physical system. Further, when such nonlinear problems admit blow up solutions, the linearisation around these solutions lead to a weighted eigenvalue problem with singular weights that do not belong to any of the Lebesgue spaces. Thus it is of interest to identify a larger class of weight functions, for which (1.1) admits a positive principal eigenvalue. The main novelty of our results is that we allow weights which are not in any of the Lebesgue spaces; however they lie in certain weak Lebesgue spaces. We also unify all the available results in the literature, by identifying a large class of weight functions that admit a positive principal eigenvalue for (1.1). In the following sections, we give a brief description of our methods and results.

1.1 A note on our function spaces

The classical function spaces, such as Lebesgue and Sobolev spaces, have played and continue to play an important role in the theory of partial differential equations and in other branches of analysis. However, for finding more general solutions and understanding the singular behaviour of the solutions one may need to refine these spaces. One of the ways to obtain more scaled refinement of function spaces are by means of so called Banach function spaces, introduced by Luxemburg in 1955. The Banach function spaces include not only the Lebesgue spaces but also their more refined variants such as the Lorentz, Orlicz, Lorentz-Zygmund and the generalized Lorentz-Zygmund spaces. In this thesis we extensively make use of some these refined variants as they serve as a better target space for the embedding of Sobolev spaces in the limiting cases. For further details on the Banach function spaces, we refer to [36].

1.1.1 The solution space $\mathcal{D}_0^{1,p}(\Omega)$

The techniques used for proving the existence of solutions for a partial differential equation sometimes require the completeness of the class of admissible solutions with respect to a norm determined by the problem. On the other hand, the space of smooth functions may not be complete with respect to this norm. The Sobolev spaces and the notion of a weak solution, allow one to consider a Hilbert space or a Banach space as an admissible class of solutions and to use certain functional analytic techniques for proving the existence of a solution.

Here we consider the space, $\mathcal{D}_0^{1,p}(\Omega)$:= completion of $\mathcal{C}_c^\infty(\Omega)$ with respect to

$$\|\nabla u\|_p = \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}$$

as the solution space for $1 < p < N$, see [32]. The Poincaré inequality shows that $\mathcal{D}_0^{1,p}(\Omega)$ coincides with the classical Sobolev space $W_0^{1,p}(\Omega)$, when Ω is bounded. We stress that the restriction on the dimension of Ω, is primarily due to the fact that the space $\mathcal{D}_0^{1,p}(\Omega)$ may not be a function space when Ω is unbounded with $p \geq N$, in the sense $\mathcal{D}_0^{1,p}(\Omega)$ may not be identified with a subspace of $L^1_{loc}(\Omega)$. For example, when $N = 1, 2$, the space $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ is not embedded in $L^1_{loc}(\mathbb{R}^N)$ continuously(see [38]). On the other hand, for $1 < p < N$, the space $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$, where $p^* = \frac{Np}{N-p}$ is the critical exponent, see [54].

Let us emphasise that the embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into the Lebesgue space $L^{p^*}(\Omega)$ played a crucial role in the results obtained in [4, 5, 45]. The main novelty of our results is that we used a finer embedding of the space $\mathcal{D}_0^{1,p}(\Omega)$ into the Lorentz space $L(p^*, p)$, which improve the usual Sobolev embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, see appendix of [1]. Indeed, the Lebesgue space $L^{p^*}(\Omega)$ is strictly contained in the Lorentz space $L(p^*, p)$.

1.1.2 The function space $\mathcal{F}_d$

In this thesis we extensively make use of certain closed subspace of weak-$L^d(\Omega)$ (Lorentz space $L(d, \infty)$) space. A primary motivation behind these subspaces has come from the fact that the closure of $\mathcal{C}_c^\infty(\Omega)$ in $L^\infty(\Omega)$ is a proper closed subspace of $L^\infty(\Omega)$. A similar situation occurs in the case of Lorentz spaces as well. Indeed for $d > 1$, the closure of $\mathcal{C}_c^\infty(\Omega)$ in $L(d, \infty)$ defines a proper subspace that will henceforth be denoted by $\mathcal{F}_d$:

$$\mathcal{F}_d := \overline{\mathcal{C}_c^\infty(\Omega)}^{\|\cdot\|_{(d,\infty)}} \subset L(d, \infty).$$

Many equivalent definitions and characterizations of $\mathcal{F}_d$ are discussed in this thesis. For example, by using certain rearrangement inequalities we obtain the following characterizations of the space $\mathcal{F}_d$:

Theorem 1.1.1. *The following statements are equivalent:*

(i) $f \in \mathcal{F}_d$,

(ii) for every $\varepsilon > 0$, *there exists* $f_\varepsilon \in L^\infty(\Omega)$ *such that* $|\ supp\ (f_\varepsilon)| < \infty$ *and*
$$\|f - f_\varepsilon\|_{(d,\infty)} < \varepsilon,$$

(iii) if f^* *is the symmetric rearrangement of* f, *then* $f^*(t) = o(t^{\frac{-1}{d}})$ *at 0 and* ∞, *i.e.*
$$\lim_{t \to 0_+} t^{\frac{1}{d}} f^*(t) = 0 = \lim_{t \to \infty} t^{\frac{1}{d}} f^*(t), \tag{1.3}$$

(iv) if α_f *is the distribution function of* f, *then* $\alpha_f(s) = o(s^{-d})$ *at 0 and* ∞, *i.e.*
$$\lim_{s \to 0_+} s(\alpha_f(s))^{\frac{1}{d}} = 0 = \lim_{s \to \infty} s(\alpha_f(s))^{\frac{1}{d}}. \tag{1.4}$$

Further, we obtain functions in the space $\mathcal{F}_d$ via a suitable interpolation of a Lebesgue and a weak-Lebesgue space. Indeed, in this thesis we show that the space $\mathcal{F}_d$ is an admissible class of weight functions for which (1.1) admits a positive principal eigenvalue, for an appropriate choice of d.

1.2 An Overview: the Laplacian

Here we focus on the linear weighted eigenvalue problem for the Laplacian. More precisely, we look for sufficient conditions on g for the existence of a positive principal eigenvalue for the following problem:
$$-\Delta u \quad = \quad \lambda g u, \quad u \in \mathcal{D}_0^{1,2}(\Omega), \tag{1.5}$$
where Ω is a non-empty connected open subset of $\mathbb{R}^N$ with $N \geq 3$ and $g \in L^1_{loc}(\Omega)$.

The case $p = 2$, is considered separately, mainly because of the simple and rich theory available for the Laplacian. Moreover, the results that hold for the p-Laplacian may hold for the Laplacian under weaker assumptions. The weighted eigenvalue problems for the Laplacian arises when one linearises semilinear equations $-\Delta u = \lambda f(x, u)$, which appear in the mathematical formulation of phenomena in physics, biology etc. In the linear stability analysis of the strictly positive steady states of some singular parabolic problem such as
$$\frac{\partial u^\beta}{\partial t} - \Delta u = f(x, u) \text{ in } \Omega,$$

with Dirichlet boundary condition boils down to weighted eigenvalue problem for the Laplacian. The above equation includes the so called porous media equation ([12]) as a particular case when $0 < \beta < 1$. Thus it is of relevance to identify a large class of weight functions for which (1.5) admits a principal eigenfunction and study its properties.

In the classical case, i.e, when $g \equiv 1$ and Ω is a bounded domain, the operator $(-\Delta)^{-1}$ defines a compact operator from $L^2(\Omega) \to L^2(\Omega)$. It is easy to see that the reciprocal of an eigenvalue of $(-\Delta)^{-1}$ gives an eigenvalue of (1.5). Moreover, using the maximum principle one can verify all the hypothesis of the Krein-Rutman theorem [18] to obtain a unique positive principal eigenvalue of (1.5) which is simple. Further, using the spectral theorem for the self adjoint compact operator one obtain the set of all eigenvalues of (1.5) and the corresponding normalized eigenfunctions form an orthonormal basis for $L^2(\Omega)$. Also by the Courant variational principle (see Theorem 6.3.14, [35]) the eigenvalues have the following expression:

$$\lambda_k = \inf_{\substack{u \perp \{u_1, \ldots, u_{k-1}\} \\ \|u\|_2 = 1}} \int_\Omega |\nabla u|^2, \ k = 1, 2, \ldots. \tag{1.6}$$

The existence of a principal eigenvalue for (1.5) and the fact that the associated eigenspace has dimension 1 are well known when $g \in C^0(\overline{\Omega})$ is positive on Ω (see [26]). If g has an indefinite sign with $g^+ \in L^r(\Omega) \backslash \{0\}$ with $r > \frac{N}{2}$ all these results can be extended thanks to the work of Manes and Micheletti [61]. In all these works, Rellich compactness theorem and Harnack's inequality were the the main ingredients for proving the existence of a positive principal eigenvalue; both of them fail when $r = \frac{N}{2}$. The development of weaker compactness criteria and a weak form the strong maximum principle (see [8, 19, 64]) allows one to consider the case $r = \frac{N}{2}$, for example see [58].

There are several sufficient conditions on the weight function g are available for the existence of a positive principal eigenvalue for (1.5) when Ω is an unbounded domain. For example when $\Omega = \mathbb{R}^N$, one such sufficient condition introduced by Brown, Cosner and Fleckinger in [21] is that the weight function g be smooth, negative and bounded away from 0 at infinity. The relaxation of the first condition, was introduced by Brown and Tertikas in [22], namely the positive part g^+ to be of compact support. Further improvement was given by Allegretto in [4], for $g^+ \in L^{\frac{N}{2}}(\Omega)$. All these sufficient conditions required either the weight g or its positive part g^+ to be in the Lebesgue space $L^{\frac{N}{2}}(\Omega)$. A sufficient condition beyond the Lebesgue spaces was obtained in [73] by Szulkin and Willem, by considering the problem (1.5) on a general domain for a weight g such that $g^+ \in L^{\frac{N}{2}}(\Omega)$ or having a faster decay than $|x|^{-2}$ at infinity and at any point in the domain.

The existence of a positive principal eigenvalue for (1.5) is closely related to the existence of a minimizer for the Rayleigh quotient associated to problem (1.5). Here we consider the following Rayleigh quotient

$$R(u) := \frac{\int_\Omega |\nabla u|^2}{\int_\Omega g\, u^2}, \tag{1.7}$$

with the domain of definition

$$\mathcal{D}^+(g) \;:=\; \left\{ u \in \mathcal{D}_0^{1,2}(\Omega) : \int_\Omega g u^2 > 0 \right\}.$$

Let

$$\lambda_1 = \inf_{u \in \mathcal{D}^+(g)} R(u). \tag{1.8}$$

Note that, if $\lambda_1 > 0$, then the following inequality holds:

$$\int_\Omega g\, u^2 \leq \frac{1}{\lambda_1} \int_\Omega |\nabla u|^2, \quad \forall\, u \in \mathcal{D}_0^{1,2}(\Omega). \tag{1.9}$$

The constant $\frac{1}{\lambda_1}$ is sharp. In addition, if λ_1 is attained at some $u \in \mathcal{D}^+(g)$, i.e. $\lambda_1 = R(u)$, then under certain integrability assumptions on g, one may be able to derive the following Euler-Lagrange equation for the minimizer:

$$\int_\Omega \nabla u \cdot \nabla v = \lambda_1 \int_\Omega g\, u\, v, \quad \forall\, v \in \mathcal{D}_0^{1,2}(\Omega). \tag{1.10}$$

Thus the question of existence of an eigenvalue of (1.5) boils down to the question of existence of a minimizer for R on $\mathcal{D}^+(g)$. Indeed, for the weights considered in [4, 22, 21, 26, 61] the best constant in the inequality (1.9) is attained.

Inequality (1.9) is known as *generalized Hardy-Sobolev inequality* in the literature. In order to enlarge the class of weight functions considered in the inequality (1.9) beyond the Lebesgue spaces, Visciglia [77] considered problem (1.5) with positive weights g lying in certain Lorentz spaces, a two parameter family of spaces introduced by Lorentz in [57] that generalizes the Lebesgue spaces. Indeed, he has shown that the generalized Hardy inequality (1.9) holds for each $g \in L(\frac{N}{2}, \infty)$. Following this direction Ramaswamy and Lucia in [60] proved the existence of a positive principal eigenvalue for (1.5), when g is such that

$$g \in \bigcup_{1 \le q < \infty} L\left(\frac{N}{2}, q\right), \qquad g^+ \not\equiv 0. \tag{1.11}$$

On the other hand the space $L(\frac{N}{2}, \infty)$ is still too large to ensure the existence of a minimizer, since the weight $g(x) = \frac{1}{|x|^2}$ belongs to the space $L(\frac{N}{2}, \infty)$ and we have the celebrated Hardy inequality

$$\int_\Omega \frac{1}{|x|^2} u^2 \le \left(\frac{2}{N-2}\right)^2 \int_\Omega |\nabla u|^2, \quad \forall\, u \in C_c^\infty(\Omega), \tag{1.12}$$

where Ω is an open set containing the origin. However, the best constant in the above inequality is not attained for any $u \in \mathcal{D}_0^{1,2}(\Omega)$ and hence (1.5) does not admit a positive principal eigenvalue. Our goal is to generalize and unify all the previous works by exhibiting a general class of admissible weights in a suitable subspace of Lorentz space $L(\frac{N}{2}, \infty)$ that ensures the existence of a positive principal eigenvalue for (1.5).

Before stating our results, we define the following:

$$\mathcal{M} := \left\{ u \in \mathcal{D}_0^{1,2}(\Omega) : \int_\Omega g u^2 = 1 \right\}, \tag{1.13}$$

$$J(u) := \frac{1}{2} \int_\Omega |\nabla u|^2. \tag{1.14}$$

Using the homogeneity of R, it is easy to see that the existence of a minimizer for R on $\mathcal{D}^+(g)$ is equivalent to the existence of a minimizer for J on M. Recall the definition of the following space,

$$\mathcal{F}_{\frac{N}{2}} = \textit{closure of } C_c^\infty(\Omega) \textit{ in the Lorentz space } L(\frac{N}{2}, \infty).$$

Now we state our results:

Theorem 1.2.1. *Let $N \ge 3$ and let $g \in L^1_{loc}(\Omega)$ such that $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Then*

$$\lambda_1 = \inf \{J(u) : u \in \mathcal{M}\} \tag{1.15}$$

is the unique positive principal eigenvalue of (1.5). *Furthermore, each eigenfunction corresponding to λ_1 is of constant sign and λ_1 is simple.*

Let us emphasise that the existence of an eigenvalue of (1.5) for more general positive weights is obtained in [74] using the concentration compactness lemma and in [68] using certain capacity conditions of Mazja [62]. However, their eigenfunctions satisfy (1.5) only in the sense of distributions and other qualitative properties of the first eigenvalue were obtained under certain additional assumptions on g. We use a standard variational method for the existence of an eigenvalue of (1.5) and under the same assumptions on the weight function g, certain qualitative properties of the first eigenvalue are also obtained.

We also obtain the existence of infinitely many eigenvalues of (1.5). Indeed, our results subsumes the results of [4, 73] which give sufficient conditions on g for the existence of infinitely many eigenvalues of (1.5).

Theorem 1.2.2. *Let $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Then (1.5) admits a sequence of positive eigenvalues going to ∞.*

In [63], Pohozaev proved an important identity for the solutions of the Dirichlet problem

$$-\Delta u + f(u) = 0, \quad \text{in } \Omega; \quad u = 0 \;\text{ on } \partial\Omega.$$

Using this identity he showed the nonexistence of non trivial solutions for the above problem, when Ω is a bounded star shaped domain and f is a continuous function on $\mathbb{R}$ satisfying

$$(N-2)tf(t) - 2NF(t) > 0, \qquad \text{for}\, t \neq 0,$$

where $F(t) = \int_0^t f(s)ds$ is the primitive of f. Thereafter many variants of Pohozaev type identities are proved and extensively used for proving the nonexistence of solution for more general semilinear partial differential equations on domains with certain geometry, see for example [42, 65] and the references therein. A Pohozaev type identity is used in [73, 74] for proving the nonexistence of an eigenvalue for (1.5). Here we obtain the same identity under some weaker assumptions.

Theorem 1.2.3. *Let $a \in L^1_{loc}(\mathbb{R}^N)$ and let $u \in \mathcal{D}^{1,2}_0(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$. Further, assume that $a(x)u^2$, $x \cdot \nabla\, a(x)u^2 \in L^1(\mathbb{R}^N)$. If u solves*

$$-\Delta u = a(x)u$$

in the weak sense, then

$$\int_{\mathbb{R}^N} \{\, x \cdot \nabla a(x) + \, 2\, a(x)\, \} u^2 = 0. \tag{1.16}$$

Indeed, this identity gives the nonexistence of an eigenvalue for (1.5) in $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ for certain class of weight functions.

1.3 An Overview: the p-Laplacian

A development parallel to that of the Laplacian was happening in the literature on the theory of weighted eigenvalue problem for the p-Laplacian in the recent years. However, due to its technical difficulty and the degenerate elliptic nature, the weighted eigenvalue problem for p-Laplacian, for $p \neq 2$, is considered separately from the Laplacian. Let us point out that unlike in the case of the Laplacian, a complete description of the set of all eigenvalues of p-Laplacian ($p \neq 2$) is widely open. The question of the discreteness and the countability of the set of all eigenvalues of the p-Laplacian is not known even in the simplest case: $g \equiv 1$ and Ω is a ball. In the linear stability analysis of a steady state of the evolution of non-Newtonian fluid motion leads to a singular weighted eigenvalue problem for the p-Laplacian. Although our results are true for certain more general quasilinear operators we restrict ourselves to p-Laplacian as its analysis is simpler and well developed. Also the p-Laplace operator is the prototype for the class of quasilinear operators in the divergence form.

In the case when Ω is bounded, Lindqvist in [56] proved the existence, the uniqueness and the simplicity of a positive principal eigenvalue for general p, when $g \equiv 1$. In [7], Anane has extended this result for g in $L^\infty(\Omega)$. Many authors have given sufficient conditions on g for the existence of a positive principal eigenvalue for (1.1), when $\Omega = \mathbb{R}^N$, for example Huang [45], Allegretto and Huang [5], Cuesta [27], Fleckinger et al. [39], studied the problem (1.1) for a general p. In all these works the sufficient conditions required that either g or g^+ is in $L^{\frac{N}{p}}(\mathbb{R}^N)$. In [73], Willem and Szulkin enlarged the class of weight functions beyond the Lebesgue space $L^{\frac{N}{p}}(\Omega)$ by proving the existence of an eigenvalue of (1.1), for the weights whose positive part has a faster decay than $\frac{1}{|x|^p}$ at infinity and at all the points in the domain.

In this thesis, we unify the sufficient conditions given in [5, 39, 45, 73] by proving a result analogous to Theorem 1.2.1. More precisely, we consider a suitable subspace of the Lorentz space $L(\frac{N}{p}, \infty)$ that serve as an admissible class of weight functions that admits positive principal eigenvalue for (1.1).

As before, for the existence of an eigenvalue, we use a direct variational principle by considering the following Rayleigh quotient

$$R_p(u) := \frac{\int_\Omega |\nabla u|^p}{\int_\Omega g|u|^p},$$

with the domain of definition

$$\mathcal{D}_p^+(g) := \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p > 0 \right\}. \tag{1.17}$$

Let

$$\begin{aligned} \mathcal{M}_p &:= \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p = 1 \right\}, \\ J_p(u) &:= \frac{1}{p}\int_\Omega |\nabla u|^p. \end{aligned}$$

If R_p is $\mathcal{C}^1$, then we arrive at equation (1.2) as the Euler-Lagrange equation corresponding to a critical point of R_p on $\mathcal{D}_p^+(g)$, with the critical values as the eigenvalues of (1.1). Moreover, there is a one-to-one correspondence between the critical points of R over $\mathcal{D}^+(g)$ and the critical points of J_p over M_p. Thus we are interested in the existence of critical points of J_p on $\mathcal{M}_p$ including the minimizer. Here we give sufficient conditions on g^+ for the existence of a minimizer. As in the case of $p = 2$, we consider the following space (see Subsection 1.1.2):

$$\mathcal{F}_{\frac{N}{p}} = \text{ closure of } \mathcal{C}_c^\infty(\Omega) \text{ in } L\left(\frac{N}{p}, \infty\right).$$

Now we state our result:

Theorem 1.3.1. *Let Ω be an open connected subset of $\mathbb{R}^N$. Let $p \in (1, N)$ and let $g \in L^1_{loc}(\Omega)$ such that $g^+ \in \mathcal{F}_{\frac{N}{p}} \setminus \{0\}$. Then*

$$\lambda_1 = \inf\{J_p(u) : u \in \mathcal{M}_p\} \tag{1.18}$$

is the unique positive principal eigenvalue of (1.1)*. Furthermore, all the eigenfunctions corresponding to λ_1 are of constant sign and λ_1 is simple.*

The main difficulty for proving different properties of λ_1 is the lack of regularity of the eigenfunctions due to the weak assumptions on the weight g. As far as we know, in general the eigenfunctions are only in $W^{1,p}_{loc}(\Omega)$. Hence the classical tools for proving the simplicity of λ_1 are not applicable here.

There are several works devoted to study the simplicity of the first eigenvalue, for example [7, 14, 50, 58, 59]. In [50], Kawohl, Lucia and Prashanth proved the simplicity of λ_1, if it exists, even for g in $L^1_{loc}(\Omega)$. Their result is well applicable in our case, once the existence of first eigenvalue is proved.

The study of the symmetries and the monotonicity properties of the solutions of certain partial differential equations has a long history of its own, see [48, 49] for an excellent exposition on this topic. In [17], Bhattacharya proved the radial symmetry of the first eigenfunctions of (1.1), when Ω is a ball and $g \equiv 1$. Thus our result is a two fold generalization of results of Bhattacharya, as we allow more general weight functions and the domain can be $\mathbb{R}^N$. Our result uses certain rearrangement inequalities. It is worth to mention that we are not assuming any condition on g that ensures λ_1 is an eigenvalue of (1.1).

Theorem 1.3.2. *Let Ω be $\mathbb{R}^N$ or a ball centred at the origin. Let g be nonnegative, radial and radially decreasing measurable function. If λ_1 is an eigenvalue of* (1.1)*, then any positive eigenfunction corresponding to λ_1 is radial and radially decreasing.*

The symmetries of the eigenfunctions corresponding to higher eigenvalues are open, even in the case of the Laplacian when $g \equiv 1$ and Ω is the unit ball. It is proved recently that the second eigenfunctions of p-Laplacian on a disc (2-dimension) are not radial by Drabek et. al. using the interval arithmetic and the computer simulations [15].

As we mentioned before, a complete picture of the set of all eigenvalues for the p-Laplacian is not known. Nevertheless, there are several methods to exhibit an infinite sequence of eigenvalues that goes to infinity. In [13], Azorero and Alonso identified infinitely many eigenvalues of (1.1) for $p \neq 2$ and $g \equiv 1$, using the Ljusternik-Schnirelmann type minmax theorem, which is a nonlinear analogue of the Courant principle. For $g \not\equiv 1$, the existence of infinitely many eigenvalues is obtained in [5, 45, 73], again by using the Ljusternik-Schnirelmann type minimax theorem. Here we obtain the existence of infinitely many eigenvalues for (1.1) under weaker integrability assumptions on g.

Theorem 1.3.3. *Let Ω be a domain in $\mathbb{R}^N$ with $N > p$. Let $g \in L^1_{loc}(\Omega)$ such that $g^+ \in \mathcal{F}_{\frac{N}{p}} \setminus \{0\}$. Then* (1.1) *admits a sequence of positive eigenvalues going to ∞.*

The classical Ljusternik-Schnirelmann minimax theorem uses a deformation homotopy which requires the set $\mathcal{M}_p$ to be at least a $\mathcal{C}^{1,1}$ manifold (i.e, transition maps are $\mathcal{C}^1$ and its derivatives are locally Lipschitz). However, a weaker version of the Ljusternik-Schnirelmann theorem is available due to Szulkin. In [72], he developed the Ljusternik-Schnirelmann theorem on $\mathcal{C}^1$ manifold using the Ekeland variational principle. Due to the weaker assumptions on the weight g, the set $\mathcal{M}_p$ may not even possess a manifold structure from the topology inherited from $\mathcal{D}_0^{1,p}(\Omega)$. However, we define a suitable topology on $\mathcal{M}_p$ that makes $\mathcal{M}_p$ a $\mathcal{C}^1$ Banach manifold and use a result of [72] to get an infinite sequence of eigenvalues that goes to infinity.

As in the case of the Laplacian, here we prove a necessary condition, a Pohozaev type identity, for the existence of an eigenvalue for (1.1) for a certain class of weights. A Pohozaev type identity is often used for proving the nonexistence of solutions for certain classes of partial differential equations on certain types of domains. Here we prove an identity that rules out the possibility of having a positive eigenvalues for (1.1) for certain classes of weight functions. In contrast to the linear case, we need certain additional regularity assumptions on a in the case of p-Laplacian, with $p \neq 2$.

Theorem 1.3.4. *Let $a \in \mathcal{C}^{\alpha}_{loc}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$ and let $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$. Further, assume that $a(x)|u|^p$, $x \cdot \nabla\, a(x)u^p \in L^1(\mathbb{R}^N)$. If u solves*

$$-\,\Delta_p u = a(x)|u|^{p-2}u \tag{1.19}$$

in the weak sense, then the following identity holds:

$$\int_{\mathbb{R}^N} \{\, x \cdot \nabla a(x) + \, p\, a(x)\,\}|u|^p = 0.$$

The standard calculations to derive a Pohozaev type identity requires the existence of the second derivative of the solution at all points. However, all the regularity theorems available in the literature guarantee that the weak solutions of (1.19) are only in $\mathcal{C}^{1,\alpha}_{loc}(\mathbb{R}^N)$, for some $\alpha \in (0,1)$. Due to this weak regularity of the solutions of (1.19), we use a weak form of divergence theorem of Cuesta and Takac [28] for deriving the identity (1.3.4).

1.4 Existence of solution branches for nonlinear PDEs

It is of great importance to study the existence of solutions for a nonlinear equation depending on a real parameter as below:

$$F(x, \lambda) = 0,$$

where x is in a suitable Banach space. In applications, most of the equations of the above form, admit the zero solution for each λ and a new solution, bifurcating from the zero solution may appear for certain values of λ. The existence of such a bifurcating branch without further structural conditions on F is too general to be answered affirmatively. Depending on the structure of the nonlinearity, there are several tools such as the implicit function theorem, Lyapunov-Schmidt method, the Morse's theory, the perturbation method and the Rabinowitz theorem to obtain a bifurcation branch of nontrivial solution for the above equation, see Chapter 4 of [52] for a detailed discussion on this topic.

We consider the existence of a nontrivial solution branch for the following types of partial differential equation:

$$-\Delta_p u \quad = \quad \lambda g\, f(u) + h\, r(u), \quad u \in \mathcal{D}_0^{1,p}(\Omega), \tag{1.20}$$

under suitable assumptions on a, b, f and r. In a suitable functional frame work, the above equation can be converted into an equation of the form $F(x, \lambda) = 0$.

1.4.1 Solution branches via bifurcation

First we consider the case: $f(u) = u$ and $p = 2$. More precisely, we have the following semilinear equation:

$$-\Delta u \quad = \quad \lambda g\, u + h\, r(u), \quad u \in \mathcal{D}_0^{1,2}(\Omega). \tag{1.21}$$

Further, we may assume that $r(0) = 0$, so that $(\lambda, 0)$ will be a trivial solution branch of the above equation. The existence of a bifurcating solution branches is studied by many authors under various regularity assumptions on the weights g and h with certain growth assumption on the nonlinearity r. See for example, [31, 34, 60] and the references therein. The main novelty of our results is that we allow the weight functions to be in weak Lebesgue spaces. More precisely, we make the following assumptions:

$$\textbf{(H1)} \begin{cases} r \in \mathcal{C}(\mathbb{R}), \quad |r(s)| \le C|s|^{\gamma-1}, \text{ for } \gamma \in [1, 2^*) \text{ and } C > 0. \\ \lim_{|s|\to 0} \frac{|r(s)|}{|s|} = 0, \text{ if } 1 \le \gamma \le 2. \end{cases}$$

$$\textbf{(H2)} \begin{cases} g \in \mathcal{F}_{\frac{N}{2}}, \quad g^+ \neq 0, \\ \\ h \in \begin{cases} \mathcal{F}_{\tilde{\gamma}} \text{ if } \gamma \geq 2, \quad \text{where } \frac{1}{\tilde{\gamma}} + \frac{\gamma}{2^*} = 1, \\ \mathcal{F}_{\frac{N}{2}} \text{ if } 1 \leq \gamma < 2. \end{cases} \end{cases}$$

For g, h and r satisfying $(H1)$ and $(H2)$, we define the following

$$\mathcal{S} := \big\{(\lambda, u) \in \mathbb{R} \times \mathcal{D}_0^{1,2}(\Omega) \,:\, (\lambda, u) \text{ solves } (1.21),\, u \not\equiv 0\big\}.$$

Further, we set

$$\sigma(g) := \big\{\lambda : \lambda \text{ is an eigenvalue of } (1.5)\big\}.$$

Indeed, by Theorem 1.2.1, the set $\sigma(g)$ is nonempty and λ_1 is a simple positive principal eigenvalue of (1.5). Further, we adapt the Rabinowitz bifurcation theorem to obtain the following result:

Theorem 1.4.1. *Assume (H1), (H2) and* $g^+ \not\equiv 0$. *Then, there exists a connected branch* $\mathcal{C}^+$ *in* $\overline{\mathcal{S}}^{\mathbb{R}\times\mathcal{D}_0^{1,2}(\Omega)}$ *bifurcating from* $(\lambda_1, 0)$. *Moreover,*

$$\begin{aligned} &(i) \text{ either } \mathcal{C}^+ \text{ is unbounded in } \mathbb{R} \times \mathcal{D}_0^{1,2}(\Omega), \\ &(ii) \text{ or } (\lambda, 0) \in \mathcal{C}^+ \text{ with } \lambda_1 \neq \lambda \in \sigma(g). \end{aligned}$$

1.4.2 Positive solution branches

We now consider the following problem:

$$-\Delta u \;=\; \lambda a\, f(u), \quad u \in \mathcal{D}_0^{1,2}(\Omega). \tag{1.22}$$

with $f(0) \neq 0$ and $a \in L^1_{loc}(\Omega)$.

In this case, we obtain a positive solution branch for (1.22) using a direct application of the implicit function theorem. Note that (1.22) is not included in the previous case as $(\lambda, 0)$ is no longer a solution branch for the above equation. There are several sufficient conditions available in the literature for the existence of a positive solution branch for (1.22) , when Ω is bounded domain, see [23, 24, 43] and the references therein.

In [3], Afrouzi and Brown established the existence of a positive solution for λ small, using the implicit function theorem for smooth f. They related the existence of positive solution of (1.22) to the positivity of the solution

v of the linearised problem

$$\begin{aligned} -\Delta v &= f(0)\, a \quad \text{ in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.23}$$

To the best of our knowledge, there are no results available for the existence of positive solutions for (1.20) in $\mathbb{R}^N$ in the case when $f(0) \neq 0$. For $N \geq 3$, in the spirit of [3], we obtain the existence of a positive weak solution for the following problem:

$$-\Delta u = \lambda\, a\, f(u), \quad u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N), \tag{1.24}$$

for small positive λ.

We assume the following:

(A1) $f \in \mathcal{C}^1(\mathbb{R})$ such that $f(0) \neq 0$ and there exists $s_0 > 0$ such that

$$|f'(s)| \leq C|s|^{\gamma-2}, \ \forall\, |s| \geq s_0 \text{ with } \gamma \in [2, 2^*).$$

(A2) $a \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \{\widetilde{\gamma}, 2\}$, where $\widetilde{\gamma}$ is the conjugate exponent of $(\frac{2^*}{\gamma})$.

The main novelty of our hypotheses is that the weight function a need not be smooth and the function f not necessarily be bounded, but we demand that a lies in certain Lebesgue spaces and f is $\mathcal{C}^1(\mathbb{R})$ with the subcritical growth at infinity.

For $\varepsilon \geq 0$, let v_ε be a weak solution of the following problem:

$$-\Delta v = f(0)(a - \epsilon\, a^-). \tag{1.25}$$

Now we state our results:

Theorem 1.4.2. *Let (A1)-(A2) hold. Furthermore, one of the following assumptions holds:*

(A3) v_0 *is positive in* $\mathbb{R}^N$ *and* $f(0)\, a \geq 0$ *a.e. near infinity.*

(A4) v_ε *is positive in* $\mathbb{R}^N$, *for some* $\varepsilon > 0$.

Then there exists a $\lambda_0 > 0$ *such that* (1.24) *has a positive weak solution* $u_\lambda \in D_0^{1,2}(\mathbb{R}^N)$ *for* $0 < \lambda < \lambda_0$.

1.4.3 Solution branches for quasilinear equations

Now we consider the case $p \neq 2$. More precisely, we consider the following equation:

$$-\Delta_p u \quad = \quad \lambda\, g\, |u|^{p-2} u + \lambda\, f\, r(u), \qquad \text{in } \mathcal{D}_0^{1,p}(\Omega), \tag{1.26}$$

where Ω is an open connected subset of $\mathbb{R}^N$ with $1{<}p{<}N$ and $g, f \in L^1_{loc}(\Omega)$. Further, we assume that $r \in \mathcal{C}(\mathbb{R})$ and $r(0) = 0$.

There are several results available in the literature on the existence of a solution branch of (1.26) that bifurcate from the trivial solution branch, for example, see [31, 34]. In all these earlier works, authors assumed that f and g are bounded and lie in $L^{\frac{N}{p}}(\Omega)$. Here we allow f, g to be in certain weak Lebesgue spaces and they are not assumed to be bounded. We make the following assumptions on the functions r, g and f,:

$$\text{(B1)} \quad \begin{cases} r \in \mathcal{C}(\mathbb{R}), \quad |r(s)| \leq |s|^{\gamma - 1}, \quad \gamma \in [1, p^*), \text{ where } p^* = \frac{Np}{N-p}, \\ \\ \lim_{|s| \to 0} \frac{|r(s)|}{|s|^{p-1}} = 0, \text{ if } 1 \leq \gamma \leq p. \end{cases}$$

$$\text{(B2)} \quad \begin{cases} g \in \mathcal{F}_{\frac{N}{p}}, \quad g^+ \neq 0, \\ \\ f \in \begin{cases} \mathcal{F}_{\widetilde{p}} \text{ if } \gamma \geq p, \quad \text{where } \frac{1}{\widetilde{p}} + \frac{1}{p^*} = 1, \\ \mathcal{F}_{\frac{N}{p}} \text{ if } 1 \leq \gamma < p. \end{cases} \end{cases}$$

We use a topological degree argument as in [34] for proving the existence of a solution branch of (1.26) bifurcating from the trivial branch of zero solutions. Leray and Schauder extended the finite dimensional degree theory to the infinite dimensional Banach spaces, by defining the degree for the compact perturbations of the identity; see [35, 52] for the definition and important properties of the Leray-Schauder degree. In order to study the bifurcation property of (1.26), using the degree theory, one needs to extend the definition of the degree for the maps between a Banach space and its dual. Indeed, the degree is defined for certain classes of maps from a Banach space to its dual, see [71] for more details on this topic.

Now we state our main result in this section.

Theorem 1.4.3. *Let Ω be an open connected subset of $\mathbb{R}^N$ with $1 < p < N$. Let r, f and g satisfy $(B1), (B2)$. Then the first eigenvalue λ_1 of* (1.1) *is a bifurcation point of* (1.26).

1.5 An overview: the Laplacian in dimension 2

Now we study the existence of a positive principal eigenvalue for the following weighted eigenvalue problem:

$$-\Delta u = \lambda\, g\, u, \qquad \text{in } \Omega, \tag{1.27}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.28}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and $g \in L^1_{loc}(\Omega)$.

The case $N = 2$, needs a separate consideration as most of the results that we obtained in higher dimension do not hold good in this case. More importantly, in [21], Brown et al proved the nonexistence of a positive principal eigenvalue for (1.27) in $\mathbb{R}^2$, even for g such that $\int_{\mathbb{R}^2} g > 0$. This completely rules out the possibility of obtaining a sufficient condition for the existence, namely a condition on g^+ alone. Further, the natural space $\mathcal{D}^{1,2}_0(\Omega)$, associated to the the weighted eigenvalue problem (1.27), may not even be a function space when Ω is unbounded. Thus, we restrict ourselves to bounded domains in $\mathbb{R}^2$ and consider the space $H^1_0(\Omega)$ as the solution space.

Our first sufficient condition make use of the embedding of $H^1_0(\Omega)$ into the Orlicz space generated by $A(t) = exp(t^2) - 1$, due to Moser and Trudinger (see [2]), which is an optimal embedding in the class of Orlicz spaces. Let $L_{\widetilde{B}}(\Omega)$ be the Orlicz space generated by the following Orlicz function:

$$\widetilde{B}(s) = \begin{cases} 0 \ \text{ if } 0 \le s \le 1, \\ \\ s \log s - s + 1 \text{ if } \ s > 1. \end{cases} \tag{1.29}$$

Now we have the following result:

Theorem 1.5.1. *Let Ω be a bounded domain in $\mathbb{R}^2$. Let $g \in L^1_{loc}(\Omega)$ be such that $g^+ \in L_{\widetilde{B}}(\Omega) \setminus \{0\}$. Then*

$$\lambda_1 = \inf \{J(u) : u \in M\} \tag{1.30}$$

is the unique positive principal eigenvalue of (1.27). *Furthermore, all the eigenfunctions corresponding to λ_1 are of the constant sign and λ_1 is simple.*

In Chapter 4, we use the optimal embedding of $\mathcal{D}^{1,2}_0(\Omega)$ in the class of Lorentz spaces that refines the usual Sobolev embedding. Analogously, for the bounded domains in $\mathbb{R}^2$, a refinement of the usual Moser-Trudinger embedding of $H^1_0(\Omega)$ is available due to Hansson [44]. Furthermore, this embedding is optimal in the class of Lorentz-Zygmund spaces. Here we also obtain another sufficient condition on g that relies on the Hansson's embedding.

$$X_3 := \left\{ u \text{ measurable} : \sup_{t>0} t\,(1 + |log(t)|)^2\, u^*(t) < \infty \right\}.$$

Motivated by the space $F_d, d > 1$ we consider the following space

$$\mathcal{F}_1 = \overline{\mathcal{C}_c^\infty(\Omega)} \text{ in } X_3.$$

Now we have the following theorem:

Theorem 1.5.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ and let $g \in L^1_{loc}(\Omega)$. If $g^+ \in \mathcal{F}_1$, then*

$$\lambda_1 = \inf \{J(u) : u \in M\} \tag{1.31}$$

is the unique positive principal eigenvalue of (1.27)*. Furthermore, all the eigenfunctions corresponding to λ_1 are of the constant sign and λ_1 is simple.*

The above theorem is applicable for certain weights that are not in the Orlicz space $L_{\widetilde{B}}(\Omega)$.

Chapter 2

Preliminaries

2.1 Symmetrization

In this section we define two types of rearrangements for a measurable function and discuss some of their properties. Further, we present some integral inequalities concerning rearrangements. For further details on this topic, we refer to the books [36, 53, 54].

Let $\Omega \subset \mathbb{R}^N$ be a domain. Let $\mathcal{M}_0(\Omega)$ denote the set of all extended real valued Lebesgue measurable functions that are finite a.e. in Ω. Let $f \in M_0(\Omega)$ and let $E_f(s) = \{x : |f(x)| > s\}, s > 0$. Now we define the *distribution function* α_f of f as follows:

$$\alpha_f(s) \quad := \quad \big|E_f(s)\big|, \;\; \text{for } s > 0, \tag{2.1}$$

where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$.

Some elementary properties of α_f are summarized in the following proposition, see Proposition 3.2.2 of [36] for a proof.

Proposition 2.1.1. *Let $f \in \mathcal{M}_0(\Omega)$ and let α_f be its distribution function. Then:*

(a) α_f is nonnegative and decreasing.

(b) α_f is right continuous.

(c) If $|f| \leq |g|$ a.e, then $\alpha_f \leq \alpha_g$.

(d) $\alpha_{af}(s) = \alpha_f(a^{-1}s),\ a \in \mathbb{R}_+$.

(e) $\alpha_{f+g}(s_1 + s_2) \leq \alpha_f(s_1) + \alpha_g(s_2)$.

Now we define the *decreasing rearrangement* f^* of f, using its distribution function α_f.

$$f^*(t) \quad := \quad \begin{cases} \inf\{s > 0 \,:\, \alpha_f(s) \le t\}, & t > 0, \\ \text{ess sup}|f|, & t = 0. \end{cases}$$

Remark 2.1.2. *Since α_f is decreasing and right continuous, using the 'infimum property', one can verify that*

$$f^*(t) = \sup\{s > 0 : \alpha_f(s) > t\} = |E_{\alpha_f}(t)|, \quad \textit{for } t > 0.$$

Thus, f^* is simply the distribution function of α_f.

In the next proposition we give the distribution function and the rearrangement of some elementary functions that we often use.

Proposition 2.1.3. *(i) Let $E \subset \mathbb{R}^N$ be measurable. Then*

$$\alpha_{\chi_E}(s) = |E|\,\chi_{[\,0,1\,)}(s), \quad {\chi_E}^*(t) = \chi_{[\,0,|E|\,)}(t).$$

(ii) Let $f = \sum_i^m c_i \chi_{E_i}$ with E_is are disjoint measurable subsets of $\mathbb{R}^N$. Then

$$\alpha_f(s) = \sum_i^m |E_i|\,\chi_{[\,0,|c_i|\,)}(s), \quad f^*(t) = \sum_i^m |c_i| \chi_{[0,|E_i|)}(t).$$

(iii) Let $d > 0$ and let $h(x) = |x|^{-d}$, $x \in \mathbb{R}^N$. Then for $s, t \in \mathbb{R}^+$,

$$\alpha_h(s) \quad = \quad \omega_N\, s^{-\frac{N}{d}}; \qquad h^*(t) = {\omega_N}^{\frac{d}{N}}\, t^{-\frac{d}{N}}, \tag{2.2}$$

where ω_N is the Lebesgue measure of the unit ball in $\mathbb{R}^N$.

(iv) More generally, let Φ be a strictly decreasing continuous function on $\mathbb{R}^+$ and let $f(x) = \Phi(|x|)$, $x \in \mathbb{R}^N$. Then

$$\begin{aligned} \alpha_f(s) \quad &= \quad \omega_N \left[\Phi^{-1}(s)\right]^N, \; s \in \text{Range}(\Phi); \\ f^*(t) \quad &= \quad \Phi\left(\left[\frac{t}{\omega_N}\right]^{\frac{1}{N}}\right), \; t \in \mathbb{R}^+. \end{aligned}$$

Next we summarize some elementary properties of f^* in the following proposition. For a proof, see Proposition 3.2.4 of [36].

Proposition 2.1.4. *Let $\Omega \subset \mathbb{R}^N$ be a domain and let $f, g \in \mathcal{M}_0(\Omega)$. Then,*

(a) f^* *is nonnegative and decreasing;*

(b) f^* *is right continuous;*

(c) if $|f| \leq |g|$ *a.e, then* $f^* \leq g^*$;

(d) $(af)^*(t) = |a| f^*(t)$;

(e) $f^*(t_1 + t_2) \leq f^*(t_1) + f^*(t_2)$.

In view of statements (e) of both Proposition 2.1.1 and Proposition 2.1.4, one may ask, do $\alpha_f(s)$ and $f^*(t)$ satisfy the subadditivity property with respect to f. More precisely, do the following inequalities hold?

$$\begin{array}{rcl} \alpha_{f+g}(s) & \leq & \alpha_f(s) + \alpha_g(s) \\ (f+g)^*(t) & \leq & f^*(t) + g^*(t) \end{array}$$

By taking $f = \chi_{[0,1)}$ and $g = \chi_{[1,2]}$ one can easily see that the answer to the above question is negative.

Next we give some inequalities connecting the distribution function and rearrangement of functions in $\mathcal{M}_0(\Omega)$.

Proposition 2.1.5. *Let $\Omega \subset \mathbb{R}^N$ be a domain and let $f, g \in \mathcal{M}_0(\Omega)$. Then,*

(a) if $s < f^*(t)$, *then* $t < \alpha_f(s)$;

(b) if $t < \alpha_f(s)$, *then* $s \leq f^*(t)$;

(c) $f^*(\alpha_f(s)) \leq s$, $\alpha_f(f^*(t)) \leq t$;

(d) $(f+g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2)$;

(e) $t < \alpha_f(s)$ *if and only if* $s < f^*(t)$;

(f) let $c, s, t > 0$ *such that* $c = st^{\frac{1}{p}}$, *then*

$$t^{\frac{1}{p}} f^*(t) \leq c \quad \text{if and only if} \quad s(\alpha_f(s))^{\frac{1}{p}} \leq c. \tag{2.3}$$

Proof. For a proof of (a)-(e), we refer to [36]. By taking $s = ct^{\frac{-1}{p}}$ in (e) we deduce that

$$t^{\frac{1}{p}} f^*(t) \leq c \quad \text{if and only if} \quad \alpha_f(s) \leq t.$$

Since $t = (\frac{c}{s})^p$, we get

$$\alpha_f(s) \le t \quad \text{if and only if} \quad s(\alpha_f(s))^{\frac{1}{p}} \le c.$$

This proves (f). $\square$

In the following proposition we discuss the integrability of a function and its rearrangement, see [36] for a proof.

Proposition 2.1.6. *Let $\Omega \subset \mathbb{R}^N$ be a domain and let $f \in \mathcal{M}_0(\Omega)$. Then,*

(a) f and f^* are equimeasurable. i.e, $\alpha_f(s) = \alpha_{f^*}(s)$, $\forall s > 0$;

(b) if $f \in L^p(\Omega)$ then $\|f\|_p^p = \int_0^{|\Omega|} p\, s^{p-1} \alpha_f(s)\, ds$, for $1 \le p < \infty$;

(c) if $f \in L^p(\Omega)$ then $f^* \in L^p(0, |\Omega|)$ and $\|f\|_p = \|f^*\|_p$, for $1 \le p \le \infty$.

