

STUDIES IN
QUANTUM MECHANICAL COLLISIONS
WITH SPECIAL REFERENCE TO
SCATTERING AND PHOTO PRODUCTION OF PIONS
FROM NUCLEI

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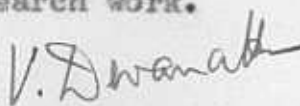
PREFACE

This thesis comprises the work done by the author during the years 1958-1962 on quantum mechanical collisions with special reference to scattering and photoproduction of pions from nuclei, under the guidance of Professor Alladi Ramakrishnan who has been till recently Professor of Physics, University of Madras and now the Director, Institute of Mathematical Sciences, Madras. It consists of three parts, the first dealing with rearrangement collisions, the second with the scattering and photoproduction of pions from nuclei and the third with some problems in electrodynamics and deuteron disintegration processes.

Six papers relating to part of the subject-matter of this thesis have been published by the author and two more are in course of publication. The available reprints are enclosed in the form of a booklet. Collaboration either with my guide Professor Alladi Ramakrishnan or with my colleagues G. Ramachandran and K. Venkatesan was necessitated by the nature and the range of problems dealt with in this thesis and due acknowledgment of this has been made in each chapter.

The author is deeply indebted to Professor Alladi Ramakrishnan for his guidance throughout the preparation of this work and very grateful to the University of Madras for providing excellent facilities for research work.

University of Madras,
January, 1963.


(V. Devanathan)

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STUDIES IN QUANTUM MECHANICAL COLLISIONS WITH SPECIAL
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Introduction

Collision problems can be broadly divided into two classes based on the nature of the complications involved; one dealing with elementary particle interactions, the study of which has assumed increasing complexities due to strong interactions involving the production of strange particles and the other with the collision of composite systems, the interaction between the constituents of which is somewhat understood. The former belongs to the domain of high energy physics wherein the perturbation theory is found to be inadequate due to the strong nature of the interactions involved and the situation, it is felt, requires radically new concepts and ideas, whereas the latter deals with comparatively low energy collisions involving complex nuclei. This thesis deals mainly with the latter class of problems and in particular the rearrangement collisions and the scattering and photoproduction of pions from nuclei.

Part I deals with rearrangement collisions and a time dependent formulation is given as a simple generalisation of the scattering theory¹⁾. In discussing the scattering process, it has been found convenient to introduce the interaction representation, and the temporal development of the state vector in the interaction representation is completely determined by a unitary transformation operator $U(t, t_0)$ which takes the state vector in the interaction representation at time t_0 to the time t . The extension to the case of rearrangement collisions²⁾ is based on the realisation that the initial and final systems are described in different interaction representations and hence it is necessary to define a new unitary transformation operator $U^{BA}(t, t_0)$ which takes the state vector at time t_0 described in one interaction representation, say A to another interaction representation, say B at time t . Once such an operator is defined, it is then possible to adopt the time-dependent formalism of the ordinary scattering theory to the case of rearrangement collisions. The general theory is discussed in Chapter I. Chapter II is devoted to the discussion of some specific processes of rearrangement collisions such as exchange scattering, pick-up and stripping reactions. The matrix

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- 1) B.A. Lippmann and J. Schwinger, Phys. Rev., 79, 469 (1950)
M. Gell-Mann and M.L. Goldberger, Phys. Rev., 91, 398 (1953)
C. Moller : Kgl. Danske Vid. Selsk, 23, No.1 (1945)
 - 2) Alladi Ramakrishnan, G. Ramachandran and V. Devanathan, Nuovo Cimento, 21, 145 (1961)

elements are presented in a form suitable for further evaluation. This chapter is included for the sake of completeness though not much of originality is claimed.

Part II is devoted to the study of the scattering and photoproduction of pions from nuclei. In Chapter III, an outline of the method is given. It includes a brief discussion of the Chew-Low scattering and photoproduction amplitudes ³⁾ from free nucleons which is found to give reliable results below 500 Mev. The extension to the study of scattering and photoproduction process in the case of complex nuclei is made possible by the impulse approximation ⁴⁾ developed and justified by Chew himself. In Chapters IV and V, we discuss the scattering and photoproduction ^{5), 6)} of pions from the lightest nuclei viz., the deuterons. The multiple scattering and the effect of the potential are neglected and the cross section is obtained in pure impulse approximation using the Chew-Low amplitude for free nucleons. Of the various possible processes, the elastic processes viz., the elastic scattering of pions and the 'elastic' photoproduction of neutral pions from deuterons are of considerable importance since a reliable calculation is possible in these cases for there exists no uncertainty in the final state of the system. Hence,

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- 3) G.F. Chew and F.E. Low, Phys. Rev., 101, 1570, 1579 (1956)
 G.F. Chew, M.L. Goldberger, F.E. Low and Y. Nambu,
 Phys. Rev., 106, 1345 (1957)
 A. Ramakrishnan, Elementary ~~and~~ Particles and Cosmic Rays,
 Pergamon Press (1962)
- 4) G.F. Chew, Phys. Rev., 80, 196 (1950)
 G.F. Chew and G.C. Wick, Phys. Rev., 85, 636 (1952)
 G.F. Chew and M.L. Goldberger, Phys. Rev., 87, 778 (1952)

(for 5) and 6), see next page)

much attention has been paid to the discussion of these elastic processes. In the case of elastic scattering of pions, numerical calculations have been made at various energies 60, 85, 140, 195, and 250 and 300 Mev and compared with the available experimental data (60, 85, 140 and 300 Mev). A good fit has been obtained at low energies 60 and 85 Mev and also at higher energy 300 Mev, but at 140 Mev there seems to be some discrepancy, the theoretical values obtained being much higher than the experimental values at large angles. The good agreement at low and high energy and the discrepancy at 140 Mev seem to suggest the failure of the method of impulse approximation at energies in the neighbourhood of the pion-nucleon resonance (190 Mev). This inference ⁶⁾ is based on the sole discordant result that we have obtained at 140 Mev. This disagreement at 140 Mev has also been corroborated by other independent calculation but they have made some sceptical remarks regarding the validity of the impulse approximation at energies in the range 140 Mev and above. But as we have observed earlier, the agreement seems to be good at higher energy (300 Mev).

In view of the earlier observation, it will be highly desirable to restrict our investigation in the case of heavier

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- 5/4) V. Devanathan and G. Ramachandran, Nuclear Physics, 23, 312 (1961)
 A. Ramakrishnan, V. Devanathan and G. Ramachandran, Nuclear Physics, 24, 163 (1961)
 A. Ramakrishnan, V. Devanathan and K. Venkatesan, Nuclear Physics, 29, 630 (1962)
- 6) V. Devanathan, Nuclear Physics (in press).

nuclei to the energy range in which a good fit has been obtained for the deuteron. This is essential to draw reliable inferences. The scattering and photoproduction of pions from nuclei may be used to probe the structure of the nucleus. For this purpose, the free nucleon amplitudes and the impulse approximation can be used. If we assume the shell model description of the nucleus, then we can calculate the potential well-depth which gives a good fit with the experimental data on the scattering and photoproduction of pions from nuclei. Conversely, if we assume the potential well depth from other considerations, we can calculate the cross sections and compare them with the experimental results. It is the latter purpose we have in mind.

7)

The early experiments on the photoproduction of charged pions from nuclei have been devoted to the measurement of the total cross section, mostly accompanied by nucleon emission and they indicate an $A^{2/3}$ dependence for the sum of the π^+ and π^- cross sections. In order to explain this gross feature, Butler has proposed the model of the surface production according to which the pion production from the core is suppressed and only the outer nucleons i.e., the nucleons beyond the boundary r_0 , the radius of the central core of constant density, are effective in the process. But we consider the transition of the nucleus

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- 7) R.M. Littauer and D. Walker, Phys. Rev., 83, 206, (1956) where further references can be found.
 8) S.T. Butler, Phys. Rev., 87, 1117, (1952)

from the given initial state to a well defined final state with the production of a charged pion by the incident photon. Such experiments ⁹⁾ have been reported recently and they are of considerable interest to us since a rigorous theoretical treatment is possible in these cases. Laing and Moorhouse ¹⁰⁾ have attempted to treat such cases using the independent particle model but they have digressed into a discussion on the surface and volume production. We hold the view that the elements of the surface production model of Butler are ingrained in any treatment based on shell-model and hence a complete and rigorous treatment of the photoproduction process based on the shell model description should precede any discussion of the surface and volume production.

The free nucleon photoproduction amplitude of Chew et al, which is used for the investigation involves the magnetic moments of the proton and the neutron. The magnetic moments of the free nucleons are not usually the same as the magnetic moments within the nuclei and consequently, the ambiguity ¹¹⁾ arises as to the sets of values that have to be taken for the purpose of calculation. This has been investigated in Chapter VI and numerical calculations are separately presented

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- 9) I. S. Hughes and P. V. March, Proc. Phys. Soc., A 72, 259 (1959)
 P. V. March and T. G. Walker, Proc. Phys. Soc., 77, 293 (1961)
- 10) E. W. R. Laing and R. G. Moorhouse, Proc. Phys. Soc.,
 A 72, 289 (1958)
- 11) V. Devanathan and G. Ramachandran, Nuclear Physics,
38, 654. (1962).

using (a) free nucleon magnetic moments and (b) magnetic moments within the nuclei. One can decide in favour of the one or the other by observing the angular distribution of the emitted pion.

In Chapter VII, we outline a complete and rigorous method of evaluating the cross section for the photoproduction of charged pions from complex ^{nuclei} by treating both the spin dependent and spin independent terms alike and summing over magnetic quantum numbers ¹²⁾. The cross section is obtained in terms of the square of single particle matrix elements using the concept of fractional parentage coefficients. The j-j coupling shell model description is used in the study.

Chapter VIII is devoted to the study of scattering of pions by complex nuclei based on the method that has been developed in the earlier chapter. The advantage of studying both the scattering and photoproduction of pions from complex nuclei by the same method is stressed.

Part III deals with some electrodynamic and deuteron disintegration processes. In Chapter IX, we discuss the equivalence of field theoretic and Feynman formalisms and make a comparative study ¹³⁾ of the methods of Wick ¹⁴⁾ and Ramakrishnan et al ¹⁵⁾. Chapter X deals with the application of Low's

12) V. Devanathan and G. Ramachandran, Nuclear Physics, in press.
 13) A. Ramakrishnan, K. Venkatesan and V. Devanathan, J. Math. Anal. and Applns. (in press)
 14) G.C. Wick, Phys. Rev., 90, 268 (1950)
 S.S. Schweber, H. A. Bethe and F.D. Hoffmann, Mesons and Fields, Vol. 1 (Now, Peterson and Co., New York) p. 207, (1956).
 (for ref.15), see next page)

procedure to the study of electromagnetic dynamic processes with a view to understand the role of the equal time commutator term. In Chapter XI, we study the disintegration processes in deuterium (photo, meso and electro disintegration) in a unified way using Low's method which brings out the similarity and interconnection between the matrix elements for these processes.

PART I

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15) A. Ramakrishnan and N.R.Ranganathan, J. Math. Anal. and Appln., 3, 261 (1961)
A. Ramakrishnan, T.K.Radha and R. Thunga, J. Math. Anal. and Appln., 4, 494 (1962)
A. Ramakrishnan, Elementary Particles and Cosmic Rays, Pergamon Press (1962)

16) F.E.Low, Phys. Rev., 97, 1392 (1955)

CHAPTER I

A FIVE-DIFFERENT APPROACH TO REARRANGEMENT COLLISIONS*

2. Introduction

The S -matrix theory of scattering is combined with the treatment of a system of particles free as initial to a final state. The case of particles number the case can be described as follows: the system of the same number of particles before and after scattering. This is a particular case though an important one, of a more general class of processes. When two particles (or a particle and a particle, for example) collide, they often give rise, after collision, to systems which are composed of particles different from those of the initial system. For instance,

PART I

REARRANGEMENT COLLISIONS

Such collisions are referred to as rearrangement collisions or collisions leading to reaction channels. It is the purpose of this chapter to suggest a simple generalization of the usual S -matrix theory to include rearrangement collisions.

- * All of these references are taken from the book "Scattering Theory and Applications" by R. G. Newton, Wiley-Interscience, New York (1975).
 [1] S. Lippman and J. Schwinger, Phys. Rev., **72**, 469 (1952).
 [2] R. Hall-John and H. L. Goldberger, Phys. Rev., **91**, 308 (1953).
 [3] R. G. Newton, Phys. Rev., **105**, 2082 (1957).
 [4] R. G. Newton, R. J. Eden and P. D. Drell, Phys. Rev., **105**, 2092 (1957).

CHAPTER I

A TIME-DEPENDENT APPROACH TO REARRANGEMENT COLLISIONS*

1. Introduction

The S -matrix theory of scattering¹⁾⁻⁴⁾ is concerned with the transition of a system of particles from an initial to a final state. The system of particles remains the same and can be described as the eigenstates of the same unperturbed Hamiltonian before and after scattering. This is a particular case though an important one, of a more general class of phenomena. When two composite systems of particles (nuclei, for example) collide, they often give rise, after collision, to systems which may comprise of nuclei different from those of the initial system. For instance,



Such collisions are referred to as rearrangement collisions or collisions leading to reaction channels. It is the purpose of this chapter to suggest a simple generalisation of the usual S -matrix theory to include rearrangement collisions.

1)* Alladi Ramakrishnan, G. Ramachandran and V. Devanathan, Nuovo Cimento, 21, 145 (1961); Proc. of the Annual Low Energy Physics Symposium, Bombay (1961).

1 Z) B. A. Lippman and J. Schwinger, Phys. Rev., 79, 469 (1950)

2 Z) M. Gell-Mann and M. L. Goldberger, Phys. Rev., 91, 398 (1953)

3) C. Moller, Kgl. Danske, vid. Selsk, 23, No.1 (1945)

4) S. S. Schweber, H. A. Bethe and F. D. Hoffmann, Mesons and Fields, Vol. 1, (Row, Peterson and Co.) p.192 (1956)

Such processes have been described in great detail using a time-independent approach, especially by the R -matrix theory of Wigner⁵⁾. The growth and the success of this theory has been reviewed in an extensive article by Lane and Thomas⁶⁾. Some attempts have also been made to develop a time-dependent theory which has claimed attention⁷⁾ for example in connection with the problem of exchange scattering. The generalised scattering matrix $\langle \psi_f^{(-)} | \psi_i^{(+)} \rangle$ which is indeed the definition of the S -matrix in the stationary state (time-independent) scattering formalism, includes also the case of rearrangement collisions and this can be obtained using a time-dependent theory.

In dealing with collisions of composite systems of particles, we seek the transition amplitude from an initial state consisting of one set of composite systems, say A_1 and A_2 to a final state consisting of a different set, say B_1 and B_2 .⁸⁾ Since the interaction representation has been found to be convenient to describe the scattering theory, we shall use the same representation for describing the rearrangement collisions also. By interaction representation, we mean a description of the system with the time-dependence associated with a specified pair of free parts

-
- 5) E.P.Wigner, Phys. Rev., 72, 29 (1947)
 6) A.M.Lane and R.G.Thomas, Rev. Mod.Phys., 30, 257 (1958)
 7) B.A.Lippmann, Phys. Rev., 102, 264 (1956)
 8) Stuckeberg, Ann. der Phys., 21, 367 (1934);
 Helv. Phys. Act, 17, 43 (1943);
 Nature, 153, 143 (1944)

removed and if the total Hamiltonian can be split into a sum of unperturbed and interaction parts in more than one way it becomes possible to define different interaction pictures for the description of the system. This is exactly the situation in the case of rearrangement collisions, wherein the initial and final states are to be described in different interaction pictures. However, in the case of ordinary scattering, we have the unique way of describing the system in a single interaction representation. So, the treatment of rearrangement collision reduces to the study of transition from one interaction picture to another.

The temporal development of the state of a system is usually described independent of the initial state by regarding the evolution as the unfolding of a unitary transformation. It is of interest to observe the relationship between such unitary operators in different interaction pictures and to study the properties of that class of operators describing the time-development of the system from one picture to another. This is done in section 2.

The S -matrix for rearrangement collisions is defined in section 3 and the adiabatic hypothesis⁹⁾ invoked to obtain well-defined limits. The matrix elements appear in forms similar to those met with in scattering theory and provide agreement with Lippmann⁷⁾, Low¹⁰⁾ and Sunakawa¹¹⁾. The alternative approach using the operator technique is also described and used to obtain the matrix elements.

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- 9) F.J. Dyson, 'An advanced Quantum Mechanics', Lithoprinted notes, Cornell University (1951);
 Phys. Rev., 82, 428 (1951); Phys. Rev., 83, 608 (1951);
 Proc. Roy. Soc. (Lond) A 207, 395 (1951);
 Phys. Rev., 83, 1207 (1951)
- 10) F.E. Low, Summer Institute Lectures (1959), Brandeis University.
- 11) S. Sunakawa, Prog. Theor. Phys., 24, 963 (1960).

2. Transition operators for multichannel processes.

Consider a system whose Schrodinger equation is given by

$$i\hbar \frac{\partial \psi(t)}{\partial t} = H \psi(t) \quad (1)$$

$\psi(t)$ represents the state of the system in Schrodinger representation and in treating scattering problems, it is of advantage to work in interaction representation wherein the Hamiltonian of the system can be separated into two parts as

$$H = H_0^A + V^A \quad (2)$$

where H_0^A represents the unperturbed Hamiltonian of the colliding parts of the system and V^A , the interaction between them.

The state $\Psi^A(t)$ of the system in interaction representation is defined by

$$\Psi^A(t) = e^{iH_0^A t/\hbar} \psi(t) \quad (3)$$

and obeys the equation

$$i\hbar \frac{\partial \Psi^A(t)}{\partial t} = e^{iH_0^A t/\hbar} V^A e^{-iH_0^A t/\hbar} \Psi^A(t) \quad (4)$$

The temporal development of the state can be represented by

$$\Psi^A(t) = U^A(t, t_0) \Psi^A(t_0) \quad (5)$$

where $U^A(t, t_0)$ is the unitary transformation operator. Expressing the evolution of the state vector in interaction representation and it obeys the differential equation

$$i\hbar \frac{\partial U^A(t, t_0)}{\partial t} = e^{iH_0^A t/\hbar} V^A e^{-iH_0^A t/\hbar} U^A(t, t_0) \quad (6)$$

with the boundary condition $U^A(t_0, t_0) = 1$. The integral form of equation (6) is given by

$$U^A(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t e^{iH_0^A t'/\hbar} V^A e^{-iH_0^A t'/\hbar} U^A(t', t_0) dt' \quad (7)$$

Suppose the total Hamiltonian H of the system can also be separated as

$$H = H_0^B + V^B \quad (8)$$

corresponding to a different pair B of interacting parts and their interaction. Then we can define a corresponding interaction state $\psi^B(t)$ as before and the time-development of the system could be expressed in terms of a unitary operator $U^B(t, t_0)$.

To make a comparison between the two unitary operators describing the evolution of the system, we observe that, starting with an initial state $\psi(t_0)$ at time t_0 , the state $\psi(t)$ at t obtained via either representation must be the same

$$\begin{aligned} \psi(t) &= e^{-iH_0^A t/\hbar} \psi^A(t) \\ &= e^{-iH_0^A t/\hbar} U^A(t, t_0) \psi^A(t_0) \\ &= e^{-iH_0^A t/\hbar} U^A(t, t_0) e^{iH_0^A t_0/\hbar} \psi(t_0) \end{aligned} \quad (9)$$

Similarly

$$\psi(t) = e^{-iH_0^B t/\hbar} U^B(t, t_0) e^{iH_0^B t_0/\hbar} \psi(t_0) \quad (10)$$

Equations (9) and (10) represent the time development of the state vector $\psi(t)$ in the Schrodinger representation obtained by different interaction pictures. Writing

$$\psi(t) = U_S(t, t_0) \psi(t_0) \quad (11)$$

we obtain

$$\begin{aligned} U_S(t, t_0) &= e^{-iH_0^A t/t} U^A(t, t_0) e^{iH_0^A t_0/t} \\ &= e^{-iH_0^B t/t} U^B(t, t_0) e^{iH_0^B t_0/t} \end{aligned} \quad (12)$$

and thus the transformation law for the operator $U(t, t_0)$ from one interaction picture to another is

$$U^B(t, t_0) = e^{iH_0^B t/t} e^{-iH_0^A t/t} U^A(t, t_0) e^{iH_0^A t_0/t} e^{-iH_0^B t_0/t} \quad (13)$$

The operator $\int_{t_0 \rightarrow -\infty}^{t \rightarrow +\infty} U^A(t, t_0)$ the limits being taken using the adiabatic hypothesis, is the usual S -matrix describing scattering of the separated parts A and it connects a state ϕ_a^A describing the initially free parts A of the system with a state ϕ_b^A describing the same finally free parts. Similar is the role of $U^B(\infty, -\infty)$. The suffices a or b denote the energy, relative momentum, angular momentum and any other attributes of the system. The problem of rearrangement collisions is clearly that of prescribing an S -matrix S^{BA} representing a transition from the state ϕ_a^A of the initially free parts A to a state ϕ_b^B of the finally separated parts B which are different from A . Therefore we seek an

operator $U^{BA}(t, t_0)$ transforming a state in interaction representation A at time t_0 to a state at time t in interaction representation B .

Since

$$\psi(t) = U_S(t, t_0) \psi(t_0)$$

or equivalently

$$\underline{\psi}^B(t) = e^{iH_0^B t/t} U_S(t, t_0) e^{-iH_0^A t/t} \underline{\psi}^A(t_0) \quad (14)$$

the relationship between the transformation operators

and $U_S(t, t_0)$ is given by

$$U^{BA}(t, t_0) = e^{iH_0^B t/t} U_S(t, t_0) e^{-iH_0^A t_0/t} \quad (15)$$

Using (12), $U^{BA}(t, t_0)$ can be represented in terms of either $U^A(t, t_0)$ or $U^B(t, t_0)$ as follows:

$$\begin{aligned} U^{BA}(t, t_0) &= e^{iH_0^B t/t} e^{-iH_0^A t/t} U^A(t, t_0) \\ &= U^B(t, t_0) e^{iH_0^B t/t} e^{-iH_0^A t_0/t} \end{aligned} \quad (16)$$

It is easily seen that $U^{BA}(t, t_0)$ satisfies also the group property

$$\begin{aligned} U^{BA}(t, t_0) &= U^{BA}(t, t_1) U^A(t_1, t_0) \\ &= U^B(t, t_1) U^{BA}(t_1, t_0) \\ &= U^{BC}(t, t_1) U^{CA}(t_1, t_0) \end{aligned} \quad (17)$$

Using the properties (16) and (17):

where C denotes another possible interaction representation of the system. The differential equation satisfied by

$U^{BA}(t, t_0)$ is

$$i\hbar \frac{\partial U^{BA}(t, t_0)}{\partial t} = e^{iH_0^B t/\hbar} V^B e^{-iH_0^B t/\hbar} U^{BA}(t, t_0) \quad (18)$$

with

$$U^{BA}(t_0, t_0) = e^{iH_0^B t_0/\hbar} e^{-iH_0^A t_0/\hbar} \quad (19)$$

The hermitian conjugate operator $U^{BA\dagger}(t, t_0)$ defined by

$$\langle \Psi^B(t) | = \langle \Psi^A(t_0) | U^{BA\dagger}(t, t_0) \quad (20)$$

obeys the differential equation

$$-i\hbar \frac{\partial U^{BA\dagger}(t, t_0)}{\partial t} = U^{BA\dagger}(t, t_0) e^{iH_0^B t/\hbar} V^B e^{-iH_0^B t/\hbar} \quad (21)$$

with the condition

$$U^{BA\dagger}(t_0, t_0) = e^{iH_0^A t_0/\hbar} e^{-iH_0^B t_0/\hbar} \quad (22)$$

It is easily seen that $U^{BA}(t_0, t_0)$ is unitary i.e.

$$\begin{aligned} U^{BA\dagger}(t, t_0) &= U^{BA^{-1}}(t, t_0) \\ &= e^{iH_0^A t_0/\hbar} e^{-iH_0^B t_0/\hbar} U^B(t, t_0) \\ &= U^{AB}(t_0, t) \end{aligned} \quad (23)$$

using the properties (16) and (17).

