THE GEOMETRY OF SOME QUANTUM HOMOGENEOUS SPACES AND THE WEAK HEAT KERNEL EXPANSION

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A thesis submitted to the board of studies of Mathematical Sciences
in partial fulfillment of the requirements for the award of

Doctor of Philosophy
of
HOMI BHABHA NATIONAL INSTITUTE

MARCH 2011

Homi Bhabha National Institute

Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we recommend that the dissertation prepared by S.Sundar entitled "THE GEOMETRY OF SOME QUANTUM HOMOGENEOUS SPACES AND THE WEAK HEAT KERNEL EXPANSION" may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Declaration

I hereby declare that the investigation presented in this thesis has been carried out by me . The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

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Abstract

We study the noncommutative geometry of some quantum homogeneous spaces associated with the quantum group $SU_q(n)$. First we consider the quantum space $SU_q(n)/SU_q(n-1)$ called the odd dimensional quantum spheres and denoted S_q^{2n-1}. We consider two spectral triples associated to the odd dimensional quantum spheres S_q^{2n-1}. We show that the spectral triples satisfy the hypothesis of the local index formula. A conceptual explanation is given by considering a property which we call the weak heat kernel asymptotic expansion property of spectral triples. We show that a spectral triple having the weak heat kernel asymptotic expansion property satisfies the hypothesis of the local index formula. We also show that this property is stable under quantum double suspension. Finally we compute the K-groups of the quantum homogeneous space $SU_q(n)/SU_q(n-2)$.

Acknowledgements

First and foremost I thank my advisor Prof.Partha Sarathi Chakraborty for his guidance in completing this thesis. This thesis would not have taken its present shape without his persevering guidance and supervision. I am extremely grateful to him for the faith that he kept on me and for lending a helping hand whenever I had trouble.

I take this oppurtunity to thank Prof.V.S.Sunder for his consistent support and encouragement throughout my stay in Matscience. It is him who introduced me to Operator Algebras and interactions with him during my early days of Ph.d helped me a lot.

I take this oppurtunity to record my thanks to Prof.Arup Kumar Pal of ISI, Delhi who helped me a lot during my stay in Delhi. A major part of this thesis was completed during my stay there and I thank all the academic and administrative members of ISI Delhi who helped me to make my stay memorable.

I thank all the academic and the administrative members of the Institute of Mathematical Sciences for giving a conducive environment to pursue research.

I thank my friends Soumya, Alok, Gopal, Madhushree and Ramachandra for keeping my spirits high throughout. I thank my office mates Krishnan, Ajay singh thakur and Umesh Dubey. It has been a pleasent experience to share office with them. I thank my friends Stalin and Senthil for coming to see me several times just to cheer me up.

I thank my brother Walters for his support and encouragement. Last but not least I thank my parents for giving me the freedom to pursue research. This thesis is dedicated to them.

S SUNDAR

List of Publications and Preprints

- ArupKumar Pal and S.Sundar. Regularity and dimension spectrum of the equivariant spectral triple for the odd dimensional quantum spheres, Journal of Noncommutative Geometry, 389-439, Vol 4, Issue 3, 2010
- Partha Sarathi Chakraborty and S.Sundar. Quantum double suspension and spectral triples, Journal of functional analysis, 2011, doi:10.1016/j.jfa.2011.01.009
- Partha Sarathi Chakraborty and S.Sundar. K-groups of the quantum homogeneous space $SU_q(n)/SU_q(n-2)$. arXiv:1006.1742/math.KT

Contents

Chapter 1

Introduction

The theory of Noncommutative geometry initiated by Alain Connes has become an active area of research in Mathematics with applications to Physics. The starting point of Noncommutative topology can be traced back to the Gelfand Naimark theorem which gives an anti-equivalence between the category of locally compact Hausdorff spaces and the category of commutative C^* algebras. The correspondence is given by the map $X \mapsto C_0(X)$ where $C_0(X)$ is the algebra of continuous complex valued functions which vanish at infinity. This says that all the information about a space is actually encoded in the algebra of continuous functions on it. Thus one thinks of noncommutative C^* algebras as noncommutative topological spaces and tries to apply topological methods to understand them. K-theory and K-homology adapts well to study C^* algebras. Elliot's classification of AF algebras using K-theory is a famous instance of this.

Elements of K-homology are made of what are called Fredholm modules. An even Fredholm module for a C^* algebra A is a quadraple $(\pi, \mathcal{H}, F, \gamma)$ such that

- the map $\pi : A \to \mathcal{L}(\mathcal{H})$ is a $*$ representation,
- $\mathcal{H}$ is a Hilbert space with a $\mathbb{Z}_2$ grading γ,
- the operator F is a selfadjoint unitary which commutes with γ,
- the commutator $[F, \pi(a)]$ is compact for every $a \in A$, and
- for every $a \in A$, $\pi(a)\gamma = \gamma\pi(a)$.

If there is no grading present, one calls it an odd Fredholm module. There is a natural pairing between K-theory and K-homology. If $(A, \mathcal{H}, F)$ is a Fredholm module, the K-theory/K-homology pairing is given by an index map $Ind_F : K_*(A) \to \mathbb{Z}$. In geometric examples, Fredholm modules arise from unbounded operators like elliptic differential operators and the unbounded Fredholm modules are called spectral triples .

In geometry, the topological space one tries to understand will usually be a smooth manifold. Connes proposed the following notion of spectral triples as the noncommutatative counterpart of smooth manifolds. An **even spectral triple** for a $*$ algebra $\mathcal{A}$ is a triple $(\pi, \mathcal{H}, D)$ together with a $\mathbb{Z}_2$ grading γ on $\mathcal{H}$ such that

1. the map $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a $*$ representation such that $\pi(a)\gamma = \gamma\pi(a)$ for every $a \in \mathcal{A}$,
2. the operator D is an unbounded operator with compact resolvent such that $D\gamma = -\gamma D$, and
3. the commutator $[D, \pi(a)]$ is bounded for every $a \in \mathcal{A}$.

If no grading is present one calls it an odd spectral triple. Usually $\mathcal{A}$ will be a dense subalgebra of a C^* algebra. If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple then $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module where $F := sign(D)$. The reason for calling spectral triples as noncommutative manifolds is due to the fact that the spectral triple $(C^\infty(M), L^2(M, S), D)$[[15]] captures all the information about the manifold M. Here

- M is a smooth Riemannian manifold,
- $S \to M$ is a spinor bundle and $L^2(M, S)$ denote the space of square integrable sections, and
- The operator D is the Dirac operator associated to the Levi-Civita connection.

Also Connes in [1] proved that if $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple which satisfies certain assumptions and if $\mathcal{A}$ is commutative then the spectral triple comes from a classical spectral triple $(C^\infty(M), L^2(M, S), D)$ for some smooth manifold M. Thus it makes good sense to think of spectral triples as non-commutative manifolds.

In [14], Connes constructed cyclic cohomology as the natural receipent of Chern character from the K-theory. He defines the Chern character of a finitely summable Fredholm module and calculates the index map as the pairing between K-theory and cyclic cohomology. But in geometric examples coming from spectral triples, the Chern character is difficult to compute as the sign of the operator D is difficult to compute. Thus one needs a manageable formula completely in terms of D. The local index formula in [12] achieves this. The formula is given in terms of residues of certain meromorphic functions associated with D. Let us briefly explain the formula before we go further.

First let us recall the notion of regularity and dimension spectrum of spectral triples. Consider an unbounded selfadjoint operator D. Let

$$\mathcal{H}_\infty := \cap_{n=1} Dom(|D|^n),$$
$$OP^0 := \{T \in \mathcal{L}(\mathcal{H}) : T \in \cap_n Dom(\delta^n)\}$$

where $\delta := [|D|, .]$ is the unbounded derivation on $\mathcal{L}(\mathcal{H})$.

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be regular if $\mathcal{A} + [D, \mathcal{A}] \subset OP^0$. Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple which is finitely summable. Assume that $|D|^{-p}$ is trace class. Let $\mathcal{B}$ be the algebra generated by $\delta^n(\mathcal{A})$ and $\delta^n([D, \mathcal{A}])$ for $n \geq 0$. We say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has **discrete dimension spectrum** $\Sigma \subset \mathbb{C}$ if Σ is discrete and for every $b \in \mathcal{B}$, the function $Trace(b|D|^{-z})$ initially defined for $Re(z) > p$ extends to a meromorphic function with poles only in Σ. We say the dimension spectrum is simple if all the poles are simple.

Let $(\mathcal{A}, \mathcal{H}, D)$ be an odd finitely summable spectral triple which is regular and has discrete dimension spectrum. Let $\mathcal{B}$ be the algebra generated by $\delta^n(\mathcal{A})$ and $\delta^n([D, \mathcal{A}])$ in $\mathcal{L}(\mathcal{H})$. For $b \in \mathcal{B}$, we let $b^{(1)} := [D^2, b]$ and $b^{(k)} := [D^2, b^{(k-1)}]$. We denote the algebra generated by $\mathcal{B}$ and $|D|^k, k \in \mathbb{N}$ by $\mathcal{D}$. Then for $b \in \mathcal{D}$ we let $\int b := Res_{z=0} Tr(b|D|^{-z})$.

For every odd n and a multiindex $k = (k_1, k_2, \cdots, k_n)$, consider the $n+1$ multilinear functional $\phi_{n,k}$ on $\mathcal{A}$ defined as

$$\phi_{n,k}(a_0, a_1, \cdots, a_n) := \int a_0 [D, a_1]^{(k_1)} [D, a_2]^{(k_1)} \cdots [D, a_n]^{(k_n)} |D|^{-n-2|k|}$$

where $|k| := \sum_{i=1}^n k_i$. Note that if $|k| + n > p$ then $\phi_{n,k} = 0$. We let $\phi_n := \sum_k c_{n,k} \phi_{n,k}$ where the constants $c_{n,k}$ are given by

$$c_{n,k} := (-1)^{|k|} \sqrt{2i} \frac{\Gamma(|k| + \frac{n}{2})}{\prod k_j! \prod (k_1 + k_2 + \cdots + k_j + j)}.$$

The local index formula states that the Chern character can be computed, upto a coboundary, by the functionals ϕ_n. If we consider the Dirac operator associated to a closed Reimannian spin manifold then the terms $\phi_{n,k}$ vanishes if $|k| \neq 0$. Thus most of the terms in the local Chern character is visible only in the case of truly noncommutative situations and should be interpreted as a signature of noncommutativity. To understand the local index formula better one would like to have some examples of simple spectral triples which satisfy the assumptions of the local index formula. Connes illustrated the local index formula for the equivariant spectral triple on $SU_q(2)$ constructed in [5]. Similar computations were done in [19],[39],[29]. In this thesis, a similar computation for the $SU_q(n+1)$ equivariant spectral triple on the odd dimensional quantum spheres is carried out. We also develop a general method of verifying the hypothesis of the local index formula.

All these examples come from quantum homogeneous spaces. The rich interplay between Lie groups and differential geometry naturally raises the question of understanding the interaction between quantum groups and noncommutative geometry. Papers [5],[39],[28] attempt to put quantum groups within the framework of Connes' noncommutativie geometry. Chakraborty and Pal in [5] produced a satisfactory spectral triple on the quantum group $SU_q(2)$ which is also equivariant. Connes made a detailed study of this spectral triple in [17] from the local index formula point of view. The work of Tuset and Neshveyev in [28] is an attempt to produce equivariant spectral triples on compact quantum groups arising from the q deformation of general simple lie algebras.

Let us recall a few basic notions regarding compact quantum groups before proceeding further. The theory of quantum groups has its origin in finding a good duality theorem, analogous to Pontraygin duality theorem, for general locally compact groups. If G is a compact group then the group multiplication in G gives rise to the comultiplication map $\Delta : C(G) \to C(G) \otimes C(G)$ defined by $\Delta(f)(x, y) = f(xy)$. It can be shown that the group G can be recovered from the pair $(C(G), \Delta)$. A compact quantum group is roughly a unital C^* algebra together with a comultiplication $\Delta : A \to A \otimes A$ which is coassociative i.e. $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$. In late 1980's, Woronowicz developed a general theory of compact quantum groups and developed a Peter-Weyl theory for them in [41], [42],[43].

Let A be a unital C^* algebra and G be a compact quantum group. An action of G on A is a unital homomorphism $\tau : A \to A \otimes C(G)$ such that $(1 \otimes \Delta_G)\tau = (\tau \otimes 1)\tau$ where Δ_G is the comultiplication on $C(G)$. We call the triple (A, G, τ) a C^* dynamical system. If G is a compact quantum group then G acts on $C(G)$ by the comultiplication. If G is a compact quantum group and H a quantum subgroup then G acts on the quotient $C(G/H)$ by the comultiplication Δ_G. A representation of a compact quantum group G on a Hilbert space $\mathcal{H}$ is a unitary element u in the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C(G))$ such that $(id \otimes \Delta)(u) = u_{12}u_{13}$. A covariant representation of a C^* dynamical system $(\mathcal{A}, G, \tau)$ consists of a pair (π, u) where π is a representation of the C^* algebra A on a Hilbert space $\mathcal{H}$, u is a unitary representation of G on $\mathcal{H}$ and they obey the condition

$$u(\pi(a) \otimes 1)u^* = (\pi \otimes id)\tau(a) \quad \text{for} \quad a \in A.$$

Let $(\mathcal{A}, G, \tau)$ be a C^* dynamical system. An **odd G equivariant spectral triple** is a quadruple $(\pi, u, D, \mathcal{H})$ such that

1. The pair (π, u) is a covariant representation of the dynamical system (A, G, τ) on the Hilbert space $\mathcal{H}$,

2. There exists a dense unital *subalgebra $\mathcal{A} \subset A$ such that the triple $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, and

3. The operator D is G equivariant i.e. $u(D \otimes 1)u^* = D$.

Now we discuss the results obtained in this thesis. Let us recall the definition of the quantum group $SU_q(n)$ due to Woronowicz. Throughout we assume that $q \in (0, 1)$. Recall that the C^* algebra $C(SU_q(n))$ is the universal unital C^* algebra generated by n^2 elements u_{ij} satisfying the following condition

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij} \quad , \quad \sum_{k=1}^{n} u_{ki}^* u_{kj} = \delta_{ij} \tag{1.0.1}$$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2\cdots i_n} u_{j_1 i_1}\cdots u_{j_n i_n} = E_{j_1 j_2\cdots j_n} \tag{1.0.2}$$

where

$$E_{i_1 i_2\cdots i_n} := \begin{cases} 0 & if \quad i_1, i_2, \cdots i_n \text{ are not distinct,} \\ (-q)^{\ell(i_1,i_2,\cdots,i_n)} & \end{cases}$$

where for a permutation σ on $\{1,2,\cdots,n\}$, $\ell(\sigma)$ denotes the number of inversions i.e. the cardinality of the set $\{(i,j) : i < j, \sigma(i) > \sigma(j)\}$. The C^* algebra $C(SU_q(n))$ has a compact quantum group structure. The comultiplication Δ is given by

$$\Delta(u_{ij}) := \sum_k u_{ik}\otimes u_{kj}.$$

Call the generators of $SU_q(n-m)$ as v_{ij} for $1 \leq m \leq n-1$. The map $\phi : C(SU_q(n)) \to C(SU_q(n-m))$ defined by

$$\phi(u_{ij}) := \begin{cases} v_{i-m,j-m} & \text{if } m+1 \leq i,j \leq n \\ \delta_{ij} & \text{otherwise} \end{cases} \tag{1.0.3}$$

is a surjective unital C^* algebra homomorphism such that $\Delta\circ\phi = (\phi\otimes\phi)\Delta$. In this way the quantum group $SU_q(n-m)$ is a subgroup of the quantum group $SU_q(n)$. The C^* algebra of the quotient $SU_q(n)/SU_q(n-m)$ denoted by $C(S_q^{n,m})$ is defined by the equation

$$C(S_q^{n,m}) := \{a \in C(SU_q(n)) : \ (\phi\otimes 1)\Delta(a) = 1\otimes a\}.$$

The C^* algebra $C(S_q^{n,1})$ is denoted by $C(S_q^{2n-1})$. The C^* algebras $C(S_q^{n,m})$ are called the quantum Steifel manifolds in [31] and the C^* algebras $C(S_q^{2n-1})$ are called the odd dimensional quantum spheres.

Let h be the Haar state on the quantum group $SU_q(n+1)$ and let $L^2(SU_q(n+1))$ be the corresponding GNS space. We denote the closure of $C(S_q^{2n+1})$ in $L^2(SU_q(n+1))$ by $L^2(S_q^{2n+1})$. Then $L^2(S_q^{2n+1})$ is invariant under the right regular representation of $SU_q(n+1)$. Thus we get a covariant representation for the dynamical system $(C(S_q^{2n+1}), SU_q(n+1), \Delta)$. In [8], $SU_q(n+1)$ equivariant spectral triples for this covariant representation were studied and a non-trivial one was constructed. The Hilbert space $L^2(S_q^{2n+1})$ is nothing but $\ell^2(\mathbb{N}^n\times\mathbb{Z}\times\mathbb{N}^n)$ upto a unitary map. Then the selfadjoint operator D_{eq} constructed in [8] is given on the orthonormal basis $\{e_\gamma : \gamma\in\mathbb{N}^n\times\mathbb{Z}\times\mathbb{N}^n\}$ by the formula $D_{eq}(e_\gamma) := d_\gamma e_\gamma$ where d_γ is given by

$$d_\gamma := \begin{cases} \sum_{i=1}^{2n+1}|\gamma_i| & \text{if} \quad (\gamma_{n+1},\gamma_{n+2},\cdots,\gamma_{2n+1}) = 0 \ \text{ and } \ \gamma_{n+1}\geq 0, \\ -\sum_{i=1}^{2n+1}|\gamma_i| & \text{else .} \end{cases}$$

In this thesis, we study the spectral triple $(C(S_q^{2n+1}), L^2(S_q^{2n+1}), D_{eq})$ from the local index formula point of view. We show in particular that it satisfies the assumptions of the local index formula namely regularity and discreteness of the dimension spectrum. In particular, we prove the following theorem.

Theorem A. *There exists a unital dense *subalgebra $C^{\infty}(S_q^{2n+1})$ of the C^* algebra $C(S_q^{2n+1})$ such that the triple $(C^{\infty}(S_q^{2n+1}), L^2(S_q^{2n+1}), D_{eq})$ is a regular spectral triple with simple and discrete dimension spectrum. In particular the dimension spectrum is the set $\{1, 2, \cdots, 2n+1\}$. Moreover the dense subalgebra $C^{\infty}(S_q^{2n+1})$ is closed under holomorphic functional calculus.*

Theorem A is the main result in [30]. But this computation and also the computations carried out in [17], [39] and in [29] are case specific. There are not many results of general nature which will imply regularity and discreteness of the dimension spectrum. To our knowledge, only Higson's results in [20] are in this direction. We need some functorial constructions on regular spectral triples with discrete dimension spectrum such that the resulting spectral triple is also regular and has discrete dimension spectrum.

We consider a property of spectral triples which we call the **weak heat kernel asymptotic expansion** property and also the construction of the quantum double suspension of spectral triples. We show that a spectral triple having the weak heat kernel expansion property is regular and has simple dimension spectrum. We also show that the weak heat kernel expansion property is stable under quantum double suspension. This gives a conceptual explanation for Theorem A.

Now let us recall the definition of the quantum double suspension as in [22]. If A is a unital C^* algebra then its quantum double suspension denoted $\Sigma^2(A)$ is the C^* algebra generated by $1 \otimes S$ and $A \otimes p$ where S is the left shift on $\ell^2(\mathbb{N})$ and $p := 1 - S^*S$. If $(\pi, \mathcal{H}, D)$ is a spectral triple for A then $(\pi \otimes 1, \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D) := (F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ is a spectral triple for the algebra $\Sigma^2(A)$. Here $F := sign(D)$ and N is the number operator on $\ell^2(\mathbb{N})$ defined on the standard orthonormal basis by $Ne_n := ne_n$.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular and finitely summable spectral triple. Let $\mathcal{B}$ be the algebra generated by $\bigcup_{n \geq 0} \delta^n(\mathcal{A} + [D, \mathcal{A}])$. We say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has the **weak heat kernel asymptotic expansion** property if the function $Tr(be^{-t|D|})$ admits an asymptotic expansion in t near 0 for every $b \in \mathcal{B}$ and we say that it has the heat kernel asymptotic expansion property if $Tr(be^{-tD^2})$ has an asymptotic expansion property for every $b \in \mathcal{B}$. We prove the following theorem in this thesis.

Theorem B. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple which has the weak heat kernel asymptotic expansion property. Then the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has a finite simple dimension spectrum contained in the set of positive integers. If the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has the weak heat kernel expansion then the spectral triple $(\Sigma^2(\mathcal{A}), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D))$ also has the weak heat kernel asymptotic expansion property. Moreover if the dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$ lies inside $\{1, 2, \cdots, n\}$ then the dimension spectrum of its quantum double suspension lies inside $\{1, 2, \cdots, n+1\}$.*

We compare the weak heat kernel expansion property with that of the classical heat kernel expansion property. The following proposition is proved in the thesis.

Proposition. *If a spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *has the heat kernel asymptotic expansion property then* $(\mathcal{A}, \mathcal{H}, D)$ *has the weak heat kernel expansion property.*

Thus the standard spectral triple $(C^\infty(M), L^2(S), D)$ on a spin manifold, where $S \to M$ is a spinor bundle and D is the Dirac operator associated with Levi-Civita connection, has the weak heat kernel asymptotic expansion property. Thus quantum double suspending this spectral triple recursively produces noncommutative examples of spectral triples which satisfies the hypothesis of the local index formula.

The K-groups of the quantum group $SU_q(n)$ were computed by Gabriel Nagy using his bivariant K theory in [27]. But one would like to obtain some explicit generators for the K-groups to construct non-trivial spectral triples on them. The final chapter of the thesis achieves this for the quantum Steifel manifolds $C(S_q^{n,2})$. The following theorem is proved in the thesis.

Theorem C. *The K groups* $K_0(C(S_q^{n,2}))$ *and* $K_1(C(S_q^{n,2}))$ *are both isomorphic to* $\mathbb{Z}^2$.

We obtain explicit generators for the K-groups.

We complete the introduction by indicating the organisation of the thesis.

In the next Chapter, we collect the preliminaries that are required to understand this thesis. We review the basics of K-theory. The definitions pertaining to spectral triples and quantum groups are recalled in this chapter. The local index formula for spectral triples is recalled.

In Chapter 3 , we consider a spectral triple called the torus equivariant one on S_q^{2n+1} and show that it is regular and has discrete dimension spectrum.

In Chapter 4, we consider the $SU_q(n)$ equivariant spectral triple on S_q^{2n+1}. Theorem A is proved in this chapter.

Chapter 5 starts with a brief discussion on asymptotic expansions and Mellin transform. We then consider quantum double suspension of C^* algebras and that of spectral triples. We prove Theorem B in this chapter.

In the final chapter, we compute the K-groups of the quantum Steifel manifold $C(S_q^{n,2})$.

The chapters are followed by an appendix. In the appendix, we discuss the results of Larry B.Schweitzer on smooth subalgebras of operator algebras obtained in [33],[34]. We have included the proofs of certain results for the sake of completeness.

Chapter 2

Preliminaries

In this chapter, we collect the preliminaries which are essential for understanding this thesis. In particular we review the K-theory of C^* algebras, definitions pertaining to spectral triples and the local index formula.

2.1 Topological K-theory

We start by recalling the definition of topological K-theory. Let X be a compact Hausdorff space. We denote the set of isomorphism classes of vector bundles of finite rank over X by $V(X)$. The Whitney sum of vector bundles makes $V(X)$ an abelian semigroup with an identity element. The abelian group $K(X)$ is defined to be the group obtained from $V(X)$ by the Grothendieck construction. The group $K(X)$ is called the K-theory of X.

Let us recall the Grothendieck construction. Suppose $(R, +)$ is an abelian semigroup with identity. Define an equivalence relation $\sim$ on $R \times R$ as follows:

$$(a, b) \sim (c, d) \text{ if there exists } e \in R \text{ such that } a + d + e = b + c + e.$$

We think of the equivalence class $[(a, b)]$ as representing the difference $a - b$. The addition $+$ on $R \times R/\sim$ is defined as

$$[(a, b)] + [(c, d)] = [(a + c, b + d)].$$

Then $+$ is well defined on $R \times R/\sim$ and $(R \times R/\sim, +)$ is an abelian group with $[(a, a)]$ as the identity element for any $a \in R$ and the inverse of $[(a, b)]$ is $[(b, a)]$.

The fact that allows one to adapt K-theory to noncommutative C^* algebras is the **Serre-Swan theorem**. Let $\pi : E \to X$ be a vector bundle over X. For $x \in X$, denotes its fibre over x by E_x i.e $E_x := \pi^{-1}(x)$. A section of $\pi : E \to X$ is a map $s : X \to E$ such that $s(x) \in E_x$. The set of all sections is denoted by $\Gamma(E)$. Then $\Gamma(E)$ is a left $C(X)$ module where the module structure is given by

$$(f.s)(x) := f(x)s(x) \text{ for } \ f \in C(X),\ s \in \Gamma(E).$$

We denote the isomorphism classes of finitely generated projective left modules over $C(X)$ by $Proj_{fin}(C(X))$. Then $(Proj_{fin}(C(X)), +)$ is a semigroup with identiy.

Theorem 2.1.1 (Serre-Swan)**.** *For a vector bundle (E, π, X), the set of its sections $\Gamma(E)$ is a finitely generated projective module over $C(X)$. Furthermore, the map*

$$Vect(X) \ni [E] \to [\Gamma(E)] \in Proj_{fin}(C(X))$$

is a semigroup isomorphism.

Thus the abelian group $K(X)$ can as well be obtained from the semigroup $Proj_{fin}(C(X))$ by the Grothendieck construction and $K(X)$ depends only on the algebra $C(X)$. We can now replace $C(X)$ by any noncommutative algebra and can obtain an invariant for it. In particular K-theory adapts well to noncommutative C^* algebras. For a more detailed account of the topological K-theory, we refer to [26].

2.2 K-theory for C^* algebras

In this section, we give a brief review of the K theory for C^* algebras. We refer to [4] for a detailed account of it. All the algebras that we consider in this thesis are over $\mathbb{C}$. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. We denote the direct sum $\underbrace{\mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{ntimes}$ by $\mathcal{A}^n$ which is naturally a $\mathcal{A} - M_n(\mathcal{A})$ bimodule.

Proposition 2.2.1. *Let $\mathcal{A}$ be an algebra over the field of complex numbers $\mathbb{C}$.*

1. *For an idempotent $p \in M_n(\mathcal{A})$, the left $\mathcal{A}$ module $\mathcal{A}^n p$ is a finitely generated projective $\mathcal{A}$ module. In fact, any finitely generated projective $\mathcal{A}$ module arises this way.*
2. *The modules $\mathcal{A}^n p$ and $\mathcal{A}^m q$ are isomorphic if and only if there exists matrices $u \in M_{n\times m}(\mathcal{A})$ and $v \in M_{m\times n}(\mathcal{A})$ such that $uv = p$ and $vu = q$.*

For an algebra $\mathcal{A}$, let $E(\mathcal{A}) := \{e \in \mathcal{A} :\ e^2 = e\}$ and $E_\infty(\mathcal{A}) := \bigcup_n E(M_n(\mathcal{A}))$. Define an equivalence relation on $E_\infty(\mathcal{A})$ as follows: Let $p \in M_m(\mathcal{A})$ and $q \in M_n(\mathcal{A})$.

$$p \sim q \Leftrightarrow \text{ there exists } u \in M_{m\times n}(\mathcal{A}), v \in M_{n\times m}(\mathcal{A}) \text{ such that } uv = p \text{ and } vu = q.$$

Then we have the following proposition.

Proposition 2.2.2. *The operation $\oplus$ defined by $[p]\oplus[q] := \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is well defined on $E_\infty(A)/\sim$. Moreover, $(E_\infty(A)/\sim, \oplus)$ is a commutative semigroup with identity.*

We define the K-group $\hat{K}_0(\mathcal{A})$ to be the Grothendieck group of the abelian semigroup $(E_\infty(\mathcal{A}), \oplus)$. In C^* algebras, we can replace the idempotents by projections and the equivalence relation by Murray-von Neumann equivalence.

Let us first introduce some notations. Let A be a C^* algebra. An element $p \in A$ is said to be a projection if $p^2 = p = p^*$. Let

$$\begin{aligned} P(A) &:= \{p \in A : \ p \text{ is a projection}\}, \\ M_n(A) &:= \text{ the algebra of } n \times n \text{ matrices over } A, \text{ and} \\ P_\infty(A) &:= \bigcup_{n \geq 1} P(M_n(A)). \end{aligned}$$

Two projections p and q in a C^* algebra A are said to be **Murray-von Neumann** equivalent if there exists $v \in A$ such that $v^*v = p$ and $vv^* = q$. It is easy to verify that Murray-von Neumann equivalence is an equivalence relation.

Define an equivalence relation on $P_\infty(A)$ as follows: Let $p \in M_m(A)$ and $q \in M_n(A)$ be projections.

$$p \sim q \Leftrightarrow \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \text{ are Murray-von Neumann equivalent in } M_N(A) \text{ for some N .}$$

Then we have the following proposition.

Proposition 2.2.3. *The operation $\oplus$ defined by $[p]\oplus[q] := \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is well defined on $P_\infty(A)/\sim$. Moreover $(P_\infty(A)/\sim, \oplus)$ is a commutative semigroup with identity.*

We define the K-group $\hat{K}_0(A)$ to be the Grothendieck group of the abelian semigroup $(P_\infty(A), \oplus)$. It can be shown that for a C^* algebra A, $\hat{K}_0(A)$ defined in terms of the projections coincide with the one defined in terms of idempotents. Note that if $\phi : A \to B$ is a $*$ homomorphism then ϕ induces a map $\hat{K}_0(\phi)$ at the level of $\hat{K}_0(A)$. Thus $\hat{K}_0$ is a functor from the category of C^* algebras to the category of abelian groups.

Let A be a C^* algebra and let A^+ denote the C^* algebra obtained by adding an unit. More precisely $A^+ := A \oplus \mathbb{C}$ and the multiplication is defined as $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. One can show that A^+ is a C^* algebra with A as an ideal. We let ϵ be the map $(a, \lambda) \to \lambda$. Then one has the following exact sequence

$$0 \longrightarrow A \longrightarrow A^+ \stackrel{\epsilon}{\longrightarrow} \mathbb{C} \longrightarrow 0.$$

We define for a C^* algebra A,

$$K_0(A) := Ker\hat{K}_0(\epsilon).$$

Then K_0 is a covariant functor from the category of C^* algebras to the category of abelian groups. For unital C^* algebras, K_0 and $\hat{K}_0$ are naturally isomorphic.

Now we recall the definition of K_1 for a C^* algebra. Let A be a unital C^* algebra. An element $u \in A$ is called a unitary element if $u^*u = uu^* = 1$. We denote the set of unitaries in $M_n(A)$ by $U_n(A)$. Then the group $U_n(A)$ is a topological group and the connected component containing 1 is denoted by $U_n^0(A)$. Then $U_n^0(A)$ is a normal subgroup of $U_n(A)$. We embedd $U_n(A)$ in $U_{n+1}(A)$ by the inclusion $u \to \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$. Thus we get a directed system of groups

$$\cdots \to U_{n-1}(A)/U_{n-1}^0(A) \to U_n(A)/U_n^0(A) \to U_{n+1}(A)/U_{n+1}^0(A) \to \cdots$$

Define

$$\hat{K}_1(A) := \lim_{n\to\infty} \frac{U_n(A)}{U_n^0(A)}$$

Then one has the following proposition.

Proposition 2.2.4. *Let A be a unital C^* algebra.*

1. *For a unitary $u \in M_n(A)$, in $\hat{K}_1(A)$ one has* $[u] = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}$.
2. *The group operation on $\hat{K}_1(A)$ is given by* $[u].[v] = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$.
3. *The group $\hat{K}_1(A)$ is abelian.*

For a C^* algebra A, we let $K_1(A) := \hat{K}_1(A^+)$. Then K_1 is a covariant functor from the category of C^* algebras to the category of abelian groups. For unital C^* algebras the functors K_1 and $\hat{K}_1$ are naturally isomorphic.

2.2.1 The six term sequence in K theory

An important computational tool in K theory is the six term exact sequence. For an exact sequence

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\pi} B \longrightarrow 0,$$

one has the following six term exact sequence

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0(\phi)} & K_0(A) & \xrightarrow{K_0(\pi)} & K_0(B) \\ \partial\uparrow & & & & \downarrow\sigma \\ K_1(B) & \xleftarrow[K_1(\pi)]{} & K_1(A) & \xleftarrow[K_1(\phi)]{} & K_1(I) \end{array}$$

The map ∂ is called the index map and σ is called the exponential map. Moreover the index and the exponential maps are functorial. The proof of the six term sequence involves the following

- Half exactness of the K-groups i.e top and bottom rows are exact.

- The construction of the index map.

- Bott periodicity and the construction of the exponential map.

For a detailed proof we refer to [4]. Here we just recall the construction of the index map. Let u be a unitary in $M_n(B^+)$. Then the unitary $U := \begin{bmatrix} u & 0 \\ 0 & u^* \end{bmatrix}$ is connected to 1_{2n}. Let V be a unitary lift in $M_{2n}(A^+)$ of U. Then $\partial([u])$ is defined as $\partial([u]) = [Vp_nV^*] - [p_n]$ where $p_n := \begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix}$.

Lemma 2.2.5. *Consider an exact sequence of C^* algebras*

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\pi} B \longrightarrow 0.$$

Let u be a unitary in $M_n(B^+)$. Suppose that there exists a partial isometry $v \in M_n(A^+)$ such that $\pi(v) = u$. Then $\partial([u]) = [1 - v^*v] - [1 - vv^*]$.

Proof. Note that the unitary $V := \begin{bmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{bmatrix}$ is a lift of $diag(u, u^*)$. We write $diag(a, b)$ to denote the matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Hence $\partial([u]) = [Vp_nV^*] - [p_n] = [diag(vv^*, 1 - v^*v] - [diag(1_n, 0)]$. Thus one has $\partial([u]) = [1 - v^*v] - [1 - vv^*]$. This completes the proof. □

2.3 Cyclic cohomology

In this section, the periodic cyclic cohomology for an algebra defined in [14] is recalled. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. We denote the set of $n + 1$ linear functionals on $\mathcal{A}$ by $C^n(\mathcal{A})$. Consider the map $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ defined by

$$\begin{aligned} b\phi(a_0, a_1, \cdots, a_{n+1}) : &= \sum_{i=0}^{n} (-1)^i \phi(a_0, a_1, \cdots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \cdots, a_{n+1}) \\ &+ (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \cdots, a_n). \end{aligned}$$

Then one can show that $b^2 = 0$ and the cohomology of the complex $(C^n(\mathcal{A}), b)_{n \geq 0}$ is called the Hochschild cohomology of $\mathcal{A}$. An $n + 1$ linear functional ϕ on $\mathcal{A}$ is said to be cyclic if $\phi(a_1, a_2, \cdots, a_n, a_0) = (-1)^n \phi(a_0, a_1, \cdots, a_n)$ for every $a_0, a_1, \cdots, a_n \in \mathcal{A}$. Let $C_\lambda^n(\mathcal{A})$ be the subspace of $n + 1$ multilinear functionals on $\mathcal{A}$ which are cyclic. It is shown in [14] that b maps the subspace $C_\lambda^n(\mathcal{A})$ to the subspace $C_\lambda^{n+1}(\mathcal{A})$. This gives rise to a complex $(C_\lambda^n(\mathcal{A}), b)_{n \geq 0}$ and its cohomology is called the cyclic cohomology defined by

$$H_\lambda^n(\mathcal{A}) := \frac{\{\tau \in C_\lambda^n(\mathcal{A}) : \ b\tau = 0\}}{\{b\phi : \ \phi \in C_\lambda^{n-1}(\mathcal{A})\}}.$$

Elements of $H^n_\lambda(\mathcal{A})$ come from what are called cycles over $\mathcal{A}$. Let us recall the following definition from [14].

Definition 2.3.1. *An n dimensional cycle is a triple $(\Omega, d, \int)$ where $\Omega := \oplus_{p=0}^n \Omega_p$ is a graded algebra, d is a graded derivation and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace i.e.*

1. *For $\omega \in \Omega^{n-1}$, the integral $\int d\omega = 0$.*
2. *If $\omega_1 \in \Omega_p$ and $\omega_2 \in \Omega_q$ then $\int \omega_1\omega_2 = (-1)^{pq} \int \omega_2\omega_1$.*

Let $(\Omega_1, d_1, \int)$ be an m dimensional cycle and $(\Omega_2, d_2, \int)$ be an n dimensional cycle. Then the tensor product $(\Omega_1 \otimes \Omega_2, d, \int)$ is an $m+n$ dimensional cycle with d and $\int$ being defined as

$$\begin{aligned} d(\omega_1 \otimes \omega_2) &= d_1(\omega_1) \otimes \omega_2 + (-1)^{deg\omega_1} \omega_1 \otimes d_2\omega_2, \text{ and} \\ \int \omega_1 \otimes \omega_2 &= \int \omega_1 \int \omega_2. \end{aligned}$$

An n dimensional cycle for an algebra $\mathcal{A}$ is an n dimensional cycle $(\Omega, d, \int)$ together with a homomorphism $\rho : \mathcal{A} \to \Omega^0$. Let $(\Omega, d, \int, \rho)$ be an n dimensional cycle for $\mathcal{A}$. Then its character is the $n+1$ linear functional τ on $\mathcal{A}$ defined by

$$\tau(a_0, a_1, \cdots, a_n) := \int \rho(a_0) d(\rho(a_1)) d(\rho(a_2)) \cdots d(\rho(a_n)).$$

Then it is shown in [14] that τ is a cyclic cocycle and any cyclic cocycle comes from a cycle. Tensor products of cycles gives rise to the cup product $\# : H^n_\lambda(\mathcal{A}) \times H^m_\lambda(\mathcal{B}) \to H^{n+m}_\lambda(\mathcal{A} \otimes \mathcal{B})$ in the cyclic cohomology . Let σ be the cyclic 2 cocycle on $\mathbb{C}$ defined by $\sigma(1,1,1) = \frac{1}{2\pi i}$. Consider the map $S : H^n_\lambda(\mathcal{A}) \to H^{n+2}_\lambda(\mathcal{A})$ defined by $S(\phi) = \phi \# \sigma$. The periodic cyclic cohomology $H^{ev}_\lambda(\mathcal{A})$ and $H^{odd}_\lambda(\mathcal{A})$ are defined as

$$\begin{aligned} H^{even}_\lambda(\mathcal{A}) &:= \lim_{n \ even} (H^n_\lambda(\mathcal{A}), S), \text{ and} \\ H^{odd}_\lambda(\mathcal{A}) &:= \lim_{n \ odd} (H^n_\lambda(\mathcal{A}), S). \end{aligned}$$

Now we present another picture of the periodic cyclic cohomology which is essential to explain the local index formula. Recall that $C^n(\mathcal{A})$ is the space of $n+1$ linear functionals on $\mathcal{A}$. Define the operator $B : C^{n+1}(\mathcal{A}) \to C^n(\mathcal{A})$ as follows:

$$B\phi(a_0, a_1, \cdots, a_n) = \sum_{j=0}^{n} (-1)^{nj} \phi(1, a_j, a_{j+1}, \cdots, a_{j-1}) + \sum_{j=0}^{n} (-1)^{n(j-1)} \phi(a_j, a_{j+1}, \cdots, a_{j-1}, 1).$$

Then it can be shown that $B^2 = 0$ and $bB + Bb = 0$. Consider an element $\phi = (\phi_0, \phi_2, \cdots)$ in the direct sum $\oplus_{n \ even} C^n(\mathcal{A})$. Then ϕ is called an even (b, B) cocycle if $b\phi_{2k} + B\phi_{2k+2} = 0$. Similarly an element $\phi = (\phi_1, \phi_3, \cdots ,)$ is called an odd (b, B) cocycle if $b\phi_{2k-1} + B\phi_{2k+1} = 0$. If τ is a cyclic n cocycle then $(0, 0, \cdots, \tau, \cdots)$ is a (b, B) cocycle.

Two even (b,B) cocycles ϕ and $\phi^{'}$ are said to be cohomologous and written $\phi \sim_{coh} \phi^{'}$ if there exists $\psi = (\psi_1, \psi_3, \cdots,)$ in the direct sum $\oplus_{n\ odd} C^n(\mathcal{A})$ such that $\phi_{2k} - \phi_{2k} = b\psi_{2k-1} + B\psi_{2k+1}$. Similarly one defines the relation of being cohomologous for odd (b,B) cocycles.

Definition 2.3.2. *For a unital algebra $\mathcal{A}$ over $\mathbb{C}$, the even(resp. odd) periodic cyclic cohomology $PH_{\lambda}^{even}(\mathcal{A})$ (resp. $PH_{\lambda}^{odd}(\mathcal{A})$) is the vector space of even(resp. odd) (b,B) cocycles modulo the relation $\sim_{coh}$.*

It is proved in [14] that the (b,B) picture of the cyclic cohomology and that defined via cyclic cocycles coincide. The map (upto a normalising constant)

$$H_{\lambda}^{even}(\mathcal{A}) \ni [\tau] \to [(0,0,\cdots,\tau,\cdots,)] \in PH_{\lambda}^{even}(\mathcal{A})$$

is infact an isomorphism. A similar statment holds for odd periodic cyclic cohomology. Henceforth we use the same notation $H_{\lambda}^{even}(\mathcal{A})$ to denote both the spaces $H_{\lambda}^{even}(\mathcal{A})$ and $PH_{\lambda}^{even}(\mathcal{A})$. Similarly we denote the odd periodic cyclic cohomology by $H_{\lambda}^{odd}(\mathcal{A})$.

2.3.1 Fredholm modules and the Chern character

The 'dual' of K-theory called the K-homology theory is made up of Fredholm modules.

Definition 2.3.3. *Let A be a unital C^* algebra. An* **even Fredholm module** *is a triple $(\pi, \mathcal{H}, F)$ where*

- *the vector space $\mathcal{H}$ is a $\mathbb{Z}_2$ graded Hilbert space with a grading γ,*
- *the map π is a unital $*$ representation of A on $\mathcal{H}$ such that $\gamma\pi(a)\gamma = \pi(a)$ for every $a \in A$,*
- *the operator F is a selfadjoint unitary which anticommutes with γ, and*
- *the commutator $[F, \pi(a)]$ is compact for every $a \in A$.*

An **odd Fredholm module** *for A is a triple $(\pi, \mathcal{H}, F)$ where π is a unital $*$ representation of A on $\mathcal{H}$ and F is a selfadjoint unitary such that $[F, \pi(a)]$ is compact for every $a \in A$.*

If $(\pi, \mathcal{H}, F)$ is an odd Fredholm module for A then $(\pi_n := \pi \otimes 1, \mathcal{H}_n := \mathcal{H} \otimes \mathbb{C}^n, F_n := F \otimes 1)$ is an odd Fredholm module for $M_n(A)$. A similar statement holds for even Fredolm modules. We will simply write a for $\pi_n(a)$ if $a \in M_n(A)$. Let $(\pi, \mathcal{H}, F)$ be an odd Fredholm module for a unital C^* algebra A. We denote the projection $\frac{(1+F_n)}{2}$ by P_n. If u is a unitary in $M_n(A)$ then $P_n u P_n : P_n\mathcal{H}_n \to P_n\mathcal{H}_n$ is Fredholm and hence has an index. This gives rise to a well defined map $Ind_F : K_1(A) \to \mathbb{Z}$ defined by

$$Ind_F([u]) := Index(P_n u P_n)$$

Let $(\pi, \mathcal{H}, F, \gamma)$ be an even Fredholm module and we let $\gamma_n = \gamma \otimes 1$. We denote the eigen spaces $Ker(\gamma_n - 1)$ and $Ker(\gamma_n + 1)$ by $\mathcal{H}_n^+$ and $\mathcal{H}_n^-$ respectively. If p is a projection in $M_n(A)$ then $pF_np : p\mathcal{H}_n^+ \to p\mathcal{H}_n^-$ is Fredholm and has an index. Thus we obtain a map $Ind_F : K_0(A) \to \mathbb{Z}$ defined by

$$Ind_F([p]) := Index(pF_np).$$

The map Ind_F is called the analytical index and in [14], Connes explains this pairing as the cyclic cohomology/K-theory pairing. Since we will be considering only odd Fredholm modules and odd spectral triples, we will explain only the odd case.

Definition 2.3.4. *An odd Fredholm module $(\pi, \mathcal{H}, F)$ for a C^* algebra is said to p summable if the $*$algebra $\mathcal{A}^\infty := \{a \in A : [F, \pi(a)] \in \mathcal{L}^p(\mathcal{H})\}$ is dense in A where $\mathcal{L}^p(\mathcal{H})$ denotes the pth Schatten ideal.*

Let (π, H, F) be a Fredholm module for a C^* algebra A and let $\mathcal{A}^\infty$ be as in Definition 2.3.4. For $a \in \mathcal{A}^\infty$ let $||a|| := ||\pi(a)||_{op} + ||[F, \pi(a)]||_p$ where $||.||_{op}$ denotes the operator norm on $\mathcal{L}(\mathcal{H})$ and $||.||_p$ denotes the norm on $\mathcal{L}^p(\mathcal{H})$. Then $(\mathcal{A}^\infty, ||.||)$ is a dense Frećhet algebra and if $a \in \mathcal{A}^\infty$ is invertible in A then $a^{-1} \in \mathcal{A}^\infty$. Such an algebra is called a smooth subalgebra and in the Appendix, some results related with smooth subalgebras of C^* algebras are reviewed. In particular if $\mathcal{A}^\infty$ is a smooth subalgebra of a C^* algebra then their K-groups coincide. If $(\pi, \mathcal{H}, F)$ is a p summable Fredholm module for A with p odd then its Chern character is the $p+1$ linear functional Ch_F on $\mathcal{A}^\infty$ defined by the formula

$$Ch_F^p(a_0, a_1, \cdots, a_p) := \frac{\Gamma(\frac{p}{2}+1)}{2p!} Trace(\pi(a_0)[F, \pi(a_1)][F, \pi(a_2)] \cdots [F, \pi(a_p)]).$$

Then Ch_F is a p cocycle. If $(\pi, \mathcal{H}, F)$ is a p summable Fredholm module then $(\pi, \mathcal{H}, F)$ is also $p+2$ summable. But it is shown in [14] that $S(Ch_F^p) = Ch_F^{p+2}$. Thus (Ch_F^p) gives a well defined element in $H^*(\mathcal{A}^\infty)$ which we denote by Ch_F.

Next we explain the cyclic cohomology/K-theory pairing. The usual trace $Tr : M_n(\mathbb{C}) \to \mathbb{C}$ is a 0 cocycle. If ϕ is a n cocyle for $\mathcal{A}$ then taking the cup product with Tr gives a n cocycle for $M_n(\mathcal{A})$ which we will denote by ϕ itself. Now let $\mathcal{A}^\infty$ be a smooth subalgebra of a C^* algebra A. Then the pairing $K_1(\mathcal{A}^\infty) \times H_\lambda^{odd}(\mathcal{A}^\infty) \to \mathbb{C}$ defined by

$$< [u], [\tau] > := \tau(u^* - 1, u - 1, u^* - 1, \cdots, u^* - 1, u - 1)$$

is called the cyclic cohomology/K-theory pairing. Since $\mathcal{A}^\infty$ is smooth in A this pairing extends to a pairing between $K_1(A)$ and $H_\lambda^{odd}(\mathcal{A}^\infty)$. Then Connes' index theorem states that if $(\pi, \mathcal{H}, F)$ is a finitely summable Fredholm module then for $[u] \in K_1(A)$

$$Ind_F([u]) = < [u], Ch_F > .$$

Remark 2.3.5. *When we consider the cyclic cohomology of a smooth subalgebra $\mathcal{A}^\infty$ of a C^* algebra A, we consider only the multilinear functionals on $\mathcal{A}^\infty$ that are continuous w.r.t to the topology on $\mathcal{A}^\infty$.*

2.3.2 Regular spectral triples and the local index formula

Let us recall the definition of an odd spectral triple.

Definition 2.3.6. *Let A be a unital C^* algebra. An odd spectral triple for A is a triple $(\pi, \mathcal{H}, D)$ such that*

1. *the map $\pi : A \to \mathcal{L}(\mathcal{H})$ is a unital $*$ representation,*
2. *the operator D is an unbounded selfadjoint operator with compact resolvent, and*
3. *there exists a dense subalgebra $\mathcal{A}$ such that $[D, \pi(a)]$ is bounded for every $a \in \mathcal{A}$.*

We sometime write $(\mathcal{A}, \mathcal{H}, D)$ to denote a spectral triple for a C^* algebra A and suppress the representation π where $\mathcal{A}$ denotes a dense subalgebra for which the commutator $[D, \mathcal{A}]$ is bounded.

Let $\mathcal{H}$ be a Hilbert space and D an unbounded selfadjoint operator on $\mathcal{H}$ with compact resolvent. Let $\mathcal{H}_s$ be the domain of the operator $|D|^s$ for each $s \geq 0$. Then $\mathcal{H}_s$ can be identified with the graph of the operator $|D|^s$ and thus $\mathcal{H}_s$ acquires a Hilbert space structure. We denote the intersection $\cap_s \mathcal{H}_s$ by $\mathcal{H}_\infty$. Note that $\mathcal{H}_\infty$ is infact a core for the operators $|D|^s$ for every $s \geq 0$.

An operator $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is said to be have analytic order r if T extends to a bounded operator from $\mathcal{H}_{s+r}$ to $\mathcal{H}_s$ for every $s, s + r \geq 0$.

Definition 2.3.7. An operator $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is said to be smoothing if for every $m, n \geq 0$ the operator $|D|^m T |D|^n$ is bounded. The vector space of smoothing operators is denoted by $OP^{-\infty}$.

For $T \in OP^{-\infty}$, define $\|T\|_{m,n} = \||D|^m T |D|^n\|$ for $m, n \geq 0$.

Lemma 2.3.8. *The vector space $OP^{-\infty}$ is an involutive subalgebra of $\mathcal{L}(\mathcal{H})$ and equipped with the family of seminorms $\| \cdot \|_{m,n}$, it is a Fréchet algebra.*

Let δ be the unbounded derivation $[|D|, \cdot]$. More precisely, $Dom(\delta)$ consists of all bounded operators T which leaves $Dom(|D|)$ invariant and for which $\delta(T) := [|D|, T]$ extends to a bounded operator. The proofs of the next two lemmas are taken from [1]. We repeat it for our convenience.