We have seen that both α_f anf f^* are not subadditive with respect to f. However we obtain a subadditive function from f^*, namely the maximal function f^{**} of f^* defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(\tau)\, d\tau, \quad t > 0, \tag{2.4}$$

The subadditivity of f^{**} with respect to f helps us to define norms in certain function spaces. Some important properties of f^{**} are listed in the following proposition:

Proposition 2.1.7. *Let $\Omega \subset \mathbb{R}^N$ be a domain and let $f, g \in \mathcal{M}_0(\Omega)$. Then,*

(a) $f^* \le f^{**}$;

(b) if $|f| < |g|$, then $f^{**} \le g^{**}$;

(c) $(af)^{**}(t) = |a| f^{**}(t)$;

(d) Subadditivity:

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad \forall\, t > 0.$$

Properties (a), (b) and (c) are immediate from the definition of f^{**}. For a proof of (d), see Theorem 3.4 of [16].

Next we define the Schwarz symmetrization of a measurable function.

Definition 2.1.8. *Let $A \subset \mathbb{R}^N$ be a measurable set. We define A_*, the symmetric rearrangement of the set A, to be the open ball centred at origin whose volume is that of A. Thus*

$$A_* = \{x : |x| < r\}, \qquad \text{with } \omega_N\, r^N = |A|.$$

If A is of infinite measure, then we take $A_* = \mathbb{R}^N$.

Definition 2.1.9. *We define the symmetric rearrangement of the characteristic function χ_A of a measurable set A of finite measure as χ_{A_*}. For a measurable function f on Ω, we define the symmetric decreasing rearrangement, f_* on Ω_*, as*

$$f_*(x) = \int_0^\infty \chi_{\{|f|>s\}_*}(x)\, ds.$$

In the next proposition we give a relation between f^* and f_*, which is very useful for constructing functions which are invariant under the Schwarz symmetrization, using functions of one variable.

Proposition 2.1.10. *Let $f : \Omega \to \mathbb{R}$ be a measurable function and f_* be the symmetric decreasing rearrangement of f. Then*

$$f_*(x) = f^*(\omega_N |x|^N). \tag{2.5}$$

Proof. For $x \in \Omega_*$, observe that

$$f_*(x) = |\{s : \chi_{\{|f|>s\}_*}(x) = 1\}| = |\{s : \alpha_f(s) > \omega_N |x|^N\}|.$$

Further, from Remark (2.1.2), we have

$$|\{s : \alpha_f(s) > \omega_N |x|^N\}| = f^*(\omega_N |x|^N).$$

□

As a consequence of the previous lemma one can see that f_* is nonnegative, lower semi-continuous, radial and radially decreasing. In the next proposition we list a few important properties f_* that are used in the subsequent chapters.

Proposition 2.1.11. *Let $f \in \mathcal{M}_0(\Omega)$.*

(a) For all $s > 0$, we have $\{f_* > s\} = \{|f| > s\}_*$. Thus f and f_* are equimeasurable. i.e, $\alpha_f(s) = \alpha_{f_*}(s)$.

(b) Let $\Omega = \Omega_*$ and let f be nonnegative, radial and radially decreasing. Then $f = f_*$ a.e.

(c) Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a positive Borel measurable function. Then

$$\int_{\mathbb{R}^N} F(f_*(x)) = \int_{\mathbb{R}^N} F(|f(x)|).$$

(d) Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non decreasing positive Borel measurable functions. Then

$$(\Phi \circ |f|)_* = \Phi \circ f_* \, a.e\,.$$

Proof. For a proof of (a), (c) and (d) we refer to [36]. Here we give a proof for (b).
Since f is radial and radially decreasing, we have

$$\alpha_f(f(x_0)) = |\{f > f(x_0)\}| \leq w_N |x_0|^N.$$

Now using property (a) of Proposition 2.1.5 and the equation (2.5) we get

$$f_*(x_0) \leq f(x_0). \tag{2.6}$$

Let x_0 be a point of continuity of f and let $\varepsilon > 0$ be given. Thus there exists y such that $|y| > |x_0|$ and $f(y) > f(x_0) - \varepsilon$. Since f is radially decreasing one can see that,

$$\alpha_f(f(x_0) - \varepsilon) = |\{\, x : f(x) > f(x_0) - \varepsilon\}| \geq w_N |y|^N > w_N |x_0|^N.$$

Now by taking $s = f(x_0) - \varepsilon$ and $t = w_N |x_0|^N$ in property (b) of Proposition 2.1.5 we have

$$f_*(x_0) \geq f(x_0) - \varepsilon. \tag{2.7}$$

Since ε is arbitrary, by using (2.6) and (2.7) we obtain $f(x_0) = f_*(x_0)$ at the point of continuity of f. Now the result follows as the set of points of discontinuity of a radially decreasing radial function is of measure zero. $\square$

The following corollary is immediate from the above proposition.

Corollary 2.1.12. *Let $f \in \mathcal{M}_0(\Omega)$. Then the following identities hold*

(a) $f^*(t) = (f_*)^*(t)$.

(b) $\int_\Omega |f(x)|^p = \int_{\Omega_*} |f_*(x)|^p$, *for* $p > 0$.

(c) $(|f|^p)_* = (f_*)^p$, *a.e for* $p > 0$.

Proof. (a) is obvious as f and f_* are equimeasurable, (b) and (c) follows by taking $F(t) = |t|^p$ in (c) and (d) of Proposition 2.1.11 respectively. $\square$

Finally, we recall an important inequality concerning the Schwarz symmetrization, see Theorem 3.2.10 of [36].

Theorem 2.1.13. *(Hardy-Littlewood inequality) Let f, $g \in \mathcal{M}_0(\Omega)$ and let f_* and g_* be their symmetric-decreasing rearrangements. Then*

$$\int_{\mathbb{R}^N} |f(x)g(x)|dx \leq \int_{\mathbb{R}^N} f_*(x)g_*(x)\,dx.$$

2.2 Lorentz spaces

In this section we define Lorentz spaces and discuss some of their properties. These spaces have been first discussed by Lorentz himself in[57]. For more details on Lorentz spaces, we refer to the books [2, 36, 41] and the article [46].

Let Ω be a domain in $\mathbb{R}^N$. Given a function $f \in \mathcal{M}_0(\Omega)$ and $p, q \in [1, \infty]$ we set

$$\|f\|_{(p,q)} := \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right\|_{q\,;(0,\infty)} = \begin{cases} \left(\displaystyle\int_0^\infty t^{\frac{q}{p}} f^*(t)^q \frac{dt}{t} \right)^{\frac{1}{q}}; \ 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t); \ q = \infty. \end{cases}$$

The Lorentz space $L(p,q)(\Omega)$ is defined as

$$L(p,q)(\Omega) := \left\{ f \in \mathcal{M}_0(\Omega) : \|f\|_{(p,q)} < \infty \right\}.$$

For the notational convenience we denote the Lorentz space $L(p,q)(\Omega)$ by $L(p,q)$, the domain under consideration will be evident from the context. The index p in $L(p,q)$ is called principal index and the index q is called secondary index.

Remark 2.2.1. *(a) For $1 \leq p \leq \infty$, $L(p,p) = L^p$ and $\|f\|_{(p,p)} = \|f\|_p$; this follows from Proposition 2.1.6-(c).*

(b) $L(\infty, q) = \{0\}$ when $q \in [1, \infty)$; this is immediate from the following inequality

$$\|f\|^q_{(\infty,q)} \geq \int_0^{t_0} \frac{f^*(t)^q}{t}\,dt \geq f^*(t_0)^q \int_0^{t_0} \frac{1}{t}\,dt, \quad \forall\, t_0 > 0.$$

□

Note that when $q=\infty$,

$$\|f\|_{(p,\infty)} = \sup_{t>0} t^{\frac{1}{p}} f^*(t).$$

Recall that for $p>1$, the weak-L^p space is defined as

$$\text{weak-}L^p := \left\{ f : \sup_{s>0} s(\alpha_f(s))^{\frac{1}{p}} < \infty \right\}.$$

Next lemma shows that the Lorentz space $L(p,\infty)$ coincides with the weak-L^p space.

Lemma 2.2.2. *Let $f \in \mathcal{M}_0(\Omega)$. Then, for $p>1$,*

$$\sup_{t>0} t^{\frac{1}{p}} f^*(t) = \sup_{s>0} s\,(\,\alpha_f(s)\,)^{\frac{1}{p}}.$$

Proof. Let

$$c_1 = \sup_{t>0} t^{\frac{1}{p}} f^*(t),\ c_2 = \sup_{s>0} s\,(\,\alpha_f(s)\,)^{\frac{1}{p}}. \tag{2.8}$$

without loss of generality assume that $c_1 < \infty$. For $s>0$, let $t = (\frac{c_1}{s})^p$. Thus $t^{\frac{1}{p}} f^*(t) \leq c_1$. Now take $c = c_1$ in (2.3), with $c_1 = s\,t^{\frac{1}{p}}$, to deduce that $s\,(\,\alpha_f(s)\,)^{\frac{1}{p}} \leq c_1$. Hence by taking supremum over s, we get $c_2 \leq c_1$. The reverse inequality follows in a similar way. □

Proposition 2.2.3. *[Inclusions of Lorentz spaces]:*

(a) **Inclusion in the secondary index:** *Let $p,q,r \in [1,\infty]$ and let $q \leq r$. Then*

$$L(p,q) \hookrightarrow L(p,r).$$

(b) **Inclusion in the primary index:** *Let $p,q,r,s \in [1,\infty]$ and let $r<p$. Then*

$$L(p,q) \hookrightarrow L_{loc}(r,s),$$

where

$$L_{loc}(r,s) := \left\{ f \in \mathcal{M}_0(\Omega) : \left\| f_{|K} \right\|_{(r,s)} < \infty,\, \forall\, K \subset\subset \Omega \right\}.$$

For a proof, see Proposition 3.4.3 and Proposition 3.4.4 of [36].

Remark 2.2.4. *The assumption $r < p$ is necessary in (b), for example $f(x) = \frac{1}{|x|} \in L(1,\infty)$ but not in $L^1_{loc}(\mathbb{R})$.*

Remark 2.2.5. *The part (b) of the above proposition asserts that $L(d,\infty)$ is contained in $L^r_{loc}(\Omega)$, for $1 \le r < d$.*

Note that the functional $\|\cdot\|_{(p,q)}$ is not a norm in general, since the triangular inequality fails. However, using (d) of Proposition 2.1.5 one can verify that

$$\|f+g\|_{(p,q)} \le 2(\|f\|_{(p,q)} + \|g\|_{(p,q)}).$$

Therefore$\|.\|_{(p,q)}$ defines a quasi-norm. In order to get a norm, we set

$$\|f\|^*_{(p,q)} := \left\| t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \right\|_{q\,;\,(0,\infty)}, \quad p,q \in [1,\infty]$$

Now we define

$$L_{(p,q)} := \{ f \in \mathcal{M}_0(\Omega) : \|f\|^*_{(p,q)} < \infty \}.$$

Since f^{**} is subadditive (see (d) of Proposition 2.1.7), the triangular inequality of $\|\cdot\|^*_{(p,q)}$ follows immediately from the Minkowski's inequality. Note that

- $L_{(1,1)} = \{0\}$. More precisely, when $\|f\|^*_{(1,1)} < \infty$, for each $t_0 > 0$ we get $f^*(t_0) = 0$. This can be seen from the following inequality:

$$\begin{aligned}\|f\|^*_{(1,1)} &= \int_0^\infty \frac{1}{t}\left(\int_0^t f^*(\tau)d\tau\right)dt \ge \int_{t_0}^\infty \frac{1}{t}\left(\int_0^t f^*(\tau)d\tau\right)dt \\ &\ge t_0 f^*(t_0)\int_{t_0}^\infty \frac{1}{t}\,dt.\end{aligned}$$

- $L_{(1,\infty)} = L^1(\Omega)$.

$$\|f\|^*_{(1,\infty)} = \sup_{t>0} t f^{**}(t) = \sup_{t>0} \int_0^t f^*(t) = \|f\|_1 .$$

However the following lemma shows that the spaces $L(p,q)$ and $L_{(p,q)}$ are one and the same when $p > 1$, see [16] for a proof.

Lemma 2.2.6. *For $p > 1$ and $q \ge 1$*

$$\|u\|^*_{(p,q)} \le \|u\|_{(p,q)} \le \frac{1}{p-1}\|u\|^*_{(p,q)} .$$

Moreover endowed with this norm $L(p,q)$ is a Banach space, for $p>1$, $q\geq 1$. In view of Lemma 2.2.6, we work with $\|\cdot\|_{(p,q)}$ rather than $\|\cdot\|^*_{(p,q)}$.

In the following proposition, we summarize some properties of $L(p,q)$ spaces (see [2, 36, 46] for proofs).

Proposition 2.2.7. *(i) (***Hölder inequality***) Given $(f,g)\in L(p_1,q_1)\times L(p_2,q_2)$ and $(p,q)\in(1,\infty)\times[1,\infty]$ such that $1/p=1/p_1+1/p_2$, $1/q\leq 1/q_1+1/q_2$, then*

$$\|fg\|_{(p,q)}\leq C\,\|f\|_{(p_1,q_1)}\;\|g\|_{(p_2,q_2)}\,, \tag{2.9}$$

where C depends only on p.

(ii) Let $(p,q)\in(1,\infty)\times(1,\infty)$. Then the dual space of $L(p,q)$ is isomorphic to $L(p',q')$ where $1/p+1/p'=1$ and $1/q+1/q'=1$.

(iii) Let $\gamma>0$. Then

$$\left\||f|^\gamma\right\|_{(\frac{p}{\gamma},\frac{q}{\gamma})}=\left\|f\right\|^\gamma_{(p,q)}. \tag{2.10}$$

The following corollary is immediate from the above properties of the Lorentz spaces.

Corollary 2.2.8. *Let $g\in L\left(\frac{N}{p},\infty\right)$ and let $p\in(1,N)$. Then the following maps are continuous*

(i) $G_1: L(p^*,p)\to\mathbb{R}$ defined by

$$G_1(u)=\frac{1}{p}\int_\Omega g|u|^p. \tag{2.11}$$

(ii) $G_2: L(p^*,p)\to[L(p^*,p)]'$ defined as

$$\langle G_2(u),v\rangle=\int_\Omega g|u|^{p-2}uv.$$

Proof. For $u\in L(p^*,p)$, using (2.10), we get $|u|^p\in L\left(\frac{p^*}{p},1\right)$. Now as $g\in L\left(\frac{N}{p},\infty\right)$, the dual space of $L\left(\frac{p^*}{p},1\right)$, using the Hölder inequality we obtain the following:

$$|G_1(u)|\leq C_1\|g\|_{\left(\frac{N}{p},\infty\right)}\left\||u|^p\right\|_{\left(\frac{p^*}{p},1\right)}\leq C_1\|g\|_{\left(\frac{N}{p},\infty\right)}\|u\|^p_{(p^*,p)}\,,$$

where C_1 is a constant that is independent of u. Thus the map G_1 is well defined. Now the continuity of G_1 is a direct consequence of the generalized dominated convergence theorem (Theorem 17, Chapter 4 of [67]). Using a similar argument one can show that G_2 is continuous. □

2.3 The space $\mathcal{D}_0^{1,p}(\Omega)$ and certain embeddings

Definition 2.3.1. *Let Ω be a domain in $\mathbb{R}^N$ and let $p \in (1, N)$. Then the Beppo-Levi space $\mathcal{D}_0^{1,p}(\Omega)$ is defined as*

$$\mathcal{D}_0^{1,p}(\Omega) := \text{ completion of } \mathcal{C}_c^\infty(\Omega) \text{ with respect to } \|\nabla .\|_p \text{ norm },$$

where $\mathcal{C}_c^\infty(\Omega)$ is the set of all $\mathcal{C}^\infty$ functions with compact support contained in Ω and

$$\|\nabla u\|_p := \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}}.$$

Remark 2.3.2. *When Ω is bounded, using the Poincaré inequality one can easily see that $\mathcal{D}_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$; in general the Sobolev space $W_0^{1,p}(\Omega)$ is contained in $\mathcal{D}_0^{1,p}(\Omega)$.*

Remark 2.3.3. *We emphasize that $\mathcal{D}_0^{1,p}(\Omega)$ is not a function space in general, when $p \geq N$, in the sense $\mathcal{D}_0^{1,p}(\Omega)$ is not continuously embedded in $L^1_{loc}(\Omega)$. For example, $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ is not a function space, when N=1, 2 (see Remark 2.2 of [38]).*

In contrast to the above remark, when $N > p$, the functions in $\mathcal{D}_0^{1,p}(\Omega)$ belong to the Lebesgue space $L^{p^*}(\Omega)$, where $p^* = \frac{Np}{N-p}$.

Theorem 2.3.4. *For $1 < p < N$, the space $\mathcal{D}_0^{1,p}(\Omega)$ is embedded continuously into $L^{p^*}(\Omega)$. In other words, there exists $C > 0$ such that*

$$\|u\|_{p^*} \leq C \|\nabla u\|_p , \forall u \in \mathcal{D}_0^{1,p}(\Omega).$$

For a proof for the case $p = 2$ we refer to Theorem 8.3 of [54].

Since $L^{p^*}(\Omega)$ is embedded continuously in $L^q_{loc}(\Omega)$, for each $q \in [1, p^*)$, using the above theorem and the Rellich's compactness theorem we have the following corollary.

Corollary 2.3.5. *For $1 < p < N$, the space $\mathcal{D}_0^{1,p}(\Omega)$ is embedded compactly in to $L^q_{loc}(\Omega)$ for $q \in [1, p^*)$.*

Remark 2.3.6. *It is easy to see that* $\mathcal{D}_0^{1,2}(\Omega)$ *is a Hilbert space, as the norm* $\|\nabla u\|_2$ *is induced by the following inner product:*

$$\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v, \quad \forall\, u, v \in \mathcal{D}_0^{1,2}(\Omega).$$

Thus, by the Riesz representation theorem, the map $-\Delta:\mathcal{D}_0^{1,2}(\Omega)\to[\mathcal{D}_0^{1,2}(\Omega)]'$ *defined as*

$$\langle -\Delta u, v\rangle = \int_\Omega \nabla u \cdot \nabla v, \quad \forall\, u \in \mathcal{D}_0^{1,2}(\Omega),$$

is an isometry onto $[\mathcal{D}_0^{1,2}(\Omega)]'$.

In the next proposition we give examples of certain test functions in $\mathcal{D}_0^{1,2}(\Omega)$ that we use later.

Proposition 2.3.7. *Let* Ω *be a domain in* $\mathbb{R}^N$ *and let* $1 < p < N$. *Let* $v \in \mathcal{D}_0^{1,2}(\Omega)$ *and* $v > 0$ *a.e. Then for* $\phi \in \mathcal{C}_c^\infty(\Omega)$, $\frac{|\phi|^p}{v+\varepsilon} \in \mathcal{D}_0^{1,2}(\Omega)$.

Proof. We prove the result for $p = 2$, the proof for $p \neq 2$ is similar. Since $H_0^1(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$, it is enough to show that $\frac{\phi^2}{v+\varepsilon} \in H_0^1(\Omega)$. Since $v > 0$, $(v+\varepsilon)^{-1}$ is bounded and hence $\frac{\phi^2}{v+\varepsilon}$ belongs to $L^2(\Omega)$. Further,

$$\nabla\left(\frac{\phi^2}{v+\varepsilon}\right) = \frac{2\phi}{(v+\varepsilon)}\nabla\phi - \frac{\phi^2}{(v+\varepsilon)^2}\nabla v.$$

Since $|\nabla v| \in L^2(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$, it is easy to see that $|\nabla\left(\frac{\phi^2}{v+\varepsilon}\right)|$ also belongs to $L^2(\Omega)$. Now as $\frac{\phi^2}{v+\varepsilon} = 0$ on $\partial\Omega$, using Theorem 2.2.4 of [51], we conclude that $\frac{\phi^2}{v+\varepsilon} \in H_0^1(\Omega)$. □

In the following proposition we give an equivalent definition for the space $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$. For a proof, see Lemma 1-(i) in Section 4.7 of [37].

Proposition 2.3.8. *Let* $1 < p < N$.

$$\mathcal{D}_0^{1,p}(\mathbb{R}^N) = \left\{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\right\}. \tag{2.12}$$

From the above theorem one can easily deduce the following properties of functions in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$.

Corollary 2.3.9. *Let* $1 < p < N$. *Then* *(i)* *If* $f, g \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *then* $\max\{f,g\},\ \min\{f,g\} \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$. *In particular* $f^+,\ f^- \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$.

(ii) *If* $f \geq 0$ *and* $f \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *then* $\min\{f,k\}, \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *for every* $k \in \mathbb{R}_+$.

(iii) *If* $f \in L^\infty(\mathbb{R}^N) \cap \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *then* $|f|^\alpha \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *for every* $\alpha \geq 1$.

The main interest of considering the Lorentz spaces is that the usual Sobolev embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ can be improved as below (See Appendix of [1] for a proof):

Theorem 2.3.10. *The space $\mathcal{D}_0^{1,p}(\Omega)$ continuously embedded in to the Lorentz space $L(p^*,p)$ i.e, we have the following Lorentz-Sobolev inequality*

$$\|u\|_{(p^*,p)} \leq C_s \|\nabla u\|_p \, , \forall u \in \mathcal{D}_0^{1,p}(\Omega),$$

where C_s is a constant depending on p and Ω.

Remark 2.3.11. *Note that the above theorem clearly implies Theorem 2.3.4 as $L(p^*,p)$ is proper subset of $L(p^*,p^*) = L^{p^*}(\Omega)$ and hence the Lorentz-Sobolev embedding gives a finer embedding than that given in Theorem 2.3.4.*

Remark 2.3.12. *From Theorem 2.3.10, we see that $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L(2^*,2)$ continuously. Since $-\Delta$ is an isometry onto $[\mathcal{D}_0^{1,2}(\Omega)]'$ we have*

$$(-\Delta)^{-1} : [L(2^*,2)]' \to \mathcal{D}_0^{1,2}(\Omega)$$

is continuous.

Chapter 3

The function space $\mathcal{F}_d$

Let Ω be a domain in $\mathbb{R}^N$. In this chapter we identify an important subspace of the weak- $L^d(\Omega)$ spaces for $d > 1$ and characterize these subspaces with the behaviour of $t^{\frac{1}{d}} f^*(t)$ at 0 and ∞. Later we show that these subspaces have a close connection with the existence of eigenvalues for certain weighted eigenvalue problems for the Laplacian and the p- Laplacian.

The main results presented in this chapter are published in [10] as a joint work with Marcello Lucia and Mythily Ramaswamy.

For $1 \le d < \infty$, it is well known that $\mathcal{C}_c^\infty(\Omega)$ is dense in the Lebesgue space $L^d(\Omega)$, whereas in $L^\infty(\Omega)$ the closure of $\mathcal{C}_c^\infty(\Omega)$ defines a proper closed subspace of $L^\infty(\Omega)$, namely, the space of functions vanishing on $\partial\Omega$ and at ∞ (if the domain is unbounded). A similar situation occurs in the Lorentz spaces. More precisely, for $(d, q) \in [1, \infty) \times [1, \infty)$, $\mathcal{C}_c^\infty(\Omega)$ is dense in the Lorentz space $L(d, q)$. However, for $d > 1$, the closure of $\mathcal{C}_c^\infty(\Omega)$ in $L(d, \infty)$ defines a proper subspace that will henceforth be denoted by $\mathcal{F}_d$:

$$\mathcal{F}_d := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{(d,\infty)}} \subset L(d, \infty).$$

Equivalently the space $\mathcal{F}_d$ can also be defined as the closure of the set of all simple functions on Ω. More precisely, let

$$S_1 := \left\{ f = \sum_{j=1}^{m} c_j \ \chi_{E_j}, m \in \mathbb{N}, c_i \in \mathbb{R} \setminus \{0\} \right\},$$

where $E_1, \ldots, E_m$ are bounded measurable sets and χ_E stands for the characteristic function of the set E. Then S_1 is dense in $L(d, q)$ for $1 \le d, q < \infty$ (see [36]). Now using the definition of $\mathcal{F}_d$ and the continuous embedding of

$L(d,q)$ into $L(d,\infty)$, we can see that

$$\overline{S_1}^{\|\cdot\|_{(d,\infty)}} = \mathcal{F}_d.$$

In certain situations we may have to consider the following type of step functions:

$$S_2 := \left\{ f = \sum_{j=1}^{m} c_j \ \chi_{E_j}, m \in \mathbb{N}, c_i \in \mathbb{R} \setminus \{0\} \right\},$$

where E_j is a measurable set of finite measure, not necessarily bounded. However, as the distribution function of χ_E depends only on the measure of E and not on the boundedness of E, we see that, $S_1 \subset S_2 \subset L(d,q)$. Hence

$$\overline{S_2}^{\|\cdot\|_{(d,\infty)}} = \overline{S_1}^{\|\cdot\|_{(d,\infty)}} = \mathcal{F}_d.$$

Throughout this chapter we assume that $d > 1$. Thus by Remark 2.2.5, $\mathcal{F}_d$ is contained in $L^1_{loc}(\Omega)$. The following proposition shows that $\mathcal{F}_d$ is bigger than the Lorentz space $L(d,q)$ for any finite q, however strictly smaller than $L(d,\infty)$.

Proposition 3.0.13. *(i) For each* $1 \le q < \infty$, $L(d,q) \subset \mathcal{F}_d$.

(ii) For each $a \in \Omega \subset \mathbb{R}^N$ *and* $d < N$, *the Hardy potential* $h(x) = |x-a|^{-d}$ *does not belong to* $\mathcal{F}_{\frac{N}{d}}$.

Proof. (i) Let $f \in L(d,q)$. Since $\mathcal{C}_c^\infty(\Omega)$ is dense in $L(d,q)$, there exists a sequence $f_n \in \mathcal{C}_c^\infty(\Omega)$ such that $\lim\limits_{n\to\infty} \|f - f_n\|_{(d,q)} = 0$. By (a) of Proposition 2.2.3 we have

$$\|f - f_n\|_{(d,\infty)} \le C \, \|f - f_n\|_{(d,q)} \, ,$$

hence by the definition $f \in \mathcal{F}_d$.

(ii) From (2.2) it is clear that $h \in L\left(\frac{N}{d}, \infty\right)$. Let $f \in S_1$, $f = \sum_{i=1}^m c_i \chi_{E_i}$ with $c_i \in \mathbb{R}$. We show that $\|h - f\|_{(d,\infty)} \ge c > 0$ for some constant c independent of f, which will conclude the proof, since S_1 is dense in $\mathcal{F}_{\frac{N}{d}}$.

Fix a ball $B(a,r) \subset \Omega$. Let $c_0 = \max_{1 \le i \le m} |c_i|$. Let $s > 0$.

$$\begin{aligned}\alpha_{h-f}(s) &\ge |\,\{x \in B(a;r) : |(h-f)(x)| > s\}\,| \\ &\ge \sum_{i=1}^{m} \left|\left\{x \in B(a,r) \cap E_i :\ \left||x-a|^{-d} - c_i\right| > s\right\}\right| \\ &\qquad + \left|\left\{x \in B(a,r) \setminus (\cup_{i=1}^{m} E_i)\ :\ |x-a|^{-d} > s\right\}\right|.\end{aligned}$$

Notice that if $x \in B(a;r) \setminus \cup_{i=1}^{m} E_i$ and

$$|x-a|^{d} < (c_0 + s)^{-1}, \tag{3.1}$$

then we have

$$s < c_0 + s < |x-a|^{-d}. \tag{3.2}$$

Similarly, if $x \in B(a;r) \cap E_i$ and if (3.1) holds, then

$$|c_i| + s \le c_0 + s < |x-a|^{-d}.$$

Thus

$$s < |x-a|^{-d} - |c_i| \le \left||x-a|^{-d} - c_i\right|. \tag{3.3}$$

Consequently, from (3.2) and (3.3), we see that

$$\alpha_{h-f}(s) \ge \left|\,\left\{x \in B(a;r) : |x-a|^{d} < (c_0+s)^{-1}\right\}\,\right|.$$

Let $s_0 > 0$ be such that $(c_0+s)^{-1} < r^d$, $\forall\, s > s_0$. Thus, from the above inequality, we get

$$\alpha_{h-f}(s) \ge \omega_N\, (c_0+s)^{\frac{-N}{d}},\ \forall\, s > s_0.$$

Now,

$$\|h-f\|_{(\frac{N}{d},\infty)} = \sup_{s>0}\left\{s\ (\alpha_{h-f}(s))^{\frac{d}{N}}\right\} \ge \sup_{s>0}\left\{\,s\,\omega_N{}^{\frac{d}{N}}\,(c_0+s)^{-1}\right\} \ge \omega_N{}^{\frac{d}{N}}.$$

$\square$

Remark 3.0.14. *From (i) it is clear that the usual Lebesgue space $L^d(\Omega)$ is contained in $\mathcal{F}_d$ and from (ii), by taking $h(x) = |x-a|^{-\frac{N}{d}}$, for some $a \in \Omega$, we see that $\mathcal{F}_d$ is a proper subspace of weak-$L^d(\Omega)$, for $d > 1$.*

In the following lemma we give an equivalent condition for a function to be in the space $\mathcal{F}_d$, in terms of splitting of the function.

Lemma 3.0.15. *Let $f \in L(d,\infty)$. Then $f \in \mathcal{F}_d$ if and only if for each $\varepsilon > 0$, there exists $f_\varepsilon \in L^\infty(\Omega)$ such that $|$ supp $(f_\varepsilon)\,| < \infty$ and $\|f - f_\varepsilon\|_{(d,\infty)} < \varepsilon$.*

Proof. Let $f \in \mathcal{F}_d$ and let $\varepsilon > 0$ be given. Then by the definition of the space $\mathcal{F}_d$, we get $f_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that $||f - f_\varepsilon||_{(d,\infty)} < \varepsilon$. Clearly $f_\varepsilon \in L^\infty(\Omega)$ and $|$ supp $(f_\varepsilon)\,| < \infty$. Conversely, for a given $\varepsilon > 0$ let $f_\varepsilon \in L^\infty(\Omega)$ such that $||f - f_\varepsilon||_{(d,\infty)} < \frac{\varepsilon}{2}$ and $|\text{supp}(f_\varepsilon)| < \infty$. Thus $f_\varepsilon \in L^d(\Omega)$ and hence we can find $\varphi_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that

$$||f_\varepsilon - \varphi_\varepsilon||_{(d,\infty)} \leq \|\varphi_\varepsilon - f_\varepsilon\|_d < \frac{\varepsilon}{2}.$$

Now

$$||f - \varphi_\varepsilon||_{(d,\infty)} \leq ||f - f_\varepsilon||_{(d,\infty)} + ||f_\varepsilon - \varphi_\varepsilon||_{(d,\infty)} < \varepsilon.$$

Hence $f \in \mathcal{F}_d$ by the definition of $\mathcal{F}_d$. □

The next lemma gives a necessary condition for a function to be in $\mathcal{F}_d$ in terms of splitting of the domain Ω.

Lemma 3.0.16. *If $f \in \mathcal{F}_d$, then for every $\varepsilon > 0$, there exists bounded set $\Omega_\varepsilon \subset \Omega$ such that $\left\|f\chi_{\Omega\setminus\Omega_\varepsilon}\right\|_{(d,\infty)} < \varepsilon$.*

Proof. For a given $\varepsilon > 0$, we choose $f_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that $\|f - f_\varepsilon\|_{(d,\infty)} < \varepsilon$. Now the set $\Omega_\varepsilon = \{x : f_\varepsilon(x) \neq 0\}$ is bounded. Further, using (iii) of Proposition 2.1.4 we obtain

$$\left\|f\chi_{\Omega\setminus\Omega_\varepsilon}\right\|_{(d,\infty)} = \left\|(f - f_\varepsilon)\chi_{\Omega\setminus\Omega_\varepsilon}\right\|_{(d,\infty)} \leq \|f - f_\varepsilon\|_{(d,\infty)} < \varepsilon.$$

□

The above characterization of the space $\mathcal{F}_d$ is very useful for determining whether a function belongs $\mathcal{F}_d$ or not. In most of the cases, we take Ω_ε to be certain level sets of the function, as it is easy to check whether the level sets is of finite measure or not.

Next we list a few useful characterization of the space $\mathcal{F}_d$ in terms of the decreasing rearrangement and the distribution function.

Proposition 3.0.17. *Let $f \in \mathcal{M}_0(\Omega)$. Then the following statements are equivalent:*

(i) $f \in \mathcal{F}_d$.

(ii) $f^*(t) = o(t^{\frac{-1}{d}})$ *at 0 and* ∞*. i.e,*

$$\lim_{t \to 0_+} t^{\frac{1}{d}} f^*(t) \ = 0 = \ \lim_{t \to \infty} t^{\frac{1}{d}} f^*(t). \tag{3.4}$$

(iii) $\alpha_f(s) = o(s^{-d})$ *at 0 and* ∞*. i.e,*

$$\lim_{s \to 0_+} s^d \alpha_f(s) \ = 0 \ = \ \lim_{s \to \infty} s^d \alpha_f(s). \tag{3.5}$$

Proof. (i)⇒(ii)
Let $f \in \mathcal{F}_d$. For a given $\varepsilon > 0$ write $f = f_\varepsilon + (f - f_\varepsilon)$ where $f_\varepsilon \in C_c^\infty(\Omega)$ is such that $\|f - f_\varepsilon\|_{(d,\infty)} < \varepsilon$. From (d) of Proposition 2.1.5, we have $(f_1 + f_2)^*(t_1 + t_2) \le f_1^*(t_1) + f_2^*(t_2)$, for any $t_1, t_2 > 0$. Thus we deduce that $f^*(2t) \le (f - f_\varepsilon)^*(t) + f_\varepsilon^*(t)$. Hence

$$(2t)^{\frac{1}{d}} f^*(2t) \ \le \ 2^{\frac{1}{d}} \left\{ t^{\frac{1}{d}} (f - f_\varepsilon)^*(t) + t^{\frac{1}{d}} f_\varepsilon^*(t) \right\} \ \le \ 2^{\frac{1}{d}} \ \left\{ \varepsilon + t^{\frac{1}{d}} f_\varepsilon^*(t) \right\}.$$

Since $f_\varepsilon \in C_c^\infty(\Omega)$, the function f_ε^* has also compact support and therefore satisfies (3.4). Thus $(2t)^{\frac{1}{d}} f^*(2t)$ can be made arbitrarily small for large and small values of t, showing that (3.4) holds for any $f \in \mathcal{F}_d$.

(ii)⇒(iii)
Let (ii) hold. Thus for given $\varepsilon > 0$, there exist $t_1, t_2 > 0$ such that

$$t^{\frac{1}{d}} f^*(t) < \varepsilon, \quad \forall \ t \in (0, t_1) \cup (t_2, \infty). \tag{3.6}$$

Let $s_1 = \varepsilon \, (t_1)^{\frac{-1}{d}}$ and $s_2 = \varepsilon \, (t_2)^{\frac{-1}{d}}$. Note that,

$$\text{If } s \in (0, s_2) \ \cup \ (s_1, \infty), \ \text{ then } t = \left(\frac{\varepsilon}{s} \right)^d \in (0, t_1) \cup (t_2, \infty).$$

Now using (3.6) and (2.3) with $c = \varepsilon$, we get

$$s(\alpha_f(s))^{\frac{1}{d}} < \varepsilon, \ \forall \ s \in (0, s_2) \cup (s_1, \infty).$$

This shows that $\alpha_f(s) = o(s^{-d})$ at 0 and ∞.

(iii)⇒ (i)
Assume that (iii) holds. Thus for given $\varepsilon > 0$, there exist s_1, s_2 such that

$$s(\alpha_f(s))^{\frac{1}{d}} < \varepsilon, \ \forall \ s \in (0, s_1] \cup [s_2, \infty). \tag{3.7}$$

Let

$$A_\varepsilon := \{x : s_1 < |f(x)| < s_2\}, \quad f_\varepsilon := f\chi_{A_\varepsilon}, \quad g = f\chi_{A_\varepsilon^c}$$

Note that $|A_\varepsilon| \leq \alpha_f(s_1) < \infty$ and $f_\varepsilon \in L^\infty(\Omega)$. Thus by Proposition 3.0.15 it is enough to prove that

$$\|f - f_\varepsilon\|_{(d,\infty)} = \|g\|_{(d,\infty)} < \varepsilon.$$

Observe that, for $s \in (s_1, s_2)$, $\alpha_g(s) = \alpha_f(s_2)$ and hence

$$s(\alpha_g(s))^{\frac{1}{d}} < s_2(\alpha_f(s_2))^{\frac{1}{d}} < \varepsilon, \ \forall\, s \in (s_1, s_2). \tag{3.8}$$

Since $|g| \leq |f|$, we have $\alpha_g(s) \leq \alpha_f(s)$, $\forall\, s > 0$. Now by combining (3.7) and (3.8) we get

$$s(\alpha_g(s))^{\frac{1}{d}} < \varepsilon, \ \forall\, s > 0.$$

Hence by Lemma 2.2.2 we get $\|g\|_{(d,\infty)} < \varepsilon$. □

Recall the following definition:

Definition 3.0.18. *We say that, a norm $\|.\|_X$ on a function space X is absolutely continuous with respect to the measure μ, if for each $g \in X$, $\|g\chi_E\| \to 0$ as $\mu(E) \to 0$.*

Now using Proposition 3.0.17, we have the following corollary:

Corollary 3.0.19. *The norm in $\mathcal{F}_d$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$.*

Proof. Let $g \in \mathcal{F}_d$. Now for $E \subset \Omega$, note that $(g\chi_E)^*(t) = 0$ for $t > |E|$ and $g^*(t) \geq (g\chi_E)^*(t)$. Therefore

$$\|g\chi_E\|_{(d,\infty)} = \sup_{t>0} \left\{ t^{\frac{1}{d}} (g\chi_E)^*(t) \right\} \leq \sup_{0<t\leq|E|} \left\{ t^{\frac{1}{d}} g^*(t) \right\},$$

Since $g \in \mathcal{F}_d$, by the above characterization $t^{\frac{1}{d}} g^*(t) \to 0$ as $t \to 0$. Thus it is immediate that

$$\|g\chi_E\|_{(d,\infty)} \to 0 \text{ as } |E| \to 0.$$

□

Remark 3.0.20. *In contrast to the above result, observe that the norm in weak-$L^d(\Omega)$, for $d > 1$, is not absolutely continuous with respect to the*

Lebesgue measure in $\mathbb{R}^N$*. Indeed, for the Hardy potential* $h(x) = |x-a|^{-\frac{N}{d}}$ *(take* $f \equiv 0$*, in the proof of (ii) of Proposition 3.0.13), we have*

$$\left\|h\chi_{B(a,r)}\right\|_{\left(\frac{N}{d},\infty\right)} \geq \omega_N{}^{\frac{d}{N}}, \quad \forall\, r > 0.$$

The following sufficient condition for a function to be in $\mathcal{F}_d$ is similar to a condition of Rozenblum [68](see (2.19) of [68]).

Lemma 3.0.21. *Let* $h \in L(d,\infty)$ *and* $h > 0$*. If* f *is such that*

$$\int_\Omega h^{d-q}|f|^q \;<\; \infty, \text{ for some } q \geq d,$$

then $f \in L(d,q)$ *and hence in* $\mathcal{F}_d$*.*

Proof. The result is obvious when $q = d$. For $q > d$, let $g = h^{\frac{d}{q}-1}f$. Then the above integrability condition yields $g \in L^q(\Omega)$. Thus by property (2.10), we get $h^{1-\frac{d}{q}} \in L(\frac{dq}{q-d},\infty)$. Now by Hölder inequality (2.9) we get $f \in L(d,q)$ and hence in $\mathcal{F}_d$ as $L(d,q) \subset \mathcal{F}_d$. $\square$

Remark 3.0.22. *Let* $g \in L^q(\mathbb{R}^N)$ *with* $q \geq d$ *and let*

$$f(x) = |x|^{\left(\frac{1}{q}-\frac{1}{d}\right)N}\, g(x).$$

Then using the above lemma one can easily verify that $f \in L(d,q)$*. In general, if* $h \in L(d,\infty)$ *with* $h > 0$*, then* $f = g\, h^{1-\frac{d}{q}} \in L(d,q)$*. Thus one can obtain Lorentz spaces by interpolating Lebesgue and weak-Lebesgue spaces suitably.*

Next we show that the functions having faster decay than the Hardy potential $\frac{1}{|x-a|^d}$ at infinity and at all points in $a \in \Omega$ are in $\mathcal{F}_{\frac{N}{d}}$. More precisely, we consider a measurable function, say g, on $\Omega \subset \mathbb{R}^N$ with $d < N$, satisfying the following conditions:

$$(i) \lim_{|x|\to\infty\,,\, x\in\Omega} |x|^d g(x) = 0, \quad (ii) \lim_{x\to a\,,\, x\in\overline{\Omega}} |x-a|^d g(x) = 0, \quad \forall a \in \overline{\Omega}. \tag{3.9}$$

First we prove the following preparatory lemma.

Lemma 3.0.23. *Let* g *be a measurable function satisfying condition* (3.9)*. Then there exist* $a_1, \ldots, a_m \in \overline{\Omega}$ *with the property: for every* $\varepsilon > 0$ *there* *exists $R := R(\varepsilon) > 0$ such that*

$$\begin{aligned} |g(x)| &< \frac{\varepsilon}{|x|^d} \ \ a.e\ x \in \Omega \setminus B(0,R), && (3.10)\\ |g(x)| &< \frac{\varepsilon}{|x-a_i|^d} \ \ a.e\ x \in \Omega \cap B(a_i, R^{-1})\,, \quad i = 1, \dots, m, && (3.11)\\ g &\in L^\infty(\Omega \setminus A_\varepsilon)\,, && (3.12)\end{aligned}$$

where $A_\varepsilon := \bigcup_{i=1}^m B(a_i, R^{-1}) \cap \Omega$.

Proof. Using condition (i) of (3.9) we can find $r > 0$ such that

$$|g(x)| < \frac{1}{|x|^d} \quad \text{a.e. } x \in \Omega \setminus B(0,r).$$

Now for each $a \in \overline{\Omega \cap B(0,r)}$, by condition (ii), there exists $r_a > 0$ such that

$$|g(x)| < \frac{1}{|x-a|^d} \quad a.e. \text{ in } \Omega \cap B(a, r_a)\,.$$

Since $\overline{\Omega \cap B(0,r)}$ is compact, there exist $a_1, \dots, a_m \in \overline{\Omega \cap B(0,r)}$ such that $\overline{\Omega \cap B(0,r)} \subset \cup_{i=1}^m B(a_i, r_{a_i})$.

Using condition (i) again, for a given $\varepsilon > 0$, we choose $R = R(\varepsilon) \geq r, r_{a_1}, \dots, r_{a_m}$ so that

$$|g(x)| < \frac{\varepsilon}{|x|^d}, \quad a.e. \text{ in } \Omega \setminus B(0,R).$$

Applying condition (ii) in each ball $B(a_i, r_{a_i})$ $(i = 1, \dots, m)$, by choosing R larger if necessary, we can satisfy

$$|g(x)| < \frac{\varepsilon}{|x-a_i|^d}, \quad a.e \text{ in } \Omega \cap B(a_i, R^{-1}) \ \text{ and } \ g \in L^\infty(\Omega \setminus A_\varepsilon)\,.$$

□

Theorem 3.0.24. *Let $\Omega \subset \mathbb{R}^N$ and $d < N$. Let $g : \Omega \to \mathbb{R}$ be as in the previous lemma. Then $g \in \mathcal{F}_{\frac{N}{d}}$.*

Proof. We use condition (iii) of Proposition 3.0.17 to show that $g \in \mathcal{F}_{\frac{N}{d}}$. First we compute the distribution function of g. For $\varepsilon > 0$, let R be given by the previous lemma. Let $s_1 := \varepsilon R^{-d}$. We will show that

$$s(\alpha_g(s))^{\frac{d}{N}} < \varepsilon, \quad \forall\, s < s_1.$$

Using (3.10), for each $s \in (0, s_1)$ we have,

$$B(0,R) \subset B(0, (\frac{\varepsilon}{s})^{1/d}) \quad \text{and} \quad |g(x)| < s, \quad \forall x \in \Omega \setminus B(0, (\frac{\varepsilon}{s})^{\frac{1}{d}}). \tag{3.13}$$

Therefore, for each $s \in (0, s_1)$, the distribution function $\alpha_g(s)$ can be estimated as follows:

$$\alpha_g(s) \;=\; \left|\{x \in \Omega \cap B(0, (\frac{\varepsilon}{s})^{\frac{1}{d}}) : \; |f(x)| > s\}\right| \;\leq\; \omega_N (\frac{\varepsilon}{s})^{\frac{N}{d}} \,,$$

where ω_N is the volume of unit ball in $\mathbb{R}^N$. Thus

$$s(\alpha_g(s))^{\frac{d}{N}} < C_1 \varepsilon, \quad \forall\, s < s_1. \tag{3.14}$$

where the constant C_1 is independent of ε. Next consider the set $A_\varepsilon = \bigcup_{i=1}^m B(a_i, R^{-1}) \cap \Omega$ and let $s_2 := \|g\|_{L^\infty(\Omega \setminus A_\varepsilon)}$. For $s > s_2$, using (3.11) the distribution function can be estimated as follows:

$$\begin{aligned}
\alpha_g(s) \;&=\; \left|\{x \in \Omega : \; |g(x)| > s\}\right| \;=\; \left|\{x \in A_\varepsilon : \; |g(x)| > s\}\right| \\
&\leq\; \sum_{i=1}^m \left|\{x \in B(a_i, R^{-1}) \cap \Omega : \; |g(x)| > s\}\right| \\
&\leq\; \sum_{i=1}^m \left|\{x \in B(a_i, R^{-1}) : \; \varepsilon |x - a_i|^{-d} > s\}\right| \\
&=\; \sum_{i=1}^m \omega_N \left(\frac{\varepsilon}{s}\right)^{\frac{N}{d}} .
\end{aligned}$$

Therefore

$$s(\alpha_g(s))^{\frac{d}{N}} \;\leq\; C_2 \varepsilon \quad \forall s > s_2, \tag{3.15}$$

where C_2 is independent of ε. Now proof follows using condition (iii) of proposition 3.0.17 together with (3.14) and (3.15). $\square$

Chapter 4

Weighted eigenvalue problems for the Laplacian

In this chapter we consider the weighted eigenvalue problem for the Laplacian. More precisely, for a given connected domain Ω in $\mathbb{R}^N$ with $N > 2$, we study the sufficient conditions on a weight function g for the existence of $\lambda \in \mathbb{R}$ and a weak solution $u \in \mathcal{D}_0^{1,2}(\Omega) \setminus \{0\}$ for the following problem:

$$-\Delta u \quad = \quad \lambda\, g\, u \text{ in } \Omega. \tag{4.1}$$

Recall that $u \in \mathcal{D}_0^{1,2}(\Omega)$ is said to be a weak solution of (4.1), if

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega g\, uv, \quad \forall\, v\, \in \mathcal{D}_0^{1,2}(\Omega). \tag{4.2}$$

There are various sufficient conditions on the weight function g available in the literature for the existence of a positive principal eigenvalue for (4.1). For example, see [61, 21, 22, 4] and the references therein. All these sufficient conditions required either the weight g or its positive part g^+ to be in the Lebesgue space $L^{\frac{N}{2}}(\Omega)$. A sufficient condition beyond the Lebesgue spaces was obtained in [73] by Szulkin and Willem, by considering the problem (4.1) for a weight g such that $g^+ \in L^{\frac{N}{2}}(\Omega)$ or having a faster decay than $|x|^{-2}$ at infinity and at any point in the domain.