Thus $U^{BA\dagger}(t_0, t)$ operating on the right on a state in interaction representation B at t_0 evolves it into the state at t in interaction representation A. Therefore $U^{BA\dagger}(t_0, t)$ obeys also the equation

$$i\hbar \frac{\partial U^{BA\dagger}(t_0, t)}{\partial t} = e^{iH_0^A t/\hbar} V^A e^{-iH_0^A t/\hbar} U^{BA\dagger}(t_0, t) \quad (24)$$

with the initial condition

$$U^{BA\dagger}(t_0, t_0) = e^{iH_0^A t_0/\hbar} e^{-iH_0^B t_0/\hbar}$$

In a similar way, $U^{BA}(t_0, t)$ operating on the 'bra' state $\langle \Psi^B(t_0) |$ at time t_0 takes it into the state at time t . Thus $U^{BA}(t_0, t)$ obeys the equation

$$-i\hbar \frac{\partial U^{BA}(t_0, t)}{\partial t} = U^{BA}(t_0, t) e^{iH_0^A t/\hbar} V^A e^{-iH_0^A t/\hbar} \quad (25)$$

with

$$U^{BA}(t_0, t_0) = e^{iH_0^B t_0/\hbar} e^{-iH_0^A t_0/\hbar}$$

It may also be observed that the initial conditions at t_0 can be set as unity by defining the interaction representations so as to coincide with the Schrodinger ~~equati~~ representation at time t_0 rather than at zero.

3. The S-matrix

The S-matrix for rearrangement collisions can be defined as

$$S^{BA} = \lim_{\substack{t \rightarrow +\infty \\ t_0 \rightarrow -\infty}} U^{BA}(t, t_0) \quad (26)$$

and it is necessary now to invoke the adiabatic hypothesis to obtain well-defined limits. This is accomplished by attaching a factor $\exp(-\varepsilon|t|)$ to the interaction Hamiltonian where ε is a small quantity which may be allowed to tend to the limit zero after a calculation is performed. An element

$\langle \phi_b^B, S^{BA} \phi_a^A \rangle$ of the S -matrix is now written as

$$\begin{aligned} \langle \phi_b^B, S^{BA} \phi_a^A \rangle &= \lim_{\substack{t \rightarrow +\infty \\ t_0 \rightarrow -\infty}} \langle \phi_b^B, U^{BA}(t, t_0) \phi_a^A \rangle \\ &= \langle U^B(0, \infty) \phi_b^B, U^A(0, -\infty) \phi_a^A \rangle \end{aligned} \quad (27)$$

using (16) and (17). $U(0, \pm\infty)$ are the wellknown Moller matrices

$$\Omega^\pm \phi_a = \phi_a + \frac{1}{E_a - H \pm i\varepsilon} V \phi_a = \psi_a^{(\pm)} \quad (28)$$

Thus

$$\langle \phi_b^B, S^{BA} \phi_a^A \rangle = \langle \psi_b^{B(-)}, \psi_a^{A(+)} \rangle \quad (29)$$

Equation (29) is the definition of the generalised S -matrix in the stationary state formalism.

The integral equation form of (19) under the adiabatic hypothesis is

$$\begin{aligned} U^{BA}(t, t_0) &= e^{iH_0^B t/\hbar} e^{-iH_0^A t_0/\hbar} V^B e^{-\varepsilon|t|} \\ &\quad - \frac{i}{\hbar} \int_{t_0}^t e^{iH_0^B t'/\hbar} V^B e^{-\varepsilon|t'|} e^{-iH_0^B t'/\hbar} U^{BA}(t', t_0) dt' \end{aligned} \quad (30)$$

The transition matrix element can now be written as

$$\langle \phi_b^B, S^{BA} \phi_a^A \rangle = \lim_{t_0 \rightarrow -\infty} \left\langle \phi_b^B, e^{iH_0^B t_0/t} e^{-iH_0^A t_0/t} \phi_a^A \right\rangle - \frac{i}{t} \int_{-\infty}^{+\infty} \phi_b^B, e^{iH_0^B t'/t} V^B e^{-\epsilon|t'|} e^{-iH_0^B t'/t} U^{BA}(t', -\infty) \phi_a^A dt' \quad (31)$$

The first term vanishes on energy integration in the limit

$t_0 \rightarrow -\infty$. However, it should be noted that in the case of ordinary scattering $H_0^A = H_0^B = H_0$, the first term reduces to $\langle \phi_b, \phi_a \rangle$ which is equal to δ_{ba} . In the case of rearrangement collisions, H_0^A is not equal to H_0^B and hence

$$\begin{aligned} \langle \phi_b^B, S^{BA} \phi_a^A \rangle_{B \neq A} &= \langle \phi_b^B, T^{BA} \phi_a^A \rangle \\ &= -\frac{i}{t} \int_{-\infty}^{+\infty} \phi_b^B, e^{iH_0^B t'/t} V^B e^{-\epsilon|t'|} e^{-iH_0^A t'/t} U^A(t', -\infty) \phi_a^A dt' \end{aligned} \quad (32)$$

Using the relation (16). We now have

$$T_{ba}^{BA} = -\frac{i}{t} \langle \phi_b^B, V^B \psi_a^{A(+)}(E_b) \rangle \quad (33)$$

where

$$\psi_a^{A(+)}(E) = \int_{-\infty}^{+\infty} e^{i(E-H_0^A)t'/t} e^{-\epsilon|t'|} U^A(t', -\infty) \phi_a^A dt' \quad (34)$$

which satisfies the well-known Lippman-Schwinger equation

$$\psi_a^{A(\pm)}(E) = 2\pi t \delta(E - E_a) \phi_a^A + \frac{1}{E - H_0^A \pm i\epsilon} V^A \psi_a^{A(\pm)}(E) \quad (35)$$

Removing the δ -function in the usual way as a common factor

$$\psi_a^{A(\pm)}(E) = 2\pi i \delta(E - E_a) \psi_a^{A(\pm)} \quad (36)$$

$\psi_a^{A(\pm)}$ is an eigen state of the total Hamiltonian and is given by (28).

We have

$$T_{ba}^{BA} = -2\pi i \delta(E_b - E_a) \langle \phi_b^B, V^B \psi_a^{A(+)} \rangle \quad (37)$$

In a similar way one can show, using the integral equation form of (25) for $U^{BA}(\infty, -\infty)$ that the transition matrix element T_{ba}^{BA} is also given by

$$T_{ba}^{BA} = -2\pi i \delta(E_b - E_a) \langle \psi_b^{B(-)}, V^A \phi_a^A \rangle \quad (38)$$

and thus the transition matrix element T_{ba}^{BA} on the energy shell is

$$\begin{aligned} T_{ba}^{BA} &= \langle \phi_b^B, V^B \psi_a^{A(+)} \rangle \\ &= \langle \psi_b^{B(-)}, V^A \phi_a^A \rangle \end{aligned} \quad (39)$$

The above relations (37) and (38) can also be obtained from the equation (29) using the technique of operator algebra. The matrix element can be written using the Moller expressions (28) in the following way:

$$\begin{aligned}
\langle \psi_b^{B(-)}, \psi_a^{A(+)} \rangle &= \langle \phi_b^B, \left\{ 1 + V^B \frac{1}{E_b - H + i\epsilon} \right\} \left\{ 1 + \frac{1}{E_a - H + i\epsilon} V^A \right\} \phi_a^A \rangle \\
&= \langle \phi_b^B, \phi_a^A \rangle + \langle \phi_b^B, \left\{ V^B \frac{1}{E_b - H + i\epsilon} + \frac{1}{E_a - H + i\epsilon} V^A \right\} \phi_a^A \rangle \\
&\quad + \langle \phi_b^B, V^B \frac{1}{E_b - H + i\epsilon} \frac{1}{E_a - H + i\epsilon} V^A \phi_a^A \rangle
\end{aligned} \tag{40}$$

Applying the familiar operator relations

$$\begin{aligned}
\frac{1}{\alpha - \beta} &= \frac{1}{\alpha} + \frac{1}{\alpha - \beta} \beta \frac{1}{\alpha} \\
&= \frac{1}{\alpha} + \frac{1}{\alpha} \beta \frac{1}{\alpha - \beta}
\end{aligned} \tag{41}$$

One can write

$$\begin{aligned}
V^B \frac{1}{E_b - H + i\epsilon} &= V^B \frac{1}{E_b - H_0^A + i\epsilon} + V^B \frac{1}{E_b - H + i\epsilon} V^A \frac{1}{E_b - H_0^A + i\epsilon} \\
\frac{1}{E_a - H + i\epsilon} V^A &= \frac{1}{E_a - H_0^B + i\epsilon} V^A + \frac{1}{E_a - H_0^B + i\epsilon} V^B \frac{1}{E_a - H + i\epsilon} V^A
\end{aligned} \tag{42}$$

We can now replace H_0^A and H_0^B by E_a and E_b and using the integral property in the limit $\epsilon \rightarrow 0$, we obtain

$$\frac{1}{E_a - E_b + i\epsilon} = \frac{P}{E_a - E_b} - i\pi \delta(E_a - E_b) \tag{43}$$

After simplification, we get

$$\begin{aligned}
\langle \psi_b^{B(-)}, \psi_a^{A(+)} \rangle &= \langle \phi_b^B, \left\{ 1 + \frac{P}{E_a - E_b} (V^A - V^B) \right\} \phi_a^A \rangle \\
&\quad - i\pi \delta(E_a - E_b) \left\{ \langle \psi_b^{B(-)}, V^A \phi_a^A \rangle + \langle \phi_b^B, V^B \psi_a^{A(+)} \rangle \right\}
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
\langle \phi_b^B, \left\{ 1 + \frac{P}{E_a - E_b} (V^A - V^B) \right\} \phi_a^A \rangle &= \delta_{ba} \quad , \quad A = B \\
&= 0 \quad , \quad A \neq B
\end{aligned} \tag{45}$$

Also

$$\langle \phi_b^B, V^B \psi_a^{A(+)} \rangle = \langle \psi_b^{B(-)}, V^A \phi_a^A \rangle \quad (46)$$

on the energy shell since

$$\begin{aligned} \langle \phi_b^B, V^B \phi_a^A \rangle - \langle \phi_b^B, V^A \phi_a^A \rangle &= \langle \phi_b^B, (H_0^A - H_0^B) \phi_a^A \rangle \\ &= 0 \end{aligned} \quad (47)$$

Thus we obtain

$$\begin{aligned} T_{ba}^{BA} &= 2\pi i \delta(E_a - E_b) \langle \phi_b^B, V^B \psi_a^{A(+)} \rangle \\ &= 2\pi i \delta(E_a - E_b) \langle \psi_b^{B(-)}, V^A \phi_a^A \rangle \end{aligned} \quad (48)$$

Using similar techniques, the following useful relations on the energy shell are also established

$$\langle \psi_b^{B(+)} , \psi_a^{A(+)} \rangle = \langle \psi_b^{B(-)} , \psi_a^{A(-)} \rangle = \delta_{ab} , \quad A=B \\ = 0 , \quad A \neq B \quad (49)$$

The transition rate W_{ba}^{BA} for rearrangement collisions is given by

$$W_{ba}^{BA} = \frac{2\pi}{\hbar} \delta(E_b - E_a) |T_{ba}^{BA}|^2 \quad (50)$$

in the usual way.

CHAPTER II.

SOME SPECIFIC PROCESSES OF REARRANGEMENT COLLISIONS*

1. Introduction

The theory that we have outlined in Chapter I is rigorous and the expression (39) for the matrix element is exact. But in order to evaluate the matrix element for any specific process, we have to resort to some approximations : Born approximation or distorted wave Born approximation. For the purpose of illustration, we shall deal with the following processes :-

- a) exchange scattering,
- b) ionization process,
- c) pick-up reaction,
- d) stripping reaction, and
- e) break-up reaction.

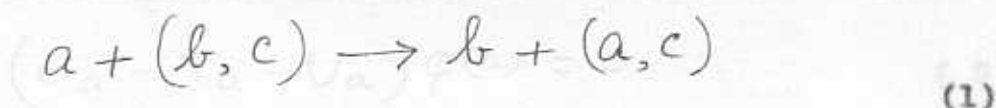
In the last section, we briefly refer to the different mechanisms of nuclear reactions that have been developed hitherto to explain the experimental data and indicate how the contribution from each of these mechanisms can be separated out and studied in a unified way starting from the expression (39) of Chapter I, obtained for rearrangement collisions.

* This chapter is included for the sake of completeness though not much of originality is claimed.

2. Specific processes

(a) Exchange scattering.

Consider a projectile a to be incident on a target consisting of a particle b bound to an inert core c . As a result of the reaction, re-arrangement takes place. The projectile a gets bound while the particle b is ejected.



The unperturbed and interaction Hamiltonians are

$$H_0^A = T_a + T_b + U_b \quad ; \quad V^A = U_a + V_{ab} \quad (2)$$

$$H_0^B = T_a + T_b \quad ; \quad V^B = U_a + U_b + V_{ab} \quad (3)$$

where T_a and T_b are the kinetic energy operators of the particles a and b , U_a and U_b the interaction of the particles a and b with the core, and V_{ab} the interaction between a and b .

The transition amplitude for the exchange scattering can be written in either the "post" or "prior"

$$= \langle \psi(b) \phi(a) | \frac{1}{E - H_0^B - U_b - V_{ab}} (V^B - V^A) \psi(a) \phi(b) \rangle \quad (17.6)$$

- 1) Goldberger and L. Goldberger, Phys. Rev., 81, 350 (1952)
- 2) H. Lipson and J. Brinkman, Phys. Rev., 75, 407 (1952)
- 3) E. Gerjuoy, Results of Physics, 2, 126 (1953)
- 4) S. Durrant, Proc. Roy. Soc., 12, 352 (1950)
- 5) L. D. Durrant, L. J. Goldberger, Advances in Theoretical Physics, 1, 151 (1951)

1) - 4)
form

$$M = \langle \varphi(a) \eta(b) | \sqrt{B} | \psi^{A(+)} \rangle \quad (4 \text{ a})$$

$$= \langle \psi^{B(-)} | \sqrt{A} | \eta(a) \varphi(b) \rangle \quad (4 \text{ b})$$

η represents the free state (plane wave) and φ the bound state wave function which satisfies the equation

$$(\epsilon_a - T_a - U_a) \varphi(a) = 0 \quad (5 \text{ a})$$

or

$$(\epsilon_b - T_b - U_b) \varphi(b) = 0 \quad (5 \text{ b})$$

ϵ_a or ϵ_b is the energy of the bound particle a or b . The outgoing wave solution $\psi^{A(+)}$ and the ingoing wave solution $\psi^{B(-)}$ of the total Hamiltonian can be written as

$$\psi^{A(+)} = \eta(a) \varphi(b) + \frac{1}{E - H_0^A + i\epsilon} (U_a + V_{ab}) \psi^{A(+)} \quad (6 \text{ a})$$

$$= \eta(a) \varphi(b) + \frac{1}{E - H_0^A - U_a - V_{ab} + i\epsilon} (U_a + V_{ab}) \eta(a) \varphi(b) \quad (6 \text{ b})$$

$$\psi^{B(-)} = \eta(b) \varphi(a) + \frac{1}{E - H_0^B - i\epsilon} (U_b + V_{ab}) \psi^{B(-)} \quad (7 \text{ a})$$

$$= \eta(b) \varphi(a) + \frac{1}{E - H_0^B - U_b - V_{ab} - i\epsilon} (U_b + V_{ab}) \eta(b) \varphi(a) \quad (7 \text{ b})$$

- 1) Gell-Mann and M.L. Goldberger, Phys. Rev., 91, 398 (1953)
- 2) B. Lippmann and J. Schwinger, Phys. Rev., 79, 469 (1950)
- 3) E. Gerjuoy, Annals of Physics, 5, 58 (1958)
- 4) S. Sunakawa, Prog. Theor. Phys., 24, 963 (1960)
- 5) T.B. Day, L.S. Rodberg, G.A. Snow and J. Sucher, Phys. Rev., 123, 1051 (1961)

Since the interaction Hamiltonian consists of two terms, we can make use of the two-potential formula of Gell-Mann and Goldberger¹⁾ and write the matrix element (4 a) in terms of the outgoing and ingoing waves distorted by the potential U due to the core

$$\chi^+(a) = \eta(a) + \frac{1}{E_a - T_a - U_a + i\epsilon} U_a \eta(a) \quad (8a)$$

$$\chi^{(-)}(b) = \eta(b) + \frac{1}{E_b - T_b - U_b - i\epsilon} U_b \eta(b) \quad (8b)$$

E_a and E_b represent the kinetic energy of a and b and therefore

$$E_a = E - E_b$$

$$E_b = E - E_a$$

Combining (6) and (8) and using the identity

$$\frac{1}{E - H_0^A + i\epsilon} = \frac{1}{E - H_0^A - U_a + i\epsilon} - \frac{1}{E - H_0^A - U_a + i\epsilon} U_a \frac{1}{E - H_0^A + i\epsilon} \quad (9)$$

we obtain

$$\psi^{A(+)} = \chi^+(a) \eta(b) + \frac{1}{E - H + i\epsilon} V_{ab} \chi^{(+)}(a) \eta(b) \quad (10)$$

The matrix element (4 a) can now be written as

$$\begin{aligned} M &= \langle \varphi(a) \eta(b) | U_b + V_{ab} | \psi^{A(+)} \rangle \\ &= \langle \varphi(a) \eta(b) | U_b | \psi^{A(+)} \rangle + \langle \varphi(a) \eta(b) | V_{ab} | \psi^{A(+)} \rangle \\ &= \langle \varphi(a) \eta(b) | U_b | \psi^{A(+)} \rangle + \langle \varphi(a) \chi^{(-)}(b) | V_{ab} | \psi^{A(+)} \rangle \\ &\quad - \langle \varphi(a) \eta(b) | U_b \frac{1}{E_b - T_b - U_b + i\epsilon} V_{ab} | \psi^{A(+)} \rangle \end{aligned} \quad (11)$$

Using equation 3 (b). Since $E_b = E - \epsilon_a$ and U_b commutes with $T_a + U_a$, we can use 5 (a) to replace the Green's function in the last term of (11) by $(E - H_0 + i\epsilon)^{-1}$

where

$$H_0 = T_a + T_b + U_a + U_b \quad (12)$$

Combining this last term with the first and making use of (10) we finally obtain

$$M = \langle \eta(b) \varphi(a) | U_b | \chi^{(+)}(a) \varphi(b) \rangle + \langle \varphi(a) \chi^{(-)}(b) | V_{ab} | \psi^{A(+)} \rangle \quad (13)$$

$\chi^{(+)}(a)$ represents the scattering solution and $\varphi(a)$ the bound state solution of the same Hamiltonian $T_a + U_a$ with different eigen values and hence they are orthogonal. So, the first term vanishes identically. Hence

$$M = \langle \varphi(a) \chi^{(-)}(b) | V_{ab} | \psi^{A(+)} \rangle \quad (14)$$

Expanding $\psi^{A(+)}$ in terms of $\chi^{(+)}(a) \eta(b)$ using equation (10), we obtain

$$M = \langle \varphi(a) \chi^{(-)}(b) | \left[V_{ab} + V_{ab} \frac{1}{E - H + i\epsilon} V_{ab} \right] | \chi^{(+)}(a) \varphi(b) \rangle \quad (15)$$

There is a great symmetry and there exists no prior or post discrepancy. The quantity in the bracket represents the effective interaction, the t matrix, representing the scattering of a by b in the presence of the core. If we neglect the second term in (15), we obtain the transition amplitude in the distorted wave Born approximation

$$M_{D.B.A} = \langle \varphi(a) \chi^{(-)}(b) | V_{ab} | \chi^{(+)}(a) \varphi(b) \rangle \quad (16)$$

Neglecting all the distortion effects, we get the lowest order Born approximation term

$$M_{B.A} = \langle \varphi(a) \eta(b) | V_{ab} | \eta(a) \varphi(b) \rangle \quad (17)$$

Instead, if we substitute the approximations in (13) directly, we will get

$$M_{B.A} = \langle \eta(b) \varphi(a) | U_b | \eta(a) \varphi(b) \rangle + \langle \varphi(a) \eta(b) | V_{ab} | \eta(a) \varphi(b) \rangle \quad (18)$$

Here the first term will contribute, since $\eta(a)$ and $\varphi(a)$ are no longer orthogonal because they are eigenfunctions of different Hamiltonians. However, expression (18) is wrong and it has been emphasized by Day et al* that the approximations should be made only at the final stage.

(b) Ionization Process.

In this process, the projectile a knocks out the particle b initially bound to the inert core c .



4)
* Day et al have pointed out that expression (18) has been used wrongly in certain calculations. A recent example is that of the calculation of the relative capture probabilities of the meson into various atomic states with the ejection of an atomic electron (R. A. Mann and M. E. Rose, Phys. Rev. 121, 293 (1961)).

The unperturbed and interaction Hamiltonians are

$$H_0^A = T_a + T_b + U_b ; V^A = U_a + V_{ab} \quad (20)$$

$$H_0^B = T_a + T_b ; V^B = U_a + U_b + V_{ab} \quad (21)$$

The transition amplitude for the ionization process can now be written as

$$M = \langle \eta^{(a)} \eta^{(b)} | V^B | \psi^{A(+)} \rangle \quad (22 a)$$

$$= \langle \psi^{B(-)} | V^A | \eta^{(a)} \varphi^{(b)} \rangle \quad (22 b)$$

If we apply the Born approximation directly, we obtain

$$M_{B.A} = \langle \eta^{(a)} \eta^{(b)} | U_a + U_b + V_{ab} | \eta^{(a)} \varphi^{(b)} \rangle \quad (23 a)$$

$$= \langle \eta^{(a)} \eta^{(b)} | U_a + V_{ab} | \eta^{(a)} \varphi^{(b)} \rangle \quad (23 b)$$

The difference between (23 a) and (23 b) is what is called the prior and post discrepancy and this arises due to inadequate

approximation. Expressions (23 a) and (23 b) are however wrong.