Lemma 2.3.9. *The unbounded derivation δ is a closed derivation i.e. if T_n is a sequence in $Dom(\delta)$ such that $T_n \to T$ and $\delta(T_n) \to S$ then $T \in Dom(\delta)$ and $\delta(T) = S$.*

Proof. Let T_n be a sequence in $Dom(\delta)$ such that T_n converges to T and $\delta(T_n)$ converges to S. Consider a vector $\xi \in Dom(|D|)$. Now note that $T_n\xi \to T\xi$ and $|D|T_n\xi = \delta(T_n)\xi + T_n|D|\xi$ which converges to $S\xi + T|D|\xi$. Since $|D|$ is a closed operator, it follows that $T\xi \in Dom(|D|)$

and $|D|T\xi = S\xi + T|D|\xi$. Thus T leaves $|D|$ invariant and $\delta(T) = S$. This completes the proof. $\Box$

Define $OP^0 := \{T \in \mathcal{L}(\mathcal{H}) : T \in \cap_n Dom(\delta^n)\}$. The following lemma says that elements of OP^0 are operators on $\mathcal{H}_\infty$.

Lemma 2.3.10. *Let T be a bounded operator on $\mathcal{H}$. Then the following are equivalent.*

(1) The operator $T \in OP^0$.

(2) The operator T leaves $\mathcal{H}_\infty$ invariant and $\delta^n(T) : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is bounded for every $n \in \mathbb{N}$.

Proof. First lets prove (2) implies (1). Let T be a bounded operator which leaves $\mathcal{H}_\infty$ invariant and $\delta^n(T)$ is bounded for every $n \geq 0$. Let ξ be a vector in $Dom(|D|)$. Since $\mathcal{H}_\infty$ is a core for $Dom(|D|)$, it follows that there exists a sequence $\xi_n \in \mathcal{H}_\infty$ such that $\xi_n \to \xi$ and $|D|\xi_n \to |D|\xi$. On $\mathcal{H}_\infty$, one has $|D|T = \delta(T) + T|D|$. Hence $T\xi_n$ converges to $T\xi$ and $|D|T\xi_n$ converges to $\delta(T)\xi + T|D|\xi$. Since $|D|$ is closed, it follows that $T\xi \in Dom(|D|)$. As $\delta(T) : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is bounded, it follows that $T \in Dom(\delta)$. The same proof applied recursively to $\delta(T)$,$\delta^2(T)\cdots$ shows that $T \in \cap_n Dom(\delta^n) = OP^0$.

To prove (1) implies (2), we first prove the following claim.

Claim: For every $m \geq 1$, if $T \in OP^0$ and $\xi \in \mathcal{H}_\infty$ then $T\xi \in Dom(|D|^m)$ and

$$|D|^m T\xi := \sum_{k=0}^{m} \binom{m}{k} \delta^k(T)|D|^{m-k}\xi.$$

The proof is by induction on m. Let $T \in OP^0$ and $\xi \in \mathcal{H}_\infty$. Since $T \in Dom(\delta)$, by definition, it follows that $T\xi \in Dom(|D|)$ as $\xi \in Dom(|D|)$ and the equation $|D|T\xi = \delta(T)\xi + T|D|\xi$ is just the definition.

Now assume the claim for $k \leq m$. Let $T \in OP^0$ and $\xi \in \mathcal{H}_\infty$. By assumption $T\xi \in Dom(|D|^m)$ and

$$|D|^m T\xi := \sum_{k=0}^{m} \binom{m}{k} \delta^k(T)|D|^{m-k}\xi.$$

As $\delta^k(T)$ leaves $Dom(|D|)$ invariant, it follows that each term $\delta^k(T)|D|^{m-k}\xi$ is in $Dom(|D|)$ and hence $|D|^m T\xi \in Dom(|D|)$. Hence $T\xi \in Dom(|D|^{m+1})$. Now

$$\begin{aligned}
|D|^{m+1}T\xi &= |D|^m\delta(T)\xi + |D|^m T|D|\xi \\
&= \sum_{k=0}^{m} \binom{m}{k} \delta^{k+1}(T)|D|^{m-k}\xi + \sum_{k=0}^{m} \binom{m}{k} \delta^k |D|^{m+1-k}\xi \\
&= (\sum_{j=1}^{m} (\binom{m}{j-1} + \binom{m}{j}) \delta^j |D|^{m+1-j}\xi) + T|D|^{m+1}\xi + \delta^{m+1}(T)\xi \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} \delta^j(T)|D|^{m+1-j}\xi.
\end{aligned}$$

This completes the proof of the claim and hence the proof. □

It is easy to see from lemma 2.3.10 that OP^0 is a $*$ algebra and that $OP^{-\infty}$ is an ideal in OP^0. To see this, let $T \in OP^0$ and $S \in OP^{-\infty}$. Now $|D|^m TS|D|^n = \sum_{k=0}^{m} \binom{m}{k} \delta^k(T)|D|^{m-k}S|D|^n$. Since for every k, $|D|^{m-k}S|D|^n$ is bounded and hence $|D|^m TS|D|^n$ is bounded for every $m, n \geq 0$. Similarly one can show that $ST \in OP^{-\infty}$.

Definition 2.3.11. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. We say that $(\mathcal{A}, \mathcal{H}, D)$ is **regular** if $\mathcal{A} + [D, \mathcal{A}] \subset OP^0$.

Let D be a selfadjoint operator with compact resolvent. Then $|D|$ has finite dimensional kernel and let P be the projection onto the kernel of $|D|$. Let $D' = D + P$. Then D' is invertible. We denote $|D'|^{-z}$ for $Re(z) > 0$ simply by $|D|^{-z}$.

Definition 2.3.12. *A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be p+summable if $|D|^{-s}$ is trace class for $Re(s) > p$. A spectral triple is said to be finitely summable if it is p+summable for some $p > 0$.*

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple which is p+summable for some p. Let $\mathcal{B}$ be the algebra generated by $\delta^n(\mathcal{A})$ and $\delta^n([D, \mathcal{A}])$. We say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has **discrete dimension spectrum** $\Sigma \subset \mathbb{C}$ if Σ is discrete and for every $b \in \mathcal{B}$, the function $Trace(b|D|^{-z})$ initially defined for $Re(z) > p$ extends to a meromorphic function with poles only in Σ. We say the dimension spectrum is simple if all the poles are simple.

Proposition 2.3.13. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular p+ summable spectral triple and let $F := sign(D)$. Then the triple $(\mathcal{A}, \mathcal{H}, F)$ is a $p + 1$ summable Fredholm module.*

Proof. We denote the orthogonal projection onto the kernel of D by P and let $D' := D + P$. Then $(\mathcal{A}, \mathcal{H}, D')$ is a regular p+ summable spectral triple and $sign(D') = sign(D)$. Thus, without loss of generality, we can assume that $(\mathcal{A}, \mathcal{H}, D)$ is a regular p+ summable spectral triple with D invertible. Let $a \in \mathcal{A}$. On $\mathcal{H}_\infty$, we have

$$\begin{aligned}
[F, a] &= [D|D|^{-1}, a] \\
&= [D, a]|D|^{-1} + D[|D|^{-1}, a] \\
&= [D, a]|D|^{-1} - D|D|^{-1}[|D|, a]|D|^{-1} \\
&= [D, a]|D|^{-1} - F\delta(a)|D|^{-1}.
\end{aligned}$$

Since $[D, a]$ and $\delta(a)$ are bounded and $|D|^{-1}$ is in the $(p+1)^{th}$ Schatten ideal, it follows that $[F, a]$ is in the $(p+1)^{th}$ Schatten class. This completes the proof. □

2.3.3 Topological tensor products

Apart from C^*-algebras and their tensor products, we will also deal with Fréchet algebras and their tensor products. Suppose A_1 and A_2 are two Fréchet algebras with topologies coming from the families of seminorms $(\|\cdot\|_\lambda)_{\lambda\in\Lambda}$ and $(\|\cdot\|_{\lambda'})_{\lambda'\in\Lambda'}$. For each pair $(\lambda,\lambda')\in\Lambda\times\Lambda'$, one forms the projective cross norm $\|\cdot\|_{\lambda,\lambda'}$ which is a seminorm on the algebraic tensor product $A_1\otimes_{alg}A_2$. The family $(\|\cdot\|_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda\times\Lambda'}$ then gives rise to a topology on $A_1\otimes_{alg}A_2$. Completion with respect to this is a Fréchet algebra and is called the projective tensor product of A_1 and A_2. Let us recall the definition of the projective cross norm. If p_1 is seminorm on A_1 and p_2 is a seminorm on A_2 then the projective cross norm $p_1\otimes p_2$ on $A_1\otimes_{alg}A_2$ is defined as

$$(p_1\otimes p_2)(x) := \inf\{\sum_i p_1(x_i)p_2(y_i) : x=\sum_i x_i\otimes y_i\}.$$

While talking about tensor product of two Fréchet algebras, we will always mean their projective tensor product and will denote it by $A_1\otimes A_2$.

We will mainly be concerned with Fréchet algebras sitting inside some $\mathcal{L}(\mathcal{H})$ with Fréchet topology finer than the norm topology. In other words, we will be dealing with Fréchet algebras with faithful continuous representations on Hilbert spaces. Let A_1, A_2 be Fréchet algebras. If $\rho_i : A_i\to\mathcal{L}(\mathcal{H}_i)$ are continuous representations for $i=0,1$ where $\mathcal{H}_i$'s are Hilbert spaces, then by the universality of the projective tensor product it follows that there exists a unique continuous representation $\rho_1\otimes\rho_2 : A_1\otimes A_2\to\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ such that $(\rho_1\otimes\rho_2)(a_1\otimes a_2)=\rho_1(a_1)\otimes\rho_2(a_2)$. If A_i's are subalgebras of $\mathcal{L}(\mathcal{H}_i)$ then we will call the tensor product representation of $A_1\otimes A_2$ on $\mathcal{H}_1\otimes\mathcal{H}_2$ as the natural representation.

Lemma 2.3.14. *Let $(A_1,\mathcal{H}_1,D_1)$ and $(A_2,\mathcal{H}_2,D_2)$ be regular spectral triples. Assume that the following conditions hold.*

1. *The algebras A_1 and A_2 are Fréchet algebras represented faithfully on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.*
2. *The selfadjoint operators D_1 and D_2 are positive with compact resolvent.*
3. *For $i=0,1$, the unbounded derivations $\delta_i=[D_i,.]$ leave A_i invariant and $\delta_i : A_i\to A_i$ is continuous.*

Let $D := D_1\otimes 1 + 1\otimes D_2$. Suppose that the natural representation of $A_1\otimes A_2$ on $\mathcal{H}_1\otimes\mathcal{H}_2$ is faithful. Then the triple $(A_1\otimes A_2,\mathcal{H}_1\otimes\mathcal{H}_2,D)$ is a regular spectral triple. More precisely the unbounded derivation $\delta := [D,.]$ leaves the algebra $A_1\otimes A_2$ invariant and the map $\delta : A_1\otimes A_2\to A_1\otimes A_2$ is continuous.

Proof: Let $\delta'=\delta_1\otimes 1+1\otimes\delta_2$. Then δ' is a continuous linear operator on $A_1\otimes A_2$. Clearly $A_1\otimes_{alg}A_2\subset Dom(\delta)$ and $\delta=\delta'$ on $A_1\otimes_{alg}A_2$. Now let $a\in A_1\otimes A_2$ be given. Choose a sequence $(a_n)\in A_1\otimes_{alg}A_2$ such that $a_n\to a$ in $A_1\otimes A_2$. Then $a_n\to a$ in $\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$. Since δ' is continuous and because the inclusion $A_1\otimes A_2\subset\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ is continuous, it follows that the sequence $(\delta'(a_n))=(\delta(a_n))$ is cauchy in $\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$. As δ is closed, it follows that $a\in Dom(\delta)$ and $\delta(a)=\delta'(a)$. Now the proposition follows. □

The above lemma can be extended to tensor product of finite number of spectral triples with the appropriate assumptions.

2.3.4 The local index formula

Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable spectral triple for a unital C^* algebra A. Then the Fredholm module $(A, \mathcal{H}, F)$ is finitely summable where $F := sign(D)$. The index map Ind_F can be computed via the Chern character of F. But in geometric examples, this Chern character is often difficult to compute and one needs an alternate formula which computes the index completely in terms of D. This is acheived by Connes and Moscovici in [12]. The formula they obtain is called the local index formula.

First let us fix some notations. Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular $p+$ summable spectral triple with simple and discrete dimension spectrum. Let $\mathcal{B}$ be the algebra generated by $\delta^n(\mathcal{A})$ and $\delta^n([D, \mathcal{A}])$ in $\mathcal{L}(\mathcal{H})$. For $b \in \mathcal{B}$, we let $b^{(1)} := [D^2, b]$ and $b^{(k)} := [D^2, b^{(k-1)}]$. We denote the algebra generated by $\mathcal{B}$ and $|D|^k, k \in \mathbb{N}$ by $\mathcal{D}$. For $b \in \mathcal{D}$, let $\int b := Res_{z=0} Tr(b|D|^{-z})$.

For n odd and a multiindex $k = (k_1, k_2, \cdots, k_n)$, consider the $n+1$ multilinear functional $\phi_{n,k}$ on $\mathcal{A}$ defined as

$$\phi_{n,k}(a_0, a_1, \cdots, a_n) := \int a_0[D, a_1]^{(k_1)}[D, a_2]^{(k_1)} \cdots [D, a_n]^{(k_n)}|D|^{-n-2|k|}$$

where $|k| := \sum_{i=1}^n k_i$. Note that if $|k| + n > p$ then $\phi_{n,k} = 0$. We let $\phi_n := \sum_k c_{n,k}\phi_{n,k}$ where the constants $c_{n,k}$ are given by

$$c_{n,k} := (-1)^{|k|}\sqrt{2i}\frac{\Gamma(|k| + \frac{n}{2})}{\prod k_j! \prod(k_1 + k_2 + \cdots + k_j + j)}.$$

Theorem 2.3.15 (Connes-Moscovici)**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular, finitely summable spectral triple with discrete and simple dimension spectrum. Then*

1. *The sequence $\phi := (\phi_1, \phi_3, \phi_5, \cdots)$ is a (b, B) cocycle.*

2. *The cocycle ϕ is cohomologous to the Chern character Ch_F of the Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ where $F := sign(D)$.*

2.4 Compact Quantum groups

In this section, we recall the definition of quantum groups defined by Wornowicz in [43]. The C^* algebras that we consider are nuclear and so no problem arises with the tensor product.

Definition 2.4.1. *A compact quantum group is a pair (A, Δ) where A is a unital C^* algebra and $\Delta : A \to A \otimes A$ is a unital C^* algebra homomorphism such that*

1. *The homomorphism Δ is coassociative i.e. $(1\otimes\Delta)\Delta = (\Delta\otimes 1)\Delta$.*

2. *The span of $\{(a\otimes 1)\Delta(b) : a, b \in A\}$ and $\{(1\otimes a)\Delta(b) : a, b \in A\}$ are dense in $A\otimes A$.*

We write $C(G)$ for the algebra A and call G the compact quantum group.

If G is a compact group then $C(G)$, the algebra of continuous complex valued functions on G, is an example of a compact quantum group. The comultiplication $\Delta : C(G) \to C(G)\otimes C(G)$ is defined as $\Delta(f)(x,y) = f(xy)$. In fact, any compact quantum group G for which $C(G)$ is commutative arises this way.

The following examples of compact quantum groups due to Wornowicz and its homogeneous spaces will occupy the major portion of this thesis. We let $q \in (0,1)$.

Example 2.4.2. *The C^* algebra $C(SU_q(2))$ is defined as the universal unital C^* algebra generated by two elements α and β such that*

1. *$\alpha\beta = q\beta\alpha$, $\alpha\beta^* = q\beta^*\alpha$,*

2. *The matrix $\begin{bmatrix} \alpha & -q\beta \\ \beta^* & \alpha^* \end{bmatrix}$ is unitary, and*

3. *The element β is normal.*

The C^* algebra $C(SU_q(2))$ has a quantum group structure. The comultiplication Δ is given by

$$\Delta(\alpha) := \alpha\otimes\alpha - q\beta\otimes\beta^*,$$
$$\Delta(\beta) := \beta\otimes\alpha^* + \alpha\otimes\beta.$$

Example 2.4.3. *Let $n \geq 3$. The C^* algebra $C(SU_q(n))$ is the universal C^* algebra generated by n^2 elements (u_{ij}) satisfying the following relations*

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij} \quad , \quad \sum_{k=1}^{n} u_{ki}^*u_{kj} = \delta_{ij} \tag{2.4.1}$$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2\cdots i_n} u_{j_1 i_1}\cdots u_{j_n i_n} = E_{j_1 j_2\cdots j_n} \tag{2.4.2}$$

where

$$E_{i_1 i_2\cdots i_n} \quad := \quad \begin{cases} 0 & if \quad i_1, i_2, \cdots i_n \text{ are not distinct} \\ (-q)^{\ell(i_1,i_2,\cdots,i_n)} & \end{cases}$$

where for a permutation σ on $\{1,2,\cdots,n\}$ $\ell(\sigma)$ denotes the number of inversions i.e. the cardinality of the set $\{(i,j) : i < j,\ \sigma(i) > \sigma(j)\}$. The C^* algebra $C(SU_q(n))$ has a compact quantum group structure with the comultiplication Δ given by

$$\Delta(u_{ij}) := \sum_k u_{ik}\otimes u_{kj}.$$

A subgroup of a compact quantum group (G, Δ_G) is a compact quantum group (H, Δ_H) together with a surjection $\phi : C(G) \to C(H)$ such that $\Delta_H \circ \phi = (\phi \otimes \phi)\Delta_G$. If H is a subgroup of G then one defines the right quotient G/H by

$$C(G/H) := \{a \in C(G) : \quad a \in C(G) : \ (\phi \otimes 1)\Delta_G(a) = 1 \otimes a\}.$$

Similarly the left quotient is defined by

$$C(H\backslash G) := \{a \in C(G) : \quad a \in C(G) : \ (1 \otimes \phi)\Delta_G(a) = a \otimes 1\}.$$

In this thesis, only right quotients are considered.

Example 2.4.4. *Let $1 \leq m \leq n-1$. Call the generators of $C(SU_q(n-m))$ as v_{ij}. The map $\phi : C(SU_q(n)) \to C(SU_q(n-m))$ defined by*

$$\phi(u_{ij}) \quad := \quad \begin{cases} v_{i-m,j-m} & \textit{if } m+1 \leq i,j \leq n, \\ \delta_{ij} & \textit{otherwise} \end{cases} \tag{2.4.3}$$

is a surjective unital C^* algebra homomorphism such that $\Delta \circ \phi = (\phi \otimes \phi)\Delta$. In this way the quantum group $SU_q(n-m)$ is a subgroup of the quantum group $SU_q(n)$.

The C^* algebra of the quotient $SU_q(n)/SU_q(n-m)$ is denoted as $C(S_q^{n,m})$. The C^* algebra $C(S_q^{n,1})$ is denoted by $C(S_q^{2n-1})$. The C^* algebras $C(S_q^{n,m})$ are called the quantum Steiffel manifolds and the C^* algebras $C(S_q^{2n-1})$ are called the odd dimensional quantum spheres. In [31], it was proved that the C^* algebra $C(S_q^{n,m})$ is generated by the first m rows of the matrix (u_{ij}) of $C(SU_q(n))$. Infact, the algebra $C(S_q^{n,m})$ is given by a presentation. In particular the C^* algebra of the odd dimensional quantum sphere S_q^{2n+1} is the universal unital C^* algebra generated by $z_1, z_2, \cdots, z_{n+1}$ satisfying the following relations

$$\begin{aligned} z_i z_j &= q z_j z_i, && 1 \leq j < i \leq n+1, \\ z_i^* z_j &= q z_j z_i^*, && 1 \leq i \neq j \leq n+1, \\ z_i z_i^* - z_i^* z_i + (1-q^2)\sum_{k>i} z_k z_k^* &= 0, && 1 \leq i \leq n+1, \\ \sum_{i=1}^{n+1} z_i z_i^* &= 1. \end{aligned}$$

The map $z_i \to q^{-i+1} u_{i1}^*$ gives the desired isomorphism.

Chapter 3

The torus equivariant spectral triple

For odd dimensional quantum spheres two families of spectral triples are studied. One is equivariant with the natural torus action and the other is equivariant with the quantum group action of $SU_q(\ell+1)$. In this chapter, we analyse the spectral triple equivariant under the torus action. We introduce a smooth subalgebra of the C^* algebra of odd dimensional quantum spheres. We prove that the torus equivariant spectral triple is regular and has discrete dimension spectrum. This computation forms the base case for the computation of the dimension spectrum of the equivariant spectral triple.

3.1 Equivariant spectral triples

Let us recall a few basic definitions.

Definition 3.1.1. *Let A be a unital C^* algebra and G be a compact quantum group. An action of G on A is a unital homomorphism $\tau : A \to A \otimes C(G)$ such that $(1 \otimes \Delta_G)\tau = (\tau \otimes 1)\tau$. We call the triple (A, G, τ) a C^* dynamical system.*

If G is a compact quantum group then G acts on $C(G)$ by the comultiplication. If G is a compact quantum group and H a subgroup then G acts on the quotient $C(G/H)$ by the comultiplication Δ_G.

A representation of a compact quantum group G on a Hilbert space $\mathcal{H}$ is a unitary element u in the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C(G))$ such that $(id \otimes \Delta)(u) = u_{12}u_{13}$. Here $\mathcal{K}(\mathcal{H})$ denotes the C^* algebra of compact operators on $\mathcal{H}$. A covariant representation of a C^* dynamical system $(\mathcal{A}, G, \tau)$ consists of a pair (π, u) where π is a representation of the C^* algebra A on a Hilbert space $\mathcal{H}$, u is a unitary representation of G on $\mathcal{H}$ and they obey the condition

$$u(\pi(a) \otimes 1)u^* = (\pi \otimes id)\tau(a) \quad a \in A.$$

We need one more definition that of an equivariant spectral triple.

Definition 3.1.2. *Let $(\mathcal{A}, G, \tau)$ be a C^* dynamical system. An **odd G equivariant spectral triple** is a quadruple $(\pi, u, D, \mathcal{H})$ such that*

1. *The pair (π, u) is a covariant representation of the dynamical system (A, G, τ) on the Hilbert space $\mathcal{H}$.*

2. *There exists a dense unital *subalgebra $\mathcal{A} \subset A$ such that the triple $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple.*

3. *The operator D is G equivariant i.e. $u(D \otimes 1)u^* = D$.*

One of the first examples of equivariant spectral triples on quantum groups is constructed in [5]. Let G be a compact quantum group and h be the Haar state on G. The C^* dynamical system $(C(G), G, \Delta)$ has a natural covariant representation on the GNS space $L^2(h)$. Now the problem is to construct equivariant spectral triples for the dynamical system $(C(G), G, \Delta)$ on $L^2(h)$. This question is studied in [5] for $G = SU_q(2)$ and a non-trivial one is constructed. To illustrate the idea behind the construction, let us explain the analysis involved when $G = \mathbb{T}$.

The C^* algebra $C(\mathbb{T})$ is represented on the Hilbert space $L^2(\mathbb{T})$ by multiplication operators. The algebra $C(\mathbb{T})$ is generated by a single unitary z and we denote U to be the unitary on $L^2(\mathbb{T})$. Let $t \to V_t$ be the right regular representation of the group $\mathbb{T}$. W.r.t the standard orthonormal basis $\{e_n\}$, the operators U and V_t are given by

$$\begin{aligned} Ue_n &= e_{n+1}, \\ V_t e_n &= t^{-n} e_n. \end{aligned}$$

If D is an unbounded operator on $L^2(\mathbb{T})$ which is $\mathbb{T}$ equivariant then D has to keep the eigen spaces of V_t invariant and thus D must diagonalise w.r.t the orthonormal basis $\{e_n\}$. Thus $De_n = d_n e_n$ for some sequence (d_n). If $(C(\mathbb{T}), L^2(\mathbb{T}), D)$ is a spectral triple then the fact that the commutator $[D, U]$ is bounded forces one to conclude that $|d_{n+1} - d_n| = O(1)$. Thus $d_n = O(n)$. Taking $d_n = n$ gives the usual spectral triple on the circle which is non-trivial. Let us denote the sign of the number operator N defined by $Ne_n = ne_n$ by F_N.

Proposition 3.1.3. *Let $(C(\mathbb{T}), L^2(\mathbb{T}), D)$ be a $\mathbb{T}$ equivariant spectral triple. If $F := sign(D)$ then upto a compact perturbation F is either ± 1 or $\pm F_N$.*

Proof. Let D be given by $De_n = d_n e_n$. Let M be such that $|d_{n+1} - d_n| \leq 2M$. Let $k \geq 1$ be such that $|d_n| > M$ if $|n| \geq k$. We claim that d_n's are of the same sign if $n \geq k$. Suppose not. Then there exists $\ell \geq k$ such that d_ℓ and $d_{\ell+1}$ are of different signs. Then $|d_{\ell+1} - d_\ell| > 2M$ which is a contradiction. Thus d_n's are of the same sign if $n \geq k$. Similarly one can prove that d_n's are of the same sign if $n \leq -k$. Thus upto a finite perturbation, F is either ± 1 or $\pm F_N$. This completes the proof. □

The analysis carried out in [5], [7] and in [8] are exactly similar in spirit. In this chapter we consider the torus equivariant spectral triple on $S_q^{2\ell+1}$ constructed in [7] and analyse it from the local index formula point of view. In particular, we prove that this spectral triple is regular and has discrete dimension spectrum. This analysis is essential for the computation carried out in the next chapter.

Notations Let us fix some notations. We denote the Hilbert space $\ell^2(\mathbb{N}^\ell \times \mathbb{Z})$ by $\mathcal{H}_\ell$. On both $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$, we denote the left shift by S and is defined by $Se_n = e_{n-1}$. We let $p_0 := 1 - S^*S$.

Let $\mathscr{T}$ be the Toeplitz algebra, i.e. the C^*-subalgebra of $\mathcal{L}(\ell_2(\mathbb{N}))$ generated by S. For a positive integer k, we will denote by $\mathscr{T}_k$ the k-fold tensor product of $\mathscr{T}$, embedded in $\mathcal{L}(\ell_2(\mathbb{N}^k))$. Denote by σ the symbol map from $\mathscr{T}$ to $C(\mathbb{T})$ that sends S^* to $\mathbf{z}$ and all compact operators to 0. Let N be the number operator on $\ell^2(\mathbb{N})$ defined on the orthonormal basis $\{e_n\}$ by $Ne_n := ne_n$.

3.2 The spectral triple

In this section we recall the spectral triple for the odd dimensional quantum spheres given in [7]. We begin with some known facts about odd dimensional quantum spheres. Let $q \in [0, 1)$. The C^*-algebra $C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^*-algebra generated by elements $z_1, z_2, \ldots, z_{\ell+1}$ satisfying the following relations (see [22]):

$$\begin{aligned} z_i z_j &= q z_j z_i, & 1 \le j < i \le \ell+1, \\ z_i^* z_j &= q z_j z_i^*, & 1 \le i \ne j \le \ell+1, \\ z_i z_i^* - z_i^* z_i + (1-q^2)\sum_{k>i} z_k z_k^* &= 0, & 1 \le i \le \ell+1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

We will denote by $\mathcal{A}(S_q^{2\ell+1})$ the *-subalgebra of A_ℓ generated by the z_j's. Note that for $\ell = 0$, the C^*-algebra $C(S_q^{2\ell+1})$ is the algebra of continuous functions $C(\mathbb{T})$ on the torus and for $\ell = 1$, it is $C(SU_q(2))$.

There is a natural torus group $\mathbb{T}^{\ell+1}$ action τ on $C(S_q^{2\ell+1})$ as follows. For $w = (w_1, \ldots, w_{\ell+1}) \in \mathbb{T}^{\ell+1}$, define an automorphism τ_w by $\tau_w(z_i) = w_i z_i$. Let $Y_{k,q}$ be the following operators on $\mathcal{H}_\ell$:

$$Y_{k,q} = \begin{cases} \underbrace{q^N \otimes \ldots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1-q^{2N}} S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-k \text{ copies}}, & \text{if } 1 \le k \le \ell, \\ \\ \underbrace{q^N \otimes \cdots \otimes q^N}_{\ell \text{ copies}} \otimes S^*, & \text{if } k = \ell+1. \end{cases} \tag{3.2.1}$$

Here for $q = 0$, q^N stands for the rank one projection $p_0 = |e_0\rangle\langle e_0|$. Then $\pi_\ell : z_k \mapsto Y_{k,q}$ gives a faithful representation of $C(S_q^{2\ell+1})$ on $\mathcal{H}_\ell$ for $q \in [0, 1)$ (see lemma 4.1 and remark 4.5, [22]). We will denote the image $\pi_\ell(C(S_q^{2\ell+1}))$ by $A_\ell(q)$ or by just A_ℓ.

Let $\{e_\gamma : \gamma \in \Gamma_{\Sigma_\ell}\}$ be the standard orthonormal basis for $\mathcal{H}_\ell$. For $w = (w_1, w_2, \cdots, w_{\ell+1}) \in \mathbb{T}^{\ell+1}$ we define the unitary U_w on $\mathcal{H}_\ell$ by $U_w(e_\gamma) = w_1^{\gamma_1} w_2^{\gamma_2} \ldots w_{\ell+1}^{\gamma_{\ell+1}} e_\gamma$ where $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_{\ell+1}) \in \Gamma_{\Sigma_\ell}$. Then (π_ℓ, U) is a covariant representation of $(C(S_q^{2\ell+1}), \mathbb{T}^{\ell+1}, \tau)$. Note that $A_\ell \subset \mathscr{T}_\ell \otimes C(\mathbb{T})$.

In [7] all spectral triples equivariant with respect to this covariant representation were characterised and an optimal one was constructed. We recall the following theorem from [7].

Theorem 3.2.1 ([7])**.** *Let D_ℓ be the operator $e_\gamma \to d(\gamma)e_\gamma$ on $\mathcal{H}_\ell$ where the d_γ's are given by*

$$d(\gamma) \quad = \quad \begin{cases} \gamma_1 + \gamma_2 + \cdots \gamma_\ell + |\gamma_{\ell+1}| & \text{if } \gamma_{\ell+1} \geq 0, \\ -(\gamma_1 + \gamma_2 + \cdots \gamma_\ell + |\gamma_{\ell+1}|) & \text{if } \gamma_{\ell+1} < 0. \end{cases}$$

Then $(\mathcal{A}(S_q^{2\ell+1}), \mathcal{H}_\ell, D_\ell)$ is a non-trivial $(\ell + 1)$ summable spectral triple. Also D_ℓ commutes with U_w for every $w \in \mathbb{T}^{\ell+1}$.

The operator D_ℓ is optimal i.e. if $(\mathcal{A}(S_q^{2\ell+1}), \mathcal{H}_\ell, D)$ is a spectral triple such that D commutes with U_w for every w, then there exist positive reals a and b such that $|D| \leq a + b|D_\ell|$.

In this section, we will introduce a dense subalgebra $\mathcal{A}_\ell^\infty$ of $A_\ell(q)$ closed under its holomorphic function calculus and establish regularity of the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$. We will also compute its dimension spectrum.

3.3 The smooth function algebra $\mathcal{A}_\ell^\infty$

In this section, we associate a dense Fréchet $*$-subalgebra of $A_\ell(q) = \pi_\ell(C(S_q^{2\ell+1}))$ which is closed under holomorphic functional calculus. We will first show that the C^*-algebra $A_\ell(q)$ is independent of q.

Lemma 3.3.1. *For any $q \in (0, 1)$, one has $A_\ell(0) = A_\ell(q)$.*

Proof: Let us first show that $A_\ell(q) \subseteq A_\ell(0)$. We denote the generators $Y_{j,q}$ of $A_\ell(q)$ by $Y_{j,q}^{\ell+1}$. The inclusion is trivial for $\ell = 0$. Note that for $j \geq 1$, $Y_{j+1,0}^{\ell+1} = p_0 \otimes Y_{j,0}^\ell$. Hence by the induction hypothesis, it follows that $p_0 \otimes Y_{j,q}^\ell \in A_\ell(0)$. Now note that for $j \geq 1$

$$Y_{j+1,q}^{\ell+1} = \sum_{n \in \mathbb{N}} q^n (Y_{1,0}^{\ell+1})^n (p_0 \otimes Y_{j,q}^\ell)(Y_{1,0}^{\ell+1})^{*n}.$$

Hence $Y_{j,q}^{\ell+1} \in A_\ell(0)$ for $j \geq 2$. Observe that $p_0 \otimes 1 = Y_{2,0}^* Y_{2,0}$ and

$$q^N \otimes 1 = \sum_n q^n (Y_{1,0}^{\ell+1})^n (p_0 \otimes 1)(Y_{1,0}^{\ell+1})^{*n}.$$

Thus $q^N \otimes 1 \in A_\ell(0)$. As $Y_{1,q}^{\ell+1} := (\sqrt{1-q^{2N}} \otimes 1)Y_{1,0}^{\ell+1}$, it follows that $Y_{1,q}^{\ell+1} \in A_\ell(0)$.

For the other inclusion, we will use the following fact: if B denotes the C^*-subalgebra of $\mathcal{L}(\ell_2(\mathbb{N}))$ generated by the operator $X = (1-q^{2N})^{\frac{1}{2}}S^*$, then B contains the shift operator S. This is because the operator $|X|$ is invertible and $S^* = X|X|^{-1}$. Using this fact for the first copy of $\ell_2(\mathbb{N})$, since $Y_{1,q} \in A_\ell(q)$, one gets $Y_{1,0} \in A_\ell(q)$. Next assume that $Y_{i,0} \in A_\ell(q)$ for $1 \le i \le j-1$, where $2 \le j \le \ell$. Then $P_{j-1} := I - \sum_{k=1}^{j-1} Y_{k,0}Y_{k,0}^* \in A_\ell(q)$. Observe that

$$P_{j-1}Y_{j,q} = \underbrace{p_0 \otimes \cdots \otimes p_0}_{j-1} \otimes X \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-j}, \qquad Y_{j,0} = \underbrace{p_0 \otimes \cdots \otimes p_0}_{j-1} \otimes S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-j}.$$

Therefore using the above fact for the $\mathrm{j^{th}}$ copy of $\ell_2(\mathbb{N})$, we get $Y_{j,0} \in A_\ell(q)$. Finally, since $Y_{\ell+1,0} = Y_{\ell+1,q}(I - \sum_{k=1}^{\ell} Y_{k,0}Y_{k,0}^*)$, one has $Y_{\ell+1,0} \in A_\ell(q)$. □

Let us write α_i for $Y_{i,0}^*$. Note that the C^*-subalgebra of A_ℓ generated by $\alpha_2, \cdots, \alpha_{\ell+1}$ is isomorphic to $A_{\ell-1}$ where the map $a \mapsto p_0 \otimes a$ gives the isomorphism. We define the Fréchet subalgebras $\mathcal{A}_\ell^\infty$ inductively as follows.

The algebra

$$\mathcal{A}_0^\infty := \left\{ \sum_{n\in\mathbb{Z}} a_n \mathbf{z}^n : (a_n) \text{ is rapidly decreasing} \right\}$$

is the algebra of smooth functions on $\mathbb{T}$ together with the increasing family of seminorms $\|\cdot\|_p$ given by $\|(a_n)\|_p = \sum (1+|n|)^p |a_n|$. Then $\mathcal{A}_0^\infty$ is a dense $*$ Fréchet subalgebra of $A_0 = C(\mathbb{T})$. Note that $\|a\| \le \|a\|_0$ for $a \in \mathcal{A}_0^\infty$. Now assume that $(\mathcal{A}_{\ell-1}^\infty, \|\cdot\|_m)$ be defined such that

1. the seminorms $\|\cdot\|_m$ are increasing and $(\mathcal{A}_{\ell-1}^\infty, \|\cdot\|_m)$ is a Fréchet algebra,
2. the subalgebra $\mathcal{A}_{\ell-1}^\infty$ is $*$ closed and dense in $A_{\ell-1}$. For every $a \in \mathcal{A}_{\ell-1}^\infty$, one has $\|a^*\|_m = \|a\|_m$,
3. for every $a \in \mathcal{A}_{\ell-1}^\infty$, one has $\|a\| \le \|a\|_0$ where $\|\cdot\|$ denotes the C^* norm of $A_{\ell-1}$.

Now define

$$\begin{aligned} \mathcal{A}_\ell^\infty \quad := \quad & \Bigg\{ \sum_{j,k\in\mathbb{N}} \alpha_1^{*j}(p_0 \otimes a_{jk})\alpha_1^k + \sum_{k\ge 0} \lambda_k \alpha_1^k + \sum_{k>0} \lambda_{-k}\alpha_1^{*k} \ : \ a_{jk} \in \mathcal{A}_{\ell-1}^\infty, \\ & \qquad \sum_{j,k} (1+j+k)^n \|a_{jk}\|_m < \infty, \ (\lambda_k) \text{ is rapidly decreasing} \Bigg\}. \end{aligned} \tag{3.3.2}$$

Let $a := \sum_{j,k} \alpha_1^{*j}(p_0 \otimes a_{jk})\alpha_1^k + \sum_{k\ge 0} \lambda_k \alpha_1^k + \sum_{k>0} \lambda_{-k}\alpha_1^{*k}$ be an element of $\mathcal{A}_\ell^\infty$. Define for $m \in \mathbb{N}$, the seminorms $\|a\|_m$ as follows:

$$\|a\|_m = \max_{r,s\le m} \Big(\sum_{j,k} (1+j+k)^r \|a_{jk}\|_s\Big) + \sum_{k\in\mathbb{Z}} (1+|k|)^m |\lambda_k|.$$

Proposition 3.3.2. *The pair* $(\mathcal{A}_\ell^\infty, \|\cdot\|_m)$ *has the following properties:*

1. *the seminorms* $\|\cdot\|_m$ *are increasing and* $(\mathcal{A}_\ell^\infty, \|\cdot\|_m)$ *is a Fréchet algebra,*
2. *the subalgebra* $\mathcal{A}_\ell^\infty$ *is* $*$ *closed and dense in* A_ℓ*. For every* $a \in \mathcal{A}_\ell^\infty$*, one has* $\|a^*\|_m = \|a\|_m$*,*
3. *for every* $a \in \mathcal{A}_\ell^\infty$*, one has* $\|a\| \leq \|a\|_0$ *where* $\|\cdot\|$ *denotes the* C^* *norm of* A_ℓ*.*

Proof: The proof is by induction on ℓ. Parts (2) and (3) and the fact that the seminorms $\|\cdot\|_m$ are increasing follow from the definition and the induction hypothesis. One verifies directly that $(\mathcal{A}_\ell^\infty, \|\cdot\|_m)$ is a Fréchet algebra using induction and the following relations.

$$\alpha_1\alpha_1^* = 1,$$
$$\alpha_1^{*j}(p_0 \otimes a_{jk})\alpha_1^k\alpha_1^{*r}(p_0 \otimes a_{rs})\alpha_1^s = \delta_{kr}\alpha_1^{*j}(p_0 \otimes a_{jk}a_{rs})\alpha_1^s,$$

$$\alpha_1^{*m}\alpha_1^n = \begin{cases} (\alpha_1^*)^{m-n} - \sum_{k=0}^{n-1}(\alpha_1^*)^{m-n+k}(p_0 \otimes 1)\alpha_1^k & \text{if } m \geq n, \\ \alpha_1^{n-m} - \sum_{k=0}^{m-1}\alpha_1^{*k}(p_0 \otimes 1)\alpha_1^{n-m+k} & \text{if } m < n. \end{cases}$$

□

Denote the generators $z_1, z_2, \cdots z_{\ell+1}$ of $C(S_q^{2\ell+1})$ by $z_1^{(\ell+1)}, z_2^{(\ell+1)}, \cdots, z_{\ell+1}^{(\ell+1)}$. Let $\sigma_\ell : C(S_q^{2\ell+1}) \to C(S_q^{2\ell-1})$ be the homomorphism given by $\sigma_\ell(z_{\ell+1}^{(\ell+1)}) = 0$ and $\sigma_\ell(z_i^{(\ell+1)}) = z_i^{(\ell)}$ for $1 \leq i \leq \ell$. Let us denote by the same symbol σ_ℓ the induced homomorphism from A_ℓ to $A_{\ell-1}$. Observe that if one applies the map σ on the ℓ^{th} copy of $\mathscr{T}$ in $\mathscr{T}_\ell \otimes C(\mathbb{T})$ followed by evaluation at 1 in the $(\ell+1)^{\text{th}}$ copy, then the restriction of the resulting map to A_ℓ is precisely σ_ℓ.

Proposition 3.3.3. *The dense Fréchet* $*$*-subalgebra* $\mathcal{A}_\ell^\infty$ *of* A_ℓ *is closed under holomorphic functional calculus in* A_ℓ*. Moreover, the algebra* $\mathcal{A}_\ell^\infty$ *contains the generators* $Y_{1,q}^{(\ell+1)}, \cdots, Y_{\ell+1,q}^{(\ell+1)}$*.*

Proof: We prove this proposition by induction on ℓ. For $\ell = 0$, by definition $\mathcal{A}_0^\infty = C^\infty(\mathbb{T})$. Hence the proposition is clear in this case. Now assume that the algebra $\mathcal{A}_{\ell-1}^\infty$ is closed under holomorphic functional calculus in $A_{\ell-1}$ and contains $Y_{1,q}^{(\ell)}, \cdots, Y_{\ell,q}^{(\ell)}$. The homomorphism $\sigma_\ell : A_\ell \to A_{\ell-1}$ gives the following exact sequence

$$0 \longrightarrow \mathcal{K}(\ell_2(\mathbb{N}^\ell)) \otimes C(\mathbb{T}) \longrightarrow A_\ell \longrightarrow A_{\ell-1} \longrightarrow 0.$$

One also has at the smooth algebra level the "sub" extension

$$0 \longrightarrow \mathcal{S}(\ell_2(\mathbb{N}^\ell)) \otimes C^\infty(\mathbb{T}) \longrightarrow \mathcal{A}_\ell^\infty \longrightarrow \mathcal{A}_{\ell-1}^\infty \longrightarrow 0.$$

Since $\mathcal{S}(\ell_2(\mathbb{N}^\ell)) \otimes C^\infty(\mathbb{T}) \subset \mathcal{K}(\ell_2(\mathbb{N}^\ell)) \otimes C(\mathbb{T})$ and $\mathcal{A}_{\ell-1}^\infty \subset A_{\ell-1}$ are closed under the respective holomorphic functional calculus, it follows Lemma A.1.4 that $\mathcal{A}_\ell^\infty$ is spectrally invariant in A_ℓ. Since $\|a\| \leq \|a\|_0$ for all $a \in \mathcal{A}_\ell^\infty$, it follows that the Fréchet topology of $\mathcal{A}_\ell^\infty$ is finer than the norm topology. Therefore $\mathcal{A}_\ell^\infty$ is closed under holomorphic functional calculus in A_ℓ. Observe that for $i \geq 2$, we have $Y_{i,q}^{(\ell+1)} = \sum_{n\geq 0} q^n \alpha_1^{*n}(p_0 \otimes Y_{i-1,q}^{(\ell)})\alpha_1^n$. Hence $Y_{i,q}^{(\ell+1)} \in \mathcal{A}_\ell^\infty$ for $i = 2, \cdots, \ell+1$. Also note that $q^N \otimes I = \sum_{n\geq 0} q^n \alpha_1^{*n}(p_0 \otimes 1)\alpha_1$. Since $\mathcal{A}_\ell^\infty$ is closed under holomorphic functional calculus, it follows that $\sqrt{1-q^{2N+2}} \otimes I \in \mathcal{A}_\ell^\infty$. As $Y_{1,q}^{(\ell+1)} = \alpha_1^*(\sqrt{1-q^{2N+2}} \otimes I)$ it follows that $Y_{1,q}^{(\ell+1)} \in \mathcal{A}_\ell^\infty$. This completes the proof. □

Next we proceed to prove that the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ is regular and compute its dimension spectrum. The proof is by induction. We start with the case $\ell = 0$ to start the induction.

3.3.1 The case $\ell = 0$

For $\ell = 0$, the spectral triple $(\mathcal{A}_0^\infty, \mathcal{H}_0, D_0)$ is unitarily equivalent to the spectral triple $(C^\infty(\mathbb{T}), L_2(\mathbb{T}), \frac{1}{i}\frac{d}{d\theta})$. For $f \in C^\infty(\mathbb{T})$, one has $[D_0, f] = \frac{1}{i}f'$. Let (e_k) be the standard orthonormal basis and let p_k be the projection onto e_k. Let $F_0 := sign(D_0)$. Note that $[F_0, \mathbf{z}] = 2p_0\mathbf{z}$ and hence by induction $[F_0, \mathbf{z}^n] = 2\sum_{k=0}^{n-1} p_k \mathbf{z}^n p_{k-n}$ for $n \geq 0$. Thus $[F_0, \mathbf{z}^n]$ is smoothing for $n \geq 0$. Also $\||D_0|^r [F_0, \mathbf{z}^n]|D_0|^s\| \leq 2(1+n)^{r+s+1}$. Since $[F_0, \mathbf{z}^{-|n|}]^* = -[F_0, \mathbf{z}^{|n|}]$, it follows that $[F_0, \mathbf{z}^n] \in OP^{-\infty}$ for every n. Also $\|[F_0, \mathbf{z}^n]\|_{r,s} \leq 2(1+|n|)^{r+s+1}$. Hence we observe that $[F_0, f] \in OP^{-\infty}$ and $\|[F_0, f]\|_{r,s} \leq 2\|f\|_{r+s+1}$. Let δ be the unbounded derivation $[|D_0|, \cdot]$.

Lemma 3.3.4. *Let $\mathcal{B} := \{f_0 + f_1F_0 + R : f_0, f_1 \in C^\infty(\mathbb{T}), R \in OP_{D_0}^{-\infty}\}$. Then*

1. *If $f_0 + f_1F_0$ is smoothing then $f_0 = f_1 = 0$. Hence $\mathcal{B}$ is isomorphic to the direct sum $C^\infty(\mathbb{T}) \oplus C^\infty(\mathbb{T}) \oplus OP_{D_0}^{-\infty}$. We give $\mathcal{B}$ the Fréchet space structure coming from this decomposition. This topology on $\mathcal{B}$ is generated by the seminorms $(\|\cdot\|_m)_{m\in\mathbb{N}}$ which are defined by $\|f_0 + f_1F_0 + R\|_m := \|f_0\|_m + \|f_1\|_m + \sum_{r+s\leq m} \|R\|_{r,s}$.*

2. *The vector space $\mathcal{B}$ is closed under δ and the derivation $[D_0, \cdot]$.*

3. *For every $b \in \mathcal{B}$, $[F_0, b] \in OP^{-\infty}$. Also the map $b \to [F_0, b] \in OP^{-\infty}$ is continuous. The derivations δ and $[D_0, \cdot]$ are continuous.*

4. *The vector space $\mathcal{B}$ is an algebra and contains $C^\infty(\mathbb{T})$.*

Proof: First observe that a bounded operator T on $\ell_2(\mathbb{Z})$ is smoothing if and only if $(\langle Te_m, e_n\rangle)_{m,n}$ is rapidly decreasing. Now suppose that $R := f_0 + f_1F_0$ be smoothing. Fix an integer r. Observe that $\langle R(e_n), e_{r+n}\rangle$ converges to $\hat{f}_0(r) + \hat{f}_1(r)$ as $n \to +\infty$ and converges to $\hat{f}_0(r) - \hat{f}_1(r)$ as $n \to -\infty$. But since R is smoothing it follows that $\hat{f}_0(r) + \hat{f}_1(r) = 0 = \hat{f}_0(r) - \hat{f}_1(r)$. Hence $\hat{f}_0(r) = \hat{f}_1(r) = 0$ for every integer r. Thus $f_0 = f_1 = 0$. This proves part (1).

Parts (2), (3) and (4) follow from the observations that $[D_0, f] = \frac{1}{i}f'$, $[F_0, f] \in OP^{-\infty}$, $\|[F_0, f]\|_{r,s} \leq 2\|f\|_{r+s+1}$ and $\delta(b) = [D_0, b]F_0 + D_0[F_0, b]$. This completes the proof. □

In particular, it follows from parts (2) and (4) of the above lemma that the spectral triple $(\mathcal{A}_0^\infty, \mathcal{H}_0, D_0)$ is regular.

Let $\mathcal{E}$ be the C^*-subalgebra of $\mathcal{L}(\ell_2(\mathbb{Z}))$ generated by $C(\mathbb{T})$ and F_0. Note that the algebra $\mathcal{B}$ plays the role of smooth function subalgebra for the C^*-algebra $\mathcal{E}$. Therefore $\mathcal{E}^\infty$ will stand for the algebra $\mathcal{B}$.

3.4 Regularity and the dimension spectrum

In this subsection we prove regularity and calculate the dimension spectrum for the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$. The proof is by induction on ℓ. Let us denote the derivation $[|D_\ell|, \cdot]$ by δ_ℓ and let F_ℓ stand for the sign of the operator D_ℓ. Observe that $F_\ell = 1^{\otimes \ell} \otimes F_0 = 1 \otimes F_{\ell-1}$.

Proposition 3.4.1. *Let* $\mathcal{B}_\ell := \{A_0 + A_1 F_\ell + R : \; A_0, A_1 \in \mathcal{A}_\ell^\infty, R \in OP^{-\infty}\}$. *Then*

1. *if* $A_0 + A_1 F_\ell$ *is smoothing then* $A_0 = A_1 = 0$. *Hence* $\mathcal{B}_\ell$ *is isomorphic to the direct sum* $\mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus OP^{-\infty}$. *Equip* $\mathcal{B}_\ell$ *with the Fréchet space structure coming from this decomposition. This topology on* $\mathcal{B}_\ell$ *is induced by the seminorms* $(\|\cdot\|_m)_{m\in\mathbb{N}}$ *which are defined by* $\|A_0 + A_1 F_\ell + R\|_m := \|A_0\|_m + \|A_1\|_m + \sum_{r+s\leq m} \|R\|_{r,s}$.