The existence of a positive principal eigenvalue for (4.1) is closely related to the existence of a minimizer for the following Rayleigh quotient

$$R(u) := \frac{\int_\Omega |\nabla u|^2}{\int_\Omega g\, u^2}, \tag{4.3}$$

with the domain of definition

$$\mathcal{D}^+(g) := \left\{ u \in \mathcal{D}_0^{1,2}(\Omega) : \int_\Omega g u^2 > 0 \right\}.$$

Let

$$\lambda_1 = \inf_{u \in \mathcal{D}^+(g)} R(u). \tag{4.4}$$

Note that, if $\lambda_1 > 0$, then the following inequality holds:

$$\int_\Omega g\, u^2 \leq \frac{1}{\lambda_1} \int_\Omega |\nabla u|^2, \quad \forall\, u \in \mathcal{D}_0^{1,2}(\Omega). \tag{4.5}$$

In addition, if λ_1 is attained for some $u \in \mathcal{D}^+(g)$, then under certain integrability assumptions on g, one may be able to derive Eq: (4.2) as the Euler-Lagrange equation for the minimizer. In order to enlarge the class of weight functions considered in the inequality (4.5) beyond the Lebesgue spaces, Visciglia [77] considered positive weights in $L(\frac{N}{2}, \infty)$. Following this direction Ramaswamy and Lucia in [60] proved the existence of a positive principal eigenvalue for (4.1), when g is such that

$$g \in \bigcup_{1 \leq q < \infty} L\left(\frac{N}{2}, q\right), \quad g^+ \not\equiv 0. \tag{4.6}$$

They have also shown that the above class of weights and those weights considered in [73] by Szulkin and Willem are independent. We unify all the previous works by proving the existence of a positive principal eigenvalue for (4.1), when $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Furthermore, under the same assumption g^+, we obtain the existence of infinitely many eigenvalues of (4.1). The results that we present in this chapter have appeared in [10].

This chapter is organized as follows. In Section 1, we give examples of weights for which (4.1) admits a positive principal eigenvalue and relate our sufficient condition with various sufficient conditions available in the literature. In order to show that (4.1) does not admit an eigenvalue certain classes of weights, we derive a Pohozaev type identity in Section 2. The existence and the other qualitative properties of the first eigenvalue are discussed in Section 3. The existence of infinitely many eigenvalues of (4.1) is studied in Section 4. Further extensions and miscellaneous remarks are given in Section 5.

4.1 Examples of good weight functions

Indeed, in Chapter 3 we had given many examples of weights that are in $\mathcal{F}_{\frac{N}{2}}$. Thus by our main theorem of this chapter, every function g such that $g^+ \in \mathcal{F}_{\frac{N}{2}}$ admits a minimizer for J on $\mathcal{M}$. In particular Proposition 3.0.13 shows that

$$\bigcup_{1\leq q<\infty} L\left(\frac{N}{2}, q\right) \subset \mathcal{F}_{\frac{N}{2}}. \tag{4.7}$$

This shows that weights considered in [4, 22, 21, 60, 61] are in $\mathcal{F}_{\frac{N}{2}}$ and hence our result subsumes all their results.

Another class of admissible weight functions that admit a minimizer for J on $\mathcal{M}$ is provided by the work of Szulkin and Willem in [73]. More specifically they consider the functions g defined by the following conditions:

$$\begin{cases} g \in L^1_{\text{loc}}(\Omega), \quad g^+ = g_1 + g_2 \not\equiv 0, \quad g_1 \in L^{\frac{N}{2}}(\Omega), \\ \lim\limits_{|x|\to\infty\ ,\ x\in\Omega} |x|^2 g_2(x) = 0, \qquad \lim\limits_{x\to a\ ,\ x\in\overline{\Omega}} |x-a|^2 g_2(x) = 0 \quad \forall a \in \overline{\Omega}. \end{cases} \tag{4.8}$$

Lemma 4.1.1. *Let g be a measurable function satisfying condition* (4.8)*. Then g^+ belongs to the space $\mathcal{F}_{\frac{N}{2}}$.*

Proof. Clearly $g_1 \in \mathcal{F}_{\frac{N}{2}}$, since $L^{\frac{N}{2}}(\Omega) \subset \mathcal{F}_{\frac{N}{2}}$ (see Proposition 3.0.13). Further, by Theorem 3.0.24, $g_2 \in \mathcal{F}_{\frac{N}{2}}$. Hence the result. $\square$

The above lemma shows that the weights that we consider here, include those that are in [73]. We emphasize that conditions (4.7) and (4.8) do not exhaust the space $\mathcal{F}_{\frac{N}{2}}$. This is illustrated by the following example:

Example 4.1.2. *There are functions lying in $\mathcal{F}_{\frac{N}{2}}$ which fail to satisfy both* (4.7) *and* (4.8)*. Let $\Omega = \{(x_1, \ldots, x_n) : |x_i| < R\}$ with $R = 2^{-\frac{N}{N-1}}$, and consider*

$$g_1(x) = \begin{cases} |x_1|^{-\frac{2}{N}} \left|\log\left(|\log(|x_1|)|\right)\right|^{-1} & \text{if } x \in \Omega \text{ and } x_1 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly g_1 does not satisfy (4.8)*, since it fails to satisfy the conclusions of Lemma 3.0.23. To show that g_1 does not satisfy* (4.7)*, we first note that the* *distribution function of* g_1 *is given by:*

$$\alpha_{g_1}(s) \;=\; 2^N R^{N-1}|x_1(s)|,$$

where $x_1(s)$ *is the first coordinate of* $x \in \Omega$*, such that* $s = g_1(x)$*. Now for* $t \in (0,(2R)^N)$*,*

$$\begin{aligned} g_1^*(t) &= \inf\{s : \alpha_{g_1}(s) \le t\} \;=\; \inf\left\{s : 2^N R^{N-1}|x_1(s)| \le t\right\} \\ &= \inf\left\{|x_1(s)|^{-\frac{2}{N}}\left|\log\left(|\log(|x_1(s)|)|\right)\right|^{-1} :\; 2^N R^{N-1}|x_1(s)| \le t\right\} \\ &= \left(\frac{t}{2^N R^{N-1}}\right)^{-\frac{2}{N}} \left|\log\left(\left|\log\left(\frac{t}{2^N R^{N-1}}\right)\right|\right)\right|^{-1} \\ &= t^{-\frac{2}{N}}|\log(|\log t|)|^{-1}. \end{aligned}$$

Observe that our choice of R *gives that* $2^N R^{N-1} = 1$ *and* $(2R)^N < 1$*. Since* $|\Omega| = (2R)^N$*,* $g_1^*(t) = 0$ *for* $t \in [(2R)^N, \infty)$*. Thus*

$$g^*(t) = \begin{cases} t^{-\frac{2}{N}}|\log(|\log t|)|^{-1}, & 0 < t < (2R)^N, \\ 0, & t \ge (2R)^N. \end{cases}$$

A straight forward computation shows

$$\int_0^\infty \left\{t^{\frac{2}{N}} g_1^*(t)\right\}^q \frac{dt}{t} \;=\; \int_0^{(2R)^N} \frac{1}{|\log(|\log t|)|^q}\frac{dt}{t} \;=\; \int_{|\log(2R)^N|}^\infty \frac{1}{|\log y|^q} dy.$$

Thus $g_1 \notin L(\frac{N}{2}, q)$ *for any* $q \in [1,\infty)$*, since the last integral is divergent. However*

$$\lim_{t\to 0} t^{-\frac{2}{N}} g_1^*(t) = \lim_{t\to 0}|\log(|\log t|)|^{-1} = 0 = \lim_{t\to\infty} t^{-\frac{2}{N}} g_1^*(t).$$

Now using characterization (3.4) *of* $\mathcal{F}_{\frac{N}{2}}$*, we conclude that* $g_1 \in \mathcal{F}_{\frac{N}{2}}$*.*

4.2 A Pohozaev type identity

In order to show that for certain classes of weights (4.1) does not admit an eigenvalue, we derive a Pohozaev type identity. We consider an elliptic partial differential equation of the form $-\Delta u = a\,u$, where a is a weight function satisfying certain integrability conditions. We derive a necessary condition, namely a Pohozaev type identity, for the existence of a solution. Our Pohozaev type identity is inspired by Proposition 4.5 of [74].

Theorem 4.2.1. *Let $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$ and $a \in L^1_{loc}(\mathbb{R}^N)$ such that $a(x)u^2,\ x \cdot \nabla a(x)u^2 \in L^1(\mathbb{R}^N)$. If u solves*

$$-\Delta u = a(x)u \tag{4.9}$$

in the weak sense, then the following identity holds:

$$\int_{\mathbb{R}^N} \{\, x \cdot \nabla a(x) + 2\,a(x)\} u^2 = 0. \tag{4.10}$$

Proof. Since $u \in H^2_{loc}(\mathbb{R}^N)$ solves Eq. (4.9), we must have

$$-\Delta u = a(x)u,\ \ a.e. \text{ in } \mathbb{R}^N. \tag{4.11}$$

First, we choose a cut-off function $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that

$$(i)\ 0 \le \phi \le 1, \quad (ii)\ \ \phi(r) = 1,\ 0 \le r \le 1, \quad (iii)\ \phi(r) = 0,\ \ r \ge 2.$$

Now for each $n \in \mathbb{N}$, we define

$$\psi_n(x) = \phi\left(\frac{|x|^2}{n^2}\right).$$

It is easy to see that there exists a constant $c > 0$, independent of n, such that

$$|\psi_n(x)|, |x \cdot \nabla\psi_n(x)| \le c, \quad \forall\, x \in \mathbb{R}^N,\ \forall\, n \in \mathbb{N}.$$

By multiplying Eq: (4.11) by $\{x \cdot \nabla u\}\psi_n(x)$ we obtain the following point wise identity:

$$-\Delta u\,\{x \cdot \nabla u\}\psi_n(x) = a(x)u\,\{x \cdot \nabla u\}\psi_n(x),\ \ a.e.\, x \in \mathbb{R}^N. \tag{4.12}$$

Now, straight forward calculations show that the following point wise identities hold:

$$\begin{aligned}
\operatorname{div}\big(\,\nabla u\,\{x \cdot \nabla u\}\,\psi_n(x)\,\big) &= \Delta u\,\{x \cdot \nabla u\}\,\psi_n(x) + \nabla u \cdot \nabla\,(\,\{x \cdot \nabla u\}\,\psi_n(x)\,)\,. \\
\nabla u \cdot \nabla\,(\,\{x \cdot \nabla u\}\,\psi_n(x)) &= \nabla u \cdot \nabla\,\{x \cdot \nabla u\}\,\psi_n(x) + \{x \cdot \nabla u\}\,\nabla u \cdot\ \nabla\psi_n(x). \\
\nabla u \cdot \nabla\,\{x \cdot \nabla u\} &= |\nabla u|^2 + \nabla u\ \cdot\,(\,\nabla^2 u\,)\,x. \\
\operatorname{div}(\,x\,|\nabla u|^2\,) &= N\,|\nabla u|^2 + x \cdot \nabla(\,|\nabla u|^2\,). \\
x \cdot \nabla(|\nabla u|^2) &= 2\,\nabla u \cdot (\nabla^2 u)\,x.
\end{aligned} \tag{4.13}$$

From the above identities we obtain the following:

$$\begin{aligned}\Delta u\,\{x\cdot\nabla u\}\psi_n(x) = \operatorname{div}\left(\nabla u\,\{x\cdot\nabla u\}\,\psi_n(x) - \frac{1}{2}(\,x\,|\nabla u|^2\,)\,\psi_n(x)\right)\\ -\left(1-\frac{N}{2}\right)|\nabla u|^2\,\psi_n(x) - \nabla u\cdot\,\{x\cdot\nabla u\}\nabla\psi_n(x).\end{aligned} \tag{4.14}$$

Next we derive an identity for the right hand side of (4.12). Let $F(u) = \frac{u^2}{2}$. One can easily verify the following identities:

$$\begin{aligned}\operatorname{div}\left(x\,a(x)F(u)\,\psi_n(x)\right) &= Na(x)F(u)\,\psi_n(x) + x\cdot\nabla\left\{a(x)F(u)\,\psi_n(x)\right\}.\\ x\cdot\nabla\left\{a(x)F(u)\,\psi_n(x)\right\} &= x\cdot\nabla a(x)F(u)\,\psi_n(x) + a(x)\,u\,\{x\cdot\nabla u\}\,\psi_n(x)\\ &\quad + a(x)\,F(u)\,x\,\cdot\nabla\psi_n(x).\end{aligned}$$

From the above identities we get

$$\begin{aligned}a(x)u\,\{x\cdot\nabla u\}\,\psi_n(x) &= \operatorname{div}\big(\,x\,a(x)F(u)\,\psi_n(x)\,\big) - N\,a(x)\,F(u)\,\psi_n(x)\\ &\quad - x\cdot\nabla a(x)F(u)\,\psi_n(x) - a(x)\,F(u)\,x\cdot\nabla\psi_n(x).\end{aligned} \tag{4.15}$$

Now using (4.12), (4.14) and (4.15) we obtain the following identity:

$$\begin{aligned}&\operatorname{div}\left\{\left(\nabla u\,\{x\,\cdot\nabla u\} + x\,a(x)\,F(u) - \frac{1}{2}(x|\nabla u|^2)\right)\psi_n(x)\right\}\\ &= \left(1-\frac{N}{2}\right)|\nabla u|^2\psi_n(x) + \{Na(x)F(u) + x\cdot\nabla a(x)F(u)\}\,\psi_n(x)\\ &\quad + \{(x\cdot\nabla u)\,\nabla u\, + a(x)\,F(u)\,x\,\}\cdot\nabla\psi_n(x).\end{aligned} \tag{4.16}$$

Since $\psi_n(x) = 0$ in the complement of the ball $B_{\sqrt{2}n}$, the classical divergence theorem on the ball $B_{\sqrt{2}n}$ yields the following:

$$\begin{aligned}\int_{\mathbb{R}^N}\left\{\left(1-\frac{N}{2}\right)|\nabla u|^2 + Na(x)F(u) + x\cdot\nabla a(x)F(u)\right\}\psi_n(x)dx\\ = -\int_{\mathbb{R}^N}\{(x\cdot\nabla u)\nabla u\, +\, a(x)\,F(u)\,x\}\cdot\nabla\psi_n(x)dx.\end{aligned} \tag{4.17}$$

Note that, the integrands of each of the above integrals are bounded by functions that are in $L^1(\mathbb{R}^N)$. Thus by applying the dominated convergence theorem to pass through the limit, noting that that $\psi_n(x)\to 1$ and $\nabla\psi_n(x)\to 0$

as $n \to \infty$, we obtain

$$\int_{\mathbb{R}^N} \left(1 - \frac{N}{2}\right) |\nabla u|^2 + \{N a(x) + x \cdot \nabla a(x)\} \frac{u^2}{2} = 0. \tag{4.18}$$

Further, multiplying the equation (4.11) by u yields

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_{\mathbb{R}^N} a(x) u^2. \tag{4.19}$$

Now by substituting (4.19) in (4.18) we get the required identity.

$$\int_{\mathbb{R}^N} \{\, x \cdot \nabla a(x) + \, 2\, a(x) \,\}\, u^2 = 0.$$

$\square$

The following corollary is an immediate consequence of the above theorem:

Corollary 4.2.2. *Let a be as in the above theorem. If $x \cdot \nabla a(x) + 2\, a(x)$ has a definite sign, then we have the nonexistence of solution for (4.9) in $\mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$.*

Example 4.2.3. *For the weight $a(x) = \frac{1}{1+|x|^2}$ one can see that*

$$x \cdot \nabla a(x) + 2\, a(x) > 0.$$

Thus we have the nonexistence of solution for (4.9) in $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$. More importantly, in this case we have the nonexistence of solution of (4.9) in $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ itself, since any $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ solution of (4.9) is in $H^2_{loc}(\mathbb{R}^N)$. This follows from the Calderon-Zygmund regularity theorem, since $a \in L^\infty(\mathbb{R}^N)$ and $u \in L^2_{loc}(\mathbb{R}^N)$.

Remark 4.2.4. *It is easy to see that $\frac{1}{1+|x|^2} \in L(\frac{N}{2}, \infty)$. Thus, by using the Hölder inequality (Theorem 2.2.7) and the Lorentz-Sobolev embedding (Theorem 2.3.10) we obtain the following Hardy inequality:*

$$\int_{\mathbb{R}^N} |\nabla u|^2 \leq \frac{1}{\lambda_1} \int_{\mathbb{R}^N} \frac{1}{1+|x|^2} u^2, \ \textit{for } u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N),$$

where

$$\lambda_1 = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 : u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N), \ \int_{\mathbb{R}^N} \frac{u^2}{1+|x|^2} = 1 \right\}.$$

Note that $\frac{1}{\lambda_1}$ *is the best constant in the above inequality. Thus we conclude that the best constant is not attained for any* $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$*, otherwise a minimizer would be a weak solution of* (4.1)*.*

In order to apply Theorem 4.2.1 for proving the nonexistence of an eigenvalue for (4.1), we must show that any solution of (4.1) is in $H^2_{loc}(\mathbb{R}^N)$. However one can obtain (4.10), even for the solutions of (4.9) that are in $H^2_{loc}\left(\mathbb{R}^N \setminus \{x_1, \ldots, x_n\}\right)$. For example, we have the following corollary:

Corollary 4.2.5. *Let* $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap H^2_{loc}\left(\mathbb{R}^N \setminus \{0\}\right)$ *and* $a \in L^1_{loc}(\mathbb{R}^N)$ *such that* $a(x)u^2$*,* $x \cdot \nabla\, a(x)u^2 \in L^1(\mathbb{R}^N)$*. If* u *solves*

$$-\Delta u = a(x)u \tag{4.20}$$

in the weak sense, then the following identity holds:

$$\int_{\mathbb{R}^N} \{\, x \cdot \nabla a(x) + \, 2\, a(x)\,\} u^2 = 0. \tag{4.21}$$

Proof. One can imitate the proof of Theorem 4.2.1, by choosing the following cut-off functions:

$$\psi_n(x) = (1 - \phi(n^2|x|^2))\phi\left(\frac{|x|^2}{n^2}\right).$$

□

The following example is considered in [73, 74].

Example 4.2.6. *For the weight* $a(x) = \frac{1}{|x|^2(1+|x|^2)}$ *one can verify that*

$$x \cdot \nabla a(x) + 2\, a(x) < 0$$

and that any $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ *solution of* (4.9) *is at least in* $H^2_{loc}(\mathbb{R}^N \setminus \{0\})$*. Thus from the above corollary we have the nonexistence of the solution for* (4.9) *in* $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$*. In particular, we have the nonexistence of an eigenvalue for* (4.1)*, when* $g(x) = \frac{1}{|x|^2(1+|x|^2)}$*.*

Remark 4.2.7. *Note that for the Hardy potential,* $a(x) = \frac{1}{|x|^2}$*, we get*

$$x \cdot \nabla a(x) + 2\, a(x) = 0.$$

Thus our identity holds. However, using the scale invariance of equation (4.1)*, when* $g(x) = \frac{1}{|x|^2}$*, one can obtain the nonexistence of a principal* *eigenvalue, see [47] for a proof. A more general result for the p-Laplacian is proved in Chapter 5.*

4.3 Existence and some properties of the first eigenvalue

4.3.1 Existence of an eigenvalue

In this section we prove the existence of an eigenvalue for (4.1) using a direct variational principle. For $g \in L^1_{loc}(\Omega)$, first of recall the definitions of R, $\mathcal{D}^+(g)$ and let us define the following:

$$\begin{aligned} \mathcal{M} &:= \left\{ u \in \mathcal{D}_0^{1,2}(\Omega) : \int_\Omega g u^2 = 1 \right\}, \\ J(u) &:= \frac{1}{2} \int_\Omega |\nabla u|^2. \end{aligned}$$

Now we consider the following minimization problems:

$$\begin{aligned} &(\mathbf{P}) \quad \text{Minimize } R \ \text{ on } \ \mathcal{D}^+(g)\,, \\ &(\widetilde{\mathbf{P}}) \quad \text{Minimize } J \text{ on } \ \mathcal{M}\,. \end{aligned}$$

Observe that due to the homogeneity of the Rayleigh quotient R, problems $(\mathbf{P})$ and $(\widetilde{\mathbf{P}})$ are equivalent. Furthermore, if problem $(\mathbf{P})$ admits a minimizer u, then we show that u is an eigenfunction of (4.1), corresponding to the eigenvalue $R(u)$. Recall that

$$\lambda_1 = \inf_{u \in \mathcal{M}} J(u).$$

Note that, if $(\mathbf{P})$ admits a minimizer, then λ_1 must be nonzero. From the definition of λ_1, it is clear that $\lambda_1 \geq 0$. Thus, for the existence of a minimizer for J on $\mathcal{M}$, we must have $\lambda_1 > 0$. In the following lemma we show that λ_1 is positive, when $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. In fact, we obtain a positive lower bound for λ_1 in terms of $\|g^+\|_{(\frac{N}{p},\infty)}$.

Lemma 4.3.1. *If $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$, then $\lambda_1 > 0$.*

Proof. Let $u \in \mathcal{M}$. Thus by Lorentz-Sobolev embedding (Theorem 2.3.10), $u \in L(2^*, 2)$. Now using property (2.10) we get $u^2 \in L(\frac{2^*}{2}, 1)$. Further, $g^+ \in L(\frac{N}{2}, \infty)$, which is the dual space of $L(\frac{2^*}{2}, 1)$. Thus by using Hölder inequality (see (2.9)) we obtain:

$$\int_\Omega g^+ u^2 \le C \left\|g^+\right\|_{\left(\frac{N}{2},\infty\right)} \left\|u^2\right\|_{\left(\frac{2^*}{2},1\right)}, \tag{4.22}$$

where C is a constant independent of u. Moreover using (2.10) and Lorentz-Sobolev embedding we get

$$\left\|u^2\right\|_{\left(\frac{2^*}{2},1\right)} \le \|u\|^2_{(2^*,2)} \le C_s \int_\Omega |\nabla u|^2, \tag{4.23}$$

where C_s is the best constant that appears in Lorentz-Sobolev embedding. Note that $\int_\Omega g\, u^2 \le \int_\Omega g^+ u^2$. Since $u \in \mathcal{M}$, we have $\int_\Omega gu^2 = 1$. Now using (4.22) and (4.23) we obtain the following inequality:

$$1 \le C\, C_s \left\|g^+\right\|_{\left(\frac{N}{2},\infty\right)} \int_\Omega |\nabla u|^2.$$

Therefore

$$\frac{1}{CC_s \left\|g^+\right\|_{\left(\frac{N}{2},\infty\right)}} \le \lambda_1.$$

□

From the definition of the space $\mathcal{D}_0^{1,2}(\Omega)$, it is obvious that J is coercive and weakly lower semi-continuous. Now using some standard results in functional analysis one may conclude the existence of a minimizer for J on $\mathcal{M}$, provided $\mathcal{M}$ is weakly closed. But, this is far from being satisfied, due to the weak assumption on g^-. Observe that the map G defined as

$$G(u) := \frac{1}{2} \int_\Omega g\, u^2 \tag{4.24}$$

may not be even continuous and hence the set $\mathcal{M}$ is not even be closed in $\mathcal{D}_0^{1,2}(\Omega)$. Nevertheless, we are not interested in the existence of weak limits of every weakly convergent sequence in $\mathcal{M}$, rather we are concerned particularly with the existence of a weak limit of the minimizing sequence of J on $\mathcal{M}$. This may hold true, even when $\mathcal{M}$ is not weakly closed.

In the sequel, for the ease of exposition, we will say that a map $T{:}X \to Y$ between Banach spaces X and Y is **completely continuous** *if $x_n \rightharpoonup x$ (weakly) in X implies that $T(x_n) \to T(x)$(strongly in Y.)*

Remark 4.3.2. *Indeed, a completely continuous map is continuous. In addition, if the Banach space X is reflexive, then the complete continuity of*

T is equivalent to the compactness of T.

Since all the solution spaces that we come across in this thesis are reflexive Banach spaces, here after we use the word compact instead of completely continuous.

Next we prove the following result.

Lemma 4.3.3. *Let $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Let*

$$G^+(u) = \frac{1}{2}\int_\Omega g^+ u^2.$$

Then $G^+ : \mathcal{D}_0^{1,2}(\Omega) \to \mathbb{R}$ is compact.

Proof. Let $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,2}(\Omega)$. We show that $\{G^+(u_n)\}$ converges to $G^+(u)$. For $\phi \in \mathcal{C}_c^\infty(\Omega)$, we have

$$2(G^+(u_n) - G^+(u)) = \int_\Omega \phi(u_n^2 - u^2) + \int_\Omega (g^+ - \phi)\,(u_n^2 - u^2). \qquad (4.25)$$

We estimate the second integral using Lorentz Sobolev embedding and Hölder inequality as below

$$\int_\Omega |g^+ - \phi|\,\left|u_n^2 - u^2\right| \le C\left\|g^+ - \phi\right\|_{(\frac{N}{2},\infty)} \left\{\,\|u_n\|^2_{(2^*,2)} + \|u\|^2_{(2^*,2)}\,,\right\} \qquad (4.26)$$

where C is a constant independent of ϕ. Since the sequence $\{u_n\}$ is bounded in $\mathcal{D}_0^{1,2}(\Omega)$, $\{u_n\}$ is also bounded in $L(2^*, 2)$ by Theorem 2.3.10. Let

$$m := \sup_n \left\{\|u_n\|^2_{(2^*,2)} + \|u\|^2_{(2^*,2)}\right\}.$$

Now using the definition of the space $\mathcal{F}_{\frac{N}{2}}$, for a given $\varepsilon > 0$, we choose $g_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ so that

$$\left\|g^+ - g_\varepsilon\right\|_{(\frac{N}{2},\infty)} < \frac{\varepsilon}{mC}.$$

Thus by taking $\phi = g_\varepsilon$ in (4.26), we obtain

$$\int_\Omega |(g^+ - g_\varepsilon)|\,\left|(u_n^2 - u^2)\right| < \varepsilon. \qquad (4.27)$$

Since $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^2_{loc}(\Omega)$ compactly, the first integral in the right hand side of (4.25) with $\phi = g_\varepsilon$ can be made arbitrary small for large $n \in \mathbb{N}$. Hence we can choose $n_0 \in \mathbb{N}$, so that

$$\int_\Omega g_\varepsilon(u_n^2 - u^2) < \varepsilon, \ \ \forall\, n > n_0. \tag{4.28}$$

Now using (4.27), (4.28) together with (4.25) we conclude that $\{G^+(u_n)\}$ converges to $G^+(u)$. $\square$

Remark 4.3.4. *If we assume that $g \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$, i.e. both g^+ and g^- are in $\mathcal{F}_{\frac{N}{2}} \setminus \{0\}$, then the map $G : \mathcal{D}_0^{1,2}(\Omega) \to \mathbb{R}$ is also compact.*

Now we are in a position to prove the existence of a minimizer for J on $\mathcal{M}$.

Theorem 4.3.5. *Let $g \in L^1_{loc}(\Omega)$ and let $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Then J admits a minimizer on $\mathcal{M}$.*

Proof. Since $g \in L^1_{loc}(\Omega)$ and $g^+ \neq 0$, there exists $\varphi \in C_c^\infty(\Omega)$ such that $\int_\Omega g\varphi^2 > 0$ (see for example, Proposition 4.2 of [50]) and hence $\mathcal{M} \neq \emptyset$. Let $\{u_n\}$ be a minimizing sequence of J on $\mathcal{M}$, i.e,

$$\lim_{n\to\infty} J(u_n) = \lambda_1 = \inf_{u\in\mathcal{M}} J(u).$$

Now by the coercivity of $J, \{u_n\}$ is bounded in $\mathcal{D}_0^{1,2}(\Omega)$. Hence using the reflexivity of $\mathcal{D}_0^{1,2}(\Omega)$ we obtain a subsequence of $\{u_n\}$ that converges weakly to some $u \in \mathcal{D}_0^{1,2}(\Omega)$. Let us denote the subsequence by $\{u_n\}$ itself. Since the map G^+ is compact, we get

$$\lim_{n\to\infty} \int_\Omega g^+ u_n^2 = \int_\Omega g^+ u^2. \tag{4.29}$$

Now as $u_n \in \mathcal{M}$ we write,

$$\int_\Omega g^- u_n^2 = \int_\Omega g^+ u_n^2 - 1.$$

Since the embedding $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^2_{loc}(\Omega)$ is compact, up to a subsequence $u_n \to u$ a.e in Ω. Now we apply the Fatou's lemma and let n goes to infinity in the above equation to obtain

$$\int_\Omega g^- u^2 \le \int_\Omega g^+ u^2 - 1.$$

This shows that $\int_\Omega gu^2 \geq 1$. Setting $\widetilde{u} := \frac{u}{(\int_\Omega gu^2)^{\frac{1}{2}}}$, the weak lower semi continuity of J yields the following,

$$\lambda_1 \leq J(\widetilde{u}) = \frac{J(u)}{\int_\Omega gu^2} \leq J(u) \leq \liminf_n J(u_n) = \lambda_1.$$

Thus equality must hold at each step and hence $\int_\Omega gu^2 = 1$, which shows that $u \in \mathcal{M}$ and $J(u) = \lambda_1$. $\square$

Remark 4.3.6. *Note that, when $g \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$ the map $G : \mathcal{D}_0^{1,2}(\Omega) \to \mathbb{R}$ is compact. Hence*

$$\lim_{n\to\infty} \int_\Omega gu_n^2 = \int_\Omega g\, u^2$$

for the minimizing subsequence $\{u_n\}$ obtained in the above proof. Since $u_n \in \mathcal{M}$, we have $\int_\Omega gu_n^2 = 1$ and hence $\int_\Omega gu^2 = 1$. Thus we conclude that $u \in \mathcal{M}$ and $J(u) = \lambda_1$.

Proposition 4.3.7. *Let $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Let u be a minimizer of R on $\mathcal{D}^+(g)$. Then u is an eigenfunction corresponding to the eigenvalue λ_1 of* (4.1).

Proof. Notice that $u \in \mathcal{D}^+(g)$ implies that $u \neq 0$. For each $\phi \in \mathcal{C}_c^\infty(\Omega)$, using dominated convergence theorem one can verify that R admits directional derivative along ϕ. Now since u is a minimizer of J on $\mathcal{D}^+(g)$ we get

$$\frac{d}{dt} R(u + t\phi)|_{t=0} = 0.$$

Therefore

$$\int_\Omega \nabla u \cdot \nabla \phi = \lambda_1 \int_\Omega g\, u\, \phi, \quad \forall\, \phi \in \mathcal{C}_c^\infty(\Omega).$$

Now we use the density of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{D}_0^{1,2}(\Omega)$ to conclude that

$$\int_\Omega \nabla u \cdot \nabla v = \lambda_1 \int_\Omega g\, u\, v, \quad \forall\, v \in \mathcal{D}_0^{1,2}(\Omega). \tag{4.30}$$

$\square$

Remark 4.3.8. *If we assume that $g \in L^1_{loc}(\Omega)$ and $g^- \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$, then by the same analysis for the weight $-g$ we obtain a maximum negative eigenvalue μ_1. In particular, when $g \in \mathcal{F}_{\frac{N}{2}}$ with $g^+, g^- \neq 0$, then we obtain a positive and a negative eigenvalue for* (4.1).

4.3.2 Sign of the eigenfunctions

In this section we discuss the sign of the minimizers that we obtained in Proposition 4.3.5. First we study the principal nature of λ_1. Since $R(u) = R(|u|)$, if u is an eigenfunction corresponding to λ_1 then $|u|$ is also an eigenfunction. However this does not mean that λ_1 is a principal eigenvalue, unless the zero set of u is of measure zero. Note that, the sign of a measurable function is well defined only when its zero set is of measure zero. Thus, if u and g are regular enough, one may use classical maximum principle to deduce that $|u|$ is positive except for a set of zero measure. However, when these functions have less regularity one may use a weaker version of the strong maximum principle due to Ancona[8] or Brezis and Ponce [19] . Here we use the following result of Brezis and Ponce, see Corollary 4 of [19]:

Theorem 4.3.9. (Strong Maximum Principle) *Let $O \subset \mathbb{R}^N$ be a nonempty open connected bounded set and $a \in L^1_{loc}(O)$ with $a \geq 0$. Assume that $au \in L^1_{loc}(O)$ and Δu is a Radon measure on O. If*

$$-\Delta u + au \geq 0 \text{ in } \mathcal{D}'(O)$$

and $u = 0$ on a subset of O with positive measure, then $u = 0$ a.e. in O.

Now using the above theorem we prove that the eigenfunctions corresponding to λ_1 are of constant sign.

Proposition 4.3.10. *Let g be as in Theorem 4.3.5. Then every eigenfunctions corresponding to λ_1 is of constant sign.*

Proof. Let u be an eigenfunction corresponding to λ_1. Clearly u^+ or u^- is nonzero. Without loss of generality assume that $u^+ \not\equiv 0$. Now by taking $v = u^+$ in (4.2) we see that u^+ minimizes R on $\mathcal{D}^+(g)$. Thus by Proposition 4.3.7, u^+ solves (4.1) in the weak sense,

$$-\Delta u^+ - \lambda_1 g u^+ = 0, \text{ in } \Omega. \tag{4.31}$$

Thus we have the following differential inequality in the sense of distribution,

$$-\Delta u^+ + \lambda_1 g^- u^+ = \lambda_1 g^+ u^+ \geq 0, \text{ in } \Omega.$$

Note that equation (4.31) shows that $\Delta u^+ \in L^1_{loc}(\Omega)$ and in particular a

Radon measure on Ω. Furthermore,

$$(g^-)^{\frac{1}{2}} \in L^2_{loc}(\Omega), \text{ since } g^- \in L^1_{loc}(\Omega),$$
$$(g^-)^{\frac{1}{2}}(u^+) \in L^2(\Omega), \text{ since } u^+ \text{ is a solution } g(u^+)^2 \in L^1(\Omega).$$

Now let us write

$$g^- u^+ = (g^-)^{\frac{1}{2}}(g^-)^{\frac{1}{2}} u^+$$

and then use Hölder inequality to obtain $g^- u^+ \in L^1_{loc}(\Omega)$. Now in view of Theorem 4.3.9, we obtain $u^+ > 0$ a.e.. Thus $u = u^+$ and moreover, the zero set of u is of measure zero. □

4.3.3 The uniqueness of the positive principal eigenvalue

Here we prove the uniqueness of the positive principal eigenvalue using a Picone's identity.

Theorem 4.3.11. *If* $\lambda > 0$ *is any eigenvalue of* (4.1) *with an eigenfunction which does not change sign, then* $\lambda = \lambda_1$.

Proof. Let v be a positive eigenfunction corresponding to the eigenvalue λ. Let $u \in \mathcal{M}$. Thus there exists a sequence $\{\phi_n\}$ such that $\phi_n \in \mathcal{C}_c^\infty(\Omega)$ and as $n \to \infty$,

$$\|u - \phi_n\|_{\mathcal{D}_0^{1,2}(\Omega)} \to 0.$$

Further, $\frac{\phi_n^2}{(v+\varepsilon)} \in \mathcal{D}_0^{1,2}(\Omega)$ (see Proposition 2.3.7) is a valid test function in (4.2). Thus we obtain

$$\int_\Omega |\nabla \phi_n|^2 - \int_\Omega \nabla \left(\frac{\phi_n^2}{v+\varepsilon} \right) \cdot \nabla v = \int_\Omega |\nabla \phi_n|^2 - \lambda \int_\Omega g\, v \, \frac{\phi_n^2}{v+\varepsilon}. \tag{4.32}$$

A straight calculation shows that the following "Picone's identity" holds:

$$|\nabla \phi_n|^2 - \nabla v \cdot \nabla \left(\frac{\phi_n^2}{v+\epsilon} \right) = \left| \nabla \phi_n - \left(\frac{\phi_n}{v+\epsilon} \right) \nabla v \right|^2. \tag{4.33}$$

By plugging (4.33) in (4.32) we deduce that

$$0 \leq \int_\Omega |\nabla \phi_n|^2 - \lambda \int_\Omega g\, v \, \frac{\phi_n^2}{v+\varepsilon}.$$

Using dominated convergence theorem, we let $\varepsilon \to 0$, to obtain

$$0 \leq \int_\Omega |\nabla \phi_n|^2 - \lambda \int_\Omega g\phi_n^2.$$

Now by letting $n \to \infty$ and by the continuity of the map G^+ (see Corollary 2.2.8) and by Fatou's lemma, we obtain

$$0 \leq \int_\Omega |\nabla u|^2 - \lambda \int_\Omega gu^2 = \int_\Omega |\nabla u|^2 - \lambda.$$

Therefore

$$\lambda \leq \int_\Omega |\nabla u|^2, \quad \forall\, u \in \mathcal{M}.$$

Hence by the definition of λ_1, we conclude that $\lambda = \lambda_1$. □

Remark 4.3.12. *In particular the above theorem shows that all the eigenfunctions corresponding to an eigenvalue $\lambda > \lambda_1$ must change sign.*

4.3.4 The simplicity of the first eigenvalue

Using the connectedness of Ω, we show that λ_1 is simple. For the proof, we adapt a connectedness argument as in [29, 61]. To show that any two eigenfunctions corresponding to λ_1 are linearly dependent, we prove the following lemma first.

Lemma 4.3.13. *Let u and v be two measurable functions on Ω and let $u > 0$ a.e. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, $u - \varepsilon v > 0$ on a set E_ε of positive measure.*

Proof. If not, we can find $\varepsilon_n \downarrow 0$ and sets E_n, each of measure zero, such that $u - \varepsilon_n v \leq 0$ on E_n^c. Since $E = \cup E_n$ also has measure zero and $u - \varepsilon_n v \leq 0$ for all n on E^c, we deduce that $u \leq 0$ a.e, a contradiction. □

Proposition 4.3.14. *Let g and λ_1 be as in Theorem 4.3.5. Then λ_1 is simple.*

Proof. Let u_1, u_2 be two nonnegative eigenfunctions corresponding to λ_1. Define

$$\begin{aligned} A^+ &:= \{\alpha \in \mathbb{R} :\ u_1 + \alpha u_2 > 0 \ \text{ a.e. } \}, \\ A^- &:= \{\alpha \in \mathbb{R} :\ u_1 + \alpha u_2 < 0 \ \text{ a.e. } \}. \end{aligned}$$

(i) Since $0 \in A^+, A^+ \neq \emptyset$.

(ii) Set $u = u_2$ and $v = u_1$ in the above lemma and choose $\varepsilon > 0$ such that $u_2 - \varepsilon u_1 > 0$ on a set E_ε of positive measure. Then by Proposition 4.3.10, $u_2 - \varepsilon u_1 > 0$ a.e. Then $u_1 - \frac{1}{\varepsilon} u_2 < 0$ a.e. and so $A^- \neq \emptyset$.

(iii) Let $\alpha \in A^+$. Then $u_1 + \alpha u_2 > 0$ a.e. Set $u = u_1 + \alpha u_2$ and $v = u_2$ in the above lemma to obtain an $\varepsilon > 0$ such that $u_1 + \alpha u_2 - \varepsilon u_2 > 0$ on a set E_ε of positive measure. Since $u_1 + (\alpha - \varepsilon) u_2$ is an eigenfunction of (4.1), by Proposition 4.3.10, $u_1 + (\alpha - \varepsilon) u_2 > 0$ a.e. Thus for every $\beta > \alpha - \varepsilon$, $u_1 + \beta u_2 > 0$ a.e. Thus A^+ is open.

(iv) Let $\alpha \in A^-$. Then $-(u_1 + \alpha u_2) > 0$ a.e. Set $u = -(u_1 + \alpha u_2)$ and $v = u_2$ in the above lemma to deduce, as in (iii), that A^- is also open.

Now from the connectedness of $\mathbb{R}$, we deduce that $\mathbb{R} \setminus (A^+ \cup A^-) \neq \emptyset$. Hence there exists α_0 such that $u_1 + \alpha u_2$ vanishes on a set of positive measure and therefore must vanish a.e. in Ω (by Proposition 4.3.10). This shows that u_1 and u_2 are linearly dependent. Thus we conclude that the first eigenfunction is unique up to a constant multiple. $\square$

Now we state the main result of this chapter. In view of Theorem 4.3.5, Proposition 4.3.10, Theorem 4.3.11 and Proposition 4.3.14 we have the following theorem:

Theorem 4.3.15. *Let $N \geq 3$ and let $g \in L^1_{loc}(\Omega)$ such that $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Then*

$$\lambda_1 = \inf \{J(u) : u \in M\} \tag{4.34}$$

is the unique positive principal eigenvalue of (4.1). *Furthermore, each eigenfunction corresponding to λ_1 is of constant sign and λ_1 is simple.*

4.4 Existence of an infinite sequence of eigenvalues

In this section, we prove the existence of an infinite sequence of eigenvalues of (4.1), when $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{2}}$. First of all we obtain an infinite orthogonal set in $\mathcal{M}$, using an idea from Proposition 4.2 of [50].