In order to obtain the correct expression under Born approximation, it is essential that the initial and final states should be represented as the eigenstates of the same Hamiltonian

$$H_0 = T_a + T_b + U_a + U_b \quad (24)$$

the common interaction Hamiltonian being V_{ab} . For this purpose, let us define

$$\psi^{B(-)} = \chi^{(-)}(a) \chi^{(-)}(b) + \frac{1}{E - H_0 - i\epsilon} V_{ab} \chi^{(-)}(a) \chi^{(-)}(b) \quad (25)$$

where

$$\chi^{(-)}(a) \chi^{(-)}(b) = \eta(a) \eta(b) + \frac{1}{E - H_0 - i\epsilon} (U_a + U_b) \eta(a) \eta(b) \quad (26)$$

$\chi^{(-)}(a)$ and $\chi^{(-)}(b)$ represent the particles a and b scattered by the potentials U_a and U_b respectively. Now the transition amplitude can be deduced in a form suitable for making Born approximation, using the procedure outlined earlier.

$$M = \langle \chi^{(-)}(a) \chi^{(-)}(b) | V_{ab} | \psi^{A(+)} \rangle \quad (27 a)$$

$$= \langle \psi^{B(-)} | V_{ab} | \chi^{(-)}(a) \phi(b) \rangle \quad (27 b)$$

The correct transition matrix element in Born approximation is given by

$$M_{B.A} = \langle \eta(a) \eta(b) | V_{ab} | \eta(a) \eta(b) \rangle \quad (28)$$

(c) Pick-up reaction

The incident particle a picks up the extra-core particle b forming a bound state



There is a symmetry in the type of reaction and hence it is desirable to take the S-matrix defined by expression (29) of Chapter I, as the starting point of our discussion

$$S^{BA} = \langle \psi^{B(-)} | \psi^{A(+)} \rangle \quad (30)$$

The unperturbed and interaction Hamiltonians of the initial and final states are

$$\begin{aligned} H_0^A &= T_a + T_b + U_b \quad ; \quad V^A = U_a + V_{ab} \\ H_0^B &= T_a + T_b + V_{ab} \quad ; \quad V^B = U_a + U_b \end{aligned} \quad (31)$$

The ingoing wave solution $\psi^{B(-)}$ obeys the integral equation

$$\psi^{B(-)} = \varphi(r) \eta(R) + \frac{1}{E_f - H_0^B - i\epsilon} (U_a + U_b) \psi^{B(-)} \quad (32)$$

where $\varphi(r)$ represents the internal wave function of the two-particle bound system and $\eta(R)$ the plane wave representing the motion of its centre of mass. Equation (32) can be rewritten as follows:

$$\psi^{B(-)} = \psi_0^{B(-)} + \frac{1}{E_f - H_0 - i\epsilon} V_{ab} \psi^{B(-)} \quad (33)$$

where

$$\psi_0^{B(-)} = \varphi(r) \eta(R) + \frac{1}{E_f - H_0 - i\epsilon} (U_a + U_b - V_{ab}) \varphi(r) \eta(R) \quad (34)$$

with

$$H_0 = T_a + T_b + U_a + U_b \quad (35)$$

It can be shown that the inhomogeneous term of equation (33) vanishes and consequently

$$\psi^{B(-)} = \frac{1}{E_f - H_0 - i\epsilon} V_{ab} \psi^{B(-)} \quad (36)$$

The outgoing wave solution obeys the equation

$$\psi^{A(+)} = \varphi(b) \chi^{(+)}(a) + \frac{1}{E_i - H + i\epsilon} V_{ab} \varphi(b) \chi^{(+)}(a) \quad (37)$$

Substituting equations (37) and (36) in equation (30), we obtain

$$\begin{aligned}
 S^{BA} &= \langle \psi^{B(-)} | \chi^{(+)}(a) \varphi(b) \rangle + \langle \psi^{B(-)} | \frac{1}{E_i - H + i\epsilon} V_{ab} | \varphi(b) \chi^{(+)}(a) \rangle \\
 &= \langle \frac{1}{E_f - H_0 - i\epsilon} V_{ab} \psi^{B(-)} | \chi^{(+)}(a) \varphi(b) \rangle \\
 &\quad + \frac{1}{E_i - E_f + i\epsilon} \langle \psi^{B(-)} | V_{ab} | \chi^{(+)}(a) \varphi(b) \rangle \\
 &= \left\{ \frac{1}{E_f - E_i - i\epsilon} + \frac{1}{E_i - E_f + i\epsilon} \right\} \langle \psi^{B(-)} | V_{ab} | \chi^{(+)}(a) \varphi(b) \rangle \\
 &= -2\pi i \delta(E_i - E_f) \langle \psi^{B(-)} | V_{ab} | \varphi(b) \chi^{(+)}(a) \rangle
 \end{aligned}$$

(38)

In this case, it is not possible to obtain the reciprocal formula which is however possible in the earlier cases (e.g. see equations (25 a) and (25 b)). This is due to the asymmetry inherent in the pick-up reaction. For calculational purposes, we can approximate $\psi^{B(-)}$ by $\varphi(r) \eta(R)$. Thus

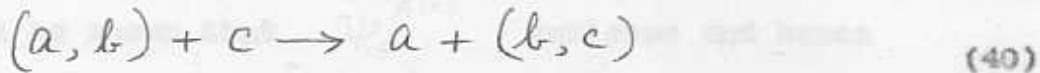
$$S^{BA} \simeq -2\pi i \delta(E_i - E_f) \langle \varphi(r) \eta(R) | V_{ab} | \varphi(b) \chi^{(+)}(a) \rangle \quad (39)$$

The integral equation (38) can be written in the form

$$\psi^{B(-)} = \psi^{B(-)} + \frac{1}{E_i - H_0 - i\epsilon} V_{ab} \psi^{B(-)} \quad (40)$$

(d) Stripping reaction

This is just the reverse process of the pick-up reaction. Initially, the particles a and b form a bound system, of which one of the particles is stripped off by the target nucleus



The unperturbed and interaction Hamiltonians of the initial system are

$$\begin{aligned} H_0^A &= T_a + T_b + V_{ab} \\ V^A &= U_a + U_b \end{aligned} \quad (41 a)$$

For the final system

$$\begin{aligned} H_0^B &= T_a + T_b + U_b \\ V^B &= U_a + V_{ab} \end{aligned} \quad (41 b)$$

The S-matrix element for the stripping process can be written as

$$S^{BA} = \langle \psi^{B(-)} | \psi^{A(+)} \rangle$$

This may be written in a form analogous to equation (39) obtained in the case of pick-up reaction. For this purpose, we define

$$\psi^{A(+)} = \varphi(a)\eta(R) + \frac{1}{E_i - H_0^A + i\epsilon} (U_a + U_b) \psi^{A(+)} \quad (42)$$

and

$$\psi^{B(-)} = \varphi(b)\chi^{(-)} + \frac{1}{E_f - H - i\epsilon} V_{ab} \varphi(b)\chi^{(-)} \quad (43)$$

The integral equation (42) can be written in the form

$$\psi^{A(+)} = \psi_0^{A(+)} + \frac{1}{E_i - H_0 - i\epsilon} V_{ab} \psi^{A(+)} \quad (44)$$

where

$$\psi_0^{A(+)} = \varphi(a)\eta(b) + \frac{1}{E_f - H_0 + i\epsilon} (V_a + V_b - V_{ab}) \varphi(a)\eta(b) \quad (45)$$

It can be shown that $\psi_0^{A(+)}$ vanishes and hence

$$\psi^{A(+)} = \frac{1}{E_i - H_0 + i\epsilon} V_{ab} \psi^{A(+)} \quad (46)$$

Now the S-matrix element for the stripping reaction can be written as

$$\begin{aligned} S^{BA} &= \langle \varphi(b) \chi^{(-)}(a) | \psi^{A(+)} \rangle \\ &+ \langle \varphi(b) \chi^{(-)}(a) | V_{ab} \frac{1}{E_f - H_0 + i\epsilon} | \psi^{A(+)} \rangle \\ &= \langle \varphi(b) \chi^{(-)}(a) | \frac{1}{E_i - H_0 + i\epsilon} V_{ab} | \psi^{A(+)} \rangle \\ &+ \langle \varphi(b) \chi^{(-)}(a) | V_{ab} \frac{1}{E_f - H_0 + i\epsilon} | \psi^{A(+)} \rangle \\ &= \left(\frac{1}{E_i - E_f + i\epsilon} + \frac{1}{E_f - E_i + i\epsilon} \right) \langle \varphi(b) \chi^{(-)}(a) | V_{ab} | \psi^{A(+)} \rangle \\ &= -2\pi i \delta(E_i - E_f) \langle \varphi(b) \chi^{(-)}(a) | V_{ab} | \psi^{A(+)} \rangle \quad (47) \end{aligned}$$

If the distortion of the initial two-particle bound system can be neglected, we obtain

- 5) K.M. Watson, Phys. Rev., 83, 1163 (1952)
H.C. Francis and K.M. Watson, Phys. Rev., 93, 313 (1954)

$$S^{BA} \approx -2\pi i \delta(E_i - E_f) \langle \varphi(b) \chi^{(a)} | V_{ab} | \varphi(a) \eta(b) \rangle \quad (48)$$

The matrix element can be interpreted as representing the distortion of two particle bound system by the nuclear potential and a subsequent impulsive break-up by their mutual interaction V_{ab} .

(e) Break-up reaction

This problem is different from the previous one only in that the final state now consists of free particles a and b .

$$\begin{aligned} H_0^B &= T_a + T_b \\ V^B &= U_a + U_b + V_{ab} \end{aligned} \quad (49)$$

As before, the matrix element for the process is

$$M = \langle \chi^{(a)} \chi^{(b)} | V_{ab} | \psi^{A(i)} \rangle \quad (50)$$

where

$$\chi^{(a)} \chi^{(b)} = \eta(a) \eta(b) + \frac{1}{E_f - H_0^B - i\epsilon} (U_a + U_b) \chi^{(a)} \chi^{(b)} \quad (51)$$

One can give a physical interpretation to the matrix element (50) that the two-particle system, after break-up by their mutual interaction, undergoes elastic scattering in the nuclear potential.

6) James E. Young, Phys. Rev., 116, 1201 (1959)

3. Direct and compound nuclear reactions

In recent years, attempts are being made to develop a unified theory of nuclear reactions. They mark a departure from the conventional R-matrix theory of nuclear reactions introduced by Wigner⁷⁾ and reviewed in an extensive article by Lane and Thomas⁸⁾. Different mechanisms of nuclear reactions - compound nucleus model⁹⁾, optical model¹⁰⁾ and direct interaction model¹¹⁾ - have been introduced to explain the experimental data.

- 7) E.P.Wigner, Phys. Rev., 72, 29 (1947)
 8) A.M.Lane and R.G.Thomas, Rev. Mod. Phys., 30, 257 (1958).
 9) N.Bohr, Nature, 137, 344 (1936)
 G. Breit and E.P.Wigner, Phys. Rev., 49, 519 (1936)
 10) H.Feshbach, C.E.Porter and V.F.Weisskopf, Phys. Rev., 96, 448 (1954).
 S. Fernbach, R. Serber and T.B.Taylor, Phys.Rev., 75, 1352 (1949).
 11) S.T.Butler, Proc. Roy. Soc., A 202, 559 (1951),
 Austern, Butler and McManus, Phys. Rev., 92, 350 (1953)
 S.T.Butler, N. Austern and C. Pearson, Phys. Rev., 112, 1227, (1958)
 C.A.Levinson and M.K.Bannerjee, Annals of Physics, 2, 471, 499 (1957); 3, 67 (1958)
 N. Austern, "Direct reaction theories" in Fast Neutron Physics (Interscience Publishers Inc., New York (1960) Chapter V.

Each model appears to be successful in a certain energy region. At low energy, the cross section exhibits closely spaced peaks called resonances and they can be explained by the compound nucleus model. At higher energy, the optical model seems to be successful. The velocity of the incident particle is still so low that the nucleons of the target nucleus rearrange very often while the incident nucleon traverses the nucleus, and it is possible to represent the effect of the target nucleus by an average potential. If the energy of the incident particle is further increased, the interaction of the nucleons of the nucleus can be neglected, and the nucleus treated as an aggregate of free particles, with which the reaction of the incident particle takes place through direct interaction. Both the latter models are sometimes called direct interaction models to express the absence of the compound state as an intermediary between the initial and final states. In some cases, the direct interaction may become significant between the resonances also at low energies.

These different mechanisms of nuclear reactions are operative under different idealized conditions and in any actual situation, one may expect each of these mechanisms to contribute to a less or greater extent. So it has been found desirable to develop a single mathematical apparatus to treat the scattering by an optical potential as well as direct interactions and processes involving the formation of a compound nucleus.

Dispersion formulae in the framework of a unified theory¹²⁾⁻¹⁶⁾
of nuclear reactions has been studied by Feshbach¹²⁾, Sano,
Yoshida and Teresawa¹³⁾, Fonda and Newton¹⁴⁾, Serdobolsky¹⁵⁾
and others.

Starting from the expressions (39) of Chapter I obtained
by the formal theory of re-arrangement collisions, it is possible
to separate the individual contributions of the different mecha-
nisms of nuclear reactions. For this purpose, let us introduce
an optical potential V_0 and the wave functions $\psi_0^{(+)}$ and
 $\psi_0^{(-)}$ describe the scattering by an optical potential in each
of these channels. The indices A, B etc. are used to distinguish
between different channels. Let $\mathcal{V} = V - V_0$. The following
identity can be obtained

$$\begin{aligned}
\langle \phi^B | V^B | \psi^{A(+)} \rangle &= \langle \phi^B | V^B | \psi_0^{A(+)} \rangle \\
&+ \langle \psi_0^{B(+)} | \mathcal{V}' | \psi_0^{A(+)} \rangle \\
&+ \langle \psi_0^{B(-)} | \mathcal{V}' \frac{1}{E - H + i\epsilon} \mathcal{V} | \psi_0^{A(+)} \rangle
\end{aligned}
\tag{52}$$

- 12) H. Feshbach, *Annals of Physics*, 5, (1958)
13) M. Sano, S. Yoshida and T. Teresawa, *Nuclear Physics*,
6, 20 (1958).
14) L. Fonda and R.G. Newton, *Annals of Physics*, 10, 490 (1960)
15) V.I. Serdobolsky, *Nuclear Physics*, 21, 245 (1960)
16) A. Agodi and E. Eberle, *Nuovo Cimento*, 18, 718 (1960).

The first term describes an optical interaction and vanishes for $B \neq A$. The second term represents the amplitude of direct processes in the distorted wave approximation method. The third term can be regarded as a resonance term.

PART II

SCATTERING AND TRANSMISSION OF WAVES

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Section III

Outline of the Report

I. Introduction

The theory of scattering and photoproduction of pions from nucleons has been the subject of study for the past few years ever since the pions were experimentally discovered and at present the theory of Chew et al.¹⁾ is widely accepted

PART II

SCATTERING AND PHOTOPRODUCTION OF PIONS

The impetus for the scattering and photoproduction of pions from free nucleons, FROM NUCLEI Chew-Low theory can be taken over and applied to the problem of scattering and photoproduction of pions from nuclei.²⁾ This extension to the case of complex nuclei is made possible by the impulse approximation³⁾ developed and justified by Chew himself.

1) G.F.Chew and F.E.Low, Phys. Rev., **101**, 1570, 1579 (1956)
 G.F.Chew, M.L.Goldberger, F.E.Low and Y.Nambu, Phys. Rev., **104**, 1495 (1957).
 G.F.Chew, Handbook for Physics, Interscience Wiley, **32** (1959).

2) Y. Imamura and M. Hara, Nuclear Physics, **23**, 512 (1951); **23**, 544 (1951) in press.
 S. Hara, Y. Imamura, Y. Kawarabayashi and M. Hara, Nuclear Physics, **23**, 123 (1951).
 S. Hara, Y. Imamura and M. Hara, Nuclear Physics, **23**, 512 (1951).

3) G.F.Chew, Phys. Rev., **95**, 500 (1954)
 G.F.Chew and M.L.Goldberger, Phys. Rev., **95**, 496 (1954).
 G.F.Chew and M.L.Goldberger, Phys. Rev., **95**, 718 (1954).

Chapter III.

Outline of the method

1. Introduction

The theory of scattering and photoproduction of pions from nucleons has been the subject of study for the past many years ever since the pions were experimentally discovered; and at present the theory of Chew et al¹⁾ is widely accepted and it is found to yield reliable results below 500 Mev. The amplitude for the scattering and photoproduction of pions from free nucleons, as given by Chew-Low theory can be taken over and applied to the problem of scattering and photoproduction of pions from nuclei²⁾. This extension to the case of complex nuclei is made possible by the impulse approximation³⁾ developed and justified by Chew himself.

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- 1) G.F.Chew and F.E.Low, Phys. Rev., 101, 1570, 1579 (1956)
G.F.Chew, M.L.Goldberger, F.E.Low and Y. Nambu,
Phys. Rev., 106, 1345 (1957).
G.F.Chew, Handbuch der Physik, Springer Verlag, 43 (1959).
 - 2) V. Devanathan and G. Ramachandran, Nuclear Physics,
23, 312 (1961); 38, 654, (1962) (1962); in press.
A. Ramakrishnan, V.Devanathan and K. Venkatesan, G.Ramachandran,
Nuclear Physics, 24, 163 (1961).
A. Ramakrishnan, V. Devanathan and K. Venkatesan
Nuclear Physics, 29, 680 (1962)
 - 3) G.F.Chew, Phys. Rev., 80, 196 (1950)
G.F.Chew and G.C.Wick, Phys. Rev., 85, 636 (1952).
G.F.Chew and M.L.Goldberger, Phys. Rev., 87, 778 (1952).

In sec. 2, a brief account of the scattering and photo-production of pions from free nucleons is given. No attempt is made to give the general theory^{1) 4)} but only the results which are relevant to our work are presented. Section 3 is devoted to a discussion on the Impulse approximation and its validity.

2. Scattering and photoproduction of pions from free nucleons*

The Chew-Low amplitude for the pion-nucleon scattering in the static approximation is given by

$$t(2,1) = \frac{2\pi}{\omega} \sum_{\alpha=1}^4 P_{\alpha}(2,1) h_{\alpha} \quad (1)$$

where

$$P_{\alpha}(2,1) = \mathcal{G}_I(2,1) \mathcal{G}_J(2,1) \quad (2)$$

$$h_{\alpha} = \frac{e^{i\delta_{\alpha}} \sin \delta_{\alpha}}{q^3}$$

\mathcal{G}_I and \mathcal{G}_J are the projection operators for the isobaric spin and angular momentum states, ω the meson energy, q its momentum and δ_{α} the p -wave phase shifts. In the above,

4) A. Ramakrishnan, Elementary Particles and Cosmic Rays, Pergamon Press (1962), where further references can be found.

* Throughout, we use natural units i.e. $\hbar = c = 1$ and the pion mass is taken as unity.

α stands for any one of the four states 33, 31, 13, 11 in Fermi's notation, according to which a state is designated by $(2I, 2J)$, I and J representing respectively the total isobaric spin and total angular momentum of the pion-nucleon system.

It is to be observed that the scattering amplitude given by experiment (1) involves only the p -wave phase shifts since the nucleon is considered to be static. Also $\delta_{13} = \delta_{31}$. The projection operators \mathcal{P}_I and \mathcal{P}_J are given by

$$\mathcal{P}_{\frac{1}{2}}(2,1) = \frac{1}{3} \delta_{q_2 q_1} + \frac{1}{3} \frac{[\tau_{q_2}, \tau_{q_1}]}{2}$$

$$\mathcal{P}_{\frac{3}{2}}(2,1) = \frac{2}{3} \delta_{q_2 q_1} - \frac{1}{3} \frac{[\tau_{q_2}, \tau_{q_1}]}{2}$$

$$\mathcal{G}_{\frac{1}{2}}(2,1) = \vec{q}_2 \cdot \vec{q}_1 + i \vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1)$$

$$\mathcal{G}_{\frac{3}{2}}(2,1) = 2 \vec{q}_2 \cdot \vec{q}_1 - i \vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1)$$

where $\vec{\sigma}$ is the spin operator for the nucleon, \vec{q}_1 and \vec{q}_2 are the momenta of the incident and scattered pion and the suffixes q_1 and q_2 are the pion indices. The projection operators may be evaluated between the specified initial and final states of the pion-nucleon system to obtain the following amplitudes for the direct and charge exchange scatterings: (3)

$$t(p\pi^+ \rightarrow p\pi^+) = a + b(\sigma^+ - \sigma^-)$$

$$t(n\pi^+ \rightarrow n\pi^+) = c + d(\sigma^+ - \sigma^-)$$

$$t(n\pi^+ \rightarrow p\pi^0) = A + B(\sigma^+ - \sigma^-)$$

(4)

where

$$a = \frac{2\pi}{\omega q} \cos\theta (2e^{i\delta_{33}} \sin\delta_{33} + e^{i\delta_{31}} \sin\delta_{31})$$

$$b = \frac{2\pi}{\omega q} \sin\theta (e^{i\delta_{33}} \sin\delta_{33} - e^{i\delta_{31}} \sin\delta_{31})$$

$$c = \frac{2\pi}{3\omega q} \cos\theta (2e^{i\delta_{33}} \sin\delta_{33} + e^{i\delta_{31}} \sin\delta_{31} + 4e^{i\delta_{13}} \sin\delta_{13} + 2e^{i\delta_{11}} \sin\delta_{11})$$

$$d = \frac{2\pi}{3\omega q} \sin\theta (e^{i\delta_{33}} \sin\delta_{33} - e^{i\delta_{31}} \sin\delta_{31} + 2e^{i\delta_{13}} \sin\delta_{13} - 2e^{i\delta_{11}} \sin\delta_{11})$$

$$A = \frac{2\sqrt{2}\pi}{3\omega q} \cos\theta (2e^{i\delta_{33}} \sin\delta_{33} + e^{i\delta_{31}} \sin\delta_{31} - 2e^{i\delta_{13}} \sin\delta_{13} - e^{i\delta_{11}} \sin\delta_{11})$$

$$B = \frac{2\sqrt{2}\pi}{3\omega q} \sin\theta (e^{i\delta_{33}} \sin\delta_{33} - e^{i\delta_{31}} \sin\delta_{31} - e^{i\delta_{13}} \sin\delta_{13} + e^{i\delta_{11}} \sin\delta_{11})$$

(5)

expression
 In experiment (5), θ represents the angle between the direction of the incident pion and that of the scattered pion. In the above, we have considered the particular case of scattering of a positively charged pion by a nucleon, but a similar expression can be obtained for the scattering of a negative pion.

Chew and his collaborators¹⁾ have also discussed the problem of photoproduction of pions from free nucleons, first non-relativistically using the cut-off form of the Yukawa theory and later on relativistically using dispersion theory. The relativistic version includes recoil corrections to the nucleon and s -wave phase shifts. The complete amplitude can be written down, introducing for convenience, the three isotopic spin projection operators, as originally suggested by Watson⁵⁾,

$$\begin{aligned} g_{\mu}^{(0)} &= \tau_{\mu} \\ g_{\mu}^{(+)} &= \frac{1}{2} (\tau_{\mu} \tau_3 + \tau_3 \tau_{\mu}) = \delta_{\mu 3} \\ g_{\mu}^{(-)} &= \frac{1}{2} (\tau_{\mu} \tau_3 - \tau_3 \tau_{\mu}) = \frac{1}{2} [\tau_{\mu}, \tau_3] \end{aligned} \quad (6)$$

where μ is the pion index and in table I are listed the values taken by $g^{(\pm)}$ for the four possible charge conjugate configurations.