2. *For every* $b \in \mathcal{B}_\ell$, $[F_\ell, b] \in OP^{-\infty}$. *Also the map* $b \to [F_\ell, b] \in OP^{-\infty}$ *is continuous.*

3. *The vector space* $\mathcal{B}_\ell$ *is closed under the derivations* δ_ℓ *and* $[D_\ell, \cdot]$. *Moreover the derivations* δ_ℓ *and* $[D_\ell, \cdot]$ *are continuous.*

4. *The vector space* $\mathcal{B}_\ell$ *is an algebra and contains* $\mathcal{A}_\ell^\infty$.

Proof: The proof is by induction on ℓ. For $\ell = 0$, the proposition is just lemma 3.3.4. Now assume that the proposition is true for $\ell - 1$. Suppose that $A_0 + A_1 F_\ell$ is smoothing for some $A_0, A_1 \in \mathcal{A}_\ell^\infty$. Then $A_0 + A_1 F_\ell \in \mathscr{T}_\ell \otimes \mathcal{E}$ and $A_0 + A_1 F_\ell$ is compact. Therefore $(\sigma \otimes id)(A_0 + A_1 F_\ell) = 0$. Now let

$$A_i = \sum_{j,k\geq 0} \alpha_1^{*j} (p_0 \otimes a_{jk}^{(i)}) \alpha_1^k + \sum_{k\geq 0} \lambda_k^{(i)} \alpha_1^k + \sum_{k>0} \lambda_{-k}^{(i)} \alpha_1^{*k}$$

for $i = 0, 1$. Let $f_i(z) = \sum_{k\in\mathbb{Z}} \lambda_k^{(i)} z^k$ for $i = 0, 1$. Now $(\sigma \otimes id)(A_0 + A_1 F_\ell) = f_0 \otimes I + f_1 \otimes F_{\ell-1}$. So we have $f_0 \otimes I + f_1 \otimes F_{\ell-1} = 0$. Writing $F_\ell = 2P_\ell - I$, it follows that $(f_0 + f_1) \otimes P_{\ell-1} + (f_0 - f_1) \otimes (1 - P_{\ell-1}) = 0$. Hence $f_0 = f_1 = 0$. This shows that $\lambda_k^{(i)} = 0$ for $i = 0, 1$. Let $b_{jk} = a_{jk}^0 + a_{jk}^1 F_{\ell-1}$. Since $R := A_0 + A_1 F_\ell$ is smoothing, it follows that for every j, k, the matrix entries $\langle e_{(j,\gamma)}, R(e_{(k,\gamma')})\rangle$ are rapidly decreasing in (γ, γ'). Hence b_{jk} is smoothing for every j, k. By induction hypothesis $a_{jk}^{(i)} = 0$ for every $j, k \geq 0$ and for $i = 0, 1$. Thus $A_0 = A_1 = 0$. This proves part (1).

Observe that

$$\delta_\ell(\alpha_1) = -\alpha_1, \qquad |D_\ell|^r\alpha_1^{*k} = \alpha_1^{*k}(|D_\ell|+k)^r, \qquad \alpha_1^k|D_\ell|^s = (|D_\ell|+k)^s\alpha_1^k.$$

Also F_ℓ commutes with α_1. To prove (2), it is enough to show that $[F_\ell, a]$ is smoothing for every $a \in \mathcal{A}_\ell^\infty$ and the map $a \mapsto [F_\ell, a]$ is continuous. Let

$$a = \sum_{m,n\geq 0} \alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n + \sum_{m\geq 0}\lambda_m\alpha_1^m + \sum_{m>0}\lambda_{-m}\alpha_1^{*m}$$

be an element in $\mathcal{A}_\ell^\infty$. Then $[F_\ell, a] = \sum_{m,n\geq 0}\alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n$. By induction hypothesis, it follows that $p_0 \otimes [F_{\ell-1}, a_{mn}]$ is smoothing for every $m, n \geq 0$. Since $(OP_{D_\ell}^{-\infty}, \|\cdot\|_{r,s})$ is a Fréchet space, to show that $[F_\ell, a]$ is smoothing it is enough to show that the infinite sum $\sum_{m,n\geq 0}\alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n$ converges absolutely in every seminorm $\|\cdot\|_{r,s}$. Now observe that

$$|D_\ell|^r\alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n|D_\ell|^s = \alpha_1^{*m}(|D_\ell|+m)^r(p_0 \otimes [F_{\ell-1}, a_{mn}])(|D_\ell|+n)^s\alpha_1^n. \quad (3.4.3)$$

Since the map $a' \in \mathcal{A}_{\ell-1}^\infty \mapsto [F_{\ell-1}, a'] \in OP^{-\infty}$ is continuous, there exist $p \in \mathbb{N}$ and $C_p > 0$ such that $\|[F_{\ell-1}, a']\|_{i,j} \leq C_p\|a'\|_p$ for every $a' \in \mathcal{A}_{\ell-1}^\infty$ and for $i, j \leq max\{r, s\}$. Hence by equation (3.4.3), it follows that

$$\begin{aligned}\sum_{m,n}\|\alpha_1^{*m}(p_0 \otimes [F_{\ell-1}, a_{mn}])\alpha_1^n\|_{r,s} &\leq \sum_{m,n}\sum_{i=0}^{r}\sum_{j=0}^{s}\binom{r}{i}\binom{s}{j}m^{r-i}n^{s-j}\|[F_{\ell-1}, a_{mn}]\|_{i,j}\\ &\leq \sum_{i=0}^{r}\sum_{j=0}^{s}\binom{r}{i}\binom{s}{j}C_p\left(\sum_{m,n}m^rn^s\|a_{mn}\|_p\right).\end{aligned}$$

This shows that $[F_\ell, a]$ is smoothing and the above inequality also shows that for every $r, s \geq 0$, there exists $t \geq 0$ and a $C_t > 0$ such that $\|[F_\ell, a]\|_{r,s} \leq C_t\|a\|_t$. Hence the map $a \mapsto [F_\ell, a]$ is continuous. This proves (2).

To show (3), it is enough to show that the map $a \mapsto \delta_\ell(a)$ from $\mathcal{A}_\ell^\infty$ to $\mathcal{B}_\ell$ makes sense and is continuous. We will use the fact that the unbounded derivation δ_ℓ is a closed derivation. Let $a = \sum_{m,n\geq 0}\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n + \sum_{m\geq 0}\lambda_m\alpha_1^m + \sum_{m>0}\lambda_{-m}\alpha_1^{*m}$ be an element in $\mathcal{A}_\ell^\infty$. Since α_1 and $p_0 \otimes a_{mn} \in Dom(\delta_\ell)$ it follows that each of the terms in the infinite sum is an element in $Dom(\delta_\ell)$. Hence in order to show $a \in Dom(\delta_\ell)$, it is enough to show that the sum

$$\sum_{m,n}\delta_\ell(\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n) + \sum_{m\geq 0}\lambda_m\delta_\ell(\alpha_1^m) + \sum_{n>0}\lambda_{-n}\delta_\ell(\alpha_1^{*n})$$

converges. Observe that $\delta_\ell(\alpha_1^{*m}) = m\alpha_1^{*m}$, $\delta_\ell(\alpha_1^n) = -n\alpha_1^n$, and

$$\delta_\ell\big(\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n\big) = (m-n)\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n + \alpha_1^{*m}(p_0 \otimes \delta_{\ell-1}(a_{mn}))\alpha_1^n.$$

Since $\delta_{\ell-1}$ is continuous, it follows that $\|\delta_{\ell-1}(a_{mn})\|$ is rapidly decreasing where $\|\cdot\|$ is the operator norm. (Note that for $b \in \mathcal{B}_\ell$, one has $\|b\| \leq \|b\|_0$.) Hence the infinite sum

$$\sum_{m,n} \delta_\ell\big(\alpha_1^{*m}(p_0 \otimes a_{mn})\alpha_1^n\big) + \sum_{m\geq 0} \lambda_m \delta_\ell(\alpha_1^m) + \sum_{n>0} \lambda_{-n}\delta_\ell(\alpha_1^{*n})$$

converges absolutely in the operator norm. Therefore $a \in Dom(\delta_\ell)$. Since $\delta_{\ell-1}$ is continuous for every r there exists p and C_p such that $\|\delta_{\ell-1}(a')\|_r \leq C_p\|a'\|_p$. Write $\delta_{\ell-1}(a_{mn})$ as $\delta_{\ell-1}(a_{mn}) = a'_{mn} + a''_{mn}F_\ell + R_{mn}$. Let

$$\begin{aligned}
A_0 &= \sum_{m,n} \alpha_1^{*m}(p_0 \otimes ((m-n)a_{mn} + a'_{mn}))\alpha_1^n + \sum_{m\geq 0} m\lambda_m\alpha_1^m + \sum_{n>0}(-n)\lambda_{-n}\alpha_1^{*n},\\
A_1 &= \sum_{m,n} \alpha_1^{*m}(p_0 \otimes a''_{mn})\alpha_1^n,\\
R &= \sum_{m,n} \alpha_1^{*m}(p_0 \otimes R_{mn})\alpha_1^n.
\end{aligned}$$

Then $\delta_\ell(a) = A_0 + A_1F_\ell + R$. In every seminorm of $\mathcal{A}^\infty_{\ell-1}$ the double sequence (a'_{mn}) and (a''_{mn}) are rapidly decreasing. Also R_{mn} is rapidly decreasing in every seminorm of $OP^{-\infty}_{D_\ell}$. Hence $A_0, A_1 \in \mathcal{A}^\infty_\ell$ and as in the proof of (2), it follows that R is smoothing and given r, s there exists t and C_t such that $\|R\|_{r,s} \leq C_t\|a\|_t$. Fix an $r \geq 0$ and choose $t > 1 + r$ and $C_t > 1$ such that $\|\delta_{\ell-1}(a')\|_r \leq C_t\|a'\|_t$ for every $a' \in \mathcal{A}^\infty_{\ell-1}$. Now $\|A_0\|_r \leq C_t\|a\|_t$ and $\|A_1\|_r \leq C_t\|a\|_t$. This shows that the map $a \to \delta_\ell(a) \in \mathcal{B}_\ell$ is continuous. Since $[D_\ell, b] = \delta_\ell(b)F_\ell + |D_\ell|[F_\ell, b]$, the second part of (3) follows as $[F_\ell, b]$ is smoothing by (2). This proves (3).

Part (4) follows from (2) and (3). □

We next prove a lemma that will be crucial in the computation of the dimension spectrum. For an r tuple $n = (n_1, n_2, \cdots, n_r) \in \mathbb{N}^r$, we will write $|n|$ for $\sum_{i=1}^r n_i$. For $r = 0$, we let $\mathbb{N}^0 = \{0\}$.

Lemma 3.4.2. *Let $r \geq 0$ and $s \geq 1$ be integers. Let $(a(n))_{n\in\mathbb{N}^r}$ be rapidly decreasing. Then the function*

$$\xi(z) := \sum_{\substack{n\in\mathbb{N}^r, m\in\mathbb{N}^s\\ |n|+|m|\geq 1}} \frac{a(n)}{(|n| + |m|)^z}$$

is meromorphic with simple poles at $\{1, 2, \cdots, s\}$ and $Res_{z=s}\xi(z) = \frac{1}{(s-1)!}\sum_n a(n)$.

Proof: First observe that for $Re\, z > r+s$,

$$\begin{aligned}\xi(z) &= \sum_{N\geq 1} \frac{1}{N^z}\left(\sum_{|n|+|m|=N} a(n)\right)\\ &= \sum_{N\geq 1} \frac{1}{N^z}\left(\sum_{|n|\leq N} a(n) \sum_{m:|m|=N-|n|} 1\right)\\ &= \sum_{N\geq 1} \frac{1}{N^z}\left(\sum_{|n|\leq N} a(n)\binom{N-|n|+s-1}{s-1}\right).\end{aligned}$$

Note that for a function $(b(n))_{n\in\mathbb{N}^r}$ of rapid decay, the sequence $\left(\sum_{|n|\geq N} b(n)\right)_{N\in\mathbb{N}}$ is of rapid decay. Now $\binom{N-|n|+s-1}{s-1} = \sum_{k=0}^{s-1} g_k(n)N^k$ where $g_k(n)$ is a polynomial in $(n_1, n_2, \ldots, n_r)$ and $g_{s-1}(n) = \frac{1}{(s-1)!}$. Hence modulo a holomorphic function $\xi(z) = \sum_{k=0}^{s-1}(\sum_n g_k(n)a(n))\zeta(z-k)$. Now the result follows from the fact that $\zeta(z)$ is meromorphic with a simple pole at $z=1$ with residue 1. □

We will next prove that the spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ is regular and has discrete dimension spectrum with simple poles at $\{1, 2, \cdots, \ell+1\}$.

Remark 3.4.3. Recall that the unitaries U_w for $w = (w_1, w_2, \cdots, w_{\ell+1}) \in \mathbb{T}^{\ell+1}$ are given by $U_w e_\gamma = w_1^{\gamma_1} w_2^{\gamma_2} \cdots w_{\ell+1}^{\gamma_{\ell+1}} e_\gamma$. A bounded operator T on $\mathcal{H}_\ell$ is said to be homogeneous of degree $(m_1, m_2, \cdots, m_{\ell+1})$ if $U_w T U_w^* = w_1^{m_1} w_2^{m_2} \cdots w_{\ell+1}^{m_{\ell+1}} T$. If T is homogeneous of degree $(m_1, m_2, \cdots, m_{\ell+1}) \neq (0, \ldots, 0)$ then $Trace(T|D|^{-z}) = 0$ if $Re(z) > \ell+1$ since U_w's commute with the operator $|D_\ell|$.

Proposition 3.4.4. *The spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ is regular and has $\{1, 2, \cdots, \ell+1\}$ as the dimension spectrum with only simple poles.*

Proof: Regularity of the spectral triple follows from proposition 3.4.1. We now prove that for $b \in \mathcal{B}_\ell$, the function $Trace(b|D|^{-z})$ is meromorphic with simple poles at $\{1, 2, \cdots, \ell+1\}$. Since *Trace* $(b|D|^{-z})$ is holomorphic for $b \in OP^{-\infty}$, we need only to show that for $a \in \mathcal{A}_\ell^\infty$, the functions $Trace(a|D|^{-z})$ and $Trace(aF_\ell|D|^{-z})$ extend to meromorphic functions with simple poles at $\{1, 2, \cdots, \ell+1\}$. Now any element $a \in \mathcal{A}_\ell^\infty$ can be written as $a = a^0 + a^1$ where a^0 is homogeneous of degree 0 and a^1 is an infinite sum of homogeneous elements of non zero degrees. Hence by remark 3.4.3, $Trace(a|D|^{-z}) = Trace(a^0|D|^{-z})$ and $Trace(aF_\ell|D|^{-z}) = Trace(a^0 F_\ell|D|^{-z})$. Thus it is enough to consider the functions $Trace(a|D|^{-z})$ and $Trace(aF_\ell|D|^{-z})$ where a is homogeneous of degree 0.

It is easy to see that the set of homogeneous elements of degree 0 in $\mathcal{A}_\ell^\infty$ is

$$\left\{\sum_{i=0}^{\ell}\left(\sum_{n\in\mathbb{N}^i} \lambda_n^i (p_{n_1} \otimes p_{n_2} \otimes \cdots p_{n_i} \otimes 1)\right) : \ (\lambda_n^i) \text{ is of rapid decay for all } i\right\}$$

where $p_k = S^{*k} p_0 S^k$. Let $a = \sum_{i=0}^{\ell} (\sum_n \lambda_n^i (p_{n_1} \otimes p_{n_2} \otimes \cdots p_{n_i} \otimes 1)$ be a homogeneous element of degree 0 in $C^\infty(S_q^{2\ell+1})$. Then

$$Trace(a|D|^{-z}) = 2\sum_{i=0}^{\ell} \sum_{\substack{n\in\mathbb{N}^i,\, t\in\mathbb{N} \\ m\in\mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n|+|m|+t)^z} + \sum_{i=0}^{\ell} \sum_{\substack{n\in\mathbb{N}^i \\ m\in\mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n|+|m|)^z}.$$

Now $\sum_{n\in\mathbb{N}^\ell} \frac{\lambda_n^\ell}{(|n|)^z}$ is holomorphic and hence modulo a holomorphic function

$$Trace(a|D|^{-z}) = 2\sum_{i=0}^{\ell} \left(\sum_{\substack{n\in\mathbb{N}^i,\, t\in\mathbb{N} \\ m\in\mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n|+|m|+t)^z} \right) + \sum_{i=0}^{\ell-1} \left(\sum_{\substack{n\in\mathbb{N}^i \\ m\in\mathbb{N}^{\ell-i}}} \frac{\lambda_n^i}{(|n|+|m|)^z} \right).$$

It follows from lemma 3.4.2 that $Trace(a|D|^{-z})$ is meromorphic with simple poles in the set $\{1, 2, \cdots, \ell+1\}$. Similarly one can show that $Trace(aF_\ell|D|^{-z})$ is meromorphic with simple poles in $\{1, 2, \cdots, \ell\}$. Fix $0 \leq i \leq \ell+1$. Let $(\lambda_n)_{n\in\mathbb{N}^i}$ be such that $\sum_n \lambda_n = 1$. Let $a = \sum_{n\in\mathbb{N}^i} \lambda_n (p_{n_1} \otimes p_{n_2} \otimes p_{n_i} \otimes 1)$. Then one has $Res_{z=\ell+1-i} Trace(a|D|^{-z}) = \frac{2}{(\ell-i)!}$ by lemma 3.4.2 and by the above equation. Hence every $k \in \{1, 2, \cdots, \ell+1\}$ is in the dimension spectrum. This completes the proof. □

Chapter 4

The $SU_q(\ell+1)$ equivariant spectral triple

In this chapter, the equivariant spectral triple on $S_q^{2\ell+1}$ constructed in [8] is studied from the local index formula point of view. We show that this spectral triple is regular and has finite simple dimension spectrum. We first analyse the case $q = 0$. We show that for $q = 0$, the spectral triple is nothing but the torus equivariant one upto a multiplication. For $q \neq 0$, we approximate the equivariant spectral triple by the torus equivariant one and thereby deducing the computation.

4.1 The quantum group $SU_q(n)$

Let us recall the definition of the quantum group $SU_q(n)$ from [42]. The C^* algebra $C(SU_q(n))$ is defined as the universal unital C^* algebra generated by n^2 elements u_{ij} satisfying the following condition

$$\sum_{k=1}^{n} u_{ik}u_{jk}^* = \delta_{ij} \quad , \quad \sum_{k=1}^{n} u_{ki}^*u_{kj} = \delta_{ij}, \tag{4.1.1}$$

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} E_{i_1 i_2\cdots i_n} u_{j_1 i_1}\cdots u_{j_n i_n} = E_{j_1 j_2\cdots j_n} \tag{4.1.2}$$

where

$$E_{i_1 i_2\cdots i_n} \quad := \quad \begin{cases} 0 & if \quad i_1, i_2, \cdots i_n \text{ are not distinct,} \\ (-q)^{\ell(i_1,i_2,\cdots,i_n)}. & \end{cases}$$

Here for a permutation σ, $\ell(\sigma)$ denotes the number of inversed pairs i.e. the cardinality of the set $\{(i,j) : i < j, \sigma(i) > \sigma(j)\}$.

The C^* algebra $C(SU_q(n))$ has a compact quantum group structure with the comultiplication Δ given by

$$\Delta(u_{ij}) := \sum_k u_{ik} \otimes u_{kj}.$$

Call the generators of $SU_q(n-1)$ by v_{ij}. Then the map $\phi : C(SU_q(n)) \to C(SU_q(n-1))$ defined as

$$\phi(u_{ij}) \quad := \quad \begin{cases} v_{i-1,j-1} & \text{if } 2 \le i, j \le n \\ \delta_{ij} & \text{otherwise} \end{cases} \tag{4.1.3}$$

is a surjective unital C^* algebra homomorphism such that $\Delta \circ \phi = (\phi \otimes \phi)\Delta$. In this way the quantum group $SU_q(n-1)$ is a subgroup of the quantum group $SU_q(n)$. The quantum homogeneous space $C(SU_q(n)/SU_q(n-1))$ is generated by the first row of the matrix (u_{ij}). Moreover the map $\theta : C(S_q^{2n-1}) \to C(SU_q(n)/SU_q(n-1))$ defined by $\theta(z_i) := q^{-i+1}u_{1i}^*$ is an isomorphism. In this way, we realise the quantum odd dimensional spheres S_q^{2n-1} as a quantum homogeneous space.

The map $\tau : C(S_q^{2n-1}) \to C(S_q^{2n-1}) \otimes C(SU_q(n))$ defined by $\tau(z_i) := \sum_k z_k \otimes u_{ki}^*$ defines an action of $SU_q(n)$ on S_q^{2n-1}. Also one has $\Delta \circ \theta = (\theta \otimes 1)\tau$. Let h be the Haar state on $SU_q(n)$. We denote the closure of $C(S_q^{2n-1})$ in $L^2(SU_q(n))$ by $L^2(S_q^{2n-1})$. Then the right regular representation of $SU_q(n)$ on $L^2(SU_q(n))$ leaves the subspace $L^2(S_q^{2n-1})$ invariant and thus one obtains a covariant representation of $(C(S_q^{2n-1}), SU_q(n), \Delta)$ on $L^2(S_q^{2n-1})$. Equivariant spectral triples for this dynamical system on $L^2(S_q^{2n-1})$ are investigated in [8] and the sign of those operators D have been classified. In this chapter, we show that the spectral triple constructed in [8] satisfies the hypothesis of the local index formula. Since the computation is involved, it is better to fix some notations. We use the same notations as in [30].

Notations We will denote by Σ the set $\{1, 2, \ldots, 2\ell + 1\}$ and by Σ_ℓ and $\Sigma_{j,\ell}$ the subsets $\{1, 2, \ldots, \ell + 1\}$ and $\{\ell - j + 1, \ell - j + 2, \ldots, \ell + 1\}$ respectively, where $0 \le j \le \ell$.

Let $\Gamma \equiv \Gamma_\Sigma$ denote the set of maps γ from Σ to $\mathbb{Z}$ such that $\gamma_i \in \mathbb{N}$ for all $i \in \Sigma \backslash \{\ell + 1\}$, i.e. $\Gamma_\Sigma = \mathbb{N}^\ell \times \mathbb{Z} \times \mathbb{N}^\ell$. For a subset A of Σ, we will denote by γ_A the restriction $\gamma|_A$ of γ to A. Let Γ_A denote the set $\{\gamma_A : \gamma \in \Gamma\}$ and $\mathcal{H}_A$ be the Hilbert space $\ell_2(\Gamma_A)$. We will denote $\mathcal{H}_\Sigma$ by just $\mathcal{H}$, and $\mathcal{H}_{\Sigma_{j,\ell}}$ by $\mathcal{H}_j$. Thus

$$\mathcal{H}_\Sigma = \underbrace{\ell_2(\mathbb{N}) \otimes \cdots \otimes \ell_2(\mathbb{N})}_{\ell \text{ copies}} \otimes \ell_2(\mathbb{Z}) \otimes \underbrace{\ell_2(\mathbb{N}) \otimes \cdots \otimes \ell_2(\mathbb{N})}_{\ell \text{ copies}}, \qquad \mathcal{H}_j = \underbrace{\ell_2(\mathbb{N}) \otimes \cdots \otimes \ell_2(\mathbb{N})}_{j \text{ copies}} \otimes \ell_2(\mathbb{Z}).$$

Note that $\mathcal{H}_j$ and $\mathcal{H}_{\{j\}}$ are different.

Let $A \subseteq \Sigma$. We will denote by $\{e_\gamma\}_\gamma$ the natural orthonormal basis for $\mathcal{H}_A = \ell_2(\Gamma_A)$ and by p_γ the rank one projection $|e_\gamma\rangle\langle e_\gamma|$. For $i \in A$, we will denote by N_i the number operator on the i^{th} coordinate on $\mathcal{H}_A$, i.e.

$$N_i \equiv \sum_\gamma \gamma_i p_\gamma : e_\gamma \mapsto \gamma_i e_\gamma (\text{defined on } \mathcal{H}_A \text{ with } i \in A).$$

We will denote by $|D_A|$ the operator $\sum_{i\in A}|N_i|$ on $\mathcal{H}_A$.

Let F_0 be the following operator on $\ell_2(\mathbb{Z})$:

$$F_0 e_k = \begin{cases} e_k & \text{if } k \geq 0, \\ -e_k & \text{if } k < 0. \end{cases}$$

For $1 \leq j \leq 2\ell+1$, let V_j be the operator on $\mathcal{H}_{\{j\}}$ defined by

$$V_j := \begin{cases} F_0 & \text{if } j = \ell+1, \\ I & \text{otherwise.} \end{cases}$$

Let F_A denote the operator $\otimes_{j\in A} V_j$ on $\mathcal{H}_A$ and let $D_A = F_A|D_A|$. Thus

$$D_A e_\gamma = \begin{cases} -\left(\sum_{i\in A}|\gamma_i|\right) e_\gamma & \text{if } \ell+1 \in A \text{ and } \gamma_{\ell+1} < 0, \\ \left(\sum_{i\in A}|\gamma_i|\right) e_\gamma & \text{otherwise.} \end{cases}$$

We will denote $F_{\Sigma_{j,\ell}}$ by F_j and $D_{\Sigma_{j,\ell}}$ by D_j.

Recall that $\mathcal{H}_{\{j\}}$ is $\ell_2(\mathbb{N})$ if $j \neq \ell+1$ and is $\ell_2(\mathbb{Z})$ if $j = \ell+1$. Suppose for each $j \in \Sigma$, $\mathcal{F}_j$ is a subspace of $\mathcal{L}(\mathcal{H}_{\{j\}})$. For $A \subseteq \Sigma$, define

$$\mathcal{F}_{j,A} = \begin{cases} \mathcal{F}_j & \text{if } j \in A, \\ \mathbb{C}.I & \text{if } j \notin A, \end{cases}$$

and $\mathcal{F}_A$ to be the tensor product $\otimes_{j\in\Sigma}\mathcal{F}_{j,A}$ in $\mathcal{L}(\mathcal{H}_\Sigma)$ (the type of the tensor product will depend on the specific $\mathcal{F}_j$'s we look at). This tensor product will often be identified with $\otimes_{j\in A}\mathcal{F}_j \subseteq \mathcal{L}(\mathcal{H}_A)$.

On both $\ell_2(\mathbb{N})$ and $\ell_2(\mathbb{Z})$, we will denote by N the number operator defined by $Ne_n = ne_n$ and by S the left shift defined by $Se_n = e_{n-1}$. For $k \in \mathbb{Z}$ (for $k \in \mathbb{N}$ in case of $\ell_2(\mathbb{N})$), let p_k denote the projection $|e_k\rangle\langle e_k|$. We will freely identify $\ell_2(\mathbb{Z})$ with $L_2(\mathbb{T})$. Thus the right shift on $\ell_2(\mathbb{Z})$ will be multiplication by the function $t \mapsto t$ and will be denoted by $\mathbf{z}$. Let $\mathscr{T}$ be the Toeplitz algebra, i.e. the C^*-subalgebra of $\mathcal{L}(\ell_2(\mathbb{N}))$ generated by S. For a positive integer k, we will denote by $\mathscr{T}_k$ the k-fold tensor product of $\mathscr{T}$, embedded in $\mathcal{L}(\ell_2(\mathbb{N}^k))$. Denote by σ the symbol map from $\mathscr{T}$ to $C(\mathbb{T})$ that sends S^* to $\mathbf{z}$ and all compact operators to 0.

4.2 Left multiplication operators

Let us recall from [8] some basic facts on representations of $C(SU_q(\ell+1))$ on $L_2(SU_q(\ell+1))$ by left multiplication. The Hilbert space $L_2(SU_q(\ell+1))$ is the GNS space of $C(SU_q(\ell+1))$ with respect to the Haar state on $SU_q(\ell+1)$. Irreducible unitary representations of the quantum group $SU_q(\ell+1)$ are indexed by Young tableaux $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$ where $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq$

$\lambda_2 \geq \cdots \geq \lambda_{\ell+1} = 0$ (Theorem 1.5, [42]). Denote by u^λ the irreducible unitary indexed by λ. Basis elements of the Hilbert space $\mathcal{H}_\lambda$ on which u^λ acts can be parametrized by arrays of the form

$$\mathbf{r} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,\ell} & r_{1,\ell+1} \\ r_{21} & r_{22} & \cdots & r_{2,\ell} & \\ & \cdots & & & \\ r_{\ell,1} & r_{\ell,2} & & & \\ r_{\ell+1,1} & & & & \end{pmatrix},$$

where r_{ij}'s are integers satisfying $r_{1j} = \lambda_j$ for $j = 1, \ldots, \ell+1$, $r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \geq 0$ for all i, j, and the top row coincides with λ. These are known as Gelfand-Tsetlin tableaux, to be abreviated as GT tableaux now onwards. Let $\{e(\lambda, \mathbf{r}) : \mathbf{r}$ is a GT tableaux with top row $\lambda\}$ be an orthonormal basis for $\mathcal{H}_\lambda$. Denote the matrix entries of u^λ with respect to this basis by $u^\lambda_{\mathbf{r},\mathbf{s}}$. Note that the generators u_{ij} of the C^*-algebra $C(SU_q(\ell+1))$ are the matrix entries of the irreducible $\mathbb{1} = (1, 0, \ldots, 0)$. The collection $\{u^\lambda_{\mathbf{r},\mathbf{s}} : \lambda, \mathbf{r}, \mathbf{s}\}$ form a complete orthogonal set of vectors in $L_2(SU_q(\ell+1))$. Denote by $e^\lambda_{\mathbf{r},\mathbf{s}}$, or by $e_{\mathbf{r},\mathbf{s}}$ for short (as $\mathbf{r}$ and $\mathbf{s}$ specify λ), the normalized $u^\lambda_{\mathbf{r},\mathbf{s}}$'s, i.e. $e_{\mathbf{r},\mathbf{s}} = \|u^\lambda_{\mathbf{r},\mathbf{s}}\|^{-1} u^\lambda_{\mathbf{r},\mathbf{s}}$. Then $\{e_{\mathbf{r},\mathbf{s}} : \mathbf{r}, \mathbf{s}\}$ form a complete orthonormal basis for $L_2(SU_q(\ell+1))$.

Let ρ be the half-sum of positive roots of $sl(\ell+1)$ and $\lambda(\mathbf{r})$ is the weight of the weight vector $e(\lambda, \mathbf{r})$. Let F_λ be the unique intertwiner in $\mathrm{Mor}\,(u^\lambda, (u^\lambda)^{cc})$ with $\mathrm{trace}\, F_\lambda = \mathrm{trace}\, F_\lambda^{-1}$ (here for a representation u, its contragradient representation is denoted by u^c; see [25] for details). Then one has $\|u^\lambda_{\mathbf{rs}}\| = d_\lambda^{-\frac{1}{2}} q^{-\psi(\mathbf{r})}$, where

$$\psi(\mathbf{r}) = (\rho, \lambda(\mathbf{r})) = -\frac{\ell}{2} \sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} r_{ij}, \qquad d_\lambda = \mathrm{trace}\, F_\lambda = \sum_{\mathbf{r}:\mathbf{r}_1=\lambda} q^{2\psi(\mathbf{r})}. \tag{4.2.4}$$

Write

$$\kappa(\mathbf{r}, \mathbf{m}) = d_\lambda^{\frac{1}{2}} d_\mu^{-\frac{1}{2}} q^{\psi(\mathbf{r}) - \psi(\mathbf{m})}. \tag{4.2.5}$$

From equation (4.19) in [8], we have

$$\pi(u_{ij}) e^\lambda_{\mathbf{rs}} = \sum_{\mu,\mathbf{m},\mathbf{n}} C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{m}) C_q(\mathbb{1}, \lambda, \mu; j, \mathbf{s}, \mathbf{n}) \kappa(\mathbf{r}, \mathbf{m}) e^\mu_{\mathbf{mn}}, \tag{4.2.6}$$

where C_q denote the Clebsch Gordon coefficients.

For our subsequent analysis, we will compute the quantities $C_q(i, \mathbf{r}, \mathbf{s})$ and $\kappa(\mathbf{r}, \mathbf{m})$ appearing in the above formula. We will use the formulae given in ([25], pp. 220), keeping in mind that for our case (i.e. for $SU_q(\ell+1)$), the top right entry of the GT tableaux is zero.

For a positive integer j with $1 \leq j \leq \ell+1$, let

$$\mathbb{M}_j := \{(m_1, m_2, \ldots, m_j) \in \mathbb{N}^j : 1 \leq m_i \leq \ell + 2 - i \text{ for } 1 \leq i \leq j\}. \tag{4.2.7}$$

For $M = (m_1, m_2, \ldots, m_i) \in \mathbb{M}_i$, denote by $M(\mathbf{r})$ the tableaux $\mathbf{s}$ defined by

$$s_{jk} = \begin{cases} r_{jk} + 1 & \text{if } k = m_j, 1 \le j \le i, \\ r_{jk} & \text{otherwise.} \end{cases} \tag{4.2.8}$$

With this notation, observe now that $C_q(i, \mathbf{r}, \mathbf{s})$ will be zero unless $\mathbf{s}$ is $M(\mathbf{r})$ for some $M \in \mathbb{M}_i$. (One has to keep in mind though that not all tableaux of the form $M(\mathbf{r})$ is a valid GT tableaux)

From ([25], pp. 220), we have

$$C_q(i, \mathbf{r}, M(\mathbf{r})) = \prod_{a=1}^{i-1} R(\mathbf{r}, a, m_a, m_{a+1}) \times R'(\mathbf{r}, i, m_i) \tag{4.2.9}$$

where e_k stands for a vector (in the appropriate space) whose k^{th} coordinate is 1 and the rest are all zero, and

$$\begin{aligned} R(\mathbf{r}, a, j, k) &= \operatorname{sign}(k-j)\, q^{\frac{1}{2}(-r_{aj} + r_{a+1,k} - k + j)} \\ &\times \left(\prod_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} \frac{[r_{a,i} - r_{a+1,k} - i + k]_q}{[r_{a,i} - r_{a,j} - i + j]_q} \prod_{\substack{i=1 \\ i \neq k}}^{\ell+1-a} \frac{[r_{a+1,i} - r_{a,j} - i + j - 1]_q}{[r_{a+1,i} - r_{a+1,k} - i + k - 1]_q} \right)^{\frac{1}{2}} \end{aligned} \tag{4.2.10}$$

$$\begin{aligned} R'(\mathbf{r}, a, j) &= q^{\frac{1}{2}\left(1 - j + \sum_{i=1}^{\ell+1-a} r_{a+1,i} - \sum_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} r_{a,i}\right)} \\ &\times \left(\frac{\prod_{i=1}^{\ell+1-a} [r_{a+1,i} - r_{aj} - i + j - 1]_q}{\prod_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} [r_{a,i} - r_{aj} - i + j]_q} \right)^{\frac{1}{2}}, \end{aligned} \tag{4.2.11}$$

where for an integer n, $[n]_q$ denotes the q-number $(q^n - q^{-n})/(q - q^{-1})$ and $\operatorname{sign}(k-j)$ is 1 if $k \ge j$ and is -1 if $k < j$.

Remark 4.2.1. Let us look at the denominators in the above expressions. The integers $r_{a,i} - r_{a,j}$ and $j - i$ are of the same sign. Therefore for $i \neq j$, the quantity $r_{a,i} - r_{a,j} - i + j$ is nonzero. Similarly $r_{a+1,i} - r_{a+1,k}$ and $k - i$ are of the same sign. So if $i \neq k$, then $r_{a+1,i} - r_{a+1,k} - i + k - 1$ can be zero only when $r_{a+1,i} = r_{a+1,k}$ and $k = i + 1$. Now if $\mathbf{r}$ and $M(\mathbf{r})$ are GT tableaux, then $M(\mathbf{r})_{a+1,m_{a+1}} = r_{a+1,m_{a+1}} + 1$ and $M(\mathbf{r})_{a+1,i} = r_{a+1,i}$ for $i \neq m_{a+1}$. Therefore if $m_{a+1} = i + 1$, then $r_{a+1,i} - (r_{a+1,m_{a+1}} + 1) \ge 0$, i.e. $r_{a+1,i} - r_{a+1,m_{a+1}} \ge 1$. Hence $r_{a+1,i} - r_{a+1,m_{a+1}} - i + m_{a+1} - 1 \ge 1$. In other words, all the q-numbers appearing in the denominator in equation (4.2.9) are nonzero. Thus no problem arises from division by zero.

Remark 4.2.2. This is essentially a repetition of remark 4.1 of [8]. The formulae (4.2.10) and (4.2.11) are obtained from equations (45) and (46), page 220, [25] by replacing q with q^{-1}. Equation (45) is a special case of the more general formula (48), page 221, [25]. However, there is a small error in equation (48) there. The correct form can be found in equations (3.1, 3.2a, 3.2b) in [2]. Here we have incorporated that correction in equations (4.2.10) and (4.2.11).

We next compute the quantities $R(\mathbf{r},a,j,k)$ and $R'(\mathbf{r},a,j)$.

For a positive integer n, denote by $Q(n)$ the number $(1-q^{2n})^{1/2}$. Then for any two integers m and n, one has

$$\left|\frac{[m]_q}{[n]_q}\right| = q^{-|m|+|n|}\left(\frac{Q(|m|)}{Q(|n|)}\right)^2.$$

The next two lemmas are obtained from equations (4.2.10) and (4.2.11) using the above equality repeatedly and the fact that $r_{a,i} \geq r_{a+1,i} \geq r_{a,i+1}$ for all a and i.

Lemma 4.2.3. *For a GT tableaux $\mathbf{r}=(r_{ab})$, denote by $H_{ab}(\mathbf{r})$ and $V_{ab}(\mathbf{r})$ the following differences: $H_{ab}(\mathbf{r}) := r_{a+1,b} - r_{a,b+1}$ and $V_{ab}(\mathbf{r}) := r_{ab} - r_{a+1,b}$. Then one has*

$$R(\mathbf{r},a,j,k) = sign\,(k-j)q^{P(\mathbf{r},a,j,k)+S(\mathbf{r},a,j,k)}L(\mathbf{r},a,j,k), \tag{4.2.12}$$

where

$$P(\mathbf{r},a,j,k) = \sum_{j\wedge k\leq i<j\vee k} H_{ai}(\mathbf{r}) + 2\sum_{k<i<j} V_{ai}(\mathbf{r}), \tag{4.2.13}$$

$$S(\mathbf{r},a,j,k) = \begin{cases} 2(j-k-1)+1 & \text{if } j>k,\\ 0 & \text{if } j\leq k,\end{cases} \tag{4.2.14}$$

$$L(\mathbf{r},a,j,k) = \prod_{\substack{i=1\\ i\neq j}}^{\ell+2-a} \frac{Q(|r_{a,i}-r_{a+1,k}-i+k|)}{Q(|r_{a,i}-r_{a,j}-i+j|)} \prod_{\substack{i=1\\ i\neq k}}^{\ell+1-a} \frac{Q(|r_{a+1,i}-r_{a,j}-i+j-1|)}{Q(|r_{a+1,i}-r_{a+1,k}-i+k-1|)}. \tag{4.2.15}$$

Lemma 4.2.4. *One has*

$$R'(\mathbf{r},a,j) = q^{P'(\mathbf{r},a,j)}L'(\mathbf{r},a,j), \tag{4.2.16}$$

where

$$P'(\mathbf{r},a,j) = \sum_{j\leq i<\ell+2-a} H_{ai}(\mathbf{r}), \tag{4.2.17}$$

$$L'(\mathbf{r},a,j) = \frac{\prod_{i=1}^{\ell+1-a} Q(|r_{a+1,i}-r_{a,j}-i+j-1|)}{\prod_{\substack{i=1\\ i\neq j}}^{\ell+2-a} Q(|r_{a,i}-r_{a,j}-i+j|)}. \tag{4.2.18}$$

Combining lemmas 4.2.3 and 4.2.4, we get the following expression for the CG coefficient $C_q(i,\mathbf{r},M)$.

Lemma 4.2.5. *For a move $M\in\mathbb{M}_i$, let $sign\,(M)$ denote the product $\prod_{a=1}^{i-1} sign\,(m_{a+1}-m_a)$. Then one has*

$$C_q(i,\mathbf{r},M) = sign\,(M)q^{B(M)+C(\mathbf{r},M)}\left(\prod_{a=1}^{i-1} L(\mathbf{r},a,m_a,m_{a+1})\right)L'(\mathbf{r},i,m_i), \tag{4.2.19}$$

where

$$B(M) = \sum_{j:m_j>m_{j+1}} (2(m_j - m_{j+1} - 1) + 1)\,, \tag{4.2.20}$$

$$C(\mathbf{r}, M) = \sum_{a=1}^{i-1} \left(\sum_{m_a \wedge m_{a+1} \le b < m_a \vee m_{a+1}} H_{ab}(\mathbf{r}) + 2 \sum_{m_{a+1} < b < m_a} V_{ab}(\mathbf{r}) \right) + \sum_{m_i \le b < \ell+2-i} H_{ib}(\mathbf{r}) \tag{4.2.21}$$

Lemma 4.2.6.

$$\left(\prod_{a=1}^{i-1} L(\mathbf{r}, a, m_a, m_{a+1}) \right) L'(\mathbf{r}, i, m_i) = 1 + o(q).$$

Proof: This is a consequence of the following two inequalities:

$$|1 - (1-x)^{\frac{1}{2}}| < x \quad \text{for } 0 \le x \le 1,$$

and for $0 < r < 1$,

$$|1 - (1-x)^{-\frac{1}{2}}| < cx \quad \text{for } 0 \le x \le r,$$

where c is some fixed constant that depends on r. □

Next we come to the computation of $\kappa(\mathbf{r}, \mathbf{m})$. Since $C_q(i, \mathbf{r}, \mathbf{m})$ is 0 unless $\mathbf{m}$ is of the form $M(\mathbf{r})$ for some move $M = (m_1, \ldots, m_i)$, we need only to compute $\kappa(\mathbf{r}, M(\mathbf{r}))$ which we will denote by $\kappa(\mathbf{r}, M)$.

Since

$$\psi(\mathbf{s}) = -\frac{\ell}{2} \sum_{j=1}^{\ell+1} s_{1j} + \sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} s_{ij},$$

we have

$$q^{\psi(\mathbf{r}) - \psi(M(\mathbf{r}))} = q^{-\frac{\ell}{2} \sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} r_{ij} + \frac{\ell}{2}(\sum_{j=1}^{\ell+1} r_{1j} + 1) - (\sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} r_{ij} + i - 1)} = q^{\frac{\ell}{2} - i + 1}.$$

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0)$ be the top row of $\mathbf{r}$. Then

$$\min\{\psi(\mathbf{s}) : \mathbf{s}_1 = \lambda\} = -\frac{\ell}{2} \sum_{1}^{\ell} \lambda_i + \sum_{k=2}^{\ell} (k-1)\lambda_k.$$

Hence

$$d_\lambda = \sum_{\mathbf{s}:\mathbf{s}_1=\lambda} q^{2\psi(\mathbf{s})} = q^{-\ell \sum_1^\ell \lambda_i + 2\sum_{k=2}^{\ell}(k-1)\lambda_k} (1 + q^2 \phi(q^2)),$$

where ϕ is a polynomial. Therefore

$$d_\lambda = q^{-\ell \sum_1^\ell \lambda_i + 2\sum_{k=2}^{\ell}(k-1)\lambda_k} (1 + o(q)).$$

It follows that

$$\left(\frac{d_\lambda}{d_{\lambda+e_{m_1}}}\right)^{\frac{1}{2}} = q^{\frac{\ell}{2}-m_1+1}(1+o(q)).$$

Thus

$$\kappa(\mathbf{r}, M(\mathbf{r})) = q^{\ell+2-i-m_1}(1+o(q)). \tag{4.2.22}$$

Next, observe that

$$\begin{aligned}
&B(M)+\ell+2-i-m_1\\
&\quad= \sum_{j:m_j>m_{j+1}} (2(m_j-m_{j+1}-1)+1)-(m_1-m_i)+\ell+2-i-m_i\\
&\quad= 2\sum_{j:m_j>m_{j+1}}(m_j-m_{j+1}) - \sum_{j:m_j>m_{j+1}} 1 - \sum_{j=1}^{i-1}(m_j-m_{j+1})+\ell+2-i-m_i\\
&\quad= 2\sum_{j:m_j>m_{j+1}}(m_j-m_{j+1}) - \sum_{j=1}^{i-1}(m_j-m_{j+1}) - \sum_{j:m_j>m_{j+1}} 1+\ell+2-i-m_i\\
&\quad= \sum_{j:m_j>m_{j+1}} |m_j-m_{j+1}| - \#\{1\le j\le i-1 : m_j>m_{j+1}\}+\ell+2-i-m_i.
\end{aligned}$$

Thus if we write

$$A(M) = \sum_{j:m_j>m_{j+1}} |m_j-m_{j+1}| - \#\{1\le j\le i-1 : m_j>m_{j+1}\}, \tag{4.2.23}$$

$$K(M) = \ell+2-i-m_i, \tag{4.2.24}$$

then both $A(M)$ and $K(M)$ are nonnegative and $B(M)+\ell+2-i-m_1 = A(M)+K(M)$. Thus we have

$$\pi(u_{ij})e^\lambda_{\mathbf{rs}} = \sum_{\substack{M\in\mathbb{M}_i\\ M'\in\mathbb{M}_j}} C_q(i,\mathbf{r},M(\mathbf{r}))\kappa(\mathbf{r},M)C_q(j,\mathbf{s},M'(\mathbf{s}))e_{M(\mathbf{r}),M'(\mathbf{s})} \tag{4.2.25}$$

$$= \sum_{\substack{M\in\mathbb{M}_i\\ M'\in\mathbb{M}_j}} \text{sign}\,(M)\text{sign}\,(M')q^{A(M)+K(M)+C(\mathbf{r},M)+B(M')+C(\mathbf{s},M')}(1+o(q))e_{M(\mathbf{r}),M'(\mathbf{s})}. \tag{4.2.26}$$

4.3 The spectral triple

Let us briefly recall from [8] the description of the L_2 space of the sphere denoted $L^2(S_q^{2\ell+1})$ sitting inside $L_2(SU_q(\ell+1))$ i.e. the closure of $C(SU_q(\ell+1)/SU_q(\ell))$ in $L_2(SU_q(\ell+1)/SU_q(\ell))$.

The following proposition shows that the "natural" representation of the dynamical system $(C(S_q^{2\ell+1}, SU_q(\ell+1),\tau)$ on $L^2(SU_q(\ell+1)$ restricts to give a covariant representation on $L^2(S_q^{2\ell+1})$. We refer to [8] for proofs.

Proposition 4.3.1 ([8]). *Assume $\ell > 1$. The right regular representation u of G keeps the subspace $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ invariant, and the restriction of u to $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$ decomposes as a direct sum of exactly one copy of each of the irreducibles given by the young tableaux $\lambda_{n,k} := (n+k, k, k, \ldots, k, 0)$, with $n, k \in \mathbb{N}$.*

Proposition 4.3.2 ([8]). *Let $\mathbf{r}^{nk}$ denote the GT tableaux given by*

$$r_{ij}^{nk} = \begin{cases} n+k & \text{if } i = j = 1, \\ 0 & \text{if } i = 1,\ j = \ell+1, \\ k & \text{otherwise}, \end{cases}$$

where $n, k \in \mathbb{N}$. Let $\mathscr{G}_0^{n,k}$ be the set of all GT tableaux with top row $(n+k, k, \ldots, k, 0)$. Then the family of vectors

$$\{e_{\mathbf{r}^{nk},\mathbf{s}} : n, k \in \mathbb{N},\ \mathbf{s} \in \mathscr{G}_0^{n,k}\}$$

form a complete orthonormal basis for $L_2(SU_q(\ell+1)\backslash SU_q(\ell))$.

We will denote $\cup_{n,k}\mathscr{G}_0^{n,k}$ by $\mathscr{G}_0$. Since the top row of $\mathbf{r}^{nk}$ determines $\mathbf{r}^{nk}$ completely and for $e_{\mathbf{r}^{nk},\mathbf{s}}$, the top row of $\mathbf{s}$ equals the top row of $\mathbf{r}^{nk}$, one can index the orthonormal basis $e_{\mathbf{r}^{nk},\mathbf{s}}$ just by $\mathbf{s} \in \mathscr{G}_0$. It was shown in [8] that the restriction of the left multiplication to $C(SU_q(\ell+1)\backslash SU_q(\ell)) \cong C(S_q^{2\ell+1})$ keeps $L_2(SU_q(\ell+1)\backslash SU_q(\ell)) \cong L_2(S_q^{2\ell+1})$ invariant. We will continue to denote this restriction by π. The operators $\pi(z_j) = q^{-j+1}\pi(u_{1,j}^*)$ will be denoted by $Z_{j,q}$. The C^*-algebra $\pi(C(S_q^{2\ell+1}))$ will be denoted by C_ℓ.

The following theorem gives a generic equivariant spectral triple for the spheres $S_q^{2\ell+1}$ constructed in [8].

Theorem 4.3.3 ([8]). *Let D_{eq} be the operator on $L_2(S_q^{2\ell+1})$ given by:*

$$D_{eq}e_{\mathbf{r}^{nk},\mathbf{s}} = \begin{cases} ke_{\mathbf{r}^{nk},\mathbf{s}} & \text{if } n = 0, \\ -(n+k)e_{\mathbf{r}^{nk},\mathbf{s}} & \text{if } n > 0. \end{cases} \tag{4.3.27}$$

Then $(\mathcal{A}(S_q^{2\ell+1}), L_2(S_q^{2\ell+1}), D_{eq})$ is an equivariant nondegenerate $(2\ell+1)$-summable odd spectral triple.

Our main aim in the rest of the chapter is to precisely formulate the smooth function algebra for this spectral triple, establish its regularity, and compute the dimension spectrum.