Lemma 4.4.1. *Let $g \in L^1_{loc}(\Omega)$ and $g^+ \neq 0$. Then for each $n \in \mathbb{N}$, there exist $v_1, v_2, \ldots, v_n \in \mathcal{M}$ with disjoint supports.*

Proof. Let $\Omega^+ = \{x : g^+(x) > 0\}$. Since $|\Omega^+| > 0$, using the Lebesgue-Besicovitch differentiation theorem, one can choose n points $x_1, x_2, \ldots x_n$ in Ω^+ such that

$$\lim_{r \to 0} \frac{1}{|B_r(x_i)|} \int_{B_r(x_i)} g(y) dy = g(x_i) > 0.$$

Thus there exists $R > 0$, such that $B_R(x_i) \cap B_R(x_j) = \emptyset$ and

$$\frac{1}{|B_r(x_i)|} \int_{B_r(x_i)} g(y)dy > \frac{g(x_i)}{2}, \forall\, r \in (0, R). \tag{4.35}$$

In particular there exists $m > 0$ such that $\int_{B_r(x_i)} g(y)dy > m$, $\forall\, r \in \left(\frac{R}{2}, R\right)$. Thus we can choose $r \in \left(\frac{R}{2}, R\right)$ so that

$$\int_{B_R(x_i)\setminus B_r(x_i)} |g(y)|dy < \int_{B_r(x_i)} g(y)dy. \tag{4.36}$$

Let $u_i \in \mathcal{C}_c^\infty(B_R(x_i))$ be such that $0 \le u_i(x) \le 1$ and $u_i \equiv 1$ on $B_r(x_i)$. Now using (4.36) we have the following

$$\int_{B_R(x_i)} gu_i^2 = \int_{B_r(x_i)} g + \int_{B_R(x_i)\setminus B_r(x_i)} gu_i^2 \ge \int_{B_r(x_i)} g - \int_{B_R(x_i)\setminus B_r(x_i)} |g| > 0$$

Take $v_i = \frac{u_i}{(\int_\Omega gu_i^2)^{\frac{1}{2}}}$. Clearly $v_i \in \mathcal{M}$ and their supports are disjoint. □

Corollary 4.4.2. *Let Y be a finite dimensional subspace of $\mathcal{D}_0^{1,2}(\Omega)$. Then $Y^\perp \cap \mathcal{M} \neq \emptyset$.*

Proof. Let dim(Y)=m. Thus by applying the above lemma for $m+1$, we get $v_1, v_2, \dots v_{m+1} \in \mathcal{M}$ with disjoint supports. Let $Z =$ span$\{Y, v_1, \dots v_{m+1}\}$. Since the set $\{v_1, v_2, \dots, v_{m+1}\}$ is linearly independent, we have

$$\dim(Z) \ge m + 1 > \dim(Y).$$

Thus, there exists $v \in Z$, $v \perp Y$ and hence $v = \sum_{i=1}^{m+1} \alpha_i v_i$, for some $\alpha_i \in \mathbb{R}$, not all zeros. Furthermore,

$$\int_\Omega g\, v^2 = \alpha_1^2 + \dots \alpha_{m+1}^2.$$

Let $\widetilde{v} = \frac{v}{\sqrt{\alpha_1^2 + \dots \alpha_{m+1}^2}}$. Then $\widetilde{v} \in \mathcal{M}$ and $\widetilde{v} \perp Y$. □

Theorem 4.4.3. *Let $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{2}} \setminus \{0\}$. Then* (4.1) *admits a sequence of positive eigenvalues going to ∞.*

Proof. For each $n \in \mathbb{N}$, using Theorem 4.3.5, we obtain $u_n \in \mathcal{M}$ such that

$$\lambda_n = J(u_n) = \inf_{\{u \in \mathcal{M} : u \perp Y_{n-1}\}} J(u),$$

where $Y_0 = \{0\}$, $Y_{n-1} = span\{u_1, u_2, \ldots, u_{n-1}\}$. From the above corollary, it is clear that $\{u \in \mathcal{M} : u \perp Y_{n-1}\} \neq \emptyset$, for each n. Now by the same argument as in Proposition 4.3.7, we see that

$$\int_\Omega \nabla u_n \cdot \nabla v = \lambda_n \int_\Omega g\, u_n\, v, \quad \forall\, v \in Y_{n-1}^{\perp}.$$

Further, for $i = 1, 2 \ldots, n-1$

$$0 = \int_\Omega \nabla u_i \cdot \nabla u_n = \lambda_i \int_\Omega g\, u_i\, u_n\,.$$

But for $v \in Y_{n-1}$, $v = \sum_{i=1}^{n-1} a_i u_i$ and hence $\int_\Omega g\, u_n\, v = 0$, for $v \in Y_{n-1}$. Thus

$$\int_\Omega \nabla u_n \cdot \nabla v = \lambda_n \int_\Omega g\, u_n\, v, \quad \forall\, v \in \mathcal{D}_0^{1,2}(\Omega).$$

Thus we obtain an infinite orthogonal set $\{u_n\}$ of eigenfunctions of (4.1) in $\mathcal{M}$. Next we show that the sequence $\{\lambda_n\}$ is unbounded. Now by setting $v_n = \frac{u_n}{\sqrt{\lambda_n}}$, we see that $\{v_n\}$ is an orthonormal sequence in $\mathcal{D}_0^{1,2}(\Omega)$ and hence $v_n \rightharpoonup 0$. Note that

$$\lambda_n^{-1} = \int_\Omega g v_n^2 \leq \int_\Omega g^+ v_n^2.$$

Now we use the compactness of G^+ to obtain

$$0 \leq \lim_{n\to\infty} \lambda_n^{-1} \leq \lim_{n\to\infty} \int_\Omega g v_n^2 = 0.$$

This shows that $\lim\limits_{n\to\infty} {\lambda_n}^{-1} = 0$ and hence the sequence $\{\lambda_n\}$ of eigenvalues of (4.1) is unbounded. $\square$

Remark 4.4.4. *If $g \in \mathcal{F}_{\frac{N}{2}}$, then using a similar technique as in Lemma 4.3.3 one can verify that the map $G' : \mathcal{D}_0^{1,2}(\Omega) \to [\mathcal{D}_0^{1,2}(\Omega)]'$ defined by*

$$\left\langle G'(u), v \right\rangle = \int_\Omega guv$$

is a compact operator. Indeed, later we show that the map G' is the derivative of G. Now as the map $(-\Delta)^{-1} : [\mathcal{D}_0^{1,2}(\Omega)]' \to \mathcal{D}_0^{1,2}(\Omega)$ is bounded and linear, the map $L = (-\Delta)^{-1} G'$ is a compact operator on $\mathcal{D}_0^{1,2}(\Omega)$. Also L is a self *adjoint operator. Note that problem* (4.2) *is equivalent to solving:*

$$u - \lambda L(u) = 0. \tag{4.37}$$

Thus the reciprocal of a nonzero eigenvalue of L is an eigenvalue of (4.1)*. By the spectral theorem for the self adjoint compact operators we have an orthonormal basis for the* $Ker(L)^{\perp}$*. Now if* $g \neq 0$*, using Lemma 4.4.1, we obtain an infinite linearly independent set in* $Ker(L)^{\perp}$*. Thus the set of all eigenvalues of* (4.1) *is infinite, since eigenspace corresponding to each eigenvalue of L is finite dimensional. Since 0 is the only possible limit point of eigenvalues of L, the set of all eigenvalues of* (4.1) *is also unbounded.*

4.5 Miscellaneous remarks

Remark 4.5.1. *In [10], we have considered the existence of positive solutions for the following nonlinear problem:*

$$-\Delta u + V(x)u = \lambda g(x)|u|^{p-2}u, \quad u|_{\partial\Omega} = 0, \tag{4.38}$$

for some appropriate value of the parameter λ*, where* Ω *is a non-empty open connected subset of* $\mathbb{R}^N$ *with* $N \geq 2$ *and* $V, g \in L^1_{loc}(\Omega)$*. Indeed we prove the existence of a positive solution of* (4.38)*, using a similar variational technique as in Theorem 4.3.5, when* $p \in [2, 2^*)$ *and*

$$V \geq 0, \quad g^+ \in \mathcal{F}_{\widetilde{p}} \setminus \{0\} \ \text{with} \ \frac{1}{\widetilde{p}} + \frac{p}{2^*} = 1. \tag{4.39}$$

Remark 4.5.2. *A natural question is whether one can get the existence of a principal eigenfunction when* $g^+ \notin \mathcal{F}_{\frac{N}{2}}$*.*

(a) Tertikas in [74] introduced the notion of subcritical potential and using the concentration compactness lemma, he showed that for every subcritical potential g, the problem (4.1) *admits a principal eigenvalue (see [74, Corollary 3.6]). He also showed that such existence still holds if* $g = \frac{1}{|x|^2} + g_1(x)$*, where* $g_1(x) > 0$ *is any subcritical potential (see [74, Theorem 1.7]). One can verify that all positive functions in* $\mathcal{F}_{\frac{N}{2}}$ *are subcritical potentials. Clearly for any positive weight* $g_1 \in \mathcal{F}_{\frac{N}{2}}$*,* $\frac{1}{|x|^2} + g_1(x)$ *cannot be in* $\mathcal{F}_{\frac{N}{2}}$*, but* (4.1) *still admits a principal eigenvalue when* $g(x) = \frac{1}{|x|^2} + g_1(x)$*.*

(b) In [25] using the concentration compactness lemma, Chabrowski proved

the existence of a positive principal eigenvalue for (4.1) *in bounded domains for certain weights of the form* $g(x) = \frac{m(x)}{|x|^2}$, *where* $m \in \mathcal{C}(\overline{\Omega})$ *and* $m(0)\lambda_1(g) < \left(\frac{N-2}{2}\right)^2$. *There are weights satisfying Chabrowski's conditions but not lie in* $\mathcal{F}_{\frac{N}{2}}$.

Chapter 5

On solution branches of semilinear elliptic equations

In this chapter, we are concerned with the existence of a nontrivial solution branch for the semilinear elliptic partial differential equations of the following type:

$$-\Delta u = \lambda f(x,u), \quad u \in \mathcal{D}_0^{1,2}(\Omega). \tag{5.1}$$

The existence of a solution branch for the above equation is too general to be answered affirmatively, unless we make further assumptions on the structure of $f(x,u)$. The functions f that we are considering here, can be put into two different classes, depending on whether $f(x,0)$ is zero or not. These two classes leads to two different kinds of behaviour of the solution branches of (5.1), namely the presence of the trivial branch of (5.1) when $f(x,0)=0$ and its absence when $f(x,0)\neq 0$. Moreover, in the first case we obtain solutions only for sufficiently large λ and in the other case we obtain a solution branch for small λ. However, in both the cases, the existence of nontrivial solution branches are obtained via two different methods, both of them rely on the implicit function theorem in its core.

The rest of this chapter is divided into two sections, each of them studies the existence of a nontrivial solution branch of (5.1) with a different set of assumptions on the structure of $f(x,u)$ that fixes a suitable functional framework for (5.1). In Section 1, we prove the existence of a nontrivial solution branch for (5.1) using the Rabinowitz bifurcation theorem. In Section 2, we obtain the existence of positive solution branch for (5.1) by a direct application of the implicit function theorem, in a suitable functional framework. In that section we improve the regularity of the solution branch using a Moser type iteration. Further, some examples of weights satisfying our conditions are given in Section 2.

5.1 Existence of a solution branch using the bifurcation theory

In this section, as an application of Theorem 4.3.15, we study the existence of a non trivial solution branch for the following semilinear equation:

$$-\Delta u = \lambda g\, u + \lambda h\, r(u), \quad u \in \mathcal{D}_0^{1,2}(\Omega), \tag{5.2}$$

where λ is a real parameter and $\Omega \subset \mathbb{R}^N$ with $N \geq 3$. The domain Ω is assumed to be connected; however no assumptions on the the boundedness of Ω are made. The weights g and h are assumed to be in certain weak Lebesgue spaces. More precisely, we assume the following:

$$\textbf{(H1)} \begin{cases} r \in \mathcal{C}(\mathbb{R}), \quad |r(s)| \leq C|s|^{\gamma-1}, \text{ for } \gamma \in [1, 2^*) \text{ and } C > 0. \\ \\ \lim_{|s| \to 0} \frac{|r(s)|}{|s|} = 0, \text{ if } 1 \leq \gamma \leq 2. \end{cases}$$

$$\textbf{(H2)} \begin{cases} g \in \mathcal{F}_{\frac{N}{2}}, \quad g^+ \neq 0, \\ \\ h \in \begin{cases} \mathcal{F}_{\tilde{\gamma}} \text{ if } \gamma \geq 2, \quad \text{where } \frac{1}{\tilde{\gamma}} + \frac{\gamma}{2^*} = 1, \\ \mathcal{F}_{\frac{N}{2}} \text{ if } 1 \leq \gamma < 2. \end{cases} \end{cases}$$

A typical nonlinearity is the subcritical power, i.e $r(u) = |u|^{\gamma-1}, \ \gamma \in [1, 2^*)$. Note that $(\lambda, 0)$ is a trivial solution branch of (5.2), since $r(0) = 0$. Here, we are interested in certain sufficient conditions that ensure the existence of a nontrivial solution branch for (5.2), bifurcating from the branch of zero solutions. Under the assumptions (H1) and (H2) we obtained a global bifurcation results that generalize the results of [60]. The results that we present in this section are published in [10].

For g, h and r satisfying $(H1)$ and $(H2)$, let S denote the set of non-trivial solutions of (5.2), i.e.,

$$\mathcal{S} = \left\{ (\lambda, u) \in \mathbb{R} \times \mathcal{D}_0^{1,2}(\Omega) \, : \, (\lambda, u) \text{ solves (5.2)}, \, u \not\equiv 0 \right\}.$$

Further, set

$$\sigma(g) = \big\{\lambda : \lambda \text{ is an eigenvalue of } (4.1)\big\}.$$

From Remark (4.4.4), it is clear that $\sigma(g)$ is non-empty. Now we state our global bifurcation result:

Theorem 5.1.1. *Assume (H1), (H2) and $g^+ \not\equiv 0$. Then, there exists a connected branch $\mathcal{C}^+$ in $\overline{\mathcal{S}}^{\mathbb{R}\times\mathcal{D}_0^{1,2}(\Omega)}$ bifurcating from $(\lambda_1, 0)$. Moreover,*

(i) either $\mathcal{C}^+$ is unbounded in $\mathbb{R}\times\mathcal{D}_0^{1,2}(\Omega)$,
(ii) or $(\lambda, 0) \in \mathcal{C}^+$ with $\lambda_1 \neq \lambda \in \sigma(g)$.

First, we formulate our problem in a suitable functional framework. Note that the map $(-\Delta)^{-1} : [L(2^*, 2)]' \to \mathcal{D}_0^{1,2}(\Omega)$ is continuous, see Remark 2.3.6. Hence by defining

$$L : \mathcal{D}_0^{1,2}(\Omega) \to \mathcal{D}_0^{1,2}(\Omega), \qquad H : \mathbb{R}\times\mathcal{D}_0^{1,2}(\Omega) \to \mathcal{D}_0^{1,2}(\Omega), \tag{5.3}$$

$$L(u) = (-\Delta)^{-1}(g\,u), \tag{5.4}$$

$$H(\lambda, u) = (-\Delta)^{-1}(\lambda\, h\, r(u)), \tag{5.5}$$

the problem (5.2) is equivalent to solving

$$u = \lambda L(u) + H(\lambda, u), \quad u \in \mathcal{D}_0^{1,2}(\Omega). \tag{5.6}$$

Under hypotheses (H1)-(H2), we show that H and L satisfy all the requirements of the following global bifurcation theorem of Rabinowitz:

Theorem 5.1.2. (Rabinowitz, [66]) *Given a Banach space $(B, \|\cdot\|)$, consider a mapping*

$$G : \mathbb{R}\times B \to B, \qquad G(\lambda, u) = \lambda L(u) + H(\lambda, u),$$

where $L : B \to B$ is a compact linear operator and $H(\lambda,\cdot) : B \to B$ is a continuous compact mapping satisfying $\lim\limits_{\|u\|\to 0} \dfrac{\|H(\lambda, u)\|}{\|u\|} = 0$. Let

$$r(L) := \big\{\mu \in \mathbb{R} : \mu^{-1} \textit{ is an eigenvalue of } L \textit{ with odd multiplicity}\big\},$$

$$\mathcal{S} := \big\{(\lambda, u) \in \mathbb{R}\times B : (\lambda, u) \textit{ is solution of } u = G(\lambda, u), u \not\equiv 0\big\}.$$

Then, given $\mu \in r(L)$, $\overline{\mathcal{S}}$ has a connected branch $\mathcal{C}_\mu$ bifurcating from $(\mu, 0)$

and

(i) *either* $\mathcal{C}_\mu$ *is unbounded in* $\mathbb{R} \times B$,
(ii) *or,* $(\hat{\mu}, 0) \in \mathcal{C}_\mu$ *with* $\mu \neq \hat{\mu} \in r(L)$.

First we prove the following compactness result, which is a nonlinear version of Lemma 4.3.3.

Lemma 5.1.3. *Let* $r \in C^0(\mathbb{R})$ *satisfying* $|r(s)| \le C|s|^{\alpha-1}$ *for some* $\alpha \in [2, 2^*)$ *and* $w \in \mathcal{F}_{\widetilde{\alpha}}$ *with* $\frac{1}{\widetilde{\alpha}} + \frac{\alpha}{2^*} = 1$. *Then the operator*

$$N : \mathcal{D}_0^{1,2}(\Omega) \to [L(2^*, 2)]', \quad N(u) := w\, r(u), \tag{5.7}$$

is compact.

Proof. Let $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,2}(\Omega)$. First we show that a subsequence of $\{N(u_n)\}$ converges to $N(u)$ in $[L(2^*, 2)]'$. For $\phi \in C_c^\infty(\Omega)$ and for $v \in L(2^*, 2)$ we write:

$$\big|N(u_n)(v) - N(u)(v)\big| \le \int_\Omega \big|\phi\big(r(u_n) - r(u)\big)\big||v| + \int_\Omega |w - \phi||r(u_n) - r(u)||v|. \tag{5.8}$$

Using the growth condition on r, we have $r(u_n) - r(u) \in L^{\frac{2^*}{\alpha-1}}(\Omega)$, since $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$. Now we estimate the first integral in the right hand side of (5.8) using the Hölder inequality in Lebesgue spaces.

$$\int_\Omega \big|\phi(r(u_n) - r(u))\big||v| \le \big\|\phi\,(r(u_n) - r(u))\,\big\|_{[2^*]'} \|v\|_{2^*}. \tag{5.9}$$

Using our growth assumption on r, we have

$$\big|\phi\big(r(u_n) - r(u)\big)\big|^{[2^*]'} \le C_2|\phi| \left(|u_n|^{(\alpha-1)[2^*]'} + |u|^{(\alpha-1)[2^*]'}\right). \tag{5.10}$$

Note that $(\alpha - 1)[2^*]' < 2^*$ and $\mathcal{D}_0^{1,2}(\Omega) \hookrightarrow L^q_{loc}(\Omega)$ is compact for $q \in [1, 2^*)$. Thus the right hand side of (5.10) converges in $L^1_{loc}(\Omega)$ up to a subsequence, say $\{u_{n_k}\}$ of $\{u_n\}$. Further, since r is continuous, $r(u_{n_k}) \to r(u)$ a.e. in Ω. Now by applying the generalized dominated convergence theorem (Theorem 17 Chapter 4 of [67]), we have $\phi\, r(u_{n_k}) \to \phi\, r(u)$ in $L^{2^{*'}}(\Omega)$. Therefore from (5.9), noting that $\|v\|_{2^*} \le C_0 \|v\|_{(2^*,2)}$, we deduce the existence of

$k_0 \in \mathbb{N}$ such that

$$\int_\Omega |\phi||r(u_{n_k}) - r(u)||v| < \varepsilon \, \|v\|_{(2^*,2)} \, , \, \forall \, k \geq k_0. \tag{5.11}$$

The second integral in (5.8) can be estimated using the growth assumption on r, Proposition 2.2.7 and using the Lorentz-Sobolev embedding as below:

$$\begin{aligned}\int_\Omega |w-\phi||r(u_n) - r(u)||v| &\leq C \int_\Omega |w-\phi|\big(|u_n|^{\alpha-1} + |u|^{\alpha-1}\big)|v| \\ &\leq C_1 \big\|w-\phi\big\|_{(\widetilde{\alpha},\infty)} \big\||u_n|^{\alpha-1} + C_1|u|^{\alpha-1}\big\|_{\left(\frac{2^*}{\alpha-1},2\right)} \|v\|_{(2^*,2)}.\end{aligned}$$

Note that,

$$\big\||u_n|^{\alpha-1} + |u|^{\alpha-1}\big\|_{\left(\frac{2^*}{\alpha-1},2\right)} \leq C_2 \left(\|u_n\|^{\alpha-1}_{(2^*,2)} + \|u\|^{\alpha-1}_{(2^*,2)}\right), \forall \, n.$$

Therefore

$$m = \sup_n \left\{ \big\||u_n|^{\alpha-1} + |u|^{\alpha-1}\big\|_{\left(\frac{2^*}{\alpha-1},2\right)} \right\} < \infty.$$

Since $w \in \mathcal{F}_{\widetilde{p}}$, we choose $w_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$, such that

$$\|w - w_\varepsilon\|_{(\widetilde{p},\infty)} \leq \frac{\varepsilon}{C_1 \, m}.$$

Now by taking $\phi = w_\varepsilon$ in (5.12), we obtain

$$\int_\Omega |w-\phi||r(u_n) - r(u)||v| \leq \varepsilon \|v\|_{(2^*,2)}. \tag{5.12}$$

Putting together (5.8), (5.12) and (5.12) we conclude

$$\int_\Omega |w||r(u_{n_k}) - r(u)||v| < \widetilde{C} \, \varepsilon \, \|v\|_{(2^*,2)}, \qquad \forall \, k \geq k_0,$$

where $\widetilde{C}$ is a constant. This shows that $\|N(u_{n_k}) - N(u)\|_{[2^*,2]'} \to 0$. Thus every subsequence of $N(u_n)$ has a convergent subsequence and it converges to $N(u)$. Thus conclude that $N(u_n)$ converges to $N(u)$ in $[L(2^*,2)]'$. This completes the proof. □

Using the above lemma one can deduce the following proposition,

Proposition 5.1.4. *Assume (H1)-(H2) hold. Then the mappings L and H*

defined by (5.3) *are compact. Furthermore,*

$$\lim_{\|u\|\to 0} \frac{\|H(\lambda,u)\|_{\mathcal{D}_0^{1,2}(\Omega)}}{\|u\|_{\mathcal{D}_0^{1,2}(\Omega)}} = 0. \tag{5.13}$$

Proof. Using (H1) and the fact that $\frac{r(s)}{s}$ is bounded on $[\delta,1]$, for any $\delta>0$, we see that

$$|r(s)| \leq \begin{cases} C_0|s| & \text{if } \gamma\in[1,2], \\ C|s|^{\gamma-1} & \text{if } \gamma>2. \end{cases} \tag{5.14}$$

Let us consider the following maps defined from $\mathcal{D}_0^{1,2}(\Omega)$ to $[L(2^*,2)]'$

$$\widetilde{L}(u) = g(x)u, \quad \widetilde{H}(u) = h(x)r(u).$$

From Lemma 5.1.3 we have $\widetilde{L}$ and $\widetilde{H}$ are compact. Thus by Remark 2.3.6 we deduce that L and H are continuous and compact. Next we prove that

$$\lim_{\|u\|\to 0} \frac{\left\|\widetilde{H}(u)\right\|}{\|u\|} = 0. \tag{5.15}$$

First we assume that $\gamma>2$. Thus by using the Hölder inequality and the Lorentz-Sobolev embedding we obtain the following:

$$\begin{aligned}
\big\|h\,r(u)\big\|_{([2^*]',2)} &\leq C\big\|h\,|u|^{\gamma-1}\big\|_{([2^*]',2)} \\
&\leq C_1\big\|h\|_{(\widetilde{\gamma},\infty)}\big\||u|^{\gamma-1}\big\|_{\left(\frac{2^*}{\gamma-1},2\right)} \\
&\leq C_2\big\|h\|_{(\widetilde{\gamma},\infty)}\|u\|_{(2^*,2)}^{\gamma-1} \\
&\leq C_3\big\|h\|_{(\widetilde{\gamma},\infty)}\|\nabla u\|_2^{\gamma-1},
\end{aligned}$$

where all the constants appearing above are independent of u. Therefore

$$\frac{\left\|\widetilde{H}(u)\right\|_{([2^*]',2)}}{\|u\|_{\mathcal{D}_0^{1,2}(\Omega)}} \leq C_3\big\|h\|_{(\widetilde{\gamma},\infty)}\|u\|^{\gamma-2}_{\mathcal{D}_0^{1,2}(\Omega)}.$$

Thus (5.15) holds when $\gamma>2$.
Next we prove the property (5.15) for $\gamma\in[1,2]$. Let us fix $\varepsilon>0$. Note that $\frac{r(s)}{|s|^{2^*-1}}$ is continuous and hence bounded on $[\delta,1]$, for any $\delta>0$. Thus using

(5.14), we get $s_0, C_1 > 0$ depending only on ε such that

$$|r(s)| \leq \varepsilon, \ \forall\, |s| < s_0 \quad \text{and} \quad |r(s)| \leq C_1 |s|^{2^*-1}, \quad \forall\, |s| \geq s_0. \tag{5.16}$$

For each $u \in \mathcal{D}_0^{1,2}(\Omega)$, let

$$E := \{x \in \Omega \,:\, |u(x)| < s_0\} \quad \text{and} \quad F := \{x \in \Omega \,:\, |u(x)| \geq s_0\}.$$

Using the triangular inequality, we obtain

$$\big\|h\, r(u)\big\|_{([2^*]',2)} \leq \big\|h\, r(u)\chi_E\big\|_{([2^*]',2)} + \big\|h\, r(u)\chi_F\big\|_{([2^*]',2)}. \tag{5.17}$$

Let us estimate each term in the right hand side of (5.17) using Hölder inequality and Lorentz-Sobolev embedding. The first term is handled as follows:

$$\begin{aligned} \big\|h\, r(u)\chi_E\big\|_{([2^*]',2)} &\leq C_2 \big\|h\big\|_{(\frac{N}{2},\infty)} \big\|r_1(u)\chi_E\big\|_{(2^*,2)} \\ &\leq C_2\, \varepsilon \big\|h\big\|_{(\frac{N}{2},\infty)} \|u\|_{(2^*,2)}. \end{aligned} \tag{5.18}$$

To estimate $\big\|h\, r(u)\chi_F\big\|_{([2^*]',2)}$, we write $h = \phi + (h - \phi)$ with $\phi \in \mathcal{C}_c^\infty(\Omega)$. Now using (5.14) and (5.16) we have the following estimates:

$$\begin{aligned} \big\|h\, r(u)\chi_F\big\|_{([2^*]',2)} &\leq C_1 \big\|\phi\, |u|^{2^*-1}\chi_F\big\|_{([2^*]',2)} + C_2 \big\|(h-\phi)|u|\chi_F\big\|_{([2^*]',2)} \\ &\leq C_3 \big\|\phi\big\|_\infty \big\|u\big\|_{(2^*,2)}^{2^*-1} + C_4 \big\|h - \phi\big\|_{(\frac{N}{2},\infty)} \big\|u\big\|_{(2^*,2)}. \end{aligned} \tag{5.19}$$

Since $h \in \mathcal{F}_{\frac{N}{2}}$, when $\gamma \in [1,2]$, we choose $h_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that

$$\|h - h_\varepsilon\|_{(\frac{N}{2},\infty)} < \varepsilon.$$

Now by taking $\phi = h_\varepsilon$ in (5.19) we deduce that

$$\big\|h\, r(u)\chi_F\big\|_{([2^*]',2)} \leq C_5 \left\{ \big\|\nabla u\big\|_2^{2^*-1} + \varepsilon \big\|\nabla u\big\|_2 \right\}. \tag{5.20}$$

From (5.17),(5.19) and (5.20), we conclude that (5.15) holds for $\gamma \in [1,2]$. $\square$

Now we are in a position to prove the existence of a global branch of solutions for problem (5.2):

Proof of Theorem 5.1.1: Theorem 4.3.15 shows that λ_1 is an eigenvalue of multiplicity one. This fact with Lemma 5.1.4 shows that all the conditions of Theorem 5.1.2 are satisfied and hence the proof follows. $\square$

5.2 A positive solution branch using the implicit function theorem

In this section, we consider the following type of semilinear elliptic equation:

$$-\Delta u = \lambda\, a\, f(u) \quad \text{in } \mathcal{D}_0^{1,2}(\mathbb{R}^N), \tag{5.21}$$

where λ is a real parameter and $N \geq 3$. Further, we assume that f is in $\mathcal{C}^1(\mathbb{R})$ with $f(0) \neq 0$ and a may change sign.

Here we are looking for sufficient conditions on the weight function a and on the nonlinearity f for the existence of a positive solution branch for (5.21) in $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$. Note that, $(\lambda, 0)$ is no longer a solution branch for the above equation as in the case considered in the previous section, since $f(0) \neq 0$. Thus none of the methods, which assume the existence of a trivial solution branch, is applicable here. However, under certain assumptions on the weight function a and on the nonlinearity f, we apply the implicit function theorem to obtain a nontrivial solution branch for (5.21). We make the following assumptions:

(A1) $f \in \mathcal{C}^1(\mathbb{R})$ such that $f(0) \neq 0$ and there exists $s_0 > 0$ such that

$$|f'(s)| \leq C|s|^{\gamma-2}, \ \forall\, |s| \geq s_0 \text{ with } \gamma \in [2, 2^*).$$

(A2) $a \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $r > \{\widetilde{\gamma}, 2\}$, where $\widetilde{\gamma}$ is the conjugate exponent of $(\frac{2^*}{\gamma})$.

The main novelty of our hypotheses is that the weight function a need not be smooth and the function f not necessarily bounded, but we demand that a lies in certain Lebesgue spaces and f is $\mathcal{C}^1(\mathbb{R})$ with subcritical growth at infinity. The results that we present in this section have appeared in [11].

First we prove the existence of a solution branch of (5.21) for small λ, under the assumptions (A1) and (A2). The positivity of the solution branch is obtained under further assumptions on a. For $\varepsilon \geq 0$, we consider the following perturbed linearised problem:

$$-\Delta v = f(0)(a - \epsilon\, a^-) \ \text{ in } \mathbb{R}^N. \tag{5.22}$$

Using a simple interpolation we see that $a \in L^q(\mathbb{R}^N)$ for $q \in \left[\frac{2N}{N+2}, r\right)$. In particular we have $a \in L^2(\mathbb{R}^N)$. Now using the Newtonian potential, the existence of a weak solution for (5.22) in $H^1(\mathbb{R}^N)$ is well known, see Theorem 9.9 of [40]. Further, from (A2), we have $a \in L^r(\mathbb{R}^N)$ with $r > \frac{N}{2}$, since $\widetilde{\gamma} \geq \frac{N}{2}$. Thus the weak solutions of (5.22) are continuous, see Corollary 9.18 of [40]. Let v_ε be a weak solution of (5.22) that is continuous. Thus for each $\varepsilon \geq 0$, $v_\varepsilon \in D_0^{1,2}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ and satisfies equation (5.22) in the weak sense, i.e. ,

$$\int_{\mathbb{R}^N} \nabla v_\varepsilon \cdot \nabla u = \int_{\mathbb{R}^N} f(0)(a - \epsilon\, a^-)u \quad \forall\, u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N). \tag{5.23}$$

We obtain the positivity of the solution branch of (5.21) under one of the following assumptions:

(A3) v_0 is positive in $\mathbb{R}^N$ and $f(0)\, a \geq 0$ a.e. near infinity.

(A4) v_ε is positive in $\mathbb{R}^N$, for some $\varepsilon > 0$.

The existence of the solution branch of (5.21) rely on the implicit function theorem as in the work of Brown and Afrouzi [3]. Further we improve the integrability of solutions of (5.21) using a Moser type iteration as in [33], where Drábek proved the integrability of the positive solutions of quasilinear partial differential equation involving p-Laplacian in $\mathbb{R}^N$. The key difference and the difficulty in our case is the presence of $f(0)$ (which is zero in [33]), that produces an additional term in the iteration and inhomogeneity in the inequalities. We show that the regularity assumptions on a are sufficient to tackle this difficulty. Moreover, we improve the integrability for a general solution, not just for the positive solutions of (5.21). Using the improved integrability of solutions, we prove the decay at infinity and the Hölder continuity of solutions of (5.21) using certain classical results available in the literature.

5.2.1 Existence of a solution branch

In [3], authors considered only bounded domains and smooth weight function a. They have used the fact that the map Δ is a linear homeomorphism between appropriate Hölder spaces. Since we are considering entire $\mathbb{R}^N$ and a is only in certain Lebesgue spaces, we look for a functional framework so that Δ can be identified as a linear homeomorphism and $a\, f(u)$ as a C^1 map between appropriate function spaces. We set $X = \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ and identify $-\Delta$ as a mapping between X and its dual X', with the obvious action

$$\langle -\Delta u, v \rangle_{(X',X)} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v.$$

From Corollary 2.3.6, we know that $-\Delta$ is a linear isometry from X onto $X^{'}$. Now we prove that $a\,f(u)$ is a $\mathcal{C}^1$ map on X.

Lemma 5.2.1. *Let a and f satisfy conditions $(A1)$ and $(A2)$. Then the map $\widetilde{F}: L^{2^*}(\mathbb{R}^N) \to [L^{2^*}(\mathbb{R}^N)]^{'}$ defined by*

$$\widetilde{F}(u) = a\,f(u)$$

is $\mathcal{C}^1$. Further $D\widetilde{F}(u): L^{2^*}(\mathbb{R}^N) \to [L^{2^*}(\mathbb{R}^N)]^{'}$ is given by

$$D\widetilde{F}(u) = a\,f^{'}(u).$$

Proof. Using (A1) and the mean value theorem one can easily deduce that

$$|f(s)| < C(1+|s|^{\gamma-1}), \quad |f^{'}(s)| < C(1+|s|^{\gamma-2}), \forall s \in \mathbb{R}, \tag{5.24}$$

for some positive constant C depending on f. Using the interpolation on Lebesgue spaces and (A2) we get $a \in L^q(\mathbb{R}^N)$ for $q \in [\frac{2N}{N+2}, r]$. Note that $2^{*'} = \frac{2N}{N+2}$. Now for $u \in L^{2^*}(\mathbb{R}^N)$, using (5.24) and Hölder inequality we get $a\,f(u) \in [L^{2^*}(\mathbb{R}^N)]^{'}$. Thus the map $\widetilde{F}(u)$ is well defined from $L^{2^*}(\mathbb{R}^N)$ to $[L^{2^*}(\mathbb{R}^N)]^{'}$.

Next, we show that $\widetilde{F}$ is continuous. Let $u_n, u \in L^{2^*}(\mathbb{R}^N)$ and $u_n \to u$ in $L^{2^*}(\mathbb{R}^N)$. Thus up to a subsequence $u_n \to u$ a.e. in $\mathbb{R}^N$ and since f is continuous, $f(u_n) \to f(u)$ a.e in $\mathbb{R}^N$.

$$\begin{aligned} |a\,f(u_n) - a\,f(u)| &\leq |a\,|\big\{|f(u_n)| + |f(u)|\big\} \\ &\leq 2C|a\,| + C|a\,|\big\{|u_n|^{\gamma-1} + |u|^{r-1}\big\}, \end{aligned}$$

using (5.24). Now as $|u_n|^{\gamma-1} \to |u|^{\gamma-1}$ in $L^{\frac{2^*}{\gamma-1}}$ and $a \in L^{\widetilde{\gamma}}$ using Hölder inequality we get $|a\,||u_n|^{\gamma-1} \to |a\,||u|^{\gamma-1}$ in $[L^{2^*}(\mathbb{R}^N)]^{'}$. Thus the right hand side of the above inequality converges to $2C|a\,|\big\{1+|u|^{\gamma-1}\big\}$ in $[L^{2^*}(\mathbb{R}^N)]^{'}$ and hence by the generalized dominated convergence theorem $\{a\,f(u_n) - a\,f(u)\} \to 0$ in $[L^{2^*}(\mathbb{R}^N)]^{'}$. By a similar calculation one can show that $\widetilde{F}$ is $\mathcal{C}^1$ and its derivative is given by $D\widetilde{F}(u) = a\,f^{'}(u)$.

Theorem 5.2.2. *Let a, f satisfy conditions $(A1)$-$(A2)$. Then there exist a $\varepsilon_0 > 0$ and a C^1 map $u: (-\varepsilon_0, \varepsilon_0) \to X$ such that for $\lambda \in (-\varepsilon_0, \varepsilon_0)$, $u(\lambda)$ satisfies*

$$-\Delta u(\lambda) = \lambda\,a\,f(u(\lambda)) \tag{5.25}$$

in the weak sense. Further the derivative $u^{'}(0) := \frac{d}{d\lambda}u(\lambda)|_{\lambda=0}$ *satisfies*

$$-\Delta(u^{'}(0)) = f(0)a \tag{5.26}$$

in the weak sense.

Proof. By Sobolev embedding, X is embedded in $L^{2^*}(\mathbb{R}^N)$ and hence the map $F : \mathbb{R} \times X \to X^{'}$ defined by

$$F(\lambda, u) = -\Delta u - \lambda a\ f(u)$$

is well defined. Now as $\widetilde{F}$ is C^1 and $-\Delta$ is a linear isometry, F is C^1 and its partial derivatives are given by

$$\frac{\partial F}{\partial u}(\lambda, u) = -\Delta - \lambda a\, f^{'}(u),\ \frac{\partial F}{\partial \lambda}(\lambda, u) = -a\, f(u).$$

Observe that $F(0,0) = 0$ and $\frac{\partial F}{\partial u}(0,0) = -\Delta$, which is invertible. Hence by applying the implicit function theorem to $F(\lambda, u) = 0$ at $(\lambda, u) = (0,\, 0)$, we get a neighbourhood $(-\varepsilon_0, \varepsilon_0)$ of 0 and a unique map $u : (-\varepsilon_0, \varepsilon_0) \to X$ such that $u(0) = 0$ and

$$F(\lambda, u(\lambda)) = 0,\ \forall \lambda \in (-\varepsilon_0, \varepsilon_0).$$

Further the map u is $\mathcal{C}^1$. It is clear that $u(\lambda)$ satisfies the Eq. (5.25) in the weak sense. Let us differentiate the above equation with respect λ and evaluate at $(0,0)$:

$$\frac{\partial F}{\partial \lambda}(0,0)\ + \frac{\partial F}{\partial u}(0,0) \circ u^{'}(0) = 0.$$

Now by substituting for $\frac{\partial F}{\partial \lambda}(0,0)$ and $\frac{\partial F}{\partial u}(0,0)$ we get $-\Delta(u^{'}(0)) = f(0)\, a$. □

Henceforth, for the convenience we denote $u(\lambda)$ by u_λ.

Remark 5.2.3. *From the uniqueness of the weak solution of* (5.23)*, we see that* $u^{'}(0) = v_0$*. Thus*

$$\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^N)} = 0.$$

5.2.2 Regularity of the solution branch

In this section we improve the regularity of solutions using Moser type iterations as in [33].

Lemma 5.2.4. *Let a, f satisfy (A1) and (A2). Let $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ satisfy*

$$-\Delta u = \lambda a\, f(u)$$

in the weak sense; then $u \in L^s$ for every $s \in [2^*, \infty)$.

Proof. For a positive function $w \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ and $k \in \mathbb{N}$, we define

$$w_k = \min\{w, k\}.$$

Now for $\alpha > 0$, let

$$v_k = [(u^+)_k]^{2\alpha+1} - [(u^-)_k]^{2\alpha+1}.$$

From Corollary 2.3.9, we have $v_k \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Thus from the equation we have the following

$$\int \nabla u \cdot \nabla v_k = \lambda \int a\, f(u) v_k. \tag{5.27}$$

First we find a lower bound for the left hand side of (5.27)

$$\begin{aligned}
\int \nabla u \cdot \nabla v_k &= \int \nabla (u^+)_k \cdot \nabla [(u^+)_k]^{2\alpha+1} + \nabla (u^-)_k \cdot \nabla [(u^-)_k]^{2\alpha+1} \\
&= (2\alpha+1) \int |\nabla (u^+)_k|^2 [(u^+)_k]^{2\alpha} + |\nabla (u^-)_k|^2 [(u^-)_k]^{2\alpha} \\
&= \frac{(2\alpha+1)}{(\alpha+1)^2} \int |\nabla [(u^+)_k]^{\alpha+1}|^2 + |\nabla [(u^-)_k]^{\alpha+1}|^2 \\
&= \frac{(2\alpha+1)}{(\alpha+1)^2} \int |\nabla (|u|_k)^{\alpha+1}|^2 \\
&\geq \frac{1}{C_s} \frac{(2\alpha+1)}{(\alpha+1)^2} \left(\int (|u|_k)^{2^*(\alpha+1)} \right)^{\frac{2}{2^*}}.
\end{aligned} \tag{5.28}$$

The equality sign in the first and fourth steps are due to the disjointness of the supports of u_k^+ and u_k^-. The last inequality comes from the Sobolev embedding, where C_s is the best constant appearing in the corresponding Sobolev inequality. Next, we get an upper bound for the right hand side of

(5.27) using the growth condition on f and the integrability of a :

$$\begin{aligned}\left|\lambda \int a\, f(u) v_k\right| &\leq \lambda \int |a\,||f(u)||u|^{2\alpha+1} \\ &\leq \lambda C \int |a\,|(1+|u|^{\gamma-1})|u|^{2\alpha+1} \\ &\leq \lambda\, C \int |a\,||u|^{2\alpha+1} + \lambda\, C \int |a\,||u|^{2\alpha+\gamma}. \qquad (5.29)\end{aligned}$$

We estimate the second term in (5.29) using Hölder inequalities and Sobolev embedding

$$\int |a\,||u|^{2\alpha+\gamma} \quad \leq \quad \|a\|_r \left\||u|^{2\alpha+\gamma}\right\|_{r'}. \qquad (5.30)$$

Now

$$\int |u|^{(2\alpha+\gamma)r'} = \int |u|^{(\gamma-2)r'} |u|^{2(\alpha+1)r'}.$$

Let $p = \frac{2^*}{(\gamma-2)r'}$. It is given that $r > (\frac{2^*}{\gamma})'$. Thus $r' < (\frac{2^*}{\gamma}) < (\frac{2^*}{\gamma-2})$ and hence $p > 1$. Now by Hölder inequality applied for p, p' we get

$$\int |u|^{(2\alpha+\gamma)r'} \leq \left(\int |u|^{2^*}\right)^{\frac{1}{p}} \left(\int |u|^{2(\alpha+1)r'p'}\right)^{\frac{1}{p'}}.$$

Let $q = 2r'p'$. Then $q = 2r'\frac{2^*}{2^*-(\gamma-2)r'}$. Now one can see that $q < 2^*$, using the fact that $r > (\frac{2^*}{\gamma})'$ and the following implications:

$$q < 2^* \Longleftrightarrow \frac{2r'}{2^*-(\gamma-2)r'} < 1 \Longleftrightarrow 2r' < 2^* - (\gamma-2)r' \Longleftrightarrow r' < \frac{2^*}{\gamma}.$$

Now

$$\left\||u|^{(2\alpha+\gamma)}\right\|_{r'} \leq \left(\int |u|^{2^*}\right)^{\frac{1}{r'p}} \left(\int |u|^{(\alpha+1)q}\right)^{\frac{2}{q}}.$$

Therefore using (5.30)

$$\int |a\,||u|^{2\alpha+\gamma} \leq C_2 \left(\int |u|^{(\alpha+1)q}\right)^{\frac{2}{q}} = C_2\, \|u\|_{(\alpha+1)q}^{2(\alpha+1)},$$

where C_2 is a constant independent of α, k.
Next, we estimate the first integral in (5.29). Let $s_\alpha = \left(\frac{\alpha+1}{2\alpha+1}\right)q$. Observe

that $\frac{\alpha+1}{2\alpha+1} > \frac{1}{2}$ and hence $s_\alpha > \frac{q}{2} = r^{'}p^{'} \geq r^{'}$. Let $\widetilde{\alpha}$ be such that $(\widetilde{\alpha}+1)q = 2^*$, then for any $\alpha \geq \widetilde{\alpha}$, $\left(\frac{\alpha+1}{2\alpha+1}\right)q \leq \left(\frac{\widetilde{\alpha}+1}{2\widetilde{\alpha}+1}\right)q < 2^*$. Thus for $\alpha \geq \widetilde{\alpha}$, we have $s_\alpha \in (r^{'}, 2^*)$ and hence $s_\alpha^{'} \in (2^{*'}, r)$.

$$\int |a|\,|u|^{2\alpha+1} \leq \|a\|_{s_\alpha'} \|u\|_{(1+\alpha)q}^{(2\alpha+1)}.$$

Thus we get an upper bound for the right hand side of (5.27) provided $\alpha \geq \widetilde{\alpha}$.

$$\left|\lambda \int a\, f(u) v_k\right| \leq \lambda\, C_3 \left\{ \|a\|_{s_\alpha'} \|u\|_{(1+\alpha)q}^{(2\alpha+1)} + \|u\|_{(\alpha+1)q}^{2(\alpha+1)} \right\}, \tag{5.31}$$

where C_3 is a constant independent of α. Now from (5.28) and (5.31)

$$\frac{1}{C_s}\frac{(2\alpha+1)}{(\alpha+1)^2}\left(\int (|u|_k^{\alpha+1})^{2^*}\right)^{\frac{2}{2^*}} \leq \lambda\, C_3 \left\{ \|a\|_{s_\alpha'} \|u\|_{(1+\alpha)q}^{(2\alpha+1)} + \|u\|_{(\alpha+1)q}^{2(\alpha+1)} \right\}.$$

Thus

$$\left(\int |u|_k^{(\alpha+1)2^*}\right)^{\frac{2(\alpha+1)}{(\alpha+1)2^*}} \leq C_s \frac{(\alpha+1)^2}{(2\alpha+1)} \lambda\, C_3 \left\{ \|a\|_{s_\alpha'} \|u\|_{(1+\alpha)q}^{(2\alpha+1)} + \|u\|_{(\alpha+1)q}^{2(\alpha+1)} \right\}.$$

Therefore

$$\||u|_k\|_{(\alpha+1)2^*} \leq \lambda\, C_4(\alpha) \left\{ \|a\|_{s_\alpha'}^{\frac{1}{2(\alpha+1)}} \|u\|_{(1+\alpha)q}^{\frac{(2\alpha+1)}{2(\alpha+1)}} + \|u\|_{(\alpha+1)q} \right\}.$$

Now by the monotone convergence theorem we get

$$\|u\|_{(\alpha+1)2^*} \leq \lambda\, C_4(\alpha) \left\{ \|a\|_{s_\alpha'}^{\frac{1}{2(\alpha+1)}} \|u\|_{(1+\alpha)q}^{\frac{(2\alpha+1)}{2(\alpha+1)}} + \|u\|_{(\alpha+1)q} \right\}. \tag{5.32}$$

Let α_1 be such that $(1+\alpha_1)q = 2^*$. Note that for this α_1 all the terms in the right hand side of (5.32) are finite and hence $u \in L^{(\alpha_1+1)2^*}(\mathbb{R}^N)$. Now set $\beta = \frac{2^*}{q}$ and hence $u \in L^{2^*\beta}$. In the next iteration, we choose $\alpha = \alpha_2$ in (5.32) so that $(1+\alpha_2)q = (1+\alpha_1)2^*$ and $u \in L^{2^*\beta^2}$. Indeed, for each $n \in \mathbb{N}$ we choose α_n so that $(1+\alpha_n)q = (1+\alpha_{n-1})2^*$. Thus we get $u \in L^{2^*\beta^n}$. Since $\beta > 1$, using the interpolations on Lebesgue spaces we get $u \in L^p$ for all $p \in [2^*, \infty)$. $\square$

As an immediate corollary we have the following

Corollary 5.2.5. *Let u_λ be as in Theorem 5.2.2, then $u_\lambda \in L^p$, for $2^* \leq p < \infty$. Moreover $\|u_\lambda\|_p \leq C_p$, $\forall \lambda \in (-\varepsilon_0, \varepsilon_0)$.*

Proof. Since u_λ solves $-\Delta u_\lambda = \lambda a\, f(u_\lambda)$, for $\lambda \in (-\varepsilon_0, \varepsilon_0)$ from Lemma 5.2.4 we obtain $u_\lambda \in L^p(\mathbb{R}^N)$, $\forall\, p \in [2^*, \infty)$. Note that right hand side of (5.2.4) is uniformly bounded for $\lambda \in (-\varepsilon_0, \varepsilon_0)$. Thus

$$\|u_\lambda\|_p \leq C_p,\ \forall\, \lambda \in (-\varepsilon_0, \varepsilon_0). \tag{5.33}$$

□

The next lemma gives a representation formula for the solutions of Poisson equation.

Lemma 5.2.6. *Let $g \in L^{2^*}(\mathbb{R}^N)$ and $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ be such that $-\Delta u = g$. Then*

$$u(x) = (\Gamma * g)(x) = \int_{\mathbb{R}^N} \Gamma(x-y) g(y) dy,$$

where Γ is the fundamental solution of $-\Delta$ on $\mathbb{R}^N$.

Proof. Since $g \in L^{2^*}(\mathbb{R}^N)$, by the Calderon-Zygmund L^p regularity theory, the Newtonian potential $\Gamma * g \in W^{2,2^*}(\mathbb{R}^N)$ and

$$-\Delta(\Gamma * g) = g \text{ a.e. in } \mathbb{R}^N.$$

In particular $\Delta(u - \Gamma * g) = 0$ in the sense of distributions. Thus $u - \Gamma * g$ is a harmonic function in $L^{2^*}(\mathbb{R}^N)$. Therefore we conclude that $u = \Gamma * g$ is the unique solution of $-\Delta u = g$ in $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$. □

Now we have the following representation formula for the solution u_λ of (5.21).