5) K.M. Watson, Phys. Rev., 95, 228 (1954)

$$\frac{1}{ef} F^{(+)} = \frac{1}{4Mf^2} \left[\vec{\sigma} \cdot \vec{\epsilon} + 2 \frac{\vec{\sigma} \cdot (\vec{\sigma} \times \vec{\sigma}) \cdot \vec{\sigma}}{(\vec{\sigma} \cdot \vec{\sigma})^2 + 4} \right]$$

Table 1

	$p \rightarrow p + \pi^0$	$n \rightarrow n + \pi^0$	$p \rightarrow n + \pi^+$	$n \rightarrow p + \pi^-$
$g_{\mu}^{(0)}$	+1	-1	+ 2	+ 2
$g_{\mu}^{(+)}$	+1	+1	0	0
$g_{\mu}^{(-)}$	0	0	+ 2	- 2

We give below the corresponding amplitudes $F^{(+)}$, $F^{(-)}$ and $F^{(0)}$ which can be considered as the coefficients of $g^{(+)}$, $g^{(-)}$ and $g^{(0)}$.

$$\begin{aligned} \frac{1}{ef} F^{(+)} = & i \vec{\sigma} \cdot \vec{\epsilon} \left\{ i \frac{2}{3} (\delta_{S_{1/2}} - \delta_{S_{3/2}}) F_S \right\} \\ & + i \vec{\sigma} \cdot \vec{\epsilon} \vec{\mu} \cdot \vec{\sigma} \left\{ - \frac{g_p - g_n}{4Mf^2} h^{(+)} - \frac{2}{3} i e^{i\delta_{33}} \sin \delta_{33} (F_A - \frac{1}{3} F_M) \right\} \\ & + i \vec{\sigma} \cdot \vec{\sigma} \vec{\mu} \cdot \vec{\epsilon} \left\{ \frac{g_p - g_n}{4Mf^2} h^{(+)} - \frac{2}{3} i e^{i\delta_{33}} \sin \delta_{33} (F_A + \frac{1}{3} F_M) \right\} \\ & + \vec{\mu} \cdot (\vec{\sigma} \times \vec{\epsilon}) \left\{ \frac{g_p - g_n}{4Mf^2} h^{(+)} + \frac{4}{9} i e^{i\delta_{33}} \sin \delta_{33} F_M \right\} \\ & + i \vec{\sigma} \cdot \vec{\mu} \vec{\mu} \cdot \vec{\epsilon} \frac{1}{2M\mu_0} \end{aligned}$$

(7)

$$\begin{aligned}
\frac{1}{e_f} F^{(v)} = & \frac{1}{1 + \frac{\mu_0}{M}} \left[i \vec{\sigma} \cdot \vec{E} + 2i \frac{\vec{\sigma} \cdot (\vec{\omega} - \vec{\mu}) \vec{\mu} \cdot \vec{E}}{(\vec{\omega} - \vec{\mu})^2 + 1} \right] \\
& + i \vec{\sigma} \cdot \vec{E} \vec{\mu} \cdot \vec{\omega} \left\{ -\frac{g_p - g_n}{4Mf^2} h^{--} + \frac{1}{3} i e^{i\delta_{33}} \sin \delta_{33} \right. \\
& \qquad \qquad \qquad \left. (F_A - \frac{1}{3} F_M) \right\} \\
& + i \vec{\sigma} \cdot \vec{\omega} \vec{\mu} \cdot \vec{E} \left\{ \frac{g_p - g_n}{4Mf^2} h^{--} + \frac{1}{3} i e^{i\delta_{33}} \sin \delta_{33} \right. \\
& \qquad \qquad \qquad \left. (F_A + \frac{1}{3} F_M) \right\} \\
& + \vec{\mu} \cdot (\vec{\omega} \times \vec{E}) \left\{ \frac{g_p - g_n}{4Mf^2} h^{+-} - \frac{2}{9} i e^{i\delta_{33}} \sin \delta_{33} F_M \right\} \\
& - i \vec{\sigma} \cdot \vec{\mu} \vec{\mu} \cdot \vec{E} \frac{1}{2M\mu_0}
\end{aligned} \tag{8}$$

$$\begin{aligned}
\frac{1}{e_f} F^{(w)} = & -i \vec{\sigma} \cdot \vec{E} \frac{g_p + g_n}{2M} \mu_0 \\
& - i \vec{\sigma} \cdot \vec{\mu} \times (\vec{\omega} \times \vec{E}) \frac{g_p + g_n}{2M\mu_0} \\
& + i \vec{\sigma} \cdot \vec{\mu} \vec{\mu} \cdot \vec{E} \frac{1}{2M\mu_0}
\end{aligned} \tag{9}$$

where $\vec{\sigma}$ is the spin operator for the nucleon and M its mass (6.65) in units of pion mass. $\vec{\omega}$ and ω_0 are the momentum and energy of the incident photon, \vec{E} its polarisation vector and $\vec{\mu}$ and μ_0 the momentum and energy of the pion. $h^{(++)}$, $h^{(+-)}$ and $h^{(--)}$ are the combinations of p -wave scattering amplitudes defined below:

$$h^{(++)} = \frac{1}{3} (h_1 + 4h_2 + 4h_3)$$

$$h^{(+-)} = \frac{1}{3} (h_1 + h_2 - 2h_3)$$

$$h^{(--) } = \frac{1}{3} (h_1 - 2h_2 + h_3)$$

(10)

where

$$h_1 = h_{11} = \frac{e^{i\delta_{11}} \sin \delta_{11}}{\mu^3}$$

$$h_2 = h_{13} = h_{31} = \frac{e^{i\delta_{13}} \sin \delta_{13}}{\mu^3}$$

$$h_3 = h_{33} = \frac{e^{i\delta_{33}} \sin \delta_{33}}{\mu^3}$$

(11)

The quantities F_S , F_M and F_a occurring in (7), (8) and (9) are given by

$$F_S = 1 - \frac{1}{2} \left(1 + \frac{1-v^2}{2v} \log \frac{1-v}{1+v} \right)$$

$$F_M = \frac{3}{4\mu^2} \left(1 + \frac{1-v^2}{2v} \log \frac{1-v}{1+v} \right)$$

$$F_a = \frac{1}{\mu^2} \left\{ 1 - \frac{3}{4v^2} \left(1 + \frac{1-v^2}{2v} \log \frac{1-v}{1+v} \right) \right\}$$

where

$$v = \frac{\mu}{\mu_0}$$

(12)

The dominant terms in the above amplitudes are those containing the 33 amplitude multiplied by the factor $g_p - g_n$ besides the first term in (8). g_p and g_n are the proton and neutron magnetic moments in units of nuclear magnetons. ($g_p = 2.78$;

$g_n = -1.91$). The amplitude $F^{(0)}$ has no large part and may

be considered as a perturbation on $F^{(+)}$ and $F^{(-)}$. The chief significance of $F^{(0)}$ is that it gives rise to a difference between the cross-sections for positive and negative meson production.

Now we can write down the transition amplitude for the nucleon with the photoproduction of neutral or charged pions

$$t(\gamma + p \rightarrow \pi^0 + p) = F^{(+)} + F^{(0)} \quad (13)$$

$$t(\gamma + n \rightarrow \pi^0 + n) = F^{(+)} - F^{(0)}$$

$$t(\gamma + p \rightarrow \pi^+ + n) = \sqrt{2}(F^{(-)} + F^{(0)})$$

$$t(\gamma + n \rightarrow \pi^- + p) = -\sqrt{2}(F^{(-)} - F^{(0)})$$

(14)

In subsequent chapters, we will be using these single nucleon amplitudes for obtaining the cross-section for the photoproduction of pions from complex nuclei, the extension to which is made possible by the impulse approximation.

3. The Impulse approximation and its validity.

According to the impulse approximation, the amplitude for the scattering or photoproduction of pions from a complex nucleus can be written as a linear superposition of the individual free nucleon amplitudes i.e.

$$T = t_1 + t_2 + t_3 + \dots \quad (15)$$

It involves the neglect of multiple scattering, off-the-energy shell matrix elements and the internucleon potential. However, the correct wave function may be used for the initial and final states in computing the matrix elements.

The assumptions under which the approximation is valid are the following:-

(1) The incident particle interacts only with one single nucleon at a time.

(2) The amplitude of the incident wave is not appreciably diminished in crossing the nucleus.

(3) The binding force has a negligible effect during the interval of strong interaction.

The errors involved in these assumptions and the ~~velocity~~ validity of the impulse approximation have been investigated by Chew and his collaborators²⁾ and we briefly reproduce the necessary corrections. To be specific, we consider the scattering of a particle by a complex nucleus containing N nucleons, the matrix element of which can be written as

$$\begin{aligned}
 T_{ba} &= \langle \Phi_b, V \Psi_a^{(+)} \rangle \\
 &= \langle \Phi_b, V \Omega^{(+)} \Phi_a \rangle
 \end{aligned}
 \tag{16}$$

where Φ_a and Φ_b are the eigen functions of the unperturbed Hamiltonian H_0 with the same eigenvalue E .

$$H_0 = K + U \tag{17}$$

where K is the total kinetic energy operator and U the potential energy of the nucleus. The total Hamiltonian of the system is

$$H = H_0 + V \tag{18}$$

with

$$V = \sum_k V_k \tag{19}$$

V_k represents the interaction of the incident particle with the k^{th} nucleon in the target nucleus. $\Psi_a^{(+)}$ is the eigenfunction of the total Hamiltonian. The operator $\Omega^{(+)}$ in (16) is given by

$$\Omega^{(+)} = 1 + \frac{1}{E_a - H_0 - V + i\epsilon} V \tag{20}$$

The operator T which when taken between the unperturbed states gives the matrix element, can be formally written as

$$T = V + V \frac{1}{E_a - H_0 - V + i\epsilon} V \tag{21}$$

This expression, though exact, is not in a form for easy evaluation. For this purpose, the two-body scattering matrices are introduced.

$$(t_k)_{mn} = \langle \chi_m, V_k \psi_{n,k}^{(+)} \rangle \quad (22)$$

where the ψ 's are solutions of the equations

$$\psi_{l,k}^{(+)} = \chi_l + \frac{1}{E_l - K + i\varepsilon} V_k \psi_{l,k}^{(+)} \quad (23)$$

The χ 's are eigenfunctions of K and E_l is the eigenvalue of K belonging to χ_l . Defining the two-body operator t_k as

$$t_k = V_k \omega_k^{(+)} \quad (24)$$

where

$$\omega_k^{(+)} = 1 + \frac{1}{E_l - K - V_k + i\varepsilon} V_k \quad (25)$$

and rewriting equation (21) in the form

$$T = \sum_{k=1}^N \left\{ V_k + V \frac{1}{E_a - H_0 - V + i\varepsilon} V_k \right\} \quad (26)$$

we can obtain the required form, if we use the identity

$$\frac{1}{E_a - H_0 - V + i\varepsilon} V_k = (\omega_k^{(+)} - 1) + \frac{1}{E_a - H_0 - V + i\varepsilon} \left\{ [U, \omega_k^{(+)}] + (V - V_k) (\omega_k^{(+)} - 1) \right\} \quad (27)$$

Substituting (27) into (26), we obtain after slight rearrangement of terms

$$T = \sum_{k=1}^N \left\{ t_k + V \frac{1}{E_a - H_0 - V + i\epsilon} [U, \omega_k^{(+)}] \right. \\ \left. + \left(1 + V \frac{1}{E_a - H_0 - V + i\epsilon} \right) (V - V_k) (\omega_k^{(+)} - 1) \right\}$$

(28)

The first term in (28) represents the linear superposition of the individual free nucleon amplitudes and it is known as the impulse approximation and the second and third terms the corrections for the same. The second term denotes the correction due to binding energy and the third term the correction due to multiple scattering. An adequate and workable approximation in many cases is thus obtained by evaluating $\sum_k t_k$ exactly and either neglecting completely the last two terms of (28) or approximating them by further iteration.

- 1) S. Fubini, T. Regge and G. Furlan, Phys. Rev., 85, 1558 (1951)
- 2) S. M. Berman, Phys. Rev., 121, 208 (1957); 122, 713 (1957)
- 3) E. E. Tompkins, Phys. Rev., 85, 204 (1951); 86, 211 (1951);
E. E. Tompkins and G. F. Lund, Phys. Rev., 92, 219 (1953);
T. de Alfaro and G. Furlan, Nuovo Journal, 11, 483 (1954)
- 4) E. C. Squires and G. M. Bickerton, Phys. Rev., 122, 167 (1957)

Chapter IVScattering of pions by deuterons1. Introduction

Fernbach, Green and Watson¹⁾ and Rockmore²⁾ have studied earlier the scattering of pions by deuterons using the impulse approximation which involves the neglect of internucleon potential, multiple scattering and off-the energy shell scattering of pion by nucleon. The correction due to the internucleon potential has been investigated by Rockmore and is found to be less than 10 %. But there seems to be a controversy as regards the multiple scattering. The multiple scattering effect³⁾ has been initially studied by Brueckner and he finds a considerable reduction in the cross-section of about 50 % at 30° and at 135 Mev. However Rockmore points out that the neglect of off-the-energy shell matrix elements in multiple scattering is certainly invalid. He has carried out the calculations at only one energy (85 Mev) using pure impulse approximation and finds an excellent agreement with the experimental results of Rogers and Lederman⁴⁾.

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- 1) S.Fernbach, T.A.Green and K.M.Watson, Phys.Rev. 80,1084 (1951)
 - 2) R.M.Rockmore, Phys. Rev., 105 , 256 (1957). 90, 715 (1953)
 - 3) K.A.Brueckner, Phys. Rev., 89, 834, (1953); 96, ~~849~~
S.D.Drell and L. Verlet, Phys. Rev., 99, 849 (1956)
V.De Alfaro and R.Stroffolini, Nuovo Cimento, 11, 447 (1959).
 - 4) K.G.Rogers and L.M.Lederman, Phys. Rev., 105, 247 (1957)

The elastic scattering of pions by deuterons has also been studied by Bransden and Moorhouse⁵⁾ employing the variational method and the numerical results are presented at various incident pion energies from 85 to 378 Mev. In the expression they have derived, they are able to identify the terms corresponding to multiple scattering and show that the inclusion of these terms alters the cross-section by less than 5 % even at the most favourable angles and energies in the energy range considered. Their numerical results agree well with the experimental results of Rogers and Lederman⁴⁾, Arase et al⁶⁾ and Perwitt et al⁷⁾.

The investigations of Rockmore and Bransden and Moorhouse seem to show that the impulse approximation is certainly an approximation valid in the energy region 85 to 300 Mev. So we thought it worth while to carry out the calculations using the Chew-Low amplitude¹¹⁾ for the pion-nucleon scattering and present numerical results for the elastic, inelastic and charge-exchange scattering of pions by deuterons at various incident pion energies 85, 140, 195 and 250 Mev. Calculations have also been done at energies 60 and 300 Mev for the elastic process alone for the specific purpose of comparing with the experimental results of Sachs et al¹²⁾ and Dul'kovo et al¹³⁾ reported at these energies.

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- 5) B.H. Bransden and R.G. Moorhouse, Nucl. Physics, 6, 310 (1958)
 6) E. Arase, G. Goldhaber and S. Goldhaber, Phys. Rev., 90, 160 (1953)
 7) E.C. Perwitt et al, Proc. of the 1960 Annual International Conference on High Energy Physics (Rochester) p. 196, (1960)
 11) G.F. Chew and F.E. Low, Phys. Rev., 101, 1570 (1956)
 12) A.M. Sachs, H. Winick and B.A. Wooten, Phys. Rev., 109, 1733 (1958)
 13) L.S. Dul'kova, I.B. Sokolova and M.G. Shapranova, Soviet Physics, JETP, 8, 217 (1959)

Recently, Frank and Spriggs⁸⁾ have calculated the on-the-energy shell multiple scattering contribution using the di-nucleon formalism developed earlier by Frank⁹⁾ and finds an agreement with the observations of Brueckner. Pendleton¹⁰⁾ has made an exhaustive study of the elastic scattering of charged pions by deuteron at laboratory energy 142 Mev in a form factor approximation which includes double scattering effects and the values that he has obtained for the angular distribution of the pions is not very much different from that of ours.

The numerical results that we have obtained agree well with the experimental values of Sachs et al at 61 Mev and that of Rogers and Lederman for elastic and inelastic scattering of positive pions by deuterons at 85 Mev; but the fit is not so good in the case of elastic scattering at 140 Mev. Earlier Green¹⁴⁾ has reported an impulse approximation calculation for elastic scattering of charged pions by deuterons at 135 Mev and he has also found too large values for the cross-sections as compared with the then available experimental result of Arase et al. As a result, he has made some sceptical remarks about the validity of the impulse approximation at the energy 135 Mev and above.

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- 8) W.M.Frank and T.Spriggs, Annals of Physics.
 9) W.M.Frank, Annals of Physics, 17, 205 (1962)
 10) Pendleton, Private communication
 14) T.A.Green, Phys. Rev., 90, 161 (1953)

The following processes viz., elastic, inelastic and charge exchange scattering of positive pions by deuterons are considered.

The only experiment that is available in the higher energy range is that of Dul'kova at 300 Mev. It is to be pointed out that the theoretical curve (reported as obtained from the impulse approximation) which they have presented for comparison with their experimental values, is not, however, correct. Their experimental values are in excellent agreement with the calculated values of ours but are in complete discordance with those of Bransden and Moorhouse.

Thus our investigation indicates that the impulse approximation is a valid approximation in the low (85 Mev) and as well as high energies (300 Mev) but seems to fail in the neighbourhood of the pion-nucleon resonance. Unfortunately there are no experimental data at energies of about 200 Mev (i.e. in the neighbourhood of resonance). The continuance of experimental investigations in this energy range is strongly suggested for it is hoped that these experiments will clearly indicate the range of validity of the impulse approximation.

The impulse approximation has also been used for analysing the various other nuclear processes and hence the investigation of the energy range in which the impulse approximation is valid assumes a greater importance.

2. General considerations

The following processes viz., elastic, inelastic and charge exchange scattering of positive pions by deuterons are considered.

$$(i) \quad \pi^+ + D \rightarrow \pi^+ + D$$

$$(ii) \quad \pi^+ + D \rightarrow \pi^+ + p + n$$

$$(iii) \quad \pi^+ + D \rightarrow \pi^0 + p + p$$

The matrix element for anyone of the above processes can be written using the impulse approximation as

$$Q = \langle f | T^{(1)} e^{i\vec{k}\cdot\vec{r}_1} + T^{(2)} e^{i\vec{k}\cdot\vec{r}_2} | i \rangle \quad (1)$$

where $|i\rangle$ is the initial state represented by

$$|i\rangle = 2^{-1/2} (p(1)n(2) - p(2)n(1))^3 \chi_m u_d(\vec{p}) (2\pi)^{-3/2} \quad (2)$$

and $|f\rangle$, the final state which is different for different processes. For process (i), the final state is

$$|f\rangle = 2^{-1/2} (p(1)n(2) - p(2)n(1))^3 \chi_{m'} u_d(\vec{p}) e^{i\vec{k}\cdot\vec{R}} (2\pi)^{-3/2} \quad (3)$$

For process (ii), the following are the possible final states:

Charge singlet

$$|f_e\rangle_s = 2^{-1/2} (p(1)n(2) - p(2)n(1))^3 \chi_{m'} u_{f_s} e^{i(\vec{k}_0\cdot\vec{p})} e^{i\vec{k}\cdot\vec{R}} (2\pi)^{-3/2} \quad (4a)$$

$$|f_o\rangle_s = 2^{-1/2} (p(1)n(2) - p(2)n(1))^3 \chi_o u_{f_s} e^{i(\vec{k}_0\cdot\vec{p})} e^{i\vec{k}\cdot\vec{R}} (2\pi)^{-3/2} \quad (4b)$$

Charge triplet

$$|f_e\rangle_T = 2^{-1/2} (p(1)n(2) + p(2)n(1))^3 \chi_o u_{f_s} e^{i(\vec{k}_0\cdot\vec{p})} e^{i\vec{k}\cdot\vec{R}} (2\pi)^{-3/2} \quad (4c)$$

$$|f_o\rangle_T = 2^{-1/2} (p(1)n(2) + p(2)n(1))^3 \chi_{m'} u_{f_s} e^{i(\vec{k}_0\cdot\vec{p})} e^{i\vec{k}\cdot\vec{R}} (2\pi)^{-3/2} \quad (4d)$$

For process (iii), only the charge triplet final states are possible.

$$|f_e\rangle_T = (2\pi)^{-3/2} p^{(1)} p^{(2)} {}^1\chi_0 u_{f,e}(\vec{k}_0, \vec{p}) e^{i\vec{k}\cdot\vec{R}} \quad (5a)$$

$$|f_o\rangle_T = (2\pi)^{-3/2} p^{(1)} p^{(2)} {}^3\chi_m u_{f,o}(\vec{k}_0, \vec{p}) e^{i\vec{k}\cdot\vec{R}} \quad (5b)$$

The subscripts e and o refer to even and odd spatial parts of the wave function respectively, and S and T denote the charge-singlet and charge-triplet states. In the above expressions, \vec{r}_1 and \vec{r}_2 refer to the position coordinates of the nucleons, and $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$ and $\vec{p} = \vec{r}_1 - \vec{r}_2$ the centre of mass and relative coordinates of the two-nucleon system. \vec{k} denotes momentum transfer to the target nucleon and is the difference between the momenta of the incident and the scattered pion and \vec{k}_0 represents the relative momentum of the two nucleons in the final state. $T^{(j)}$ is the transition operator for the scattering of pions by the j^{th} nucleon and is taken from the theory of Chew et al.

3. Differential crosssections

(a) Elastic scattering

The transition operator $T^{(j)}$ in (1) can be written as

$$\begin{aligned} T^{(j)} &= t_p^{(j)} \frac{1 + \tau_3^{(j)}}{2} + t_n^{(j)} \frac{1 - \tau_3^{(j)}}{2} \\ &= \frac{t_p^{(j)} + t_n^{(j)}}{2} + \frac{t_p^{(j)} - t_n^{(j)}}{2} \tau_3^{(j)} \end{aligned} \quad (6)$$

$\frac{1+\tau_3^{(j)}}{2}$ and $\frac{1-\tau_3^{(j)}}{2}$ are the projection operators for the proton and the neutron respectively. The τ_3 part in (6) does not connect the charge singlet states and the matrix element for the elastic scattering (process (ii)) after separating the spatial parts and performing the isotopic spin operation reduces to

$$Q = \left\langle {}^3\chi_{m'} \left| \frac{t_p^{(1)} + t_n^{(1)}}{2} + \frac{t_p^{(2)} + t_n^{(2)}}{2} \right| {}^3\chi_m \right\rangle E \quad (7)$$

where

$$E = \int |u_d(\vec{p})|^2 e^{i \frac{\hbar}{2} \cdot \vec{p}} d\vec{p} \quad (8)$$

t_p and t_n represent the amplitude for direct scattering of π^+ by proton and neutron. These amplitudes have been obtained by Chew et al and discussed in the preceding chapter. They can be written in the form

$$t_p = a + b(\sigma^+ - \sigma^-) \quad (9a)$$

$$t_n = c + d(\sigma^+ - \sigma^-) \quad (9b)$$

where a and b are the coefficients involving the pion energy, its momentum, angle of scattering and scattering phase shifts and they are defined in Chapter III.