4.4 The case $q = 0$

The L_2 spaces $L_2(S_q^{2\ell+1})$ for different values of q can be identified by identifying the elements of their canonical orthonormal bases which are parametrized by the same set. Thus we will assume we are woring with one single Hilbert space $\mathcal{H}$ with orthonormal basis given by $e_{\mathbf{r}^{n,k}\mathbf{s}}$ where $\mathbf{r}^{n,k}$ is as defined earlier and $\mathbf{s}$ is given by

$$\mathbf{s} = \begin{pmatrix} c_1 = n+k & k & k & \cdots & k & k & d_1 = 0 \\ c_2 & k & k & \cdots & k & d_2 & \\ \cdots & & \cdots & & & & \\ c_{\ell-1} & k & d_{\ell-1} & & & & \\ c_\ell & d_\ell & & & & & \\ c_{\ell+1} = d_{\ell+1} & & & & & & \end{pmatrix} \tag{4.4.28}$$

where $c_1 \geq c_2 \geq \ldots \geq c_\ell \geq k$, $d_1 \leq d_2 \leq \ldots \leq d_\ell \leq k$ and $d_\ell \leq d_{\ell+1} \leq c_\ell$. Since specifying the GT tableaux $\mathbf{s}$ specifies $\mathbf{r}^{n,k}$ also and thus completely specifies the basis element $e_{\mathbf{r}^{n,k}\mathbf{s}}$, we will sometimes use just $\mathbf{s}$ in place of the basis element $e_{\mathbf{r}^{n,k}\mathbf{s}}$.

Let us denote by $\mathbb{M}_j^{\pm}$ the following subsets of $\mathbb{M}_j$:

$$\begin{aligned} \mathbb{M}_j^{+} &= \{(m_1, \ldots, m_j) \in \mathbb{M}_j : m_i \in \{1, \ell+2-i\} \text{ for } 1 \leq i \leq j,\ m_1 = 1\}, \\ \mathbb{M}_j^{-} &= \{(m_1, \ldots, m_j) \in \mathbb{M}_j : m_i \in \{1, \ell+2-i\} \text{ for } 1 \leq i \leq j,\ m_1 = \ell+1\}. \end{aligned}$$

Let us denote by $N_{i,j}$ the following element of $\mathbb{M}_j$:

$$N_{i,j} = (\underbrace{1, \ldots, 1}_{i}, \ell+1-i, \ell-i, \ldots, \ell+2-j), \qquad 0 \leq i \leq j \leq \ell+1.$$

We will denote $N_{i,\ell+1}$ by just N_i. Then from (4.2.26), we get

$$\begin{aligned} \pi(u_{1j}) e_{\mathbf{r}^{n,k}\mathbf{s}} &= \sum_{M \in \mathbb{M}_j^{+}} \operatorname{sign}(M) q^{\ell+k+B(M)+C(\mathbf{s},M)} (1+o(q)) e_{\mathbf{r}^{n+1,k}, M(\mathbf{s})} \\ &\quad + \sum_{M \in \mathbb{M}_j^{-}} \operatorname{sign}(M) q^{B(M)+C(\mathbf{s},M)} (1+o(q)) e_{\mathbf{r}^{n,k-1}, M(\mathbf{s})} \end{aligned} \tag{4.4.29}$$

Therefore

$$\begin{aligned} Z_{j,q}^{*} e_{\mathbf{r}^{n,k}\mathbf{s}} &= \sum_{M \in \mathbb{M}_j^{+}} \operatorname{sign}(M) q^{-j+1+\ell+k+B(M)+C(\mathbf{s},M)} (1+o(q)) e_{\mathbf{r}^{n+1,k}, M(\mathbf{s})} \\ &\quad + \sum_{M \in \mathbb{M}_j^{-}} \operatorname{sign}(M) q^{-j+1+B(M)+C(\mathbf{s},M)} (1+o(q)) e_{\mathbf{r}^{n,k-1}, M(\mathbf{s})} \end{aligned} \tag{4.4.30}$$

Let us first look at the cases $1 \leq j \leq \ell$. In this case, the power of q in the first summation is positive. Therefore none of the terms would survive for $q = 0$. For terms in the second summation, assume $M \in \mathbb{M}_j$ with $m_1 = \ell+1$ and $m_i = 1$ for some $i \leq j$. Let $a = \min\{2 \leq i \leq j : m_i = 1\}$. Then $m_i = \ell+2-i$ for $1 \leq i \leq a-1$ so that

$$\begin{aligned} B(M) &\geq \sum_{i=1}^{a-2} (2((\ell+2-i) - (\ell+1-i) - 1) + 1) + 2(\ell+3-a-1-1) + 1 \\ &= a - 2 + 2(\ell - a + 1) + 1 \\ &= 2\ell - a + 1. \end{aligned}$$

Hence $B(M)+1-j>0$ and so such terms will not survive for $q=0$. Thus the only term that will survive is the one corresponding to $M=N_{0,j}=(\ell+1,\ell,\ell-1,\ldots,\ell+2-j)$. In this case we have $B(M)=j-1$, $C(\mathbf{s},M)=d_j$ and $\operatorname{sign}(M)=(-1)^{j-1}$. Therefore

$$Z_{j,0}^{*}e_{\mathbf{r}^{n,k}\mathbf{s}}=\begin{cases}(-1)^{j-1}e_{\mathbf{r}^{n,k-1},N_{0,j}(\mathbf{s})} & \text{if } d_j=0,\\ 0 & \text{if } d_j>0.\end{cases} \tag{4.4.31}$$

Next let us look at the case $j=\ell+1$. Here the first sum will be over all M with $m_1=1=m_{\ell+1}$. If $m_i\neq 1$ for some i, then $B(M)>0$ and therefore the power of q will be positive, so that the term will not survive for $q=0$. If $m_i=1$ for all i, i.e. if $M=N_\ell$, then we have $B(M)=0=C(\mathbf{s},M)$ and $\operatorname{sign}(M)=1$. Therefore for $q=0$, the first summation will become $e_{\mathbf{r}^{n+1,k},N_\ell(\mathbf{s})}$ provided $k=0$.

The second sum is over all M with $m_1=\ell+1$. Let $a=\min\{2\leq i\leq \ell+1: m_i=1\}$. Then as before, $B(M)\geq 2\ell-a+1$. Therefore if $a\leq\ell$, then $-\ell+B(M)\geq \ell-a+1>0$, so that the term will not survive for $q=0$. If $a=\ell+1$, i.e. if $M=N_0$, then $B(M)=\ell$, $C(\mathbf{s},M)=d_{\ell+1}$ and $\operatorname{sign}(M)=(-1)^\ell$. So for $q=0$, the second summation will become $(-1)^\ell e_{\mathbf{r}^{n,k-1},N_0(\mathbf{s})}$ if $k>0$ and $d_{\ell+1}=0$. Thus we have

$$Z_{\ell+1,0}^{*}e_{\mathbf{r}^{n,k}\mathbf{s}}=\begin{cases}e_{\mathbf{r}^{n+1,k},N_\ell(\mathbf{s})} & \text{if } k=0,\\ (-1)^\ell e_{\mathbf{r}^{n,k-1},N_0(\mathbf{s})} & \text{if } k>0,\ d_{\ell+1}=0,\\ 0 & \text{if } k>0,\ d_{\ell+1}>0.\end{cases} \tag{4.4.32}$$

Next we will establish a natural unitary map between $L_2(S_q^{2\ell+1})$ and

$$\mathcal{H}_\Sigma\equiv\underbrace{\ell_2(\mathbb{N})\otimes\cdots\otimes\ell_2(\mathbb{N})}_{\ell\text{ copies}}\otimes\ell_2(\mathbb{Z})\otimes\underbrace{\ell_2(\mathbb{N})\otimes\cdots\otimes\ell_2(\mathbb{N})}_{\ell\text{ copies}}.$$

For $t\in\mathbb{R}$, let t_+ denote the positive part $\max\{t,0\}$ and let t_- denote the negative part $\max\{(-t),0\}$ of t. Let us now observe that for any $\gamma\in\Gamma_\Sigma$, the tableaux

$$\mathbf{s}(\gamma):=\begin{pmatrix}\sum_1^{2\ell+1}|\gamma_i| & \sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+ & \cdots & \sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+ & 0\\ \sum_1^{2\ell}|\gamma_i| & \sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+ & \cdots & \gamma_1 & \\ \cdots & \cdots & & & \\ \sum_1^{\ell+3}|\gamma_i| & \sum_1^\ell\gamma_i+(\gamma_{\ell+1})_+ & \sum_1^{\ell-2}\gamma_i & & \\ \sum_1^{\ell+2}|\gamma_i| & \sum_1^{\ell-1}\gamma_i & & & \\ \sum_1^\ell\gamma_i+(\gamma_{\ell+1})_- & & & & \end{pmatrix}$$

is in $\mathscr{G}_0$. Conversely, let $\mathbf{s}\in\mathscr{G}_0^{n,k}$ for some $n,k\in\mathbb{N}$ so that $e_{\mathbf{r}^{n,k}\mathbf{s}}$ is a basis element of $L_2(S_q^{2\ell+1})$. Note that $\mathbf{s}$ is of the form (4.4.28). Define $\gamma\in\Gamma_\Sigma$ as follows:

1. if $k > d_{\ell+1}$, then

$$\begin{aligned}
&\gamma_i = d_{i+1} - d_i && \text{for } 1 \le i \le \ell - 1,\\
&\gamma_i = c_{2\ell+2-i} - c_{2\ell+3-i} && \text{for } \ell + 3 \le i \le 2\ell + 1,\\
&\gamma_\ell = d_{\ell+1} - d_\ell, \quad \gamma_{\ell+1} = k - d_{\ell+1}, \quad \gamma_{\ell+2} = c_\ell - k,
\end{aligned}$$

2. if $k \le d_{\ell+1}$, then

$$\begin{aligned}
&\gamma_i = d_{i+1} - d_i && \text{for } 1 \le i \le \ell - 1,\\
&\gamma_i = c_{2\ell+2-i} - c_{2\ell+3-i} && \text{for } \ell + 3 \le i \le 2\ell + 1,\\
&\gamma_\ell = k - d_\ell, \quad \gamma_{\ell+1} = k - d_{\ell+1}, \quad \gamma_{\ell+2} = c_\ell - d_{\ell+1}.
\end{aligned}$$

Then $\mathbf{s}(\gamma) = \mathbf{s}$. Thus we have a bijective correspondence between $\mathscr{G}_0$ and Γ_Σ. We will often denote a basis element $e_{\mathbf{r}^{n,k}\mathbf{s}}$ by ξ_γ using this bijective correspomdence.

Lemma 4.4.1. *Let $\gamma \in \Gamma_\Sigma$. For $n \in \mathbb{Z}$, let*

$$Z^{(n)}_{\ell+1,0} := \begin{cases} Z^n_{\ell+1,0} & \text{if } n \ge 0,\\ (Z^*_{\ell+1,0})^{-n} & \text{if } n < 0. \end{cases}$$

Define

$$\xi'_\gamma := Z^{\gamma_1}_{1,0} \dots Z^{\gamma_\ell}_{\ell,0} Z^{(\gamma_{\ell+1})}_{\ell+1,0} \begin{pmatrix} \sum_{\ell+2}^{2\ell+1} \gamma_i & 0 & \cdots & 0 & 0\\ \sum_{\ell+2}^{2\ell} \gamma_i & 0 & \cdots & 0 & \\ & \cdots & & & \\ \gamma_{\ell+2} & 0 & & & \\ 0 & & & & \end{pmatrix}.$$

Then $\{\xi'_\gamma : \gamma \in \Gamma_\Sigma\}$ is an orthonormal basis for $L_2(S_q^{2\ell+1})$.

Proof: It follows from equations (4.4.31) and (4.4.32) that the actions of $Z_{j,0}$ for $1 \le j \le \ell$ on the basis elements $e_{\mathbf{r}^{n,k}\mathbf{s}}$ are as follows:

$$Z_{j,0} : \begin{pmatrix} n+k & k & & \cdots & & k & k & 0 \\ c_2 & k & & \cdots & & k & 0 & \\ \cdots & & \cdots & & & & & \\ c_j & k & \cdots & k & 0 & & & \\ c_{j+1} & k & \cdots & d_{j+1} & & & & \\ \cdots & & \cdots & & & & & \\ c_\ell & d_\ell & & & & & & \\ d_{\ell+1} & & & & & & & \end{pmatrix} \longrightarrow$$

$$(-1)^{j-1} \begin{pmatrix} 1+n+k & 1+k & & \cdots & & 1+k & 1+k & 0 \\ 1+c_2 & 1+k & & \cdots & & 1+k & 0 & \\ \cdots & & \cdots & & & & & \\ 1+c_j & 1+k & \cdots & 1+k & 0 & & & \\ 1+c_{j+1} & 1+k & \cdots & 1+d_{j+1} & & & & \\ \cdots & & \cdots & & & & & \\ 1+c_\ell & 1+d_\ell & & & & & & \\ 1+d_{\ell+1} & & & & & & & \end{pmatrix}$$

and is 0 for $\mathbf{s}$ with $d_j > 0$.

Similarly the action of $Z_{\ell+1,0}$ on the basis elements are as follows:

$$\begin{pmatrix} n & 0 & & \cdots & 0 & 0 \\ c_2 & 0 & & \cdots & 0 & \\ \cdots & & \cdots & & & \\ c_{\ell-1} & 0 & 0 & & & \\ c_\ell & 0 & & & & \\ d_{\ell+1} & & & & & \end{pmatrix} \longrightarrow \begin{pmatrix} n-1 & 0 & & \cdots & 0 & 0 \\ c_2-1 & 0 & & \cdots & 0 & \\ \cdots & & \cdots & & & \\ c_{\ell-1}-1 & 0 & 0 & & & \\ c_\ell-1 & 0 & & & & \\ d_{\ell+1}-1 & & & & & \end{pmatrix}$$

if $d_{\ell+1} > 0$, and

$$\begin{pmatrix} n+k & k & & \cdots & k & 0 \\ c_2 & k & & \cdots & 0 & \\ \cdots & & \cdots & & & \\ c_{\ell-1} & k & 0 & & & \\ c_\ell & 0 & & & & \\ 0 & & & & & \end{pmatrix} \longrightarrow (-1)^\ell \begin{pmatrix} 1+n+k & 1+k & & \cdots & 1+k & 0 \\ 1+c_2 & 1+k & & \cdots & 0 & \\ \cdots & & \cdots & & & \\ 1+c_{\ell-1} & 1+k & 0 & & & \\ 1+c_\ell & 0 & & & & \\ 0 & & & & & \end{pmatrix}$$

if $d_{\ell+1} = 0$. Similarly the action of $Z^*_{\ell+1,0}$ on the basis elements are as follows:

$$\begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ c_2 & 0 & \cdots & 0 & \\ \cdots & & \cdots & & \\ c_{\ell-1} & 0 & 0 & & \\ c_\ell & 0 & & & \\ d_{\ell+1} & & & & \end{pmatrix} \longrightarrow \begin{pmatrix} 1+n & 0 & \cdots & 0 & 0 \\ 1+c_2 & 0 & \cdots & 0 & \\ \cdots & & \cdots & & \\ 1+c_{\ell-1} & 0 & 0 & & \\ 1+c_\ell & 0 & & & \\ 1+d_{\ell+1} & & & & \end{pmatrix}$$

and for $k > 0$,

$$\begin{pmatrix} n+k & k & \cdots & k & 0 \\ c_2 & k & \cdots & 0 & \\ \cdots & & \cdots & & \\ c_{\ell-1} & k & 0 & & \\ c_\ell & 0 & & & \\ 0 & & & & \end{pmatrix} \longrightarrow (-1)^\ell \begin{pmatrix} n+k-1 & k-1 & \cdots & k-1 & 0 \\ c_2-1 & k-1 & \cdots & 0 & \\ \cdots & & \cdots & & \\ c_{\ell-1}-1 & k-1 & 0 & & \\ c_\ell-1 & 0 & & & \\ 0 & & & & \end{pmatrix}$$

Then it follows from the above that

$$\begin{aligned} & Z_{1,0}^{\gamma_1} \ldots Z_{\ell,0}^{\gamma_\ell} Z_{\ell+1,0}^{(\gamma_{\ell+1})} \begin{pmatrix} \sum_{\ell+2}^{2\ell+1} \gamma_i & 0 & \cdots & 0 & 0 \\ \sum_{\ell+2}^{2\ell} \gamma_i & 0 & \cdots & 0 & \\ & \cdots & & & \\ \gamma_{\ell+2} & 0 & & & \\ 0 & & & & \end{pmatrix} \\ = \quad & (-1)^{\eta(\gamma)} \begin{pmatrix} \sum_1^{2\ell+1} |\gamma_i| & \sum_1^\ell \gamma_i + (\gamma_{\ell+1})_+ & \cdots & \sum_1^\ell \gamma_i + (\gamma_{\ell+1})_+ & 0 \\ \sum_1^{2\ell} |\gamma_i| & \sum_1^\ell \gamma_i + (\gamma_{\ell+1})_+ & \cdots & \gamma_1 & \\ \cdots & \cdots & & & \\ \sum_1^{\ell+3} |\gamma_i| & \sum_1^\ell \gamma_i + (\gamma_{\ell+1})_+ & \sum_1^{\ell-2} \gamma_i & & \\ \sum_1^{\ell+2} |\gamma_i| & \sum_1^{\ell-1} \gamma_i & & & \\ \sum_1^\ell \gamma_i + (\gamma_{\ell+1})_- & & & & \end{pmatrix}, \end{aligned} \tag{4.4.33}$$

where $\eta(\gamma) := \sum_{i=1}^{\ell}(i-1)\gamma_i + \ell(\gamma_{\ell+1})_+$. Thus $\xi'_\gamma = (-1)^{\eta(\gamma)}\xi_\gamma$. Therefore it follows that $\{\xi'_\gamma : \gamma \in \Gamma_\Sigma\}$ is an orthonormal basis for $L_2(S_q^{2\ell+1})$. $\square$

The map $U : L_2(S_q^{2\ell+1}) \to \mathcal{H}_\Sigma$ given by $U\xi'_\gamma = e_\gamma$ sets up a unitary isomorphism between $L_2(S_q^{2\ell+1})$ and $\mathcal{H}_\Sigma$. Let P denote the projection onto the span of $e_0 \otimes \cdots \otimes e_0$ in $\ell_2(\mathbb{N}^\ell)$. Then we have

$$UZ_{j,0}U^* = Y_{j,0} \otimes I = Y_{j,0} \otimes P + Y_{j,0} \otimes (I-P), \tag{4.4.34}$$

and

$$UD_{eq}U^* = D_\ell \otimes P - |D_\ell| \otimes (I - P) - I \otimes \widetilde{N}, \tag{4.4.35}$$

where $\widetilde{N}$ is the operator $e_{m_1} \otimes \cdots \otimes e_{m_\ell} \mapsto (\sum m_i)\, e_{m_1} \otimes \cdots \otimes e_{m_\ell}$. In other words, with respect to the decomposition

$$\mathcal{H}_\Sigma = \mathcal{H}_\ell \oplus \left(\mathcal{H}_\ell \otimes \ell_2(\mathbb{N}^\ell \backslash \{0, \ldots, 0\})\right),$$

one has

$$UZ_{j,0}U^* = Y_{j,0} \oplus (Y_{j,0} \otimes I),$$

and

$$UD_{eq}U^* = D_\ell \oplus \left(-|D_\ell| \otimes I - I \otimes \widetilde{N}\right).$$

Next we will define the smooth function algebra $C^\infty_{eq}(S_0^{2\ell+1})$ and prove that the spectral triple $(C^\infty_{eq}(S_0^{2\ell+1}), \mathcal{H}, D_{eq})$ is regular with simple dimension spectrum $\{1, 2, \ldots, 2\ell + 1\}$.

It follows from decomposition (4.4.34) that if we identify $L_2(S_q^{2\ell+1})$ with $\mathcal{H}_\Sigma$, then the C^*-algebra generated by the $Z_{j,0}$'s is $A_\ell \otimes I$, where A_ℓ is the C^*-algebra generated by the Y_j's in $\mathcal{L}(\mathcal{H}_\ell)$. In view of the decomposition (4.4.34–4.4.35), it is natural to define

$$C^\infty_{eq}(S_0^{2\ell+1}) = \{a \otimes I : a \in \mathcal{A}_\ell^\infty\}. \tag{4.4.36}$$

Theorem 4.4.2. *The triple $(C^\infty_{eq}(S_0^{2\ell+1}), \mathcal{H}_\Sigma, D_{eq})$ is a regular spectral triple with simple dimension spectrum $\{1, 2, \ldots, 2\ell + 1\}$.*

Proof: Since $\mathcal{A}_\ell^\infty$ is closed under holomorphic function calculus in A_ℓ, it follows that $C^\infty_{eq}(S_0^{2\ell+1})$ is closed under holomorphic function calculus in $C^*(\{Z_{j,0} : 1 \le j \le \ell + 1\}) = A_\ell \otimes I$. In order to show regularity, let us introduce the algebra

$$\mathcal{B}_{eq} := \{a \otimes P + b \otimes (I - P) : a, b \in \mathcal{B}_\ell\}. \tag{4.4.37}$$

Clearly $\mathcal{B}_{eq}$ contains $C^\infty_{eq}(S_0^{2\ell+1})$. We will show that $\mathcal{B}_{eq}$ is closed under derivations with both $|D_{eq}|$ as well as D_{eq}. This will prove regularity of the spectral triple $(C^\infty_{eq}(S_0^{2\ell+1}), \mathcal{H}, D_{eq})$.

Note that $|D_{eq}| = |D_\ell| \otimes I + I \otimes \widetilde{N}$. Since $I \otimes \widetilde{N}$ commutes with every element of $\mathcal{B}_{eq}$, we get $\delta(a \otimes P + b \otimes (I - P)) = [|D_\ell|, a] \otimes P + [|D_\ell|, b] \otimes (I - P)$ and $[D_{eq}, a \otimes P + b \otimes (I - P)] = [D_\ell, a] \otimes P - [|D_\ell|, b] \otimes (I - P)$. Since $\mathcal{B}_\ell$ is closed under derivations with $|D_\ell|$ and D_ℓ, it follows that $\mathcal{B}_{eq}$ is closed under derivations with $|D_{eq}|$ and D_{eq}.

Next we compute the dimension spectrum of the spectral triple. For $w \in \mathbb{T}^{\ell+1}$, let $\tilde{U}_w := U_w \otimes I$. Then $|D_{eq}|$ commutes with $\tilde{U}_w$. Hence again it is enough to consider homogeneous elements of degree 0. Now by lemma 3.4.2 it follows that for $b \in \mathcal{B}_{eq}$ with b homogeneous of degree 0, the function $Trace(b|D_{eq}|^{-z})$ is meromorphic with simple poles and the poles lie in $\{1, 2, \cdots, 2\ell + 1\}$. To show that every point of $\{1, 2, \cdots, 2\ell + 1\}$ is in the dimension spectrum, observe that

$$Trace(|D_{eq}|^{-z}) = \Sigma_{k=0}^{2\ell}(2c_k^{2\ell} - c_k^{2\ell-1})\zeta(z - k) \tag{4.4.38}$$

where c_k^r is defined as the coefficient of N^k in $\binom{N+r}{r}$. Note that for $0 \leq k \leq r$ one has $c_k^r > 0$. Also note the recurrence $rc_k^r = c_{k-1}^{r-1} + rc_k^{r-1}$. Hence $c_k^r \geq c_k^{r-1}$. Now from equation (4.4.38) it follows that $Res_{z=k+1}Trace(|D_{eq}|^{-z}) = 2c_k^{2\ell} - c_k^{2\ell-1} > 0$ for $0 \leq k \leq 2\ell$. This proves that every point of $\{1, 2, \cdots, 2\ell + 1\}$ is in the dimension spectrum. This completes the proof. □

We will need the fact that $Trace(|D_{eq}|^{-z})$ is meromorphic with simple poles lying inside $\{1, 2, \cdots, 2\ell + 1\}$ with non-zero residue and hence we state it as a separate lemma.

Lemma 4.4.3. *The function $Trace(|D_{eq}|^{-z})$ is meromorphic with simple poles at $\{1, 2, \cdots, 2\ell + 1\}$. Also for $k \in \{1, 2, \cdots, 2\ell + 1\}$, the residue $Res_{z=k}Trace(|D_{eq}|^{-z})$ is non-zero.*

4.5 Regularity and dimension spectrum for $q \neq 0$

Consider the smooth subalgebra of the Toeplitz algebra defined as:

$$\mathscr{T}^\infty = \left\{ \sum_{j,k \in \mathbb{N}} \lambda_{jk} \ S^{*j} p_0 S^k + \sum_{k \geq 0} \lambda_k S^k + \sum_{k > 0} \lambda_{-k} S^{*k} : \lambda_{jk}, (\lambda_k) \text{ are rapidly decreasing .} \right\}$$

For $a := \sum_{j,k\in\mathbb{N}} \lambda_{jk} \ S^{*j} p_0 S^k + \sum_{k\geq 0} \lambda_k S^k + \sum_{k>0} \lambda_{-k} S^{*k} \in \mathscr{T}^\infty$, define the seminorm $\| \cdot \|_m$ by $\|a\|_m := \sum (1 + |j| + |k|)^m |\lambda_{kl}| + \sum (1 + |k|)^m |\lambda_k|$. Equipped with this family of seminorms, $\mathscr{T}^\infty$ is a Fréchet algebra. We will denote by $\mathscr{T}_k^\infty$ the k-fold tensor product of $\mathscr{T}^\infty$.

Lemma 4.5.1. *The triple $(\mathscr{T}^\infty, \ell_2(\mathbb{N}), N)$ is a regular spectral triple. More precisely, $\mathscr{T}^\infty$ is contained in $Dom(\delta)$ where δ is the unbounded derivation $[N, .]$ and δ leaves the algebra $\mathscr{T}^\infty$ invariant. Also the map $\delta : \mathscr{T}^\infty \to \mathscr{T}^\infty$ is continuous.*

Proof: Note that $[N, S] = -S$ and $[N, p] = 0$. Now the lemma follows from the fact that the unbounded derivation δ is closed. □

For $\alpha \in \mathbb{N}^2 \cup \mathbb{Z}$, let

$$W_\alpha = \begin{cases} S^{*m} p_0 S^n & \text{if} \quad \alpha = (m, n), \\ S^r & \text{if} \quad \alpha = r \geq 0, \\ S^{*r} & \text{if} \quad \alpha = r < 0. \end{cases}$$

For $\alpha \in \mathbb{N}^2 \cup \mathbb{Z}$, define $|\alpha|$ to be $|m| + |n|$ if $\alpha = (m, n) \in \mathbb{N}^2$ and the usual absolute value $|\alpha|$ if $\alpha \in \mathbb{Z}$. For an ℓ tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\ell)$ in $(\mathbb{N}^2 \cup \mathbb{Z})^\ell$, let $|\alpha| = \sum |\alpha_i|$ and $W_\alpha := W_{\alpha_1} \otimes W_{\alpha_2} \otimes \cdots W_{\alpha_\ell}$. We need the following simple lemma whose proof we omit as it is easy to prove.

Lemma 4.5.2. *The natural tensor product representation of $\mathscr{T}_\ell^\infty$ on $\ell_2(\mathbb{N})^{\otimes \ell}$ is injective. Thus we identify $\mathscr{T}_\ell^\infty$ with it's range which is $\{\sum x_\alpha W_\alpha : \sum (1 + |\alpha|)^p |x_\alpha| < \infty$ for every $p\}$.*

Remark 4.5.3. The tensor product representation of $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$ on $\mathcal{L}(\mathcal{H}_\Sigma)$ is injective since $OP^{-\infty}_{D_\ell} := \mathcal{S}(\mathcal{H}_\ell)$ and hence we identify $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$ with its image.

For an operator T, let L_T denote the left multiplication map $X \mapsto TX$. Then for $T \in OP^0_{D_\ell}$, the map $L_T : OP^{-\infty}_{D_\ell} \to OP^{-\infty}_{D_\ell}$ is continuous. Note that if A is a Fréchet algebra and $a \in A$, then L_a is a continuous linear operator.

Lemma 4.5.4. *Let $T \in OP^0_{D_\ell}$ and $a \in \mathscr{T}^\infty_\ell$. Then the map $L_{T\otimes a}$ leaves the algebra $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$ invariant. Moreover $L_{T\otimes a} = L_T \otimes L_a$ on the algebra $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$.*

Proof: Clearly $L_{T\otimes a} = L_T \otimes L_a$ on the algebraic tensor product $OP^{-\infty}_{D_\ell} \otimes_{alg} \mathscr{T}^\infty_\ell$. Now let $a \in OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$. Then there exists a sequence $a_n \in OP^{-\infty}_{D_\ell} \otimes_{alg} \mathscr{T}^\infty_\ell$ which converges to a in $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$. Also a_n converges to a in the operator norm. Now the result follows from the continuity of $L_{T\otimes a}$ and $L_T \otimes L_a$. □

Proposition 4.5.5. *Let*

$$\mathscr{B} := \mathcal{B}_{eq} + OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell \tag{4.5.39}$$

Then one has the following.

1. *The vector space $\mathscr{B}$ is an algebra.*
2. *The algebra $\mathscr{B}$ is invariant under the derivations $\delta := [|D_{eq}|, .]$ and $[D_{eq}, .]$.*
3. *For $b \in \mathscr{B}$, the commutator $[F_{eq}, b] \in OP^{-\infty}_{D_{eq}}$.*
4. *For $b \in \mathscr{B}$, the function $Trace(b|D_{eq}|^{-z})$ is meromorphic with only simple poles and the poles lie in $\{1, 2, \cdots, 2\ell + 1\}$.*

Proof: Lemma 4.5.4 and the fact that $\mathcal{B}_\ell \subset OP^0$ implies that $\mathscr{B}$ is an algebra. As seen in Theorem 4.4.2, it follows that $\mathcal{B}_{eq}$ is invariant under δ and $[D_{eq}, .]$. Also (3) and (4) holds for $b \in \mathcal{B}_{eq}$. Hence to complete the proof it is enough to consider (2), (3) and (4) for the algebra $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$.

Lemma 2.3.14 and the decomposition $|D_{eq}| = |D_\ell| \otimes 1 + 1 \otimes \tilde{N}$ implies that δ leaves the algebra $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$ invariant. Now note that $P \in OP^{-\infty}_{\tilde{N}}$, it follows that left and right multiplication by $F_\ell \otimes P$ and $1 \otimes P$ sends $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$ to $OP^{-\infty}_{D_{eq}} \equiv OP^{-\infty}_{D_\ell} \otimes OP^{-\infty}_{\tilde{N}}$. Now since $F_{eq} = F_\ell \otimes P - I \otimes (I - P)$, it follows that $[F_{eq}, b]$ is smoothing for every $b \in OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty$. Now the invariance of $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$ under $[D_{eq}, .]$ follows from the equation $[D_{eq}, b] = \delta(b)F_{eq} + |D_{eq}|[F_{eq}, b]$ and the fact that $OP^{-\infty}_{D_{eq}} := OP^{-\infty}_{D_\ell} \otimes OP^{-\infty}_{\tilde{N}}$ is contained in $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$.

Now we will prove that for $b \in OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$, the function $Trace(b|D_{eq}|^{-z})$ is meromorphic with simple poles and the poles lie in $\{1, 2, \cdots, \ell\}$. For $w \in \mathbb{T}^{2\ell+1}$, let $U_w = U_{w_1} \otimes U_{w_2} \otimes \cdots U_{w_{2\ell+1}}$ be the unitary operator on $\mathcal{H}_\Sigma$. Clearly $U_w|D_{eq}|U^*_w = |D_{eq}|$ for $w \in \mathbb{T}^{2\ell+1}$. Hence it is enough to consider $Trace(b|D_{eq}|^{-z})$ with b homogeneous of degree 0.

An element b is homogeneous if and only if it commutes with the operators U_w for all $w \in \mathbb{T}^{2\ell+1}$. This implies b must be of the form $e_\gamma \mapsto \phi(\gamma)e_\gamma$ for some function ϕ, i.e. $b = \sum_\gamma \phi(\gamma)p_\gamma$. An operator of the form $\sum_{\gamma \in \Gamma_{\Sigma_\ell}} \phi(\gamma)p_\gamma$ is in $OP^{-\infty}_{D_\ell}$ if and only if $\phi(\gamma)$ is rapidly decaying on Γ_{Σ_ℓ}. Also, using the description of $\mathscr{T}^\infty$, it follows that an operator of the form $\sum_{n\in\mathbb{N}} \phi(n)p_n$ belongs to $\mathscr{T}^\infty$ if and only if $\phi(\cdot) - \lim_{n\to\infty} \phi(n)$ is rapidly decreasing. Thus combining these, one can see that the operator $\sum_\gamma \phi(\gamma)p_\gamma$ belongs to $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$ if and only if ϕ is a linear combination of ϕ_A with A varying over subsets of Σ containing Σ_ℓ, where each $\phi_A(\gamma)$ depends only on γ_A and $\phi_A(\gamma_A)$ is rapidly decreasing on Γ_A. For an element $b = \sum_\gamma \phi_A(\gamma)p_\gamma$, one has

$$Trace(b|D_{eq}|^{-z}) = \sum_\gamma \frac{\phi_A(\gamma)}{|\gamma|^z} = \sum_\gamma \frac{\phi_A(\gamma_A)}{(|\gamma_A| + |\gamma_{\Sigma\backslash A}|)^z}.$$

By lemma 3.4.2 it follows that $Trace(b|D_{eq}|^{-z})$ is meromorphic with simple poles and the poles lie in $\{1, 2, \cdots, |\Sigma\backslash A|\} \subseteq \{1, 2, \ldots, \ell\}$. This completes the proof. □

4.6 The smooth function algebra $C^\infty(S_q^{2\ell+1})$

In this subsection, we will define a dense $*$ Fréchet algebra $C^\infty(S_q^{2\ell+1})$ of $C_\ell = \pi(C(S_q^{2\ell+1}))$ and show that it is closed under holomorphic functional calculus. Let B_ℓ be the C^* algebra generated by A_ℓ and F_ℓ. Note that B_ℓ contains $\mathcal{K}(\ell_2(\mathbb{N})^{\otimes \ell} \otimes \ell_2(\mathbb{Z}))$. Recall that $\mathcal{E}$ denotes the C^* algebra generated by $C(\mathbb{T})$ and F_0.

Lemma 4.6.1. *The C^* algebra $\mathcal{E}$ contains $\mathcal{K}$ and $\mathcal{E}/\mathcal{K}$ is isomorphic to the C^* algebra $C(\mathbb{T}) \oplus C(\mathbb{T})$.*

Proof: Let $|e_m\rangle\langle e_n|$ be the matrix units in $\mathcal{K}(\ell_2(\mathbb{Z}))$. Note that $[F_0, S^*]S = 2|e_0\rangle\langle e_0|$. Hence $p_0 \equiv |e_0\rangle\langle e_0| \in \mathcal{E}$. Now $S^{*m}p_0S^n = |e_m\rangle\langle e_n|$. Hence $\mathcal{K} \subset \mathcal{E}$. Let $P_0 := \frac{1+F_0}{2}$. Then $[P_0, f]$ is compact for every f. Thus $\mathcal{E}/\mathcal{K}$ is generated by $C(\mathbb{T})$ and a projection P_0 which is in the center of $\mathcal{E}/\mathcal{K}$. Now consider the map

$$C(\mathbb{T}) \oplus C(\mathbb{T}) \ni (f, g) \mapsto fP_0 + g(1 - P_0) \quad (mod\ \mathcal{K}) \in \mathcal{E}/\mathcal{K}.$$

We claim that this map is an isomorphism. To prove we need to show that if fP_0 is compact then $f = 0$ and if $g(1 - P_0)$ is compact then $g = 0$.

Assume that fP_0 is compact for $f \in C(\mathbb{T})$. Fix an $r \in \mathbb{Z}$. Since fP_0 is compact, it follows that $|\langle fP_0(e_n), e_{n+r}\rangle| = |\hat{f}(r)|$ converges to 0 as $n \to +\infty$. Hence $\hat{f}(r) = 0$ for every r. This proves that $f = 0$. Similarly one can show that if $g(1 - P_0)$ is compact then $g = 0$. This completes the proof. □

Lemma 4.6.2. *The C^*-algebra B_ℓ contains $\mathcal{K}(\mathcal{H}_\ell)$ and the map $(a, b) \mapsto aP_\ell + b(1 - P_\ell)$ mod $\mathcal{K}$ from $C(S_q^{2\ell+1}) \oplus C(S_q^{2\ell+1})$ to $B_\ell/\mathcal{K}(\mathcal{H}_\ell)$ is an isomorphism.*

Proof: For $\ell = 0$ this is just lemma 4.6.1. So let us prove the statement for $\ell \geq 1$. Since A_ℓ contains $\mathcal{K}(\ell_2(\mathbb{N}^\ell)) \otimes C(\mathbb{T})$, it follows that B_ℓ contains $\mathcal{K}(\mathcal{H}_\ell)$. Observe that $[P_\ell, \alpha_i] = 0$ for $1 \leq i \leq \ell$ and $[P_\ell, \alpha_{\ell+1}]$ is compact. Therefore it follows that $[P_\ell, a]$ is compact for every $a \in A_\ell$. Hence the map $(a, b) \mapsto aP_\ell + b(1 - P_\ell)$ *mod* $\mathcal{K}$ from $A_\ell \oplus A_\ell$ to $B_\ell/\mathcal{K}$ is a $*$ algebra homomorphism onto $B_\ell/\mathcal{K}$. We will show that the map is one-one. For that we have to show if aP_ℓ is compact with $a \in A_\ell$ then $a = 0$ and if $b(1 - P_\ell)$ is compact with $b \in A_\ell$ then $b = 0$.

Suppose now that aP_ℓ is compact. Observe that $B_\ell \subset \mathscr{T}_\ell \otimes \mathcal{E}$ and $aP_\ell = a(I \otimes P_0)$. Since $aP_\ell = 0$, if we apply the symbol map σ on the ℓ^{th} copy of $\mathscr{T}$, we get $\sigma_\ell(a) \otimes P_0 = 0$. Hence a is in the ideal $\mathcal{K}(\ell_2(\mathbb{N})^{\otimes \ell}) \otimes C(\mathbb{T})$. For $m, n \in \mathbb{N}^\ell$, let e_{mn} be the "matrix" units. Let $a_{mn} = (e_{mm} \otimes 1)a(e_{nn} \otimes 1)$. Then $a_{mn} = e_{mn} \otimes f_{mn}$ for some $f_{mn} \in C(\mathbb{T})$. Since aP_ℓ is compact, it follows that $f_{mn}P_0$ is compact as $P_\ell = I \otimes P_0$ commutes with $e_{nn} \otimes I$. By the $\ell = 0$ case, it follows that $f_{mn} = 0$ and hence $a_{mn} = 0$ for every m, n. Thus $a = 0$. Similarly one can show that if $b(1 - P_\ell)$ is compact then $b = 0$. This completes the proof. □

Let $\mathbb{B}$ be the C^* algebra on $\mathcal{H}_\Sigma$ generated by $A_\ell \otimes I$, $P_\ell \otimes 1$ and $1 \otimes P$ and $J := \mathcal{K}(\mathcal{H}_\ell) \otimes \mathscr{T}_\ell$. Note that J is an ideal since $\mathbb{B}_\ell$ is contained in $\mathscr{T}_\ell \otimes \mathcal{E} \otimes \mathscr{T}_\ell$. The next proposition identifies the quotient $\mathbb{B}/J$.

Proposition 4.6.3. *Let $\rho : A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell \to \mathbb{B}/J$ be the map*

$$(a_1, a_2, a_3, a_4) \mapsto a_1P_\ell \otimes P + a_2P_\ell \otimes (1 - P) + a_3(1 - P_\ell) \otimes P + a_4(1 - P_\ell) \otimes (1 - P)$$

from $A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell$ into $\mathbb{B}$ composed with the canonical projection from $\mathbb{B}$ onto $\mathbb{B}/J$. Then ρ is an isomorphism.

Proof: First note that since $[P_\ell, a] \in \mathcal{K}$ for $a \in A_\ell$, it follows that $P_\ell \otimes I$ and $I \otimes P$ are in the center of $\mathbb{B}/J$. Hence the map ρ is an algebra homomorphism. By the definition of $\mathbb{B}$ it follows that ρ is onto. Thus we have to show ρ is one-one.

Suppose that $a = a_1P_\ell \otimes P + a_2P_\ell \otimes (1 - P) + a_3(1 - P_\ell) \otimes P + a_4(1 - P_\ell) \otimes (1 - P) \in J$. Let $\epsilon : \mathscr{T} \to \mathbb{C}$ be the map $ev_1 \circ \sigma$, where ev_1 is evaluation at the point 1. Now consider the map $id \otimes \epsilon^{\otimes \ell} : \mathscr{T}_\ell \otimes \mathcal{E} \otimes \mathscr{T}_\ell \to \mathscr{T}_\ell \otimes \mathcal{E}$. Note that $I \otimes \epsilon^{\otimes \ell}$ sends J to $\mathcal{K}(\mathcal{H}_\ell)$. Hence $(I \otimes \epsilon^{\otimes \ell})(a) = a_2P_\ell + a_4(1 - P_\ell) \in \mathcal{K}(\mathcal{H}_\ell)$. Hence by lemma 4.6.2, it follows that $a_2 = 0 = a_4$. Since left multiplication by $I \otimes P$ sends the ideal J to $\mathcal{K}(\mathcal{H}_\Sigma)$. It follows that $(I \otimes P)a = a_1P_\ell \otimes P + a_3(1 - P_\ell) \otimes P$ is compact. Hence $a_1P_\ell + a_3(1 - P_\ell)$ is compact. Thus again by lemma 4.6.2, it follows that $a_1 = 0 = a_3$. This completes the proof. □

Now we prove that $\mathscr{B}$ is closed under holomorphic functional calculus in $\mathbb{B}$. Let $\mathscr{J} := OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$. Note that

$$\begin{aligned}\mathscr{B} :=\{&a_1P_\ell \otimes P + a_2P_\ell \otimes (1 - P) + a_3(1 - P_\ell) \otimes P + a_4(1 - P_\ell) \otimes (1 - P) + R : \\ &a_1, a_2, a_3, a_4 \in A^\infty_\ell, R \in \mathscr{J}\}\end{aligned}$$

Proposition 4.6.4. *The algebra $\mathscr{B}$ has the following properties:*

1. *If $a_1 P_\ell \otimes P + a_2 P_\ell \otimes (1-P) + a_3(1-P_\ell) \otimes P + a_4(1-P_\ell) \otimes (1-P) \in \mathscr{J}$ then $a_i = 0$ for $i = 1,2,3,4$. Hence $\mathscr{B}$ is isomorphic to the direct sum $A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \oplus \mathscr{J}$. Equip $\mathscr{B}$ with the Fréchet space structure coming from this direct sum decomposition.*

2. *The algebra $\mathscr{B}$ is a $*$-Fréchet algebra contained in $\mathbb{B}$. Moreover the inclusion $\mathscr{B} \subset \mathbb{B}$ is continuous.*

3. *The algebra $\mathscr{B}$ is closed under holomorphic functional calculus in $\mathbb{B}$.*

Proof: Proposition 4.6.3 implies (1). Parts (2) and (3) follows from proposition 3.4.1. Now by proposition 4.6.3 one has the exact sequence

$$0 \to J \to \mathbb{B} \to A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell \to 0.$$

At the smooth algebra level we have the following exact sequence

$$0 \to \mathscr{J} \to \mathscr{B} \xrightarrow{\theta} A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \oplus A_\ell^\infty \to 0.$$

Since $\mathscr{J} \subset J$ and $A_\ell^\infty \subset A_\ell$ are closed under holomorphic functional calculus, it follows from Lemma A.1.4 that $\mathscr{B}$ is spectrally invariant in $\mathbb{B}$. Since by part (2), the Fréchet topology of $\mathscr{B}$ is finer than the norm topology, it follows that $\mathscr{B}$ is closed in the holomorphic function calculus of $\mathbb{B}$. $\square$

Remark 4.6.5. One can prove that $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ is closed under holomorphic functional calculus in $\mathcal{K}(\mathcal{H}_\ell) \otimes \mathscr{T}_\ell$ in the same manner by applying theorem 3.2, part 2, [34] and by using the extension (after tensoring suitably)

$$0 \to \mathcal{K} \to \mathscr{T} \to C(\mathbb{T}) \to 0$$

at the C^* algebra level and the extension

$$0 \to \mathcal{S}(\ell_2(\mathbb{N})) \to \mathscr{T}^\infty \to C^\infty(\mathbb{T}) \to 0$$

at the Fréchet algebra level.

Corollary 4.6.6. *Define the smooth function algebra $C^\infty(S_q^{2\ell+1})$ by*

$$C^\infty(S_q^{2\ell+1}) = \{a \in \mathscr{B} \cap C_\ell : \theta(a) \in \iota(\mathcal{A}_\ell^\infty)\},$$

where θ is as in the proof of proposition 4.6.4 and $\iota : A_\ell \to A_\ell \oplus A_\ell \oplus A_\ell \oplus A_\ell$ is the inclusion map $a \mapsto a \oplus a \oplus a \oplus a$. Then the algebra $C^\infty(S_q^{2\ell+1})$ is closed in $\mathscr{B}$ and it is closed under holomorphic functional calculus in C_ℓ.

Proof: Let $j : \mathscr{B} \to \mathcal{L}(\mathcal{H}_\Sigma)$ denote the inclusion map. Then by definition $C^\infty(S_q^{2\ell+1}) = \theta^{-1}(\iota(A_\ell^\infty)) \cap j^{-1}(C_\ell)$. Since θ and j are continuous and as $\iota(A_\ell^\infty)$ and C_ℓ are closed, it follows that $C^\infty(S_q^{2\ell+1})$ is closed in $\mathscr{B}$. Hence $C^\infty(S_q^{2\ell+1})$ is a Fréchet algebra. Also $C^\infty(S_q^{2\ell+1})$ is $*$-closed as ρ is $*$-preserving. Now let $a \in C^\infty(S_q^{2\ell+1})$ be invertible in C_ℓ. Then a is invertible in $\mathcal{L}(\mathcal{H}_\Sigma)$. By proposition 4.6.4, it follows that $a^{-1} \in \mathscr{B}$. By the closedness of A_ℓ^∞ under holomorphic functional calculus, it follows that $\theta(a^{-1}) \in \iota(A_\ell^\infty)$. Thus one has $a^{-1} \in C^\infty(S_q^{2\ell+1})$. We have already seen that the Fréchet topology of $\mathscr{B}$ is finer than the norm topology. Same is therefore true for the topology of $C^\infty(S_q^{2\ell+1})$. Hence it is closed under holomorphic functional calculus in C_ℓ. □

Proposition 4.6.7. *The operators $Z_{j,q} \in C^\infty(S_q^{2\ell+1})$. Hence $C^\infty(S_q^{2\ell+1})$ is a dense subalgebra of C_ℓ that contains $\pi(\mathcal{A}(S_q^{2\ell+1}))$.*

The proof of this proposition will be given in the next subsection.

We are now in a position to prove the main theorem.

Theorem 4.6.8. *The triple $(C^\infty(S_q^{2\ell+1}), \mathcal{H}_\Sigma, D_{eq})$ is a regular spectral triple with simple dimension spectrum $\{1, 2, \cdots, 2\ell+1\}$.*

Proof: Since $C^\infty(S_q^{2\ell+1}) \subset \mathscr{B}$ the regularity of the spectral triple $(C^\infty(S_q^{2\ell+1}), \mathcal{H}_\Sigma, D_{eq})$ follows from the regularity of the spectral triple $(\mathscr{B}, \mathcal{H}_\Sigma, D_{eq})$ which is proved in proposition **??**. Proposition **??** also implies that the spectral triple has simple dimension spectrum which is a subset of $\{1, 2, \cdots, 2\ell+1\}$. The fact that every point in $\{1, 2, \cdots, 2\ell+1\}$ is in the dimension spectrum follows from lemma 4.4.3. This completes the proof. □

4.7 The operators $Z_{j,q}$

We will give a proof of proposition 4.6.7 in this subsection. The main idea will be to exploit the isomorphism between the Hilbert spaces $L_2(S_q^{2\ell+1})$ and $\mathcal{H}_\Sigma$ and a detailed analysis of the operators $Z_{j,q}$ to show that certain parts of these operators can be ignored for the purpose of establishing regularity and computing dimension spectrum. Deciding and establishing which parts of these operators can be ignored is the key step here. It should be noted here that a similar analysis has been done by D'Andrea in [18], where he embeds $L_2(S_q^{2\ell+1})$ in a bigger Hilbert space and proves certain approximations for the operators $Z_{j,q}$. But the approximation there is not strong enough to enable the computation of dimension spectrum. Here we prove stronger versions of those approximations, which have made it possible to use them to compute the dimension spectrum in the previous subsection.

We start with a few simple lemmas that we will use repeatedly during the computations in this subsection.

Lemma 4.7.1. *Let $A \subseteq B \subseteq \Sigma$. Then one has $OP^{-\infty}_{D_B} \otimes \mathcal{E}^{\infty}_{\Sigma\setminus B} \subseteq OP^{-\infty}_{D_A} \otimes \mathcal{E}^{\infty}_{\Sigma\setminus A}$.*

Proof: Since

$$OP^{-\infty}_{D_B} = \mathcal{S}(\mathcal{H}_B) = \mathcal{S}(\mathcal{H}_A) \otimes \mathcal{S}(\mathcal{H}_{B\setminus A}) = OP^{-\infty}_{D_A} \otimes \mathcal{S}(\mathcal{H}_{B\setminus A}),$$

and $\mathcal{S}(\mathcal{H}_{B\setminus A}) \subseteq \mathcal{E}^{\infty}_{B\setminus A}$, we have the required inclusion. □

Let $A \subseteq \Sigma$. Let $\mathcal{P}$ be a polynomial in $|A|$ variables and let T be the operator on $\mathcal{H}_A$ given by

$$Te_\gamma = \mathcal{P}(\{\gamma_i, i \in A\})q^{|\gamma_A|}e_\gamma.$$

Since the function $\gamma \mapsto \mathcal{P}(\{\gamma_i, i \in A\})q^{|\gamma_A|}$ is a rapid decay function on Γ_A, it follows that $T \in OP^{-\infty}_{D_A}$.

Lemma 4.7.2. *Let $A \subseteq \Sigma$. Let T and T_0 be the following operators on $\mathcal{H}_A$:*

$$Te_\gamma = q^{\phi(\gamma_A)}Q(\psi(\gamma_A))e_\gamma, \qquad T_0e_\gamma = q^{\phi(\gamma_A)}e_\gamma,$$

where ϕ and ψ are some nonnegative functions. If $\phi(\gamma_A) + \psi(\gamma_A) > |\gamma_A|$, then then $T - T_0 \in OP^{-\infty}_{D_A}$.