Proposition 5.2.7. *Let u_λ be as in Theorem 5.2.2. Then*

$$u_\lambda(x) = \int_{\mathbb{R}^N} \Gamma(x-y) a(y) f\big(u_\lambda(y)\big) dy.$$

Moreover $u_\lambda \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [(2^*)', r)$ and

$$\|u_\lambda\|_{W^{2,p}} \leq \widetilde{C}_p,\ \forall\, \lambda \in (-\varepsilon_0, \varepsilon_0). \tag{5.34}$$

Proof. From Corollary 5.2.5, we get $|u_\lambda|^{\gamma-1} \in L^s$ for $s \in [\frac{2^*}{\gamma-1}, \infty)$ and by the Hölder inequality $|a\,||u_\lambda|^{\gamma-1} \in L^q$ with $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$. Now using hypotheses

(A1) and (A2), one can easily see that $a\, f(u_\lambda) \in L^q$ for $q \in [(2^*)', r)$ and also $2^* \in [(2^*)', r)$. Since $-\Delta u_\lambda = \lambda a\, f(u_\lambda)$, from Lemma 5.2.6, we obtain

$$u_\lambda(x) = \lambda \int_{\mathbb{R}^N} f\big(u_\lambda(y)\big) a(y) \Gamma(x-y) dy.$$

Further, using the Calderon-Zygmund L^p regularity theory for the Laplacian, we get $u_\lambda \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [(2^*)', r)$. Now as the map $f \to f * \Gamma$ is continuous from $L^p(\mathbb{R}^N)$ into $W^{2,p}(\mathbb{R}^N)$, using (5.33) we obtain a uniform bound for $\|u_\lambda\|_{W^{2,p}}$. □

Using Morrey's inequality, we obtain the following Hölder regularity of solutions u_λ.

Proposition 5.2.8. *Let u_λ be as in Theorem 5.2.2, then $u_\lambda \in \mathcal{C}^\alpha(\mathbb{R}^N)$. Moreover*

$$\|u_\lambda\|_{\mathcal{C}^\alpha} \le C_\alpha, \forall\, \lambda \in (-\varepsilon_0, \varepsilon_0). \tag{5.35}$$

Proof. From Morrey's inequality, for $p > \frac{N}{2}$, we know that $W^{2,p}(\mathbb{R}^N) \hookrightarrow \mathcal{C}^\alpha(\mathbb{R}^N)$ with $\alpha =$ fractional part of $\{2 - \frac{N}{p}\}$. Since $r > \frac{N}{2}$, using the above proposition, we conclude that $u_\lambda \in \mathcal{C}^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. The uniform bound for $\|u_\lambda\|_\alpha$ follows as the embedding of $W^{2,p}(\mathbb{R}^N)$ is continuous and u_λ is uniformly bounded in $W^{2,p}(\mathbb{R}^N)$. □

To get the decay of the solutions, we first recall the following version of Theorem 8.17 of [40]:

Lemma 5.2.9. *Let $u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ satisfy $-\Delta u = g$ in the weak sense. If $g \in L^q(\mathbb{R}^N)$ with $q > \frac{N}{2}$ then for every $y \in \mathbb{R}^N$ and $R > 0$,*

$$\sup_{x \in B_R(y)} |u(x)| \le C(R, 2^*) \left\{ \|u\|_{L^{2^*}(\mathbb{R}^N)(B_{2R}(y))} + \|g\|_{L^q(B_{2R}(y))} \right\}. \tag{5.36}$$

In the next proposition we prove the decay of u_λ at infinity.

Proposition 5.2.10. *Let u_λ be as in the Theorem 5.2.2. Then $u_\lambda(x) \to 0$ as $|x| \to \infty$ uniformly for $\lambda \in (-\varepsilon_0, \varepsilon_0)$.*

Proof. Observe that $r > \frac{N}{2}$ and $a\, f(u_\lambda) \in L^q(\mathbb{R}^N)$ with $q > \frac{N}{2}$. Now by Lemma 5.2.9 we have the following

$$\sup_{x \in B_R(y)} |u_\lambda(x)| \le C(R, 2^*) \left\{ \|u_\lambda\|_{L^{2^*}(\mathbb{R}^N)(B_{2R}(y))} + \|a\, f(u_\lambda)\|_{L^q(B_{2R}(y))} \right\}.$$

Note that the right hand side of above inequality goes to 0 as $|y| \to \infty$. Moreover the convergence is uniform with respect λ as the estimates in the proof of Lemma 5.2.4 can be made independent of λ by choosing an upper bound for λ. □

5.2.3 Positivity of the solution branch

In this section we prove the positivity of the solution curve. Here we obtain the positivity of u_λ under the additional assumptions (A3) or (A4). The regularity of the solution curve at 0, in $\mathcal{C}^\alpha$ topology, is one of the essential ingredients the of our proof. More precisely, we make use of the differentiability of the map $u : (-\varepsilon, \varepsilon) \to \mathcal{C}^\alpha(\mathbb{R}^N)$ at 0. The C^1 regularity of the map $u : (-\varepsilon, \varepsilon) \to \mathcal{D}_0^{1,2}(\mathbb{R}^N)$, is a part of the implicit function theorem. However, it is not clear whether the map $u : (\varepsilon, \varepsilon) \to \mathcal{C}^\alpha(\mathbb{R}^N)$ is $\mathcal{C}^1$. Here, using a similar technique as in (5.2.4), we show that the map $u : (\varepsilon, \varepsilon) \to L^p(\mathbb{R}^N)$ is continuously differentiable at 0, for every $p \geq 2^*$. From Remark 5.2.3, we know that $u^{'}(0) = v_0$, where v_0 is the continuous weak solution of (5.23) for $\varepsilon = 0$. Further,

$$\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^N)} = 0.$$

Lemma 5.2.11. *Let u_λ be as in Theorem (5.2.2). Then for each $p \in [2^*, \infty)$ the map $u : (\varepsilon, \varepsilon) \to L^p(\mathbb{R}^N)$ is continuously differentiable at 0. i.e*

$$\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_p = 0.$$

Proof. First observe that $(\frac{u_\lambda}{\lambda} - v)$ satisfies the following:

$$-\Delta \left(\frac{u_\lambda}{\lambda} - v_0 \right) = a\ (f(u_\lambda) - f(0))\,. \tag{5.37}$$

Now we set $w = (\frac{u_\lambda}{\lambda} - v_0)$. For $k, \alpha > 0$, let

$$v_k = [(w^+)_k]^{2\alpha+1} - [(w^-)_k]^{2\alpha+1}.$$

Multiply Eq: (5.37) by v_k to obtain the following:

$$\int \nabla w \cdot \nabla v_k = \int a\,(f(u_\lambda) - f(0)) v_k. \tag{5.38}$$

Using the calculations similar to those that yield (5.28), we obtain a lower bound for the left hand side of (5.38) as below:

$$\int \nabla w \cdot \nabla v_k \quad \geq \quad \frac{1}{C_s}\frac{(2\alpha+1)}{(\alpha+1)^2}\left(\int (|w|_k^{\alpha+1})^{2^*}\right)^{\frac{2}{2^*}}, \tag{5.39}$$

where C_s is the best constant appearing in the corresponding Sobolev inequality. Next we estimate the right hand side of 5.38. Using the growth condition on $f^{'}$ and the mean value theorem, we obtain the following:

$$\left|\int a\,(f(u_\lambda) - f(0))v_k\right| \leq \int |a\,||u_\lambda|^{\gamma-1}|v_k|. \tag{5.40}$$

Since $|v_k| \leq |w|^{2\alpha+1}$ and $|u_\lambda|^{\gamma-1} \leq C, \forall\lambda \in (-\varepsilon, \varepsilon)$, see (5.35), from the above inequality, we deduce the following

$$\left|\int a\,(f(u_\lambda) - f(0))v_k\right| \leq C\int |a\,||w|^{2\alpha+1}. \tag{5.41}$$

Note that the right hand side of (5.41) is similar to the first integral in the right hand side of (5.29). Here also we make use of the integrability assumptions on a to estimate the above integral. As in the proof of Theorem 5.2.4, we set $q = \frac{2\,r^{'}\,2^*}{2^*-(\gamma-2)r^{'}}$ and $s_\alpha = \left(\frac{\alpha+1}{2\alpha+1}\right)q$. Let $\widetilde{\alpha}$ be such that $(\widetilde{\alpha}+1)q = 2^*$. For any $\alpha \geq \widetilde{\alpha}$, one can verify that ${s_\alpha}^{'} \in ({2^*}^{'}, r)$ and hence $a \in L^{{s_\alpha}^{'}}(\mathbb{R}^N)$. Now from the Hölder inequality we get,

$$\int |a\,||w|^{2\alpha+1} \leq \|a\|_{s_\alpha^{'}}\,\|w\|_{(1+\alpha)q}^{(2\alpha+1)}. \tag{5.42}$$

By combining (5.39), (5.41), (5.42) and using the monotone convergence theorem we obtain:

$$\|w\|_{2^*(\alpha+1)}^{2(\alpha+1)} \leq C\,\|a\|_{s_\alpha^{'}}\,\|w\|_{(1+\alpha)q}^{(2\alpha+1)}. \tag{5.43}$$

Therefore

$$\left\|\frac{u_\lambda}{\lambda} - v_0\right\|_{2^*(\alpha+1)}^{2(\alpha+1)} \leq C\,\|a\|_{s_\alpha^{'}}\left\|\frac{u_\lambda}{\lambda} - v_0\right\|_{(1+\alpha)q}^{(2\alpha+1)}. \tag{5.44}$$

Now, if we choose $\alpha_1 = \widetilde{\alpha}$, then $(1+\alpha)q = 2^*$ and hence when $\lambda \to 0$, the right hand side of the above inequality converges to zero. Thus for $\beta = \frac{2^*}{q}$, we conclude that the map $u : (\varepsilon_0, \varepsilon_0) \to L^{2^*\beta}$ is differentiable at 0. In the next step, we choose $\alpha = \alpha_2$ in the above inequality so that $(1+\alpha_2)q = (1+\alpha_1)2^*$ and hence we deduce the differentiability of the map

u at 0, in $L^{2^*\beta^2}$ topology. Since $\beta > 1$ and by interpolation on Lebesgue spaces, for all $p \in [2^*, \infty)$, we get u is differentiable at 0, in L^p topology. $\square$

For large p, the embedding of $W^{2,p}(\mathbb{R}^N)$ in $\mathcal{C}^\alpha(\mathbb{R}^N)$ is continuous and linear. Thus the map $u : (-\varepsilon_0, \varepsilon_0) \to \mathcal{C}^\alpha(\mathbb{R}^N)$ is differentiable at 0. In particular we have the following result.

Corollary 5.2.12. *Let u_λ be as in Theorem 5.2.2. Then*

$$\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_\infty = 0. \tag{5.45}$$

Now we prove the solution branch u_λ is positive for small positive λ.

Theorem 5.2.13. *Let u_λ be as in Theorem 5.2.2. Let $v_0 \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ be such that*

$$-\Delta v_0 = f(0)\, a\,. \tag{5.46}$$

If v_0 is positive in $\mathbb{R}^N$ and $f(0)a \geq 0$ a.e. near infinity, then there exists $\lambda_0 > 0$, such that $u_\lambda > 0$, for $0 < \lambda < \lambda_0$.

Proof. The idea is to use the positivity of v_0 to obtain the positivity of the solution branch in a ball for small positive λ and then use the definite sign of the weight at infinity to get the same in the complement. Without loss of generality we assume that $f(0) > 0$. Thus by the continuity of f, there exists $\delta > 0$ such that $f(s) > 0$ for $|s| < \delta$. Let us fix $R > 0$ sufficiently large, so that $a \geq 0$ and $|u_\lambda(x)| < \delta$ in B_R^c for $|\lambda| < \varepsilon_0$. This is possible since $u_\lambda(x) \to 0$ as $|x| \to \infty$ uniformly with respect to λ (thanks to Proposition 5.2.10). Hence

$$a\, f(u_\lambda(x)) \geq 0 \text{ in } B_R^c.$$

Further, from (5.45),

$$\lim_{\lambda \to 0} \left\| \frac{u_\lambda}{\lambda} - v_0 \right\|_\infty = 0. \tag{5.47}$$

Set $m = \min_{\overline{B_R}} v_0$. Note that m is positive, since v_0 is continuous and $v_0 > 0$. Further, by (5.47) there exists a λ_0 such that $0 < \lambda_0 < \varepsilon_0$ and

$$\left| \frac{u_\lambda}{\lambda}(x) - v_0(x) \right| < \frac{m}{2}, \forall\, x \in \overline{B_R},\ \forall\, \lambda \in (-\lambda_0, \lambda_0).$$

Thus $\frac{u_\lambda}{\lambda}(x) > 0$ for $x \in \overline{B_R}$ and hence $u_\lambda(x) > 0$ for $0 < \lambda < \lambda_0$ in $\overline{B_R}$. Next, we show that u_λ is positive also in B_R^c, for $0 < \lambda < \lambda_0$. Since u_λ is a weak solution of (5.25), we have the following

$$\int_{\mathbb{R}^N} \nabla u_\lambda . \nabla w = \lambda \int_{\mathbb{R}^N} a\, f(u_\lambda) w, \quad w \in \mathcal{D}_0^{1,2}(\mathbb{R}^N).$$

In particular, for $0 < \lambda < \lambda_0$, choose $w = u_\lambda^-$. Observe that $\operatorname{supp}(u_\lambda^-) \subseteq B_R^c$. Therefore

$$-\int_{B_R^c} |\nabla u_\lambda^-|^2 = \lambda \int_{B_R^c} a\, f(u_\lambda^-) u_\lambda^-. \tag{5.48}$$

Since the right hand side is nonnegative, we get $|\nabla u_\lambda^-| = 0$ and hence u_λ^- is a constant on B_R^c. Since $u_\lambda^- \in \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ we get $u_\lambda^- \equiv 0$ in B_R^c. Observe that u_λ is continuous and superharmonic in B_R^c. Since $\inf_{B_R^c} u_\lambda = 0$, by the strong minimum principle, applicable to the continuous version of superharmonic functions in L^1_{loc} (see, Theorem 9.4, [54]), we have either

$$u_\lambda > 0 \ \text{ or } \ u_\lambda \equiv 0, \ \text{ in } B_R^c,$$

but $u_\lambda \equiv 0$ leads to a contradiction to (5.25) as $a\, f(0) \not\equiv 0$ in B_R^c. Thus we conclude that

$$u_\lambda > 0 \ \text{ in } \mathbb{R}^N, \ \text{for}\, 0 < \lambda < \lambda_0.$$

This completes the proof of the theorem. □

In the next theorem we relax the sign restriction of a at infinity, by assuming the existence of a positive solution for the perturbed linearised problem (5.22).

Theorem 5.2.14. *Let u_λ be as in Theorem 5.2.2. Let $v_\varepsilon \in \mathcal{D}_0^{1,2}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ be such that*

$$-\Delta v_\varepsilon = f(0)\,(a - \varepsilon\, a^-). \tag{5.49}$$

If $v_\varepsilon > 0$ in $\mathbb{R}^N$ for some $\varepsilon > 0$, then there exists $\lambda_0 > 0$ such that $u_\lambda > 0$, for $0 < \lambda < \lambda_0$.

Proof. Here we adapt an idea of Cac et al. [24]. Note that from Proposition 5.2.7, we have the following representation formula for v_ε :

$$v_\varepsilon(x) = f(0) \int_{\mathbb{R}^N} \Gamma(x-y)(a^+(y) - (1+\varepsilon)a^-(y))dy.$$

Now using the continuity of f at 0, for any $\mu > 0$, there exists $\delta > 0$ such that

$$(1-\mu) f(0) < f(s) < (1+\mu) f(0), \ \forall\, s \in (-\delta, \delta). \tag{5.50}$$

Since $\frac{u_\lambda}{\lambda} \to v$ uniformly in $\mathbb{R}^N$ and v is bounded, we have $u_\lambda \to 0$ uniformly in $\mathbb{R}^N$ as $\lambda \to 0$. Thus there exists $\lambda_0 > 0$ such that $\|u_\lambda\|_\infty < \delta$, $0 < \lambda < \lambda_0$. Now from (5.50), for each $\lambda \in (0, \lambda_0)$, we obtain

$$(1-\mu)f(0) < f(u_\lambda(y)) < (1+\mu)f(0), \ \forall\, y \in \mathbb{R}^N. \tag{5.51}$$

Further, from Proposition 5.2.7, we have

$$u_\lambda(x) = \lambda \int_{\mathbb{R}^N} \Gamma(x-y)a(y)f(u_\lambda(y))dy.$$

Therefore using (5.51), we obtain the following inequalities

$$\begin{aligned}
u_\lambda &= \lambda \int_{\mathbb{R}^N} \Gamma(x-y)(a^+(y) - a^-(y))f(u_\lambda(y))dy \\
&> \lambda f(0) \int_{\mathbb{R}^N} \Gamma(x-y)\left[(1-\mu)a^+(y) - (1+\mu)a^-(y)\right] dy \\
&= \lambda(1-\mu)f(0) \int_{\mathbb{R}^N} \Gamma(x-y)\left[a^+(y) - \frac{(1+\mu)}{(1-\mu)}a^-(y)\right] dy \\
&= \lambda(1-\mu)f(0) \int_{\mathbb{R}^N} \Gamma(x-y)\left[a(y) - \frac{2\mu}{1-\mu}a^-(y)\right] dy.
\end{aligned} \tag{5.52}$$

Now for the choice of $\mu = \frac{\epsilon}{2+\epsilon}$, we see that $\frac{2\mu}{1-\mu} = \varepsilon$. Thus from (5.52) we conclude that, for $0 < \lambda < \lambda_0$

$$u_\lambda(x) > \frac{2\,\lambda}{2+\varepsilon}\, v_\varepsilon(x), \ \forall\, x \in \mathbb{R}^N. \tag{5.53}$$

This completes the proof, $v_\varepsilon > 0$. □

Remark 5.2.15. *It is worth noting that under the same assumptions on f and a, if the solution of the linearised problem v is positive, then we get a $\lambda_0 > 0$ such that $u_\lambda < 0$ a.e. in $\mathbb{R}^N$ for $-\lambda_0 < \lambda < 0$.*

The assumption $f(0) \neq 0$ is necessary for obtaining a solution for small λ as shown in the following lemma.

Lemma 5.2.16. *Let $|f(s)| \leq C|s|$ and let $a \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Then* (5.21) *does not admit a nontrivial solution for small λ.*

Proof. Let u be a nontrivial weak solution of (5.21) with $\lambda > 0$. Then using

Hölder inequality and Sobolev embedding we have the following:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \le C\lambda \int_{\mathbb{R}^N} |a|\,|u|^2 \le C\lambda \,\|a\|_{\frac{N}{2}} \,\|u\|_{2^*}^2 \le C_s C\lambda \,\|a\|_{\frac{N}{2}} \int_{\mathbb{R}^N} |\nabla u|^2,$$

where c_s is the constant that appears in the Sobolev inequality. Thus λ must be greater than $\frac{1}{CC_s\|a\|_{\frac{N}{2}}}$ and hence there is no nontrivial solution for λ smaller than $\frac{1}{CC_s\|a\|_{\frac{N}{2}}}$. □

Now we give some examples of nonlinear functions that satisfy our assumptions (A1) and (A2).

Example 5.2.17. *Let f be one of the following functions*

$$1+s;\ \cos(s);\ e^{-\sin s};\ \frac{s+1}{s^2+1};\ \log(2+s^2).$$

Then $f^{'}$ is a bounded function with $f(0) \neq 0$. Using the mean value theorem it is easy to see that

$$|f(s)| \le C(1+|s|),$$

for some $C > 0$. In general, one can have a nonlinearity, like

$$1+|s|^{\gamma-1};\ \left(\log(a+|s|^b)\right)^{\gamma-1}, a, b > 1,$$

where $\gamma \in [2, 2^*)$ with f' having a growth as in (A1). Now when $f^{'}$ is bounded, one can choose $a \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \bigcap L^r(\mathbb{R}^N)$ with $r > \frac{N}{2}$; otherwise depending on γ choose $a \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \bigcap L^r(\mathbb{R}^N)$ with $r > \widetilde{\gamma}$ in order to satisfy (A2).

Remark 5.2.18. *In order to obtain a positive solution for* (5.21)*, when $f(0) > 0$, we may assume that a satisfies*

$$\int_{\mathbb{R}^N} \frac{a^+(x)}{|x-y|^{N-2}}dy > \int_{\mathbb{R}^N} \frac{a^-(x)}{|x-y|^{N-2}}dy,\ \forall x \in \mathbb{R}^N.$$

for the existence of a positive solution for the linearised problem (5.21)*. Similarly for getting a positive solution for* (5.22)*, we may assume*

$$\int_{\mathbb{R}^N} \frac{a^+(x)}{|x-y|^{N-2}}dy > \int_{\mathbb{R}^N} (1+\varepsilon)\frac{a^-(x)}{|x-y|^{N-2}}dy,\ \forall x \in \mathbb{R}^N.$$

Chapter 6

Weighted eigenvalue problems for the $p-$Laplacian

In this chapter, we consider a nonlinear analogue of the linear weighted eigenvalue problem for the Laplacian, that we discussed in Chapter 4. Here, instead of the Laplace operator, we consider the p-Laplace operator Δ_p, where

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

For an open connected subset Ω in $\mathbb{R}^N$ with $p \in (1, N)$, we study the sufficient conditions for a weight function g to ensure the existence of $\lambda \in \mathbb{R}$ and $u \in \mathcal{D}_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$-\Delta_p u \quad = \quad \lambda g|u|^{p-2}u, \quad \text{ in } \Omega. \tag{6.1}$$

We say that $u \in \mathcal{D}_0^{1,p}(\Omega)$, solves (6.1), if :

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \lambda \int_\Omega g\,|u|^{p-2}u\,v, \quad \forall\, v \in \mathcal{D}_0^{1,p}(\Omega). \tag{6.2}$$

Note that, problem (6.1) is a nonlinear analogue and a natural generalization of the linear weighted eigenvalue problem for the Laplacian that we considered in Chapter 4.

There are several sufficient conditions on the weight function g available in the literature for the existence of a principal eigenvalue for (6.1). In [56] Lindqvist proved the existence, uniqueness and simplicity of a principal eigenvalue for general p, when $g \equiv 1$ and Ω is bounded. Many authors have given sufficient conditions on g for the existence of a positive principal eigenvalue for (6.1), when $\Omega = \mathbb{R}^N$: for example Huang [45], Allegretto and Huang [5], Fleckinger et al. [39], studied the problem (6.1) for general p. All these earlier results assume that either g or g^+ is in $L^{\frac{N}{p}}(\mathbb{R}^N)$. In [73], Willem and Szulkin enlarged the class of weight functions beyond the Lebesgue space $L^{\frac{N}{p}}(\Omega)$ by proving the existence of an eigenvalue of (6.1), for the weights whose positive part has a faster decay than $\frac{1}{|x|^p}$ at infinity and at all the points in the domain.

Here we prove a result analogous to Theorem 4.3.15. More precisely, we prove that for the weights whose positive part is in $\mathcal{F}_{\frac{N}{p}}$, (6.1) admits a positive principal eigenvalue. Further, we prove that this principal eigenvalue is the unique principal eigenvalue of (6.1). The results presented in this chapter have appeared [9] in Electronic Journal of Differential Equations.

As in the case of Laplacian, here also we use a direct variational principal to prove the existence of eigenvalues. Indeed, there is a one to one correspondence between the eigenvalues of (6.1) and the critical values of the following Rayleigh quotient

$$R_p(u) = \frac{\int_\Omega |\nabla u|^p}{\int_\Omega g|u|^p},$$

on the set

$$\mathcal{D}_p^+(g) = \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p > 0 \right\}. \tag{6.3}$$

Under suitable assumptions on the weight function g, one can arrive at equation (6.2) as the Euler-Lagrange equation for the critical points of R_p on $\mathcal{D}_p^+(g)$ with eigenvalues taken to be the corresponding critical values. Let

$$\begin{aligned}
\mathcal{M}_p &= \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p = 1 \right\}, \\
J_p(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p, \\
\lambda_1 &= \inf \{ J_p(u) : u \in \mathcal{M}_p \}.
\end{aligned}$$

Due to the homogeneity of the Rayleigh quotient R_p, a critical value of R_p on $\mathcal{D}_p^+(g)$ is a critical value of J_p on $\mathcal{M}_p$ and vice versa. For the existence of an eigenvalue of (6.1), we show the existence of a minimizer of J_p on $\mathcal{M}_p$. Later we prove the existence of infinitely many critical values for J_p on $\mathcal{M}_p$, using the Ljusternik Schnirelmann theorem due to Szukin [72].

This chapter is organized as follows. In Section 1, we give examples and counterexamples of weights for which (6.1) admits a positive principal eigenvalue and we relate our sufficient conditions with various sufficient conditions available in the literature. The existence and other qualitative properties such as the simplicity, the uniqueness of the first eigenvalue are discussed in Section 2. The radial symmetry of the first eigenfunctions of (6.1), under certain symmetry assumption on the weight function g is given in Section 2. In Section 3, we discuss the Ljusternik-Schnirelmann theory on $\mathcal{C}^1$ Banach manifold and a proof for the existence of infinitely many eigenvalues of (6.1) is given. Further extensions and applications are indicated in Section 4.

6.1 Examples and counterexamples.

First we give some examples of classes of weight functions for which the functional J_p admits a minimizer on $\mathcal{M}_p$. Further, we relate our results with various sufficient conditions available in the literature, for the existence of a positive principal eigenvalue for (6.1). The nonexistence of eigenvalues of (6.1) for certain weights are obtained using a Pohozaev type identity that we derive here.

6.1.1 Examples

In this chapter, we will prove that, for g such that $g^+ \in \mathcal{F}_{\frac{N}{p}}$, J_p admits a minimizer on $\mathcal{M}_p$ (Theorem 6.2.2) and this minimizer is an eigenfunction of (6.1) corresponding to λ_1. Thus Proposition 3.0.13 shows that our result subsumes results of [5, 39, 45]. In the next lemma, we show that weights considered by Szulkin and Willem in [73] for the weighted eigenvalue problems for the p-Laplacian are in $\mathcal{F}_{\frac{N}{p}}$. More specifically, they considered g satisfying the following conditions:

$$\begin{cases} g \in L^1_{\mathrm{loc}}(\Omega), \quad g^+ = g_1 + g_2 \not\equiv 0, \quad g_1 \in L^{\frac{N}{p}}(\Omega), \\ \lim\limits_{|x|\to\infty\ ,\ x\in\Omega} |x|^p g_2(x) = 0, \qquad \lim\limits_{x\to a\ ,\ x\in\overline{\Omega}} |x-a|^p g_2(x) = 0 \quad \forall a \in \overline{\Omega}. \end{cases} \tag{6.4}$$

In the next lemma we show that the positive part of a function satisfying (6.4) belongs to the space $\mathcal{F}_{\frac{N}{p}}$. Thus our results imply the results of [73].

Lemma 6.1.1. *Let Ω be a domain in $\mathbb{R}^N$ and let $1 < p < N$. Let g satisfies* *condition* (6.4). *Then* $g^+ \in \mathcal{F}_{\frac{N}{p}}$.

Proof. Let g be a function satisfying (6.4) with $g^+ = g_1 + g_2$. Clearly $g_1 \in \mathcal{F}_{\frac{N}{p}}$, since $L^{\frac{N}{p}}(\Omega) \subset \mathcal{F}_{\frac{N}{p}}$ (see Proposition 3.0.13). Further, by Theorem 3.0.24, $g_2 \in \mathcal{F}_{\frac{N}{p}}$. Hence the result. □

In the next example, we show that $\mathcal{F}_{\frac{N}{p}}$ contains weights that fail to satisfy (6.4).

Example 6.1.2. *Let* $1 < p < N$. *In the cube* $\Omega = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : |x_i| < R\}$ *with* $0 < R < 1$ *consider the function*

$$g_3(x) = \left|x_1 \log(|x_1|)\right|^{-\frac{p}{N}}, \quad x_1 \neq 0. \tag{6.5}$$

Using (3.0.21) *one can verify that* $g_3 \in L(\frac{N}{p}, q)$ *for* $q > \frac{N}{p}$ *and hence* $g_3 \in \mathcal{F}_{\frac{N}{p}}$. *However,* g_3 *does not satisfy* (6.4), *indeed along the curve* $x_2 = (x_1)^{\frac{1}{2N}}$, *the limit of* $|x|^p g_3(x)$ *is infinity as* x *tends to* 0 *and this limit is zero as* x *tends to* 0 *along the* x_1 *axis.*

6.1.2 Counterexamples

Now we derive a Pohozaev type identity for p- Laplacian, analogous to (4.10). More precisely, for the solutions of (6.1) we prove the following identity:

$$\int_{\mathbb{R}^N} \{\, x \cdot \nabla a(x) + \, p\, a(x)\,\} |u|^p = 0, \tag{6.6}$$

under certain regularity assumptions on a and u.

The Pohozaev identity is known for proving the nonexistence of solutions for certain class of partial differential equations on certain type of domains. Here we prove an identity in $\mathbb{R}^N$, similar to Pohozaev identity. Further, we use this identity to prove the nonexistence of eigenvalues for (6.1) for certain classes of weight functions. In contrast to the linear case, we prove (6.6) under additional assumptions on a and we use a weak form of divergence theorem of Cuesta and Takac [28].

Theorem 6.1.3. *Let* $a \in \mathcal{C}^{\alpha}_{loc}(\mathbb{R}^N)$ *for some* $\alpha \in (0,1)$ *and let* $u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N)$. *Further, assume that* $a(x)|u|^p$, $x \cdot \nabla\, a(x) u^p \in L^1(\mathbb{R}^N)$. *If* u *solves*

$$-\,\Delta_p u = a(x)|u|^{p-2}u \tag{6.7}$$

in the weak sense, then the following identity holds:

$$\int_{\mathbb{R}^N} \{\, x \cdot \nabla a(x) + \, p\, a(x) \,\} |u|^p = 0.$$

Proof. First we prove a point wise identity valid in the complement of zero set of ∇u. For each $\eta > 0$ let $\Omega_\eta := \{x \in \mathbb{R}^N : |\nabla u| > \eta\}$. Note that $a \in L^q_{loc}(\mathbb{R}^N)$ with $q > \frac{N}{p}$. Thus by Serrin's local regularity results ([70]) available for quasilinear operators, the solutions of (6.7) are in $\mathcal{C}^{\alpha}_{loc}(\mathbb{R}^N)$. Now Since $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is an uniformly elliptic operator on Ω_η and $a(x)|u|^{p-2}u \in \mathcal{C}^{\alpha}_{loc}(\mathbb{R}^N)$, using the standard elliptic regularity theory, we get $u \in \mathcal{C}^{2,\alpha}_{loc}(\Omega_\eta)$, see [40]. Thus one has the following point wise identity in Ω_η :

$$-\, \Delta_p u = a(x)|u|^{p-2}u \quad \text{a.e in } \Omega_\eta,\ \eta > 0. \tag{6.8}$$

First, we choose a cut-off function $\zeta \in \mathcal{C}^\infty_c(\mathbb{R})$ such that

$$(i)\ 0 \le \zeta \le 1, \quad (ii)\ \ \zeta(r) = 1,\ 0 \le r \le 1, \quad (iii)\ \zeta(r) = 0,\ \ r \ge 2.$$

and for each $n \in \mathbb{N}$ we define

$$\psi_n(x) = \zeta\left(\frac{|x|^2}{n^2}\right).$$

Then there exists $c > 0$ independent of n such that

$$|\psi_n(x)|, |x||\nabla \psi_n(x)| \le c, \quad \forall x \in \mathbb{R}^N,\ \forall n \in \mathbb{N}. \tag{6.9}$$

Now we multiply equation (6.8) by $\{x \cdot \nabla u\}\psi_n$ to obtain the following point wise identity

$$-\, \Delta_p u\ \{x \cdot \nabla u\}\psi_n = a(x)|u|^{p-2}u\ \{x \cdot \nabla u\}\psi_n \quad \text{a.e in } \Omega_\eta,\ \eta > 0. \tag{6.10}$$

It is easy to verify the following point wise identities valid in Ω_η, for each $\eta > 0$:

$$\mathrm{div}\left\{|\nabla u|^{p-2}\nabla u\,\{x \cdot \nabla u\}\psi_n\right\} = \Delta_p u\ \{x \cdot \nabla u\}\psi_n \tag{6.11}$$
$$+\, |\nabla u|^{p-2}\nabla u \cdot \nabla(\{x \cdot \nabla u\}\psi_n).$$
$$|\nabla u|^{p-2}\nabla u \cdot \nabla(\{x \cdot \nabla u\}\psi_n) = |\nabla u|^{p-2}\nabla u \cdot \nabla\{x \cdot \nabla u\}\psi_n \tag{6.12}$$
$$+\, \{x \cdot \nabla u\}|\nabla u|^{p-2}\nabla u \cdot \nabla \psi_n. \tag{6.13}$$

$$
\begin{aligned}
|\nabla u|^{p-2}\nabla u\cdot\nabla\{x\cdot\nabla u\} &= |\nabla u|^p + |\nabla u|^{p-2}\sum_j x_j \sum_i \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j} \\
&= |\nabla u|^p + |\nabla u|^{p-2}\, x\cdot(\nabla^2 u)\nabla u. \qquad (6.14)\\
\operatorname{div}(x|\nabla u|^p) &= |\nabla u|^p + x\cdot\nabla(|\nabla u|^p)\\
&= N|\nabla u|^p + p\,|\nabla u|^{p-2}\, x\cdot(\nabla^2 u)\nabla u. \qquad (6.15)
\end{aligned}
$$

From (6.14) and (6.15) we get the following identity in Ω_η:

$$
|\nabla u|^{p-2}\nabla u\cdot\nabla\{x\cdot\nabla u\} = \left(1-\frac{N}{p}\right)|\nabla u|^p + \frac{1}{p}\operatorname{div}(x|\nabla u|^p). \qquad (6.16)
$$

By combining all the above identities we obtain the following

$$
\begin{aligned}
\Delta_p u\,\{x\cdot\nabla u\}\psi_n(x) &= \operatorname{div}\left\{|\nabla u|^{p-2}\nabla u\,\{x\cdot\nabla u\}\psi_n - \frac{1}{p}(x|\nabla u|^p)\psi_n\right\}\\
&\quad -\left(1-\frac{N}{p}\right)|\nabla u|^p\psi_n - \{x\cdot\nabla u\}|\nabla u|^{p-2}\nabla u\cdot\,\nabla\psi_n \ \text{ a.e. in } \Omega_\eta.
\end{aligned}
$$

We use (6.10) in the above identity to obtain the following identity a.e. in Ω_η :

$$
\begin{aligned}
\operatorname{div}\left\{|\nabla u|^{p-2}\nabla u\,\{x\cdot\nabla u\}\psi_n - \frac{1}{p}(x|\nabla u|^p)\psi_n\right\} &= -a(x)|u|^{p-2}u\,\{x\cdot\nabla u\}\psi_n\\
&\quad + \left(1-\frac{N}{p}\right)|\nabla u|^p\psi_n + \{x\cdot\nabla u\}|\nabla u|^{p-2}\nabla u\cdot\,\nabla\psi_n.
\end{aligned}
$$

For the convenience we use the following notations:

$$
\begin{aligned}
G_n &= |\nabla u|^{p-2}\nabla u\,\{x\cdot\nabla u\}\psi_n - \frac{1}{p}(x|\nabla u|^p)\psi_n.\\
H_n &= -a(x)|u|^{p-2}u\,\{x\cdot\nabla u\}\psi_n + \left(1-\frac{N}{p}\right)|\nabla u|^p\psi_n \qquad (6.17)\\
&\quad +\{x\cdot\nabla u\}|\nabla u|^{p-2}\nabla u\cdot\,\nabla\psi_n.
\end{aligned}
$$

In the new notations, the point wise identity (6.17) reads as below

$$
\operatorname{div} G_n = H_n, \quad \text{a.e. in } \Omega_\eta. \qquad (6.18)
$$

Next we show that (6.18) holds in $\mathbb{R}^N$, in the sense of distributions. i.e.,

$$\int_{\mathbb{R}^N} \operatorname{div}(G_n)\,\phi = \int_{\mathbb{R}^N} H_n\,\phi, \quad \forall\,\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N).$$

For a given $\varepsilon > 0$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, it is enough to prove that

$$\left| \int_{\mathbb{R}^N} \operatorname{div}(G_n)\,\phi - \int_{\mathbb{R}^N} H_n\,\phi \right| < \varepsilon.$$

Because of (6.18), the above inequality will follow, provided there exists $\eta > 0$ such that

$$\left| \int_{{\Omega_\eta}^c} \operatorname{div}(G_n)\,\phi - \int_{{\Omega_\eta}^c} H_n\,\phi \right| < \varepsilon.$$

Indeed, we prove that each of the integrals in the above inequality goes to zero as $\eta \to 0$. First, we estimate $\int_{{\Omega_\eta}^c} \operatorname{div}(G_n)\,\phi$. Note that support of G_n is compact and the boundary $\partial\Omega_\eta$ is of class $\mathcal{C}^1$, since u is $\mathcal{C}^2(\Omega_\delta)$ for $\delta < \eta$. Thus by the integration by parts, we get

$$\int_{{\Omega_\eta}^c} \operatorname{div}(G_n)\,\phi = \int_{\partial\,\Omega_\eta} G_n(y)\phi(y)\cdot\gamma(y)ds(y) - \int_{\overline{\Omega_\eta}^c} G_n\ \cdot\nabla\phi.$$

From the definition of G_n, see (6.17), we obtain

$$|G_n(x)| \le \eta^p|x||\psi_n(x)| + \frac{1}{p}\eta^p|x||\psi_n| \le 2\sqrt{2}\,n\,\eta^p, \quad \forall\,x \in \mathbb{R}^N.$$

Thus

$$\left| \int_{{\Omega_\eta}^c} \operatorname{div}(G_n)\,\phi \right| \le 2\sqrt{2}\,n\,\eta^p(\|\phi\|_1 + \|\nabla\phi\|_1). \tag{6.19}$$

Now we estimate H_n. We have

$$\begin{aligned} H_n = -a(x)|u|^{p-2}u\ \{x\cdot\nabla u\}\psi_n + \left(1 - \frac{N}{p}\right)|\nabla u|^p\psi_n \\ +|\nabla u|^{p-2}\nabla u\cdot\ \{x\cdot\nabla u\}\nabla\psi_n. \end{aligned} \tag{6.20}$$

Note that from (6.9), $|x||\nabla\psi_n(x)| < C$. Thus

$$\begin{aligned}|H_n(x)| &\leq \sqrt{2}n\eta|a(x)||u|^{p-1}|\psi_n(x)| + \left(1-\frac{N}{p}\right)\eta^p + C\eta^p \\ &\leq \widetilde{C}\eta\left(1+\left\|a(x)|u|^{p-1}\psi_n\right\|_1\right). \end{aligned} \tag{6.21}$$

Therefore,

$$\left|\int_{\Omega_\eta{}^c} H_n\,\phi\right| \leq \widetilde{C}\eta\left(1+\left\|a(x)|u|^{p-1}\psi_n(x)\right\|_1\right)\|\phi\|_1. \tag{6.22}$$

It is easy to verify that $a(x)|u|^{p-1} \in L^1_{loc}(\mathbb{R}^N)$, since $a \in \mathcal{C}^\alpha_{loc}(\mathbb{R}^N)$ and $a(x)|u|^p \in L^1(\mathbb{R}^N)$. Thus from (6.19) and (6.22), we obtain the following inequality,

$$\left|\int_{\Omega_\eta{}^c} \operatorname{div}(G_n)\,\phi - \int_{\Omega_\eta{}^c} H_n\,\phi\right| < \varepsilon. \tag{6.23}$$

for sufficiently small $\eta > 0$. Therefore we conclude that

$$\operatorname{div} G_n = H_n,$$

in the sense of distributions. Note that by the regularity results of Tolksdorf [75], the solutions of (6.7) are $\mathcal{C}^{1,\alpha}_{loc}(\mathbb{R}^N)$. Thus G_n is continuous in $\overline{B_{\sqrt{2}n}}$. Further, G_n vanishes on the boundary of $B_{\sqrt{2}n}$. Now we use the weak divergence theorem due to Cuesta and Takac [28] to obtain

$$\int_{B_{\sqrt{2}n}} H_n(x)dx = 0. \tag{6.24}$$

Next we find a point wise identity for $a(x)|u|^{p-2}u\,\{x\cdot\nabla u\}\psi_n$. Let $F(u) = \frac{|u|^p}{p}$. Thus we obtain the following identity, for a.e. in $\mathbb{R}^N$:

$$\begin{aligned}\operatorname{div}\{x\,a(x)\,F(u)\psi_n(x)\} &= N\,a(x)\,F(u)\,\psi_n(x) + x\cdot\nabla a(x)\,F(u)\,\psi_n(x) \\ &\quad + a(x)\,|u|^{p-2}u\,\{x\cdot\nabla u\}\,\psi_n(x) + a(x)\,F(u)\,x\cdot\nabla\psi_n(x).\end{aligned}$$

Therefore

$$H_n(x) = N\,a(x)\,F(u)\,\psi_n + x\cdot\nabla a(x)\,F(u)\,\psi_n - \operatorname{div}\{x\,a(x)\,F(u)\psi_n\} \\ + \left(1-\frac{N}{p}\right)|\nabla u|^p\psi_n + \{x\cdot\nabla u\}|\nabla u|^{p-2}\nabla u\cdot\nabla\psi_n. \tag{6.25}$$

In particular the above identity holds in the sense of distributions. Now again by applying weak divergence theorem of [28], by noting that ψ_n vanishes on the boundary we obtain:

$$\int_{B_{\sqrt{2}n}} \left\{N\,a(x)\,F(u) + x\cdot\nabla a(x)\,F(u) + \left(1-\frac{N}{p}\right)|\nabla u|^p\right\}\psi_n(x) \\ + \int_{B_{\sqrt{2}n}} \big\{\{x\cdot\nabla u\}|\nabla u|^{p-2}\nabla u + a(x)\,F(u)\,x\big\}\cdot\nabla\psi_n(x) = 0. \tag{6.26}$$

Note that each term in the above integrals is integrable in entire $\mathbb{R}^N$. Now by letting n tend to infinity, using dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^N} N\,a(x)\,F(u) + x\cdot\nabla a(x)\,F(u) + \left(1-\frac{N}{p}\right)|\nabla u|^p = 0. \tag{6.27}$$

Further, from (6.7), we have

$$\int_{\mathbb{R}^N} |\nabla u|^p = \int_{\mathbb{R}^N} a(x)|u|^p. \tag{6.28}$$

Now by substituting the above identity in (6.27) we obtain the required identity

$$\int_{\mathbb{R}^N} \{x\cdot\nabla a(x) + p\,a(x)\}\,|u|^p = 0.$$

□

Remark 6.1.4. *Let a and u be as in the above theorem. If $x\cdot\nabla a(x)+p\,a(x)$ has a definite sign, then we have the nonexistence of solution for* (6.7). *In particular, we have the nonexistence of an eigenvalue for* (6.1), *when $x\cdot\nabla g(x)+p\,g(x)$ has a definite sign.*

Example 6.1.5. *For example, when $a(x) = \frac{1}{(1+|x|^2)^{\frac{p}{2}}}$, one can verify that*

$$x\cdot\nabla a(x) + p\,a(x) > 0.$$

Thus no solution for (6.7) *and hence* (6.1) *does not admit an eigenvalue, when* $g(x) = \frac{1}{(1+|x|^2)^{\frac{p}{2}}}$.

Remark 6.1.6. *We emphasize that the above identity is only a necessary condition and we may not be able to make any conclusion, when the quantity* $x \cdot \nabla a(x) + p\, a(x)$ *is zero. For example, for the Hardy potential* $a(x) = \frac{1}{|x|^p}$, *we have*

$$x \cdot \nabla a(x) + p\, a(x) = 0.$$

Thus our identity does not prove the nonexistence of an eigenvalue for (6.1).

Later we show that (6.1) does not admit a positive principal eigenvalue for the weight $g(x) = \frac{1}{|x|^p}$.

6.2 Existence of the first eigenvalue and its properties

6.2.1 Existence of an eigenvalue

As in the case Laplacian, here also we use a variational technique for proving the existence of an eigenvalue for (6.1). First let us recall the following definitions:

$$\begin{aligned}
R_p(u) &= \frac{\int_\Omega |\nabla u|^p}{\int_\Omega g|u|^p}, \\
\mathcal{D}_p^+(g) &= \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p > 0 \right\}, \\
\mathcal{M}_p &= \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p = 1 \right\}, \\
J_p(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p.
\end{aligned}$$

From the definition of J_p, it is obvious that J_p is coercive and weakly lower semi-continuous on $\mathcal{D}_0^{1,p}(\Omega)$. Now, in addition if $\mathcal{M}_p$ is weakly closed, then standard theorem in functional analysis gives the existence of a minimizer for J_p on $\mathcal{M}_p$. However, $\mathcal{M}_p$ is far from being weakly closed. The weak closedness of $\mathcal{M}_p$ is related to the compactness of the following nonlinear functional:

$$G_p(u) = \frac{1}{p} \int_\Omega g|u|^p.$$

Due to our weak assumptions on g^-, the map G_p may not be even continuous. However, for our objective, weak limits of all weakly convergent sequences are not required to be in $\mathcal{M}_p$; it is sufficient that only the weak limits of all sequence minimizing J_p over $\mathcal{M}_p$ lie in $\mathcal{M}_p$. We show this is indeed true, under our assumptions on g^+. First we prove the following preparatory lemma:

Lemma 6.2.1. *Let $g^+ \in F_{\frac{N}{p}} \setminus \{0\}$ and let*

$$G_p^+(u) = \frac{1}{p}\int_\Omega g^+|u|^p.$$

Then G_p^+ is compact.

Proof. Let $\{u_n\}$ converge weakly to u in $\mathcal{D}_0^{1,p}(\Omega)$. First we show that a subsequence of $\{G_p^+(u_n)\}$ converges to $G_p^+(u)$. Let $\phi \in \mathcal{C}_c^\infty(\Omega)$ be arbitrary. Now we write:

$$p(G_p^+(u_n) - G_p^+(u)) = \int_\Omega \phi\,(|u_n|^p - |u|^p) + \int_\Omega (g^+ - \phi)\,(|u_n|^p - |u|^p). \tag{6.29}$$

First we estimate the second integral using Lorentz Sobolev embedding and Hölder inequality as below

$$\int_\Omega |(g^+ - \phi)|\,\left|(|u_n|^p - |u|^p)\right| \quad \leq C\,\|g^+ - \phi\|_{(\frac{N}{p},\infty)}\left(\|u_n\|^p_{(p^*,p)} + \|u\|^p_{(p^*,p)}\right).$$

where C is a constant which depends only on N and p. Clearly u_n is a bounded sequence in $L(p^*,p)$. Let

$$m := \sup_n \left\{\|u_n\|^p_{(p^*,p)} + \|u\|^p_{(p^*,p)}\right\}.$$

Now using the definition of the space $F_{\frac{N}{p}}$, for a given $\varepsilon > 0$, we choose $g_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ so that

$$\left\|g^+ - g_\varepsilon\right\|_{(\frac{N}{p},\infty)} < \frac{\varepsilon}{m\,C}.$$

Thus by taking $\phi = g_\varepsilon$, we obtain

$$\int_\Omega |(g^+ - g_\varepsilon)|\,\left|(|u_n|^p - |u|^p)\right| < \varepsilon. \tag{6.30}$$

Since $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p_{loc}(\Omega)$ compactly, the first integral in (6.29) converges to zero with $\phi = g_\varepsilon$. Thus there exists $n_0 \in \mathbb{N}$, so that

$$\int_\Omega g_\varepsilon \left(|u_n|^p - |u|^p\right) < \varepsilon, \quad \forall n > n_0.$$

Now using (6.30), we get

$$|G_p^+(u_n) - G_p^+(u)| < C\varepsilon, \quad \forall n > n_0, \tag{6.31}$$

where C is a constant independent of n. Thus we conclude that the sequence $G_p^+(u_n)$ converges to $G_p^+(u)$. □

Now we are in a position to prove the existence of a minimizer for J_p on $\mathcal{M}_p$.