We choose for our frame of reference the direction of the incident pion as Z -axis, and the plane containing the momentum vectors of the incident and scattered pions as $X-Z$ plane. Let θ be the angle between the two momentum vectors.



Squaring, summing over final states and averaging over initial states, we obtain

$$|Q|^2 = \left\{ |a+c|^2 + \frac{2}{3} |b+d|^2 \right\} |E|^2 \quad (10)$$

where

$$|a+c|^2 = \frac{4\pi^2}{9\omega^2 q^2} \cos^2 \theta \left[\left(4 \sin 2\delta_{33} + 4 \sin 2\delta_{31} + \sin 2\delta_{11} \right)^2 + \left(8 \sin^2 \delta_{33} + 8 \sin^2 \delta_{31} + 2 \sin^2 \delta_{11} \right)^2 \right]$$

$$|b+d|^2 = \frac{16\pi^2}{\omega^2 q^2} \sin^2 \theta \left[\left(\sin 2\delta_{33} - \frac{1}{2} \sin 2\delta_{31} - \frac{1}{2} \sin 2\delta_{11} \right)^2 + \left(2 \sin^2 \delta_{33} - \sin^2 \delta_{31} - \sin^2 \delta_{11} \right)^2 \right] \quad (11)$$

The overlap integral E can be evaluated using the Hulthen wave function for the deuteron

$$u_d(\vec{p}) = \left[\frac{\alpha}{2\pi(1-\alpha p_1)} \right]^{1/2} \frac{e^{-\alpha r} - e^{-\beta p}}{p} \quad (12)$$

where $\alpha^2 = M E$, $p_1 = 1.74 \times 10^{-13}$ cm, the effective triplet scattering range and β is given by

$$\frac{3}{\beta} = p_1 \left[1 + \frac{4}{9} \alpha p_1 \right] \quad (13)$$

M is the mass of the nucleon and E , the binding energy of the deuteron. Using (12) in (8), we obtain

$$E = \frac{1}{1-\alpha\rho_1} \frac{4\alpha}{k} \left\{ \tan^{-1}\left(\frac{k}{4\alpha}\right) + \tan^{-1}\left(\frac{k}{4\beta}\right) - 2 \tan^{-1}\left(\frac{k}{2(\alpha+\beta)}\right) \right\} \quad (14)$$

It may be mentioned that in the system of units ($\hbar = c = \text{pion mass} = 1$) that we use, the constants α , β and ρ_1 assume the following values:

$$\alpha = 0.3274; \quad \beta = 2.068 \quad \text{and} \quad \rho_1 = 1.231 \quad (15)$$

The overlap integrals have been evaluated at various angles and energies and they are tabulated in Table I.

The differential cross-section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} = (2\pi)^{-2} \frac{q\omega}{v} |Q|^2 \quad (16)$$

where ω is the energy of the pion, q its momentum and v the velocity of the incident pion. If we take into account only the dominant phase shift δ_{33} and neglect others, the expression for the cross section assumes a very simple form.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} = \frac{32}{27q^2} \sin^2 \delta_{33} (1 + 5 \cos^2 \theta) |E|^2 \quad (17)$$

In the above, we have made the justifiable assumption that the recoiling deuteron receives only momentum but no appreciable energy.

(b) Inelastic scattering

In the case of inelastic scattering (process ii) four different final states enumerated in sec. 2 are possible; and so the square of the matrix element is a sum of four terms i.e.

$$|Q|^2 = |Q_e|_S^2 + |Q_o|_S^2 + |Q_e|_T^2 + |Q_o|_T^2 \quad (18)$$

where

$$|Q_e|_S = \langle {}^3\chi_{m'} | \frac{1}{2} (t_p^{(1)} + t_n^{(1)} + t_p^{(2)} + t_n^{(2)}) | {}^3\chi_m \rangle \mathcal{E} \quad (19 a)$$

$$|Q_o|_S = \langle {}^1\chi_o | \frac{1}{2} (t_p^{(1)} + t_n^{(1)} - t_p^{(2)} - t_n^{(2)}) | {}^3\chi_m \rangle 0 \quad (19 b)$$

$$|Q_e|_T = \langle {}^1\chi_o | \frac{1}{2} (t_p^{(1)} - t_n^{(1)} - t_p^{(2)} + t_n^{(2)}) | {}^3\chi_m \rangle \mathcal{E} \quad (19 c)$$

$$|Q_o|_T = \langle {}^3\chi_{m'} | \frac{1}{2} (t_p^{(1)} - t_n^{(1)} + t_p^{(2)} - t_n^{(2)}) | {}^3\chi_m \rangle 0 \quad (19 d)$$

In the above

$$\mathcal{E} = \int u_{f,e}^* (\vec{k}_o \cdot \vec{p}) \exp(i \frac{\vec{k}}{2} \cdot \vec{p}) u_d(\vec{p}) d\vec{p} \quad (20 a)$$

$$0 = \int u_{f,o}^* (\vec{k}_o \cdot \vec{p}) \exp(i \frac{\vec{k}}{2} \cdot \vec{p}) u_d(\vec{p}) d\vec{p} \quad (20 b)$$

\mathcal{E} and 0 are functions of \vec{k}_o , the relative momentum of the outgoing nucleons. The cross-section is given by

$$\frac{d\sigma}{d\vec{u}} = \frac{(2\pi)^{-2}}{v} \int |Q|^2 \delta(\mathcal{E} + \omega_2 + \frac{k^2/4 + k_o^2}{M} - \omega_1) d\vec{k} \quad (21)$$

where ω_1 and ω_2 are respectively the energies of the incident and the outgoing pion. The integration over \vec{k}_0 can easily be performed if we neglect the binding energy of the deuteron and invoke the closure approximation according to which all possible relative momentum of the final states of the two nucleons may be taken into account without considering the relative energy involved. This is a useful approximation which simplifies the numerical calculations enormously but the cross-sections we get are not for the inelastic scattering but for the sum of elastic and inelastic scattering. Since we have earlier calculated the cross-section for the elastic scattering separately, we can easily find the contribution from the inelastic scattering alone.

Neglecting the binding energy of the deuteron and invoking the closure approximation the integral $\int |Q|^2 d\vec{k}_0$ can easily be evaluated since

$$\begin{aligned} \int |\mathcal{E}|^2 d\vec{k}_0 &= \frac{1}{2}(1+I) \\ \int |O|^2 d\vec{k}_0 &= \frac{1}{2}(1-I) \end{aligned} \quad (22)$$

where

$$\begin{aligned} I &= \int \cos(k \cdot p) |u_d(\vec{p})|^2 d\vec{p} \\ &= \frac{1}{1-\alpha p} \cdot \frac{2\alpha}{k} \left[\tan^{-1}\left(\frac{k}{2\alpha}\right) + \tan^{-1}\left(\frac{k}{2\beta}\right) - 2 \tan^{-1}\left(\frac{k}{\alpha+\beta}\right) \right] \end{aligned} \quad (23)$$

Finally we obtain the differential cross section for the sum of elastic and inelastic scattering as

$$\left(\frac{d\sigma}{dn}\right)_{\text{elastic} + \text{inelastic}} = (2\pi)^{-2} \omega^2 \left[|a|^2 + |b|^2 + |c|^2 + |d|^2 + 2\text{I} \left\{ \text{Re} a \text{Re} c + \text{Im} a \text{Im} c + \frac{1}{3} (\text{Re} b \text{Re} d + \text{Im} b \text{Im} d) \right\} \right] \quad (24)$$

The quantities a , b , c and d are listed in Chapter
The differential cross section assumes a simple form if we
retain only the dominant phase shift δ_{33}

$$\left(\frac{d\sigma}{dn}\right)_{\text{elastic} + \text{inelastic}} = \frac{2}{9q^2} \sin^2 \delta_{33} \left\{ 5(1 + 3 \cos^2 \theta) + (1 + 11 \cos^2 \theta) \text{I} \right\} \quad (25)$$

(c) Charge exchange scattering

In the case of charge exchange scattering (process iii)
the transition operator in (1) is given by

$$T^{(j)} = t_c^{(j)} \tau_+^{(j)} \quad (26)$$

where τ_+ is the operator which changes the neutron into the
proton but yields zero when acting on the proton. t_c is the
amplitude for the charge exchange scattering which can be written
in the form

$$t_c = t(\pi^+ n \rightarrow \pi^0 p) = A + B(\sigma^+ - \sigma^-) \quad (27)$$

the quantities A and B having been defined in the preceding
chapter.

There are two possible final states (charge triplet states) and the square of the matrix element can be written as

$$|Q|^2 = |Q_e|^2 + |Q_o|^2 \quad (29)$$

where

$$Q_e = 2^{-1/2} \langle {}^1\chi_o | t_c^{(1)} - t_c^{(2)} | {}^3\chi_m \rangle |\epsilon| \quad (30 a)$$

$$Q_o = 2^{-1/2} \langle {}^3\chi_m | t_c^{(1)} + t_c^{(2)} | {}^3\chi_m \rangle |0| \quad (30 b)$$

Following the procedure adopted earlier, the differential cross section can be obtained using the closure approximation and neglecting the binding energy of the deuteron.

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{charge exchange}} = (2\pi)^{-2} \omega^2 \left[|A|^2 + |B|^2 - \frac{I}{3} \{ 3|A|^2 + |B|^2 \} \right] \quad (31)$$

The quantity I has already been defined in exp. (23). If we take into account only the dominant δ_{33} phase shift, the differential cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{charge exchange}} = \frac{2}{9q^2} \sin^2 \delta_{33} \left\{ 1 + 3 \cos^2 \theta - \frac{I}{3} (1 + 11 \cos^2 \theta) \right\} \quad (32)$$

It will be instructive to compare the cross-sections for the deuteron with the free nucleon cross sections

$$\left(\frac{d\sigma}{d\Omega}\right)_{\pi^+p \rightarrow \pi^+p} = (2\pi)^{-2} \omega^2 (|a|^2 + |b|^2) \quad (33)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\pi^+n \rightarrow \pi^+n} = (2\pi)^{-2} \omega^2 (|c|^2 + |d|^2) \quad (34)$$

$$\left(\frac{d\sigma}{d\nu}\right)_{\pi^+n \rightarrow \pi^0p} = (2\pi)^{-2} \omega^2 (|A|^2 + |B|^2) \quad (35)$$

We find that in the case of the sum of the elastic and inelastic scattering by deuteron, the cross-section is larger than the sum of the free-proton and the free-neutron cross-sections. In the case of charge exchange scattering, to which only the neutron contributes, we find that the cross-section is reduced by the presence of the other particle (proton). This effect is to be attributed to the Pauli exclusion principle. In the former case (elastic and inelastic scattering by deuteron) this effect is shrouded by the presence of the interference terms. The overlap integrals E and I are decreasing functions of both energy and angle as shown in Table I. Hence at high energies and at large angles, the differential cross-section for the process (i) and (ii) taken together, approaches the sum of the free-proton and the free-neutron cross-sections and for the process (iii) the cross-section approaches that of the charge-exchange scattering of π^+ by free neutrons.

4. Numerical results and discussion

The differential cross-sections for the elastic, inelastic and charge-exchange scattering of pions by deuterons have been calculated numerically using (16), (24) and (31) at various incident pion energies 85, 140, 195 and 250 Mev and also at 60 and 300 Mev

for the elastic process alone. It may be noted that the expressions (16), (24) and (31) have to be evaluated in the c.m. system of the pion-nucleon system (and not the pion-deuteron system), since in the impulse approximation, the incident pion "sees" only the individual nucleons and the interaction is described as the sum of the individual scattering amplitudes. The results are tabulated (Tables 2, 3 and 4) and they are given in the laboratory system to facilitate direct comparison with the experimental data. Only the dominant δ_{33} phase shifts have been taken into account in the numerical calculation and in the case of elastic scattering at 85 and 140 Mev, the effect of including the other p -wave phase shifts have also been studied and as shown in figures 2 and 4, the effect is to reduce the cross-sections slightly.

At pion energies 60, 85 and 300 Mev, there exists good agreement between the numerical results that we have obtained and the experimental data of Sachs et al, Rogers and Lederman and Dul'kova et al. Fig. 1 represents the differential cross section in laboratory system for the elastic scattering of π^+ by deuterons at 60 Mev along with the experimental points of Sachs et al. Figs. 2 and 3 correspond to the energy 85 Mev and the experimental points marked therein are those of Rogers and Lederman. The elastic scattering cross section is presented in Fig. 2 and the

sum of elastic and inelastic scattering cross section in fig. 3. The solid curves represent the theoretical values calculated using only the dominant δ_{33} phase shifts. Figs. 1, 2 and 3 depict a good agreement between the calculated and experimental values. However, in the case of charge exchange scattering, the values that we have obtained are much lower than those of Rogers and Lederman at large angles and that may be due to the neglect of S and ^{other} p -wave phase shifts.

The agreement at 140 Mev between theory and experiment seems to be poor and the large cross-sections at backward angles cannot be explained away by the on-the-energy shell multiple scattering effects and the inclusion of S -wave phase shifts and D state wave function. An exhaustive study of the elastic scattering at 142 Mev and the effect of various corrections has been made by Pendleton using the Form factor approximation; but his final result does not seem to differ much from the numerical values we have obtained by this simple approach.

In fig. 5, the calculated cross section (solid curve) for the elastic scattering at 300 Mev is presented together with the experimental results of Dul'kova et al. The agreement is excellent. However, Dul'kova et al have reported that their experimental results are in complete disagreement with the theoretical values obtained from the impulse approximation calculation. That conclusion is based on the *theor.* calculation reported herein (shown by the dotted line in fig.5). It may be observed that the values of

Branden and Moorhouse* are also in complete disagreement with the experimental results at 300 Mev and also with that of ours. We present in Table V, the three sets of values for the purpose of comparison. We obtain good agreement not only with regard to the angular distribution but also with the integrated cross-section (Table VI). Although we have considered the scattering of π^+ by deuterons, our results are essentially the same for the π^- scattering by deuterons and we do not distinguish between them.

Unfortunately there are no experimental data to compare with for the inelastic and charge-exchange scattering at 140 Mev and above. The recent experiments of Pewitt et al and that of Dul'kova et al are on the elastic scattering alone and that too at energies 140 and 300 Mev only. In fact, the experiments on the elastic scattering are the most important ones for testing the validity of the impulse approximation since a reliable calculation can be made in this case, the final state of the system being well defined. Our investigation seems to indicate that the impulse approximation is a valid approximation in the low (85 Mev) as well as high energies (300 Mev) but fails in the neighbourhood of the pion-nucleon resonance. This conclusion is based on the discordant

* Pendleton observes that Branden and Moorhouse "use an inconsistent static model which assumes that nucleons are infinitely massive within the deuteron but which gives the deuteron its correct mass." Their results agree well with the experimental data at 85 and 140 Mev but fails completely at 300 Mev, yielding too low values.

results that we obtain at 140 Mev. The continuance of experimental investigations at energies in the close neighbourhood of resonance (200 Mev) is strongly suggested for it is hoped that these experiments will clearly decide the issue and, if an conjecture is confirmed, will stimulate the theoretical investigation of the influence of the pion-nucleon resonance on the impulse approximation. One plausible explanation is that in the resonance region, the pion-nucleon forms a quasi-bound state and consequently the interaction time is much longer, thus invalidating the impulse assumption that the other nucleon is a spectator.

	1	1	0.7962	0.3125	0.3541	0.2770	0.2425	0.2313
	1	1	0.9280	0.7415	0.8027	0.5356	0.4772	0.4525
	1	1	0.7156	0.4277	0.3845	0.2895	0.2704	0.2525
	1	1	0.8394	0.4173	0.4825	0.3104	0.3208	0.3244
	1	1	0.5380	0.0797	0.1402	0.1120	0.0228	0.0075
	1	1	0.3042	0.3295	0.3707	0.7835	0.2481	0.2418
	1	1	0.4572	0.1975	0.2048	0.0635	0.0575	0.0541
	1	1	0.7834	0.3911	0.7907	0.2253	0.1807	0.1807
	1	1	0.6126	0.1154	0.0757	0.0457	0.0227	0.0238
	1	1	0.7020	0.8159	0.9370	0.1860	0.1541	0.1512

Table I

The overlap integrals E and I as a function of pion kx kinetic energy (lab.) and the lab. angle.

Pion kinetic energy (lab) in Mev.	Lab. angle	The differential cross section $\frac{d\sigma}{d\Omega}$ for the elastic							
		0	30	60	90	120	150	180	
60	E	1	0.9519	0.8110	0.6915	0.6146	0.5741	0.5619	
	I	1	0.7902	0.5105	0.3541	0.2770	0.2419	0.2313	
85	E	1	0.9200	0.7415	0.6027	0.5196	0.4772	0.4646	
	I	1	0.7158	0.4127	0.2665	0.1995	0.1704	0.1623	
140	E	1	0.8598	0.6179	0.4623	0.3786	0.3388	0.3264	
	I	1	0.5880	0.2797	0.1609	0.1128	0.0928	0.0875	
195	E	1	0.8002	0.5235	0.3709	0.2815	0.2461	0.2418	
	I	1	0.4922	0.1973	0.1068	0.0635	0.0573	0.0541	
250	E	1	0.7454	0.3911	0.2987	0.2283	0.1960	0.1867	
	I	1	0.4186	0.1194	0.0753	0.0493	0.0382	0.0355	
300	E	1	0.7020	0.3859	0.2520	0.1869	0.1591	0.1510	

Table II

The differential cross section $\frac{d\sigma}{d\Omega}$ for the elastic scattering of π^+ by deuterons in units of m.b./sterad

Pion kinetic energy in Lab. (MeV)	Lab. angle	0°	30°	60°	90°	120°	150°	180°
60		5.044	3.238	0.8351	0.3346	0.5808	0.8026	0.8742
85		12.62	7.568	1.747	0.6359	1.039	1.388	1.495
140		45.19	23.66	4.342	1.340	1.974	2.503	2.641
195		68.71	31.17	4.737	1.311	1.650	2.008	2.186
250		38.44	15.13	1.480	0.4756	0.6106	0.7131	0.7352
300		22.79	7.954	0.8543	0.2007	0.2426	0.2785	0.2852

Table III

The differential cross section $\frac{d\sigma}{d\Omega}$ for the sum of elastic and inelastic scattering of π^+ by deuterons in units of

m.b. / sterad

Pion kinetic energy in lab. (MeV)	Lab. angle	0°	30°	60°	90°	120°	150°	180°
85		12.62	8.104	2.844	1.674	2.928	4.263	4.751
140		45.19	27.54	9.754	5.853	10.08	14.63	16.31
195		68.71	40.18	14.43	8.794	14.99	21.82	24.34
250		38.44	21.75	7.857	4.884	8.333	12.08	13.47

Table IV

The differential cross section $\frac{d\sigma}{d\Omega}$ for the charge exchange scattering of π^+ by deuteron in units of

m.b. / sterad

Pion kinetic energy in lab. (MeV)	Lab. angle.	0°	30°	60°	90°	120°	150°	180°
	85		0	0.3865	0.3708	0.2826	0.4489	0.6488
140		0	1.878	1.471	1.058	1.740	2.530	2.829
195		0	3.417	2.371	1.644	2.760	3.990	4.460
250		0	2.153	1.397	.9315	1.563	2.277	2.544

Table V

The differential cross section $\frac{d\sigma}{d\Omega}$ in laboratory system for the elastic scattering of charged pions from deuterons at 300 Mev in units of mb/sterad

Laboratory angle	0°	30°	60°	90°	120°	150°	180°
B.M.	10.44	2.72	0.25	0.03	0.02	0.03	0.03
D.S.S. (Theoretical)	--	18.00	4.00	0	0	0	0
D.S.S. (Experimental)	--	7.5 ± 3	1 ± 0.5	1 ± 0.5	0.5 ± 0.25	0.25 ± 0.1	-
P.C.	22.79	7.954	0.8543	0.2007	0.2426	0.2785	-

B.M. = The theoretical values of Bransden and Moorhouse given in ref. 5 at energy 298 Mev.

D.S.S. (Theoretical) = The theoretical values at 300 Mev obtained on the impulse approximation as reported in ref. 13. The values are taken from the curve given for the purpose of comparison with their experimental results.

D.S.S. (Experimental) = The experimental values at 300 Mev taken from the experimental curve given in ref. 13.

P.C. = The present calculation at 300 Mev based on the impulse approximation using the Chew-Low amplitude for the scattering of pions from free nucleons.

Table VI

The integrated cross section for the elastic scattering of charged pions by deuterons at 300 Mev in the angular range $15^\circ - 170^\circ$ in the laboratory system in units of mb.

Experimental $\pi^+ D \rightarrow \pi^+ D$	Experimental $\pi^- D \rightarrow \pi^- D$	Theoretical
21 ± 6	14 ± 4	14

0° 30° 60° 90° 120° 150° 180°
 θ LAB
 FIG. 1. ELASTIC SCATTERING OF CHARGED PIONS BY
 D AT 300 MEV

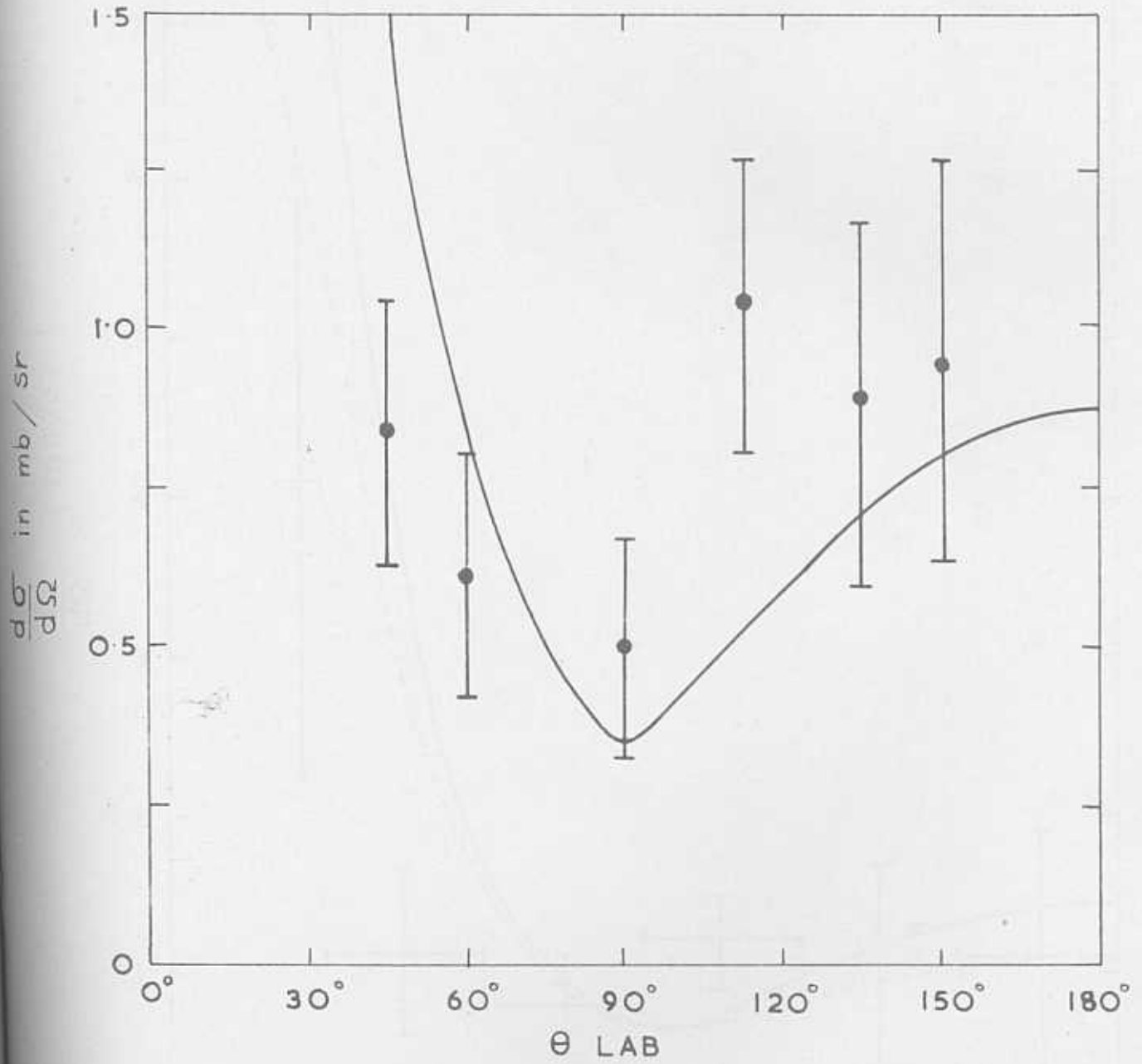


FIG. 1. ELASTIC SCATTERING OF CHARGED PIONS BY D AT 60 MEV.