Proof: This is a consequence of the inequality $|1-(1-x)^{\frac{1}{2}}| < x$ for $0 \le x \le 1$. □

Lemma 4.7.3. *Let $A \subseteq \Sigma$. Let T and T_0 be operators on $\mathcal{H}_A$ given by:*

$$Te_\gamma = q^{\phi(\gamma_A)}Q(\psi(\gamma_A))^{-1}e_\gamma, \qquad T_0e_\gamma = q^{\phi(\gamma_A)}e_\gamma$$

for some nonnegative functions ϕ and ψ. If $\phi(\gamma_A) + \psi(\gamma_A) > |\gamma_A|$, then $T - T_0 \in OP^{-\infty}_{D_A}$.

Proof: For $0 < r < 1$, one has

$$|1-(1-x)^{-\frac{1}{2}}| < cx \quad \text{for } 0 \le x \le r,$$

where c is some fixed constant that depends on r. Using this, it follows that the map $\gamma \mapsto q^{\phi(\gamma)}|1-(1-q^{2\psi(\gamma)})^{-\frac{1}{2}}|$ is a rapid decay function on Γ_A. □

For $j \in \Sigma$, we will denote by $\mathcal{E}_j$ the C^*-algebra $\mathscr{T}$ if $j \neq \ell+1$ and the C^*-algebra $\mathcal{E}$ if $j = \ell+1$. Thus $\mathcal{E}^{\infty}_j$ will be $\mathscr{T}^{\infty}$ for $j \neq \ell+1$ and $\mathcal{E}^{\infty} = \mathcal{B}$ for $j = \ell+1$. Thus $\mathcal{E}^{\infty}_{\Sigma}$ will stand for the space $\mathscr{T}^{\infty}_{\ell} \otimes \mathcal{E}^{\infty} \otimes \mathscr{T}^{\infty}_{\ell}$. Note that for any subset A of Σ, one has $OP^{-\infty}_{D_A} \subseteq \mathcal{E}^{\infty}_{\Sigma}$.

Lemma 4.7.4. *Let $A \subseteq \Sigma$, $a, b, m, n \in \mathbb{N}$ and $n > 0$. Let T_1 and T_2 be the operators on $\mathcal{H}_\Sigma$ given by*

$$T_1e_\gamma = Q(|\gamma_A| + a(\gamma_{\ell+1})_+ + b(\gamma_{\ell+1})_- + m)e_\gamma, \qquad T_2e_\gamma = Q(|\gamma_A| + a(\gamma_{\ell+1})_+ + b(\gamma_{\ell+1})_- + n)^{-1}e_\gamma.$$

Then T_1 and T_2 are in $\mathcal{E}^{\infty}_{\Sigma}$.

Proof: First note that if T_1' and T_1'' are operators given by

$$T_1' e_\gamma = Q(|\gamma_A| + a|\gamma_{\ell+1}| + m)e_\gamma, \qquad T_1'' e_\gamma = Q(|\gamma_A| + b|\gamma_{\ell+1}| + m)e_\gamma,$$

then $T_1 = P_\Sigma T_1' + (I - P_\Sigma)T_1''$, where $P_\Sigma = \frac{I+F_\Sigma}{2}$. By the two previous lemmas, $I - T_1'$ and $I - T_2'$ are in $OP^{-\infty}_{D_B}$ where $B = A \cup \{\ell+1\}$. Since $OP^{-\infty}_{D_B}$ is contained in $\mathcal{E}^\infty_\Sigma$, it follows that $T_1', T_2' \in \mathcal{E}^\infty_\Sigma$. Since $P_\Sigma \in \mathcal{E}^\infty_\Sigma$, we get $T_1 \in \mathcal{E}^\infty_\Sigma$.

Proof for T_2 is exactly similar. □

We next proceed with a detailed analysis of the operators $Z_{j,q}$. First recall that

$$U^* e_\gamma = \xi'_\gamma = (-1)^{\sum_{i=1}^{\ell}(i-1)\gamma_i + \ell(\gamma_{\ell+1})_+} e_{\mathbf{r}^{n,k},\mathbf{s}}, \tag{4.7.40}$$

where $\mathbf{s}$ is given by

$$n = \sum_{i=\ell+2}^{2\ell+1} \gamma_i, \quad k = \sum_{i=1}^{\ell} \gamma_i + (\gamma_{\ell+1})_+, \tag{4.7.41}$$

$$d_m = \sum_{i=1}^{m-1} \gamma_i, \qquad c_m = \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}| + \sum_{i=\ell+2}^{2\ell+2-m} \gamma_i \quad \text{for } 1 \le m \le \ell. \tag{4.7.42}$$

$$d_{\ell+1} = c_{\ell+1} = \sum_{i=1}^{\ell} \gamma_i + (\gamma_{\ell+1})_-. \tag{4.7.43}$$

We will use this correspondence between $e_{\mathbf{r}^{n,k},\mathbf{s}}$ and ξ'_γ freely in what follows.

From equation (4.2.25), we get

$$\begin{aligned}\pi(u_{1j})e_{\mathbf{r}^{n,k}\mathbf{s}} &= \sum_{M\in\mathbb{M}_j^+} C_q(1,\mathbf{r}^{n,k},N_{1,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{1,1})e_{\mathbf{r}^{n+1,k},M(\mathbf{s})}\\ &\quad + \sum_{M\in\mathbb{M}_j^-} C_q(1,\mathbf{r}^{n,k},N_{0,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{0,1})e_{\mathbf{r}^{n,k-1},M(\mathbf{s})}.\end{aligned}$$

Therefore

$$\begin{aligned}Z_{j,q}^* e_{\mathbf{r}^{n,k}\mathbf{s}} &= q^{-j+1} \sum_{M\in\mathbb{M}_j^+} C_q(1,\mathbf{r}^{n,k},N_{1,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{1,1})e_{\mathbf{r}^{n+1,k},M(\mathbf{s})}\\ &\quad + q^{-j+1} \sum_{M\in\mathbb{M}_j^-} C_q(1,\mathbf{r}^{n,k},N_{0,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{0,1})e_{\mathbf{r}^{n,k-1},M(\mathbf{s})}.\end{aligned}$$

Thus we have $Z_{j,q}^* = \sum_{M\in\mathbb{M}_j^+} S_M^+ T_M^+ + \sum_{M\in\mathbb{M}_j^-} S_M^- T_M^-$, where the operators $S_M^\pm$ and $T_M^\pm$ are given by

$$S_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = e_{\mathbf{r}^{n+1,k},M(\mathbf{s})}, \quad M \in \mathbb{M}_j^+, \tag{4.7.44}$$

$$S_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} = e_{\mathbf{r}^{n,k-1},M(\mathbf{s})}, \quad M \in \mathbb{M}_j^-, \tag{4.7.45}$$

$$T_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = q^{-j+1} C_q(1,\mathbf{r}^{n,k},N_{1,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{1,1})e_{\mathbf{r}^{n,k}\mathbf{s}}, \quad M \in \mathbb{M}_j^+, \tag{4.7.46}$$

$$T_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} = q^{-j+1} C_q(1,\mathbf{r}^{n,k},N_{0,1})C_q(j,\mathbf{s},M)\kappa(\mathbf{r}^{n,k},N_{0,1})e_{\mathbf{r}^{n,k}\mathbf{s}}, \quad M \in \mathbb{M}_j^-. \tag{4.7.47}$$

Lemma 4.7.5. *Let $S_M^{\pm}$ be as above. Then $US_M^{\pm}U^* \in \mathcal{E}_\Sigma^\infty$.*

Proof: Let us first look at the case $M \in \mathbb{M}_j^{\pm}$ where $1 \le j \le \ell$. In this case, one has $S_M^{\pm}\xi_\gamma = \xi_{\gamma'}$ where γ' is given by

$$\gamma'_i = \begin{cases} \gamma_i + 1 & \text{if } \begin{cases} m_i = 1 \text{ and } m_{i+1} = \ell + 1 - i, \\ m_{2\ell+2-i} = 1 \text{ and } m_{2\ell+3-i} = \ell + 2 - (2\ell + 3 - i), \end{cases} \\ \\ \gamma_i - 1 & \text{if } \begin{cases} m_i = \ell + 2 - i \text{ and } m_{i+1} = 1, \\ m_i = \ell + 2 - i \text{ and } i = j, \\ m_{2\ell+2-i} = \ell + 2 - (2\ell + 2 - i) \text{ and } m_{2\ell+3-i} = 1, \end{cases} \\ \\ \gamma_i & \text{otherwise.} \end{cases}$$

Note that since $1 \le j \le \ell$, we have $\gamma'_{\ell+1} = \gamma_{\ell+1}$, and $\eta(\gamma') - \eta(\gamma)$ depends just on M and not on γ. Therefore $US_M^{\pm}U^*$ is a constant times simple tensor product of shift operators. Thus in this case $US_M^{\pm}U^* \in \mathscr{T}_\ell^\infty \otimes I \otimes \mathscr{T}_\ell^\infty \subseteq \mathcal{E}_\Sigma^\infty$.

Next we look at the case $M \in \mathbb{M}_{\ell+1}^{\pm}$. In this case, define γ' and γ'' as follows:

$$\gamma'_i = \begin{cases} \gamma_i + 1 & \text{if } \begin{cases} m_i = 1 \text{ and } m_{i+1} = \ell + 1 - i, \\ m_{2\ell+2-i} = 1 \text{ and } m_{2\ell+3-i} = \ell + 2 - (2\ell + 3 - i), \end{cases} \\ \\ \gamma_i - 1 & \text{if } \begin{cases} m_i = \ell + 2 - i \text{ and } m_{i+1} = 1, \\ m_{2\ell+2-i} = \ell + 2 - (2\ell + 2 - i) \text{ and } m_{2\ell+3-i} = 1, \\ i = \ell + 1 \end{cases} \\ \\ \gamma_i & \text{otherwise.} \end{cases}$$

$$\gamma_i'' = \begin{cases} \gamma_i + 1 & \text{if } \begin{cases} m_i = 1 \text{ and } m_{i+1} = \ell + 1 - i, \\ m_{2\ell+2-i} = 1 \text{ and } m_{2\ell+3-i} = \ell + 2 - (2\ell + 3 - i), \\ i = \ell, \end{cases} \\ \\ \gamma_i - 1 & \text{if } \begin{cases} m_i = \ell + 2 - i \text{ and } m_{i+1} = 1, \\ m_{2\ell+2-i} = \ell + 2 - (2\ell + 2 - i) \text{ and } m_{2\ell+3-i} = 1, \\ i = \ell + 1 \end{cases} \\ \\ \gamma_i & \text{otherwise.} \end{cases}$$

Then one has

$$S_M^{\pm}\xi_\gamma = \begin{cases} \xi_{\gamma'} & \text{if } \gamma_{\ell+1} \le 0, \\ \xi_{\gamma''} & \text{if } \gamma_{\ell+1} > 0. \end{cases}$$

Therefore in this case, one will have $US_M^{\pm}U^* \in \mathscr{T}_\ell^\infty \otimes \mathcal{E}^\infty \otimes \mathscr{T}_\ell^\infty \subseteq \mathcal{E}_\Sigma^\infty$.

□

We will next take a closer look at the operators $T_M^{\pm}$. For this, we will need to compute the quantitites involved in equations (4.7.46) and (4.7.47) more precisely than we have done earlier. We start with the computation of κ. From equation (4.2.4), we get

$$\begin{aligned} \psi(\mathbf{r}^{n,k}) &= -\frac{\ell}{2}(n + k + (\ell-1)k) + \frac{\ell(\ell+1)}{2}k \\ &= -\frac{\ell}{2}(n-k). \end{aligned}$$

Therefore

$$\psi(\mathbf{r}^{n,k}) - \psi(N_{1,1}(\mathbf{r}^{n,k})) = \psi(\mathbf{r}^{n,k}) - \psi(\mathbf{r}^{n+1,k}) = \frac{\ell}{2}, \tag{4.7.48}$$

$$\psi(\mathbf{r}^{n,k}) - \psi(N_{0,1}(\mathbf{r}^{n,k})) = \psi(\mathbf{r}^{n,k}) - \psi(\mathbf{r}^{n,k-1}) = \frac{\ell}{2}. \tag{4.7.49}$$

Let us write $\lambda = (n+k, k, \ldots, k, 0)$. We will next compute d_λ, where d_λ is given by (4.2.4). One has $d_\lambda = \sum_{\mathbf{s}} q^{2\psi(\mathbf{s})}$ where the sum is over all those $\mathbf{s}$ for which the top row is λ. Such an $\mathbf{s}$ is of the form (4.4.28) and one has

$$\psi(\mathbf{s}) = -\frac{1}{2}\ell(n+\ell k) + \frac{1}{2}(\ell-1)(\ell-2)k + \sum_{i=2}^{\ell}(c_i + d_i) + d_{\ell+1}.$$

Thus we have

$$d_\lambda = q^{-\ell(n+k)-2(\ell-1)k} \sum_{\substack{k \le c_\ell \le c_{\ell-1} \le \ldots \le c_2 \le n+k \\ 0 \le d_2 \le d_3 \le \ldots \le d_\ell \le k \\ d_\ell \le d_{\ell+1} \le c_\ell}} q^{2\left(\sum_{i=2}^{\ell}(c_i+d_i) + d_{\ell+1}\right)}$$

Now for any x, we have

$$\begin{aligned}
&\sum_{\substack{k\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k\\ 0\le d_2\le d_3\le\ldots\le d_\ell\le k\\ d_\ell\le d_{\ell+1}\le c_\ell}} x^{\left(\sum_{i=2}^{\ell}(c_i+d_i)+d_{\ell+1}\right)}\\
&= \left(\sum_{k\le d_{\ell+1}\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k} x^{\left(\sum_{i=2}^{\ell}c_i+d_{\ell+1}\right)}\right)\left(\sum_{0\le d_2\le d_3\le\ldots\le d_\ell\le k} x^{\left(\sum_{i=2}^{\ell}d_i\right)}\right)\\
&\quad+\left(\sum_{k\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k} x^{\left(\sum_{i=2}^{\ell}c_i\right)}\right)\left(\sum_{0\le d_2\le d_3\le\ldots\le d_\ell\le d_{\ell+1}<k} x^{\left(\sum_{i=2}^{\ell}d_i+d_{\ell+1}\right)}\right)
\end{aligned}\tag{4.7.50}$$

If we now use the identity

$$\sum_{k\le t_1\le t_2\le\ldots\le t_j\le n} x^{\left(\sum_{i=1}^{j}t_i\right)} = x^{jk}\prod_{i=1}^{j}\left(\frac{1-x^{n-k+i}}{1-x^i}\right),$$

we get

$$\begin{aligned}
&\sum_{\substack{k\le c_\ell\le c_{\ell-1}\le\ldots\le c_2\le n+k\\ 0\le d_2\le d_3\le\ldots\le d_\ell\le k\\ d_\ell\le d_{\ell+1}\le c_\ell}} x^{\left(\sum_{i=2}^{\ell}(c_i+d_i)+d_{\ell+1}\right)}\\
&= x^{\ell k}\prod_{i=1}^{\ell}\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^{\ell-1}\left(\frac{1-x^{k+i}}{1-x^i}\right)+x^{(\ell-1)k}\prod_{i=1}^{\ell-1}\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^{\ell}\left(\frac{1-x^{k-1+i}}{1-x^i}\right)\\
&= x^{(\ell-1)k}\prod_{i=1}^{\ell-1}\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^{\ell-1}\left(\frac{1-x^{k+i}}{1-x^i}\right)\frac{1}{1-x^i}\frac{1}{1-x^\ell}\left(x^k(1-x^{n+\ell})+1-x^k\right)\\
&= x^{(\ell-1)k}\prod_{i=1}^{\ell-1}\left(\frac{1-x^{n+i}}{1-x^i}\right)\prod_{i=1}^{\ell-1}\left(\frac{1-x^{k+i}}{1-x^i}\right)\left(\frac{1-x^{n+k+\ell}}{1-x^\ell}\right).
\end{aligned}$$

Thus

$$d_\lambda^{\frac{1}{2}} = q^{-\frac{\ell(n+k)}{2}}\prod_{i=1}^{\ell-1}\left(\frac{Q(n+i)}{Q(i)}\frac{Q(k+i)}{Q(i)}\right)\frac{Q(n+k+\ell)}{Q(\ell)}.\tag{4.7.51}$$

Write

$$\lambda' = (n+1+k, k, \ldots, k, 0), \qquad \lambda'' = (n+k-1, k-1, \ldots, k-1, 0).$$

Then one has

$$\begin{aligned}
d_\lambda^{\frac{1}{2}}d_{\lambda'}^{-\frac{1}{2}} &= q^{\ell/2}\frac{Q(n+1)}{Q(n+\ell)}\frac{Q(n+k+\ell)}{Q(n+k+\ell+1)},\\
d_\lambda^{\frac{1}{2}}d_{\lambda''}^{-\frac{1}{2}} &= q^{-\ell/2}\frac{Q(k+\ell-1)}{Q(k)}\frac{Q(n+k+\ell)}{Q(n+k+\ell-1)}.
\end{aligned}$$

Combining these with (4.7.48) and (4.7.49), we get

$$\kappa(\mathbf{r}^{n,k}, N_{1,1}(\mathbf{r}^{n,k})) = q^{\ell}\frac{Q(n+1)}{Q(n+\ell)}\frac{Q(n+k+\ell)}{Q(n+k+\ell+1)}, \tag{4.7.52}$$

$$\kappa(\mathbf{r}^{n,k}, N_{0,1}(\mathbf{r}^{n,k})) = \frac{Q(k+\ell-1)}{Q(k)}\frac{Q(n+k+\ell)}{Q(n+k+\ell-1)}. \tag{4.7.53}$$

Lemma 4.7.6. *Let $M \in \mathbb{M}_j^+$ and T_M^+ be as in equation (4.7.46). Then $UT_M^+U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ if $j \le \ell$ or if $j = \ell+1$ and $M \ne N_\ell$.*

Proof: From lemma 4.2.5 and equations (4.7.46) and 4.7.52 we get, for $M = (m_1, \ldots, m_j) \in \mathbb{M}_j^+$,

$$\begin{aligned} T_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} &= \operatorname{sign}(M) q^{\ell-j+1+C(\mathbf{r}^{n,k},N_{1,1})+B(N_{1,1})+C(\mathbf{s},M)+B(M)} \frac{Q(n+1)}{Q(n+\ell)}\frac{Q(n+k+\ell)}{Q(n+k+\ell+1)} \\ &\times L'(\mathbf{r}^{n,k},1,1)\left(\prod_{a=1}^{j-1} L(\mathbf{s},a,m_a,m_{a+1})\right) L'(\mathbf{s},j,m_j)\, e_{\mathbf{r}^{n,k}\mathbf{s}}. \end{aligned} \tag{4.7.54}$$

Since $C(\mathbf{r}^{n,k}, N_{1,1}) = k$ and $B(N_{1,1}) = 0$, we get

$$T_M^+ e_{\mathbf{r}^{n,k}\mathbf{s}} = \operatorname{sign}(M) q^{\ell-j+1+B(M)+k+C(\mathbf{s},M)}\phi(\mathbf{s},M) e_{\mathbf{r}^{n,k}\mathbf{s}},$$

with $\phi(\mathbf{s},M)$ a product of terms of the form $Q(\psi(\gamma))^{\pm 1}$ where $\psi(\gamma) = |\gamma_A| + c(\gamma_{\ell+1})_\pm + m$ for some subset $A \subseteq \Sigma$, $c \in \{0,1\}$ and some integer m that does not depend on $\mathbf{s}$. Therefore

$$UT_M^+U^* e_\gamma = \operatorname{sign}(M) q^{\ell-j+1+B(M)+k+C(\mathbf{s},M)}\phi(\mathbf{s},M) e_\gamma,$$

where k and $\mathbf{s}$ are given by equations (4.7.41–4.7.43). Since $\phi(\mathbf{s},M)$ a product of terms of the form $Q(\psi(\gamma))^{\pm 1}$, it follows from lemma 4.7.4 that the operator $e_\gamma \mapsto \phi(\mathbf{s},M)e_\gamma$ is in $\mathcal{E}_\Sigma^\infty$. Next look at the operator $e_\gamma \mapsto q^{k+C(\mathbf{s},M)}e_\gamma$. Assume that there is some $i \le j$ such that $m_i \ne 1$. Let $p = \min\{2 \le i \le j : m_i \ne 1\}$. Then $C(\mathbf{s},M) \ge H_{p-1,1}(\mathbf{s}) \ge (\gamma_{\ell+1})_-$. Therefore

$$k + C(\mathbf{s},M) \ge k + (\gamma_{\ell+1})_- = \sum_{i=1}^{\ell}\gamma_i + |\gamma_{\ell+1}|.$$

Hence $UT_M^+U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Next assume that $j \le \ell$ and $m_i = 1$ for all $i \le j$. In this case, $C(\mathbf{s},M) \ge H_{j,1}(\mathbf{s}) \ge (\gamma_{\ell+1})_-$. Therefore again we have

$$k + C(\mathbf{s},M) \ge k + (\gamma_{\ell+1})_- = \sum_{i=1}^{\ell}\gamma_i + |\gamma_{\ell+1}|.$$

and hence $UT_M^+U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Combining the two cases, we have the required result. □

Lemma 4.7.7. *Let $M \in \mathbb{M}_j^-$ and T_M^- be as in equation (4.7.47). Then $UT_M^-U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$ if $M \ne N_{0,j}$.*

Proof: From lemma 4.2.5 and equations (4.7.47) and (4.7.53), we get, for $M = (m_1, \ldots, m_j) \in \mathbb{M}_j^-$,

$$\begin{aligned} T_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} &= \operatorname{sign}(M) q^{-j+1+C(\mathbf{r}^{n,k},N_{0,1})+B(N_{0,1})+C(\mathbf{s},M)+B(M)} \frac{Q(k+\ell-1)}{Q(k)} \frac{Q(n+k+\ell)}{Q(n+k+\ell-1)} \\ &\times \quad L'(\mathbf{r}^{n,k}, 1, \ell+1) \left(\prod_{a=1}^{j-1} L(\mathbf{s}, a, m_a, m_{a+1}) \right) L'(\mathbf{s}, j, m_j) e_{\mathbf{r}^{n,k}\mathbf{s}}. \end{aligned} \tag{4.7.55}$$

Since $C(\mathbf{r}^{n,k}, N_{0,1}) = 0$ and $B(N_{0,1}) = 0$, we get

$$T_M^- e_{\mathbf{r}^{n,k}\mathbf{s}} = \operatorname{sign}(M) q^{-j+1+C(\mathbf{s},M)+B(M)} \phi(\mathbf{s}, M) e_{\mathbf{r}^{n,k}\mathbf{s}},$$

with $\phi(\mathbf{s}, M)$ a product of terms of the form $Q(\psi(\gamma))^{\pm 1}$ where $\psi(\gamma) = |\gamma_A| + c(\gamma_{\ell+1})_\pm + m$ for some subset $A \subseteq \Sigma$, $c \in \{0, 1\}$ and some integer m that does not depend on $\mathbf{s}$. Therefore

$$U T_M^- U^* e_\gamma = \operatorname{sign}(M) q^{-j+1+C(\mathbf{s},M)+B(M)} \phi(\mathbf{s}, M) e_\gamma,$$

where k and $\mathbf{s}$ are given by equations (4.7.41–4.7.43). As in the proof of lemma 4.7.6, it is now enough to prove that $C(\mathbf{s}, M) \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|$. Now assume that $m_i = 1$ for some $i \leq \ell$. Let $p = \min\{2 \leq i \leq j : m_i = 1\}$. Then $p \leq \ell$. We then have

$$\begin{aligned} C(\mathbf{s}, M) &\geq \sum_{i=1}^{p-2} H_{i,\ell+1-i}(\mathbf{s}) + H_{p-1,1}(\mathbf{s}) + H_{p-1,\ell+2-p}(\mathbf{s}) + V_{p-1,\ell+2-p}(\mathbf{s}) \\ &\geq \sum_{i=1}^{p-2} \gamma_i + (\gamma_{\ell+1})_- + \gamma_{p-1} + \left(\sum_{i=1}^{\ell} \gamma_i + (\gamma_{\ell+1})_+ - \sum_{i=1}^{p-1} \gamma_i \right) \\ &= \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|. \end{aligned}$$

So the result follows. □

Remark 4.7.8. As mentioned in the beginning of this subsection, weaker versions of the two lemmas above has been proved by D'Andrea in [18]. In our notation, he proves that the part of $Z_{j,q}$ that be ignored is of the order $q^k = q^{\sum_{i=1}^{\ell} \gamma_i + (\gamma_{\ell+1})_+}$, whereas we prove here that one can actually ignore terms of a slightly higher order, namely $q^{\sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|}$, which makes it possible to compute $Z_{j,q}$ modulo the ideal $OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty$.

Lemma 4.7.9. *Define operators X_j on $L_2(S_q^{2\ell+1})$ by*

$$e_{\mathbf{r}^{n,k},\mathbf{s}} \mapsto \begin{cases} (-1)^{j-1} q^{d_j} Q(d_{j+1} - d_j) e_{\mathbf{r}^{n,k-1}, N_{0,j}(\mathbf{s})} & \text{if } 1 \leq j \leq \ell - 1, \\ (-1)^{\ell-1} q^{d_\ell} Q(d_{\ell+1} - d_\ell) Q(k - d_\ell) e_{\mathbf{r}^{n,k-1}, N_{0,\ell}(\mathbf{s})} & \text{if } j = \ell. \end{cases} \tag{4.7.56}$$

Then one has

$$U Z_{j,q}^* U^* - U X_j U^* \in OP_{D_\ell}^{-\infty} \otimes \mathscr{T}_\ell^\infty.$$

Proof: In view of the two forgoing lemmas, it is enough to show that

$$US^-_{N_{0,j}}T^-_{N_{0,j}}U^* - UX_jU^* \quad\in\quad OP^{-\infty}_{D_\ell}\otimes\mathscr{T}^\infty_\ell, \quad \text{for } 1\leq j\leq \ell, \tag{4.7.57}$$

Let us first look at the case $1\leq j\leq \ell-1$. Observe that

$$\text{sign}\,(N_{0,j}) = (-1)^{j-1}, \quad C(\mathbf{r}^{n,k},N_{0,1}) = 0 = B(N_{0,1}), \quad C(\mathbf{s},N_{0,j}) = d_j, \quad B(N_{0,j}) = j-1.$$

Therefore from (4.7.55), we get

$$\begin{aligned} UT^-_{N_{0,j}}U^*e_\gamma &= (-1)^{j-1}q^{d_j}\frac{Q(k+\ell-1)}{Q(k)}\frac{Q(n+k+\ell)}{Q(n+k+\ell-1)}L'(\mathbf{r}^{n,k},1,\ell+1)\\ &\times \left(\prod_{a=1}^{j-1}L(\mathbf{s},a,\ell+2-a,\ell+1-a)\right)L'(\mathbf{s},j,\ell+2-j)\;e_\gamma. \end{aligned}\tag{4.7.58}$$

From (4.2.18), one gets

$$\begin{aligned} L'(\mathbf{r}^{n,k},1,\ell+1) &= \left(\prod_{i=2}^{\ell}\frac{Q(|k-0-i+\ell+1-1|)}{Q(|k-0-i+\ell+1|)}\right)\frac{Q(|k-0-1+\ell+1-1|)}{Q(|n+k-0-1+\ell+1|)}\\ &= \left(\prod_{i=2}^{\ell}\frac{Q(k+\ell-i)}{Q(k+\ell-i+1)}\right)\frac{Q(k+\ell-1)}{Q(n+k+\ell)}\\ &= \frac{Q(k)}{Q(n+k+\ell)}. \end{aligned}\tag{4.7.59}$$

Similarly, from (4.2.15) one gets, for $1\leq a\leq \ell-1$,

$$\begin{aligned} &L(\mathbf{s},a,\ell+2-a,\ell+1-a)\\ &= \prod_{i=1}^{\ell+1-a}\frac{Q(|s_{a,i}-s_{a+1,\ell+1-a}-i+\ell+1-a|)}{Q(|s_{a,i}-s_{a,\ell+2-a}-i+\ell+2-a|)}\prod_{i=1}^{\ell-a}\frac{Q(|s_{a+1,i}-s_{a,\ell+2-a}-i+\ell+2-a-1|)}{Q(|s_{a+1,i}-s_{a+1,\ell+1-a}-i+\ell+1-a-1|)}\\ &= \frac{Q(c_a-d_{a+1}+\ell-a)}{Q(c_a-d_a+\ell+1-a)}\frac{Q(c_{a+1}-d_a+\ell-a)}{Q(c_{a+1}-d_{a+1}+\ell-a-1)}\\ &\quad\times\prod_{i=2}^{\ell+1-a}\frac{Q(k-d_{a+1}-i+\ell+1-a)}{Q(k-d_a-i+\ell+2-a)}\prod_{i=2}^{\ell-a}\frac{Q(k-d_a-i+\ell+1-a)}{Q(k-d_{a+1}-i+\ell-a)}\\ &= \frac{Q(c_a-d_{a+1}+\ell-a)}{Q(c_a-d_a+\ell+1-a)}\frac{Q(c_{a+1}-d_a+\ell-a)}{Q(c_{a+1}-d_{a+1}+\ell-a-1)}\\ &\quad\times\prod_{i=1}^{\ell-a}\frac{Q(k-d_{a+1}-i+\ell-a)}{Q(k-d_a-i+\ell+1-a)}\prod_{i=2}^{\ell-a}\frac{Q(k-d_a-i+\ell+1-a)}{Q(k-d_{a+1}-i+\ell-a)}\\ &= \frac{Q(c_a-d_{a+1}+\ell-a)}{Q(c_a-d_a+\ell+1-a)}\frac{Q(c_{a+1}-d_a+\ell-a)}{Q(c_{a+1}-d_{a+1}+\ell-a-1)}\frac{Q(k-d_{a+1}+\ell-a-1)}{Q(k-d_a+\ell-a)}, \end{aligned}\tag{4.7.60}$$

and from (4.2.18), for $j \leq \ell - 1$,

$$\begin{aligned} & L'(\mathbf{s}, j, \ell+2-j) \\ = & \prod_{i=1}^{\ell+1-j} \frac{Q(|s_{j+1,i} - s_{j,\ell+2-j} - i + \ell + 2 - j - 1|)}{Q(|s_{j,i} - s_{j,\ell+2-j} - i + \ell + 2 - j|)} \\ = & \frac{Q(c_{j+1} - d_j + \ell - j)}{Q(c_j - d_j + \ell + 1 - j)} \left(\prod_{i=2}^{\ell-j} \frac{Q(k - d_j + \ell + 1 - j - i)}{Q(k - d_j + \ell + 2 - j - i)} \right) \frac{Q(d_{j+1} - d_j)}{Q(k - d_j + 1)} \\ = & \frac{Q(c_{j+1} - d_j + \ell - j)}{Q(c_j - d_j + \ell + 1 - j)} \frac{Q(d_{j+1} - d_j)}{Q(k - d_j + \ell - j)} \end{aligned}$$

From the above two equations, we get

$$\begin{aligned} & \left(\prod_{a=1}^{j-1} L(\mathbf{s}, a, \ell+2-a, \ell+1-a) \right) L'(\mathbf{s}, j, \ell+2-j) \\ = & \frac{Q(d_{j+1} - d_j)}{Q(k + \ell - 1)} \left(\prod_{a=1}^{j-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \right) \left(\prod_{a=1}^{j} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \right). \end{aligned}$$

Now substituting all these in equation (4.7.58), we get

$$\begin{aligned} UT^{-}_{N_{0,j}} U^* e_\gamma = & (-1)^{j-1} q^{d_j} \frac{Q(d_{j+1} - d_j)}{Q(n + k + \ell - 1)} \left(\prod_{a=1}^{j-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \right) \\ & \times \left(\prod_{a=1}^{j} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \right) e_\gamma. \end{aligned} \tag{4.7.61}$$

Now note that for $1 \leq a \leq j-1$,

$$d_j + c_a - d_{a+1} + \ell - a \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|, \qquad d_j + c_{a+1} - d_{a+1} + \ell - a - 1 \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|,$$

and for $1 \leq a \leq j$,

$$d_j + c_{a+1} - d_a + \ell - a \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|, \qquad d_j + c_a - d_a + \ell + 1 - a \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|,$$

and $d_j + n + k + \ell - 1 \geq \sum_{i=1}^{\ell} \gamma_i + |\gamma_{\ell+1}|$. Therefore by using lemmas 4.7.2 and 4.7.3, we can write, modulo an operator in $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^{\infty}_{\ell}$,

$$UT^{-}_{N_{0,j}} U^* e_\gamma = (-1)^{j-1} q^{d_j} Q(d_{j+1} - d_j)\, e_\gamma.$$

Using equation (4.7.45), we get

$$US^{-}_{N_{0,j}} U^* e_\gamma = (-1)^{j-1} e_{\gamma'},$$

where

$$\gamma_i' = \begin{cases} \gamma_i & \text{if } i \neq j, \\ \gamma_i - 1 & \text{if } i = j. \end{cases}$$

Observe also that

$$UX_j U^* e_\gamma = q^{d_j} Q(d_{j+1} - d_j)\, e_{\gamma'},$$

where γ' is as above. Therefore we get (4.7.57) for $j \leq \ell - 1$.

In the case $j = \ell$, one has

$$L'(\mathbf{s}, \ell, 2) = \frac{Q(|s_{\ell+1,1} - s_{\ell,2}|)}{Q(|s_{\ell,1} - s_{\ell,2} + 1|)} = \frac{Q(d_{\ell+1} - d_\ell)}{Q(c_\ell - d_\ell + 1)}.$$

and as a result, one has

$$\begin{aligned} &\left(\prod_{a=1}^{\ell-1} L(\mathbf{s}, a, \ell + 2 - a, \ell + 1 - a)\right) L'(\mathbf{s}, \ell, 2) \\ &= \frac{Q(k - d_\ell)}{Q(k + \ell - 1)} \left(\prod_{a=1}^{\ell-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)}\right) \left(\prod_{a=1}^{\ell} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)}\right). \end{aligned}$$

As before, substituting all these in equation (4.7.58), one gets

$$\begin{aligned} UT^-_{N_{0,\ell}} U^* e_\gamma &= (-1)^{\ell-1} q^{d_\ell} \frac{Q(k - d_\ell)}{Q(n + k + \ell - 1)} \left(\prod_{a=1}^{\ell-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)}\right) \\ &\quad \times \left(\prod_{a=1}^{\ell} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)}\right) e_\gamma. \end{aligned} \tag{4.7.62}$$

Application of lemmas 4.7.2 and 4.7.3, now enable us to write the following equality modulo an operator in $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$:

$$UT^-_{N_{0,\ell}} U^* e_\gamma = (-1)^{\ell-1} q^{d_\ell} Q(k - d_\ell) Q(d_{\ell+1} - d_\ell)\, e_\gamma.$$

Using equation (4.7.45), we get

$$US^-_{N_{0,\ell}} U^* e_\gamma = (-1)^{\ell-1} e_{\gamma'},$$

where

$$\gamma_i' = \begin{cases} \gamma_i & \text{if } i \neq \ell, \\ \gamma_i - 1 & \text{if } i = \ell. \end{cases}$$

Observe also that

$$UX_\ell U^* e_\gamma = q^{d_\ell} Q(k - d_\ell) Q(d_{\ell+1} - d_\ell)\, e_{\gamma'},$$

where γ' is as above. Therefore we get (4.7.57) for $j = \ell$. □

Lemma 4.7.10. *Let X_j be as in lemma 4.7.9. Then for $1 \leq j \leq \ell$, one has $UX_jU^* - Y^*_{j,q} \otimes I \in OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$.*

Proof: It follows from equations (4.7.40–4.7.43) that for $j \leq \ell - 1$, one in fact has $UX_jU^* - Y^*_{j,q} \otimes I = 0$. For $j = \ell$, one has

$$\left(UX_jU^* - Y^*_{j,q} \otimes I\right) e_\gamma = \left(q^{\sum_{i=1}^{\ell-1}\gamma_i} Q\left(\gamma_\ell + (\gamma_{\ell+1})_-\right) Q\left(\gamma_\ell + (\gamma_{\ell+1})_+\right)\right) e_{\hat{\gamma}},$$

where $\hat{\gamma}_i = \gamma_i - 1$ if $i = \ell$ and $\hat{\gamma}_i = \gamma_i$ for all other i. Thus

$$\left|UX_jU^* - Y^*_{j,q} \otimes I\right| \in OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell \qquad \text{sign}\,\left(UX_jU^* - Y^*_{j,q} \otimes I\right) \in \mathcal{E}^\infty_\Sigma.$$

Therefore $UX_jU^* - Y^*_{j,q} \otimes I \in OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$. □

From the two lemmas above (lemmas 4.7.9 and 4.7.10), it follows that for $1 \leq j \leq \ell$, one has $UZ^*_{j,q}U^* \in C^\infty(S^{2\ell+1}_q)$. Thus we now need only to take care of the case $j = \ell + 1$.

Lemma 4.7.11. $UZ^*_{\ell+1,q}U^* \in C^\infty(S^{2\ell+1}_q)$.

Proof: Using lemmas 4.7.6 and 4.7.7, it is enough to show that

$$U(S^+_{N_\ell}T^+_{N_\ell} + S^-_{N_{0,\ell+1}}T^-_{N_{0,\ell+1}})U^* - UX_{\ell+1}U^* \quad \in \quad C^\infty(S^{2\ell+1}_q). \tag{4.7.63}$$

From (4.7.54), we get

$$T^+_{N_\ell} e_{\mathbf{r}^{n,k}\mathbf{s}} = q^k \frac{Q(n+1)}{Q(n+\ell)} \frac{Q(n+k+\ell)}{Q(n+k+\ell+1)} L'(\mathbf{r}^{n,k}, 1, 1) \left(\prod_{a=1}^{\ell} L(\mathbf{s}, a, 1, 1)\right)\; e_{\mathbf{r}^{n,k}\mathbf{s}}. \tag{4.7.64}$$

From (4.2.15), we get for $1 \leq a \leq \ell - 1$,

$$\begin{aligned}
&L(\mathbf{s}, a, 1, 1)\\
&\quad= \prod_{i=2}^{\ell+2-a} \frac{Q(|s_{a,i} - s_{a+1,1} - i + 1|)}{Q(|s_{a,i} - s_{a,1} - i + 1|)} \prod_{i=2}^{\ell+1-a} \frac{Q(|s_{a+1,i} - s_{a,1} - i + 1 - 1|)}{Q(|s_{a+1,i} - s_{a+1,1} - i + 1 - 1|)}\\
&\quad= \prod_{i=2}^{\ell+1-a} \frac{Q(c_{a+1} - k + i - 1)}{Q(c_a - k + i - 1)} \prod_{i=2}^{\ell-a} \frac{Q(c_a - k + i)}{Q(c_{a+1} - k + i)}\\
&\qquad \times \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)}\\
&\quad= \prod_{i=1}^{\ell-a} \frac{Q(c_{a+1} - k + i)}{Q(c_a - k + i)} \prod_{i=2}^{\ell-a} \frac{Q(c_a - k + i)}{Q(c_{a+1} - k + i)}\\
&\qquad \times \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)}\\
&\quad= \frac{Q(c_{a+1} - k + 1)}{Q(c_a - k + 1)} \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)},
\end{aligned}$$

and for $a = \ell$,

$$L(\mathbf{s}, \ell, 1, 1) = \frac{Q(|s_{\ell,2} - s_{\ell+1,1} - 2 + 1|)}{Q(|s_{\ell,2} - s_{\ell,1} - 2 + 1|)} = \frac{Q(d_{\ell+1} - d_\ell + 1)}{Q(c_\ell - d_\ell + 1)}.$$

Also from (4.2.18), we have

$$\begin{aligned} L'(\mathbf{r}^{n,k}, 1, 1) &= \left(\frac{\prod_{i=1}^{\ell} Q(|k - n - k - i + 1 - 1|)}{\prod_{i=2}^{\ell} Q(|k - n - k - i + 1|)} \right) \frac{1}{Q(|0 - n - k - \ell - 1 + 1|)} \\ &= \left(\prod_{i=2}^{\ell} \frac{Q(n+i)}{Q(n+i-1)} \right) \frac{Q(n+1)}{Q(n+k+\ell)} \\ &= \frac{Q(n+\ell)}{Q(n+k+\ell)}. \end{aligned}$$

Plugging these in equation (4.7.64) and using (4.7.40), we get

$$\begin{aligned} UT^{+}_{N_\ell} U^* e_\gamma &= q^k \frac{Q(n+1)}{Q(n+\ell)} \frac{Q(n+k+\ell)}{Q(n+k+\ell+1)} \frac{Q(n+\ell)}{Q(n+k+\ell)} \\ &\times \left(\prod_{a=1}^{\ell-1} \frac{Q(c_{a+1} - k + 1)}{Q(c_a - k + 1)} \frac{Q(c_{a+1} - d_a + \ell + 1 - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_a - d_{a+1} + \ell + 1 - a)}{Q(c_{a+1} - d_{a+1} + \ell + 1 - a)} \right) \\ &\times \frac{Q(d_{\ell+1} - d_\ell + 1)}{Q(c_\ell - d_\ell + 1)} \, e_\gamma. \end{aligned}$$

Thus as earlier, modulo an operator in $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^{\infty}_{\ell}$, we have the equality

$$UT^{+}_{N_\ell} U^* e_\gamma = q^k e_\gamma. \tag{4.7.65}$$

Next note that $B(N_0) = \ell$, $C(\mathbf{s}, N_0) = d_{\ell+1}$ and $\operatorname{sign}(N_0) = (-1)^\ell$ so that we get from (4.7.55)

$$\begin{aligned} T^{-}_{N_0} e_{\mathbf{r}^{n,k}\mathbf{s}} &= (-1)^\ell q^{d_{\ell+1}} \frac{Q(k+\ell-1)}{Q(k)} \frac{Q(n+k+\ell)}{Q(n+k+\ell-1)} \\ &\times \; L'(\mathbf{r}^{n,k}, 1, \ell+1) \left(\prod_{a=1}^{\ell} L(\mathbf{s}, a, \ell+2-a, \ell+1-a) \right) e_{\mathbf{r}^{n,k}\mathbf{s}}. \end{aligned} \tag{4.7.66}$$

Now using (4.7.40), (4.7.59), (4.7.60) and the fact that

$$L(\mathbf{s}, \ell, 2, 1) = \frac{Q(|s_{\ell,1} - s_{\ell+1,1} - 1 + \ell + 1 - \ell|)}{Q(|s_{\ell,1} - s_{\ell,2} - 1 + \ell + 2 - \ell|)} = \frac{Q(c_\ell - d_{\ell+1})}{Q(c_\ell - d_\ell + 1)},$$

we get

$$\begin{aligned} UT^{-}_{N_0} U^* e_\gamma &= (-1)^\ell q^{d_{\ell+1}} \frac{Q(k+\ell-1)}{Q(k)} \frac{Q(n+k+\ell)}{Q(n+k+\ell-1)} \frac{Q(k)}{Q(n+k+\ell)} \\ &\times \left(\prod_{a=1}^{\ell-1} \frac{Q(c_a - d_{a+1} + \ell - a)}{Q(c_a - d_a + \ell + 1 - a)} \frac{Q(c_{a+1} - d_a + \ell - a)}{Q(c_{a+1} - d_{a+1} + \ell - a - 1)} \frac{Q(k - d_{a+1} + \ell - a - 1)}{Q(k - d_a + \ell - a)} \right) \\ &\times \frac{Q(c_\ell - d_{\ell+1})}{Q(c_\ell - d_\ell + 1)} \, e_\gamma. \end{aligned}$$

Thus modulo $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$, we have the equality

$$UT^-_{N_0}U^* e_\gamma = (-1)^\ell q^{d_{\ell+1}} e_\gamma. \tag{4.7.67}$$

Define operators $T^\pm$ on $L_2(S_q^{2\ell+1})$ by

$$T^+\xi_\gamma = q^k \xi_\gamma, \qquad T^-\xi_\gamma = (-1)^\ell d_{\ell+1}\xi_\gamma.$$

By equations (4.7.65) and (4.7.67), it is enough to look at the operators $S^+_{N_\ell}T^+ + S^-_{N_0}T^-$.

Now observe that

$$S^-_{N_0}\xi_\gamma = \begin{cases} \xi_{\gamma'} & \text{if } \gamma_{\ell+1} > 0, \\ \xi_{\gamma''} & \text{if } \gamma_{\ell+1} \le 0, \end{cases} \qquad S^+_{N_\ell}\xi_\gamma = \begin{cases} \xi_{\gamma'''} & \text{if } \gamma_{\ell+1} > 0, \\ \xi_{\gamma'} & \text{if } \gamma_{\ell+1} \le 0, \end{cases}$$

where

$$\gamma'_i = \begin{cases} \gamma_i - 1 & \text{if } i = \ell+1, \\ \gamma_i & \text{otherwise}, \end{cases} \qquad \gamma''_i = \begin{cases} \gamma_i - 1 & \text{if } \ell \le i \le \ell+2, \\ \gamma_i & \text{otherwise}. \end{cases}$$

and

$$\gamma'''_i = \begin{cases} \gamma_i + 1 & \text{if } i = \ell \text{ or } i = \ell+2, \\ \gamma_i - 1 & \text{if } i = \ell+1, \\ \gamma_i & \text{otherwise}. \end{cases}$$

Therefore

$$\left(S^+_{N_\ell}T^+ + S^-_{N_0}T^-\right)\xi_\gamma = \begin{cases} q^k\xi_{\gamma'''} + (-1)^\ell q^{d_{\ell+1}}\xi_{\gamma'} & \text{if } \gamma_{\ell+1} > 0, \\ q^k\xi_{\gamma'} + (-1)^\ell q^{d_{\ell+1}}\xi_{\gamma''} & \text{if } \gamma_{\ell+1} \le 0. \end{cases}$$

So if we now define

$$T\xi_\gamma = \begin{cases} (-1)^\ell q^{\sum_{i=1}^\ell \gamma_i}\xi_{\gamma'} & \text{if } \gamma_{\ell+1} > 0, \\ q^{\sum_{i=1}^\ell \gamma_i}\xi_{\gamma'} & \text{if } \gamma_{\ell+1} \le 0, \end{cases}$$

then one gets from the above equation that $U\left(S^+_{N_\ell}T^+ + S^-_{N_0}T^- - T\right)U^*$ is in $OP^{-\infty}_{D_\ell} \otimes \mathscr{T}^\infty_\ell$. Thus it is enough to show that $UTU^* \in C^\infty(S_q^{2\ell+1})$. Now note that

$$\eta(\gamma) - \eta(\gamma') = \begin{cases} \ell & \text{if } \gamma_{\ell+1} > 0, \\ 0 & \text{if } \gamma_{\ell+1} \le 0. \end{cases}$$

Therefore it follows that $UTU^*e_\gamma = q^{\sum_{i=1}^\ell \gamma_i}e_{\gamma'}$, i.e. $UTU^* = Y^*_{\ell+1,q} \otimes I$. Thus we get the required result. □

Putting together lemmas 4.7.9, 4.7.10 and 4.7.11, we get proposition 4.6.7.

4.8 The Chern character of the equivariant spectral triple

We end this chapter by comparing the Chern character of the equivariant spectral triple with that of the torus equivariant one. Consider a spectral triple $(\mathcal{B}, \mathcal{H}, D)$ with the following properties:

- The algebra $\mathcal{B}$ is invariant under $\delta := [|D|, .]$ and $[D, .]$. Assume that $F := Sign(D) \in \mathcal{B}$.
- The dimension spectrum Σ is finite and simple and does not contain 0.
- For $b \in \mathcal{B}$ the commutator $[F, b]$ is smoothing.

Then the Fredholm module $(\mathcal{B}, \mathcal{H}, F)$ is 1 summable. In this section, we associate a 1 dimensional cycle whose character is cohomologous to the character of the the Fredholm module $(\mathcal{B}, \mathcal{H}, F)$. Let $P := \frac{1+F}{2}$. Define

$$\begin{aligned} \tau(b_0, b_1) :=& \frac{1}{2} Tr(b_0[F, b_1]), \\ \psi_0(b) :=& Tr(bP|D|^{-z})_{z=0}, \\ \psi_1(b_0, b_1) :=& \tau + b\psi_0. \end{aligned}$$

We will describe a cycle for $\mathcal{B}$ whose character is ψ_1.

Remark 4.8.1. *The cochain ψ_0 makes sense as Σ does not contain 0.*

First we define the differential graded algebra. Define

$$\begin{aligned} \Omega^0 :&= \mathcal{B}, \\ \Omega^1 :&= \prod_{i=1}^{\infty} \mathcal{B}, \\ \Omega :&= \Omega^0 \oplus \Omega^1. \end{aligned}$$

We will define a Ω^0 bimodule structure on Ω^1 such that the linear map $d : \Omega^0 \to \Omega^1$ defined by $d(b) := (\delta(b), \delta^2(b), \delta^3(b), \cdots)$ becomes a differential. The left multiplication is the usual one inherited from the algebra multiplication of $\mathcal{B}$. The right module structure is defined by

$$\begin{aligned} (b_1, b_2, b_3, \cdots).b :=& \ (b_1^{'}, b_2^{'}, b_3^{'}, \cdots) \quad \text{where } b_r^{'}\text{s are given by} \\ b_r^{'} :=& \ \Sigma_{i=1}^{r} \binom{r}{i} b_i \delta^{r-i}(b). \end{aligned}$$

Lemma 4.8.2. *The vector space Ω^1 is a Ω^0 bimodule and (Ω, d) is a differential graded algebra.*

The proof is by direct verification by using the fact that δ is derivation and the lebnitz rule.

$$\delta^n(bc) = \Sigma_{i=0}^{n} \binom{n}{i} \delta^i(b) \delta^{n-i}(c).$$

Define for $r \in \mathbb{N}_+$, the functional τ_r on $\mathcal{B}$ by $\tau_r(b) := \quad Res_{z=r}Tr(\ bP|D|^{-z})$. Since the dimension spectrum is finite it follows that there exists N such that the functionals τ_r vanishes for $r \geq N$. Note that the functionals τ_r are δ invariant i.e. $\tau_r(\delta(b)) = 0$. Now define a functional $\int : \Omega^1 \to \mathbb{C}$ by

$$\int (b_1, b_2, b_3, ...) := \Sigma_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \tau_r(b_r).$$

The functional $\int$ makes sense as τ_r is zero for sufficiently large r.