Theorem 6.2.2. *Let Ω be a domain in $\mathbb{R}^N$ with $p \in (1, N)$ and let $g \in L^1_{loc}(\Omega)$. If $g^+ \in \mathcal{F}_{\frac{N}{p}} \setminus \{0\}$, then J_p admits a minimizer on $\mathcal{M}_p$.*

Proof. Since $g \in L^1_{loc}(\Omega)$ and $g^+ \neq 0$, there exists $\varphi \in C_c^\infty(\Omega)$ such that $\int_\Omega g|\varphi|^p > 0$ (see Lemma 6.3.9) and hence $\mathcal{M}_p \neq \emptyset$. Let $\{u_n\}$ be a minimizing sequence of J_p on $\mathcal{M}_p$. Thus

$$\lim_{n\to\infty} J_p(u_n) = \lambda_1 = \inf_{u \in \mathcal{M}_p} J_p(u).$$

By the coercivity of $J_p, \{u_n\}$ is bounded and hence using the reflexivity of $\mathcal{D}_0^{1,p}(\Omega)$ we obtain a subsequence of $\{u_n\}$ that converges weakly to some $u \in \mathcal{D}_0^{1,p}(\Omega)$. We denote the subsequence by $\{u_n\}$ itself. Now using the compactness of G_p^+, we get

$$\lim_{n\to\infty} \int_\Omega g^+|u_n|^p = \int_\Omega g^+|u|^p. \tag{6.32}$$

Now as $u_n \in \mathcal{M}_p$ we write,

$$\int_\Omega g^-|u_n|^p = \int_\Omega g^+|u_n|^p - 1.$$

Since the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p_{loc}(\Omega)$ is compact, up to a subsequence $u_n \to u$ a.e in Ω. Hence using (6.32) and the Fatou's lemma, we obtain

$$\int_\Omega g^-|u|^p \leq \int_\Omega g^+|u|^p - 1,$$

this shows that $\int_\Omega g|u|^p \geq 1$. Set $\widetilde{u} := \frac{u}{(\int_\Omega g|u|^p)^{\frac{1}{p}}}$. Now the weak lower semi continuity of J_p yields the following:

$$\lambda_1 \leq J_p(\widetilde{u}) = \frac{J_p(u)}{\int_\Omega g|u|^p} \leq J_p(u) \leq \liminf_n J_p(u_n) = \lambda_1$$

Thus the equality must hold at each step and hence $\int_\Omega g|u|^p = 1$, which shows that $u \in \mathcal{M}_p$ and $J_p(u) = \lambda_1$. □

Remark 6.2.3. *It is easy to see that there is a one to one correspondence between the minimizers J_p on $\mathcal{M}_p$ and the minimizers of R_p on $\mathcal{D}_p^+(g)$.*

Note that R_p is not regular enough to conclude that u is an eigenfunction of (6.2) corresponding to λ_1, using the critical point theory.

Proposition 6.2.4. *Let g be as in the above theorem and let u be a minimizer of J_p on $\mathcal{M}_p$. Then u is an eigenfunction of* (6.1) *corresponding to λ_1.*

Proof. For each $\phi \in \mathcal{C}_c^\infty(\Omega)$, using the dominated convergence theorem one can verify that R_p admits the directional derivative along ϕ. Since u is a minimizer of J_p on $\mathcal{D}_p^+(g)$ we get

$$\frac{d}{dt} R_p(u + t\phi)|_{t=0} = 0.$$

Therefore

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi = \lambda_1 \int_\Omega g\, |u|^{p-2} u\, \phi, \quad \forall\, \phi \in \mathcal{C}_c^\infty(\Omega).$$

Now using the density of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{D}_0^{1,p}(\Omega)$ we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \lambda_1 \int_\Omega g\, |u|^{p-2} u\, v, \quad \forall\, v \in \mathcal{D}_0^{1,p}(\Omega).$$

This shows that u is an eigenfunction of (6.1) corresponding to λ_1. □

6.2.2 Sign and the uniqueness of minimizer

First we prove that eigenfunctions corresponding to λ_1 are of constant sign. Due to the lack of regularity of first eigenfunctions, we can't use the classical strong maximum principle to prove that the first eigenfunctions are of constant sign. For $p = 2$, we used the strong maximum principle due to

Brezis and Ponce [19] to show that first eigenfunctions are of constant sign (see Chapter 4). In [50] a similar strong maximum principle has been proved for more general quasilinear operators. From Proposition 3.2 of [50] one can obtain the following lemma.

Lemma 6.2.5 **(Strong Maximum principle for Δ_p).** *Let $u \in \mathcal{D}_0^{1,p}(\Omega)$ and $V \in L^1_{loc}(\Omega)$ be such that $u, V \geq 0$ a.e in Ω. If $V\, u^{p-1} \in L^1_{loc}(\Omega)$ and u satisfies the following differential inequality(in the sense of distributions)*

$$-\Delta_p u + V(x) u^{p-1} \geq 0 \quad \text{ in } \Omega.$$

Then either $u \equiv 0$ or $u > 0$ a.e.

Now from the above lemma we have the following result.

Lemma 6.2.6. *Let g be as in Theorem 6.2.2. Then each eigenfunction corresponding to λ_1 is of constant sign.*

Proof. It is clear that the eigenfunctions corresponding to λ_1 are the minimizers of R_p on $\mathcal{D}_p^+(g)$. Let u be a minimizer of R_p on $\mathcal{D}_p^+(g)$. Since $u \neq 0$ either u^+ or u^- is non zero. Without loss of generality we may assume that $u^+ \neq 0$. Now by taking u^+ as a test function in (6.2), we see that u^+ also minimizes R_p on $\mathcal{D}_p^+(g)$. Thus by Proposition 6.2.4, we see that u^+ also solves (6.1) in the weak sense,

$$-\Delta_p u^+ - \lambda_1\, g\, (u^+)^{p-1} = 0, \text{ in } \Omega.$$

In particular we have the following differential inequality in the sense of distributions,

$$-\Delta_p u^+ + \lambda_1 g^- \, (u^+)^{p-1} = \lambda_1 g^+ (u^+)^{p-1} \geq 0, \text{ in } \Omega.$$

It is clear that g^- and u^+ satisfy all the assumptions of Lemma 6.2.5, if we show that $g^-(u^+)^p \in L^1_{loc}(\Omega)$. Since $g|u|^p \in L^1(\Omega)$, we have $(g^-)^{\frac{1}{q}}(u^+)^{p-1} \in L^q(\Omega)$, where q is the conjugate exponent of p. Further, $(g^-)^{\frac{1}{p}} \in L^p_{loc}(\Omega)$, since $g \in L^1_{loc}(\Omega)$. Let us write

$$g^-(u^+)^{p-1} = (g^-)^{\frac{1}{p}}(g^-)^{\frac{1}{q}}(u^+)^{p-1}.$$

Now we use Hölder inequality to conclude that $g^-(u^+)^{p-1} \in L^1_{loc}(\Omega)$. Now in view of Lemma 6.2.5, we obtain $u^+ > 0$ a.e. and hence $u = u^+$. Moreover, the zero set of u is of measure of zero. $\square$

From the above lemma, it is clear that λ_1 is a principal eigenvalue of (6.1). Next we prove the uniqueness of the positive principal eigenvalue of (6.1), using the Picone's identity for the p-Laplacian. In [6], Picone's identity is proved for $\mathcal{C}^1$ functions. However it is not difficult to obtain a similar identity for less regular functions.

Lemma 6.2.7. *(Picone's identity) Let $u \geq 0, v > 0$ a.e. and let $|\nabla v|, |\nabla u|$ exist as measurable functions. Then the following identity holds a.e.*

$$\begin{aligned} |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p &- p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \\ &= |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right) \cdot |\nabla v|^{p-2}\nabla v. \end{aligned}$$

Furthermore, the left and the right hand sides of the above identity are non-negative.

Now we prove the uniqueness of the positive principal eigenvalue of (6.1).

Lemma 6.2.8. *Let $g \in L(\frac{N}{p}, \infty)$ and let $\lambda > 0$ be a positive principal eigenvalue of* (6.1). *Then*

$$\lambda = \lambda_1 = \inf\left\{\int_\Omega |\nabla u|^p : u \in \mathcal{M}_p\right\}.$$

Proof. Let $v \in \mathcal{D}_0^{1,p}(\Omega)$ be a positive eigenfunction of (6.1) corresponding to λ. Let $u \in \mathcal{M}_p$. Thus there exists a sequence $\{\phi_n\}$ in $\mathcal{C}_c^\infty(\Omega)$ such that $\|u - \phi_n\|_{\mathcal{D}_0^{1,p}(\Omega)} \to 0$ and $\int_\Omega g|u|^p = 1$. Note that $\frac{|\phi_n|^p}{v+\varepsilon} \in \mathcal{D}_0^{1,p}(\Omega)$ (see Proposition 2.3.7). Now by applying the Picone's identity for $|\phi_n|$ and $v + \varepsilon$, we obtain the following inequality:

$$0 \leq \int_\Omega |\nabla|\phi_n||^p - \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\left(\frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}\right). \tag{6.33}$$

Since v is an eigenfunction of (6.1) corresponding to λ, we have

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\left(\frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}\right) = \lambda\int_\Omega g v^{p-1}\frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}. \tag{6.34}$$

Now from (6.33) and (6.34) we obtain the following:

$$0 \leq \int_\Omega |\nabla|\phi_n||^p - \lambda\int_\Omega g v^{p-1}\frac{|\phi_n|^p}{(v+\varepsilon)^{p-1}}. \tag{6.35}$$

By letting $\varepsilon \to 0$, the dominated convergence theorem yields:

$$0 \le \int_\Omega |\nabla|\phi_n||^p - \lambda \int_\Omega g\phi_n^p. \tag{6.36}$$

Note that $\int_\Omega |\nabla|\phi_n||^p = \int_\Omega |\nabla\phi_n|^p$. Now using Corollary 2.2.8 and the Fatou's lemma we let $n \to \infty$ to obtain the following inequality:

$$0 \le \int_\Omega |\nabla u|^p - \lambda \int_\Omega gu^p. \tag{6.37}$$

Therefore

$$\lambda \le \int_\Omega |\nabla u|^p, \quad \forall u \in \mathcal{M}_p. \tag{6.38}$$

This completes the proof. □

Remark 6.2.9. *As a consequence of the above lemma, we see that the eigenfunctions corresponding to other eigenvalues of* (6.1) *must change sign.*

Remark 6.2.10. *The continuity argument that we used for proving the simplicity of the first eigenvalue of Laplacian (see Proposition 4.3.14) is not applicable here, since the sum of two eigenfunctions corresponding to an eigenvalue of p-Laplacian is no longer an eigenfunction. In [59], Lucia and Prashanth obtained the simplicity of the first eigenvalue of* (6.1)*, if it exit, even when $g \in L^r(\Omega), r > 1$ and Ω is connected. Later in [50], Kawohl, Lucia and Prashanth extended this results for g which are only in $L^1_{loc}(\Omega)$. For a proof of simplicity of λ_1 we refer to Theorem 1.3 of [50].*

From Theorem 6.2.2, Lemma 6.2.4, Remark 6.2.9 and Theorem 1.3 of [50] we have the following result:

Theorem 6.2.11. *Let Ω be an open connected subset of $\mathbb{R}^N$. Let $p \in (1, N)$ and let $g \in L^1_{loc}(\Omega)$ such that $g^+ \in \mathcal{F}_{\frac{N}{p}} \setminus \{0\}$. Then*

$$\lambda_1 = \inf \{J_p(u) : u \in \mathcal{M}_p\} \tag{6.39}$$

is the unique positive principal eigenvalue of (6.1)*. Furthermore, each eigenfunction corresponding to λ_1 is of constant sign and λ_1 is simple.*

6.2.3 Radial solution

Now we give sufficient conditions for the radial symmetry of eigenfunctions, if exists, corresponding to the eigenvalue λ_1 of (6.1). Here we assume that the domain Ω is a ball centred at origin or $\mathbb{R}^N$. In [17], Bhattacharya proved the radial symmetry of the first eigenfunctions of (6.1), when $g \equiv 1$ and Ω is ball. Here we prove that all positive eigenfunctions corresponding to λ_1 are radial and radially decreasing, provided g is nonnegative, radial and radially decreasing. Thus our result is a two fold generalization of results of Bhattacharya, as we allow more general weight functions and the domain can be $\mathbb{R}^N$. Our result uses certain rearrangement inequalities. We emphasize that here we are not assuming any conditions to ensure that λ_1 is an eigenvalue.

Theorem 6.2.12. *Let Ω be a ball centred at origin or $\mathbb{R}^N$. Let g be nonnegative, radial and radially decreasing measurable function. If λ_1 is an eigenvalue of* (6.1)*, then any positive eigenfunction corresponding to λ_1 is radial and radially decreasing.*

Proof. Let u be a positive eigenfunction of (6.1) corresponding to λ_1. Let u_* and g_* be the symmetric decreasing rearrangement of u and g respectively. Since g is nonnegative, radial and radially decreasing, we use property (a) of Proposition 2.1.11 to conclude that $g = g_*$ a.e. Further, as u is positive by property (c) of Proposition 2.1.11 we obtain $(u^p)_* = (u_*)^p$ a.e. Now by the Hardy-Littlewood inequality (see Theorem 2.1.13),

$$\int_\Omega g\, u^p \leq \int_\Omega g_*(u^p)_* = \int_\Omega g(u_*)^p.$$

Further due to Polya-Szego we have the following inequality:

$$\int_\Omega |\nabla u_*|^p \leq \int_\Omega |\nabla u|^p.$$

Thus

$$\frac{1}{\int_\Omega g(u_*)^p} \int_\Omega |\nabla u_*|^p \leq \frac{1}{\int_\Omega g(u)^p} \int_\Omega |\nabla u|^p. \tag{6.40}$$

Since u is a minimizer of R_p on $\mathcal{D}_p^+(g)$, equality holds in (6.40) and hence u_* also minimizes R_p on $\mathcal{D}_p^+(g)$. Now as λ_1 is simple, we get $u_* = \alpha u$ a.e. for some $\alpha > 0$. This shows that u is radial and radially decreasing. □

Next we prove that the Hardy potential, $\frac{1}{|x|^p}$ does not admit a positive principal eigenvalue for (6.1).

Proposition 6.2.13. *Let $g(x) = \frac{1}{|x|^p}$, $x \in \mathbb{R}^N$. Then* (6.1) *does not admit a positive principal eigenvalue.*

Proof. By the above lemma, it is enough to show that λ_1 is not an eigenvalue of (6.1), when $g(x) = \frac{1}{|x|^p}$. If λ_1 is an eigenvalue of (6.1), then by Theorem 1.3 of [50] we must have λ_1 is simple. Further, if u is an eigenfunction of (6.1) corresponding λ_1, then using the scale invariance of (6.1), for each $\alpha \in \mathbb{R}$, one can verify that

$$v_\alpha(x) = u(\alpha x)$$

is also an eigenfunction of (6.1), corresponding to λ_1. Now using the simplicity of λ_1 and the radial symmetry of u. one can show that

$$u(x) = |x|^{1-\frac{N}{p}} u(1).$$

This is a contradiction as $|x|^{1-\frac{N}{p}} \notin \mathcal{D}_0^{1,p}(\Omega)$. $\square$

6.3 Existence of infinitely many eigenvalues

In this section we discuss the existence of infinitely many eigenvalues of (6.1). Our proof relies on the Ljusternik-Schnirelmann theory on a $\mathcal{C}^1$ manifold due to Szulkin [72]. The classical Ljusternik-Schnirelmann minimax theorem uses a deformation homotopy which requires the set $\mathcal{M}_p$ to be at least a $\mathcal{C}^{1,1}$ manifold (i.e, transition maps are $\mathcal{C}^1$ and its derivatives are locally Lipschitz). However Szulkin, in [72] developed Ljusternik-Schnirelmann theorem on a $\mathcal{C}^1$ manifold using the Ekeland variational principle. It is worth mentioning that due to the weaker assumptions on the weight g, the set $\mathcal{M}_p$ that we are considering does not even possess a manifold structure from the topology inherited from $\mathcal{D}_0^{1,p}(\Omega)$. However, we define a suitable topology on $\mathcal{M}_p$ that makes $\mathcal{M}_p$ a $\mathcal{C}^1$ Banach manifold and use a result of [72] to get an infinite sequence of eigenvalues tending to infinity.

First we give the following definitions: Let $\mathcal{M}$ be a $\mathcal{C}^1$ manifold and $f \in \mathcal{C}^1(\mathcal{M};\mathbb{R})$. Denote the differential of f at u by $df(u)$, an element of $(T_u\mathcal{M})^*$, the cotangent space of $\mathcal{M}$ at u (see section 27.4 of [30] for definition and properties).

Let A be a closed and symmetric (i.e, $-A = A$) subset of $\mathcal{M}$. The *Krasnoselskii* genus $\gamma(A)$ is defined to be the smallest integer k for which there exists a non-vanishing odd continuous mapping from A to $\mathbb{R}^k$. If there exists no such map for any k, then we define $\gamma(A) = \infty$ and we set $\gamma(\emptyset) = 0$. For more details and properties of genus we refer to Chapter 7 of [66].

We say that a map $f \in \mathcal{C}^1(\mathcal{M};\mathbb{R})$ satisfies the Palais-smale (P.S. for short) condition on $\mathcal{M}$, if a sequence $\{u_n\} \subset \mathcal{M}$ is such that $f(u_n) \to \lambda$

and $df(u_n) \to 0$ then $\{u_n\}$ possesses a convergent subsequence.

From Corollary 4.1 of [72] one can deduce the following theorem:

Theorem 6.3.1 (Szulkin's Theorem). *Let $\mathcal{M}$ be a closed symmetric $\mathcal{C}^1$ submanifold of a real Banach space X and $0 \notin \mathcal{M}$. Let $f \in \mathcal{C}^1(\mathcal{M}; R)$ be an even function which satisfies P.S. condition on $\mathcal{M}$ and bounded below. Define*

$$c_j := \inf_{A \in \Gamma_j} \sup_{x \in A} f(x),$$

where $\Gamma_j = \{A \subset \mathcal{M} : A \text{ is compact and symmetric about origin}, \gamma(A) \geq j\}$. If for a given j, $c_j = c_{j+1} \ldots = c_{j+p} \equiv c$, then $\gamma(K_c) \geq p+1$, where $K_c = \{x \in \mathcal{M} : f(x) = c\,, df(x) = 0\}$.

Note that the set, $\mathcal{M}_p = \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p = 1 \right\}$, may not even possess a manifold structure from the topology of $\mathcal{D}_0^{1,p}(\Omega)$, due to the weak assumptions on g^-. However we show that $\mathcal{M}_p$ admits a $\mathcal{C}^1$ Banach manifold structure from a subspace of $\mathcal{D}_0^{1,p}(\Omega)$.

For $g^- \in L^1_{loc}(\Omega)$, we define

$$\|u\|_X^p := \int_\Omega |\nabla u|^p + \int_\Omega g^- |u|^p.$$

$$X := \{u \in \mathcal{D}_0^{1,p}(\Omega) : \|u\|_X < \infty\}.$$

Then one can easily verify the following:

- X is a Banach space with the norm $\|\,.\,\|_X$ and X is reflexive.
- Since g^- is locally integrable, $\mathcal{C}_c^\infty(\Omega)$ is contained in X.
- Let $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{p}}$. Then $\mathcal{D}_p^+(g)$ is contained in X. This can be seen as below:

$$\int_\Omega g^- |u|^p < \int_\Omega g^+ |u|^p \leq C \left\|g^+\right\|_{(\frac{N}{p},\infty)} \|u\|^p_{\mathcal{D}_0^{1,p}(\Omega)} < \infty, \tag{6.41}$$

  where C is the constant involving the constants that are appearing in the Lorentz-Sobolev embedding and the Hölder inequality. Note that the first inequality follows as $\int_\Omega g|u|^p > 0$, for $u \in \mathcal{D}_p^+(g)$.

- X is continuously embedded into $\mathcal{D}_0^{1,p}(\Omega)$. Thus X embedded continuously into the Lorentz space $L(p^*, p)$ and embedded compactly into $L^p_{loc}(\Omega)$.

We denote the dual space of X by $X^{'}$ and the duality action by $\langle\, , \rangle$.

Using the definition of the norm one can easily see that, the map G_p^-, defined by

$$G_p^-(u) := \frac{1}{p}\int_\Omega g^-|u|^p,$$

is continuous on X. Further, using the dominated convergence theorem one can verify that G_p^- is continuously differentiable on X and its derivative is given by

$$\left\langle {G_p^-}^{'}(u), v\right\rangle = \int_\Omega g^-|u|^{p-2}u\, v.$$

Similarly using the Sobolev embedding and the Hölder inequality one can easily verify that G_p^+ is $\mathcal{C}^1$ in $\mathcal{D}_0^{1,p}(\Omega)$ and in particular on X. The derivative of G_p^+ is given by

$$\left\langle {G_p^+}^{'}(u), v\right\rangle = \int_\Omega g^+|u|^{p-2}u\, v.$$

Note that for $u \in \mathcal{M}_p$, $\left\langle G_p^{'}(u), u\right\rangle = p$ and hence the map $G_p^{'}(u) \neq 0$. Recall that, $c \in \mathbb{R}$ is called a regular value of G_p, if $G_p^{'}(u) \neq 0$ for all u such that $G_p(u) = c$. Thus we have the following lemma:

Lemma 6.3.2. *Let g be as in theorem 6.2.11. Then the map G_p is in $\mathcal{C}^1(X;\mathbb{R})$ and $G_p^{'} : X \to X^{'}$ is given by*

$$\left\langle G_p^{'}(u), v\right\rangle = \int_\Omega g|u|^{p-2}u\, v.$$

Further, 1 is a regular value of G_p.

Remark 6.3.3. *In view of Example 27.2 of [30], the above lemma shows that $\mathcal{M}_p$ is a $\mathcal{C}^1$ Banach submanifold of X. Note that $\mathcal{M}_p$ is symmetric about the origin as the map G_p is even.*

Next we show that J_p satisfies all the conditions to apply Theorem 6.3.1.

Lemma 6.3.4. *J_p is a $\mathcal{C}^1$ functional on $\mathcal{M}_p$ and the derivative of J_p is given by*

$$\left\langle J_p^{'}(u), v\right\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v$$

The proof is straight forward.

Remark 6.3.5. *Using Proposition 6.4.35 of [35], one can deduce that*

$$\|dJ_p(u)\| = \min_{\lambda\in\mathbb{R}} \left\| J_p^{'}(u) - \lambda G_p^{'}(u) \right\|. \tag{6.42}$$

Thus $dJ_p(u_n) \to 0$ if and only if there exists a sequence $\{\lambda_n\}$ of real numbers such that $J_p^{'}(u_n) - \lambda_n G_p^{'}(u_n) \to 0$.

In the next lemma we prove the compactness of the map G_p^+, that we use for showing that the map J_p satisfies P.S. condition on $\mathcal{M}_p$.

Lemma 6.3.6. *The map $G_p^{+'} : X \to X^{'}$ is compact.*

Proof. Let $u_n \rightharpoonup u$ in X and $v \in X$. Let q be the conjugate exponent of p. Now using the Lorentz-Sobolev embedding and the Hölder inequality available for the Lorentz spaces, one can verify the following:

$$\begin{aligned}
\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) &\in L\left(\frac{p^*}{p-1}, \frac{p}{p-1}\right),\\
(g^+)^{\frac{1}{q}}\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) &\in L\left(\frac{p}{p-1}, \frac{p}{p-1}\right),\\
(g^+)^{\frac{1}{p}}|v| &\in L(p\,,p),\\
\left\|(g^+)^{\frac{1}{p}}v\right\|_p &\leq C\|g^+\|^{\frac{1}{p}}_{(\frac{N}{p},\infty)}\|v\|_{(p^*,\,p)},
\end{aligned}$$

where C is a constant that depends only on p, N. Now by using usual Hölder inequality we get,

$$\begin{aligned}
|\langle G_p^{'}(u_n) - G_p^{'}(u), v\rangle| &\leq \int_\Omega g^+ \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right| |v|\\
&\leq \left(\int_\Omega g^+ \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_\Omega g^+|v|^p\right)^{\frac{1}{p}}\\
&\leq \|g^+\|^{\frac{1}{p}}_{(\frac{N}{p},\infty)}\|v\|_{(p^*,p)} \left(\int_\Omega g^+ \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Thus

$$\|G_p^{'}(u_n) - G_p^{'}(u)\| \leq \|g^+\|^{\frac{1}{p}}_{(\frac{N}{p},\infty)} \left(\int_\Omega g^+ \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Now it is enough to show that

$$\left(\int_\Omega g^+ \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \to 0, \text{ as } n \to \infty.$$

Let $\varepsilon > 0$ and $g_\varepsilon \in C_c^\infty(\Omega)$ be arbitrary.

$$\begin{aligned}\int_\Omega g^+ \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}} &= \int_\Omega g_\varepsilon \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}} \\ &\quad + \int_\Omega (g^+ - g_\varepsilon) \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}}\end{aligned} \tag{6.43}$$

First we estimate the second integral. Note that $\left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}}$ is bounded in $L(\frac{p^*}{p}, 1)$. Let

$$m = \sup_n \left\| \left|\left(| u_n |^{p-2}u_n - | u |^{p-2}u \right)\right|^{\frac{p}{p-1}} \right\|_{(\frac{p^*}{p},1)}.$$

$$\int_\Omega |(g^+ - g_\varepsilon)| \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}} \leq C\, m \left\| \left(g^+ - g_\varepsilon\right) \right\|_{(\frac{N}{p},\infty)},$$

where the constant C includes all the constants that appear in the Hölder inequality and the Lorentz-Sobolev embedding. Now since $g^+ \in F_{\frac{N}{p}}$, from the definition of $F_{\frac{N}{p}}$, we can choose $g_\varepsilon \in C_c^\infty(\Omega)$ such that

$$m \left\| \left(g^+ - g_\varepsilon\right) \right\|_{(\frac{N}{p},\infty)} < \frac{\varepsilon}{2C}.$$

Thus we can make the second integral in (6.43) smaller than $\frac{\varepsilon}{2}$ for a suitable choice of g_ε. Since X is embedded compactly into $L^p_{loc}(\Omega)$, the first integral converges to zero up to a subsequence $\{u_{n_k}\}$ of $\{u_n\}$. Hence we get $k_0 \in \mathbb{N}$ so that,

$$\int_\Omega g^+ \left|\left(|u_{n_k}|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}} < \varepsilon, \quad \forall\, k > k_0.$$

Now the uniqueness of limit of subsequence helps us to conclude, as in Lemma 6.2.1, that $\left(\int_\Omega g^+ \left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \to 0$ as $n \to \infty$. Hence the proof. $\square$

Definition 6.3.7. *For $\lambda \in \mathbb{R}^+$, we define $A_\lambda : X \to X^{'}$ as*

$$A_\lambda = J_p^{\,'} + \lambda\, G_p^{-\,'}.$$

In the next proposition we show that the map J_p indeed satisfies P.S. condition on the $\mathcal{M}_p$.

Proposition 6.3.8. *J_p satisfies P.S. condition on $\mathcal{M}_p$.*

Proof. Let $\{u_n\}$ be a sequence in $\mathcal{M}_p$, such that $J_p(u_n) \to \lambda$ and $dJ_p(u_n) \to 0$. Thus there exists a sequence $\{\lambda_n\}$ such that

$$J_p^{'}(u_n) - \lambda_n G_p^{'}(u_n) \to 0 \text{ as } n \to \infty. \tag{6.44}$$

Since $J_p(u_n)$ is bounded, using the estimate (6.41), we see that $\{G_p^-(u_n)\}$ is bounded. Thus the sequence $\{u_n\}$ is bounded in X and hence by the reflexivity we may assume that $u_n \rightharpoonup u$, by passing to a subsequence. Since G_p^+ is compact, we get $G_p^+(u_n) \to G_p^+(u)$. Now by Fatou's lemma

$$\int_\Omega g^-|u|^p \leq \liminf \int_\Omega g^+|u_n|^p - 1 = \int_\Omega g^+|u|^p - 1. \tag{6.45}$$

Thus $\int_\Omega g|u|^p \geq 1$ and hence $u \neq 0$. Further, $\lambda_n \to \lambda$ as $n \to \infty$, since

$$p(J_p(u_n) - \lambda_n) = \left\langle J_p^{'}(u_n) - \lambda_n G_p^{'}(u_n), u_n \right\rangle \to 0.$$

Now we write (6.44) as

$$A_{\lambda_n}(u_n) - \lambda_n G_p^{+\prime}(u_n) \to 0.$$

Since $\lambda_n \to \lambda$, we obtains $A_{\lambda_n}(u_n) - A_\lambda(u_n) \to 0$. Now the compactness of $G_p^{+'}$ yields the strong convergence of $A_\lambda(u_n)$ and hence $\langle A_\lambda(u_n), u_n - u\rangle \to 0$. Since $u_n \rightharpoonup u$, using Lemma 4.3 of [73] one obtain $u_n \to u$. □

Next we show that $\Gamma_n \neq \emptyset$, using the same argument as in Lemma 4.4.1

Lemma 6.3.9. *For each $n \in \mathbb{N}$, the set $\Gamma_n \neq \emptyset$.*

Proof. The idea is to construct odd continuous maps from $S^{n-1} \to M$, for each $n \in \mathbb{N}$. Let $\Omega^+ = \{x : g^+(x) > 0\}$. Since $|\Omega^+| > 0$, using the Lebesgue-Besicovitch differentiation theorem, one can choose n points $x_1, x_2, \ldots x_n$ in

Ω^+ such that

$$\lim_{r\to 0} \frac{1}{|B_r(x_i)|} \int_{B_r(x_i)} g(y)dy = g(x_i) > 0.$$

Thus there exists $R > 0$, such that $B_R(x_i) \cap B_R(x_j) = \emptyset$ and

$$\int_{B_r(x_i)} g(y)dy > 0, \text{ for } 0 < r < R.$$

Now one can choose r such that $0 < r < R$ and

$$\int_{B_R(x_i)\setminus B_r(x_i)} |g(y)|dy < \int_{B_r(x_i)} g(y)dy. \tag{6.46}$$

Let $u_i \in \mathcal{C}_c^\infty(B_R(x_i))$ such that $0 \leq u_i(x) \leq 1$ and $u_i \equiv 1$ on $B_r(x_i)$. Now using (6.46) we have the following

$$\begin{aligned}\int_{B_R(x_i)} g|u_i|^p &= \int_{B_r(x_i)} g + \int_{B_R(x_i)\setminus B_r(x_i)} g|u_i|^p \\ &\geq \int_{B_r(x_i)} g - \int_{B_R(x_i)\setminus B_r(x_i)} |g| > 0.\end{aligned}$$

Thus we get $v_i = \frac{u_i}{(\int_\Omega g|u_i|^p)^{\frac{1}{p}}} \in M$. Note that the support of v_is are disjoint. Now for $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n) \in \mathbb{R}^n$ with $\sum |\alpha_i|^p = 1$, we have $\sum \alpha_i v_i \in C_c^\infty(\Omega)$ and $\int_\Omega g|\sum \alpha_i v_i|^p = 1$. It is easy to see that the map $\phi(\alpha) = \sum \alpha_i u_i$ is an odd continuous map from S^{n-1} into $\mathcal{M}_p$. Thus $\phi(S^{n-1})$ is compact and symmetric about origin. Now from the definition of genus it follows that $\gamma(\phi(S^{n-1})) \geq \gamma(S^{n-1}) = n$. □

Now we are in a position to adapt the Ljusternik-Schnirelmann theorem available for $\mathcal{C}^1$ manifold in our situation and prove the existence of infinitely many eigenvalues for (6.1). In fact we prove the following theorem:

Theorem 6.3.10. *Let $p \in (1, N)$, $g \in L^1_{loc}(\Omega)$ and $g^+ \in \mathcal{F}_{\frac{N}{p}} \setminus \{0\}$. Then* (6.1) *admits a sequence of positive eigenvalues going to ∞.*

Proof. From all the above J_p and $\mathcal{M}_p$ satisfy all the requirements of Theorem 6.3.1, for each $j \in \mathbb{N}$ and so we have $\gamma(K_{c_j}) \geq 1$. Thus $K_{c_j} \neq \emptyset$ and hence there exists $u_j \in M$ such that $dJ_p(u_j) = 0$ and $J_p(u_j) = c_j$. Therefore c_j is an eigenvalue of (6.1) and u_j is an eigenfunction corresponding to c_j.

A proof for the unboundedness of the sequence $\{c_n\}$ is given in [45], see Theorem 2. For the sake of completeness we adapt the same idea in our situation. Recall that the space X is separable (see 3.5 of [2]) and hence X

admits a biorthogonal system $\{e_m, e_m^*\}$, (see Proposition 1.f.3 of [55]) such that

$$\overline{\{e_m, m :\in \mathbb{N}\}} = X, \quad e_m^* \in X', \quad \langle e_m^*, e_n\rangle = \delta_{n,m}$$

$$\langle e_m^*, x\rangle = 0, \ \forall m \Rightarrow x = 0.$$

Let

$$E_n = \text{ span } \{e_1, e_2, \dots, e_n\},$$

and let

$$E_n^{\perp} = \overline{\text{ span } \{e_{n+1}, e_{n+2}, \dots\}}.$$

Since $E_{n-1}^{\perp}$ is of codimension $n-1$, for any $A \in \Gamma_n$ we have $A \cap E_{n-1}^{\perp} \neq \emptyset$ (see Proposition 7.8 of [66]).

Let

$$\mu_n = \inf_{A\in\Gamma_n} \sup_{A\cap E_{n-1}^{\perp}} J_p(u), \quad n = 1, 2, \dots$$

Now we show that $\mu_n \to \infty$. If possible let $\{\mu_n\}$ be bounded, then there exists $u_n \in E_{n-1}^{\perp} \cap M$ such that $\mu_n \le J_p(u_n) < c$ for some constant $c > 0$. Since $u_n \in M$, the estimate (6.41) shows that u_n is indeed bounded in X. Thus $u_n \rightharpoonup u$ for some $u \in X$. Now by the choice of biorthogonal system, for each m, $\langle e_m^*, u_n\rangle \to 0$ as $n \to \infty$. Thus $u_n \rightharpoonup 0$, in X and hence $u = 0$, a contradiction to $\int_\Omega g|u|^p \ge 1$ (see the conclusion followed by the estimate (6.45)). Therefore $\mu_n \to \infty$ and hence $c_n \to \infty$ as $\mu_n \le c_n$. □

Remark 6.3.11. *If $g \in \mathcal{F}_{\frac{N}{p}}$ and $g^- \not\equiv 0$, then there exists a sequence μ_n of negative eigenvalues of* (6.1) *tending to $-\infty$.*

6.4 Miscellaneous Remarks

In this section we study some extensions of Theorem 6.2.11.

One can study the existence of ground states for Δ_p with more general subcritical nonlinearities on the right hand side of (6.1). Precisely, for given V, g locally integrable on a domain $\Omega \subset \mathbb{R}^N$ with $V \ge 0$ but g allowed to change sign, one can look for the positive solutions in $\mathcal{D}_0^{1,p}(\Omega)$ for the following problem

$$\begin{aligned} \Delta_p u + V|u|^{p-2}u &= \lambda g|u|^{q-2}u, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{6.47}$$

where $q \in [p, p^*)$ and $1 < p < N$.

Theorem 6.4.1. *If $g^+ \in \mathcal{F}_{\widetilde{p}} \setminus \{0\}$ with $\frac{1}{\widetilde{p}} + \frac{q}{p^*} = 1$, then* (6.47) *has a positive solution.*

One has to just verify that $G_p(u) = \int_\Omega g^+|u|^q$ is compact, then by arguing as in the proposition 6.2.11, it is immediate that $\int_\Omega \{|\nabla u|^p + V|u|^p\}$ has a positive minimizer on $\mathcal{M}_q = \{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^q = 1\}$. Then by the homogeneity of the Rayleigh quotient $R = \frac{\int_\Omega \{|\nabla u|^p + V|u|^p\}}{(\int_\Omega g|u|^q)^{\frac{p}{q}}}$ corresponding to (6.47) we get a minimizer of R_p on $\{u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^q > 0\}$. For the positivity of this minimizer one can use Lemma (6.2.5).

Remark 6.4.2. *Let g be as in the above remark. Then the following generalized Hardy-Sobolev inequality holds*

$$\left(\int_\Omega g|u|^q\right)^{\frac{q}{p}} \leq \frac{1}{\lambda_1}\int_\Omega \{|\nabla u|^p + V|u|^p\}, \quad \forall\, u \in \mathcal{D}_0^{1,p}(\Omega), \int_\Omega g|u|^q > 0 \tag{6.48}$$

where λ_1 is the minimum of $\int_\Omega \{|\nabla u|^p + V|u|^p\}$ on $\mathcal{M}_q$. Further the best constant is attained. This extends the results of Visciglia [77] for $p \neq 2$.

Chapter 7

Bifurcation for the p-Laplacian

In this chapter we study the existence of a bifurcation branch for the following type of equation:

$$-\Delta_p u \;=\; \lambda g\,|u|^{p-2}u + \lambda f\,r(u), \quad \text{ in } \mathcal{D}_0^{1,p}(\Omega). \tag{7.1}$$

where Ω is an open connected subset of $\mathbb{R}^N$ with $1<p<N$ and $g, f \in L^1_{loc}(\Omega)$. Further, we assume that $r \in \mathcal{C}(\mathbb{R})$ and $r(0) = 0$.

As before we look for the weak solutions of (7.1). Note that for each λ, $u = 0$ is a trivial solution for (7.1). We say that a real number λ_0 is a bifurcation point for the equation (7.1), if for any $\varepsilon>0$ there exist $u_\varepsilon \in \mathcal{D}_0^{1,p}(\Omega)\setminus\{0\}$, $\lambda_\varepsilon \in \mathbb{R}$ such that $|u_\lambda| < \varepsilon, |\lambda_\varepsilon - \lambda_0| < \varepsilon,$ and the pair $(\lambda_\varepsilon, u_\varepsilon)$ satisfies (7.1) in the weak sense, i.e,

$$\int_\Omega |\nabla u_\varepsilon|^{p-1}\nabla u_\varepsilon \cdot \nabla v = \lambda_\varepsilon \int_\Omega g\,|u_\varepsilon|^{p-2}u_\varepsilon\, v + f\,r(u_\varepsilon)\,v, \quad \forall\, v \in \mathcal{D}_0^{1,p}(\Omega). \tag{7.2}$$

There are several results available in the literature on the existence of bifurcating branches of (7.1) for both bounded and unbounded domain Ω, for example, [31, 34]. In all these earlier works, authors assumed that f and g are bounded and lie in $L^{\frac{N}{p}}(\Omega)$. Here we allow f, g to be in certain weak Lebesgue spaces and they are not assumed to be bounded.

Here we make the following assumptions on the functions r, g and f:

$$\text{(B1)} \quad \begin{cases} r \in \mathcal{C}(\mathbb{R}), \quad |r(s)| \le |s|^{\gamma-1}, \quad \gamma \in [1, p^*), \text{ where } p^* = \frac{Np}{N-p}, \\[2ex] \lim_{|s|\to 0} \frac{|r(s)|}{|s|^{p-1}} = 0, \text{ if } 1 \le \gamma \le p. \end{cases}$$

$$
\text{(B2)} \begin{cases} g \in \mathcal{F}_{\frac{N}{p}}, \quad g^+ \neq 0, \\ \\ f \in \begin{cases} \mathcal{F}_{\widetilde{p}} \text{ if } \gamma \geq p, \quad \text{where } \frac{1}{\widetilde{p}} + \frac{1}{p^*} = 1, \\ \mathcal{F}_{\frac{N}{\widetilde{p}}} \text{ if } 1 \leq \gamma < p. \end{cases} \end{cases}
$$

We use a topological degree argument as in [34] for proving the existence of a solution branch of (7.1) bifurcating from the trivial branch of zero solutions. Leray and Schauder extended the finite dimensional degree theory for certain maps from an infinite dimensional Banach space to itself. More precisely they define the degree for the compact perturbations of the identity. See [52, 35] for the definition and the important properties of the Leray-Schauder degree. We emphasize that the Leray-Schauder degree is defined for the maps from a Banach space to itself. Here, the functional frame work for the equations of the type (7.1) leads to an equation involving certain maps between $\mathcal{D}_0^{1,p}(\Omega)$ and its dual. In order to study the bifurcation property of (7.1) via topological degree theory, one needs to extend the definition of the degree for certain maps between a Banach space and its dual. In Appendix A, the degree is defined for certain classes of maps from a Banach space to its dual, see [71] for more details on this topic.

Under the assumptions (B1) and (B2), using Theorem A.0.23 we show that the first eigenvalue λ_1 of the following problem

$$
-\Delta_p u = \lambda g|u|^{p-2}u \quad \text{in } \mathcal{D}_0^{1,p}(\Omega). \tag{7.3}
$$

is a bifurcation point of equation (7.1).

In order to apply A.0.23 we need to introduce a suitable functional frame work for equation (7.1). Throughout this chapter we assume that Ω is an open connected subset of $\mathbb{R}^N$ with $1 < p < N$ and $g, f \in L^1_{loc}(\Omega)$. We denote $\mathcal{D}_0^{1,p}(\Omega)$ by X and its dual by X^*. Under the assumptions $(B1), (B2)$ recall the definition of the following functionals on X :

$$
\begin{aligned}
J_p(u) &:= \frac{1}{p}\int_\Omega |\nabla u|^p, \\
G_p(u) &:= \frac{1}{p}\int_\Omega g|u|^p.
\end{aligned}
$$

It is easy to see that J_p is differentiable and the derivative ${J_p}' : X \to X^*$ is given by

$$
\left\langle {J_p}'(u), v \right\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v, \quad \forall\, v \in X. \tag{7.4}
$$

Similarly using the assumption $g \in \mathcal{F}_{\frac{N}{p}}$, one can show that the map G_p is differentiable and the derivative ${G_p}^{'} : X \to X^*$ is given by

$$\left\langle {G_p}^{'}(u), v \right\rangle = \int_\Omega g\, |u|^{p-2}\, u\, v, \quad \forall\, v \in X. \tag{7.5}$$

Next we define a class of maps from X to X^*, for which we can define the degree.

Definition 7.0.3. *Let $A : X \to X^*$, we say that A is of class $\alpha(X)$, if whenever $u_n \rightharpoonup u_0$ (weakly) in X and*

$$\overline{\lim_{n\to\infty}}\, \langle A u_n, u_n - u_0 \rangle \leq 0 \tag{7.6}$$

then the sequence $\{u_n\}$ converges to u_0 (strongly).

Remark 7.0.4. *The class $\alpha(X)$ is invariant under perturbations by compact functions. i.e. if A is of class $\alpha(X)$ and if K is compact map from X to X^*, then $A + K$ is also of class $\alpha(X)$.*

The definition of degree for the maps in $\alpha(X)$ and its properties are given in Appendix A . Next we prove the the following proposition:

Proposition 7.0.5. *The map ${J_p}^{'}$ is of class $\alpha(X)$.*

Proof. First we show that ${J_p}^{'}$ is monotone. For $u, v \in X$, we have

$$\left\langle {J_p}^{'}(u), v \right\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \leq \|u\|^{p-1} \|v\|.$$

$$\begin{aligned} \left\langle {J_p}^{'}(u) - {J_p}^{'}(v), u - v \right\rangle &= \|u\|^p + \|v\|^p - \left\langle {J_p}^{'}(v), u \right\rangle - \left\langle {J_p}^{'}(u), v \right\rangle \\ &\geq \|u\|^p + \|v\|^p - \|u\|\, \|v\|^{p-1} - \|u\|^{p-1} \|v\| \\ &= (\|u\| - \|v\|) \left(\|u\|^{p-1} - \|v\|^{p-1} \right). \end{aligned} \tag{7.7}$$

Next we show that ${J_p}^{'}$ is of class $\alpha(X)$. Let $u_n \rightharpoonup u_0$ in $\mathcal{D}_0^{1,p}(\Omega)$ and $\overline{\lim} \left\langle {J_p}^{'}(u_n), u_n - u_0 \right\rangle \leq 0$. Note that

$$\begin{aligned} \overline{\lim} \left\langle {J_p}^{'}(u_n) - {J_p}^{'}(u_0), u_n - u_0 \right\rangle \leq \overline{\lim} \left\langle {J_p}^{'}(u_n), u_n - u_0 \right\rangle \\ - \underline{\lim} \left\langle {J_p}^{'}(u_0), u_n - u_0 \right\rangle. \end{aligned}$$

Also we have $\underline{\lim} \left\langle {J_p}'(u_0), u_n - u_0 \right\rangle = 0$, since $u_n \rightharpoonup u_0$. Therefore

$$\overline{\lim} \left\langle {J_p}'(u_n) - {J_p}'(u_0), u_n - u_0 \right\rangle \leq 0. \tag{7.8}$$

From the monotonicity of the map ${J_p}'$, we have

$$\underline{\lim} \left\langle {J_p}'(u_n) - {J_p}'(u_0), u_n - u_0 \right\rangle \geq 0. \tag{7.9}$$

Thus from (7.8) and (7.9) we get $\left\langle {J_p}'(u_n) - {J_p}'(u_0), u_n - u_0 \right\rangle \to 0$. Now using (7.7), we conclude that $u_n \to u_0$ (strongly).