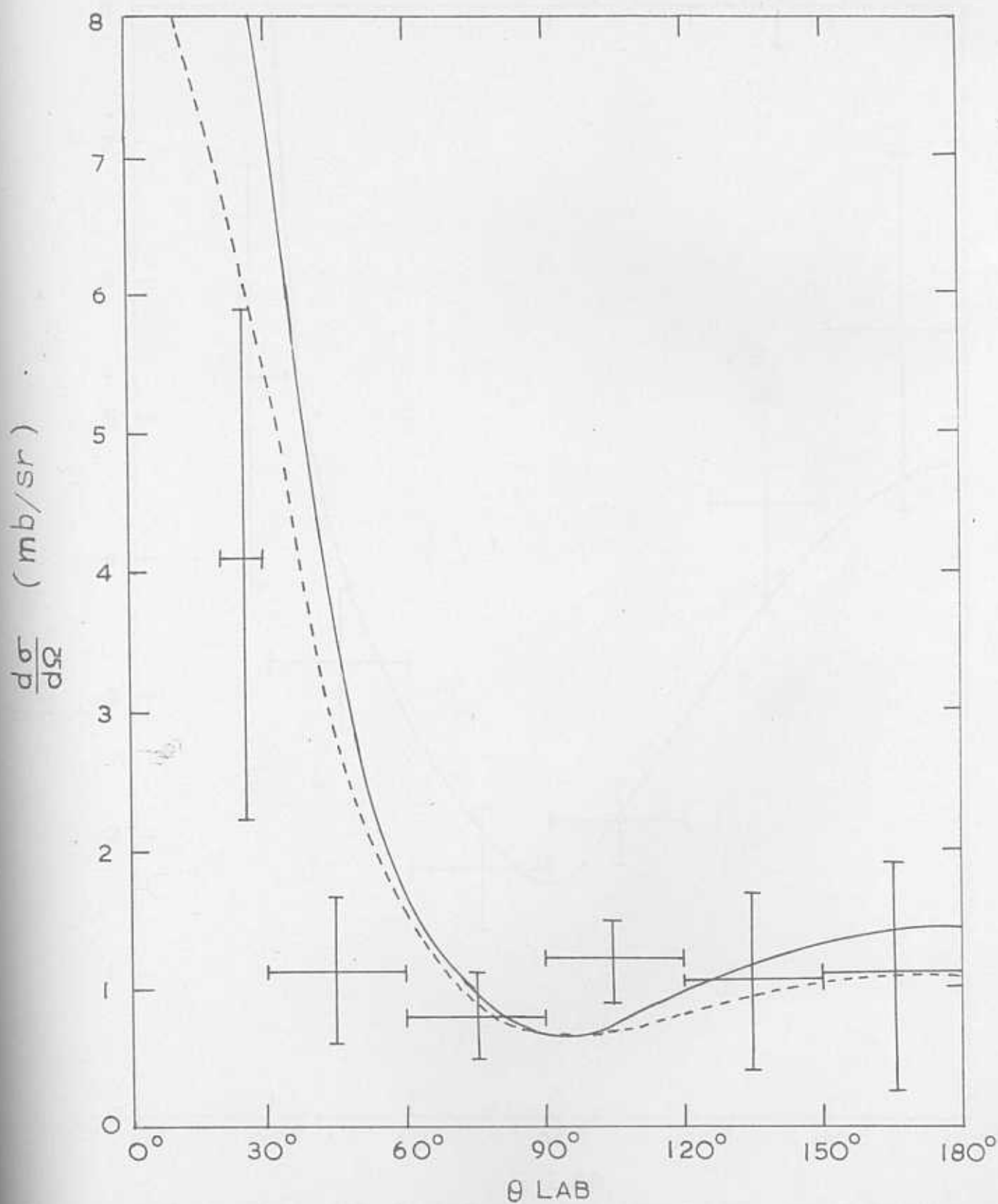


FIG. 2. ELASTIC SCATTERING OF π^+ BY D AT LAB ENERGY 85 MEV.

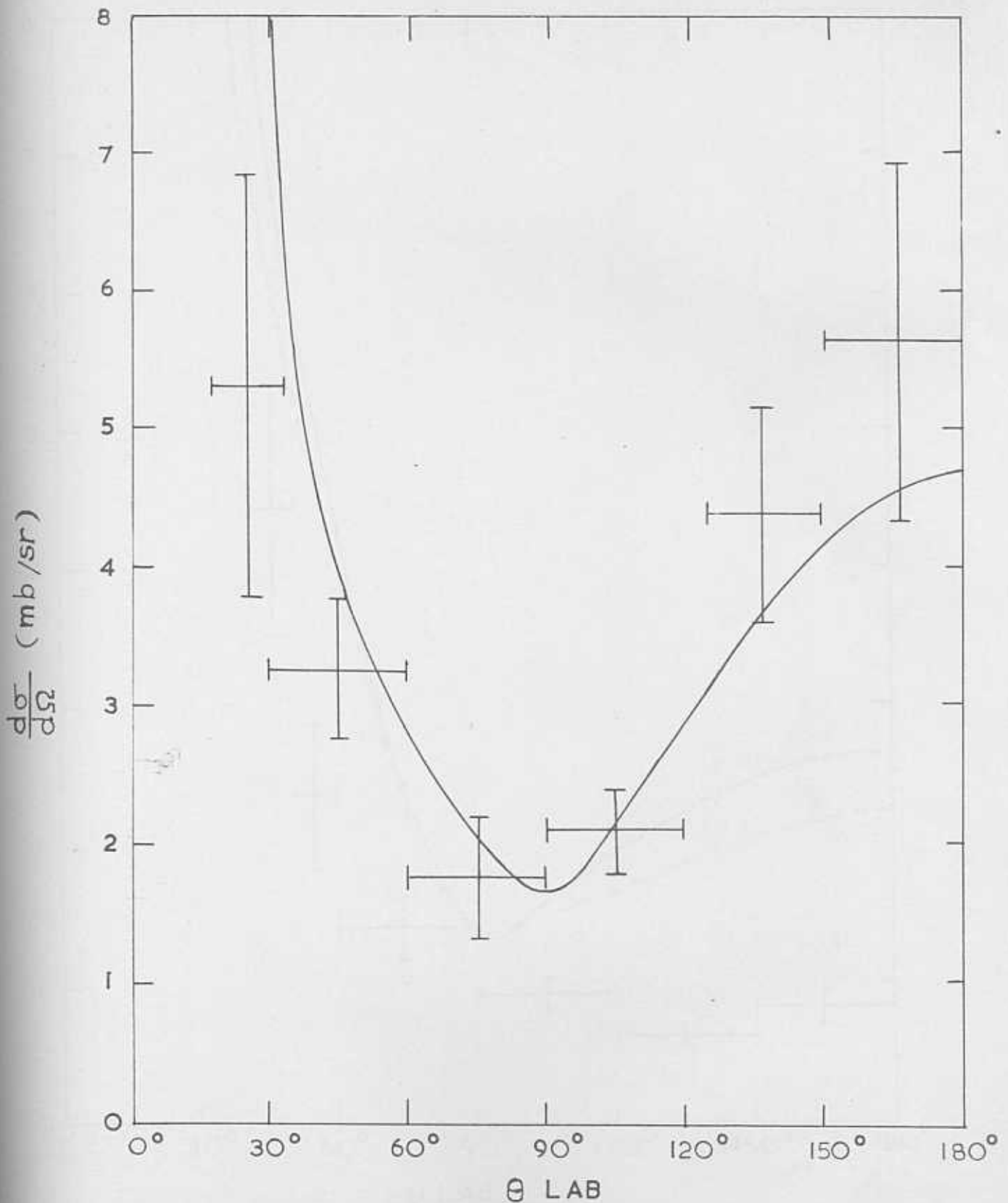


FIG. 3. SUM OF ELASTIC AND INELASTIC SCATTERING OF π^+ BY D AT LAB ENERGY 85 MEV.

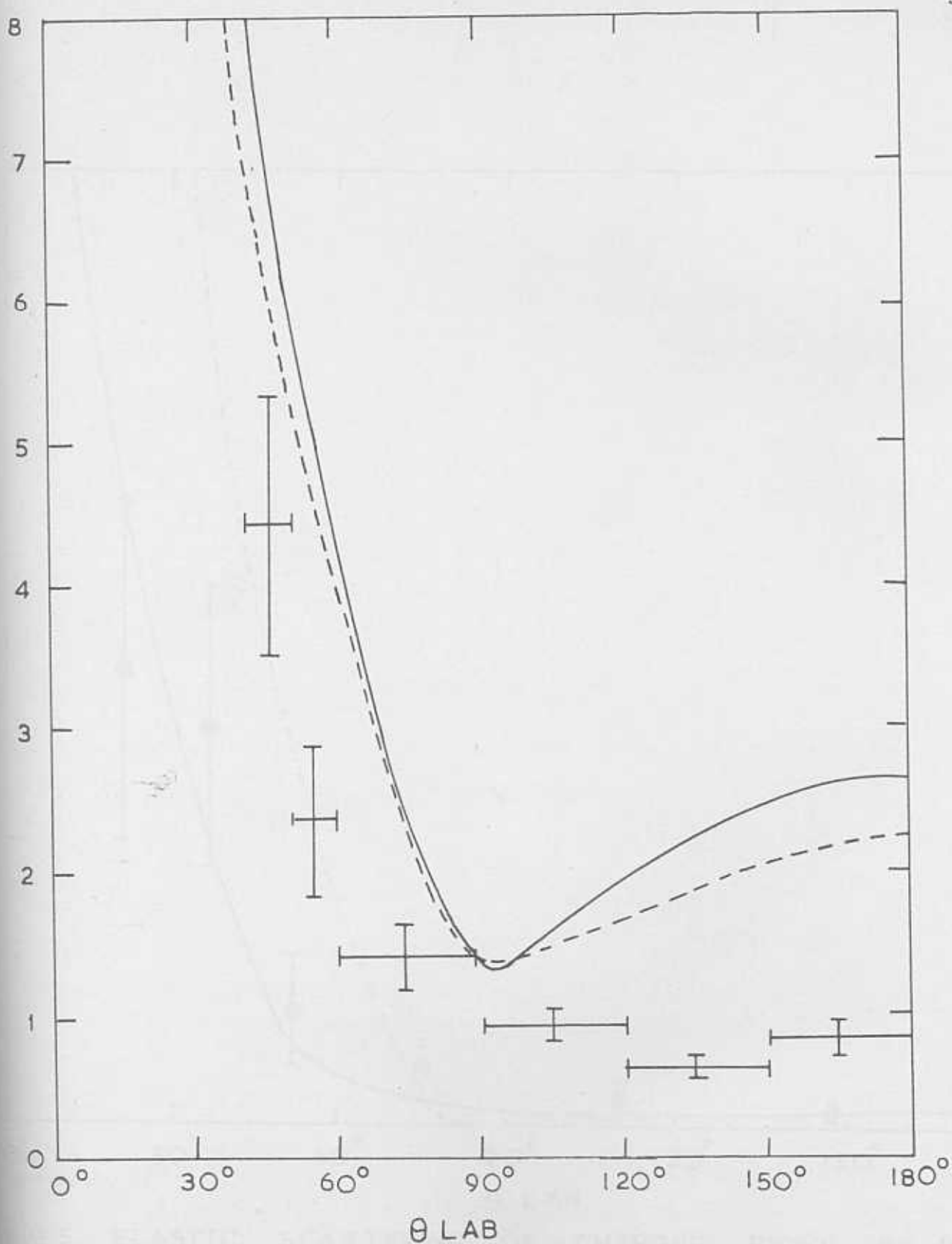


FIG. 4. ELASTIC SCATTERING OF π^+ BY D AT LAB ENERGY 140 MEV.

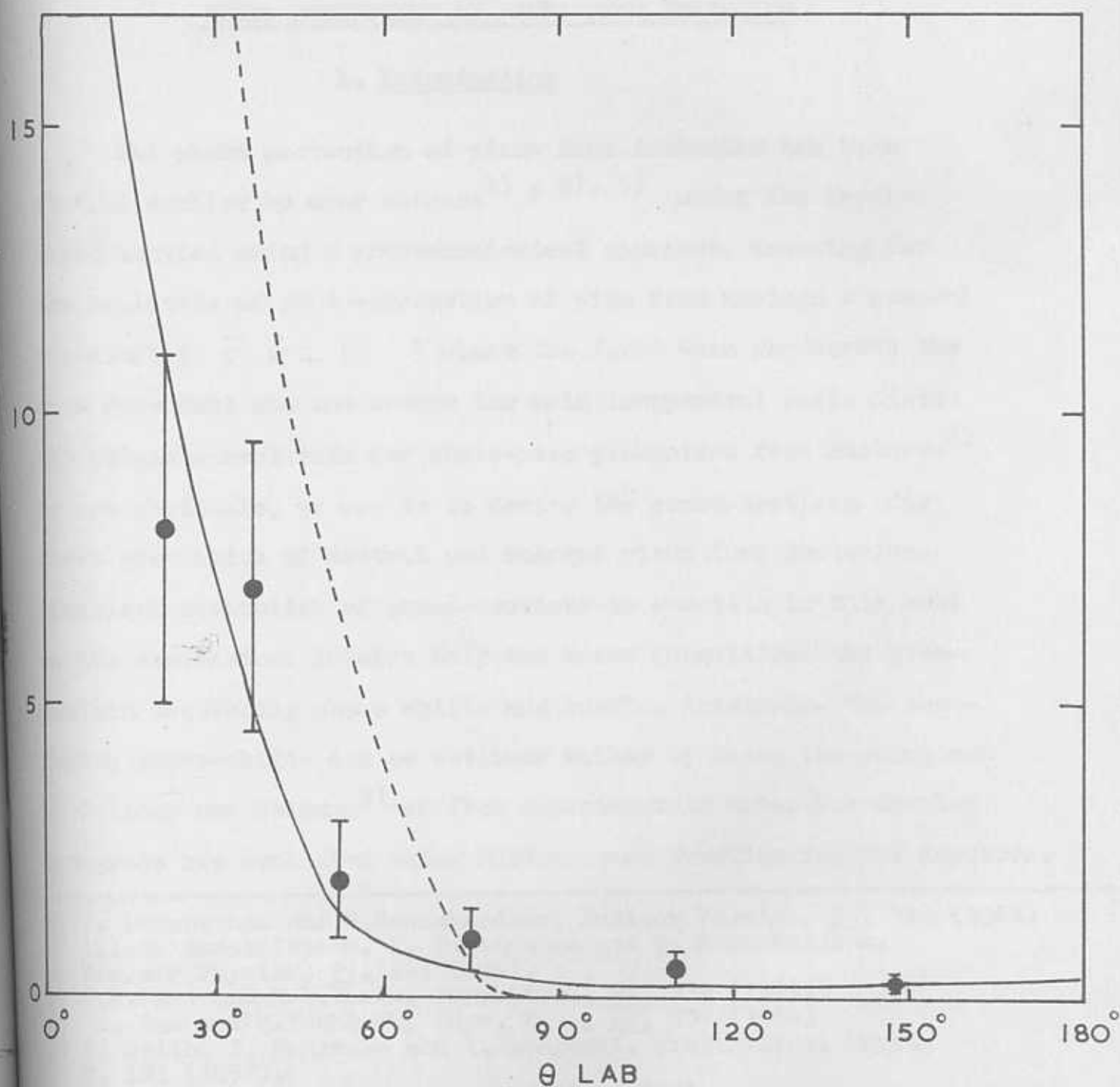


FIG.5. ELASTIC SCATTERING OF CHARGED PIONS BY D AT 300 MEV.

CHAPTER V

PHOTO PRODUCTION OF PIONS FROM DEUTERIUM*

1. Introduction

The photo production of pions from deuterium has been studied earlier by many authors^{1), 2), 3)} under the impulse approximation using a phenomenological approach, assuming for the amplitude of photo-production of pion from nucleon a general structure ($\vec{\sigma} \cdot \vec{K} + L$) where the first term represents the spin dependent and the second the spin independent part. Since the complete amplitude for photo-pion production from nucleons⁴⁾ is now available, we use it to derive the cross-sections for photo production of neutral and charged pions from deuterium. Numerical evaluation of cross-sections is possible in this case as the expressions involve only the known quantities: the pion-nucleon scattering phase shifts and overlap integrals. The scattering phase-shifts can be obtained either by using the solutions of Salzman and Salzman⁵⁾ or from experimental data. The overlap integrals are evaluated using Hulthen wave function for the deuteron.

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- * V. Devanathan and G. Ramachandran, Nuclear Physics, 23, 312 (1961)
 Alladi Ramakrishnan, V. Devanathan and G. Ramachandran,
 Nuclear Physics, 24, 163 (1961)
- 1) G.F.Chew and H.W.Lewis, Phys. Rev., 84, 779 (1951)
 2) M. Lax and H.Feshbach, Phys. Rev., 88, 509 (1952)
 3) Y. Saito, Y. Watanabe and Y. Yamaguchi, Prog. Theor. Phys.,
7, 103 (1952).
 John Chapplear, Phys. Rev., 99, 254 (1955).
 4) G.F.Chew, F.E.Low, M.L.Goldberger and Y. Nambu, Phys. Rev.
106, 1345 (1957).
 5) G.Salzman and F. Salzman, Phys. Rev., 108, 1619 (1957)

In section 2, we consider the photo production of neutral pions from deuterium, the 'elastic' process of which is of particular importance for it presents no uncertainty regarding the final state of the system. A reliable calculation is possible in this case since the spatial wave function of the deuteron is known and the complete amplitude for the photo production of π^0 from nucleons is given by Chew et al. The differential cross-sections for the elastic photo production of π^0 from deuterium are obtained at various photon energies 1.5, 2.0 and 2.5 in units of pion mass. A detailed comparison is made with the experimental results of Wolfe, Silverman and De Wire⁶⁾, Davis and Corson⁷⁾ and Rosengren and Baron⁸⁾.

As Chew and Lewis¹⁾ have pointed out, it would be of interest to study the final spin state of the recoiling deuteron in the case of 'elastic process'. The differential cross-sections for the various final spin states are also given separately in sec. 2.3 for the incident photon of energy 2 pion mass and it is found that they are nearly equal at large angles of pion emission but at forward angles $m_z = 0$ state predominates.

Section 3 is devoted to the charged pion production. The cross section for the meson spectrum is obtained using

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- 6) B. Wolfe, A. Silverman and J.W. DeWire, Phys. Rev., 99, 268 (1955).
 7) H.L. Davis and D.R. Corson, Phys. Rev., 99, 273 (1955)
 8) J.W. Rosengren and N. Baron, Phys. Rev., 101, 410 (1956)

the overlap integrals of Lax and Feshbach²⁾. The energy spectrum of mesons for a given production angle and also the angular distribution of mesons are obtained for the incident photon of energy 320 Mev and the agreement with the experimental results is satisfactory.

In evaluating the overlap integral, Lax and Feshbach²⁾ have taken into account the binding energy of the deuteron and also the relative kinetic energy of the two nucleons in the final state. But the calculations become very much simplified if the deuteron binding energy is neglected and the closure approximation is used for the integration of the momentum of the two nucleon system in the final state.

2. Photo production of neutral pions from deuterium

2.1. Evaluation of the differential cross section

Using the notation followed in the earlier chapter, the matrix element for the photo production of π^0 from deuterons may be written as

$$Q = \langle f | T^{(1)} \exp(i\vec{k}\cdot\vec{r}_1) + T^{(2)} \exp(i\vec{k}\cdot\vec{r}_2) | i \rangle \quad (1)$$

where $|i\rangle$ is the initial state represented by

$$|i\rangle = 2^{-1/2} \left\{ p^{(1)} n^{(2)} - p^{(2)} n^{(1)} \right\} \sqrt{3} \chi_m U_d(\vec{p}) (2\pi)^{-3/2} \quad (2)$$

and $|f\rangle$ the final state which may either be charge singlet or charge triplet. For the 'elastic' process $\gamma + D \rightarrow \pi^0 + D$, the final state is

$$|f\rangle = 2^{-1/2} \{ p(1) n(2) - p(2) n(1) \} {}^3\chi_m u_d(\vec{p}) \exp(i\vec{k}\cdot\vec{R}) (2\pi)^{-3/2} \quad (3)$$

For the 'inelastic' process $\gamma + D \rightarrow \pi^0 + p + n$,

we have the following final states:

(a) charge singlet

$$|f_e\rangle_s = 2^{-1/2} \{ p(1) n(2) - p(2) n(1) \} {}^3\chi_m u_{f,e}(\vec{k}_0, \vec{p}) \exp(i\vec{k}\cdot\vec{R}) (2\pi)^{-3/2} \quad (4a)$$

$$|f_o\rangle_s = 2^{-1/2} \{ p(1) n(2) - p(2) n(1) \} {}^1\chi_o u_{f,o}(\vec{k}_0, \vec{p}) \exp(i\vec{k}\cdot\vec{R}) (2\pi)^{-3/2} \quad (4b)$$

(b) charge triplet

$$|f_e\rangle_T = 2^{-1/2} \{ p(1) n(2) + p(2) n(1) \} {}^1\chi_o u_{f,e}(\vec{k}_0, \vec{p}) \exp(i\vec{k}\cdot\vec{R}) (2\pi)^{-3/2} \quad (5a)$$

$$|f_o\rangle_T = 2^{-1/2} \{ p(1) n(2) + p(2) n(1) \} {}^3\chi_m u_{f,o}(\vec{k}_0, \vec{p}) \exp(i\vec{k}\cdot\vec{R}) (2\pi)^{-3/2} \quad (5b)$$

The subscripts e and o refer to even and odd spatial parts of the wave function respectively. It can be shown that the contribution from the charge triplet final state is almost negligible since the Chew-Low amplitudes for the photo production of π^0 from proton and neutron are nearly equal. The major contribution to the cross-section comes from the charge singlet final state (both elastic and inelastic). We shall, however, deal in some detail the 'elastic process' wherein

the final state is well-defined and presents no uncertainties and for which the experimental data are available for comparison.