Proposition 4.8.3. *The triple $(\Omega, d, \int)$ is a 1-dimensional cycle for $\mathcal{B}$ whose character is ψ_1 and hence is cohomologous to τ which is the character of the Fredholm module $(\mathcal{B}, \mathcal{H}, F)$.*

Proof. First let us compute the coboundaries $b\tau_r$. Note the asymptotic expansion ([17])

$$|D|^{-z}a \approx \Sigma_{i=0}^{\infty} \binom{-z}{i} \ \delta^i(a)|D|^{-(i+z)}. \tag{4.8.68}$$

From the above equation and the fact that $[P, b]$ is smoothing it follows that

$$\tau_r(bc - cb) = \ \Sigma_{i=1}^{\infty}(-1)^{i+1}\binom{r+i-1}{i} \ \tau_{r+i}(b\delta^i(c)).$$

Now we will show that $\int$ is a closed graded trace. The fact that the functional $\int$ is closed follows from the invariance of τ_r under δ. Now let $b \in \Omega^0$ and $\omega := (b_1, b_2, b_3, \cdots) \in \Omega^1$. Let $b\omega - \omega b := (b_1^{'}, b_2^{'}, b_3^{'}, \cdots)$. Then

$$b_r^{'} := bb_r - b_r b - \Sigma_{i=1}^{r-1}\binom{r}{i} b_i \delta^{r-i}(b).$$

Hence

$$\begin{aligned}
\int b\omega - \omega b =& \Sigma_{r=1} \frac{(-1)^r}{r} \tau_r(bb_r - b_r b) - \Sigma_{r=1} \frac{(-1)^r}{r} \tau_r \big(\Sigma_{i=1}^{r-1}\binom{r}{i} b_i \delta^{r-i}\big) \\
=& \Sigma_{r=1}\Big(\Sigma_{i=1}^{\infty} \frac{(-1)^{r+i}}{r} \tau_{r+i}(b_r\delta^i(b))\binom{r+i-1}{i}\Big) \ - \Sigma_{r=1}^{\infty} \frac{(-1)^r}{r}\Big(\Sigma_{i=1}^{r-1}\tau_r(b_i\delta^{r-i}(b))\binom{r}{i}\Big) \\
=& I_1 - I_2
\end{aligned}$$

where I_1 denote the first sum and I_2 the second. After changing the order of the summation in I_2, it follows that

$$\begin{aligned}
I_2 &= \Sigma_{i=1}^{\infty}\left(\Sigma_{r>i} \frac{(-1)^r}{r} \tau_r(b_i\delta^{r-i}(b))\binom{r}{i}\right) \\
&= \Sigma_{m=1}\left(\Sigma_{n=1}^{\infty} \frac{(-1)^{n+m}}{n+m} \tau_{m+n}(b_m\delta^n(b))\binom{m+n}{m}\right) \\
&= I_1
\end{aligned}$$

since $\frac{1}{m+n}\binom{m+n}{m} = \frac{1}{m}\binom{m+n-1}{n}$. This proves that $\int$ is a trace. Thus $(\Omega, d, \int)$ is a 1-dimensional cycle. Now we show that it's character χ is ψ_1 by explicit computation using the asymptotic expansion 4.8.68. By the asymptotic expansion 4.8.68 one has

$$\begin{aligned} Tr(bP|D|^{-z}c) \approx\, & Tr(\ bcP|D|^{-z}) \ + \ Tr(\ b[P,c]|D|^{-z}) \\ & + \Sigma_{i=1}^{\infty}\binom{-z}{i}\Big(Tr(\ b\delta^i(c)P|D|^{-(i+z)}) + Tr(\ b[P,\delta^i(c)]|D|^{-(i+z)}\Big). \end{aligned}$$

Since $[P, b]$ is smoothing, it follows that

$$\begin{aligned} b\psi_0(b,c) + Tr(b[P,c]) &= \Sigma_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\tau_i(b\delta^i(c)), \\ b\psi_0(b,c) + \tau(b,c) &= \chi(b,c) \quad \text{(by the defn of } \chi). \end{aligned}$$

Hence, by definition $\chi := \psi_1$ and is cohomologous to the Chern character of the Fredholm module $(\mathcal{B}, \mathcal{H}, F)$. This completes the proof. □

Now we show that the Chern character of both the equivariant and the torus equivariant spectral triple coincide in $H^{odd}(C^\infty(S_q^{2\ell+1}))$. Let $(\pi_{eq}, \mathcal{H}, D)$ be the equivariant spectral triple and $(\pi_{torus}, \mathcal{H}, D)$ be the torus equivariant one with amplification. We have $F := F_\ell \otimes p - 1 \otimes (1-p)$ where $F := sign(D)$. Note that for $a \in C^\infty(S_q^{2\ell+1})$ one has $\rho(a) := \pi_{eq}(a) - \pi_{torus}(a) \in OP^{-\infty} \otimes \mathscr{T}_\ell^\infty$.

Lemma 4.8.4. *Let $\phi(a) := Tr(P\rho(a))$ for $a \in C^\infty(S_q^{2\ell+1})$ where $P := \frac{1+F}{2}$. Then one has $b\phi = ch(\pi_{torus}, \mathcal{H}, F) - ch(\pi_{eq}, \mathcal{H}, F)$.*

Proof. Since $P \in OP^0 \otimes_{alg} OP^{-\infty}$, it follows that $P\rho(a)$ is smoothing for every $a \in C^\infty(S_q^{2\ell+1})$ and hence trace class and thus the cochain ϕ makes sense. Now we compute the coboundary $b\phi$.

$$\begin{aligned} b\phi(a,b) &= \phi(ab) - \phi(ba) \\ &= Tr(P(\rho(ab) - \rho(ba))). \end{aligned}$$

Note that $\rho(ab) - \rho(ba) = (\pi_{eq}(a)\rho(b) + \rho(a)\pi_{torus}(b)) - (\rho(b)\pi_{eq}(a) + \pi_{torus}(b)\rho(a))$. Hence

$$\begin{aligned} b\phi(a,b) &= Tr([P,\pi_{eq}(a)]\rho(b)) + Tr([P,\rho(a)]\pi_{torus}(b))) \\ &= \frac{1}{2}Tr([F,\pi_{eq}(a)]\pi_{eq}(b)) - \frac{1}{2}Tr([F,\pi_{torus}(a)]\pi_{torus}(b)). \end{aligned}$$

Hence $b\phi = ch(\pi_{torus}, \mathcal{H}, F) - ch(\pi_{eq}, \mathcal{H}, F)$. This completes the proof. □

Remark 4.8.5. *The above lemma also follows more easily from the following observation. Consider the local cocyle ψ_1 for the spectral triple $(B_{eq}, \mathcal{H}, D)$. The functionals τ_i vanishes on the ideal $OP^{-\infty} \otimes \mathscr{T}_\ell^\infty$. Hence $\psi_1(b,c)$ vanishes if b or c is in $OP^{-\infty} \otimes \mathscr{T}_\ell^\infty$ which implies $\psi_1^{torus} = \psi_1^{eq}$. Therefore the Chern characters of the torus equivariant and the equivariant spectral triples differ by the coboundary $b(\psi_0^{eq} - \psi_0^{torus})$. Observe that $\phi = \psi_0^{eq} - \psi_0^{torus}$.*

Remark 4.8.6. *Let* $(\pi_\ell, \mathcal{H}_\ell, D_\ell)$ *be the torus equivariant spectral triple. Since* $(\pi_{torus}, \mathcal{H}, F)$ *is unitary equivalent to* $(\pi_\ell \oplus \pi^{'}, \mathcal{H}_\ell \oplus \mathcal{H}^{'}, F_\ell \oplus -1)$ *it follows that the Chern character of* $(\pi_{torus}, \mathcal{H}, F)$ *and that of* $(\pi_\ell, \mathcal{H}_\ell, F_\ell)$ *coincide. The same is true of the local cocycle* ψ_1*.*

Remark 4.8.7. *Thus to prove that the index map* $ind_D : K_1(C(S_q^{2\ell+1})) \to \mathbb{Z}$ *is non-trivial for the equivariant* D *it is enough to prove that the index map* ind_{D_ℓ} *is non-trivial which we do by using the local cocycle* ψ_1*. Let* $U := p \otimes S^* + (1-p) \otimes 1$*. The local cocycle* ψ_1 *on* $C^\infty(U)$ *is given by* $\psi_1(f, g) := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) g^{'}(\theta) d\theta$*. Hence* $\psi_1(U, U^{-1}) = -1$*.*

Chapter 5

The weak heat kernel expansion

The computations carried out in the last two chapters are by direct methods. But one can offer an easier conceptual explanation for it. Observe that the Mellin transform of the function $Tr(be^{-t|D|})$ is $\Gamma(s)Tr(b|D|^{-s})$. Thus if the function $Tr(be^{-t|D|})$ has an asymptotic expansion near 0, the meromorphic continuation of $Tr(b|D|^{-s})$ would follow. We show that the spectral triples considered in the earlier two chapters have this property.

We consider a property called the weak heat kernel expansion property and show that it is stable under quantum double suspension. We also show that if a spectral triple has the weak heat kernel expansion property then it is regular and has finite simple dimension spectrum lying in the set of positive integers. Since the torus equivariant spectral triple is obtained by quantum double suspending the standard spectral triple on the circle recursively, the result in Chapter 3 follows. We also discuss some examples of spectral triples which have this property. This gives a way to construct some more examples for which the local index formula holds. We begin this chapter with a brief discussion about asymptotic expansions and the Mellin transform. Then we consider the weak heat kernel expansion property and show its stability under quantum double suspension.

5.1 Asymptotic expansions and the Mellin transform

Let $\phi : (0,\infty) \to \mathbb{C}$ be a continuous function. We say that ϕ has an asymptotic power series expansion near 0 if there exists a sequence $(a_r)_{r=0}^{\infty}$ of complex numbers such that given N there exists $\epsilon, M > 0$ such that if $t \in (0,\epsilon)$

$$|\phi(t) - \sum_{r=0}^{N} a_r t^r| \leq M t^{N+1}.$$

We write $\phi(t) \sim \sum_0^\infty a_r t^r$ as $t \to 0+$. Note that the coefficients (a_r) are unique. For,

$$a_N = \lim_{t\to 0+} \frac{\phi(t) - \sum_{r=0}^{N-1} a_r t^r}{t^N}. \tag{5.1.1}$$

If $\phi(t) \sim \sum_{r=0}^{\infty} a_r t^r$ as $t \to 0+$ then ϕ can be extended continuously to $[0, \infty)$ simply by letting $\phi(0) := a_0$.

Let X be a topological space and $F : [0, \infty) \times X \to \mathbb{C}$ be a continuous function. Suppose that for every $x \in X$, the function $t \to F(t, x)$ has an asymptotic expansion near 0

$$F(t, x) \sim \sum_{r=0}^{\infty} a_r(x) t^r. \tag{5.1.2}$$

Let $x_0 \in X$. We say that Expansion 5.1.2 is uniform at x_0 if given N there exists an open set U containing x_0, $\epsilon > 0$ and an $M > 0$ such that for $0 < t < \epsilon$ and $x \in U$, one has

$$|F(t, x) - \sum_{r=0}^{N} a_r(x) t^r| \leq M t^{N+1}.$$

We say that Expansion 5.1.2 is uniform if it is uniform at every point of X.

Proposition 5.1.1. *Let X be a topological space and $F : [0, \infty) \times X \to \mathbb{C}$ be a continuous function. Suppose that F has a uniform asymptotic power series expansion, say*

$$F(t, x) \sim \sum_{r=0}^{\infty} a_r(x) t^r.$$

Then for every $r \geq 0$, the function a_r is continuous.

Proof. It is enough to show that the function a_0 is continuous. Let $x_0 \in X$ be given. Since the expansion of F is uniform at x_0, it follows that there exists an open set U containing x_0 and $\delta, M > 0$ such that

$$|F(t, x) - a_0(x)| \leq M t \text{ for } t < \delta \text{ and } x \in U. \tag{5.1.3}$$

Let $F_n(x) := F(\frac{1}{n}, x)$. Then Equation 5.1.3 says that F_n converges uniformly to a_0 on U. Thus a_0 is continuous on U and hence at x_0. This completes the proof. □

The following two lemmas are easy to prove and we leave the proof to the reader.

Lemma 5.1.2. *Let X, Y be topological spaces. Let $F : [0, \infty) \times X \to \mathbb{C}$ and $G : [0, \infty) \times Y \to \mathbb{C}$ be continuous. Suppose that F and G have uniform asymptotic power series expansion. Then the function $H : [0, \infty) \times X \times Y \to \mathbb{C}$ defined by $H(t, x, y) := F(t, x)G(t, y)$ has a uniform asymptotic power series expansion.*
Moreover if

$$F(t, x) \sim \sum_{r=0}^{\infty} a_r(x) t^r, \text{ and } G(t, y) \sim \sum_{r=0}^{\infty} b_r(y) t^r$$

then

$$H(t, x, y) \sim \sum_{r=0}^{\infty} c_r(x, y) t^r$$

where

$$c_r(x,y) := \sum_{m+n=r} a_m(x) b_n(y).$$

Lemma 5.1.3. *Let $\phi : [1,\infty) \to \mathbb{C}$ be a continuous function. Suppose that for every N, $\sup_{t\in[1,\infty)} |t^N \phi(t)| < \infty$. Then the function $s \mapsto \int_1^\infty \phi(t) t^{s-1} dt$ is entire.*

5.1.1 The Mellin transform

In this section, we recall the definition of the Mellin transform of a function defined on $(0,\infty)$ and analyse the relationship between the asymptotic expansion of a function and the meromorphic continuation of its Mellin transform. Let us introduce some notations. We say that a function $\phi : (0,\infty) \to \mathbb{C}$ is of rapid decay near infinity if for every $N > 0$, $\sup_{t\in[1,\infty)} |t^N \phi(t)|$ is finite. We let $\mathcal{M}_\infty$ to be the set of continuous complex valued functions on $(0,\infty)$ which has rapid decay near infinity. For $p \in \mathbb{R}$, we let

$$\mathcal{M}_p((0,1]) := \{\phi : (0,1] \to \mathbb{C} : \ \phi \text{ is continuous and } \sup_{t\in(0,1]} t^p |\phi(t)| < \infty\},$$
$$\mathcal{M}_p := \{\phi \in \mathcal{M}_\infty : \phi|_{(0,1]} \in \mathcal{M}_p((0,1])\}.$$

Note that if $p \le q$ then $\mathcal{M}_p \subset \mathcal{M}_q$ and $M_p((0,1]) \subset M_q((0,1])$.

Definition 5.1.4. *Let $\phi : (0,\infty) \to \mathbb{C}$ be a continuous function. Suppose that $\phi \in \mathcal{M}_p$ for some p. Then the Mellin transform of ϕ, denoted $M\phi$, is defined as follows: For $Re(s) > p$,*

$$M\phi(s) := \int_0^\infty \phi(t) t^{s-1} dt.$$

One can show that if $\phi \in \mathcal{M}_p$ then $M\phi$ is analytic on the right half plane $Re(s) > p + 2$. Also if $\phi \in \mathcal{M}_p((0,1])$ then $s \mapsto \int_0^1 \phi(t) t^{s-1}$ is analytic on $Re(s) > p + 2$.

For $a < b$ and $K > 0$, let $H_{a,b,K} := \{\sigma + it : \ a \le \sigma \le b, \ |t| > K\}$.

Definition 5.1.5. *Let F be a meromorphic function on the entire complex plane with simple poles lying inside the set of integers. We say that F has decay of order $r \in \mathbb{N}$ along the vertical strips if the function $s \mapsto s^r F(s)$ is bounded on $H_{a,b,K}$ for every $a < b$ and $K > 0$. We say that F is of rapid decay along the vertical strips if F has decay of order r for every $r \in \mathbb{N}$.*

Proposition 5.1.6. *Let $\phi : (0,\infty) \to \mathbb{C}$ be a continuous function of rapid decay. Assume that $\phi(t) \sim \sum_0^\infty a_r t^r$ as $t \to 0+$. Then*

(1) The function $\phi \in \mathcal{M}_0$,

(2) The Mellin transform $M\phi$ of ϕ extends to a meromorphic function to the whole of complex plane with simple poles in the set of negative integers. $\{0, -1, -2, -3, \cdots\}$,

(3) The residue of $M\phi$ at $s=-r$ is given by $Res_{s=-r}M\phi(s)=a_r$, and

(4) The meromorphic continuation of the Mellin transform $M\phi$ has decay of order 0 along the vertical strips.

Proof. By definition, it follows that $\phi \in \mathcal{M}_0$. Since ϕ has rapid decay at infinity, by lemma 5.1.3, it follows that the function $s \mapsto \int_1^\infty \phi(t)t^{s-1}dt$ is entire. Thus, modulo a holomorphic function, $M\phi(s) \equiv \int_0^1 \phi(t)t^{s-1}$. For $N \in \mathbb{N}$, let $R_N(t) := \phi(t) - \sum_{r=0}^N a_r t^r$. Thus modulo a holomorphic function, we have

$$M\phi(s) \equiv \sum_{r=0} \frac{a_r}{s+r} + \int_0^1 R_N(t)t^{s-1}dt.$$

As $R_N \in \mathcal{M}_{-(N+1)}((0,1])$ the function $s \mapsto \int_0^1 R_N(t)t^{s-1}dt$ is holomorphic on $Re(s) > -N+1$. Thus on $Re(s) > -N+1$, modulo a holomorphic function, one has

$$M\phi(s) \equiv \sum_{r=0}^{N} \frac{a_r}{s+r}. \tag{5.1.4}$$

This shows that $M\phi$ admits a meromorphic continuation to the whole of complex plane and has simple poles lying in the set of negative integers $\{0,-1,-2,\cdots,\}$. Also (3) follows from Equation 5.1.4.

Let $a<b$ and $K>0$ be given. Choose $N \in \mathbb{N}$ such that $N+a>0$. Then one has

$$M\phi(s) = \sum_{r=0}^{N} \frac{a_r}{s+r} + \int_0^1 R_N(t)t^{s-1}dt + \int_1^\infty \phi(t)t^{s-1}dt.$$

As the function $s \mapsto \frac{1}{s+r}$ is bounded for every $r \geq 0$ on $H_{a,b,K}$, it is enough to show that the functions $\psi(s) := \int_0^1 R_N(t)t^{s-1}dt$ and $\chi(s) := \int_1^\infty \phi(t)t^{s-1}dt$ are bounded on $H_{a,b,K}$.

Choose an $M>0$ such that for $t \in (0,1]$, $|R_N(t)| \leq Mt^{N+1}$. Hence for $s := \sigma + it \in H_{a,b,K}$,

$$|\psi(s)| \leq \frac{M}{\sigma+N+1} \leq \frac{M}{a+N+1} \leq M.$$

Thus ψ is bounded on $H_{a,b,K}$.

Now for $s := \sigma + it \in H_{a,b,K}$, we have

$$|\chi(s)| \leq \int_1^\infty |\phi(t)|t^{\sigma-1}dt \leq \int_1^\infty |\phi(t)|t^{b-1}dt.$$

Since ϕ is of rapid decay, the integral $\int_1^\infty |\phi(t)|t^{a-1}dt$ is finite. Thus χ is bounded on $H_{a,b,K}$. This completes the proof. □

Corollary 5.1.7. *Let $\phi : (0,\infty) \mapsto \mathbb{C}$ be a smooth function. Assume that for every n, the n^{th} derivative $\phi^{(n)}$ has rapid decay at infinity and admits an asymptotic power series expansion near 0.*

(1) For every n, the Mellin transform $M\phi^{(n)}$ of $\phi^{(n)}$ extends to a meromorphic function to the whole of complex plane with simple poles in the set of negative integers $\{0, -1, -2, -3, \cdots\}$.

(2) The meromorphic continuation of the Mellin transform $M\phi$ is of rapid decay along the vertical strips.

Proof. (1) follows from Proposition 5.1.6. To prove (2), observe that $M\phi^{'}(s+1) = -sM\phi(s)$. For $Re(s) \gg 0$,

$$\begin{aligned} M\phi^{'}(s+1) :&= \int_0^{\infty} \phi^{'}(t)t^s dt \\ &= -\int_0^{\infty} s\phi(t)t^{s-1}dt \text{ follows from integration by parts)} \\ &= -sM\phi(s). \end{aligned}$$

As $M\phi^{'}$ and $M\phi$ are meromorphic, it follows that $M\phi^{'}(s+1) = -sM\phi(s)$. Now a repeated application of this equation gives

$$M\phi(s) := (-1)^n \frac{M\phi^{(n)}(s+n)}{s(s+1)\cdots(s+n-1)}. \tag{5.1.5}$$

Now let $a < b, K > 0$ and $r \in \mathbb{N}$ be given. Now (3) of Proposition 5.1.6 applied to $\phi^{(r)}$, together with Equation 5.1.5, implies that the function $s \mapsto s^r M\phi(s)$ is bounded on $H_{a,b,K}$. This proves (2) and the proof is complete. □

The following proposition shows how to pass from the decay properties of the Mellin transform of a function to the asymptotic expansion property of the function.

Proposition 5.1.8. *Let $\phi \in \mathcal{M}_p$ for some p. Assume that the Mellin transform $M\phi$ is meromorphic on the entire complex plane with poles lying in the set of negative integers $\{0, -1, -2, \cdots\}$. Suppose that the meromorphic continuation of the Mellin transform $M\phi$ is of rapid decay along the vertical strips. Then the function ϕ has an asymptotic expansion near 0.*

Moreover if $a_r := Res_{s=-r}M\phi(s)$ then $\phi(t) \sim \sum_{r=0}^{\infty} a_r t^r$ near 0.

Proof. The proof is a simple application of the inverse Mellin transform. Let $M \gg 0$. Then, by the inversion formula,

$$\phi(t) = \int_{M-i\infty}^{M+\infty} M\phi(s)t^{-s}ds.$$

Define $F_t(s) := M\phi(s)t^{-s}$. Suppose $N \in \mathbb{N}$ be given. Let $\sigma \in (-N-1, -N)$ be given. For every $A > 0$, by Cauchy's integral formula, we have

$$\int_{M-iA}^{M+iA} F_t(s)ds + \int_{M+iA}^{\sigma+iA} F_t(s)ds + \int_{\sigma+iA}^{\sigma-iA} F_t(s)ds + \int_{\sigma-iA}^{M-iA} F_t(s)ds = \sum_{r=0}^{N} Res_{s=-r}F_t(s). \tag{5.1.6}$$

For a fixed t, F_t has rapid decay along the vertical strips. Thus, when $A \to \infty$, the second and fourth integrals in Equation 5.1.6 vanishes and we obtain the following equation

$$\phi(t) - \sum_{r=0}^{N} a_r t^r = \int_{\sigma - i\infty}^{\sigma + i\infty} M\phi(s) t^{-s} ds. \tag{5.1.7}$$

But $M\phi(\sigma+it)$ has rapid decay in t. Let $M_\sigma := \int_{-\infty}^{\infty} |M\phi(\sigma+it)|$. Then Equation 5.1.7 implies that

$$|\phi(t) - \sum_{r=0}^{N} a_r t^r| \leq M_\sigma t^{-\sigma} \leq M_\sigma t^N \text{ for } t \leq 1.$$

Thus we have shown that for every N, $R_N(t) := \phi(t) - \sum_{r=0}^{N} a_r t^r = O(t^N)$ as $t \to 0$ and hence $R_{N-1}(t) = R_N(t) + a_N t^N = O(t^N)$ as $t \to 0$. This completes the proof. □

5.2 The weak heat kernel expansion

Now, we consider a property of spectral triples which we call the weak heat kernel asymptotic expansion property. We show that a spectral triple having the weak heat kernel asymptotic expansion property is regular and has finite simple dimension spectrum lying in the set of positive integers.

Definition 5.2.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a $p+$ summable spectral triple for a C^* algebra A where $\mathcal{A}$ is a dense $*$ subalgebra of A. We say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has the weak heat kernel asymptotic expansion property if there exists a $*$ subalgebra $\mathcal{B} \subset B(\mathcal{H})$ such that*

(1) the algebra $\mathcal{B}$ contains $\mathcal{A}$,

(2) the unbounded derivations $\delta := [|D|, .]$ leaves $\mathcal{B}$ invariant. Also the unbounded derivation $d := [D, .]$ maps $\mathcal{A}$ into $\mathcal{B}$,

(3) the algebra $\mathcal{B}$ is invariant under the left multiplication by F where $F := sign(D)$, and

(4) for every $b \in \mathcal{B}$, the function $\tau_{p,b} : (0, \infty) \mapsto \mathbb{C}$ defined by $\tau_{p,b}(t) = t^p Tr(be^{-t|D|})$ has an asymptotic power series expansion.

If the algebra $\mathcal{A}$ is unital and the representation of $\mathcal{A}$ on $\mathcal{H}$ is unital then (3) can be replaced by the condition $F \in \mathcal{B}$. The next proposition proves that an odd spectral triple that has the heat kernel asymptotic expansion property is regular and has simple dimension spectrum.

Proposition 5.2.2. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a $p+$ summable spectral triple which has the weak heat kernel asymptotic expansion property. Then the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular and has finite simple dimension spectrum. Moreover the dimension spectrum is contained in $\{1, 2, \cdots, p\}$.*

Proof. Let $\mathcal{B} \subset B(\mathcal{H})$ be a $*$ algebra for which $(1) - (4)$ of Definition 5.2.1 is satisfied. The fact that $\mathcal{B}$ satisfies (1) and (2) implies that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular. First we assume that D is invertible. Let $b \in \mathcal{B}$ be given.

Since $|D|^{-q}$ is trace class for $q > p$, it follows that for every $N > p$ there exists an $M > 0$ such that $Tr(e^{-t|D|}) \leq Mt^{-N}Tr(|D|^{-N})$. Now for $1 \leq t < \infty$ and $N \geq p$ one has

$$\begin{aligned} |Tr(be^{-t|D|})| &\leq \|b\|Tr(e^{-t|D|}) \\ &\leq \|b\|Mt^{-N}Tr(|D|^{-N}) \end{aligned}$$

Thus the function $t \mapsto Tr(be^{-t|D|})$ is of rapid decay near infinity. Now observe that for $Re(s) \gg 0$

$$Tr(b|D|^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty Tr(be^{-t|D|})t^{s-1}dt. \tag{5.2.8}$$

By assumption, the function $\phi(t) := t^pTr(be^{-t|D|})$ has an asymptotic power series expansion near 0. By Equation 5.2.8, it follows that $M\phi(s) = \Gamma(s+p)Tr(b|D|^{-s-p})$. Now Proposition 5.1.6 implies that the function $s \mapsto \Gamma(s)Tr(b|D|^{-s})$ is meromorphic with simple poles lying inside $\{n \in \mathbb{Z} : n \leq p\}$. As $\frac{1}{\Gamma(s)}$ is entire and has simple zeros at $\{k : k \leq 0\}$, it follows that the function $s \to Tr(b|D|^{-s})$ is meromorphic and has simple poles with poles lying in $\{1, 2, \cdots, p\}$.

Suppose D is not invertible. Let P denote the projection onto the kernel of D which is finite dimensional. Let $D^{'} := D + P$ and b be an element in $\mathcal{B}^\infty$. Now note that

$$Tr(be^{-t|D^{'}|}) = Tr(PbP)e^{-t} + Tr(be^{-t|D|}).$$

Hence the function $t \to t^pTr(be^{-t|D^{'}|})$ has an asymptotic power series expansion. Thus the function $s \to Tr(b|D^{'}|^{-s})$ is meromorphic with simple poles lying in $\{1, 2, \cdots, p\}$. Observe that for $Re(s) \gg 0$, $Tr(b|D^{'}|^{-s}) = Tr(b|D|^{-s})$. Hence the function $s \to Tr(b|D|^{-s})$ is meromorphic with simple poles lying in $\{1, 2, \cdots, p\}$. This completes the proof. □

Remark 5.2.3. *If $Tr(be^{-t|D|}) \sim \sum_{r=-p}^{\infty} a_r(b)t^r$ then (3) of Proposition 5.1.6 implies that*

$$\begin{aligned} Res_{z=k}Tr(b|D|^{-z}) &= \frac{1}{k!}a_{-k}(b) \text{ for } 1 \leq k \leq p, \\ Tr(b|D|^{-z})_{z=0} &= a_0(b). \end{aligned}$$

Remark 5.2.4. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple which has the weak heat kernel asymptotic expansion property. Then the dimension spectrum Σ is finite and lies in the set of positive integers. We call the greatest element in the dimension spectrum as the dimension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. If Σ is empty, we set the dimension to be* 0.

In the next proposition, we show that the usual heat kernel asymptotic expansion implies the weak heat kernel asymptotic expansion.

Proposition 5.2.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a $p+$ summable spectral triple for a C^* algebra A. Suppose that $\mathcal{B}$ is a $*$ subalgebra of $B(\mathcal{H})$ satisfying $(1)-(4)$ of Definition 5.2.1. Assume that for every $b \in \mathcal{B}$, the function $\sigma_{p,b} : (0,\infty) \to \mathbb{C}$ defined by $\sigma_{p,b}(t) := t^p Tr(be^{-t^2D^2})$ has an asymptotic power series expansion.*

Then for every $b \in \mathcal{B}$, the function $\tau_{p,b} : t \mapsto t^p Tr(be^{-t|D|})$ has an asymptotic power series expansion.

Proof. It is enough to consider the case where D is invertible. Let $b \in \mathcal{B}$ be given. Let ψ denotes the Mellin transform of the function $t \mapsto Tr(be^{-t^2D^2})$ and χ denote the Mellin transform of the function $t \mapsto Tr(be^{-t|D|})$. Then a simple change of variables shows that $\psi(s) = \frac{\Gamma(\frac{s}{2})}{2} Tr(b|D|^{-s})$. But then $\chi(s) = \Gamma(s)Tr(b|D|^{-s})$. Thus we obtain the equation

$$\chi(s) = \frac{2\Gamma(s)}{\Gamma(\frac{s}{2})}\psi(s).$$

But we have following duplication formula([40]) for the gamma function

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

Hence one has

$$\chi(s) = \frac{1}{\sqrt{\pi}}2^s\Gamma(\frac{s+1}{2})\psi(s).$$

Now Proposition 5.1.6 implies that ψ has decay of order 0 along the vertical strips and has simple poles lying inside $\{n \in \mathbb{Z} : n \leq p\}$. Since the gamma function has rapid decay along the vertical strips, it follows that χ has rapid decay along the vertical strips and has poles lying in $\{n \in \mathbb{Z} : n \leq p\}$. If $\tilde{\chi}$ denotes the Mellin transform of $\tau_{p,b}$ then $\tilde{\chi}(s) = \chi(s+p)$. Hence $\tilde{\chi}$ has rapid decay along the vertical strips and has poles lying in the set of negative integers. Now Proposition 5.1.8 implies that the map $\tau_{p,b}$ has an asymptotic power series expansion near 0. This completes the proof. □

5.3 Stability of the weak heat kernel expansion and the quantum double suspension

Let us recall the definition of the quantum double suspension of a unital C^* algebra . The quantum double suspension is first defined in [22] and our equivalent definition is as in [23]. Let us fix some notations. We denote the left shift on $\ell^2(\mathbb{N})$ by S which is defined on the standard orthonormal basis (e_n) as $Se_n = e_{n-1}$ and p denote the projection $|e_0\rangle\langle e_0|$. The number operator on $\ell^2(\mathbb{N})$ is denoted by N and defined as $Ne_n := ne_n$. We denote the C^* algebra generated by S in $B(\ell^2(\mathbb{N}))$ by $\mathscr{T}$ which is the Toeplitz algebra. Note that $SS^* = 1$ and $p = 1 - S^*S$. Let $\sigma : \mathscr{T} \to C(\mathbb{T})$ be the symbol map which sends S to the generating unitary z. Then one has the following exact sequence

$$0 \to \mathcal{K} \to \mathscr{T} \xrightarrow{\sigma} C(\mathbb{T}) \to 0.$$

Definition 5.3.1. *Let A be a unital C^* algebra. Then the quantum double suspension of A denoted $\Sigma^2(A)$ is the C^* algebra generated by $A \otimes p$ and $1 \otimes S$ in $A \otimes \mathscr{T}$.*

Let A be a unital C^* algebra. One has the following exact sequence.

$$0 \to A \otimes \mathcal{K}(\ell^2(\mathbb{N})) \to \Sigma^2(A) \xrightarrow{\rho} C(\mathbb{T}) \to 0.$$

where ρ is just the restriction of $1 \otimes \sigma$ to $\Sigma^2(A)$.

Remark 5.3.2. *It can be easily shown that $\Sigma^2(C(\mathbb{T})) = C(SU_q(2))$ and more generally one can show that $\Sigma^2(C(S_q^{2n-1})) = C(S_q^{2n+1})$. We refer to [22] or Lemma 3.3.1 of Chapter 3 for the proof. Thus the odd dimensional quantum spheres can be obtained from the circle $\mathbb{T}$ by applying the quantum double suspension recursively.*

Let $\mathcal{A}$ be a dense $*$ subalgebra of a C^* algebra A. Define

$$\Sigma^2_{alg}(\mathcal{A}) := span\{a \otimes k, 1 \otimes S^n, 1 \otimes S^{*m} : a \in \mathcal{A}, k \in \mathcal{S}(\ell^2(\mathbb{N})), n, m \geq 0\}$$

where $\mathcal{S}(\ell^2(\mathbb{N})) := \{(a_{mn}) : \sum_{m,n}(1+m+n)^p|a_{mn}| < \infty$ for every $p\}$.

Then $\Sigma^2_{alg}(\mathcal{A})$ is just the $*$ algebra generated by $\mathcal{A} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N})$ and $1 \otimes S$. Clearly $\Sigma^2_{alg}(\mathcal{A})$ is a dense subalgebra of $\Sigma^2(A)$.

Definition 5.3.3. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and denote the sign of the operator D by F. Then the spectral triple $(\Sigma^2_{alg}(\mathcal{A}), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D) := ((F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ is called the quantum double suspension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.*

Remark 5.3.4. *Note that the torus equivariant spectral triple on $S_q^{2\ell+1}$ is obtained from the spectral $(C^\infty(\mathbb{T}), L^2(\mathbb{T}), \frac{1}{i}\frac{d}{d\theta})$ by applying the double suspension recursively.*

5.3.1 Stability of the weak heat kernel expansion

We consider the stability of the weak heat kernel expansion under quantum double suspension. First observe that the following are easily verifiable.

(1) The spectral triple $(\mathcal{S}(\ell^2(\mathbb{N})), \ell^2(\mathbb{N}), N)$ has the weak heat kernel asymptotic expansion with dimension 0.

(2) Let $(\mathcal{A}_i, \mathcal{H}_i, D_i)$ be a spectral triple with the weak heat kernel asymptotic expansion property with dimension p_i for $1 \leq i \leq n$. Then the spectral triple $(\oplus_{i=1}^n \mathcal{A}_i, \oplus_{i=1}^n \mathcal{H}_i, \oplus_{i=1}^n D_i)$ has the weak heat kernel expansion property with dimension $p := max\{p_i : 1 \leq i \leq n\}$.

(3) If $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with the weak heat kernel asymptotic expansion property and has dimension p then $(\mathcal{A}, \mathcal{H}, |D|)$ also has the weak heat kernel asymptotic expansion with dimension p.

(4) Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with the weak heat kernel asymptotic expansion property with dimension p. Then the amplification $(\mathcal{A} \otimes 1, \mathcal{H} \otimes \ell^2(\mathbb{N}), |D| \otimes 1 + 1 \otimes N)$ also has the asymptotic expansion property with dimension $p + 1$.

We start by proving the stability of the weak heat kernel expansion under tensoring by compacts.

Proposition 5.3.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with the weak heat kernel asymptotic expansion property of dimension p. Then $(\mathcal{A} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N})), \mathcal{H} \otimes \ell^2(\mathbb{N}), D_0 := (F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ also has the weak heat kernel asymptotic expansion property with dimension p.*

Proof. Let $\mathcal{B} \subset B(\mathcal{H})$ be a $*$ subalgebra for which $(1) - (4)$ of Definition 5.2.1 are satisfied. We denote $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ by $\mathcal{B}_0$. We show that $\mathcal{B}_0$ satisfies $(1) - (4)$ of Definition 5.2.1. Clearly (1) holds.

We denote the unbounded derivation $[|D_0|, .]$,$[|D|, .]$ and $[N, .]$ by δ_{D_0}, δ_D and δ_N respectively. By assumption δ_D leaves $\mathcal{B}$ invariant. Clearly $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ is contained in the domain of δ_{D_0} and $\delta_{D_0} = \delta_D \otimes 1 + 1 \otimes \delta_N$ on $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$. Similarly one can show that the unbounded derivation $[D_0, .]$ maps $\mathcal{A} \otimes_{alg} S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0$ invariant.

As $F_0 := sign(D_0) = F \otimes 1$, (3) is clear. Now (4) follows from Lemma 5.1.2 and the equality $t^p Tr((b \otimes k)e^{-t|D_0|} = t^p Tr(be^{-t|D|}) Tr(ke^{-tN})$. This completes the proof. □

Now we consider the stability of the heat kernel asymptotic expansion under the double suspension.

Proposition 5.3.6. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with the weak heat kernel asymptotic expansion property of dimension p. Assume that the algebra $\mathcal{A}$ is unital and the representation on $\mathcal{H}$ is unital. Then the spectral triple $(\Sigma^2_{alg}(\mathcal{A}), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D))$ also has the weak heat kernel asymptotic expansion property with dimension $p + 1$.*

Proof. We denote $\Sigma^2(D)$ by D_0. Let $\mathcal{B}$ be a $*$ subalgebra of $B(\mathcal{H})$ for which $(1) - (4)$ of Definition 5.2.1 are satisfied. For $f = \sum_n \lambda_n z^n \in C^\infty(\mathbb{T})$, let $\sigma(f) := \sum_{n \geq 0} \lambda_n S^n + \sum_{n>0} \lambda_{-n} S^{*n}$. We denote the projection $\frac{1+F}{2}$ by P. We let $\mathcal{B}_0$ to denote the algebra $\mathcal{B} \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ as in Proposition 5.3.5. As in Proposition 5.3.5, we let $\delta_{D_0}, \delta_D, \delta_N$ to denote the unbounded derivations $[|D_0|, .], [|D|, .]$ and $[N, .]$ respectively. Define

$$\tilde{\mathcal{B}} := \{b + P \otimes \sigma(f) + (1 - P) \otimes \sigma(g) : \ b \in \mathcal{B}_0, f, g \in C^\infty(\mathbb{T})\}.$$

Now it is clear that $\tilde{\mathcal{B}}$ satisfies (1) of Definition 5.2.1.

We have already shown in Proposition 5.3.5 that $\mathcal{B}_0$ is closed under δ_{D_0} and $d_0 := [D_0, .]$ maps $\mathcal{A} \otimes S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0$. Now note that

$$\begin{aligned}\delta_{D_0}(P \otimes \sigma(f)) &= P \otimes \sigma(if'),\\ \delta_{D_0}((1-P) \otimes \sigma(g)) &= (1-P) \otimes \sigma(ig'),\\ [D_0, P \otimes \sigma(f)] &= P \otimes \sigma(if'),\\ [D_0, (1-P) \otimes \sigma(g)] &= -(1-P) \otimes \sigma(ig').\end{aligned}$$

Thus it follows that δ_{D_0} leaves $\tilde{\mathcal{B}}$ invariant and $d_0 := [D_0, .]$ maps $\Sigma_2(\mathcal{A})$ into $\tilde{\mathcal{B}}$.

Since $F_0 := sign(D_0) = F \otimes 1$, it follows from definition that $F_0 \in \tilde{\mathcal{B}}$. Now we show that $\tilde{\mathcal{B}}$ satisfies (4).

We have already shown in Proposition 5.3.5 that given $b \in \mathcal{B}_0$, the function $\tau_{p,b}(t) = t^p Tr(be^{-t|D_0|})$ has an asymptotic expansion. Hence the function $\tau_{p+1,b}$ has an asymptotic expansion for every $b \in \mathcal{B}_0$. Now note that

$$\tau_{p+1,P\otimes\sigma(f)}(t) = (\int f(\theta)d\theta) t^p Tr(Pe^{-t|D|}) tTr(e^{-tN}), \tag{5.3.9}$$

$$\tau_{p+1,(1-P)\otimes\sigma(g)}(t) = (\int g(\theta)d\theta) t^p Tr((1-P)e^{-t|D|}) tTr(e^{-tN})). \tag{5.3.10}$$

We have assumed that $\mathcal{A}$ is unital and hence $P \in \mathcal{B}$. Hence $t^p Tr(xe^{-t|D|})$ has an asymptotic power series expansion for $x \in \{P, 1-P\}$. Also $tTr(e^{-tN})$ has an asymptotic power series expansion. From Equation 5.3.9, Equation 5.3.10 and from the earlier observation that $\tau_{p+1,b}$ has an asymptotic power series expansion for $b \in \mathcal{B}_0$, it follows that for every $b \in \tilde{\mathcal{B}}$, the function $\tau_{p+1,b}$ has an asymptotic power series expansion. This completes the proof. □

5.3.2 Higson's differential pair and the heat kernel expansion

Now we discuss some examples of spectral triples which satisfy the weak heat kernel asymptotic expansion property. In particular we discuss the spectral triple associated to noncommutative torus and the classical spectral triple associated to a spin manifold. Let us recall Higson's notion of a differential pair as defined in [21].

Consider a Hilbert space $\mathcal{H}$ and a positive, selfadjoint and an unbounded operator Δ on $\mathcal{H}$. We assume that Δ has compact resolvent. For $k \in \mathbb{N}$, let $\mathcal{H}_k$ be the domain of the operator $\Delta^{\frac{k}{2}}$. The vector space $\mathcal{H}_k$ is given a Hilbert space structure by identifying $\mathcal{H}_k$ with the graph of the operator $\Delta^{\frac{k}{2}}$. Denote the intersection $\bigcap_k \mathcal{H}_k$ by $\mathcal{H}_\infty$. An operator $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is said to be of analytic order $\leq m$ where $m \in \mathbb{Z}$ if T extends to a bounded operator from $\mathcal{H}_{k+m} \to \mathcal{H}_k$ for every k. We say an operator T on $\mathcal{H}_\infty$ has analytic order $-\infty$ if T has analytic order less than $-m$ for every $m > 0$. The following definition is due to Higson. ([21])

Definition 5.3.7. *Let Δ be a positive, unbounded, selfadjoint operator on a Hilbert space $\mathcal{H}$ with compact resolvent. Suppose that $\mathcal{D} := \bigcup_{p\geq 0} \mathcal{D}_q$ is a filtered algebra of operators on $\mathcal{H}_\infty$. The pair $(\mathcal{D}, \Delta)$ is called a differential pair if the following conditions hold.*

1. *The algebra $\mathcal{D}$ is invariant under the derivation $T \to [\Delta, T]$.*

2. *If $X \in \mathcal{D}_q$, then $[\Delta, X] \in \mathcal{D}_{q+1}$.*

3. *If $X \in \mathcal{D}_q$, then the analytic order of $X \leq q$.*

Now let us recall Higson's definition of pseudodifferential operators.

Definition 5.3.8. *Let $(\mathcal{D}, \Delta)$ be a differential pair. We denote the orthogonal projection onto the kernel of Δ by P. Then P is of finite rank as Δ has compact resolvent. Let $\Delta_1 := \Delta + P$. Then Δ_1 is invertible.*

A linear operator T on $\mathcal{H}_\infty$ is called a basic pseudodifferential operator of order $\leq k$ if for every $\ell \geq 0$ there exists m and $X \in \mathcal{D}_{m+k}$ such that

$$T = X\Delta_1^{-\frac{m}{2}} + R$$

where R has analytic order less than or equal to ℓ.

A finite linear combinations of basic pseudodifferential operators of order $\leq k$ is called a pseudodifferential operator of order $\leq k$.

We denote the set of pseudodifferential operators of order ≤ 0 by $\Psi_0(\mathcal{D}, \Delta)$. It is proved in [21] that the pseudodifferential operators of order ≤ 0 is in fact an algebra. We need the following proposition due to Higson. Denote the derivation $T \mapsto [\Delta^{\frac{1}{2}}, T]$ by δ.

Proposition 5.3.9. *Let $(\mathcal{D}, \Delta)$ be a differential pair. The derivation δ leaves the algebra $\Psi_0(\mathcal{D}, \Delta)$ invariant.*

Let $(\mathcal{D}, \Delta)$ be a differential pair. Assume that $\Delta^{-\frac{r}{2}}$ is trace class for some $r > 0$. We say that the analytic dimension of $(\mathcal{D}, \Delta)$ is p if

$$p := \inf\{q > 0 : \Delta^{\frac{-r}{2}} \text{ is trace class for every } r > q\}.$$

Let us make the following definition of the heat kernel expansion for a differential pair.

Definition 5.3.10. *Let $(\mathcal{D}, \Delta)$ be a differential pair of analytic dimension p. We say that $(\mathcal{D}, \Delta)$ has a heat kernel expansion if for $X \in \mathcal{D}_m$, the function $t \mapsto t^{p+m}Tr(Xe^{-t^2\Delta})$ has an asymptotic expansion near 0.*

Now we show that if $(\mathcal{D}, \Delta)$ has the heat kernel expansion then the algebra $\Psi_0(\mathcal{D}, \Delta)$ has the weak heat kernel expansion.

Proposition 5.3.11. *Let $(\mathcal{D}, \Delta)$ be a differential pair of analytic dimension p having the heat kernel expansion. Denote the operator $\Delta^{\frac{1}{2}}$ by $|D|$. Then for every $b \in \Psi_0(\mathcal{D}, \Delta)$, the function $t \mapsto t^p Tr(be^{-t|D|})$ has an asymptotic power series expansion.*

Proof. First observe that if $R : \mathcal{H}_\infty \to \mathcal{H}_\infty$ is an operator of analytic order $< -p-n-1$ then $R|D|^{n+1}$ is trace class and hence by Taylor's series

$$Tr(Re^{-t|D|}) = \sum_{k=0}^{n} \frac{(-1)^k Tr(R|D|^k)}{k!} t^k + O(t^{n+1})$$

for t near 0. Thus it is enough to show the result when $b = X\Delta_1^{-\frac{m}{2}}$. For an operator T on $\mathcal{H}_\infty$, let $\zeta_T(s) := Tr(T|D|^{-s})$. Then $\zeta_b(s) := \zeta_X(s+m)$. As in Proposition 5.2.5, one can show that $\Gamma(s)\zeta_X(s)$ has rapid decay along the vertical strips. Now

$$\Gamma(s)\zeta_b(s) = \frac{\Gamma(s)}{\Gamma(s+m)} \Gamma(s+m)\zeta_X(s+m).$$

Hence $\Gamma(s)\zeta_b(s)$ has rapid decay along the vertical strips. But $\Gamma(s)\zeta_b(s)$ is the Mellin transform of $Tr(be^{-t|D|})$. Hence by Proposition 5.1.8, it follows that $t^p Tr(be^{-t|D|})$ has an asymptotic power series expansion. This completes the proof. □

We make use of the following proposition to prove that the spectral triple associated to the NC torus and that of a spin manifold posses the weak heat kernel expansion property.

Proposition 5.3.12. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable spectral triple and $\Delta := D^2$. Suppose that there exists an algebra of operators $D := \bigcup_{p \geq 0} \mathcal{D}_p$ such that (D, Δ) is a differential pair of analytic dimension p. Assume that $(\mathcal{D}, \Delta)$ satisfies the following*

1. *The algebra $\mathcal{D}_0$ contains $\mathcal{A}$ and $[D, \mathcal{A}]$.*

2. *The differential pair $(\mathcal{D}, \Delta)$ has the heat kernel expansion property.*

3. *The operator $D \in \mathcal{D}_1$.*

Then the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has the weak heat kernel asymptotic expansion property.

Proof. Without loss of generality, we can assume that D is invertible. We let $\mathcal{B}$ be the algebra of pseudodifferential operators of order 0 associated to $(\mathcal{D}, \Delta)$. Now Proposition 5.3.9 together with the fact that $\mathcal{D}_0 \subset \mathcal{B}$ shows that $\mathcal{B}$ contains $\mathcal{A}$ and $[D, \mathcal{A}]$ and is invariant under $\delta := [|D|, .]$. Since $D \in \mathcal{D}_1$, it follows that $F := D\Delta^{\frac{-1}{2}} \in \mathcal{B}$. Now (4) of Definition 5.2.1 follows from Proposition 5.3.11. This completes the proof. □

5.3.3 Examples

Now we discuss some examples of spectral triples which satisfy the weak heat kernel asymptotic expansion. We start with the classical example.

Let M be a Riemannian spin manifold and $S \to M$ be a spinor bundle. We denote the Hilbert space of square integrable sections $L^2(M, S)$ by $\mathcal{H}$. We represent $C^\infty(M)$ on $\mathcal{H}$ by multiplication operators. Let D be the Dirac operator associated with the Levi-Civita connection. Then the triple $(C^\infty(M), \mathcal{H}, D)$ is a spectral triple. Also the operator D^2 is then a generalised Laplacian ([3]). Let $\mathcal{D}$ denote the usual algebra of differential operators on S. Then $(\mathcal{D}, \Delta)$ is a differential pair. Moreover Proposition 2.4.6 in [3] implies that $(\mathcal{D}, \Delta)$ has the heat kernel expansion. Also $D \in \mathcal{D}_1$. Now Proposition 5.3.12 implies that the spectral triple $(C^\infty(M), \mathcal{H}, D)$ has the weak heat kernel asymptotic expansion.

The spectral triple associated to the NC torus

Let us recall the definition of the noncommutative torus which we abbreviate as NC torus. Throughout we assume that $\theta \in [0, 2\pi)$.

Definition 5.3.13. *The C^* algebra A_θ is defined as the universal C^* algebra generated by two unitaries u and v such that $uv = e^{i\theta} vu$.*

Define the operators U and V on $\ell^2(\mathbb{Z}^2)$ as follows:

$$
\begin{aligned}
Ue_{m,n} &:= e_{m+1,n}, \\
Ve_{m,n} &:= e^{-in\theta} e_{m,n+1},
\end{aligned}
$$

where $\{e_{m,n}\}$ denotes the standard orthonormal basis on $\ell^2(\mathbb{Z}^2)$. Then it is well known that $u \to U$ and $v \to V$ gives a faithful representation of the C^* algebra A_θ.