Next we study the compactness of certain nonlinear maps.

Theorem 7.0.6. *Let $\gamma \in [p, p^*)$ and let $r \in C_0(\mathbb{R})$ such that $|r(s)| \leq c|s|^{\gamma-1}$, for some $c > 0$. Let $g \in \mathcal{F}_{\widetilde{p}}$, where $\widetilde{p}$ is the conjugate exponent of $\frac{p^*}{\gamma}$. Then the map $N : \mathcal{D}_0^{1,p}(\Omega) \to [L(p^*, p)]^*$ defined by*

$$\langle N(u), v \rangle = \int_\Omega g\, r(u)\, v$$

is compact.

Proof. First we show that N is well defined. Since $p \leq \gamma$, using the Lorentz-Sobolev embedding and the monotonicity of the Lorentz spaces in the second index, we get $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L(p^*, \gamma)$. Thus, for $u \in \mathcal{D}_0^{1,p}(\Omega)$, using property (iv) of Proposition (2.2.7), we get $|u|^{\gamma-1} \in L\left(\frac{p^*}{\gamma-1}, \frac{\gamma}{\gamma-1}\right)$. Now from the growth condition of r we obtain

$$r(u) \in L\left(\frac{p^*}{\gamma - 1}, \frac{\gamma}{\gamma - 1}\right).$$

Note that

$$\frac{1}{\widetilde{p}} + \frac{\gamma - 1}{p^*} + \frac{1}{p^*} = 1, \quad \frac{\gamma - 1}{\gamma} + \frac{1}{\gamma} = 1. \tag{7.10}$$

Thus for $v \in [L(p^*, p)]$ by using the Hölder inequality and the growth assumption on r, we get the following:

$$\begin{aligned} |\langle N(u), v \rangle| &\leq \int_\Omega |g|\,|r(u)|\,|v| \\ &\leq C_1 \|g\|_{(\widetilde{p},\infty)} \|u\|_{(p^*,p)}^{\gamma-1} \|v\|_{(p^*,\gamma)}, \end{aligned} \tag{7.11}$$

where the constant C_1 depends only on N, p and γ. The well definedness and the continuity of N is evident from (7.11). Next we show that N is compact. Let $v \in L(p^*, p)$,

$$
\begin{aligned}
|\left\langle N(u_n) - N(u)\ , v\right\rangle| &\leq \int_\Omega |g||r(u_n) - r(u)||v| \\
&= \int_\Omega \left\{ |g|^{\frac{\gamma-1}{\gamma}} |r(u_n) - r(u)| \right\} \left\{ |g|^{\frac{1}{\gamma}} |v| \right\}. \qquad (7.12)
\end{aligned}
$$

Note that

$$
\begin{aligned}
|g|^{\frac{\gamma-1}{\gamma}} &\in L\left(\frac{\gamma\,\widetilde{p}}{\gamma-1}\ , \infty \right), \\
|r(u_n) - r(u)| &\in L\left(\frac{p^*}{\gamma-1}\ , \frac{\gamma}{\gamma-1} \right), \\
\frac{\gamma-1}{\gamma\,\widetilde{p}} + \frac{\gamma-1}{p^*} &= \frac{\gamma-1}{\gamma}\left[\frac{1}{\widetilde{p}} + \frac{\gamma}{p^*} \right] = \frac{\gamma-1}{\gamma}.
\end{aligned}
$$

Thus using the Hölder inequality (Proposition (2.2.7)), we get

$$
|g|^{\frac{\gamma-1}{\gamma}} |r(u_n) - r(u)| \in L\left(\frac{\gamma}{\gamma-1}\ , \frac{\gamma}{\gamma-1} \right) = L^{\frac{\gamma}{\gamma-1}}(\Omega). \qquad (7.13)
$$

Similarly by noting that

$$
\begin{aligned}
|g|^{\frac{1}{\gamma}} \in L(\gamma\,\widetilde{p}\ , \infty), &\quad |v| \in L(p^*\ , \gamma), \\
\frac{1}{\gamma\,\widetilde{p}} + \frac{1}{p^*} &= \frac{1}{\gamma}\left[\frac{1}{\widetilde{p}} + \frac{\gamma}{p^*} \right] = \frac{1}{\gamma},
\end{aligned}
$$

we get $|g|^{\frac{1}{\gamma}}|v| \in L(\gamma\ , p) \subset L^{\gamma}(\Omega)$ and

$$
\begin{aligned}
\left\| |g|^{\frac{1}{\gamma}} |v| \right\|_{\gamma} &\leq C_2 \left\| |g|^{\frac{1}{\gamma}} \right\|_{(\gamma\,\widetilde{p}\ , \infty)} \|v\|_{(p^*\ , \gamma)} \\
&\leq C_3 \|g\|_{(\widetilde{p}\ , \infty)}^{\frac{1}{\gamma}} \|v\|_{(p^*\ , p)}\ , \qquad (7.14)
\end{aligned}
$$

where the constants C_2, C_3 depends only on N, p and γ. Now by the classical

Hölder inequalities available for the Lebesgue spaces, we obtain

$$\begin{aligned} |\langle N(u_n) - N(u)\ , v\rangle| &\le \left\| |g|^{\frac{\gamma-1}{\gamma}} |r(u_n) - r(u)| \right\|_{\frac{\gamma}{\gamma-1}} \left\| |g|^{\frac{1}{\gamma}} |v| \right\|_{\gamma} \\ &\le C_3 \, \|g\|^{\frac{1}{\gamma}}_{(\widetilde{p}\,,\infty)} \left\| |g|^{\frac{\gamma-1}{\gamma}} |r(u_n) - r(u)| \right\|_{\frac{\gamma}{\gamma-1}} \|v\|_{(p^*\ ,p)}\,. \end{aligned}$$

Thus

$$\|N(u_n) - N(u)\| \le C_4 \left\| |g|^{\frac{\gamma-1}{\gamma}} |r(u_n) - r(u)| \right\|_{\frac{\gamma}{\gamma-1}} ,$$

where the constant C_4 depends only on N, p, γ and $\|g\|_{(\widetilde{p}\,,\infty)}$. Now for the compactness of N, it is enough to prove the following:

$$\lim_{n\to\infty} \left\| |g|^{\frac{\gamma-1}{\gamma}} |r(u_n) - r(u)| \right\|_{\frac{\gamma}{\gamma-1}} = 0. \tag{7.15}$$

Using the assumption $g \in \mathcal{F}_{\widetilde{p}}$, we show that

$$\lim_{n\to\infty} \int_\Omega |g||r(u_n) - r(u)|^{\frac{\gamma}{\gamma-1}} = 0. \tag{7.16}$$

Let $\varepsilon > 0$ and $h \in C_c^\infty(\Omega)$ be arbitrary. Now we write

$$\int_\Omega |g||r(u_n)-r(u)|^{\frac{\gamma}{\gamma-1}} = \int_\Omega h|r(u_n)-r(u)|^{\frac{\gamma}{\gamma-1}} + \int_\Omega (|g| - h)\, |r(u_n)-r(u)|^{\frac{\gamma}{\gamma-1}} \tag{7.17}$$

First we estimate the second integral on the right hand side of the above equation, using the Hölder inequality and the Lorentz-Sobolev embedding.

$$\int_\Omega (|g| - h)\, |r(u_n)-r(u)|^{\frac{\gamma}{\gamma-1}} \le C \, \|\, |g| - h \,\|_{(\widetilde{p}\,,\infty)} \left\| |r(u_n) - r(u)|^{\frac{\gamma}{\gamma-1}} \right\|_{\left(\frac{p^*}{\gamma}\,,\,\frac{p}{\gamma}\right)} . \tag{7.18}$$

Further,

$$\begin{aligned} \left\| |r(u_n) - r(u)|^{\frac{\gamma}{\gamma-1}} \right\|_{\left(\frac{p^*}{\gamma}\,,\,\frac{p}{\gamma}\right)} &\le \|r(u_n) - r(u)\|^{\frac{\gamma}{\gamma-1}}_{\left(\frac{p^*}{\gamma-1}\,,\,\frac{p}{\gamma-1}\right)} \\ &\le \left\{ \|r(u_n)\|_{\left(\frac{p^*}{\gamma-1}\,,\,\frac{p}{\gamma-1}\right)} + \|r(u)\|_{\left(\frac{p^*}{\gamma-1}\,,\,\frac{p}{\gamma-1}\right)} \right\}^{\frac{\gamma}{\gamma-1}} \\ &\le C_6 \left\{ \|u_n\|^{\gamma-1}_{(p^*\,,p)} + \|u\|^{\gamma-1}_{(p^*\,,p)} \right\}. \end{aligned} \tag{7.19}$$

Note that $m = \sup_n \left\{ \|u_n\|^{\gamma-1}_{(p^*,p)} + \|u\|^{\gamma-1}_{(p^*,p)} \right\}$ is finite, since the sequence $\{u_n\}$ is bounded in $\mathcal{D}^{1,p}_0(\Omega)$ and the embedding of $\mathcal{D}^{1,p}_0(\Omega)$ into $L(p^*,p)$ is continuous. Thus from (7.19) into(7.18), we get

$$\int_\Omega (|g| - h)\, |r(u_n) - r(u)|^{\frac{\gamma}{\gamma-1}} \leq C \,\| \,|g| - h \,\|_{(\widetilde{p}\,,\infty)}\,, \quad \forall n \in \mathbb{N}. \qquad (7.20)$$

where the constant C depends only on N, p and γ. Now since $g \in \mathcal{F}_{\widetilde{p}}$, we have $|g| \in \mathcal{F}_{\frac{N}{p}}$. Thus we can choose $g_\varepsilon \in \mathcal{C}^\infty_c(\Omega)$ such that

$$\| \,|g| - g_\varepsilon \,\|_{(\widetilde{p}\,,\infty)} < \frac{\varepsilon}{C}.$$

Now by taking $h = g_\varepsilon$ in (7.20), we obtain

$$\int_\Omega (|g| - g_\varepsilon)\, |r(u_n) - r(u)|^{\frac{\gamma}{\gamma-1}} < \varepsilon, \quad \forall n \in \mathbb{N}. \qquad (7.21)$$

Next we estimate the first integral on the right hand side of (7.17). As $\gamma < p^*$, the embedding of $\mathcal{D}^{1,p}_0(\Omega)$ into $L^\gamma_{loc}(\Omega)$ is compact. Thus we have $u_n \to u$ in $L^\gamma_{loc}(\Omega)$. Since r is continuous, using the generalized dominated convergence theorem and the growth condition on r, one can easily deduce that

$$r(u_n) \to r(u) \text{ in } L^{\frac{\gamma}{\gamma-1}}_{loc}(\Omega).$$

Now since $g_\varepsilon \in \mathcal{C}^\infty_c(\Omega)$, we conclude:

$$\lim_{n\to\infty} \int_\Omega g_\varepsilon |r(u_n) - r(u)|^{\frac{\gamma}{\gamma-1}} = 0. \qquad (7.22)$$

Therefore, by taking $h = g_\varepsilon$ (7.17), (7.21) together with (7.22) yields

$$\int_\Omega |g||r(u_n) - r(u)|^{\frac{\gamma}{\gamma-1}} < 2\varepsilon,$$

for large $n \in \mathbb{N}$. This completes the proof. □

As a consequence of the above theorem we have the following corollary:

Corollary 7.0.7. *Under the assumption (B2), the map $G_p^{'} : X \to X^*$ is compact.*

Proof. Note that ${G_p}' = i^* \circ N$, where i^* is the adjoint of the inclusion map

$$i : X \to L(p^*, p),$$

given by the Lorentz Sobolev embedding and N as in the above theorem with $r(t) = |t|^{p-2}t$. Now the compactness of ${G_p}'$ is immediate from the continuity of i^* and the compactness of N. $\square$

Proposition 7.0.8. *Let $(B1), (B2)$ hold and let $H : X \to X^*$ be defined as*

$$\langle H(u), v\rangle = \int_\Omega f \; r(u)\, v.$$

Then the map H is compact. Moreover

$$\frac{\|H(u)\|_{X^*}}{\|u\|_X^{p-1}} \to 0 \quad \textit{as} \;\; \|u\|_X \to 0. \tag{7.23}$$

Proof. For $\gamma \in [p, p^*)$, the compactness of H follows from Theorem 7.0.6. For $\gamma \in [1, p)$, using $(B1)$, $|r(s)| \le C|s|^{p-1}$, for some $C > 0$. Also by (B2), $f \in \mathcal{F}_{\frac{N}{p}}$ and hence H is compact, by Theorem 7.0.6.
The proof of the second part of the theorem is divided into two cases:
Case 1: $\gamma \in (p, p^*)$
In this case, by the same calculations that yields (7.11), we obtain

$$|\,\langle H(u)\,, v\rangle\,| \le C\,\|f\|_{(\widetilde{p},\infty)}\,\|u\|^{\gamma-1}_{(p^*\,,\,p)}\,\|v\|_{(p^*,p)}\,.$$

Hence by the Lorentz-Sobolev embedding, we get

$$\|H(u)\|_{X^*} \le \widetilde{C}\,\|f\|_{(\widetilde{p},\infty)}\,\|u\|_X^{\gamma-1}\,.$$

Therefore,

$$\frac{\|H(u)\|_{X^*}}{\|u\|_X^{p-1}} \le \widetilde{C}\,\|f\|_{(\widetilde{p},\infty)}\,\|u\|_X^{\gamma-p}\,.$$

Since $\gamma > p$, the above inequality shows that,

$$\frac{\|H(u)\|_{X^*}}{\|u\|_X^{p-1}} \to 0 \quad \text{as} \;\; \|u\|_X \to 0.$$

Case 2: $\gamma \in [1, p\,]$
In this case we have,

$$\lim_{|s|\to 0} \frac{|r(s)|}{|s|^{p-1}} = 0.$$

Thus for a given $\varepsilon > 0$, there exists $s_0 > 0$, such that

$$|r(s)| \leq \frac{\varepsilon}{\|f\|_{\left(\frac{N}{p},\infty\right)}} |s|^{p-1}, \ \ \forall\, |s| \leq s_0.$$

Moreover, using $(B1)$, there exist constants C_1, C_2 depending on s_0, such that

$$|r(s)| \leq C_1 |s|^{p^*-1}, \quad |r(s)| \leq C_2 |s|^{p-1}, \quad \forall\, |s| \geq s_0.$$

Let $A = \{x : |u(x)| \leq s_0\}$ and $B = A^c \cap \Omega$. Now for $v \in \mathcal{D}_0^{1,p}(\Omega)$, we compute the following integral:

$$\langle H(u)\,,\, v\rangle = \int_\Omega f\, r(u) v = \int_A f\ r(u)\, v + \int_B f\ r(u)\, v. \tag{7.24}$$

The first integral in the right hand side of (7.24) can be estimated using the Hölder inequality as below:

$$\begin{aligned}
\left| \int_A f\ r(u)\, v \right| &\leq \int_A |f\,|\,|r(u)|\,|v| \\
&\leq \frac{\varepsilon}{\|f\|_{\left(\frac{N}{p},\infty\right)}} \int_A |f\,|\,|u|^{p-1}\,|v| \\
&\leq \frac{\varepsilon}{\|f\|_{\left(\frac{N}{p},\infty\right)}} \|f\|_{\left(\frac{N}{p},\infty\right)} \|u\|_{(p^*,p)}^{p-1} \|v\|_{(p^*,p)} \\
&\leq \varepsilon \|u\|_{(p^*,p)}^{p-1} \|v\|_{(p^*,p)}\,.
\end{aligned} \tag{7.25}$$

To estimate the second integral we note that $f \in \mathcal{F}_{\frac{N}{p}}$. Thus for $\varepsilon > 0$, we choose $f_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that $\|\,|f| - f_\varepsilon\|_{\left(\frac{N}{p}\,,\infty\right)} < \varepsilon$. Thus

$$\begin{aligned}
\left| \int_B f\ r(u)\, v \right| &\leq \int_B |f\,|\,|r(u)|\,|v| \\
&= \int_B f_\varepsilon\,|r(u)|\,|v| + \int_B (\,|f| - f_\varepsilon)\,|r(u)|\,|v|.
\end{aligned} \tag{7.26}$$

Now

$$\int_B f_\varepsilon\,|r(u)|\,|v| \leq C_1 \int_B f_\varepsilon\,|u|^{p^*-1}\,|v|.$$

Note that $|u|^{p^*-1} \in L\left(\frac{p^*}{p^*-1}\,,\,\frac{p}{p^*-1}\right) \subset L\left(\frac{p^*}{p^*-1}\,,\,\frac{p}{p-1}\right)$ and $q(p^*-1) > p$, where q is the conjugate exponent of p. Now by the Hölder inequality and the monotonicity of the Lorentz spaces in the second index yields:

$$\begin{aligned}
\int_B f_\varepsilon\,|r(u)|\,|v| &\leq C_3\,\|f_\varepsilon\|_\infty \left\|\,|u|^{p^*-1}\,\right\|_{\left(\frac{p^*}{p^*-1}\,,\,\frac{p}{p-1}\right)} \|v\|_{(p^*\,,\,p)}\\
&\leq C_3\,\|f_\varepsilon\|_\infty\;\|\,u\,\|^{p^*-1}_{(p^*\,,\,q(p^*-1))}\,\|v\|_{(p^*\,,\,p)}\\
&\leq C_4\,\|f_\varepsilon\|_\infty\;\|\,u\,\|^{p^*-1}_{(p^*\,,\,p))}\,\|v\|_{(p^*\,,\,p)}\,,
\end{aligned} \tag{7.27}$$

where all the constants depends only on N, p and γ. Next we estimate

$$\begin{aligned}
\int_B (\,|f| - f_\varepsilon)\,|r(u)|\,|v| &\leq C_2 \int_B |(\,|f| - f_\varepsilon)|\,|u|^{p-1}\,|v|\\
&\leq C_5\,\|\,|f| - f_\varepsilon\|_{\left(\frac{N}{p}\,,\,\infty\right)} \left\|\,|u|^{p-1}\,\right\|_{\left(\frac{p^*}{p-1}\,,\,\frac{p}{p-1}\right)} \|v\|_{(p^*\,,\,p)}\\
&\leq C_6\varepsilon\,\|\,u\,\|^{p-1}_{(p^*\,,\,p)}\,\|v\|_{(p^*\,,\,p)}\,.
\end{aligned} \tag{7.28}$$

Therefore by substituting (7.27), (7.28) in (7.26) we obtain

$$\left|\int_B f\;r(u)\,v\right| \leq C_7\left\{\varepsilon + \|\,u\,\|^{p^*-p}_{(p^*\,,\,p)}\right\}\|\,u\,\|^{p-1}_{(p^*\,,\,p)}\,\|v\|_{(p^*\,,\,p)}\,. \tag{7.29}$$

Now (7.25) and (7.28) together implies that

$$\frac{\|H(u)\|_{X^*}}{\|u\|_X^{p-1}} \leq C_8\left\{\varepsilon + \|\,u\,\|^{p^*-p}_{(p^*\,,\,p)}\right\}. \tag{7.30}$$

Since ε is arbitrary, the right hand side of the above inequality goes to zero as $\|u\|_X \to 0$. This completes the proof. □

Henceforth, for convenience, we denote J_p, G_p by J, G respectively.

Remark 7.0.9. *In conclusion, under the assumptions $(B1), (B2)$ the maps G' and H are compact. Since the map J' is of class $\alpha(X)$, by the Remark 7.0.4, for each $\lambda \in \mathbb{R}$, the maps*

$$A_\lambda = J' - \lambda(G' + H), \quad \widetilde{A}_\lambda = J' - \lambda\,G'$$

are in $\alpha(X)$. Further, both A_λ and $\widetilde{A}_\lambda$ are continuous. Thus the degree can be defined for A_λ and $\widetilde{A}_\lambda$ (see Appendix A).

Let λ_1 be the first eigenvalue of (7.3). Using Theorem A.0.23 we prove that λ_1 is a bifurcation point of (7.1). For this we need to calculate the index of A_λ at zero, for λ in some neighbourhood of λ_1 (see Appendix A). First we calculate the $\operatorname{Ind}(\widetilde{A}_\lambda, 0)$ for λ in some neighbourhood of λ_1 and then we use the homotopy invariance property of degree to calculate the $\operatorname{Ind}(A_\lambda, 0)$.

In order to define $\operatorname{Ind}(\widetilde{A}_\lambda, 0)$, we must prove that the trivial solution, $u \equiv 0$, is an isolated zero of $\widetilde{A}_\lambda$. In the next proposition we show that, for each $\lambda \in (0, \lambda_1)$, $u \equiv 0$ is an isolated zero of $\widetilde{A}_\lambda$.

Proposition 7.0.10. *Let $(B1), (B2)$ hold and let λ_1 be the first eigenvalue of (7.3). Then for each $\lambda \in (0, \lambda_1)$, 0 an isolated zero of $\widetilde{A}_\lambda$ and*

$$Ind(\widetilde{A}_\lambda, 0) = 1.$$

Proof. Let $\lambda \in (0, \lambda_1)$. Since, λ_1 is the minimum of the Rayleigh quotient,

$$\lambda < R(u) = \frac{J(u)}{G(u)}, \quad u \in \mathcal{D}^+(g),$$

where

$$\mathcal{D}^+(g) = \left\{ u \in \mathcal{D}_0^{1,p}(\Omega) : \int_\Omega g|u|^p > 0 \right\}.$$

Further,

$$\frac{\left\langle J^{'}(u), u \right\rangle}{\langle G^{'}(u), u \rangle} = \frac{J(u)}{G(u)}, \quad \forall\, u \in \mathcal{D}^+(g).$$

Therefore

$$\left\langle \widetilde{A}_\lambda(u), u \right\rangle = \left\langle J^{'}(u) - \lambda\, G^{'}(u), u \right\rangle > 0,\ \forall\, u \in \mathcal{D}^+(g). \tag{7.31}$$

Also for $u \notin \mathcal{D}^+(g) \cup \{0\}$, note that $G(u) \le 0$ and $J(u) > 0$. Thus, for each $u \in \mathcal{D}^+(g)$ with $u \neq 0$, we have

$$\left\langle \widetilde{A}_\lambda(u), u \right\rangle = \left\langle J^{'}(u) - \lambda\, G^{'}(u), u \right\rangle = p\,(J(u) - \lambda G(u)) > 0. \tag{7.32}$$

Now from (7.31) and (7.32), for $\lambda \in (0, \lambda_1)$, it is clear that,

$$\left\langle \widetilde{A}_\lambda(u), u \right\rangle > 0,\ \forall\, u \in \mathcal{D}_0^{1,p}(\Omega), u \neq 0.$$

Hence $u \equiv 0$ is the only zero of $\widetilde{A}_\lambda$ and hence is isolated. Further, using

Theorem A.0.18, we conclude that,

$$\operatorname{Ind}(\widetilde{A}_\lambda, 0) = 1, \ \ \forall\, \lambda \in (0, \lambda_1).$$

□

Now we need to show that $u = 0$ is an isolated zero of $\widetilde{A}_\lambda$ for each $\lambda \in (\lambda_1, \lambda_1 + \delta)$, for some $\delta > 0$. For this, first we prove the isolatedness of the first eigenvalue λ_1 of (7.3). When g is in suitable Lebesgue spaces, this has been proved by several authors, for example see [27, 34]. Here we adapt the idea of [34] for the weights in $\mathcal{F}_{\frac{N}{p}}$.

Proposition 7.0.11. *Let Ω be a connected domain in $\mathbb{R}^N$ with $1 < p < N$. Let $g \in \mathcal{F}_{\frac{N}{p}} \setminus \{0\}$. Then the first eigenvalue λ_1 of (7.3) is isolated.*

Proof. Assume that λ_1 is not isolated. Let $\{\mu_n\}$ be a sequence of eigenvalues of (7.3) such that $\mu_n \to \lambda_1$. Let v_n be an eigenfunction corresponding to μ_n normalized as $\int g|v_n|^p = 1$. Thus

$$\int |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla v = \mu_n \int g |v_n|^{p-2} v_n v, \quad \forall v \in X. \tag{7.33}$$

Let $\Omega_n^- = \{x \, : \, v_n(x) < 0\}$. From Remark 6.2.9, v_n changes sign and hence $|\Omega_n^-| \neq 0$. Now by taking $v = v_n^-$ in (7.33) and using Hölder inequality, we obtain

$$\begin{aligned}
\int_{\Omega_n^-} |\nabla v_n^-|^p = \mu_n \int_{\Omega_n^-} g |v_n^-|^p & \\
\leq C_1 \mu_n \left\| g \chi_{\Omega_n^-} \right\|_{(\frac{N}{p}, \infty)} \left\| v_n^- \right\|_{(p^*, p)}^p & \\
\leq C_2 \mu_n \left\| g \chi_{\Omega_n^-} \right\|_{(\frac{N}{p}, \infty)} \int_{\Omega_n^-} |\nabla v_n^-|^p. & \qquad (7.34)
\end{aligned}$$

The last inequality is obtained using Lorentz-Sobolev embedding. Now from (7.34), it is clear that, since $\mu_n \lambda_1 > 0$,

$$\left\| g \chi_{\Omega_n^-} \right\|_{(\frac{N}{p}, \infty)} > C_3, \tag{7.35}$$

where C_3 depends only on p and Ω. Now using Lemma 3.0.16, there exists a bounded set $\widetilde{\Omega} \subset \Omega$ such that

$$\left\| g\chi_{\Omega \setminus \widetilde{\Omega}} \right\|_{(\frac{N}{p},\infty)} < \frac{C_3}{2}. \tag{7.36}$$

Therefore

$$\begin{aligned} \left\| g\chi_{\Omega_n^-} \right\|_{(\frac{N}{p},\infty)} &\leq \left\| g\chi_{\Omega_n^- \cap \widetilde{\Omega}} \right\|_{(\frac{N}{p},\infty)} + \left\| g\chi_{\Omega_n^- \cap \widetilde{\Omega}^c} \right\|_{(\frac{N}{p},\infty)} \\ &\leq \left\| g\chi_{\Omega_n^- \cap \widetilde{\Omega}} \right\|_{(\frac{N}{p},\infty)} + \frac{C_3}{2}. \end{aligned}$$

Substituting (7.35) in the above inequality yields:

$$\left\| g\chi_{\Omega_n^- \cap \widetilde{\Omega}} \right\|_{(\frac{N}{p},\infty)} > \frac{C_3}{2}, \quad \forall\, n \in \mathbb{N}. \tag{7.37}$$

Now as the norm $\|.\|_{(\frac{N}{p},\infty)}$ is absolutely continuous in $\mathcal{F}_{\frac{N}{p}}$ with respect to the Lebesgue measure in $\mathbb{R}^N$, we must have

$$|\Omega_n^- \cap \widetilde{\Omega}| > C, \tag{7.38}$$

for some positive constant C independent of n.

Note that

$$J(v_n) = \mu_n \to \lambda_1 \text{ and } \int_\Omega g\, v_n^p = 1.$$

Now by the same argument as in Theorem 6.2.2, up to a subsequence, denoted by v_n itself, $v_n \rightharpoonup \pm\phi_1$ in $\mathcal{D}_0^{1,p}(\Omega)$, where ϕ_1 is the positive eigenfunction corresponding to λ_1 such that $\int g\phi_1^p = 1$. Further,

$$\|v_n\| = \mu_n{}^{\frac{1}{p}} \to \lambda_1{}^{\frac{1}{p}} = \|\phi_1\|\,.$$

Thus, as $\mathcal{D}_0^{1,p}(\Omega)$ is uniformly convex, we have

$$v_n \to \pm\phi_1 \text{ strongly in } \mathcal{D}_0^{1,p}(\Omega).$$

Without loss of generality, we assume that $v_n \to \phi_1$ strongly in $\mathcal{D}_0^{1,p}(\Omega)$. Thus we obtain a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$v_{n_k} \to \phi_1, \quad \text{a.e. in } \Omega,$$

since $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Now as $|\widetilde{\Omega}| < \infty$, by Egoroff's theorem (see page

73 of [69]), $v_n \to \phi_1$ uniformly on $\widetilde{\Omega}$ except on a set of arbitrarily small measure. Thus for the C in (7.38), there exists $\Omega_0 \subset \widetilde{\Omega}$ such that $v_n \to \phi_1$ uniformly on $\widetilde{\Omega} \setminus \Omega_0$ and $|\Omega_0| < C$. Hence for large n, v_n must be positive in $\widetilde{\Omega} \setminus \Omega_0$ and therefore

$$|\Omega_n^- \cap \widetilde{\Omega}| \leq |\Omega_0| < C.$$

This is a contradiction to (7.38). Hence λ_1 must be isolated. $\square$

Remark 7.0.12. *Since λ_1 is isolated there exists $\delta > 0$ such that* (7.3) *does not admit an eigenvalue in $(\lambda_1, \lambda_1 + \delta)$. Further, if $\widetilde{A}_\lambda$ has a nontrivial zero, then λ is an eigenvalue of* (7.3)*.Thus we conclude that $u \equiv 0$ is the unique zero of $\widetilde{A}_\lambda$, for $\lambda \in (\lambda_1, \lambda_1 + \delta)$.*

Next we compute the index $\widetilde{A}_\lambda$ at zero, for each $\lambda \in (\lambda_1, \lambda_1 + \delta)$.

Proposition 7.0.13. *Let* $(B1), (B2)$ *hold. Then, for $\lambda \in (\lambda_1, \lambda_1 + \delta)$,*

$$Ind(\widetilde{A}_\lambda, 0) = -1.$$

Proof. Here we adapt the idea of the proof of Theorem 4.1. of [34] for the weights in $\mathcal{F}_{\frac{N}{p}}$. For a fixed $k > 0$, we define a $\mathcal{C}^1$ function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = \begin{cases} 0, \ t \leq k, \\ \frac{2\delta}{\lambda_1}(t - 2k), \ t \geq 3k. \end{cases} \tag{7.39}$$

and ψ is strictly convex in $(k, 3k)$. For $\lambda \in (\lambda_1, \lambda_1 + \delta)$, let

$$\psi_\lambda(u) = J(u) - \lambda G(u) + \psi(J(u))$$

be a functional defined on X. Note that critical points of ψ_λ are given by the solutions of the following equation:

$$J'(u) - \lambda G'(u) + \psi'(J(u))J'(u) = 0.$$

Let u_0 be a nontrivial critical point of ψ_λ. Then u_0 must satisfy the following:

$$J'(u_0) - \frac{\lambda}{1 + \psi'(J(u_0))} G'(u_0) \quad = \quad 0. \tag{7.40}$$

From the above equation, it is clear that u_0 is an eigenfunction of (7.3) corresponding to the eigenvalue $\frac{\lambda}{1+\psi'(J(u_0))}$ of (7.3). Since λ_1 is the unique eigenvalue of (7.3) in $(0, \lambda_1 + \delta)$ and $0 < \frac{\lambda}{1+\psi'(J(u_0))} < \lambda_1 + \delta$, we must have

$$\frac{\lambda}{1 + \psi'(J(u_0))} = \lambda_1.$$

Therefore,

$$u_0 = r\phi_1, \quad \psi^{'}(J(u_0)) = \frac{\lambda}{\lambda_1} - 1,$$

for some $r \in \mathbb{R}$, where ϕ_1 is the first eigenvalue of (7.3) with $\int_\Omega g|\phi_1|^p = 1$ and $\phi_1 > 0$. Note that

$$0 < \frac{\lambda}{\lambda_1} - 1 < \frac{2\delta}{\lambda_1} \quad \text{and} \quad \psi^{'}(t) = \begin{cases} 0, & t \leq k, \\ \frac{2\delta}{\lambda_1}, & t \geq 3k. \end{cases}$$

Thus we must have $J(u_0) \in (k, 3k)$. Since ψ is strictly convex in $(k, 3k)$, there exist a unique t_0 such that $\psi^{'}(t_0) = \frac{\lambda}{\lambda_1} - 1$. Hence there exists a unique $r > 0$ such that $u_0 = \pm r\phi_1$, since J is even. Thus, for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, $\psi_\lambda^{'}$ has precisely three isolated zeros $-u_1, 0, u_1$, where u_1 is a positive eigenfunction corresponding to λ_1 and normalized as $J(u_1) \in (k, 3k)$.

If we show that $\pm u_1$ are local minima of ψ_λ, then $\text{Ind}(\psi_\lambda^{'}, \pm u_1)$ can be obtained using Theorem A.0.21. Since $\pm u_1$ are the only nontrivial critical points of ψ_λ, if we prove that ψ_λ attains its minimum, then it is necessary that $\pm u_1$ are the minimizers of ψ_λ. Thus it is enough to prove that ψ_λ is bounded below, weakly sequentially lower semi-continuous and coercive.

(i) ψ_λ **is weakly sequentially lower semi-continuous:** Let $u_n \rightharpoonup u$ in X. Since G is compact and J is lower semi-continuous, we get

$$\begin{aligned} \underline{\lim}_n \psi_\lambda(u_n) &= \underline{\lim}_n (J(u_n) - \lambda G(u_n)) + \underline{\lim}_n \psi(J(u_n) \\ &\geq J(u) - \lambda G(u) + \underline{\lim}_n \psi(J(u_n). \end{aligned}$$

Since ψ is increasing and continuous we have, $\underline{\lim}_n \psi(J(u_n) = \psi(J(u))$ and hence

$$\underline{\lim}_n \psi_\lambda(u_n) \geq \psi_\lambda(u).$$

(ii) ψ_λ **is coercive:**

$$\begin{aligned} \psi_\lambda(u) &= J(u) - \lambda G(u) + \psi(J(u)) \\ &= J(u) - \lambda_1 G(u) + (\lambda_1 - \lambda)G(u) + \psi(J(u)) \end{aligned} \tag{7.41}$$

Since $J(u) - \lambda_1 G(u) \geq 0$, we get

$$\begin{aligned} \psi_\lambda(u) &\geq (\lambda_1 - \lambda)G(u) + \psi(J(u)) \\ &\geq \frac{\lambda_1 - \lambda}{\lambda_1} J(u) + \psi(J(u)) \left(\because \frac{J(u)}{\lambda_1} \geq G(u),\ \lambda_1 - \lambda < 0 \right) \\ &\geq \frac{-\delta}{\lambda_1} J(u) + \frac{2\delta}{\lambda_1} \left(J(u) - 2k \right), \quad \text{for large } J(u) \\ &= \frac{\delta}{\lambda_1} J(u) - \frac{4\delta k}{\lambda_1} \to \infty \text{ as } \|u\|_X \to \infty. \end{aligned} \tag{7.42}$$

(iii) ψ_λ **is bounded below:**
From (7.41) we obtain,

$$\psi_\lambda(u) \geq \frac{-\delta}{\lambda_1} J(u) + \psi(J(u).$$

Now we use the definition of ψ_λ to obtain a lower bound for ψ_λ. If $J(u) \geq 3k$, then

$$\psi_\lambda(u) \geq \frac{\delta}{\lambda_1} J(u) - \frac{4\delta k}{\lambda_1} \geq \frac{-\delta k}{\lambda_1}.$$

If $J(u) \leq 3k$, then

$$\psi_\lambda(u) \geq \frac{-\delta}{\lambda_1} 3k + \psi(J(u)) \geq \frac{-3\delta k}{\lambda_1}.$$

Thus by a standard variational argument we conclude that ψ_λ attains its minimum. Now by Theorem A.0.21 we have

$$\text{Ind}(\psi_\lambda', u_1) = \text{Ind}(\psi_\lambda', -u_1) = 1. \tag{7.43}$$

Next we show that

$$\left\langle \psi_\lambda'(u), u \right\rangle > 0, \quad \text{for } \|u\| = r.$$

for sufficiently large r.

$$\begin{aligned} p \left\langle \psi_\lambda'(u), u \right\rangle &= J(u) - \lambda G(u) + \psi'(J(u))J(u) \\ &= \frac{-\delta}{\lambda_1} J(u) + \psi'(J(u))J(u) \\ &\geq \left(\frac{2\delta}{\lambda} - \frac{\delta}{\lambda} \right) J(u) > 0, \quad \text{if } J(u) \geq 3k. \end{aligned}$$

Now for $r > 3k$, using Theorem A.0.18, we get

$$\mathrm{Deg}[\psi_\lambda', B_r(0), 0] = 1.$$

As k is arbitrary, we can choose $r_0 > 0$, so that both u_1 and $-u_1$ are in $B_{r_0}(0)$ and $r_0 > 3k$. Thus

$$\mathrm{Deg}[\psi_\lambda', B_{r_0}(0), 0] = 1.$$

Further, $-u_0, 0, u_0$ are the only critical points of ψ_λ, we get

$$\mathrm{Ind}\ (\psi_\lambda', 0) = -1.$$

From the definition of ψ_λ, it is clear that

$$\mathrm{Deg}[\psi_\lambda', B_r(0), 0] = \mathrm{Deg}[\widetilde{A}_\lambda, B_r(0), 0] \tag{7.44}$$

for small $r > 0$. Thus we obtain $\mathrm{Ind}(\widetilde{A}_\lambda, 0) = -1$. □

Remark 7.0.14. *From Proposition 7.0.10 and Proposition 7.0.13 we have,*

$$\mathit{Ind}\ (\widetilde{A}_\lambda, 0) = \begin{cases} 1\,, & \lambda \in (0, \lambda_1), \\ -1\,, & \lambda \in (\lambda_1, \lambda_1 + \delta). \end{cases}$$

Next we prove that $\mathrm{Ind}\ (\widetilde{A}_\lambda, 0) = \mathrm{Ind}\ (A_\lambda, 0)$.

Proposition 7.0.15. *Let $(B1), (B2)$ hold. Let λ_1 be the first eigenvalue of* (7.3)*. Then*

$$\mathit{Ind}\ (A_\lambda, 0) = \mathit{Ind}\ (\widetilde{A}_\lambda, 0), \quad \forall\, \lambda \in (0, \lambda_1 + \delta) \setminus \lambda_1.$$

Proof. In view of homotopy invariance of degree, we show that $\widetilde{A}_\lambda$ and A_λ are homotopic for $\lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}$ in $B_r(0)$ and on $\partial B_r(0)$ for sufficiently small $r > 0$. For a fixed $\lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}$, we define

$$A_\lambda^t = \widetilde{A}_\lambda - \lambda t H, \quad t \in [0, 1],$$

where H is the map defined in Proposition 7.0.8. Note that $A_\lambda^0 = \widetilde{A}_\lambda$ and $A_\lambda^1 = A_\lambda$. Next we show that for each $t \in [0, 1]$, A_λ^t does not vanish on $\partial B_r(0)$ for sufficiently small $r > 0$. Suppose not, then there exist $t_n \in [0, 1]$, $u_n \neq 0$ such that $u_n \to 0$ and

$$A_\lambda^{t_n}(u_n) = J'(u_n) - \lambda G'(u_n) - \lambda t_n H(u_n) = 0.$$

Let $\widetilde{u}_n = \frac{u_n}{\|u_n\|}$. Since $J^{'}$ and $G^{'}$ are $p-1$ homogeneous, by dividing the above equation with $\|u\|^{p-1}$, we get

$$J^{'}(\widetilde{u}_n) - \lambda\, G^{'}(\widetilde{u}_n) - \lambda\, t_n \,\frac{H(u_n)}{\|u_n\|^{p-1}} = 0.$$

By Proposition 7.0.8 $\frac{H(u_n)}{\|u_n\|^{p-1}} \to 0$ and hence $J^{'}(\widetilde{u}_n) - \lambda G^{'}(\widetilde{u}_n) \to 0$. Since $\widetilde{u}_n$ is bounded, up to a subsequence $\widetilde{u}_n \rightharpoonup u_0$. Therefore, by the compactness of $G^{'}$ and the continuity of $J^{'-1}$ we get

$$\lim_{n\to\infty} J^{'}(\widetilde{u}_n) = \lambda G^{'}(\widetilde{u}_0).$$
$$\lim_{n\to\infty} u_n = (J^{'})^{-1}(\lambda G^{'}(\widetilde{u}_0)) = u_0.$$

Thus $J^{'}(\widetilde{u}_0) - \lambda G^{'}(\widetilde{u}_0) = 0$, a contradiction as λ is not an eigenvalue for (7.3). Thus by the homotopy invariance of the degree we get

$$\deg\,[A_\lambda, B_r(0), 0] = \begin{cases} 1, \ \lambda \in (0, \lambda_1), \\ -1, \ \lambda \in (\lambda_1, \lambda_1 + \delta). \end{cases}$$

this completes the proof. $\square$

Now we have the following result:

Theorem 7.0.16. *Let Ω be an open connected subset of $\mathbb{R}^N$ with $1 < p < N$. Let r, f and g satisfy $(B1), (B2)$. Then the first eigenvalue λ_1 of (7.3) is a bifurcation point of (7.1).*

The proof follows from the above proposition, using Theorem A.0.23.

Chapter 8

Weighted eigenvalue problems for the Laplacian in $\mathbb{R}^2$

In this chapter we study the existence of a positive principal eigenvalue for the following weighted eigenvalue problem:

$$-\Delta u = \lambda g\, u \quad \text{in } \Omega, \tag{8.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{8.2}$$

where $\Omega \subset \mathbb{R}^2$ is open, connected and $g \in L^1_{loc}(\Omega)$.

In contrast to Chapter 4, where $\Omega \subset \mathbb{R}^N$ (with $N \geq 3$) can be unbounded, here most of our results hold only for bounded Ω. This is mainly because, the natural space $\mathcal{D}_0^{1,2}(\Omega)$, associated to the weighted eigenvalue problem (8.1) may not even be a function space when Ω is an unbounded subset of $\mathbb{R}^2$ (See [38]). If we take $H_0^1(\Omega)$ as the solution space, then we lose the coercivity of the functional $\|\nabla u\|_2$ when Ω is an unbounded subset of $\mathbb{R}^2$. Our proof for the existence of an eigenvalue for the weighted eigenvalue problem that we considered in Chapter 4, relies on the coercivity of $\|\nabla u\|_2$. Brown et al, in [21], proved the nonexistence of a positive principal eigenvalue for (8.1), when $\Omega = \mathbb{R}^2$, even for g such that $\int_{\mathbb{R}^2} g > 0$. This completely rules out the possibility of obtaining a sufficient condition for the existence, similar to one result of Chapter 4, namely a condition on g^+ alone. In the first part of this chapter, we obtain existence of a positive eigenvalue for (8.1), when Ω is bounded, when g^+ is in certain function spaces.

For $N \geq 3$, earlier existence results available in the literature make use of the embedding of $\mathcal{D}_0^{1,2}(\Omega)$ into $L^{2^*}(\Omega)$, which is optimal in the class of

Lebesgue spaces. However, in Chapter 4, we make use of the finer embedding of $\mathcal{D}_0^{1,2}(\Omega)$ into the Lorentz space $L(2^*, 2)$, which is optimal in the class of Lorentz spaces. However, when $N = 2$, we don't have an optimal embedding in the class of Lebesgue space since $H_0^1(\Omega)$ is embedded in $L^r(\Omega)$ for all finite $r \geq 1$ and not in $L^\infty(\Omega)$. Nevertheless, there is an optimal embedding in the class of Orlicz spaces, due to Moser and Trudinger (see [2]). Also there is an optimal embedding of $H_0^1(\Omega)$ in the class of Lorentz-Zygmund spaces, see [36]. In this chapter, we study the existence of an eigenvalue for (8.1), by making use of these embeddings.

This chapter is organized as follows. In Section 1, we consider weights in a suitable Orlicz space and obtain a positive principal eigenvalue for (8.1). In Section 2, we prove the existence of positive principal eigenvalue for weights in a Lorentz-Zygmund space. In Section 3, we consider a general unbounded domain and obtain the existence of a positive principal eigenvalue for (8.1), for certain weight functions which are not integrable.

8.1 Minimizer for weights in Orlicz spaces

We use a direct variational method to obtain the existence of an eigenvalue of 8.1. Let $\Omega \subset \mathbb{R}^2$ be open, connected and bounded. First, recall the following definitions:

$$R(u) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega g u^2}.$$

$$J(u) = \int_\Omega |\nabla u|^2.$$

$$\mathcal{M} = \left\{ u \in H_0^1(\Omega) : \int_\Omega g u^2 = 1 \right\}.$$

Since Ω is bounded, by Poincaŕe's inequality, $\sqrt{J}$ is a norm in $H_0^1(\Omega)$. Thus the functional J is coercive in $H_0^1(\Omega)$. As in Chapter 4, we prove the existence of an eigenvalue for (8.1), by proving the existence of minimizer for J on $\mathcal{M}$. First we consider g in certain Orlicz spaces. The definitions of Orlicz functions and Orlicz spaces are given in Appendix B. Let us consider the Orlicz function $A(t) = \exp(t^2) - 1$ and the Orlicz space, $L_A(\Omega)$ generated by A. We have the following embedding of $H_0^1(\Omega)$ into $L_A(\Omega)$ (see Theorem 8.27 [2]), due to Trudinger [76] which is optimal in the class of Orlicz spaces:

Theorem 8.1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then the space $H_0^1(\Omega)$ is continuously embedded into $L_A(\Omega)$. i.e. there exists a constant $C_T > 0$*

such that

$$\|u\|_A \le C_T \|\nabla u\|_2 , \quad \forall\, u \in H_0^1(\Omega). \tag{8.3}$$

Motivated by the above embedding, we consider the following Orlicz function:

$$B(t) = \exp(t) - 1,\ t > 0.$$

Let L_B be the Orlicz space generated by B. Now we have the following lemma.

Lemma 8.1.2. *Let $u \in L_A(\Omega)$. Then $u^2 \in L_B(\Omega)$ and*

$$\left\|u^2\right\|_B = \|u\|_A^2 . \tag{8.4}$$

Proof. For $u \in L_A(\Omega)$, we compute the Orlicz norm of u^2 :

$$\begin{aligned}
\left\|u^2\right\|_B &= \inf\left\{k : \int_\Omega B\left(\frac{|u(x)|^2}{k}\right) dx \le 1\right\} \\
&= \inf\left\{k : \int_\Omega \exp\left(\frac{|u(x)|^2}{k}\right) - 1 \le 1\right\} \\
&= \inf\left\{k : \int_\Omega A\left(\frac{|u(x)|}{\sqrt{k}}\right) dx \le 1\right\} = \|u\|_A^2 .
\end{aligned}$$

□

Now the following corollary is immediate from the Moser-Trudinger embedding.