The transition operator $T^{(j)}$ in (1) can be written as

$$T^{(j)} = \frac{1}{2} t_p^{(j)} (1 + \tau_3^{(j)}) + \frac{1}{2} t_n^{(j)} (1 - \tau_3^{(j)}) \quad (6a)$$

or as

$$T^{(j)} = \frac{1}{2} (t_p^{(j)} + t_n^{(j)}) + \frac{1}{2} (t_p^{(j)} - t_n^{(j)}) \tau_3^{(j)} \quad (6b)$$

where t_p and t_n are the amplitudes for the photo-production of π^0 from proton and neutron and $\frac{1}{2}(1 + \tau_3)$ and $\frac{1}{2}(1 - \tau_3)$ the projection operators for the proton and neutron respectively. The τ_3 part in (5) does not connect the charge singlet states and the matrix element (1) for the elastic process ultimately reduces to

$$Q = \langle {}^3\chi_m | \frac{1}{2}(t_p^{(j)} + t_n^{(j)}) + \frac{1}{2}(t_p^{(j)} - t_n^{(j)}) \tau_3^{(j)} | {}^3\chi_m \rangle E \quad (7)$$

where

$$E = \int |u_d(\vec{p})|^2 \exp(i \frac{\vec{k}}{2} \cdot \vec{p}) d\vec{p} \quad (8)$$

$$\vec{k} = \vec{\omega} - \vec{\mu} \quad ; \quad \vec{p} = \vec{k}_1 - \vec{k}_2 \quad (9)$$

Here, $\vec{\omega}$ is the momentum of the incident photon and $\vec{\mu}$ the momentum of the outgoing meson.

We shall use t_p and t_n the complete amplitudes as given by Chew et al .

$$t_p = \frac{2\pi}{\sqrt{\mu_0 \omega_0}} e_f (F^+ + F^0) \quad (10)$$

$$t_n = \frac{2\pi}{\sqrt{\mu_0 \omega_0}} e_f (F^+ - F^0) \quad (11)$$

In the above, F^0 is a small term which we shall neglect and retaining only the dominant terms, F^+ can be written as

$$F^+ = i \vec{\sigma} \cdot \vec{u} \times (\vec{u} \times \vec{E}) \frac{g_p - g_n}{4Mf^2} h^{+-} + \vec{u} \cdot \vec{u} \times \vec{E} \frac{g_p - g_n}{4Mf^2} h^{++} \quad (12)$$

where

$$h^{+-} = -\frac{2}{3\mu^2} e^{i\delta_{33}} \sin \delta_{33}$$

$$h^{++} = \frac{4}{3\mu^3} e^{i\delta_{33}} \sin \delta_{33}$$

g_p , g_n are the magnetic moments of the proton and the neutron respectively, ω_0 is the energy of the incident photon,

E its polarisation, μ_0 is the energy of the outgoing meson, $\vec{\sigma}$ is the spin operator for the nucleons and δ_{33} is the phase shift for the meson-nucleon scattering in the (3.3) state.

The other phase shifts are neglected.

To evaluate the matrix element (7), we neglect the motion of the nucleons in the deuteron and choose for the operator t_p and t_n the expressions (10) and (11), which for a particular state of polarisation, say E_x , can be written as (neglecting the F_0 term)

$$t_p = t_n = a_x \sigma_+ + b_x \sigma_- + c_x \sigma_z + d_x \quad (13)$$

We choose for our frame of reference the direction of the incident photon as Z axis and the plane which contains the vectors \vec{v} and $\vec{\mu}$ as x-z plane. Let θ be the angle between \vec{v} and $\vec{\mu}$.

Squaring, summing over final states and averaging over initial spin states and photon polarisations, we obtain

$$|Q|^2 = \frac{4}{3} (a^2 + b^2 + 2c^2 + 2d^2) \quad (14)$$

where

$$|a|^2 = \frac{|a_x|^2 + |a_y|^2}{2}, \quad |b|^2 = \frac{|b_x|^2 + |b_y|^2}{2}$$

$$|c|^2 = \frac{|c_x|^2 + |c_y|^2}{2}, \quad |d|^2 = \frac{|d_x|^2 + |d_y|^2}{2} \quad (15)$$

Subscripts x and y refer to photon polarisations ϵ_x and ϵ_y .

Substituting the values of a, b, c and d, we finally obtain

$$|Q|^2 = \frac{32}{27\mu^4} c^2 \lambda^2 \omega^2 \sin^2 \delta_{33} \left(1 + \frac{5}{2} \sin^2 \theta\right) |E|^2 \quad (16)$$

where

$$c = \frac{2\pi e f}{\sqrt{\mu_0 \omega_0}}, \quad \lambda = \frac{g_p - g_n}{4Mf^2}$$

and E is given by (8). Using the Hulthen wave function for the deuteron, E can be evaluated and it is identical with the exp. (14) obtained in chapter IV.

$$E = \frac{1}{1 - \alpha \rho_1} \left(\frac{4\alpha}{k} \right) \left\{ \tan^{-1} \left(\frac{k}{4\alpha} \right) + \tan^{-1} \left(\frac{k}{4\beta} \right) - 2 \tan^{-1} \left(\frac{k}{2(\alpha + \beta)} \right) \right\} \quad (17)$$

The cross-section is given by

$$\frac{d\sigma}{d\vec{u}} = (2\pi)^{-2} |Q|^2 \delta\left(\mu_0 + \frac{|\vec{D} - \vec{u}|^2}{M} - \omega_0\right) \quad (18)$$

or

$$\frac{d\sigma}{d\mu_0 d\mu} = (2\pi)^{-2} |Q|^2 \mu \mu_0 \delta\left(\mu_0 + \frac{|\vec{D} - \vec{u}|^2}{M} - \omega_0\right) \quad (19)$$

The integration over $d\mu_0$ can easily be performed if we make the justifiable assumption that the recoiling deuteron receives only momentum but no appreciable energy. Then

$$\frac{d\sigma}{d\mu} = (2\pi)^{-2} \mu \mu_0 |Q|^2 \quad (20)$$

wherein $\mu_0 = \omega_0$, $\mu = \sqrt{\mu_0^2 - 1}$ and $|Q|^2$ is given by (16).

If the final state of the deuteron is not observed, we obtain the total cross section for the π^0 photoproduction which includes both the elastic and inelastic events. For this, we have to sum over the four different final states that are possible. However if $t_p = t_n$, the contribution for the charge triplet final states vanishes since it is the τ_3 part of T that connects the charge singlet to the charge triplet state.

The differential cross-section for the total π^0 photoproduction from the deuterons can be obtained easily neglecting the binding energy of the deuteron and using the closure approximation

$$\frac{d\sigma}{d\Omega} = (2\pi)^{-2} \mu \mu_0 \int |Q|^2 d\vec{k} \quad (21)$$

where

$$\int |Q|^2 d\vec{k} = \frac{8}{27\mu^4} \alpha^2 \lambda^2 k^2 \sin^2 \delta_{33} \left[3\left(1 + \frac{3}{2} \sin^2 \theta\right) + \left(1 + \frac{11}{2} \sin^2 \theta\right) I \right] \quad (22)$$

In the above, we have assumed that the entire energy of the photon is given to the meson ($\mu_0 = \mu$). The overlap integral I has been evaluated in the previous chapter (eq. (23) of Chapter IV).

$$I = \frac{1}{1-\alpha\beta} \cdot \frac{2\alpha}{k} \left\{ \tan^{-1} \frac{k}{2\alpha} + \tan^{-1} \frac{k}{2\beta} - 2 \tan^{-1} \frac{k}{\alpha+\beta} \right\} \quad (23)$$

2.2. Numerical results and comparison with experiments

Using (20) the differential cross-sections for the "elastic" process have been calculated at three photon energies 1.5, 2.0 and 2.5 in units of pion mass and the results are represented in fig. 1. For completeness we also give the numerical results in Table 1. Experimental results are available at photon energy of about 2 pion mass and they are plotted in fig. 2. The experimental values of Wolfe, Silverman and Dewire⁶⁾ are for photons of 50 Mev spread centred at 280 Mev and those of Davis and Corson⁷⁾ at 270 Mev. The results of Rosengren and Baron⁸⁾ are the latest and they were obtained for photons

Table I

The differential cross-section $\frac{d\sigma}{d\Omega}$ for the elastic photo production of π^0 from deuterons as a function of photon energy and laboratory angle of meson emission in $10^{-30} \text{ (cm)}^2 / \text{sterad}$

ν_0 \ θ_{Lab}	0°	30°	60°	90°	120°	150°	180°
1.5	1.155	1.481	2.346	1.109	0.5961	0.2568	0.1439
2.0	11.44	11.50	8.549	4.642	2.055	0.8044	0.4360
2.5	21.18	16.66	6.198	4.085	1.028	0.5474	0.2912

Table II

The differential cross-section $\frac{d\sigma}{d\Omega}$ for the elastic photo production of π^0 from deuterons at photon energy 2 pion mass as a function of final spin state of the deuteron and laboratory angle of mass emission in $10^{-30} \text{ (cm)}^2 / \text{sterad}$

θ_{Lab} m_z	0°	30°	60°	90°	120°	150°	180°
1	2.861	3.538	2.974	1.658	0.7183	0.2475	0.1090
0	5.722	4.423	2.602	1.326	0.6235	0.3094	0.2180
-1	2.861	3.538	2.974	1.658	0.7183	0.2475	0.1090

The subscripts $+1, 0, -1$ indicate the final m_z state of the deuteron. The numerical values for the differential cross-sections are obtained for the incident photon energy 2 pion mass and they are presented in Table I. Other values are also

of 30 Mev spread centred at about 280 Mev. The standard statistical errors are indicated in figure. They apply only to relative cross-section at various angles and the absolute cross-section is subjected to a further experimental error of 20 % . Curve 1 of fig. 2 represents the theoretical values obtained at photon energy of 2 pion mass. The general agreement of the theory with the experiments is good and especially the experiments with the results of Rosengren and Corson is remarkable.

2.3. Final spin state of the deuteron in the 'elastic' process

Formula (20) was obtained after summing over the three final spin states ($m_z = \pm 1, 0 - 1$) of the recoiling deuteron. It would of interest to study the differential cross-section for elastic π^0 production for each of the final spin states separately. The squares of the matrix elements are as follows:

$$|Q_{+1}|^2 = |Q_{-1}|^2 = \frac{8}{27\mu^4} c^2 \lambda^2 \omega^2 \sin^2 \delta_{33} (1 + 4 \sin^2 \theta) \quad (24)$$

$$|Q_0|^2 = \frac{16}{27\mu^4} c^2 \lambda^2 \omega^2 \sin^2 \delta_{33} (1 + \sin^2 \theta) \quad (25)$$

The subscripts + 1 , 0 , -1 indicate the final m_z state of the deuteron. The numerical values for the differential cross-sections are obtained for the incident photon of energy 2 pion mass and they are presented in Table 2 . Curves are also drawn

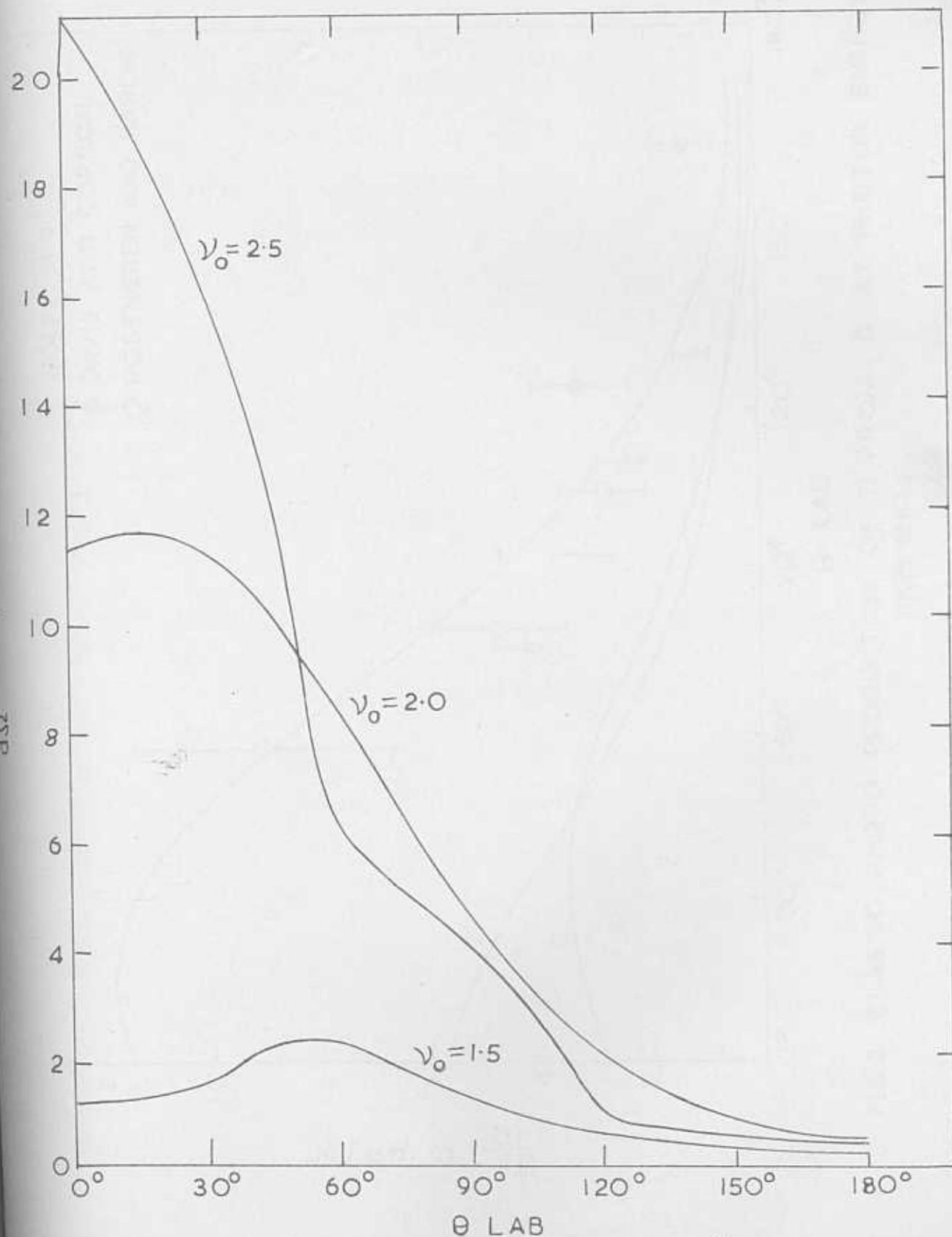


FIG. 1. ELASTIC PHOTO PRODUCTION OF π^0 FROM D AT PHOTON ENERGIES 1.5, 2.0 AND 2.5 IN PION MASS UNITS.

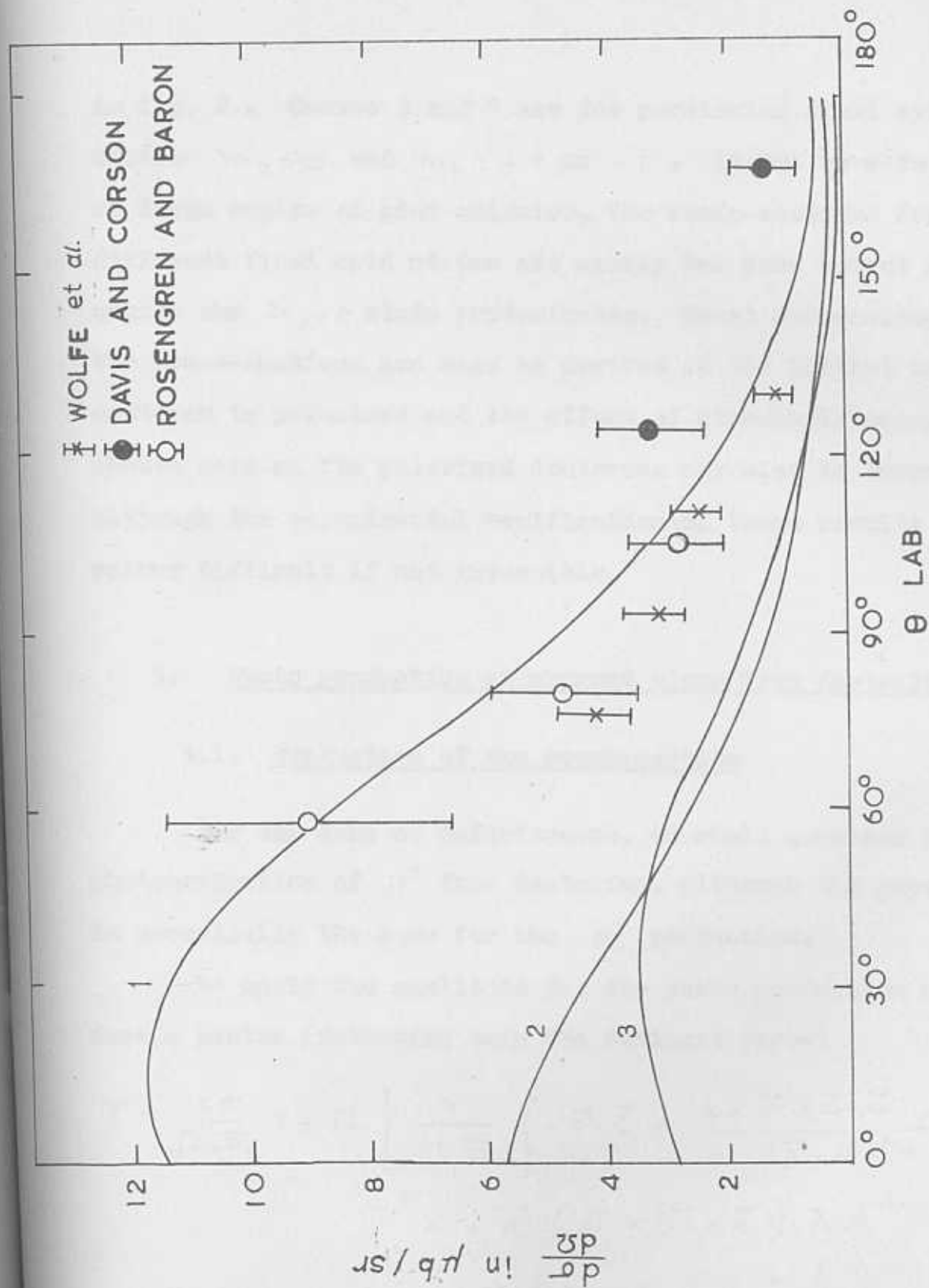


FIG. 2. ELASTIC PHOTO PRODUCTION OF π^0 FROM D AT PHOTON ENERGY 2 PION MASS.

in fig. 2 . Curves 2 and 3 are for particular final spin states $m_2 = 0$ and $m_2 = +1$ or -1 . It can be seen that at large angles of pion emission, the cross-sections for the different final spin states are nearly the same but at forward angles the $m_2 = 0$ state predominates. Exact expressions for the cross-sections can also be derived if the initial target deuteron is polarized and the effect of circularly polarised photon beam on the polarized deuterons may also be investigated; although the experimental verification of these results will be rather difficult if not impossible.

3. Photo production of charged pions from deuterium

3.1. Evaluation of the cross-section

For the sake of definiteness, we shall consider the photoproduction of π^+ from deuterium, although the procedure is essentially the same for the π^- production.

We apply the amplitude for the photo production of π^+ from a proton (retaining only the dominant terms)

$$T = \frac{2\pi}{\sqrt{\mu_0 v_0}} e_f \sqrt{2} \left[\frac{1}{1 + \frac{\mu_0}{M}} \left\{ i \vec{\sigma} \cdot \vec{\epsilon} + \frac{2i \vec{\sigma} \cdot (\vec{\Delta} - \vec{\mu}) \vec{\mu} \cdot \vec{\epsilon}}{(\vec{\Delta} - \vec{\mu})^2 + 1} \right\} \right. \\ \left. + i \vec{\sigma} \cdot (\vec{\mu} \times (\vec{\Delta} \times \vec{\epsilon})) \lambda h^{--} \right. \\ \left. + \vec{\mu} \cdot (\vec{\Delta} \times \vec{\epsilon}) \lambda h^{-+} \right] \quad (26)$$

to the problem of deuteron and follow essentially the approach of Chew and Lewis¹⁾. In the impulse approximation, the matrix element for the process can be written as

$$\langle f | T_1 e^{i(\vec{\omega}-\vec{\mu})\cdot\vec{R}_1} \tau_1^- + T_2 e^{i(\vec{\omega}-\vec{\mu})\cdot\vec{R}_2} \tau_2^- | i \rangle \quad (27)$$

where τ^- is the isotopic spin operator which converts the proton into neutron but yields zero when it acts on the neutron. The ~~initial~~ final state is that of the deuteron and the final state consists of two neutrons which may either be symmetric or antisymmetric in space

$$|f_e\rangle = (2\pi)^{-3/2} n(1)n(2) {}^1\chi_0 u_{f,e}(\vec{k}_0\cdot\vec{P}) e^{i(\vec{\omega}-\vec{\mu})\cdot\vec{R}} \quad (28 a)$$

$$|f_o\rangle = (2\pi)^{-3/2} n(1)n(2) {}^3\chi_m u_{f,o}(\vec{k}_0\cdot\vec{P}) e^{i(\vec{\omega}-\vec{\mu})\cdot\vec{R}} \quad (28 b)$$

In the above, \vec{k}_0 is the relative momentum of the two neutrons and \vec{P} and \vec{R} are the relative and centre of mass coordinates of the di-neutron system.