Consider the positive selfadjoint operator Δ on $\mathcal{H} := \ell^2(\mathbb{Z}^2)$ defined on the orthonormal basis $\{e_{m,n}\}$ by $\Delta(e_{m,n}) = (m^2 + n^2)e_{m,n}$. For a polynomial $P = p(m, n)$, define the operator T_P on $\mathcal{H}_\infty$ by $T_P(e_{m,n}) := p(m, n)e_{m,n}$. The group $\mathbb{Z}^2$ acts on the algebra of polynomials as follows. For $x := (a, b) \in \mathbb{Z}^2$ and $P := p(m, n)$, define $x.P := p(m - a, n - b)$. We denote $(1, 0)$ by e_1 and $(0, 1)$ by e_2.

Note that if P is a polynomial of degree $\leq k$, then $T_P \Delta^{-\frac{k}{2}}$ is bounded on $Ker(\Delta)^\perp$. Thus it follows that if P is a polynomial of degree $\leq k$ then T_P has analytic order $\leq k$.

Also note that

$$
\Delta_1^{\frac{k}{2}} U \Delta_1^{\frac{-k}{2}} e_{m,n} := \frac{((m+1)^2 + n^2)^{\frac{k}{2}}}{(m^2 + n^2)^{\frac{k}{2}}} e_{m+1,n} \text{ if } (m, n) \neq 0.
$$

Thus it follows that U is of analytic order ≤ 0. Similary one can show that V is of analytic order ≤ 0. Now note the following commutation relationship

$$UT_P := T_{e_1.P}U, \tag{5.3.11}$$

$$VT_P := T_{e_1.P}V. \tag{5.3.12}$$

Thus it follows that $[\Delta, U^\alpha V^\beta] = T_Q U^\alpha V^\beta$ for some degree 1 polynomial Q.

Let us define $\mathcal{D}_p := span\{T_{P_{\alpha,\beta}} U^\alpha V^\beta : deg(P_{\alpha,\beta}) \leq k\}$ and let $\mathcal{D} := \bigcup_p \mathcal{D}_p$. The above observations can be rephrased into the following proposition.

Proposition 5.3.14. *The pair* $(\mathcal{D}, \Delta)$ *is a differential pair of analytic dimension* 2.

Now we show that the differential pair $(\mathcal{D}, \Delta)$ has the heat kernel expansion.

Proposition 5.3.15. *The differential pair* $(\mathcal{D}, \Delta)$ *has the heat kernel expansion property.*

Proof. Let $X \in \mathcal{D}_q$ be given. It is enough to consider the case when $X := T_P U^\alpha V^\beta$. First note that $Tr(Xe^{-t\Delta}) = 0$ unless $(\alpha, \beta) = 0$. Now let $X := T_P$. Again it is enough to consider the case when P is a monomial. Let $P = p(m,n) = m^{k_1} n^{k_2}$. Now

$$Tr(T_P e^{-t\Delta}) = (\sum_{m \in \mathbb{Z}} m^{k_1} e^{-tm^2})(\sum_{n \in \mathbb{Z}} n^{k_2} e^{-tn^2}).$$

Now the asymptotic expansion follows from applying Proposition 2.4.6 in [3] to the standard Laplacian on the circle. This completes the proof. □

Let $\mathcal{A}_\theta$ be the $*$ algebra generated by U and V. We consider the direct sum representation of A_θ on $\mathcal{H} \oplus \mathcal{H}$. Define $D := \begin{bmatrix} 0 & T_{m-in} \\ T_{m+in} & 0 \end{bmatrix}$. Then D is selfadjoint on $\mathcal{H} \oplus \mathcal{H}$ and $D^2 = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$. It is well known that $(\mathcal{A}_\theta, \mathcal{H} \oplus \mathcal{H}, D)$ is a 2+ summable spectral triple.

Proposition 5.3.16. *The spectral triple* $(\mathcal{A}_\theta, \mathcal{H} \oplus \mathcal{H}, D)$ *has the weak heat kernel asymptotic expansion property.*

Proof. Let $(\mathcal{D}, \Delta)$ be the differential pair considered in Proposition 5.3.14. Then the amplification $(\mathcal{D}' := M_2(\mathcal{D}), D^2)$ is a differential pair. Note that $D \in \mathcal{D}'_1$. Clearly $\mathcal{A}_\theta \subset \mathcal{D}'_0$. Note the commutation relations

$$[T_{m \pm in}, U] = U,$$

$$[T_{m \pm in}, V] = \pm iV.$$

This implies that $[D, \mathcal{A}_\theta] \subset \mathcal{D}'_0$. Since $(\mathcal{D}, \Delta)$ has the heat kernel expansion, it follows that the differential pair $(M_2(\mathcal{D}), D^2)$ also has the heat kernel expansion. Now Proposition 5.3.12 implies that the spectral triple $(\mathcal{A}_\theta, \mathcal{H} \oplus \mathcal{H}, D)$ has the weak heat kernel expansion. This completes the proof. □

But to deduce that the spectral triple $(\mathcal{A}(S_q^{2\ell+1}), \mathcal{H}_\ell, D_\ell)$ satisfies the weak heat kernel asymptotic expansion, we need a topological version of Definition 5.2.1 and Proposition 5.3.6. We do this in the next section.

5.4 Smooth subalgebras and the weak heat kernel asymptotic expansion

First we recall the definition of smooth subalgebras of C^* algebras. For an algebra A (possibly non-unital), we denote the algebra obtained by adjoining a unit to A by A^+.

Definition 5.4.1. *Let A be a unital C^* algebra. A dense unital $*$ subalgebra $\mathcal{A}^\infty$ is called a smooth subalgebra of A if*

1. *The algebra $\mathcal{A}^\infty$ is a Fréchet $*$ algebra.*
2. *The unital inclusion $\mathcal{A}^\infty \subset A$ is continuous.*
3. *The algebra $\mathcal{A}^\infty$ is spectrally invariant in A i.e. if an element $a \in \mathcal{A}^\infty$ is invertible in A then $a^{-1} \in \mathcal{A}^\infty$.*

Suppose A is a non-unital C^* algebra. A dense Fréchet $*$ subalgebra $\mathcal{A}^\infty$ is said to be smooth in A if $(\mathcal{A}^\infty)^+$ is smooth in A^+.

We also assume that our smooth subalgebras satisfy the condition that if $\mathcal{A}^\infty \subset A$ is smooth then $\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k)) \subset A \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ is smooth. Here $\hat{\otimes}_\pi$ denotes the projective tensor product.

Let A be a unital C^* algebra and $\mathcal{A}^\infty$ be a smooth unital $*$ subalgebra of A. Assume that the topology on $\mathcal{A}^\infty$ is given by a countable family of seminorms $(\|\cdot\|_p)$. Let us denote the operator $1 \otimes S$ by α. Define the smooth quantum double suspension of $\mathcal{A}^\infty$ as follows

$$\Sigma^2_{smooth}(\mathcal{A}^\infty) := \Bigg\{ \sum_{j,k \in \mathbb{N}} \alpha^{*j}(a_{jk} \otimes p)\alpha^k + \sum_{k \geq 0} \lambda_k \alpha^k + \sum_{k>0} \lambda_{-k}\alpha^{*k} \ : \ a_{jk} \in \mathcal{A}^\infty, \\ \sum_{j,k}(1+j+k)^n \|a_{jk}\|_p < \infty, \ (\lambda_k) \text{ is rapidly decreasing} \Bigg\} \tag{5.4.13}$$

Now let us topologize $\Sigma^2_{smooth}(\mathcal{A}^\infty)$ by defining a seminorm $\| \ \|_{n,p}$ for every $n, p \geq 0$. For an element

$$a := \sum_{j,k \in \mathbb{N}} \alpha^{*j}(a_{jk} \otimes p)\alpha^k + \sum_{k \geq 0} \lambda_k \alpha^k + \sum_{k>0} \lambda_{-k}\alpha^{*k}$$

in $\Sigma^2_{smooth}(\mathcal{A}^\infty)$, define $\|a\|_{n,p}$ by

$$\|a\|_{n,p} := \sum_{j,k \in \mathbb{N}} (1+|j|+|k|)^n \|a_{jk}\|_p + \sum_{k \in \mathbb{Z}} (1+|k|)^n |\lambda_k|.$$

It is easily verifiable that

1. The subspace $\Sigma^2_{smooth}(\mathcal{A}^\infty)$ is a dense $*$ subalgebra of $\Sigma^2(A)$.

2. The topology on $\Sigma^2_{smooth}(\mathcal{A}^\infty)$ induced by the seminorms $(\| \, \|_{n,p})$ makes $\Sigma^2_{smooth}(\mathcal{A}^\infty)$ a Fréchet $*$ algebra.

3. The unital inclusion $\Sigma^2_{smooth}(\mathcal{A}^\infty) \subset \Sigma^2(A)$ is continuous.

The next proposition proves that the Fréchet algebra $\Sigma^2_{smooth}(\mathcal{A}^\infty)$ is in fact smooth in $\Sigma^2(A)$.

Proposition 5.4.2. *Let A be a unital C^* algebra and let $\mathcal{A}^\infty \subset A$ be a unital smooth subalgebra such that $\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k)) \subset A \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ is smooth for every $k \in \mathbb{N}$. Then the algebra $\Sigma^2_{smooth}(\mathcal{A}^\infty)\hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k))$ is smooth in $\Sigma^2(A) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ for every $k \geq 0$.*

Proof. Let us denote the restriction of $1 \otimes \sigma$ to $\Sigma^2(A)$ by ρ. Recall the $\sigma : \mathscr{T} \to C(\mathbb{T})$ is the symbol map sending S to the generating unitary. Then one has the following exact sequence at the C^* algebra level

$$0 \to A \otimes \mathcal{K}(\ell^2(\mathbb{N})) \to \Sigma^2(A) \xrightarrow{\rho} C(\mathbb{T}) \to 0.$$

At the subalgebra level one has the following "sub" exact sequence

$$0 \to \mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N})) \to \Sigma^2_{smooth}(\mathcal{A}^\infty) \xrightarrow{\rho} C^\infty(\mathbb{T}) \to 0.$$

Since $\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N})) \subset A \otimes \mathcal{K}(\ell^2(\mathbb{N}))$ and $C^\infty(\mathbb{T}) \subset C(\mathbb{T})$ are smooth, it follows from Lemma A.1.4 that $\Sigma^2_{smooth}(\mathcal{A}^\infty)$ is smooth in $\Sigma^2(A)$. Similarly one can show along the same lines first by tensoring the C^* algebra exact sequence by $\mathcal{K}(\ell^2(\mathbb{N}^k))$ and then by tensoring the Fréchet algebra exact sequence by $\mathcal{S}(\ell^2(\mathbb{N}^k))$ that $\Sigma^2_{smooth}(\mathcal{A}^\infty)\hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}^k))$ is smooth in $\Sigma^2(A) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ for every $k > 0$. This completes the proof. □

5.4.1 The topological weak heat kernel expansion

We need the following version of the weak heat kernel expansion.

Definition 5.4.3. *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a $p+$ summable spectral triple for a C^* algebra A where $\mathcal{A}^\infty$ is smooth in A. We say that the spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ has the topological weak heat kernel asymptotic expansion property if there exists a $*$ subalgebra $\mathcal{B}^\infty \subset B(\mathcal{H})$ such that*

(1) The algebra $\mathcal{B}^\infty$ has a Fréchet space structure and endowed with it it is a Fréchet $*$ algebra,

(2) The algebra $\mathcal{B}^\infty$ contains $\mathcal{A}^\infty$,

(3) The inclusion $\mathcal{B}^\infty \subset B(\mathcal{H})$ is continuous,

(4) The unbounded derivations $\delta := [|D|, .]$ leaves $\mathcal{B}^\infty$ invariant and is continuous. Also the unbounded derivation $d := [D, .]$ maps $\mathcal{A}^\infty$ into $\mathcal{B}^\infty$ in a continuous fashion,

(5) The left multiplication by the operator $F := sign(D)$ denoted L_F leaves $\mathcal{B}^\infty$ invariant and is continuous, and

(6) The function $\tau_p : (0,\infty) \times \mathcal{B}^\infty \to \mathbb{C}$ defined by $\tau_p(t,b) = t^p Tr(be^{-t|D|})$ has a uniform asymptotic power series expansion.

We need an analog of Proposition 5.3.5 and Proposition 5.3.6. First we need the following two lemmas.

Lemma 5.4.4. *Let E be a Fréchet space and $F \subset E$ be a dense subspace. Let $\phi : (0,\infty) \times E \to \mathbb{C}$ be a continuous function which is linear in the second variable. Suppose that $\phi : (0,\infty) \times F \to \mathbb{C}$ has a uniform asymptotic power series expansion then $\phi : (0,\infty) \times E \to \mathbb{C}$ has a uniform asymptotic power series expansion.*

Proof. Suppose that $\phi(t,f) \sim \sum_{r=0}^{\infty} a_r(f)t^r$. Then $a_r : F \to \mathbb{C}$ is linear and is continuous for every $r \in \mathbb{N}$. Since F is dense in E, for every $r \in \mathbb{N}$, the function a_r admits a continuous extension to the whole of E which we still denote by a_r. Now fix $N \in \mathbb{N}$. Then there exists a neighbourhood U of E containing 0 and $\epsilon, M > 0$ such that

$$|\phi(t,f) - \sum_{r=0}^{N} a_r(f)t^r| \leq Mt^{N+1} \text{ for } 0 < t < \epsilon, f \in U \cap F. \tag{5.4.14}$$

Since $\phi(t,.)$ and $a_r(.)$ are continuous and as F is dense in E, Equation 5.4.14 continues to hold for every $f \in U$. This completes the proof. □

Lemma 5.4.5. *Let E_1, E_2 be Fréchet spaces and let $F_i : (0,\infty) \times E_i \to \mathbb{C}$ be continuous and linear in the second variable for $i = 1, 2$. Consider the function $F : (0,\infty) \times E_1 \hat{\otimes}_\pi E_2 \to \mathbb{C}$ defined by $F(t, e_1 \otimes e_2) = F_1(t,e_1)F(t,e_2)$. Assume that F is continuous. If F_1 and F_2 has uniform asymptotic expansions then F has a uniform asymptotic power series expansion.*

Proof. By Lemma 5.4.4, it is enough to show that $F : (0,\infty) \times E_1 \otimes_{alg} E_2 \to \mathbb{C}$ has a uniform asymptotic power series expansion. Let $\theta : E_1 \times E_2 \to E_1 \otimes_{alg} E_2$ be defined by $\theta(e_1,e_2) = e_1 \otimes e_2$. Consider the map $G : (0,\infty) \times E_1 \times E_2 \to \mathbb{C}$ defined by $G(t,e_1,e_2) := F(t, \theta((e_1,e_2)))$. By Lemma 5.1.2, it follows that G has a uniform asymptotic power series expansion say

$$G(t,e) \sim \sum_{r=0}^{\infty} a_r(e)t^r.$$

The maps $a_r : E_1 \times E_2 \to \mathbb{C}$ are continuous bilinear. We let $\tilde{a}_r : E_1 \hat{\otimes}_\pi E_2 \to \mathbb{C}$ be the linear maps such that $\tilde{a}_r \circ \theta := a_r$. Let $N \in \mathbb{N}$ be given. Then there exists $\epsilon, M > 0$ and open sets U_1, U_2 containing 0 in E_1, E_2 such that

$$|G(t,e) - \sum_{r=0}^{N} a_r(e)t^r| \leq Mt^{N+1} \text{ for } 0 < t < \epsilon, e \in U_1 \times U_2. \tag{5.4.15}$$

Without loss of generality, we can assume that $U_i := \{x \in E_i : \ p_i(x) < 1\}$ for a seminorm p_i of E_i. Now Equation (5.4.15) implies that

$$|F(t, \theta(e)) - \sum_{r=0}^{N} \tilde{a}_r(\theta(e))t^r| \leq Mt^{N+1} \ \text{ for } 0 < t < \epsilon, e \in U_1 \times U_2. \tag{5.4.16}$$

Hence for $t \in (0, \epsilon)$ and $x \in \theta(U_1 \times U_2)$,

$$|F(t, x) - \sum_{r=0}^{N} \tilde{a}_r(x)t^r| \leq Mt^{N+1}. \tag{5.4.17}$$

Since $\tilde{a}_r$ is linear and F is linear in the second variable, it follows that Equation 5.4.17 continues to hold for x in the convex hull of $\theta(U_1 \times U_2)$ which is nothing but the unit ball determined by the seminorm $p_1 \otimes p_2$ in $E_1 \otimes_{alg} E_2$. This completes the proof. □

In the next proposition, we consider the stability of the weak heat kernel asymptotic expansion property for tensoring by smooth compacts.

Proposition 5.4.6. *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple where the algebra $\mathcal{A}^\infty$ is a smooth subalgebra of C^* algebra. Assume that $(\mathcal{A}^\infty, \mathcal{H}, D)$ has the topological weak heat kernel expansion property with dimension p. Then the spectral triple $(\mathcal{A}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N})), \mathcal{H} \otimes \ell^2(\mathbb{N}), D_0 := (F \otimes 1)(|D| \otimes 1 + 1 \otimes N))$ also has the weak heat kernel asymptotic expansion property with dimension p where $F := sign(D)$.*

Proof. Let $\mathcal{B}^\infty \subset B(\mathcal{H})$ be a $*$ subalgebra for which $(1)-(6)$ of Definition 5.4.3 are satisfied. We denote $\mathcal{B}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}))$ by $\mathcal{B}_0^\infty$. We show that $\mathcal{B}_0^\infty$ satisfies $(1)-(6)$ of Definition 5.4.3. First note that the natural representation of $\mathcal{B}_0^\infty$ in $\mathcal{H} \otimes \ell^2(\mathbb{N})$ is injective. Thus (3) is clear. Also (1) and (2) are obvious. Let us now prove (4).

We denote the unbounded derivation $[|D_0|, .]$,$[|D|, .]$ and $[N, .]$ by δ_{D_0}, δ_D and δ_N respectively. By assumption δ_D leaves $\mathcal{B}$ invariant and is continuous. It is also easy to see that δ_N leaves $\mathcal{S}(\ell^2(\mathbb{N}))$ invariant and is continuous. Let $\delta' := \delta_D \otimes 1 + 1 \otimes \delta_N$. Then $\delta' : \mathcal{B}_0^\infty \to \mathcal{B}_0^\infty$ is continuous. Clearly $\mathcal{B}^\infty \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$ is contained in the domain of δ and $\delta = \delta'$ on $\mathcal{B}^\infty \otimes_{alg} \mathcal{S}(\ell^2(\mathbb{N}))$. Now let $a \in \mathcal{B}_0^\infty$ be given. Then there exists a sequence (a_n) in $\mathcal{B}^\infty \otimes_\pi \mathcal{S}(\ell^2(\mathbb{N}))$ such that (a_n) converges to a in $\mathcal{B}_0^\infty$. Since δ' is continuous on $\mathcal{B}_0^\infty$ and the inclusion $\mathcal{B}_0^\infty \subset B(\mathcal{H})$ is continuous, it follows that $\delta_{D_0}(a_n) = \delta'(a_n)$ converges to $\delta'(a)$. As δ_{D_0} is a closed derivation, it follows that $a \in Dom(\delta_{D_0})$ and $\delta_{D_0}(a) = \delta'(a)$. Hence we have shown that δ_{D_0} leaves $\mathcal{B}_0^\infty$ invariant and is continuous. Similarly one can show that the unbounded derivation $d_0 := [D_0, .]$ maps $\mathcal{A}\hat{\otimes}_\pi S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0^\infty$ invariant in a continuous manner.

As $F_0 := sign(D_0) = F \otimes 1$, (5) is clear. Consider the function $\tau_p : (0, \infty) \times \mathcal{B}_0^\infty \to \mathbb{C}$ defined by $\tau_p(t, b) := t^p Tr(be^{-t|D_0|})$. Then $\tau_p(t, b \otimes k) = \tau_p(t, b)\tau_0(t, k)$. Hence by Lemma 5.4.5, it follows that τ_p has a uniform asymptotic power series expansion. This completes the proof. □

Now we consider the stability of the weak heat kernel asymptotic expansion under the double suspension.

Proposition 5.4.7. *Let $(\mathcal{A}^\infty, \mathcal{H}, D)$ be a spectral triple with the topological weak heat kernel asymptotic expansion property of dimension p. Assume that the algebra $\mathcal{A}^\infty$ is unital and the representation of $\mathcal{A}^\infty$ on $\mathcal{H}$ is unital. Then the spectral triple $(\Sigma^2_{smooth}(\mathcal{A}^\infty), \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2(D))$ also has the topological weak heat kernel asymptotic expansion property with dimension $p+1$.*

Proof. We denote the operator $\Sigma^2(D)$ by D_0. Let $\mathcal{B}^\infty$ be a $*$ subalgebra of $B(\mathcal{H})$ for which $(1)-(6)$ of Definition 5.4.3 are satisfied. For $f = \sum_{n\in\mathbb{Z}} \lambda_n z^n \in C^\infty(\mathbb{T})$, let $\sigma(f) := \sum_{n\geq 0} \lambda_n S^n + \sum_{n>0} \lambda_{-n} S^{*n}$. We denote the projection $\frac{1+F}{2}$ by P. We assume here that $P \neq \pm 1$ as the case $P = \pm 1$ is similar. We let $\mathcal{B}_0^\infty$ to denote the algebra $\mathcal{B}^\infty \hat{\otimes}_\pi \mathcal{S}(\ell^2(\mathbb{N}))$ as in Proposition 5.4.6. As in Proposition 5.4.6, we let $\delta_{D_0}, \delta_D, \delta_N$ to denote the unbounded derivations $[|D_0|, .], [|D|, .]$ and $[N, .]$ respectively. Define

$$\tilde{\mathcal{B}}^\infty := \{b + P \otimes \sigma(f) + (1-P) \otimes \sigma(g) : \ b \in \mathcal{B}_0^\infty, f, g \in C^\infty(\mathbb{T})\}.$$

Then $\tilde{\mathcal{B}}^\infty$ is isomorphic to the direct sum $\mathcal{B}_0^\infty \oplus C^\infty(\mathbb{T}) \oplus C^\infty(\mathbb{T})$. We give $\tilde{\mathcal{B}}^\infty$ the Fréchet space structure coming from this decomposition. It is easy to see that $\tilde{\mathcal{B}}^\infty$ is a Fréchet $*$ subalgebra of $B(\mathcal{H} \otimes \ell^2(\mathbb{N}))$. Clearly $(\pi \otimes 1)(\Sigma^2(\mathcal{A}^\infty)) \subset \tilde{\mathcal{B}}^\infty$. Thus we have shown that (1) and (2) of Definition 5.4.3 are satisfied. Since $\mathcal{B}_0^\infty$ is represented injectively on $\mathcal{H} \otimes \ell^2(\mathbb{N})$, it follows that $\tilde{\mathcal{B}}$ satisfies (3).

We have already shown in Proposition 5.4.6 that $\mathcal{B}_0^\infty$ is closed under δ_{D_0} and is continuous. Also we have shown that $d_0 := [D_0, .]$ maps $\mathcal{A}\hat{\otimes}_\pi S(\ell^2(\mathbb{N}))$ into $\mathcal{B}_0^\infty$ continuously. Now note that

$$\begin{aligned}
\delta_{D_0}(P \otimes \sigma(f)) &= P \otimes \sigma(if'), \\
\delta_{D_0}((1-P) \otimes \sigma(g)) &= (1-P) \otimes \sigma(ig'), \\
[D_0, P \otimes \sigma(f)] &= P \otimes \sigma(if'), \\
[D_0, (1-P) \otimes \sigma(g)] &= -(1-P) \otimes \sigma(ig').
\end{aligned}$$

Thus it follows that δ_{D_0} leaves $\tilde{\mathcal{B}}^\infty$ invariant and is continuous. Also, it follows that $d_0 := [D_0, .]$ maps $\Sigma^2(\mathcal{A}^\infty)$ into $\tilde{\mathcal{B}}$ in a continuous manner.

Since $F_0 := sign(D_0) = F \otimes 1$, it follows from definition that $F_0 \in \tilde{\mathcal{B}}^\infty$. Now we show that $\tilde{\mathcal{B}}^\infty$ satisfies (6).

We have already shown in Proposition 5.4.6 that the function $\tau_p : (0, \infty) \otimes \mathcal{B}_0^\infty \to \mathbb{C}$ defined by $\tau_p(t, b) := t^p Tr(be^{-t|D_0|})$ has a uniform asymptotic power series expansion. Hence τ_{p+1} restricted to $\mathcal{B}_0^\infty$ has a uniform asymptotic power series expansion. Now note that

$$\tau_{p+1}(P \otimes \sigma(f)) = (\int f(\theta)d\theta) t^p Tr(Pe^{-t|D|}) t Tr(e^{-tN}), \tag{5.4.18}$$

$$\tau_{p+1}((1-P) \otimes \sigma(g)) = (\int g(\theta)d\theta) t^p Tr((1-P)e^{-t|D|}) t Tr(e^{-tN})). \tag{5.4.19}$$

We have assumed that $\mathcal{A}^\infty$ is unital and hence $P \in \mathcal{B}^\infty$. Thus $t^pTr(xe^{-t|D|})$ has an asymptotic power series expansion for $x \in \{P, 1-P\}$. Also $tTr(e^{-tN})$ has an asymptotic power series expansion. Now Equation 5.4.18 and Equation 5.4.19, together with the earlier observation that τ_{p+1} restricted to $\mathcal{B}_0^\infty$ has a uniform asymptotic power series expansion, imply that the function $\tau_{p+1} : (0,\infty) \times \tilde{\mathcal{B}}^\infty \to \mathbb{C}$ has a uniform asymptotic power series expansion. This completes the proof. □

Remark 5.4.8. *If we start with the canonical spectral triple $(C^\infty(\mathbb{T}), L^2(\mathbb{T}), \frac{1}{i}\frac{d}{d\theta})$ on the circle and apply the double suspension recursively one obtains the torus equivariant spectral triple for the odd dimensional quantum spheres studied in Chapter 3. Now Theorem 5.4.7 implies that the torus equivariant spectral triple on $S_q^{2\ell+1}$ satisfies the weak heat kernel expansion. Also Theorem 5.4.7, along with Theorem 5.2.2, gives a proof of Proposition 3.4.4*

We end this chapter by showing that the equivariant spectral triple considered in Chapter 4 has the topological weak heat kernel expansion.

Proposition 5.4.9. *The equivariant spectral triple $(C^\infty(S_q^{2\ell+1}), \mathcal{H}, D_{eq})$ has the topological weak heat kernel expansion.*

Proof. Let $\mathcal{J} := OP_{D_\ell}^{-\infty} \hat{\otimes} \mathscr{T}_\ell^\infty$. Recall the definition of the algebra $\mathscr{B}$ considered in Proposition 4.6.4.

$$\begin{aligned}\mathscr{B} :=& \{a_1 P_\ell \otimes P + a_2 P_\ell \otimes (1-P) + a_3(1-P_\ell) \otimes P + a_4(1-P_\ell) \otimes (1-P) + R :\\ & a_1, a_2, a_3, a_4 \in \mathcal{A}_\ell^\infty, R \in \mathcal{J}\}\end{aligned}$$

The algbera $\mathscr{B}$ is isomorphic to $\mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus \mathcal{A}_\ell^\infty \oplus \mathcal{J}$. We give $\mathscr{B}$ the Fréchet space structure coming from this decomposition. Proposition 4.6.4 and 4.6.5 implies that $\mathscr{B}$ contains $C^\infty(S_q^{2\ell+1})$ and is closed under $\delta := [|D_{eq}|, .]$ and $d := [D, .]$. Moreover it is shown that δ and d are continuous on $\mathscr{B}$. Note that $F_{eq} := F_\ell \otimes P - 1 \otimes (1-P)$. Hence by definition $F_{eq} \in \mathcal{B}$. Now note that the torus equivariant spectral triple $(\mathcal{A}_\ell^\infty, \mathcal{H}_\ell, D_\ell)$ has the topological weak heat kernel asymptotic expansion. Thus it is enough to show that the map $\tau_{2\ell+1} : (0,\infty) \times \mathcal{J} \to \mathbb{C}$ defined by $\tau_{2\ell+1}(t,b) := t^{2\ell+1}Tr(be^{-t|D_{eq}|})$ has uniform asymptotic expansion.

But this follows from the fact that $(OP_{D_\ell}^{-\infty}, \mathcal{H}_\ell, D_\ell)$ and $(\mathscr{T}^\infty, \ell^2(\mathbb{N}), N)$ have the topological weak heat kernel expansion and by using Lemma 5.4.5. This completes the proof. □

Remark 5.4.10. *The method in Chapter 4 can be applied to show that the equivariant spectral triple on the quantum $SU(2)$ constructed in [5] has the weak heat kernel asymptotic expansion property with dimension 3 and hence deducing the dimension spectrum computed in [17]. It has been shown in [6] that the isospectral triple studied in [39] differs from the equivariant one (with multiplicity 2) constructed in [5] only be a smooth perturbation. As a result it will follow that (Since the extension $\mathcal{B}^\infty$ for the equivariant spectral triple satisfying Definition 5.4.3 contains* *the algebra of smoothing operators) the isospectral spectral triple also has the weak heat kernel expansion with dimension 3.*

Chapter 6

The K-groups of the quantum Steifel manifold $SU_q(n)/SU_q(n-2)$

In the final chapter of this thesis, we compute the $K-$groups of the quantum homogeneous space $SU_q(n)/SU_q(n-2)$. The K-groups of the quantum group $SU_q(n)$ were computed by Nagy in [27] by using his bivariant K-theory. To construct non-trivial spectral triples on these quantum homogeneous spaces, an explicit knowledge of the generators is essential. In this chapter, we compute the K-groups of the space $SU_q(n)/SU_q(n-2)$. We make use of the irreducible representations of the C^* algebra $C(SU_q(n)/SU_q(n-2))$ obtained in [31]. We construct certain exact sequences as in [35]. Then we perform the six term sequence computation to obtain the K-groups.

6.1 The quantum Steifel manifold $S_q^{n,m}$

First let us recall the definition of the quantum Steifel manifold $S_q^{n,m}$ as defined in [31].

Let $1 \leq m \leq n-1$. Call the generators of $SU_q(n)$ as u_{ij} and that of $SU_q(n-m)$ as v_{ij}. The map $\phi : C(SU_q(n)) \to C(SU_q(n-m))$ defined by

$$\phi(u_{ij}) \quad := \quad \begin{cases} v_{ij} & \text{if } 1 \leq i,j \leq n-m, \\ \delta_{ij} & \text{otherwise} \end{cases} \tag{6.1.1}$$

is a surjective unital C^* algebra homomorphism such that $\Delta \circ \phi = (\phi \otimes \phi)\Delta$. In this way the quantum group $SU_q(n-m)$ is a subgroup of the quantum group $SU_q(n)$. The C^* algebra of the quotient $SU_q(n)/SU_q(n-m)$ is defined as

$$C(SU_q(n)/SU_q(n-m)) := \{a \in C(SU_q(n)) : \ (\phi \otimes 1)\Delta(a) = 1 \otimes a\}$$

We refer to [31] for the proof of the following proposition.

Proposition 6.1.1. *The C^* algebra $C(SU_q(n)/SU_q(n-m))$ is generated by the last m rows of the matrix (u_{ij}) i.e. by the set $\{u_{ij} : n-m+1 \leq i \leq n\}$.*

In [31], the quotient space $SU_q(n)/SU_q(n-m)$ is called the quantum Steifel manifold and is denoted by $S_q^{n,m}$. We will also use the same notation from now on. In Chapter 3, we have used a different embedding of $SU_q(n-1)$ inside $SU_q(n)$. We first show that for both the embeddings the C^* algebras of the quotient are isomorphic.

Let σ be the permutation on $\{1, 2, \cdots, n\}$ defined by $\sigma(i) := n-i+1$. Thus σ just reverses the order. Recall the for a permutation $\tau \in S_n$, $\ell(\tau)$ denotes the number of inversed pairs i.e. the cardinality of the set $\{(i, j) : i < j, \tau(i) > \tau(j)\}$. Note that for a permutation $\tau \in S_n$, one has

$$\ell(\tau\sigma) + \ell(\tau) = \frac{n(n-1)}{2},$$
$$\ell(\sigma\tau) + \ell(\tau) = \frac{n(n-1)}{2}.$$

In this section, we prove that there exists a unique quantum group isomorphism $\phi : C(SU_q(n)) \to C(SU_q(n))$ such that $\phi(u_{ij}) = q^{j-i}u^*_{\sigma(i),\sigma(j)}$

Let us make the following definition which will help to state a few proposition in a neat fashion.

Definition 6.1.2. *Let $\mathcal{A}$ be a $*$ algebra. An element $u = (u_{ij})$ in $M_n(\mathcal{A})$ is called a $q-$matrix if it satisfies the relation*

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} E(q, i_1, i_2, \cdots i_n) u_{j_1 i_1} \cdots u_{j_n i_n} = E(q, j_1, j_2, \cdots j_n) \tag{6.1.2}$$

where

$$E(q, i_1, i_2, \cdots i_n) \;:=\; \begin{cases} 0 & if \quad i_1, i_2, \cdots i_n \text{ are not distinct}, \\ (-q)^{\ell(i_1, i_2, \cdots, i_n)} & else. \end{cases}$$

Here for a permutation τ, $\ell(\tau)$ denotes the number of inversed pairs i.e. the cardinality of the set $\{(i, j) : i < j, \sigma(i) > \sigma(j)\}$.

For a matrix $u = (u_{ij}) \in M_n(\mathcal{A})$, let us denote the matrix (u^*_{ij}) by $\bar{u}$, (u_{ji}) by u^t and $(u_{\sigma(i)\sigma(j)})$ by u^σ. We need the following proposition.

Proposition 6.1.3. *Let $\mathcal{A}$ be a $*$ algebra and $u = (u_{ij})$ be a q-matrix. Then*

(1) The matrix $\bar{u}$ is a q^{-1}-matrix.

(2) The matrix u^σ is a q^{-1}-matrix.

(3) The matrix $v = (q^{j-i}u_{ij})$ is a q-matrix.

Proof. First note that for a permutation J, $E(q, J) = (-q)^{\frac{n(n-1)}{2}} E(q^{-1}, J\sigma)$. Let u be a q-matrix with entries in $\mathcal{A}$.

$$\begin{aligned}
\sum_I E(q^{-1}, I)\bar{u}_{j_1i_1}\bar{u}_{j_2i_2}\cdots\bar{u}_{j_ni_n} &= \sum_I E(q^{-1}, I)u^*_{j_1i_1}u^*_{j_2i_2}\cdots u^*_{j_ni_n} \\
&= (\sum_I E(q^{-1}, I)u_{j_ni_n}u_{j_{n-1}i_{n-1}}\cdots u_{j_1i_1})^* \\
&= (\sum_I (-q^{-1})^{\frac{n(n-1)}{2}} E(q, I\sigma)u_{j_ni_n}u_{j_{n-1}i_{n-1}}\cdots u_{j_1i_1})^* \\
&= (-q^{-1})^{\frac{n(n-1)}{2}} E(q, I\sigma) \\
&= (-q^{-1})^{\frac{n(n-1)}{2}} (-q)^{\frac{n(n-1)}{2}} E(q^{-1}, I).
\end{aligned}$$

This proves (1). The proof of (2) and (3) are similar. □

We need one more fact. The proof can be found in [38]. We repeat the proof here for our convenience.

Proposition 6.1.4. *Let A be a compact quantum group and let h be its Haar state. Consider a finite dimensional irreducible representation u of A. Define $E := (1 \otimes h)(u^t\bar{u})$. Then $E = u^tE\bar{u}$. Also the matrix E is positive, invertible and the matrix $\sqrt{E}\bar{u}\sqrt{E^{-1}}$ is unitary.*

Proof. Let Δ be the comultiplication on A. Recall the h is a Haar state implies that $(h \otimes 1) \circ \Delta(a) = h(a)1$. Now observe that

$$\begin{aligned}
E_{ij} &= h((u^t\bar{u})_{ij}) \\
&= h(\sum_k u_{ki}u^*_{kj}) \\
&= \sum_k h(u_{ki}u^*_{kj}) \\
&= \sum_k (h \otimes 1)\Delta(u_{ki}u^*_{kj}) \\
&= \sum_{k,r,s} (h \otimes 1)(u_{kr} \otimes u_{ri})(u^*_{ks} \otimes u^*_{sj}) \\
&= \sum_{r,s} u_{ri}(\sum_k h(u_{kr}u^*_{ks}))u^*_{ks} \\
&= \sum_{r,s} u^t_{ir}E_{rs}\bar{u}_{sj}.
\end{aligned}$$

Hence $E = u^tE\bar{u}$.

Since u is a finite dimensional irreducible representation of A it follows that u^t is invertibe [[43]]. Also $u^{t*} = \bar{u}$. Thus $u^t\bar{u}$ is positive and invertibe. Hence $u^t\bar{u} \geq \delta$ for some $\delta > 0$ and thus E is positive and invertible. Let $v := \sqrt{E}\bar{u}\sqrt{E^{-1}}$. Now observe that

$$\begin{aligned} vv^* &= \sqrt{E}\bar{u}E^{-}\bar{u}^*\sqrt{E} \\ &= \sqrt{E}\bar{u}E^{-}u^t\sqrt{E} \\ &= \sqrt{E}E^{-1}(u^t)^{-1}u^t\sqrt{E} \\ &= \sqrt{E}E^{-1}\sqrt{E} \\ &= 1 \end{aligned}$$

Similarly one can show that $v^*v = 1$. This completes the proof. □.

Now we compute E for the fundamental representation u corresponding to the Young tableau $(1, 0, 0, \cdots, 0)$ of $SU_q(\ell + 1)$. We make use of the notations in Section 4.2 of Chapter 4. Let us denote the Young tableau $(0, 0, \cdots, 0)$ by 0 itself and there is only one GT-tableau with 0 as its top row which we again denote by 0. Then the basis vector $e^0_{0,0}$ represents the vector $1 \in C(SU_q(\ell + 1))$ in $L_2(SU_q(\ell + 1))$. For $1 \le i \le \ell + 1$, Define the GT tableau $r^{(i)}$ as

$$r^{(i)}_{ab} \;:=\; \begin{cases} 0 \text{ if } 1 \le a \le i, b = \ell + 2 - a, \\ 1 \text{ otherwise.} \end{cases}$$

Let r be a GT tableau and $M := (m_1, m_2, \cdots, m_i)$. Observe that $M(r) = 0$ if and only if $r = r^{(i)}$ and $M := (\ell + 1, \ell, \cdots, \ell + 2 - i)$. Let us now compute the action of u^*_{ij} on $e^0_{0,0}$. Note that

$$\begin{aligned} < \pi(u^*_{ij})e^0_{00}|e^\lambda_{rs} > &=< e^0_{00}|\pi(u_{ij})e^\lambda_{rs} > \\ &= \sum_{\mu,m,n} C_q(i, r, m)C_q(j, s, n)\kappa(r, m) < e^0_{00}|e^\mu_{mn} > \\ &= C_q(i, r, 0)C_q(j, s, 0)\kappa(r, 0). \end{aligned}$$

Thus $\pi(u^*_{ij})e^0_{00} = C_q(i, r^{(i)}, 0)C_q(j, r^{(j)}, 0)\kappa(r^{(i)}, 0)e_{r^{(i)}r^{(j)}}$.

Now we compute $C_q(i, r^{(i)}, 0)$. Let $M := (\ell + 1, \ell, \cdots, \ell + 2 - i)$. We omit the superscript i from $r^{(i)}$ and simply denote it by r. We use Lemma 4.2.5 in Chapter 4. Clearly

$$\begin{aligned} sign(M) &= (-1)^{i-1}, \\ B(M) &= i - 1, \\ C(r, M) &= 0. \end{aligned}$$

For $1 \leq a \leq i-1$, one has

$$\begin{aligned}
L(\mathbf{r}, a, m_a, m_{a+1}) &= \prod_{\substack{i=1\\ i\neq m_a}}^{\ell+2-a} \frac{Q(|r_{a,i} - r_{a+1,m_{a+1}} - i + m_{a+1}|)}{Q(|r_{a,i} - r_{a,m_a} - i + m_a|)} \prod_{\substack{i=1\\ i\neq m_{a+1}}}^{\ell+1-a} \frac{Q(|r_{a+1,i} - r_{a,m_a} - i + m_a - 1|)}{Q(|r_{a+1,i} - r_{a+1,m_{a+1}} - i + m_{a+1} - 1|)} \\
&= \prod_{i=1}^{m_{a+1}} \frac{Q(|1-0-i+m_a-1|)}{Q(|1-0-i+m_a|)} \prod_{i=1}^{m_{a+2}} \frac{Q(|1-0-i+m_a-1|)}{Q(|1-0-i+m_{a+1}-1|)} \\
&= \prod_{i=1}^{m_{a+1}} \frac{Q(m_a - i)}{Q(m_a - i + 1)} \prod_{i=1}^{m_{a+2}} \frac{Q(m_{a+1} - i + 1)}{Q(m_{a+1} - i)} \\
&= \frac{Q(m_{a+1})}{Q(m_a)}.
\end{aligned}$$

Now let us compute $L^{'}(r, i, m_i)$.

$$\begin{aligned}
L'(\mathbf{r}, i, m_i) &= \frac{\prod_{j=1}^{\ell+1-i} Q(|r_{i+1,j} - r_{i,m_i} - j + m_i - 1|)}{\prod_{\substack{j=1\\ j\neq m_i}}^{\ell+2-i} Q(|r_{i,j} - r_{i,m_i} - j + m_i|)}. \\
&= \frac{\prod_{j=1}^{\ell+1-i} Q(|1-0-j+m_i-1|)}{\prod_{j=1}^{\ell+1-i} Q(|1-0-j+m_i|)} \\
&= \frac{1}{Q(m_i)}.
\end{aligned}$$

Thus from 4.2.5, we get $C_q(i, r^{(i)}, 0) = (-q)^{i-1}\frac{1}{Q(\ell+1)}$. Now we compute the matrix E for the fundamental representation u of $C(SU_q(\ell+1))$.

Proposition 6.1.5. *Let u be the fundamental unitary defining the C^* algebra $C(SU_q(\ell+1))$. Let $E := (1 \otimes h)(u^t\bar{u})$. Then E is a diagonal matrix. Moreover there exists a constant C depending only on q such that $E_{ii} = Cq^{2i}$.*

Proof. By definition

$$\begin{aligned}
E_{ij} &= \sum_k h(u_{ki}u_{kj}^*) \\
&= \sum_k < \pi(u_{ki}^*)e_{00}^0 | \pi(u_{kj}^* e_{00}^0 > \\
&= \sum_k C_q(k, r^{(k)}, 0)^2 C_q(i, r^{(i)}, 0) C_q(j, r^{(j)}, 0) \kappa(r^{(k)}, 0)^0 < e_{r^{(k)}r^{(i)}} | e_{r^{(k)}r^{(j)}} > .
\end{aligned}$$

Thus E is a diagonal matrix and $E_{ii} := Cq^{2i}$ where $C := q^{-2}\sum_k C_q(k, r^{(k)}, 0)^2 \kappa(r^{(k)}, 0)^2$. This completes the proof. □

We prove the following theorem which is the main point of this section.

Theorem 6.1.6. *There exists a unique quantum group isomorphism $\theta : C(SU_q(n)) \to C(SU_q(n))$ such that $\theta(u_{ij}) = q^{j-i}u^*_{n-i+1,n-j+1}$.*

Proof. Let σ be the permutation on $\{1, 2, \cdots, n\}$ such that $\sigma(i) = n - i$. By Proposition 6.1.3, it follows that $(q^{i-j}u^*_{ij})$ is a q^{-1}-matrix. Hence, again by Proposition 6.1.3, $(q^{\sigma(i)-\sigma(j)}u^*_{\sigma(i),\sigma(j)})$ is a q matrix.

It follows from Proposition 6.1.4 that the matrix $E^{\frac{1}{2}}\bar{u}E^{\frac{-1}{2}}$ is a unitary where E is the matrix considered in Proposition 6.1.5. Thus the matrix $(q^{i-j}u^*_{ij})$ is a unitary matrix. Hence $(q^{\sigma(i)-\sigma(j)}u^*_{\sigma(i)\sigma(j)})$ is a unitary q matrix. Thus there exists a C^* algebra homomorphism θ : $C(SU_q(n)) \to C(SU_q(n))$ such that $\theta(u_{ij}) = q^{j-i}u^*_{n-i,n-j}$. Since $\theta^2 = 1$, it follows that θ is a C^* algebra isomorphism.

Now we check that θ is a quantum group homomorphism. For,

$$\begin{aligned}\Delta \circ \theta(u_{ij}) &= q^{j-i}\sum_k u^*_{\sigma(i)k} \otimes u^*_{k\sigma(j)} \\ &= \sum_k q^{k-i}u^*_{\sigma(i)\sigma(k)} \otimes q^{j-k}u^*_{\sigma(k)\sigma(j)} \\ &= (\theta \otimes \theta) \circ \Delta(u_{ij}).\end{aligned}$$

Thus we have θ is a quantum group isomorphism. This completes the proof. □

Let us denote the isomorphism on $C(SU_q(n))$ by θ^n. Let ϕ^m and ψ^m be the embeddings of $SU_q(n-m)$ in $SU_q(n)$ defined as follows:

$$\phi^m(u_{ij}) := \begin{cases} v_{ij} & \text{if } 1 \le i, j \le n-m, \\ \delta_{ij} & \text{otherwise.} \end{cases} \tag{6.1.3}$$

$$\psi^m(u_{ij}) := \begin{cases} v_{i-m,j-m} & \text{if } m+1 \le i, j \le n, \\ \delta_{ij} & \text{otherwise.} \end{cases} \tag{6.1.4}$$

Then the following commutative diagram is clear.

$$\begin{array}{ccc} C(SU_q(n)) & \xrightarrow{\phi^m} & C(SU_q(n-m)) \\ \theta^n \big\downarrow & & \big\downarrow \theta^{n-m} \\ C(SU_q(n)) & \xrightarrow{\psi^m} & C(SU_q(n-m)) \end{array}$$

Thus the quotient with respect to the embeddings ϕ^m and ψ^m are isomorphic. In this chapter we consider the embedding ϕ^m.

6.2 Irreducible representations of $C(S_q^{n,m})$

In this section, we recall the irreducible representations of the C^* algebra $C(S_q^{n,m})$ as described in [31]. First we recall the irreducible representations of $C(SU_q(n))$ as in [37]. The one dimensional representations of $C(SU_q(n))$ are paramatrised by the torus $\mathbb{T}^{n-1}$. We consider $\mathbb{T}^{n-1}$ as a subset of $\mathbb{T}^n$ under the inclusion $(t_1, t_2, \cdots t_{n-1}) \to (t_1, t_2, \cdots, t_{n-1}, t_n)$ where $t_n := \prod_{i=1}^{n-1} \bar{t}_i$. For $t := (t_1, t_2, \cdots, t_n) \in \mathbb{T}^{n-1}$, let $\tau_t : C(SU_q(n)) \to \mathbb{C}$ be defined as $\tau_t(u_{ij}) := t_{n-i+1}\delta_{ij}$. Then τ_t is a $*$ algebra homomorphism. Moreover the set $\{\tau_t : t \in \mathbb{T}^{n-1}\}$ forms a complete set of mutually inequivalent one dimensional representations of $C(SU_q(n))$.

Let us denote the transposition $(i, i+1)$ by s_i. The map $\pi_{s_i} : C(SU_q(n)) \to B(\ell^2(\mathbb{N}))$ defined on the generators u_{rs} as follows

$$\pi_{s_i}(u_{rs}) \;:=\; \begin{cases} \sqrt{1-q^{2N+2}}S & \text{if } r = i, s = i, \\ -q^{N+1} & \text{if } r = i, s = i+1, \\ q^N & \text{if } r = i+1, s = i, \\ S^*\sqrt{1-q^{2N+2}} & \text{if } r = i+1, s = i+1, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

is a $*$ algebra homomorphism. For any two representations ϕ and ξ of $C(SU_q(n))$, let $\phi * \xi := (\phi \otimes \xi)\Delta$. For $\omega \in S_n$, let $\omega = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression. Then the representation $\pi_\omega := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \cdots * \pi_{s_{i_k}}$ is an irreducible representation and upto unitary equivalence the representation π_ω is independent of the reduced expression. For $t \in \mathbb{T}^{n-1}$ and $\omega \in S_n$ let $\pi_{t,\omega} := \tau_t * \pi_\omega$. We refer to [37] for the proof of the following theorem.

Theorem 6.2.1. *The set $\{\pi_{t,\omega} : t \in \mathbb{T}^{n-1}, \omega \in S_n\}$ forms a complete set of mutually inequivalent irreducible representations of $C(SU_q(n))$.*

In [31] the irreducible representations of $C(S_q^{n,m})$ were studied and we recall them here. We embedd $\mathbb{T}^m$ into $\mathbb{T}^{n-1}$ via the map $t = (t_1, t_2, \cdots, t_m) \to (t_1, t_2, \cdots, t_m, 1, 1, \cdots, 1, t_n)$ where $t_n := \prod_{i=1}^{m} \bar{t}_i$. For a permutation $\omega \in S_n$, let ω^s be the permutation in the coset $S_{n-m}\omega$ with the least possible length. We denote the restriction of the representation $\pi_{t,\omega}$ to the subalgbera $C(S_q^{n,m})$ by $\pi_{t,\omega}$ itself. Then we have the following theorem whose proof can be found in [31]

Theorem 6.2.2. *The set $\{\pi_{t,\omega^s} : t \in \mathbb{T}^m, \omega \in S_n\}$ forms a complete set of mutually inequivalent irreducible representations of $C(S_q^{n,m})$.*

Before we proceed further, let us recall some notations which we have used in the earlier chapters. Let $\mathscr{T}$ denote the Toeplitz algebra and $\sigma : \mathscr{T} \to C(\mathbb{T})$ be the symbol map which sends the generating isometry to the generating unitary. Define $\epsilon := ev_1 \circ \sigma$ where $ev_1 : C(\mathbb{T}) \to \mathbb{C}$ is the evaluation map at '1'.

6.3 Composition sequences

In this section, we derive certain exact sequences analogous to that of Theorem 4 in [35]. We then apply the six term sequence in K-theory to compute the K-groups of $C(S_q^{n,2})$. We begin with the following lemma.