Corollary 8.1.3. *For $u \in H_0^1(\Omega)$, $u^2 \in L_B(\Omega)$, moreover*

$$\left\|u^2\right\|_B \le {C_T}^2 \|\nabla u\|_2^2 . \tag{8.5}$$

Remark 8.1.4. *By a straight forward calculation, we obtain $\widetilde{B}$, the conjugate Orlicz function of B as below:*

$$\widetilde{B}(s) = \max_{t \ge 0} \{st - B(t)\} = \begin{cases} 0, & 0 \le s \le 1, \\ s\log s - s + 1, & s > 1. \end{cases} \tag{8.6}$$

Moreover, one can verify that $\widetilde{B}$ is Δ regular (see Appendix B for the definition of conjugate Orlicz function).

Now by Remark 8.14 and Theorem 8.21 of [2] we have the following proposition:

Proposition 8.1.5. *Let $\widetilde{B}$ as in* (8.6) *and let $L_{\widetilde{B}}(\Omega)$ be the Orlicz space generated by $\widetilde{B}$. Then*

(i) $L_{\widetilde{B}}(\Omega) = \left\{ u \text{ measurable} : \int_\Omega \widetilde{B}(|u(x)|)dx < \infty \right\}$.

(ii) $C_c^\infty(\Omega)$ *is dense in* $L_{\widetilde{B}}(\Omega)$.

If A is an Orlicz function, then the requirement $\lim\limits_{t\to\infty} \dfrac{A(t)}{t} = \infty$ guarantee that $L_A(\Omega) \subset L^1(\Omega)$, when Ω is bounded. Now using the above proposition we prove a lemma similar to Lemma 4.3.3.

Lemma 8.1.6. *Let Ω be a bounded domain in $\mathbb{R}^2$ and let $g^+ \in L_{\widetilde{B}}(\Omega)$. Let*

$$G^+(u) := \int_\Omega g^+ u^2.$$

Then $G^+ : H_0^1(\Omega) \to \mathbb{R}$ is completely continuous.

Proof. Let $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. For $\phi \in \mathcal{C}_c^\infty(\Omega)$, we write the following:

$$G^+(u_n) - G^+(u) = \int_\Omega \phi\,(u_n^2 - u^2) + \int_\Omega (g^+ - \phi)\,(u_n^2 - u^2). \tag{8.7}$$

First we estimate the second integral using the Hölder inequality, (ii) of Proposition B.0.36, as below:

$$\left| \int_\Omega (g^+ - \phi)\,(u_n^2 - u^2) \right| \le 2\, \left\| g^+ - \phi \right\|_{\widetilde{B}} \left\| u_n^2 - u^2 \right\|_B . \tag{8.8}$$

Thus by Corollary 8.1.3, we obtain the following:

$$\left| \int_\Omega (g^+ - \phi)\,(u_n^2 - u^2) \right| \le 2\, C_T^2 \left\| g^+ - \phi \right\|_{\widetilde{B}} \left(\|\nabla u_n\|_2^2 + \|\nabla u\|_2^2 \right). \tag{8.9}$$

As $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $m = \sup\limits_n \left\{ \|\nabla u_n\|_2^2 + \|\nabla u\|_2^2 \right\} < \infty$. Since $\mathcal{C}_c^\infty(\Omega)$ is dense in $L_{\widetilde{B}}(\Omega)$, we choose $g_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that,

$$\left\| g_\varepsilon - g^+ \right\|_{\widetilde{B}} < \frac{\varepsilon}{2{C_T}^2 m}.$$

Now by taking $\phi = g_\varepsilon$ in (8.9) we get

$$\left| \int_\Omega (g^+ - g_\varepsilon)\,(u_n^2 - u^2) \right| < \varepsilon. \tag{8.10}$$

Since $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ compactly and $g_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$, one can easily deduce that the first integral converges to 0 as $n \to \infty$. Therefore there exists $n_0 \in \mathbb{N}$, so that

$$\left| \int_\Omega g_\varepsilon \left(u_n^2 - u^2\right) \right| < 2\varepsilon, \quad \forall\, n \geq n_0. \tag{8.11}$$

Now from (8.10) and (8.11) we conclude that $G(u_n) \to G(u)$. $\square$

Next we prove the existence of minimizer of J on $\mathcal{M}$.

Theorem 8.1.7. *Let $g \in L^1_{loc}(\Omega)$ and let $g^+ \in L_{\widetilde{B}}(\Omega) \setminus \{0\}$. Then J admits a minimizer on $\mathcal{M}$.*

Proof. Since $g \in L^1_{loc}(\Omega)$ and $g^+ \neq 0$, there exists $\varphi \in C_c^\infty(\Omega)$ such that $\int_\Omega g\varphi^2 > 0$ (see for example, Proposition 4.2 of [50]) and hence $\mathcal{M} \neq \emptyset$. Let

$$\lambda_1 = \inf_{u \in \mathcal{M}} J(u).$$

Let $\{u_n\}$ be a minimizing sequence of J on $\mathcal{M}$, i.e,

$$\lim_{n \to \infty} J(u_n) = \lambda_1 = \inf_{u \in \mathcal{M}} J(u).$$

Now by the coercivity of J, the sequence $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Now by the reflexivity of $H^1_0(\Omega)$ we obtain a subsequence of $\{u_n\}$ that converges weakly to some $u \in H^1_0(\Omega)$. Let us denote the subsequence by $\{u_n\}$ itself. Since the map G^+ is compact we get

$$\lim_{n \to \infty} \int_\Omega g^+ u_n^2 = \int_\Omega g^+ u^2. \tag{8.12}$$

Now as $u_n \in \mathcal{M}$ we write,

$$\int_\Omega g^- u_n^2 = \int_\Omega g^+ u_n^2 - 1.$$

Since the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, up to a subsequence $u_n \to u$ a.e in Ω. Hence, we can apply the Fatou's lemma to obtain

$$\int_\Omega g^- u^2 \leq \int_\Omega g^+ u^2 - 1.$$

This shows that $\int_\Omega g u^2 \geq 1$. Setting $\widetilde{u} := \frac{u}{\left(\int_\Omega g u^2\right)^{\frac{1}{2}}}$, the weak lower semi continuity of J yields the following,

$$\lambda_1 \leq J(\widetilde{u}) = \frac{J(u)}{\int_\Omega gu^2} \leq J(u) \leq \liminf_n J(u_n) = \lambda_1.$$

Thus equality must hold at each step and hence $\int_\Omega gu^2 = 1$, which shows that $u \in \mathcal{M}$ and $J(u) = \lambda_1$. □

Now similar to Lemma 4.3.7 we have the following lemma:

Lemma 8.1.8. *Let g be as in the above theorem and let u be a minimizer of J on $\mathcal{M}$. Then u is an eigenfunction of* (8.1) *corresponding to λ_1.*

Remark 8.1.9. *Let Ω be a bounded domain in $\mathbb{R}^2$. Let $L(t) = t\log^+(t)$ and for $p > 1$, let $A_p(t) = t^p$. Then*

- *L, A_p are Orlicz functions.*
- *L is equivalent to $\widetilde{B}$ in a bounded domain. Thus, when Ω is bounded, the Orlicz space $L_{\widetilde{B}}(\Omega)$ coincides with the $L\log(L)$ space, where*

$$L\,\log(L) = \left\{ u \text{ measurable } : \int_\Omega |u(x)|\,\log^+(|u(x)|)dx < \infty \right\}.$$

- *Since A_p dominates L near infinity and Ω is bounded, for each $p > 1$, $L^p(\Omega) \subset L_{\widetilde{B}}(\Omega)$. Indeed, $L_{\widetilde{B}}(\Omega)$ contain the functions that are not not in $L^p(\Omega), p > 1$. For example, for $\beta > 2$ the function $f(x) = \frac{1}{|x|^2|\log(|x|)|^\beta} \notin L^p(B_1(0))$ for $p > 1$, however f lies in $L_{\widetilde{B}}(\Omega)$.*
- *$L_{\widetilde{B}}(\Omega) \subsetneq L^1(\Omega)$. Let $w_1(x) = |x\,\log(|x|)|^{-2}$ and $\Omega = B(0;\frac{1}{2})$. Then $w_1 \in L_1(\Omega)$, but $w_1 \notin L_{\widetilde{B}}(\Omega)$. In particular $w_1 \notin L_p(\Omega)$, $\forall\, p > 1$.*

8.2 Minimizer for weights in Lorentz-Zygmund spaces

It is known from the work of Hansson [44], Brezis and Wainger [20], $H_0^1(\Omega)$ is embedded continuously into the Lorentz-Zygmund space $L_{\infty,2;-1}(\Omega)$, when Ω is a bounded subset of $\mathbb{R}^2$. Further $L_{\infty,2;-1}(\Omega)$ is a smaller space than the Orlicz space given by Trudinger (see Hansson [44] page 101). Thus it is natural to look for weights in the dual of appropriate Lorentz-Zygmund space associated to (8.1). For further reading on Lorentz-Zygmund spaces we refer to the books [16, 36].

Definition 8.2.1. *Let Ω be a domain in $\mathbb{R}^N$. Let $l_1(t) = 1 + |\log(t)|$. For $p,\ q \in (0,\infty]$, $\alpha \in \mathbb{R}$, we define*

$$\|f\|_{p,q;\alpha} := \|t^{\frac{1}{p}-\frac{1}{q}} l_1(t)^{\alpha} f^*(t)\|_{L_q(0,\infty)}.$$

Now we define the Lorentz-Zygmund space $L_{p,q;\alpha}(\Omega)$ as below:

$$L_{p,q;\alpha}(\Omega) = \left\{ u \text{ measurable} : \|u\|_{p,q;\alpha} < \infty \right\}.$$

For a proof of the following lemma, see Lemma 3.4.39 of [36].

Lemma 8.2.2. *For $p \in (1,\infty],\ q \in [1,\infty]\ \alpha \in \mathbb{R}$, let*

$$\|f\|^*_{p,q;\alpha} := \|t^{\frac{1}{p}-\frac{1}{q}} l_1(t)^{\alpha} f^{**}(t)\|_{L_q(0,\infty)},$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$. Then $\|.\|^*_{p,q;\alpha}$. is equivalent to $\|.\|_{p,q;\alpha}$ and $L_{p,q;\alpha}(\Omega)$ is a Banach space with the norm $\|.\|^*_{p,q;\alpha}$.

Next we list some of the Lorentz-Zygmund spaces that we often use in this chapter:

$$X_1 := L_{\infty,\,2\,;\,-1}(\Omega) = \left\{ u \text{ measurable} : \int_0^{|\Omega|} \left(\frac{u^*(t)}{1+|\log(t)|} \right)^2 \frac{dt}{t} < \infty \right\}.$$

$$X_2 := L_{\infty,\,1\,;\,-2}(\Omega) = \left\{ u \text{ measurable} : \int_0^{|\Omega|} \frac{u^*(t)}{(1+|\log(t)|)^2} \frac{dt}{t} < \infty \right\}.$$

$$X_3 := L_{1,\,\infty\,;\,2}(\Omega) = \left\{ u \text{ measurable} : \sup_{t>0} t\,(1+|\log(t)|)^2\, u^*(t) < \infty \right\}.$$

Remark 8.2.3. *Let $\mathcal{S}$ be the collection of all subsets of Ω that are Lebesgue measurable. Then for $E \in S$, $\chi_E \in X_i$, for $i = 1, 2, 3$ and*

$$\|\chi_E\|_{X_i} = \frac{1}{(1+|\log(|E|)|)^{\frac{i}{2}}}, \quad i = 1, 2.$$

Theorem 8.2.4 (Hansson's embedding,[44])**.** *Let Ω be a bounded domain in $\mathbb{R}^2$. Then $H^1_0(\Omega)$ embedded into the Lorentz-Zygmund space $L_{\infty,2;-1}(\Omega)$. i.e, there exist $C_H > 0$, such that*

$$\|u\|_{X_1} \le C_H \|\nabla u\|_2, \quad \forall\, u \in H^1_0(\Omega).$$

Remark 8.2.5. *Hansson's embedding is finer than that of Moser-Trudinger,* *in the sense that X_1 is strictly contained in $L_A(\Omega)$, the Orlicz space generated by $e^{t^2}-1$. This can be seen as below: for $f \in X_1$,*

$$\frac{f^*(t)^2}{1+|\log(t)|} = f^*(t)^2 \int_0^t \frac{1}{s(1+|\log(s)|)^2} ds \leq \int_0^t \frac{f^*(s)^2}{s(1+|\log(s)|)^2} ds = \|f\|_{X_1}.$$

Let $c = \|f\|_{X_1}$.

$$\int_\Omega e^{k\,f(x)^2} dx = \int_0^{|\Omega|} e^{k\,f^*(t)^2} dt \leq \int_0^{|\Omega|} e^{c\,k\,(1+|\log(t)|)} dt < \infty, \quad \textit{for } ck < 1.$$

Recall that, a measurable function $g \in L_A(\Omega)$, if $\int_\Omega e^{k\,g(x)^2} dx < \infty$, for some $k > 0$. Thus $X_1 \subseteq L_A(\Omega)$. Further, let $\Omega = B_1(0)$ and $f^*(t) = [\log(\frac{|\Omega|}{t})]^{\frac{1}{2}}$. Then

$$\int_0^{|\Omega|} e^{k\,f^*(t)^2} \quad = \quad \int_0^{|\Omega|} e^{k\,\log(\frac{|\Omega|}{t})} < \infty, \quad \textit{for } k < 1.$$

Hence $f \in L_A(\Omega)$. Note that $\frac{\log(\frac{|\Omega|}{t})}{t(1+|\log(t)|)} > c$, for some $c > 0$. Thus

$$\int_0^{|\Omega|} \frac{f^*(t)^2}{t(1+|\log(t)|)^2} dt = \int_0^{|\Omega|} \frac{\log(\frac{|\Omega|}{t})}{t(1+|\log(t)|)^2} dt \geq c \int_0^{|\Omega|} \frac{1}{t(1+|\log(t)|)} dt.$$

Since the last integral in the above inequality divergent, $f \notin X_1$. Therefore $X_1 \subsetneq L_A(\Omega)$.

The following proposition is immediate from the definition of norms in X_1 and X_2.

Proposition 8.2.6. *Let $u \in X_1$. Then $u^2 \in X_2$ and*

$$\left\|u^2\right\|_{X_2} = \|u\|_{X_1}.$$

Proof. For $u \in L_{\infty,2;-1}(\Omega)$,

$$\|u\|_{X_1} = \int_0^{|\Omega|} \left(\frac{u^*(t)}{1+|\log(t)|}\right)^2 \frac{dt}{t}.$$

Since $(u^2)^* = (u^*)^2$ we get,

$$\|u\|_{X_1} = \int_0^{|\Omega|} \frac{(u^2)^*(t)}{(1+|\log(t)|)^2} \frac{dt}{t} = \left\|u^2\right\|_{X_2}.$$

Thus $u^2 \in X_2$. $\square$

In the next proposition we obtain a Hölder inequality between X_2 and X_3:

Proposition 8.2.7. *Let $f \in X_2$ and $g \in X_3$. Then $f\,g \in L^1(\Omega)$ and*

$$\int_\Omega |f\,g| \leq \|f\|_{X_2} \|g\|_{X_3}\,.$$

Proof.

$$\begin{aligned}
\int_\Omega |fg| &\leq \int_0^{|\Omega|} f^*(t)g^*(t) \\
&= \int_0^{|\Omega|} g^*(t)t\,(1+|\log(t)|)^2 \,\frac{f^*(t)}{(1+|\log(t)|)^2}\frac{dt}{t} \\
&\leq \sup_{t>0}\left\{t\,\,(1+|\log(t)|)^2\,g^*(t)\right\}\|f\|_{X_2} \leq \|f\|_{X_2}\|g\|_{X_3}\,.
\end{aligned}$$

$\square$

Remark 8.2.8. *From the above proposition, it is clear that X_3 is contained in the dual space of X_2.*

In the next lemma we give some examples of functions in X_3.

Lemma 8.2.9. *Let Ω be a bounded domain in $\mathbb{R}^2$. For $p > 1$, $L^p(\Omega) \hookrightarrow X_3$ continuously,i.e, there exist $C_p > 0$, such that*

$$\|f\|_{X_3} \leq C_p\,\|f\|_p\,, \quad \forall\, f \in L^p(\Omega).$$

Proof. Let $f \in L^p(\Omega)$. Then by (c) of Proposition 2.1.6, $f^* \in L^p(0, |\Omega|)$. Let q be the conjugate exponent of p. Now using the monotonicity of f^* and Hölder inequality we have the following:

$$t\,f^*(t) \leq \int_0^t f^*(\tau)d\tau \leq \left(\int_0^t f^*(\tau)^p d\tau\right)^{\frac{1}{p}} t^{\frac{1}{q}}. \tag{8.13}$$

Therefore

$$t\,(1+|\log(t)|)^2 f^*(t) \leq \|f\|_p\,(1+|\log(t)|)^2\,t^{\frac{1}{q}}. \tag{8.14}$$

Note that $(1+|\log(t)|)^2\, t^{\frac{1}{q}}$ is bounded in $[0,|\Omega|]$. Thus we see that $f \in X_3$ and

$$\|f\|_{X_3} := \sup_{t>0} t\,(1+|\log(t)|)^2\, f^*(t) \le C_p\, \|f\|_p\,.$$

$\square$

Now motivated by the definition of the space $\mathcal{F}_{\frac{N}{2}}$, we consider the following space:

$$\mathcal{F}_1 = \overline{\mathcal{C}_c^\infty(\Omega)} \text{ in } X_3. \tag{8.15}$$

We give examples of certain classical spaces that are in $\mathcal{F}_1$.

Lemma 8.2.10. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then $L^p(\Omega) \subset \mathcal{F}_1$. Moreover $L^p(\Omega)$ is dense in $\mathcal{F}_1$.*

Proof. Since $\mathcal{C}_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ and $L^p(\Omega)$ is continuously embedded in X_3 (see Lemma 8.2.9), for $p>1,\ L^p(\Omega)$ is contained in $\mathcal{F}_1$ and $L^p(\Omega)$ is dense in $\mathcal{F}_1$. $\square$

Next we give a useful characterization for the space $\mathcal{F}_1$, similar to that for the space $\mathcal{F}_p,\ p>1$.

Proposition 8.2.11. *Let Ω be bounded subset of $\mathbb{R}^2$ and let $f \in X_3$. Then the following statements are equivalent:*

(i) $f \in \mathcal{F}_1$.

(ii) $\lim_{t\to 0} t(1+|\log(t)|)^2 f^*(t) = 0$.

Proof. Let $f \in \mathcal{F}_1$. Let $\varepsilon > 0$ be given. Using the definition of $\mathcal{F}_1$, we get $g_\varepsilon \in \mathcal{C}_c^\infty(\Omega)$ such that $\|f - g_\varepsilon\|_{X_3} < \varepsilon$. i.e,

$$\sup_{t>0} t(1+|\log(t)|)^2\,(f-g_\varepsilon)^*(t)\ < \varepsilon. \tag{8.16}$$

Since g_ε is bounded, it is easy to see that

$$\lim_{t\to 0} t(1+|\log(t)|)^2\, g_\varepsilon^*(t) = 0. \tag{8.17}$$

Since $(f+g)^*(2t) \le f^*(t) + g^*(t)$ (see property (d) of Proposition 2.1.5), we have

$$\begin{aligned} 2t(1+|\log(2t)|)^2\, f^*(2t) &\le 2t\,(1+|\log(2t)|)^2 [\,(f-g_\varepsilon)^*(t) + g_\varepsilon^*(t)] \\ &\le Ct\,(1+|\log(t)|)^2 [\,(f-g_\varepsilon)^*(t) + g_\varepsilon^*(t).] \end{aligned} \tag{8.18}$$

Now using (8.16) and (8.17) we obtain condition (ii)
Conversely assume that $f \in X_3$ and

$$\lim_{t \to 0} t(1 + |\log(t)|)^2 f^*(t) = 0. \tag{8.19}$$

Let $\varepsilon > 0$ be given. We show that, there exists $g_\varepsilon \in L^p(\Omega)$ such that

$$\|f - g_\varepsilon\|_{X_3} < \varepsilon.$$

Using (8.19), we obtain $t_0 > 0$ such that

$$t(1 + |\log(t)|)^2 f^*(t) < \varepsilon, \quad \forall\, t \leq t_0. \tag{8.20}$$

Let $s_0 = f^*(t_0)$ and

$$A = \{x \in \Omega : |f(x)| \leq s_0\}.$$

Clearly $g_\varepsilon = f\chi_A \in L^p(\Omega)$. Let $h = f - g_\varepsilon$. Note that

$$h(x) = \begin{cases} f(x) & : x \in A^c \\ 0 & : x \in A \end{cases}$$

Therefore

$$\alpha_h(s) = \begin{cases} \alpha_f(s) & : s \geq s_0 \\ \alpha_f(s_0) & : s < s_0 \end{cases}$$

Now set $t_1 = \alpha_f(s_0)$. Note that

$$t_1 = \alpha_f(f^*(t_0)) \leq t_0. \tag{8.21}$$

Since h is supported in A^c and $|A^c| = \alpha_f(s_0)$ we have $h^*(t) = 0$ for $t \geq t_1$. Moreover, $h^*(t) = f^*(t),\ 0 < t \leq t_1$. Thus

$$h^*(t) = \begin{cases} f^*(t) & : t \leq t_1 \\ 0 & : t > t_1 \end{cases} \tag{8.22}$$

Now from (8.20), (8.21) and (8.22), we see that

$$\|h\|_{X_3} < \varepsilon.$$

Therefore from (i) of Lemma 8.2.10, we conclude that $f \in \mathcal{F}_1$.

$\square$

Using the above characterization of the space $\mathcal{F}_1$, we show that there are weights not belonging to $L_{\widetilde{B}}(\Omega)$ which are however, contained in $\mathcal{F}_1$.

Example 8.2.12. *For δ small, the weight*

$$g(x) = \frac{\chi_{B(0;\delta)}(x)}{|x|^2(\log(x))^2(\log|\log(x)|)^\beta}$$

is not in $L \log L$ for $0 < \beta \leq 1$, but satisfies (ii) *of the above proposition.*

Theorem 8.2.13. *Let Ω be a bounded domain in $\mathbb{R}^2$ and let $g \in L^1_{loc}(\Omega)$. If $g^+ \in \mathcal{F}_1$, then the map*

$$G^+(u) := \int_\Omega g^+ u^2$$

is compact.

One can prove the above by a similar argument as in Lemma 8.1.6. Now we have the following result.

Theorem 8.2.14. *Let Ω be a bounded domain in $\mathbb{R}^2$ and let $g \in L^1_{loc}(\Omega)$. If $g^+ \in \mathcal{F}_1$, then J admits a minimum on $\mathcal{M}$.*

Now if we take $g^+ \in \mathcal{F}_1$, then as before we can prove the existence of positive principal eigenvalue for (8.1) using Hansson's embedding, our splitting method and a strong maximum principle of [19] as in chapter 4 .

8.3 An existence result for the unbounded domains in $\mathbb{R}^2$

In this section we consider a general domain in $\mathbb{R}^2$. In view of the nonexistence result of Brown et al, here we consider weights that are not integrable.

Theorem 8.3.1. *Let Ω be an open connected subset of $\mathbb{R}^2$. Let $g = g_1 - g_2$, such that $g \in L^1_{loc}(\Omega),\ g_1 \in \mathcal{F}_{\frac{N}{2}} \bigcap L^\infty(\Omega),\ g_2 \geq b > 0$, where N is an integer strictly greater than 2. Then*

$$\lambda_1 = \inf\left\{J(u) : u \in H^1_0(\Omega),\ \int_\Omega gu^2 > 0\right\}$$

is the unique positive principal eigenvalue (8.1). *Furthermore λ_1 is simple.*

First we prove the following lemma.

Lemma 8.3.2. *Let $f \in L(\frac{N}{2}, \infty)$ and let $\lambda(a)$ be the first eigenvalue of the following problem*

$$\begin{aligned} -\Delta u &= \lambda u \quad in\ A_N = (-a,a)^{(N-2)}, \\ u &= 0 \quad on\ \partial A_N. \end{aligned} \tag{8.23}$$

Then for each $u \in H_0^1(\Omega)$, we have the following inequality:

$$\int_\Omega fu^2 \leq C(2a)^{\frac{2(N-2)}{N}} \|f\|_{(\frac{N}{2},\infty)} \left\{ \int_\Omega |\nabla u|^2 + \lambda(a) \int_\Omega u^2 \right\}. \tag{8.24}$$

Proof. Let ψ be the eigenfunction corresponding to $\lambda(a)$ such that $\int_{A_N} \psi^2 = 1$. By regularity $\psi \in C^\infty(A_N)$ and $\psi = 0$ on the boundary of A_N. Hence $\psi \in C_c(\mathbb{R}^m)$, where $m = N-2$. Let $\phi \in C_c^\infty(\Omega)$. Define the following functions on $\Omega \times \mathbb{R}^m$ as below,

$$h(x,y) = f(x)\chi_{A_N}(y), \quad \tau(x,y) = \phi(x)\psi(y).$$

Note that τ is a bounded function with the support in $\mathrm{supp}(\phi) \times \overline{A_N}$. Thus $\tau^2 \in L(\frac{N}{N-2}, 1)$. Using the Hölder inequality and the Sobolev embedding we have the following,

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^m} |h(x,y)||\tau(x,y)|^2 &\leq C\|h\|_{(\frac{N}{2},\infty)} \|\tau^2\|_{(\frac{N}{N-2},1)} \\ &\leq C_0.C_s(2a)^{\frac{2m}{N}} \|f\|_{(\frac{N}{2},\infty)} \int_{\Omega\times\mathbb{R}^m} |\nabla\tau|^2. \end{aligned} \tag{8.25}$$

Note that $\int_{\Omega\times\mathbb{R}^m} |\nabla\tau|^2 = \int_\Omega |\nabla\phi|^2 + \lambda(a)\int_\Omega \phi^2$. Thus (8.24) holds for each $\phi \in C_c^\infty(\Omega)$. Now by the density of $\mathcal{C}_c^\infty(\Omega)$ in $H_0^1(\Omega)$, we obtain inequality (8.24) for any $u \in H_0^1(\Omega)$. $\square$

In the next lemma we give an upper bound for $\lambda(a)$ in terms of a, see [5], Lemma 4 for a proof.

Lemma 8.3.3. *Let $\lambda(a)$ be as in the Lemma 8.3.2. Then $\lambda(a) \leq \frac{6}{a^2}$.*

Lemma 8.3.4. *There exist $a > 0$ such that for any $u \in \mathcal{M}$, we have the following estimate:*

$$\int_\Omega u^2 \leq Ck(a, g_1) \int_\Omega |\nabla u|^2, \tag{8.26}$$

where

$$k(a,g) \quad = \quad (2a)^{\frac{2(N-2)}{N}} \|g\|_{(\frac{N}{2},\infty)}$$

and C is a constant that depends only on b, C_s, N.

Proof. Since g_2 is bounded below by b, from Lemma 8.3.2 we get,

$$\begin{aligned} b^2 \int_\Omega u^2 &\leq \int_\Omega g_2 u^2 \leq \int_\Omega g_1 u^2 \\ &\leq C k(a, g_1) \left\{ \int_\Omega |\nabla u|^2 + \lambda(a) \int_\Omega u^2 \right\}. \end{aligned}$$

Now by Lemma 8.3.3, $\lambda(a)(2a)^{\frac{2(N-2)}{N}} \to 0$ as $a \to \infty$. Thus we can choose a large enough so that,

$$C\lambda(a)(2a)^{\frac{2(N-2)}{N}} \|g_1\|_{(\frac{N}{2},\infty)} < \frac{b^2}{2}.$$

Remark 8.3.5. *Note that for any $u \in \mathcal{M}$ and for a as in the previous lemma, using estimate* (8.26) *we have the following:*

$$\begin{aligned} \int_\Omega g u^2 \leq \int_\Omega g_1 u^2 &\leq C k(a, g_1) \left\{ \int_\Omega |\nabla u|^2 + \lambda(a) \int_\Omega u^2 \right\} \\ &\leq C k(a, g_1) \left(1 + \frac{2\lambda(a)}{b^2} \, C \, k(a, g_1) \right) \int_\Omega |\nabla u|^2. \end{aligned}$$

the above inequality shows that J is bounded below on $\mathcal{M}$.

Theorem 8.3.6. *Let g be as in Theorem 8.3.1. Then J admits as minimizer on $\mathcal{M}$.*

Proof. Let $\{u_n\}$ be a sequence that minimizes J on $\mathcal{M}$. Since $\{J(u_n)\}$ is bounded, the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$ (from (8.26)). Hence up to a subsequence $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$. Now a similar argument as in Theorem 4.3.5 shows that $u_0 \in \mathcal{M}$ and

$$J(u_0) = \inf_{u \in \mathcal{M}} J(u).$$

□

Remark 8.3.7. *As in Chapter 4, one can show that the minimizer is an eigenfunction corresponding to λ_1. Then the strong maximum principle due to Brezis-Ponce ensures that eigenfunctions corresponding to λ_1 are of constant signs. Furthermore, λ_1 is simple and it is the unique positive principal eigenvalue of* (8.1).

Appendix A

Degree Theory

In this section we define the degree for certain nonlinear map defined on a subset $D \subset X$, with values in X^*. First we give definitions of certain classes of functions useful for defining the degree for mapping between X and X^*. For more detailed readings in this topic we refer to the book of Skrypnik [71].

Definition A.0.8. *Let X be a Banach space and let A be a map defined on a subset $D \subset X$, with values in X^*. Then*

(a) A is called monotone mapping, if the following inequality

$$\langle Au - Av, u - v \rangle \geq 0;$$

holds for all $u, v \in D$

(b) The map A is said to be demicontinuous on D, if for any sequence $u_n \in D$ converging strongly to $u_0 \in D$ then

$$\lim_{n \to \infty} \langle Au_n, v \rangle = \langle Au_0, v \rangle, \ \forall\, v \in X.$$

(c) A is said to be bounded if it carries bounded subsets of D into bounded subsets of X^*.

Definition A.0.9. *Let X be a Banach space, A be a map defined on a subset $\overline{D} \subset X$, with values in X^* and let $F \subset \overline{D}$. Then*

(a) A is said to be satisfy the condition $\alpha_0(F)$, if $u_n \in F$ such that

$$u_n \rightharpoonup u, \quad Au_n \rightharpoonup 0$$

and

$$\overline{\lim_{n\to\infty}} \langle Au_n, u_n - u_0\rangle \le 0 \tag{A.1}$$

then the sequence u_n *converges strongly to* u_0.

(b) *A satisfies the condition* $\alpha(F)$, *if* $u_n \in F$ *such that* $u_n \rightharpoonup u$ *and*

$$\overline{\lim_{n\to\infty}} \langle Au_n, u_n - u_0\rangle \le 0 \tag{A.2}$$

then the sequence u_n *converges strongly to* u_0.

Note that (b) is a stronger condition than (a).

Definition A.0.10. *Let* $F \subset \overline{D}$. *Then*

$$A_0(D,F) = \{A : \overline{D} \to X^* : A \text{ bounded, demicontinuous, satisfies } \alpha_0(F)\},$$
$$A(D,F) = \{A : \overline{D} \to X^* : A \text{ bounded, demicontinuous, satisfies } \alpha(F)\}.$$

DEFINITION OF THE DEGREE

We define $\mathrm{Deg}(A, \overline{D}, 0)$-the degree of a mapping A on the set $\overline{D}$ with respect to the origin of the space X^*- under the following conditions:
(a) $A \in A_0(D, \partial D)$;

(b) $Au \neq 0$ for any $u \in \partial D$.

First we define the degree when X is real separable reflexive Banach space. Let $\{v_i : i \in \mathbb{N}\}$ be a Schauder basis for X and let F_n be the linear span of $v_1, \dots v_n$.

Now for each $n \in \mathbb{N}$, we define the finite-dimensional approximation A_n of the mapping A in the following way:

$$A_n u = \sum_{i=1}^{n} \langle Au, v_i\rangle\, v_i, \text{ for } u \in \overline{D_n},\ D_n = D \cap F_n. \tag{A.3}$$

Note that A_n is mapping from $\overline{D_n} \subset F_n$ into F_n. Next theorem shows that under the assumptions (a) and (b) one can define the Brouwer degree of A_n.

Theorem A.0.11. *Let* A *be a mapping satisfying conditions* (a) *and* (b). *Then there exists* n_0 *such that for* $n \ge n_0$ *the following assertions hold:*

1. *the equation* $A_n u = 0$ *has no solutions belonging to* ∂D_n,

2. *the degree* $deg(A_n, \overline{D_n}, 0)$ *of the mapping* A_n *on the set* D_n *with respect* $0 \in F_n$ *is defined and independent of* n.

For a proof see Theorem 2.1 of [71]. □

By the above theorem, $\lim_{n\to\infty} \deg(A_n, \overline{D_n}, 0)$ exists and we denote it by $\deg(A, \overline{D}, 0, \{v_i\})$. Next we show that $\deg(A, \overline{D}, 0, \{v_i\})$. is independent of the choice of the Schauder basis. For a proof of next theorem, see Theorem 2.2 [71].

Theorem A.0.12. *Suppose that the conditions* (a) *and* (b) *are satisfied. Then the limit*

$$deg(A, \overline{D}, 0, \{v_i\}) = \lim_{n\to\infty} deg(A_n, \overline{D_n}, 0)$$

does not depend on the choice of the Schauder basis $\{v_i : i \in \mathbb{N}\}$.

Using the above two theorems one can justify the following definition of degree when X is a reflexive separable Banach space.

Definition A.0.13. *Let* A *be a map satisfying conditions* (a) *and* (b). *Then the degree on the set* $\overline{D}$ *with respect to the point* $0 \in X^*$ *is defined as*

$$deg\ (A_n, \overline{D_n}, 0),$$

where A_n, D_n *are as defined in* (A.3) *and this limit is denoted by Deg* $(A, \overline{D}, 0)$.

Next we define degree when X is nonseparable reflexive Banach space. We denote the set of all finite-dimensional subspaces of X by $F(X)$. Let $F \in F(X)$ and let $v_1, v_2, \ldots, v_n$ be a basis for F. We define the finite-dimensional mapping

$$A_F(u) = \sum_{i=1}^{n} \langle Au, v_i \rangle\, v_i, \quad D_F = (D \cap F). \tag{A.4}$$

Next we state a theorem without a proof, see Theorem 3.1 in [71] for a proof.

Theorem A.0.14. *Let* $A : \overline{D} \to X^*$ *be a demicountinuous operator satisfying condition* $\alpha(\partial D)$, *where* ∂D *is the boundary of a bounded open set* $D \subset X$ *and* $Au \neq 0$ *for* $u \in \partial D$. *Then there exists a subspaces* $F_0 \in F(X)$ *such that any subspace* $F \in X$ *containing* F_0 *satisfies the properties:*

1. *the equation* $A_F(u) = 0$ *has no solution on* ∂D_F,

2. *deg*$(A_F, \overline{D_F}, 0) =$*deg*$(A_{F_0}, \overline{D_{F_0}}, 0)$, *where deg is the degree of the finite-dimensional mapping.*

The above theorem justifies the following definition of degree.

Definition A.0.15. *Under the conditions of Theorem 3.1, the number*

$$Deg(A, \overline{D}, 0) = \ deg(A_{F_0}, \overline{D_{F_0}}, 0)$$

is called the degree of the mapping A on the set $\overline{D}$ with respect to the point $0 \in X^*$. Here A_F, D_F are defined according to (A.4) *and F_0 is the finite dimensional subspace determined by Theorem 3.1.*

Properties of the degree

The degree of a mapping, introduced above, possesses all the natural properties of Brouwer degree of finite dimensional mapping and Leray-Schauder degree for the compact perturbation of the identity in the infinite dimensional spaces. In this sections we discuss some important properties that are useful for proving the existence of bifurcation branches for certain nonlinear maps.

Theorem A.0.16. *Let $A : \overline{D} \to X^*$ be a mapping of class $\alpha_0(D)$ and suppose that*

$$Au \neq 0 \text{ for } u \in \overline{D}.$$

Then $Deg(A, \overline{D}, 0) = 0$.

For a proof of theorem see Theorem 4.3 of [71]. From this theorem one can obtain the following corollary which states sufficient conditions for the solvability of the equation $Au = 0$.

Corollary A.0.17. *Let $A : \overline{D} \to X^*$ be a mapping of class $\alpha_0(D)$. If $Au \neq 0$ for $u \in \partial D$ and $Deg(A, , \overline{D}, 0) \neq 0$, then equation $Au = 0$ has at least one solution in $\overline{D}$.*

The above corollary answers the solvability of nonlinear operator equations and nonlinear boundary value problems for which one can show that the degree is nonzero. Next theorem gives a condition under which the degree is nonzero.

Theorem A.0.18. *Let $A : \overline{D} \to X^*$ be a mapping of class $\alpha_0(D, \partial D)$. Suppose that $0 \in \overline{D} \setminus \partial D$ and*

$$\langle Au, u \rangle \geq 0, \quad Au \neq 0 \tag{A.5}$$

for $u \in \partial D$. *Then* $Deg(A, \overline{D}, 0) = 1$.

Index of a map

Let D be a bounded open set in a separable reflexive Banach space $X, A : \overline{D} \to X^*$ a mapping of class $A_0(D)$.

Definition A.0.19. *(i) A point* $u_0 \in D$ *is a zero of the mapping* A *if* $Au_0 = 0$ *and it is called isolated zero, if there exist* $r_0 > 0$ *such that the ball* $B_{r_0}(u_0)$ *do not contain any other zeros of* A.

(ii) The map A *is called a potential operator if* A *is the gradient of a functional.*

If u_o is an isolated zero, then one can establish that the equality

$$\text{Deg}\left(A, \overline{B_{r_0}(u_0)}, 0\right) = \text{Deg}\left(A, \overline{B_r(u_0)}, 0\right)$$

holds for $0 < r < r_0$. Thus we have the following definition:

Definition A.0.20. *The number*

$$\lim_{r \to 0} Deg(A, \overline{B_r(u_0)}, 0)$$

is called the index of the mapping A *at the isolated zero* u_0 *and it is denoted by* $Ind(A, u_0)$.

Theorem A.0.21. *Suppose that a functional* $F : X \to \mathbb{R}$ *has a local minimum at* u_0 *and it is an isolated critical point of* F. *If the derivative* F' *is of class* $\alpha(X)$ *then* $Ind(F', u_0) = 1$.

See Theorem 6.1 of [71] The index of the mapping is useful concept to study the existence of solution branches of certain nonlinear operator equations. Let U be a neighbourhood of the origin in a separable reflexive Banach space $X, A, T : X \to \overline{U} \to X^*$ nonlinear mappings satisfying the conditions:

(a) A is a mapping of class $A(U)$ and $A(0) = 0$,

(b) T is weakly compact and $T(0) = 0$.

Bifurcation theorem

We study the existence of bifurcation point for the following nonlinear operator equation

$$Au + \lambda Tu = 0. \tag{A.6}$$

Definition A.0.22. *A real number λ_0 is called a bifurcation point for the equation* (A.6) *if for any $\varepsilon > 0$ there exist $u_\varepsilon \in U, \lambda_\varepsilon \in \mathbb{R}$ such that $|\lambda_\varepsilon - \lambda_0| < \varepsilon,\ 0 < \|u_\varepsilon\| < \varepsilon$ and*

$$Au_\varepsilon + \lambda_\varepsilon Tu_\varepsilon = 0.$$

Without loss of generality one may assume that there exists a $\delta_0 > 0$ such that zero is an isolated critical point of the mapping $A+\lambda T$ for $|\lambda-\lambda_0| < \delta_0$, since otherwise λ_0 itself would be a bifurcation point. Then the index at zero, $\mathrm{Ind}(A+\lambda T,\ 0)$ is well defined for the mapping $A+\lambda T$ for $|\lambda-\lambda_0| < \delta_0$. Let

$$\begin{aligned} \bar{i}^{\pm} &= \limsup_{\lambda\to\lambda_0\pm} \mathrm{Ind}(A+\lambda T, 0). \\ \underline{i}^{\pm} &= \liminf_{\lambda\to\lambda_0\pm} \mathrm{Ind}(A+\lambda T, 0). \end{aligned}$$

Next theorem states sufficient conditions under which λ_0 is a bifurcation point for the equation (A.6):

Theorem A.0.23. *Let mappings A, T satisfy the conditions a),b) and assume that at least two of the numbers*

$$\bar{i}^{-}, \quad \underline{i}^{-}, \quad \bar{i}^{+}, \quad \underline{i}^{+}, \qquad Ind(A+\lambda T, 0)$$

are distinct. Then λ_0 is a bifurcation point of the Eq: (A.6)*.*

Appendix B

Orlicz spaces

Here we recall the definition and some of the properties of an N-function, Orlicz function, Orlicz spaces etc. For more detailed discussions on these topics, we refer to [2].

Definition B.0.24. (N-function): *An N- function is a real valued function A defined on* $[0, \infty)$ *satisfying the following conditions:*

(i) A is continuous on $[0, \infty)$*;*

(ii) A is strictly increasing;

(iii) A is convex;

(iv) $\lim_{t \to 0} \frac{A(t)}{t} = 0$ *and* $\lim_{t \to \infty} \frac{A(t)}{t} = \infty$*;*

(v) if $s > t > 0$*, then* $\frac{A(s)}{s} > \frac{A(t)}{t}$*.*

Example B.0.25. *The following are some of the classical examples of N-functions:*

$$(i)\ A_p(t) = \frac{t^p}{p}, \ \ \textit{for } 1 < p < \infty, \ \ (ii)\ t\exp(t) - t, \ \ (iii)\ \exp(t^2) - 1.$$

Definition B.0.26. (Orlicz function): *A continuous and convex real-valued function A, defined on* $[0, \infty)$ *is called an Orlicz function if,*

(i) $A(0) = 0$,

(ii) $\lim_{t \to \infty} \frac{A(t)}{t} = \infty$.

A is said to be degenerate, if $A(t) = 0$*, for some* $t > 0$*.*

Example B.0.27. *The function* $A(t) = \exp(t) - 1$, *is not an N-function as condition (iv) is not satisfied. Similarly* $B(t) = t\,log^{+}(t)$ *is not an N-function, since condition (ii) is violated. However one can easily verify that* A *and* B *satisfy all the conditions of an Orlicz function. Further,* B *is an example of a degenerate Orlicz function.*

Definition B.0.28. (Conjugate convex function): *Let* A *be an Orlicz function, we define its conjugate convex function* $\tilde{A}$ *as below:*

$$\tilde{A}(s) = \max_{t \geq 0}\{st - A(t)\}. \tag{B.1}$$

One can verify that $\widetilde{A}$ also satisfies all the conditions for an Orlicz function. Further, from the definition of the conjugate convex function, one can obtain the following Young's inequality:

Definition B.0.29. (Young's inequality): *Let* $\tilde{A}$ *be the conjugate convex function of the Orlicz function* A*. Then*

$$st \leq A(t) + \tilde{A}(s), \quad s, t > 0.$$

Definition B.0.30. (Dominance and Equivalence of Orlicz functions): *Let* A *and* B *are two Orlicz functions, we say that* B *dominates* A *globally if there exists a positive constant* k *such that*

$$A(t) \leq B(kt) \tag{B.2}$$

holds for all $t \geq 0$*. Similarly,* B *dominates* A *near infinity if there exist positive constants* t_0 *and* k *such that* (B.2) *holds for all* $t \geq t_0$*. The two Orlicz functions* A *and* B *are equivalent globally (resp. near infinity) if each dominates the other one globally (resp. near infinity).*

Definition B.0.31. (The Δ_2 Condition): *An Orlicz function,* A *is said to satisfy global* Δ_2*-condition if there exists a positive constant* k *such that for every* $t \geq 0$,

$$A(2t) \leq kA(t). \tag{B.3}$$

Similarly, A *satisfies a* Δ_2 *condition near infinity if there exists* $t_0 > 0$ *such that* (B.3) *holds for all* $t \geq t_0$.

Definition B.0.32. (The Orlicz Class $K_A(\Omega)$): *Let* Ω *be a domain in* $\mathbb{R}^n$ *and let* A *be an* N*-function. The Orlicz class* $K_A(\Omega)$ *is the set of all (equivalence classes modulo equality a.e. in* Ω *of) measurable functions* u

defined on Ω *that satisfy*

$$\int_{\Omega} A(|u(x)|)dx < \infty.$$

Since A *is convex,* $K_A(\Omega)$ *is always a convex set of functions but it may not be a vector space; for instance, there may exist* $u \in K_A(\Omega)$ *and* $\lambda > 0$ *such that* $\lambda u \notin K_A(\Omega)$.

Definition B.0.33. (Δ-regular): *We say that the pair* (A, Ω) *is* Δ-*regular if either*

(a) A *satisfies a global* Δ_2*-condition, or*

(b) A *satisfies a* Δ_2*-condition near infinity and* $|\Omega|$ *is finite .*

Definition B.0.34. (Orlicz space $L_A(\Omega)$): *The Orlicz space* $L_A(\Omega)$ *is the linear hull of the Orlicz class* $K_A(\Omega)$*. Let*

$$\|u\|_A := \inf\left\{k > 0 : \int_{\Omega} A\left(\frac{|u(x)|}{k}\right) dx \ \leq 1\right\}.$$

The functional $\|u\|_A$ *is a norm (Luxemburg norm) on* $L_A(\Omega)$*, and* $L_A(\Omega)$ *is complete with respect to the Luxemburg norm.*

Definition B.0.35. (The Space $E_A(\Omega)$): $E_A(\Omega)$ *is the closure of* $C_c(\Omega)$ *in* $L_A(\Omega)$.

Proposition B.0.36 (Some Properties of Orlicz spaces)**.** *Let* A *be an Orlicz function and let* $\widetilde{A}$ *be its conjugate Orlicz function.*

(i) $K_A(\Omega) = E_A(\Omega) = L_A(\Omega)$ *if and only if* (A, Ω) *is* Δ*- regular.*

(ii) Hölder inequality: Let $f \in L_A(\Omega), g \in L_{\tilde{A}}(\Omega)$*. Then*

$$\int_{\Omega} |fg| \leq 2\|f\|_A \|g\|_{\tilde{A}}.$$

(iii) $C_c^{\infty}(\Omega)$ *is dense is* $E_A(\Omega)$.

(iv) $E_A(\Omega)$ *is separable.*

Theorem B.0.37 (An Embedding Theorem for Orlicz Spaces)**.**
The embedding

$$L_B(\Omega) \to L_A(\Omega)$$

holds if and only if either

(a) B dominates A globally, or

(b) B dominates A near infinity and $|\Omega| < \infty$.

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