Lemma 6.3.1. *Let $t \in \mathbb{T}^m$ and $\omega := s_{n-1}s_{n-2}\cdots s_{n-k}$. Then $\pi_{t,\omega}(C(S_q^{n,m}))$ contains the algebra of compact operators $\mathcal{K}(\ell^2(\mathbb{N}^k))$.*

Proof. Since $\pi_{t,\omega}(C(S_q^{n,m})) = \pi_\omega(C(S_q^{n,m}))$, it is enough to show that $\mathcal{K}(\ell^2(\mathbb{N}^k)) \subset \pi_\omega(C(S_q^{n,m}))$. We prove this result by induction on n. Since $\pi_\omega(u_{nn}) := S^*\sqrt{1-q^{2N+2}} \otimes 1$, it follows that $S \otimes 1 \in \pi_\omega(C(S_q^{n,m})$. Hence $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1 \subset \pi_w(C(S_q^{n,m})$. Thus the result is true if $n = 2$. Next observe that for $1 \leq i \leq n-1$, $(p \otimes 1)\pi_\omega(u_{n,i}) := p \otimes \pi_{\omega'}(v_{n-1,i})$ where $\omega' := s_{n-2}s_{n-3}\cdots s_{n-k}$ and (v_{ij}) denotes the generators of $C(SU_q(n-1))$. Hence $\pi_\omega(C(S_q^{n,m}))$ contains the algebra $p \otimes \pi_{\omega'}(C(S_q^{n-1,m}))$. Now by induction hypothesis, it follows that $\pi_\omega(C(S_q^{n,m}))$ contains $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$. Since $\pi_\omega(C(S_q^{n,m}))$ contains $\mathcal{K}(\ell^2(\mathbb{N})) \otimes 1$ and $p \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$, it follows that $\pi_\omega(C(S_q^{n,m}))$ contains the algebra of compact operators. This completes the proof. □

Let w be a word on $s_1, s_2, \cdots s_n$ say $w := s_{i_1}s_{i_2}\cdots s_{i_n}$ (not necessarily a reduced expression). Define $\psi_w := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \cdots \pi_{s_{i_r}}$ and for $t \in \mathbb{T}^n$, let $\psi_{t,w} := \tau_t * \psi_w$. Observe that the image of $\psi_{t,w}$ is contained in $\mathscr{T}^{\otimes r}$. We prove that if w' is a 'subword' of w then $\psi_{t,w'}$ factors through $\psi_{t,w}$.

Proposition 6.3.2. *Let $w = w_1 s_k w_2$ be a word on $s_1, s_2, \cdots s_n$. Denote the word w_1w_2 by w'. Let $t \in \mathbb{T}^m$ be given. Then there exists a * homomorphism $\tilde{\epsilon} : \psi_{t,w}(C(S_q^{n,m})) \to \psi_{t,w'}(C(S_q^{n,m}))$ such that $\psi_{t,w'} = \tilde{\epsilon} \circ \psi_{t,w}$.*

Proof. For a word u on $s_1, s_2, \cdots, s_n$, let $\ell(u)$ denote it's length. Then $\psi_{t,w}(C(S_q^{n,m}))$ is contained in $\mathscr{T}^{\otimes \ell(w_1)} \otimes \mathscr{T} \otimes \mathscr{T}^{\otimes \ell(w_2)}$. Let $\tilde{\epsilon}$ denote the restriction of $1 \otimes \epsilon \otimes 1$ to $\psi_{t,w}(C(S_q^{n,m}))$ where $\sigma : \mathscr{T} \to \mathbb{C}$ is the homomorphism for which $\epsilon(S) = 1$.

$$\psi_{t,w}(u_{rs}) = \sum_{j_1,j_2} \psi_{t,w_1}(u_{rj_1}) \otimes \pi_{s_k}(u_{j_1j_2}) \otimes \psi_{w_2}(u_{j_2s}).$$

Since $\epsilon(\pi_{s_k}(u_{j_1j_2})) = \delta_{j_1j_2}$, it follows that

$$\tilde{\epsilon} \circ \psi_{t,w}(u_{rs}) = \sum_j \psi_{t,w_1}(u_{rj}) \otimes \psi_{w_2}(u_{js}) = \psi_{t,w'}(u_{rs}).$$

This completes the proof. □

Let w be a word on $s_1, s_2, \cdots s_n$. Then for $n-m+1 \leq i \leq n$ and $1 \leq j \leq n$, the map $\mathbb{T}^m : t \to \psi_{t,w}(u_{ij}) \in \mathscr{T}^{\otimes \ell(w)}$ is continuous. Thus we get a homomorphism $\chi_w : C(S_q^{n,m}) \to C(\mathbb{T}^m) \otimes \mathscr{T}^{\otimes \ell(w)}$ such that $\chi_w(a)(t) = \psi_{t,w}(a)$ for all $a \in C(S_q^{n,m})$.

Remark 6.3.3. *Clearly for a word w on $s_1, s_2, \cdots s_n$ the representations $\psi_{t,w}$ factors through χ_w. One can also prove as in lemma 6.3.2 that if w' is a 'subword' of w then $\chi_{w'}$ factors through χ_w.*

Let us introduce some notations. Denote the permutation $s_js_{j-1}\cdots s_i$ by $\omega_{j,i}$ for $j \geq i$. If $j > i$ we let $\omega_{j,i} := 1$. For $1 \leq k \leq n$, let $\omega_k := \omega_{n-m,1}\omega_{n-m+1,1}\cdots\omega_{n-1,n-k+1}$.

Theorem 6.3.4. *The homomorphism $\chi_{\omega_n} : C(S_q^{n,m}) \to C(\mathbb{T}^m) \otimes \mathscr{T}^{\otimes \ell(\omega_n)}$ is faithful.*

Proof. If $\omega_0 \in S_n$ then ω_0^s(the representative in $S_{n-m}\omega_0$ with the shortest length) is a 'subword' of ω_n. Hence by remark 6.3.3 every irreducible representation of $C(S_q^{n,m})$ factors through χ_{ω_n}. Thus it follows that χ_{ω_n} is faithful. This completes the proof. □

For $1 \leq k \leq n$, Let $C(S_q^{n,m,k}) := \chi_{\omega_k}(C(S_q^{n,m}))$. Then $C(S_q^{n,m,k}) \subset C(S_q^{n,m,1}) \otimes \mathscr{T}^{\otimes (k-1)}$. For $2 \leq k \leq n$, let σ_k denote the restriction of $(1 \otimes 1^{\otimes (k-2)} \otimes \epsilon)$ to $C(S_q^{n,m,k})$. Then the image of σ_k is $C(S_q^{n,m,k-1})$. We determine the kernel of σ_k in the next proposition. We need the following two lemmas.

Lemma 6.3.5. *The algebra $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$ contains $C^*(t_1) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$ which is isomorphic to $C(\mathbb{T}) \otimes \mathcal{K}(\ell^2(\mathbb{N}^k))$.*

Proof. Note that $\chi_{\omega_{n-1,n-k}}(u_{nn}) = t_1 \otimes S^* \sqrt{1 - q^{2N+2}} \otimes 1$. Hence it follows that the operator $1 \otimes \sqrt{1 - q^{2N+2}} \otimes 1 = \chi_{\omega_{n-1,n-k}}(u_{nn}^* u_{nn})$ lies in the algebra $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. As $\sqrt{1 - q^{2N+2}}$ is invertible, one has $t_1 \otimes S^* \otimes 1 \in \chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. Thus the projection $1 \otimes p \otimes 1$ is in the algebra $C(S_q^{n,1,k+1})$. Now observe that for $1 \leq s \leq n-1$, one has

$$(1 \otimes p \otimes 1)\chi_{\omega_{n-1,n-k}}(u_{ns}) = t_1 \otimes p \otimes \pi_{\omega_{n-2,n-k}}(v_{n-1,s}) \tag{6.3.5}$$

where (v_{ij}) are the generators of $C(SU_q(n-1))$. If $n = 2$ then $k = 1$ and what we have shown is that $C(S_q^{2,1,2})$ contains $t_1 \otimes S^*$ and $t_1 \otimes p$. Hence one has $C^*(t_1) \otimes \mathcal{K}$ is contained in the algebra $C(S_q^{2,1,2})$.

Now we can complete the proof by induction on n. Equation 6.3.5 shows that $C^*(t_1) \otimes p \otimes \mathcal{K}^{\otimes (k-1)}$ is contained in the algebra $C(S_q^{n,1,k+1})$ and we also have $t_1 \otimes S^* \otimes 1 \in C(S_q^{n,1,k+1})$. Hence it follows that $C^*(t_1) \otimes \mathcal{K}^{\otimes k}$ is contained in the algebra $C(S_q^{n,1,k+1})$. This completes the proof. □

Lemma 6.3.6. *Given $1 \leq s \leq n$, there exists compact operators x_s, y_s such that*

$$x_s \pi_{\omega_{n-1,n-k}}(u_{js}) y_s = \delta_{js}(p \otimes p \otimes \cdots \otimes p)$$

where $p := 1 - S^*S$.

Proof. Let $1 \leq s \leq n$ be given. Note that the operator $\omega_{n-1,n-k}(u_{ss}) = z_1 \otimes z_2 \otimes \cdots z_k$ where $z_i \in \{1, \sqrt{1 - q^{2N+2}}S, S^*\sqrt{1 - q^{2N+2}}\}$. Define x_i, y_i as follows

$$x_i := \begin{cases} p & \text{if} \quad z_i = 1, \\ p & \text{if} \quad z_i = \sqrt{1 - q^{2N+2}}S, \\ (1 - q^2)^{-\frac{1}{2}} pS & \text{if} \quad z_i = S^*\sqrt{1 - q^{2N+2}}. \end{cases}$$

$$y_i := \begin{cases} p & \text{if} \quad z_i = 1, \\ (1 - q^2)^{-\frac{1}{2}} S^* p & \text{if} \quad z_i = \sqrt{1 - q^{2N+2}}S, \\ p & \text{if} \quad z_i = S^*\sqrt{1 - q^{2N+2}}. \end{cases}$$

Then $x_i z_i y_i = p$ for $1 \leq i \leq k$. Now let $x_s := x_1 \otimes x_2 \otimes \cdots x_k$ and $y_s := y_1 \otimes y_2 \otimes \cdots y_k$. Then $x_s \chi_{\omega_{n-1,n-k}}(u_{ss}) = \underbrace{p \otimes p \otimes \cdots \otimes p}_{ktimes}$. Let $j \neq s$ be given. Then $\chi_{\omega_{n-1,n-k}}(u_{js}) = a_1 \otimes a_2 \otimes \cdots \otimes a_k$ where $a_i \in \{1, \sqrt{1-q^{2N+2}}S, S^*\sqrt{1-q^{2N+2}}, -q^{N+1}, q^N\}$. Since $j \neq s$, there exists an i such that $a_i \in \{q^N, -q^{N+1}\}$. Let r be the largest integer for which $a_r \in \{q^N, -q^{N+1}\}$. Then $z_r \neq 1$. Hence $x_r a_r y_r = 0$. Thus $x_s \chi_{\omega_{n-1,n-k}}(u_{js}) y_s = 0$. This completes the proof. □

Proposition 6.3.7. *Let $2 \leq k \leq n$. Then $C(S_q^{n,m,1}) \otimes \mathcal{K}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$ is contained in the algebra $C(S_q^{n,m,k})$. Moreover the kernel of the homomorphism σ_k is exactly $C(S_q^{n,m,1}) \otimes \mathcal{K}(\ell^2(\mathbb{N}))^{\otimes(k-1)}$. Thus we have the exact sequence*

$$0 \longrightarrow C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)} \longrightarrow C(S_q^{n,m,k}) \xrightarrow{\sigma_k} C(S_q^{n,m,k-1}) \longrightarrow 0.$$

Proof. First we prove that $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$ is contained in the algebra $C(S_q^{n,m,k})$. For $a \in C(S_q^{n,1})$ one has $\chi_{\omega_k}(a) := 1 \otimes \chi_{\omega_{n-1,n-k+1}}(a)$, it follows from lemma 6.3.5 that $C(S_q^{n,m,k})$ contains $1 \otimes \mathcal{K}(\ell^2(\mathbb{N}^{k-1}))$. Let $n-m+1 \leq r \leq m$ and $1 \leq s \leq n$ be given. Then note that

$$\chi_{\omega_k}(u_{rs}) = \sum_{j=1}^{n} \chi_{\omega_1}(u_{rj}) \otimes \pi_{\omega_{n-1,n-k+1}}(u_{js}).$$

Hence by lemma 6.3.6, there exists $x_s, y_s \in C(S_q^{n,m,k})$ such that $x_s \chi_{\omega_k}(u_{rs}) y_s := \chi_{\omega_1}(u_{rs}) \otimes p^{\otimes(k-1)}$ where $p^{\otimes(k-1)} := p \otimes p \otimes \cdots \otimes p$. Thus we have shown that $C(S_q^{n,m,k})$ contains $1 \otimes \mathcal{K}^{\otimes(k-1)}$ and $C(S_q^{n,m,1}) \otimes p^{\otimes(k-1)}$. Hence $C(S_q^{n,m,k})$ contains $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$.

Clearly σ_k vanishes on $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. Let π be an irreducible representation of $C(S_q^{n,m,k})$ which vanishes on the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. Then $\pi \circ \chi_{\omega_k}$ is an irreducible representation of $C(S_q^{n,m})$. Hence $\pi \circ \chi_{\omega_k} = \pi_{t,\omega}$ for some ω of the form $\omega_{n-m,i_1}\omega_{n-m+1,i_2} \cdots \omega_{n-1,i_{n-m}}$ and $t \in \mathbb{T}^m$. Since $\pi \circ \chi_{\omega_k}(u_{n,n-k+1}) = 0$, it follows that $\pi_{t,w}(u_{n,n-k+1}) = 0$. But one has $\pi_{t,\omega}(u_{n,n-k+1}) = t_n(1 \otimes \pi_{\omega_{n-1,i_{n-m}}}(u_{n,n-k+1}))$. Hence $i_{n-m} > n-k+1$. In other words ω is a subword of ω_{k-1}. Thus $\pi \circ \chi_{\omega_k}$ factors through $\chi_{\omega_{k-1}}$. In other words there exists a representation ρ of $C(S_q^{n,m,k-1})$ such that $\pi \circ \chi_{\omega_k} = \rho \circ \chi_{\omega_{k-1}}$. Since $\chi_{\omega_{k-1}} = \sigma_k \circ \chi_{\omega_k}$, it follows that $\pi = \rho \circ \sigma_k$. Thus we have shown that every irreducible representation of $C(S_q^{n,m,k})$ which vanishes on the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$ factors through σ_k. Hence the kernel of σ_k is exactly the ideal $C(S_q^{n,m,1}) \otimes \mathcal{K}^{\otimes(k-1)}$. This completes the proof. □

We apply the six term exact sequence in K-theory to the exact sequence in proposition 6.3.7 to compute the K-groups of $C(S_q^{n,2,k})$ for $1 \leq k \leq n$. In the next section we briefly recall the product operation in K-theory.

6.4 The operation P

Let A and B be C^* algebras. (All the C^* algebras that we consider are nuclear. Thus no problem arises with the tensor product.) We have the following product maps.

$$
\begin{aligned}
K_0(A) \otimes K_0(B) &\to K_0(A \otimes B), \\
K_1(A) \otimes K_0(B) &\to K_1(A \otimes B), \\
K_0(A) \otimes K_1(B) &\to K_1(A \otimes B), \\
K_1(A) \otimes K_1(B) &\to K_0(A \otimes B).
\end{aligned}
$$

The first map is defined as $[p] \otimes [q] \to [p \otimes q]$. The second one is defined as $[u] \otimes [p] \to [u \otimes p + 1 - 1 \otimes p]$. The third map is defined in the same manner and the fourth one is defined using Bott periodicity and using the first product. In fact we have the following formula for the last product. We refer to the appendix of [11].

Let $h : \mathbb{T}^2 \to P_1(\mathbb{C}) := \{p \in Proj(M_2(\mathbb{C}) : trace(p) = 1\}$ be a degree one map. Then given unitaries $u \in M_p(A)$ and $v \in M_q(B)$ the product $[u] \otimes [v]$ is given by $[h(u,v)] - [e_0]$ where $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(M_{pq}(A \otimes B))$ and $h(u,v)$ is the matrix with entries $h_{ij}(u \otimes 1, 1 \otimes v)$. We denote the image of $[x] \otimes [y]$ by $[x] \otimes [y]$ itself. Now let A be a unital commutative C^* algebra. Then the multiplication $m : A \otimes A \to A$ is a C^* algebra homomorphism. Hence we get a map at the K-theory level from $K_1(A) \otimes K_1(A) \to K_0(A)$.

Suppose U and V are two commuting unitaries in a C^* algebra B. Let $A := C^*(U, V)$. Then A is commutative. Define

$$P(U, V) := K_0(m)([U] \otimes [V])$$

which is an element in $K_0(A)$ which we can think of as an element in $K_0(B)$ by composing with the inclusion map. From the formula that we just recalled from [11] the following properties are clear

1. If U and V are commuting unitaries in A and p is a rank one projection in $\mathcal{K}$ we have $P(U \otimes p + 1 - 1 \otimes p, V \otimes p + 1 - 1 \otimes p) := P(U, V) \otimes p$

2. If U and V are commuting unitaries and p is a projection that commutes with U and V then $P(U, Vp + 1 - p) = P(Up + 1 - p, Vp + 1 - p)$.

3. If $\phi : A \to B$ is a unital homomorphism and if U and V are commuting unitaries in A then $K_0(\phi)(P(U, V)) = P(\phi(U), \phi(V))$.

4. If U is a unitary in A then $P(U, U) = 0$. For $P_1(\mathbb{C})$ is simply connected, it follows that the matrix $h(U, U)$ is path connected to a rank one projection in $M_2(\mathbb{C})$. Hence $P(U, U) = 0$.

We need the following lemma in the six term computation. Let $z_1 \otimes 1$ and $1 \otimes z_2$ be the generating unitaries of $C(\mathbb{T}) \otimes C(\mathbb{T})$. Then $K_0(C(\mathbb{T}^2))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $1, P(z_1 \otimes 1, 1 \otimes z_2)$.

Lemma 6.4.1. *Consider the exact sequence*

$$0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K} \longrightarrow C(\mathbb{T}) \otimes \mathscr{T} \longrightarrow C(\mathbb{T}) \otimes C(\mathbb{T}) \longrightarrow 0$$

and the six term sequence in K theory.

$$\begin{array}{ccccc} K_0(C(\mathbb{T}) \otimes \mathcal{K}) & \longrightarrow & K_0(C(\mathbb{T}) \otimes \mathscr{T}) & \longrightarrow & K_0(C(\mathbb{T}) \otimes C(\mathbb{T})) \\ \partial \uparrow & & & & \downarrow \delta \\ K_1(C(\mathbb{T}) \otimes C(\mathbb{T})) & \longleftarrow & K_1(C(\mathbb{T}) \otimes \mathscr{T}) & \longleftarrow & K_1(C(\mathbb{T}) \otimes \mathcal{K}) \end{array}$$

Then the subgroup generated by $\delta(P(z_1 \otimes 1, 1 \otimes z_2))$ coincides with the group generated by $z_1 \otimes p + 1 - 1 \otimes p$ which is $K_1(C(\mathbb{T}) \otimes \mathcal{K}) \cong \mathbb{Z}$.

Proof. The Toeplitz map $\sigma : \mathscr{T} \to C(\mathbb{T})$ induces isomorphism at the K_0 level. Thus by Kunneth theorem, it follows that the image of $K_0(1 \otimes \sigma)$ is $K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))$ which is the subgroup generated by [1]. Now the inclusion $0 \to \mathcal{K} \to \mathscr{T}$ induces the zero map at the K_0 level and hence again by Kunneth theorem the inclusion $0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(\mathbb{T}) \otimes \mathscr{T}$ induces zero map at the K_1 level. Hence the image of δ is $K_1(C(\mathbb{T}) \otimes \mathscr{T})$. This completes the proof.

Corollary 6.4.2. *Let*

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0$$

be a short exact sequence of C^* algebras. Consider the six term sequence in K theory.

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \xrightarrow{K_0(\phi)} & K_0(B) \\ \partial \uparrow & & & & \downarrow \delta \\ K_1(B) & \underset{K_1(\phi)}{\longleftarrow} & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

Suppose that U and V are two commuting unitaries in B such that there exists a unitary X and an isometry Y such that $\phi(X) = U$ and $\phi(Y) = V$. Also assume that X and Y commute. Then the subgroup generated by $\delta(P(U, V))$ coincides with the subgroup generated by the unitary $X(1 - YY^*) + YY^*$ in $K_1(I)$.

Proof. Since $C(\mathbb{T})$ is the universal C^* algebra generated by a unitary and $\mathscr{T}$ is the universal C^* algebra generated by an isometry, there exists homomorphisms $\Phi : C(\mathbb{T}) \otimes \mathscr{T} \to A$ and $\Psi : C(\mathbb{T}) \otimes C(\mathbb{T}) \to B$ such that

$$\begin{aligned} \Phi(z_1 \otimes 1) &:= X, \\ \Phi(1 \otimes S^*) &:= Y, \\ \Psi(z_1 \otimes 1) &:= U, \\ \Psi(1 \otimes z_2) &:= V. \end{aligned}$$

Hence we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\mathbb{T})\otimes\mathcal{K} & \longrightarrow & C(\mathbb{T})\otimes\mathscr{T} & \longrightarrow & C(\mathbb{T})\otimes C(\mathbb{T}) & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Psi} & & \\ 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\ \phi\ } & B & \longrightarrow & 0 \end{array}$$

Now by the functoriality of δ and P, it follows that $\delta(P(U,V)) = K_1(\Phi)(\delta(P(z_1\otimes 1, 1\otimes z_2)))$. Hence by lemma 6.4.1, it follows that the subgroup generated by $\delta(P(U,V))$ is the subgroup generated by $\Phi(z_1\otimes p+1-1\otimes p)$ in $K_1(I)$. Note that $\Phi(z_1\otimes p+1-1\otimes p) = X(1-YY^*)+YY^*$. This completes the proof. $\square$.

6.5 K-groups of $C(S_q^{n,2,k})$ for $k < n$

In this section, we compute the $K-$groups of $C(S_q^{n,2,k})$ for $1 \leq k < n$ by applying the six term sequence in $K-$theory to the exact sequence in 6.3.7. Let us fix some notations. If q is a projection in $\ell^2(\mathbb{N})$ then q_r denotes the projection $\underbrace{q\otimes q\otimes\cdots q}_{r\ times}$ in $\ell^2(\mathbb{N}^r)$. Let us define the unitaries U_k, V_k, u_k, v_k as follows.

$$\begin{aligned} U_k &:= \; t_1\otimes 1_{n-2}\otimes p_{k-1} + 1 - 1\otimes 1_{n-2}\otimes p_{k-1}, \\ V_k &:= \; t_2\otimes p_{n-2}\otimes 1_{k-1} + 1 - 1\otimes p_{n-2}\otimes 1_{k-1}, \\ u_k &:= \; t_1\otimes p_{n-2}\otimes p_{k-1} + 1 - 1\otimes p_{n-2}\otimes p_{k-1}, \\ v_k &:= \; t_2\otimes p_{n-2}\otimes p_{k-1} + 1 - 1\otimes p_{n-2}\otimes p_{k-1}. \end{aligned}$$

First let us note that the operators U_k, V_k, u_k, v_k lies in the algebra $C(S_q^{n,2,k})$. For,

$$\begin{aligned} U_k &= \; 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1})u_{n,n-k+1} + 1 - 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}), \\ V_k &= \; 1_{\{1\}}(u_{n-1,1}u^*_{n-1,1})u_{n-1,1} + 1 - 1_{\{1\}}(u_{n-1,1}u^*_{n-1,1}), \\ u_k &= \; 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1})u_{n,n-k+1} + 1 - 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1}), \\ v_k &= \; 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1})u_{n-1,1} + 1 - 1_{\{1\}}(u_{n,n-k+1}u^*_{n,n-k+1}u_{n-1,1}u^*_{n-1,1}). \end{aligned}$$

Note that the unitaries U_n, u_n, v_n lies in the algebra $C(S_q^{n,2,n})$. We start with the computation of the K groups of $C(S_q^{n,2,1})$.

Lemma 6.5.1. *The K-groups $K_0(C(S_q^{n,2,1}))$ and $K_1(C(S_q^{n,2,1}))$ are both isomorphic to $\mathbb{Z}^2$. In fact, $[U_1]$ and $[V_1]$ form a $\mathbb{Z}$ basis for $K_1(C(S_q^{n,2,1}))$ and $[1]$ and $P(u_1,v_1)$ form a $\mathbb{Z}$ basis for $K_0(C(S_q^{n,2,1}))$.*

Proof. First note that $C(S_q^{n,2,1})$ is generated by $t_1\otimes 1_{n-2}$ and $t_2\otimes\pi_{\omega_{n-2,1}}(u_{n-1,j})$ where $1\leq j\leq n-1$. But the C^* algebra generated by $\{t_2\otimes\pi_{\omega_{n-2,1}}(u_{n-1,j}) :\ 1\leq j\leq n-1\}$ is isomorphic to $C(S_q^{2n-3})$. Hence $C(S_q^{n,2,1})$ is isomorphic to $C(\mathbb{T})\otimes C(S_q^{2n-3})$. Also $K_0(C(S_q^{2n-3})$

and $K_1(C(S_q^{2n-3}))$ are both isomorphic to $\mathbb{Z}$ with [1] generating $K_0(C(S_q^{2n-3}))$ and $[t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}]$ generating $K_1(C(S_q^{2n-3}))$.

Now by the Kunneth theorem for tensor product of C^* algebras(See [4]), it follows that $C(S_q^{n,2,1})$ has both K_1 and K_0 isomorphic to $\mathbb{Z}^2$ with $[U_1]$ and $[V_1]$ generating $K_1(C(S_q^{n,2,1}))$ and [1] and $P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2})$ generating $K_0(C(S_q^{n,2,1}))$. Note that the projection $1 \otimes p_{n-2} = 1_{\{1\}}(\chi_{\omega_{n-2,1}}(u_{n-1,1}u_{n-1,1}^*))$ is in $C(S_q^{n,2,1})$ and commutes with the unitaries $t_1 \otimes 1_{n-2}$ and $t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$. Hence

$$P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}) = P(u_1, v_1).$$

This completes the proof. □

Proposition 6.5.2. *Let $1 \leq k < n$ be given. Then the K-groups $K_0(C(S_q^{n,2,k}))$ and $K_1(C(S_q^{n,2,k}))$ are both isomorphic to $\mathbb{Z}^2$. In particular, $[U_k]$ and $[V_k]$ form a $\mathbb{Z}$ basis for $K_1(C(S_q^{n,2,k}))$ and [1] and $P(u_k, v_k)$ form a $\mathbb{Z}$ basis for $K_0(C(S_q^{n,2,k}))$.*

Proof. We prove this result by induction on k. The case $k = 1$ is just lemma 6.5.1. Now assume the result to be true for k. From proposition 6.3.7 we have the short exact sequence

$$0 \longrightarrow C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(k)} \longrightarrow C(S_q^{n,2,k+1}) \xrightarrow{\sigma_{k+1}} C(S_q^{n,2,k}) \longrightarrow 0$$

which gives rise to the following six term sequence in K-theory.

$$\begin{array}{ccccc}
K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}) & \longrightarrow & K_0(C(S_q^{n,2,k+1})) & \xrightarrow{K_0(\sigma_{k+1})} & K_0(C(S_q^{n,2,k})) \\
\partial \uparrow & & & & \downarrow \delta \\
K_1(C(S_q^{n,2,k}) & \xleftarrow[K_1(\sigma_{k+1})]{} & K_1(C(S_q^{n,2,k+1})) & \longleftarrow & K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k})
\end{array}$$

We determine δ and ∂ to compute the six term sequence. As $\sigma_{k+1}(V_{k+1}) = V_k$, it follows that $\partial([V_k]) = 0$. Since $C(S_q^{n,2,k+1})$ contains the algebra $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}$, it follows that the operator $\tilde{X} := t_1 \otimes 1_{n-2} \otimes \underbrace{q^N \otimes q^N \otimes \cdots q^N}_{(k-1) times} \otimes S^*$ is in the algebra $C(S_q^{n,2,1})$ as the difference $X - \chi_{\omega_{k+1}}(u_{n,n-k+1})$ lies in the ideal $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k}$. Let $X := 1_{\{1\}}(\tilde{X}^*\tilde{X})\tilde{X} + 1 - 1_{\{1\}}(\tilde{X}^*\tilde{X})$. Then X is an isometry such that $\sigma_{k+1}(X) = U_k$. Hence $\partial([U_k]) = [1 - X^*X] - [1 - XX^*]$. Thus $\partial([U_k]) = -[1 \otimes 1_{n-2} \otimes p_k]$. Thus the image of ∂ is the subgroup of $K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes k})$ generated by $[1 \otimes 1_{n-2} \otimes p_k]$ and the kernel is $[V_k]$.

Next we compute δ. Since $\sigma_{k+1}(1) = 1$, it follows that $\delta([1]) = 0$. Let

$$Y := (1 \otimes p_{n-2} \otimes 1_k)(1 \otimes 1_{n-2} \otimes p_{k-1} \otimes 1)\tilde{X} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Since $1 \otimes p_{n-2} \otimes 1 = 1_{\{1\}}(\chi_{\omega_k}(u_{n-1,1}^* u_{n-1,1})$ and $1 \otimes 1_{n-2} \otimes p_{k-1} = 1_{\{1\}}(\tilde{X}^*\tilde{X})$ it follows that the operator $Y \in C(S_q^{n,2,k+1})$. Also

$$Y = t_1 \otimes p_{n-2} \otimes p_{k-1} \otimes S^* + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$

Note that Y is an isometry such that $\sigma_{k+1}(Y) = u_k$. One has $\sigma_{k+1}(v_{k+1}) = v_k$. Note that Y and v_{k+1} commute. Hence by corollary 6.4.1, it follows that the image of δ is the subgroup generated by $[v_{k+1}(1 - YY^*) + YY^*] = [V_1 \otimes p_k + 1 - 1 \otimes p_k]$.

Thus the above computation with the six term sequence implies that $K_0(C(S_q^{n,2,k+1}))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $P(u_1, v_1) \otimes p_k = P(u_k, v_k)$ and $[1]$ and $K_1(C(S_q^{n,2,k+1}))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $[V_{k+1}]$ and $[U_1 \otimes p_k + 1 - 1 \otimes p_k] = [U_{k+1}]$. This completes the proof. □

6.6 K-groups of $C(S_q^{n,2})$

In this section, we compute the K-groups of $C(S_q^{n,2})$. We start with a few observations.

Lemma 6.6.1. *In the permutation group S_n one has $\omega_{n-2,1}\omega_{n-1,1} = \omega_{n-1,1}\omega_{n-1,2}$.*

Proof. First note that $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ if $|i - j| \geq 2$. Hence one has $\omega_{n-1,k}\omega_{n-1,1} = \omega_{n-1,k+1}\omega_{n-1,1}s_{k+1}$. Now the result follows by induction on k. □

We denote the representation $\chi_{\omega_{n-1,1}} * \pi_{\omega_{n-1,2}}$ by $\tilde{\chi}_{\omega_n}$. Since $\omega_{n-1,1}\omega_{n-1,2}$ is a reduced expression for ω_n it follows that the representations $\tilde{\chi}_{\omega_n}$ and χ_{ω_n} are equivalent. Let U be a unitary such that $U\chi_{\omega_n}(.)U^* = \tilde{\chi}_{\omega_n}(.)$. It is clear that $\tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \subset C(\mathbb{T}^m) \otimes \mathscr{T} \otimes \mathscr{T}^{\otimes \ell(\omega_{n-1})}$. Let $\tilde{\sigma}_n$ denote the restriction of $1 \otimes \sigma \otimes 1^{\otimes(2(n-2)}$ to $\tilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Since $\tilde{\sigma}_n(\tilde{\chi}_{\omega_n}(u_{ij})) = \chi_{\omega_{n-1}}(u_{ij})$, one has the following commutative diagram

$$\begin{array}{ccc} \chi_{\omega_n}(C(S_q^{n,2})) & \xrightarrow{\;U(.)U^*\;} & \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \\ & {\scriptstyle \sigma_n} \searrow \quad \swarrow {\scriptstyle \tilde{\sigma}_n} & \\ & C(S_q^{n,2,n-1}) & \end{array}$$

Lemma 6.6.2. *There exists a coisometry $X \in \chi_{\omega_n}(C(S_q^{n,2}))$ such that $\sigma_n(X) = V_{n-1}$ and $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$.*

Proof. By the above commutative diagram, it is enough to show that there exists a coisometry $\widetilde{X} \in \widetilde{\chi}_{\omega_n}(C(S_q^{n,2}))$ such that $\widetilde{\sigma_n}(X) = V_{n-1}$ and $X^*X = 1 - 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u_{n1}^* u_{n1})$. Now note that $\widetilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1}) - q^2 u_{n1} u_{n1}) = 1 \otimes 1 \otimes \underbrace{q^{2N} \otimes q^{2N} \otimes \cdots q^{2N}}_{(n-2)times} \otimes 1_{n-2}$. Hence the projection $1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2} = 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1} - q^2 u_{n1}^* u_{n1}))$ is in the algebra $\widetilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Now let $Y := (1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2})\widetilde{\chi}_{\omega_n}(u_{n-1,1})$. Then $Y := t_2 \otimes \sqrt{1 - q^{2N+2}} S \otimes p_{n-2} \otimes 1_{n-2}$. Hence the operator $Z := t_2 \otimes S \otimes p_{n-2} \otimes 1_{n-2}$ is in the algebra $\widetilde{\chi}_{\omega_n}(C(S_q^{n,2}))$. Now let $\widetilde{X} := Z + 1 - ZZ^*$. Then $\widetilde{X}$ is a coisometry such that $\widetilde{\sigma}_n(\widetilde{X}) = V_{n-1}$ and $\widetilde{X}^*\widetilde{X} = 1 - 1 \otimes p_{n-1} \otimes 1_{n-2}$ which is $1 - 1_{\{1\}}(\widetilde{\chi}_{\omega_n}(u_{n1}^* u_{n1}))$. This completes the proof. □

Observe that the operator $\tilde{Z} := t_1 \otimes 1_{n-1} \otimes \underbrace{q^N \otimes q^N \otimes \cdots q^N}_{(n-2) times} \otimes S^*$ lies in the algebra $C(S_q^{n,2,n})$ since the difference $\tilde{Z} - \chi_{\omega_n}(u_{n,2})$ lies in the ideal $C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)})$. Let $Z := 1_{\{1\}}(\tilde{Z}^*\tilde{Z})\tilde{Z}$ and $Y_n := Z + 1 - Z^*Z$. Then

$$\begin{aligned} Z_n &= t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^*, & (6.6.6)\\ Y_n &= t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^* + 1 - 1 \otimes 1_{n-2} \otimes p_{n-2} \otimes 1. & (6.6.7) \end{aligned}$$

Hence Y is an isometry and $YY^* = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$. Let X be a coisometry in $C(S_q^{n,2,n})$ such that $\sigma_n(X) = v_{n-1}$ and $X^*X := 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{n1}^* u_{n1}))$. The existence of such an X was shown in lemma 6.6.2. Then XY is a unitary.

Proposition 6.6.3. *The K-groups $K_0(C(S_q^{n,2})$ and $K_1(C(S_q^{n,2})$ are both isomorphic to $\mathbb{Z}^2$. In particular we have the following.*

1. *The projections $[1]$ and $P(u_n, v_n)$ generate $K_0(C(S_q^{n,2}))$.*
2. *The unitaries U_n and XY_n generate $K_1(C(S_q^{n,2}))$ where X is a coisometry in $C(S_q^{n,2})$ such that $\sigma_n(X) = V_{n-1}$ and $X^*X = 1 - 1_{\{1\}}(u_{n1}^* u_{n1})$ and Y_n is as in equation 6.6.7.*

Proof. By proposition 6.3.7, we have the following exact sequence.

$$0 \longrightarrow C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)} \longrightarrow C(S_q^{n,2,n}) \xrightarrow{\sigma_n} C(S_q^{n,2,n-1}) \longrightarrow 0$$

which gives rise to the following six term sequence in K-theory.

$$\begin{array}{ccccc} K_0(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes n-1}) & \longrightarrow & K_0(C(S_q^{n,2,n})) & \xrightarrow{K_0(\sigma_n)} & K_0(C(S_q^{n,2,k})) \\ \partial \uparrow & & & & \downarrow \delta \\ K_1(C(S_q^{n,2,n-1})) & \xleftarrow[K_1(\sigma_n)]{} & K_1(C(S_q^{n,2,n})) & \longleftarrow & K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes n-1}) \end{array}$$

Now we compute ∂ and δ to compute the six term sequence. First note that since $[U_{n-1}]$ and $[V_{n-1}]$ generate $K_1(C(S_q^{n,2,n-1}))$, it follows that $[U_{n-1}]$ and $[V_{n-1}U_{n-1}]$ generate $K_1(C(S_q^{n,2,n-1}))$. As XY_n is a unitary for which $\sigma_n(XY_n) = V_{n-1}U_{n-1}$, it follows that $\partial([V_{n-1}U_{n-1}]) = 0$. Next Y_n is an isometry for which $\sigma_n(Y_n) = U_{n-1}$. Hence $\partial([U_{n-1}]) = [1 - Y^*Y] - [1 - YY^*]$. Thus $\partial([U_{n-1}]) = -[1 \otimes 1_{n-2} \otimes p_{n-1}]$.

Now we compute δ. Since $\sigma_n(1) = 1$, it follows that $\delta([1]) = 0$. Now one observes that $p_{n-2} \otimes S^* \pi_{\omega_{n-1,1}}(u_{j1}) = 0$ if $j > 1$. Hence $Z_n \chi_{\omega_n}(u_{n-1,1}) = t_1 t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes \sqrt{1 - q^{2N+2}}$ where Z_n is as defined in 6.6.6. Thus the operator $R_n := t_1 t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$ lies in the algebra $C(S_q^{n,2,n})$ as the difference $R_n - Z_n \chi_{\omega_n}(u_{n-1,1})$ lies in the ideal $C(\mathbb{T}^2) \otimes \mathcal{K}^{\otimes(2n-3))}$. Hence projection $1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1$ lies in the algebra $C(S_q^{n,2,n})$. Now define

$$\begin{aligned} &S_n := R_n + 1 - R_n R_n^*, \\ &T_n := (1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1) Z_n + 1 - 1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1. \end{aligned}$$

Then S_n is a unitary and T_n is an isometry such that $\sigma_n(S_n) = u_{n-1}v_{n-1}$ and $\sigma_n(T_n) = u_{n-1}$. Moreover S_n and T_n commute. Now note that $P(u_{n-1}, v_{n-1}) = P(u_{n-1}, u_{n-1}v_{n-1})$. Hence by corollary 6.4.1, it follows that the image of δ is the subgroup generated by $S_n(1-T_nT_n^*)+T_nT_n^*$ in $K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)})$. Now

$$S_n(1 - T_nT_n^*) + T_nT_n^* = t_1t_2 \otimes p_{n-2} \otimes p_{n-1} + 1 - 1 \otimes p_{n-2} \otimes p_{n-1}.$$

Since $1 \otimes p_{n-2}$ is a trivial in $K_0(C(S_q^{2n-3}))$ it follows that the unitary $t_1 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$ is trivial in $K_1(C(S_q^{n,2,1})) = K_1(C(\mathbb{T}) \otimes C(S_q^{2n-3})$. Hence one has $[S_n(1 - T_nT_n^*) + T_nT_n^*] = [V_1 \otimes p_{n-1} + 1 - 1 \otimes p_{n-1}]$ in $K_1(C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes(n-1)})$.

Thus the above computation with the exactness of the six term sequence completes the proof. □

6.7 K-groups of $C(SU_q(3))$

In this section, we show that for $n = 3$ the unitary XY_n in proposition 6.6.3 can be replaced by the fundamental 3×3 matrix (u_{ij}) of $C(SU_q(3))$. First note that for $n = 3$ we have $C(S_q^{n,2}) = C(SU_q(3))$. The algebra $C(S_q^{3,2,1})$ is denoted $C(U_q(2))$ in [35]. Then $C(U_q(2)) = C(\mathbb{T}) \otimes C(SU_q(2))$. Let $ev_1 : C(\mathbb{T}) \to \mathbb{C}$ be the evaluation at the point '1'. Then $\phi = (ev_1 \otimes 1)\sigma_2\sigma_3$ where $\phi : C(SU_q(3)) \to C(SU_q(2))$ is the subgroup homomorphism defined in equation 6.1.4.

Proposition 6.7.1. *The K-group $K_1(C(SU_q(3))$ is isomorphic to $\mathbb{Z}^2$ generated by the unitary $U_3 := t_1 \otimes p \otimes p + 1 - 1 \otimes p \otimes p$ and the fundamental unitary $U = (u_{ij})$.*

Proof. By proposition 6.6.3, we know that $K_1(C(SU_q(3))$ is isomorphic to $\mathbb{Z}^2$ and is generated by $[U_3]$ and $[XY_3]$ where X is a coisometry such that $\sigma_3(X) = V_2$ and $X^*X = 1 - 1_{\{1\}}(\chi_{\omega_3}(u_{31}^*u_{31}))$. Now observe that $\phi(X) = t_2 \otimes p + 1 - 1 \otimes p$ and $\phi(Y_3) = 1$. Hence $\phi(XY_3) = t_2 \otimes p + 1 - 1 \otimes p$. Also note that $\phi(U_3) = 0$ and $\phi(U) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ where u denote the fundamental unitary of $C(SU_q(2))$. Since $K_1(C(SU_q(2))$ is isomorphic to $\mathbb{Z}$ the proof is complete if we show that $t_2 \otimes p + 1 - 1 \otimes p$ and $[u]$ represents the same element in $K_1(C(SU_q(2))$ which we do in the next lemma. □

We denote the 2×2 fundamental unitary $u = (u_{ij})$ of $C(SU_q(2))$ by u_q. Consider the representation $\chi_{s_1} : C(SU_q(2)) \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}))$. We let the unitary t act on $\ell^2(\mathbb{Z})$ as the right shift i.e $te_n = e_{n+1}$. Let $\{e_{n,m} : n \in \mathbb{Z}, m \in \mathbb{N}\}$ be the standard orthonormal basis for the Hilbert space $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N})$. For an integer k, denote the orthogonal projection onto the closed subspace spanned by $\{e_{n,m} : n + m \leq k\}$ by P_k and set $F_k := 2P_k - 1$. Note that F_k is a selfadjoint unitary.

Proposition 6.7.2. *For any integer k, the triple $(\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k)$ is an odd Fredholm module for $C(SU_q(2))$ and we have the pairing*

1. $\langle [u_q], F_k \rangle = -1,$

2. $\langle t \otimes p + 1 - 1 \otimes p, F_k \rangle = -1$ *where* $p = 1 - S^*S$.

Proof. By lemma 3.3.1, it follows that $C(SU_q(2))$ is generated by $t \otimes S$ and $t \otimes p$. Now it is easy to see that $[t \otimes S, P_k] = 0$ and $[t \otimes p, P_k]$ is a finite rank operator. Hence the triple $(\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k)$ is an odd Fredholm module for $C(SU_q(2))$. Since $C(SU_q(2))$ is generated by $t \otimes S$ and $t \otimes p$, it follows that $u_p \in C(SU_q(2))$ for every $p > 0$. Also as $p \to 0$, u_p approaches to u in norm where u is given by

$$u := \begin{pmatrix} t \otimes S & 0 \\ \bar{t} \otimes p & \bar{t} \otimes S^* \end{pmatrix}.$$

Hence $[u_q] = [u]$ in $K_1(C(SU_q(2)))$. It is easy to check the following

$$\langle [u], F_k \rangle = -1,$$
$$\langle [t \otimes p + 1 - 1 \otimes p], F_k \rangle = -1.$$

This completes the proof. □

Appendix A

Smooth subalgebras

In this appendix, we discuss the results in [33] and [34]. We have made use of these results to show that some subalgebras of the C^* algebra of odd dimensional quantum spheres are closed under holomorphic functional calculus. We have reproduced the proofs here for the sake of completeness.

A.1 Spectral invariance

Definition A.1.1. *Let $\mathcal{A} \subset A$ be a unital inclusion of algebras. We say that $\mathcal{A}$ is spectrally invariant in A if given $a \in \mathcal{A}$ with a invertible in A then $a^{-1} \in \mathcal{A}$.*

Now let $\mathcal{A}$ be a sublagebra of A. We say that $\mathcal{A}$ is spectrally invariant in A if $\mathcal{A}^+$ is spectrally invariant in A^+ where $\mathcal{A}^+$ is the algebra with an unit adjoined to $\mathcal{A}$. It is easy to see that in the case of unital inclusion of algebras the two definitions coincide. Let us make a definition which will help in stating the results in an easier fashion.

Definition A.1.2. *Let $\mathcal{A}$ be a $*$ subalgebra of a C^* algebra A. We say that the pair $\mathcal{A} \subset A$ is admissible if for every irreducible representation ρ of $\mathcal{A}$ there exists an irreducible representation π of A on a Hilbert space H and an $\mathcal{A}$ invariant subspace V such that ρ is equivalent to (π, V)*

Lemma A.1.3. *Let $\mathcal{A} \subset A$ be an unital inclusion of algebras where A is a C^* algebra. The following are equivalent.*

1. *The pair $\mathcal{A} \subset A$ is admissible.*
2. *The algebra $\mathcal{A}$ is spectrally invariant in A.*

Proof. First we prove (1) implies (2). Let $a \in \mathcal{A}$ be an element invertible in A. Suppose assume that a^{-1} is not in $\mathcal{A}$. First observe that the admissibility of the pair $\mathcal{A} \subset A$ implies that for every irreducible representation ρ of $\mathcal{A}$, $ker\rho(a) = \{0\}$

Now a is not left invertible in $\mathcal{A}$. Hence $\mathcal{A}a$ is a proper left ideal of $\mathcal{A}$ which contains a. Choose a maximal left ideal N which contains a. Then $\mathcal{A}/N$ is an irreducible representation of $\mathcal{A}$ where $\mathcal{A}$ acts on $\mathcal{A}/N$ by left multiplication. In this irreducible representation $ker(a)$ is non-trivial as $a(1+N) = a+N = N$ which is a contradiction. Hence $a^{-1} \in \mathcal{A}$. This completes the proof of (1) implies (2).

Now suppose that $\mathcal{A}$ is spectrally invariant in A. Let ρ be an irreducible representation of $\mathcal{A}$ acting on a vector space V. Let v be a non-zero vector in V. Since the representation is irreducible, it follows that the map $a \to \rho(a)v$ is onto. Let N be it's kernel. Then V is isomorphic as an $\mathcal{A}$ module to $\mathcal{A}/N$. Since $\mathcal{A}/N$ is irreducible, it follows that N is a maximal left ideal. Now $\bar{N}$ is a proper left ideal in A. If not, then there exists $a \in N$ close to 1 such that a is invertible in A. Hence a is invertible in $\mathcal{A}$ which then implies $N = \mathcal{A}$. Hence $\bar{N}$ is a proper left ideal in A. Let M be a maximal left ideal such that $\bar{N} \subset M$. By corollary 1 and theorem 2 of [24], it follows that A/M can be given a Hilbert space structure such that the left multiplication representation of A on A/M is an irreducible $*$ representation. Since $M \cap \mathcal{A} = N$, it follows that the natural map $\mathcal{A}/N \to A/M$ is one-one. This completes the proof of (2) implies (1). □

Lemma A.1.4. *Let $0 \to I \to A \to B \to 0$ be an exact sequence of C^* algebras and $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B}$ be a subexact sequence of $*$ algebras. If $\mathcal{I} \subset I$ and $\mathcal{B} \subset B$ are admissible then $\mathcal{A} \subset A$ is admissible.*

Proof. Let us denote the map $A \to B$ by σ. Let ρ be an irreducible representation of $\mathcal{A}$ on V_ρ. Suppose that ρ vanishes on $\mathcal{I}$. Then ρ factors through $\mathcal{B}$ to give an irreducible representation which we denote by $\tilde{\rho}$. Since $\mathcal{B} \subset B$ is admissible, it follows that there exists an irreducible representation π of B on a Hilbert space H and a $\mathcal{B}$ invariant subspace V such that $\tilde{\rho}$ is equivalent to (π, V). Then ρ is equivalent to $(\pi \circ \sigma, V)$. Now suppose that ρ does not vanish on $\mathcal{I}$. We claim that $\rho_{|\mathcal{I}}$ is irreducible. Note that since $\mathcal{I}$ is a two sided ideal, it follows that $\bigcap_{x \in \mathcal{I}} Ker\rho(x)$ is $\mathcal{A}$ invariant. Since ρ does not vanish on $\mathcal{I}$, it follows that given a non zero vector v there exists $x \in \mathcal{I}$ such that $\rho(x)v \neq 0$. Now let W be a nonzero $\mathcal{I}$ invariant subspace. Then $\mathcal{I}W \subset W$ is an $\mathcal{A}$ invariant non-zero subspace and hence $W = V_\rho$. Hence $\rho_{|\mathcal{I}}$ is irreducible.

Now the admissibility of the pair $\mathcal{I} \subset I$ implies that there exists an irreducible representation π of I on a Hilbert space H and an $\mathcal{I}$ invariant subspace V such that $\rho_{|\mathcal{I}}$ is equivalent to (π, V). Let $F : V_\rho \to V$ be an interwiner. Now π can be extended to an irreducible representation of A. As $\mathcal{I}V = V$, it follows that V is $\mathcal{A}$ invariant. Since $\mathcal{I}V = V$ and $\mathcal{I}V_\rho = V_\rho$, it follows that F interwines ρ and (π, V). This completes the proof. □

Proposition A.1.5. *Let $\mathcal{A}$ be a $*$ subalgebra of a C^* algebra A. Then the following are equivalent*

1. *The pair $\mathcal{A} \subset A$ is admissible*

2. The subalgebra $\mathcal{A}$ is spectrally invariant in A.

Proof. First we prove (1) implies (2). Observe that the map $(a, \lambda) \to \lambda$ gives the exact sequence $0 \to A \to A^+ \to \mathbb{C} \to 0$. By Lemma A.1.4, it follows that $\mathcal{A}^+ \subset A^+$ is admissible. Hence by Lemma A.1.3, it follows that $\mathcal{A}$ is spectral invariant in A if $\mathcal{A} \subset A$ is admissible.

Now assume that $\mathcal{A}$ is spectrally invariant in A. Then by Lemma A.1.3, it follows that $\mathcal{A}^+ \subset A^+$ is admissible. It is easy to show that $\mathcal{A} \subset A$ is admissible. $\square$

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