The matrix elements for the symmetric and antisymmetric cases reduce to

$$Q_e = 2^{-1/2} \langle {}^1\chi_0 | T_1 - T_2 | {}^3\chi_m \rangle \varepsilon \quad (29 a)$$

$$Q_o = 2^{-1/2} \langle {}^3\chi_m | T_1 + T_2 | {}^3\chi_m \rangle 0 \quad (29 b)$$

where

$$\mathcal{E} = \int u_{f,e}^* (\vec{k}_0 \cdot \vec{p}) \cos\left(\frac{1}{2} \vec{k}_0 \cdot \vec{p}\right) \left[\frac{u(p)}{p}\right] d\vec{p} \quad (30 a)$$

$$0 = \int u_{f,o}^* (\vec{k}_0 \cdot \vec{p}) \sin\left(\frac{1}{2} \vec{k}_0 \cdot \vec{p}\right) \left[\frac{u(p)}{p}\right] d\vec{p} \quad (30 b)$$

To evaluate the above matrix elements, we choose for the operator T the expression (26) which for a particular state of polarisation of the photon, say ϵ_x can be conveniently written in the form

$$T = A_x \sigma_+ + B_x \sigma_- + C_x \sigma_z + D_x \quad (31)$$

Squaring, summing over final states and averaging over initial spin states and photon polarisations, we obtain

$$|\langle {}^1\chi_0 | T_1 - T_2 | {}^3\chi_m \rangle|^2 = \frac{2}{3} (|A|^2 + |B|^2 + 2|C|^2) \quad (32 a)$$

$$|\langle {}^3\chi_m | T_1 + T_2 | {}^3\chi_m \rangle|^2 = \frac{4}{3} (|A|^2 + |B|^2 + 2|C|^2 + 3|D|^2) \quad (32 b)$$

$$\begin{aligned} |Q|^2 &= |Q_e|^2 + |Q_o|^2 \\ &= \frac{1}{3} (|A|^2 + |B|^2 + 2|C|^2) |\mathcal{E}|^2 \\ &\quad + \frac{2}{3} (|A|^2 + |B|^2 + 2|C|^2 + 3|D|^2) |O|^2 \end{aligned} \quad (33)$$

where

$$\begin{aligned} |A|^2 &= \frac{|A_x|^2 + |A_y|^2}{2}, & |B|^2 &= \frac{|B_x|^2 + |B_y|^2}{2} \\ |C|^2 &= \frac{|C_x|^2 + |C_y|^2}{2}, & |D|^2 &= \frac{|D_x|^2 + |D_y|^2}{2} \end{aligned} \quad (34)$$

Subscripts x and y correspond to photon polarisation

ϵ_x and ϵ_y . The cross-section is given by

$$\frac{d\sigma}{d\vec{u}d\vec{k}} = (2\pi)^{-2} |Q|^2 \delta\left(\epsilon + \mu_0 + \frac{k^2/4 + k_0^2}{M} - \omega_0\right) \quad (35 a)$$

or

$$\frac{d\sigma}{d\mu.d\nu} = (2\pi)^{-2} \mu \mu_0 \int |Q|^2 \delta\left(\epsilon + \mu_0 + \frac{k^2/4 + k_0^2}{M} - \omega_0\right) d\vec{k} \quad (35 b)$$

where ϵ is the binding energy of the deuteron (2.226 Mev) and $\frac{k_0^2}{M}$ is the final relative kinetic energy of the two neutrons in the final state.

$|E|^2$ and $|O|^2$ have been evaluated by Lax and Feshbach choosing the Hulthen function for the initial spacial wave function of the deuteron and

$$U_{f,e}(\vec{k}_0 \cdot \vec{P}) = (2\pi)^{-3/2} \cos(\vec{k}_0 \cdot \vec{P}) \quad (36 a)$$

$$U_{f,o}(\vec{k}_0 \cdot \vec{P}) = (2\pi)^{-3/2} \sin(\vec{k}_0 \cdot \vec{P}) \quad (36 b)$$

for the final states of the di-neutron.

Using the results of Lax and Feshbach²⁾, the cross-section for the meson spectrum can be written in the final form

$$\frac{d\sigma}{d\nu d\mu} = (2\pi)^{-2} \frac{\mu \mu_0}{2} \left[Z I_1 - \frac{Z+4D^2}{3} I_2 \right] \quad (37)$$

where

$$\begin{aligned}
 Z &= |A|^2 + |B|^2 + 2|C|^2 + 2|D|^2 \\
 &= \frac{8\pi^2 e^2 f^2}{\mu_0 \omega_0} \left[\frac{2}{\left(1 + \frac{\mu_0}{M}\right)^2} \left(1 - \frac{2\mu^2 \sin^2 \theta}{\{(\vec{\omega} - \vec{\mu})^2 + 1\}^2}\right) \right. \\
 &\quad + \frac{1}{1 + \frac{\mu_0}{M}} \frac{4\omega\lambda}{3\mu} \left(\frac{2\sin^2 \theta}{(\vec{\omega} - \vec{\mu})^2 + 1} - \frac{\cos \theta}{\mu} \right) \cos \delta_{33} \sin \delta_{33} \\
 &\quad \left. + \frac{2\omega^2 \lambda^2}{9\mu^4} \left(1 + \frac{3}{2} \sin^2 \theta\right) \sin^2 \delta_{33} \right] \quad (38)
 \end{aligned}$$

$$4D^2 = \frac{8\pi^2 e^2 f^2}{\mu_0 \omega_0} \left[\frac{8}{9\mu^4} \omega^2 \lambda^2 \sin^2 \theta \sin^2 \delta_{33} \right]$$

$$I_1 = \frac{2M\alpha k_R}{\pi(1-\alpha P_1)} \left[\frac{1}{P_\alpha^4 - k_R^2 k^2} + \frac{1}{P_\beta^4 - k_R^2 k^2} \right] \quad (39)$$

$$- \frac{1}{k_R k (\beta^2 - \alpha^2)} \ln \frac{(P_\beta^2 - k_R k)(P_\alpha^2 + k_R k)}{(P_\beta^2 + k_R k)(P_\alpha^2 - k_R k)} \quad (40)$$

$$I_2 = \frac{M\alpha}{\pi(1-\alpha P_1)} \frac{\beta^2 - \alpha^2}{k(P_\alpha^2 + P_\beta^2)} \left[\frac{1}{P_\alpha^2} \ln \frac{P_\alpha^2 + k_R k}{P_\alpha^2 - k_R k} \right] \quad (41)$$

$$- \frac{1}{P_\beta^2} \ln \frac{P_\beta^2 + k_R k}{P_\beta^2 - k_R k} \quad (42)$$

$$k_R^2 = M(\omega_0 - \mu_0 - \epsilon) - \frac{k^2}{4} \quad (43)$$

$$P_\alpha^2 = M(\omega_0 - \mu_0 - \epsilon) + \alpha^2 = M(\omega_0 - \mu_0) \quad (44)$$

$$P_\beta^2 = M(\omega_0 - \mu_0 - \epsilon) + \beta^2 = M(\omega_0 - \mu_0) + \beta^2 - \alpha^2 \quad (45)$$

3.2. Numerical Results and Comparison with experiments

Expression (37) for the meson spectrum has been evaluated numerically for an incident photon of energy 320 Mev at angles 0° , 45° , 90° , 135° and 180° and the results are given in the form of curves in fig. 1. The angular distribution of mesons is given by curve 1 of fig. 2.

The experimental results of White, Jacobson and Schultz⁹⁾ are included in fig. 2 for comparison. They have made measurements at three laboratory angles 45° , 90° and 135° using a "spread-out" bremsstrahlung photon beam of 318 ± 10 Mev and hence their results represented by points in fig. 2 are for an "equivalent quantum"^{*}. Curve 2 (e) is a smooth curve drawn by them through the experimental points by extrapolation and the curve 2 (b) is obtained from the curve 2 (a) after making corrections for nuclear interactions. It is to be noted that the curve 2 (e) does not include the uncertainties of experimental observations and so also does the curve 2 (b).

Robert H. Land¹⁰⁾ has studied the π^+ meson spectra at 120° using a monochromatic photon beam of 292 Mev and his results are given in arbitrary units. We reproduce in

* The number of equivalent quanta is defined as the total energy in the beam divided by the maximum photon energy.

9) R.H.White, M.J.Jacobson and A.G.Schultz, Phys. Rev. **88**, 836 (1952).

10) Robert H. Land, Phys. Rev., **113**, 1141 (1959).

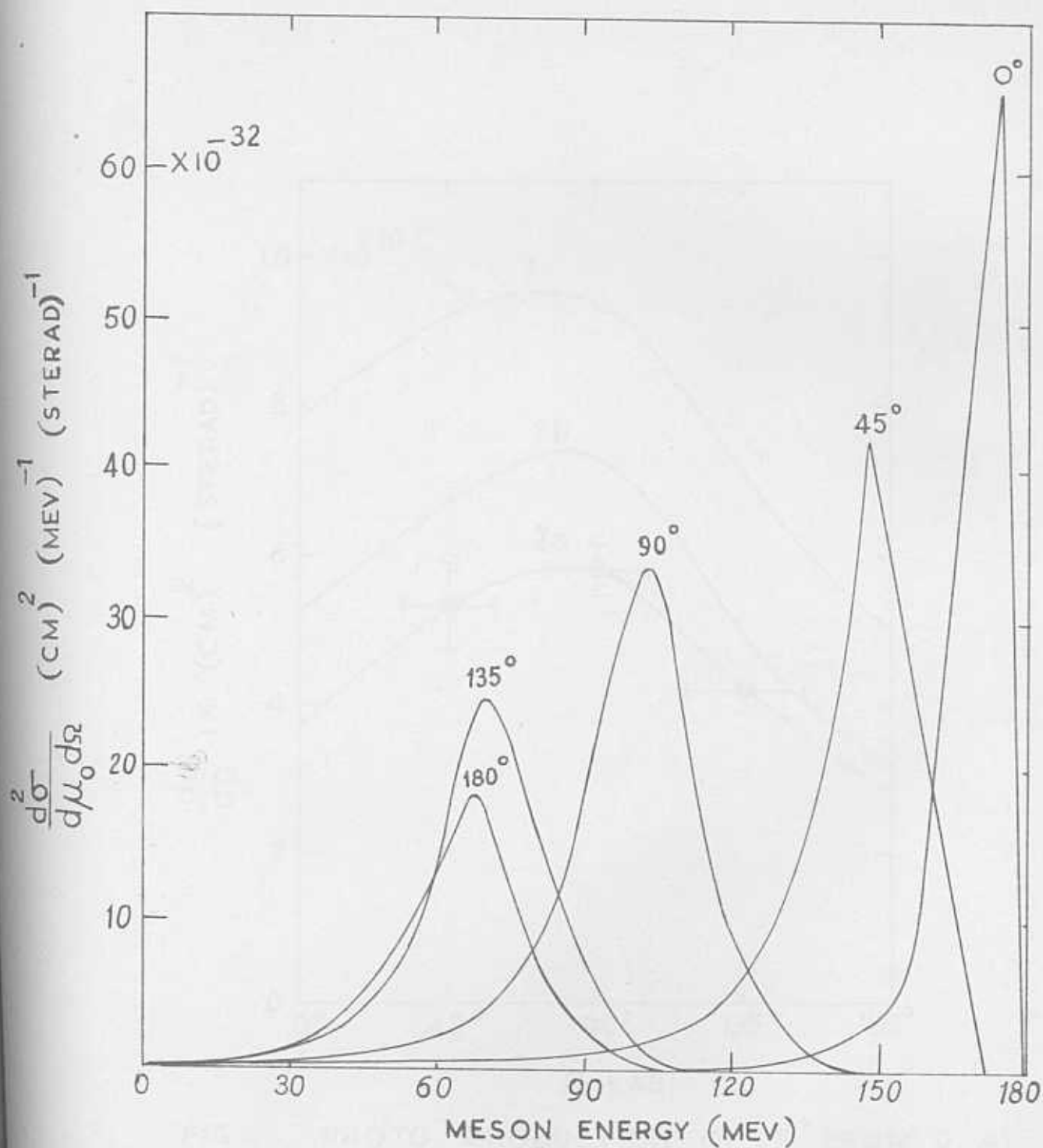


FIG. 1. ENERGY SPECTRA OF π^+ FROM DEUTERONS AT VARIOUS LAB ANGLES FOR INCIDENT PHOTON OF 320 MEV.

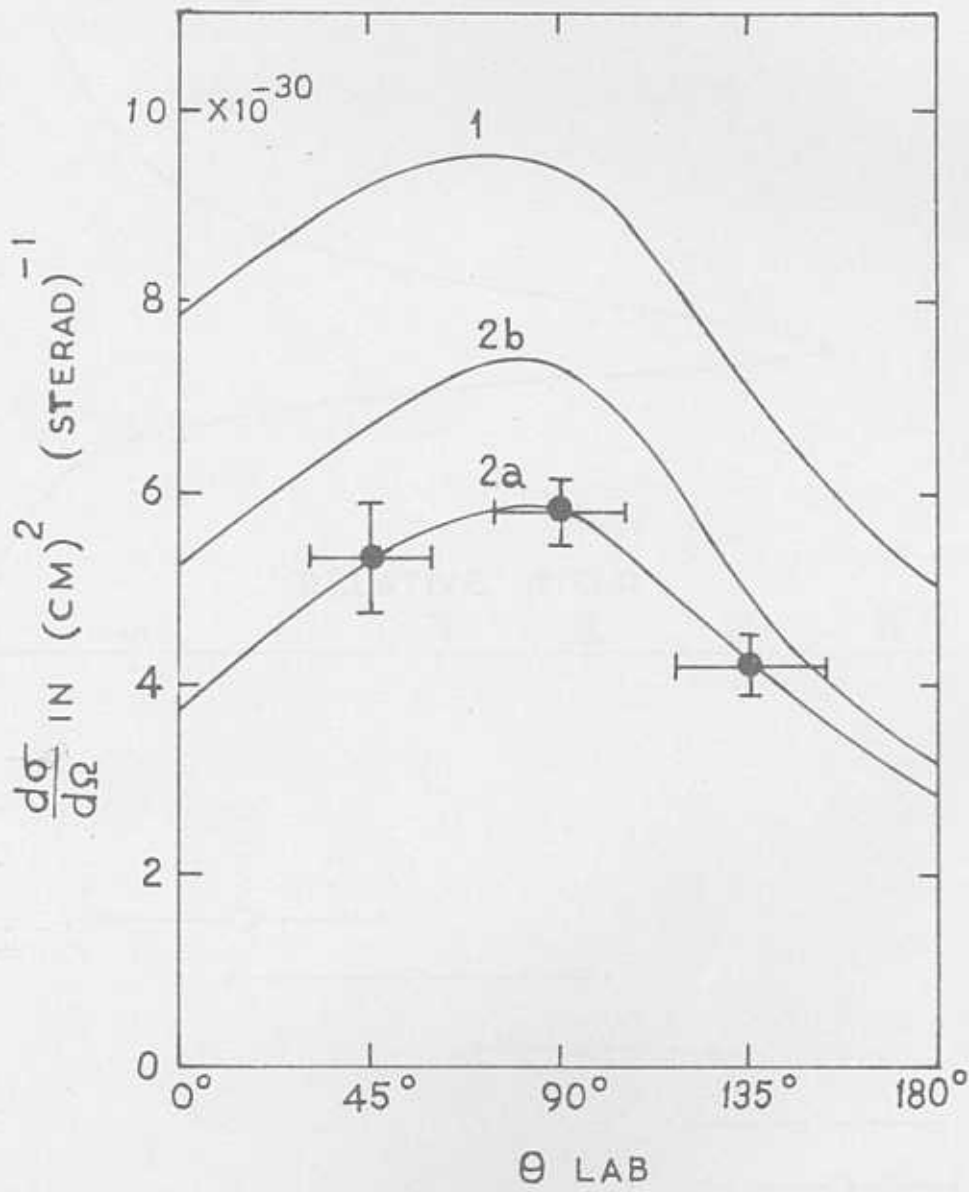


FIG. 2 . PHOTO PRODUCTION OF π^+ FROM D AT PHOTON ENERGY 320 MEV.

$$\frac{d^2\sigma}{d\mu_0 d\Omega} \text{ (CM)}^2 \text{ (MEV)}^{-1} \text{ (STERAD)}^{-1}$$

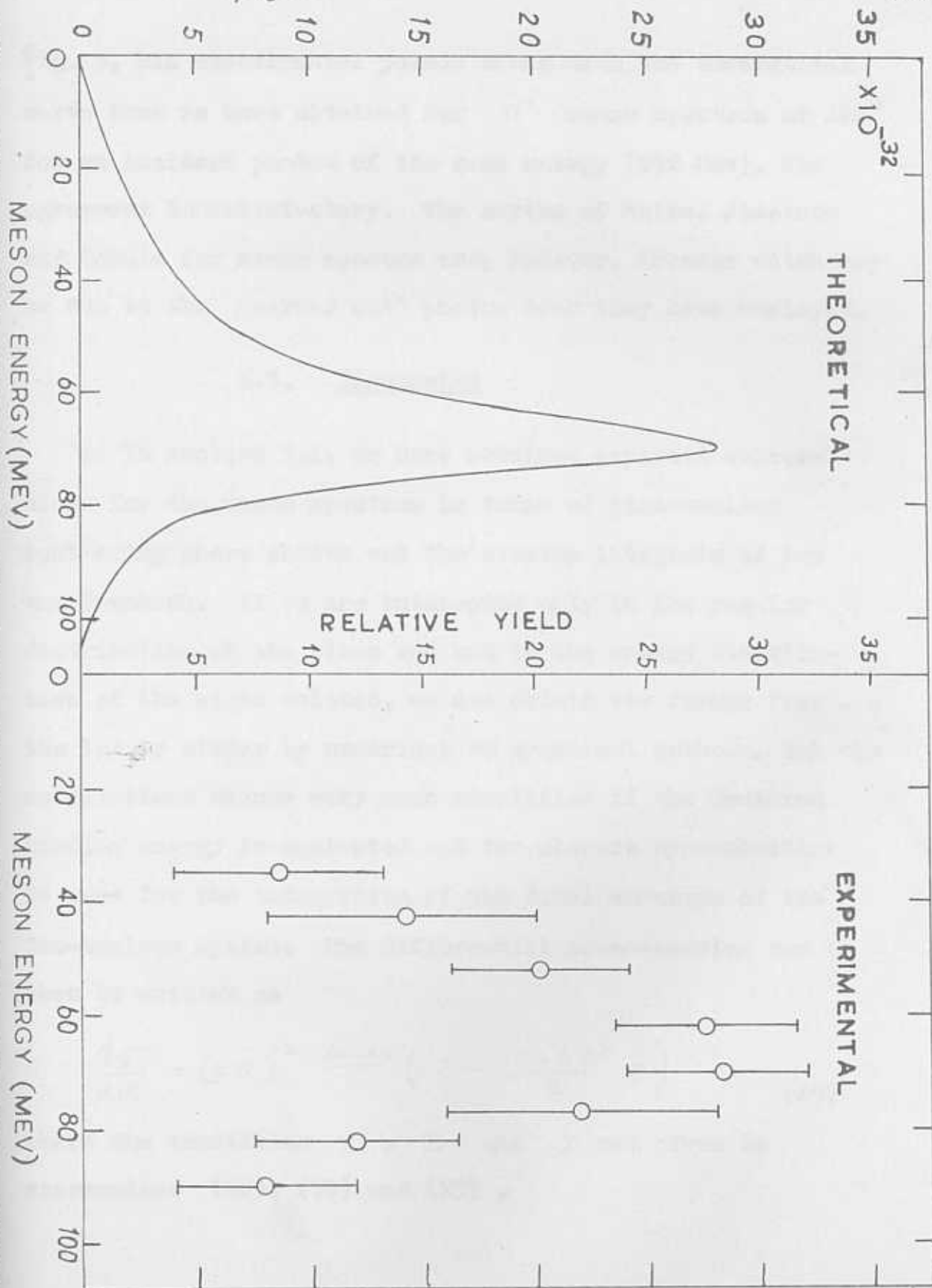


FIG.3. THE MESON SPECTRUM AT LAB ANGLE 120° FOR INCIDENT PHOTON OF 292 MEV.

Fig. 3, his experimental points along with the theoretical curve that we have obtained for π^+ meson spectrum at 120° for an incident photon of the same energy (292 Mev). The agreement is satisfactory. The curves of White, Jacobson and Schulz for meson spectra are, however, broader which may be due to the "spread out" photon beam they have employed.

3.3. Discussion

In section 3.1, we have obtained explicit expressions for the meson spectrum in terms of pion-nucleon scattering phase shifts and the overlap integrals of Lax and Feshbach. If we are interested only in the angular distribution of the pions and not in the energy distribution of the pions emitted, we can obtain the former from the latter either by numerical or graphical methods. But the calculations become very much simplified if the deuteron binding energy is neglected and the closure approximation is used for the integration of the final momentum of the two-nucleon system. The differential cross-section can then be written as

$$\frac{d\sigma}{d\Omega} = (2\pi)^{-2} \frac{\mu\mu_0}{2} \left(Z - \frac{Z+4D^2}{3} I \right) \quad (45)$$

where the quantities Z , D and I are given in expressions (38), (39) and (23).

It will be instructive to compare the deuteron cross-section with the single nucleon cross section viz.,

$$\frac{d\sigma}{d\Omega} = (2\pi)^{-2} \frac{\mu \mu_0}{2} Z \quad (46)$$

The reduction in the cross section is to be attributed to the Pauli principle. It may, however, be pointed out that we have not taken into account the final state interactions of the two outgoing nucleons. This, in principle, may be studied by analysing the final state of the two nucleons into partial waves which are distorted by the nucleon-nucleon phase shifts.

in which only the other nucleon... the final one which we shall call the... special interest since it... final state of the... calculation. Such experiments... nuclei are...

* T. Darden and G. K... also presented at the... (1957).

1) E.F. ... and T.F. ... 2) E.F. ... and G. ...

Chapter VIPhotoproduction of charged pions from Nuclei - I*
on the Magnetic Moment term.1. Introduction

In earlier chapters, we have discussed the photo-production of charged and neutral pions from deuterons and calculated the cross-sections using the Chew-Low amplitude for the photo-production from free nucleons which has been found to give reliable results below 500 Mev. In the present chapter, we extend our results to include complex nuclei. First we shall restrict ourselves to the case of nuclei with one nucleon outside the closed shell or subshell. Of the following processes

$$\begin{aligned}
 & {}^A X_Z (\gamma, \pi^\pm) {}^A Y_{Z \mp 1}, \quad {}^A X_Z (\gamma, \pi^\pm) {}^A Y_{Z \mp 1}^*, \\
 & {}^A X_Z (\gamma, \pi^+ n) {}^{A-1} Y_{Z-1}, \quad {}^A X_Z (\gamma, \pi^- p) {}^{A-1} Y_Z,
 \end{aligned}$$

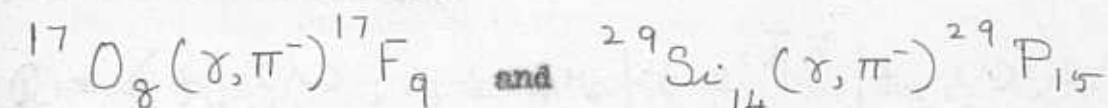
in which only the outer nucleon participates in the reaction, the first one which we shall call the "elastic" process, is of special interest since it presents no uncertainty regarding the final state of the nucleus^u and hence amenable to reliable calculation. Such experiments^{1), 2)} are also feasible if the final nuclei are β -emitters for the cross-sections of such "elastic"

* V. Devanathan and G. Ramachandran, Nuclear Physics, 38, 654 (1962)
Also presented at the Summer School in Theoretical Physics, Simla (1962).

- 1) D.V.Hughes and P.V.March, Proc. Phys. Soc., A 72, 259 (1958)
- 2) P.V.March and T.G.Walker, Proc. Phys. Soc., A 77, 293 (1961)

processes can be found by measuring the β -activity. As pointed out by March and Walker²⁾, the experimental study of the reaction is much easier if the residual nucleus is a positron-emitter for advantage can be taken of the annihilation quanta to use coincidence technique.

We give below explicit expressions for the cross-sections for such reactions e.g.,



but carry out the numerical computation only for the second case which is simpler. In both the cases cited above, the residual nucleus is a positron emitter and hence amenable to experimental results to compare with. We use the independent particle model and represent the initial and final state by the harmonic oscillator wave functions. The Chew-Low photo production amplitude which has been used for calculation involves the magnetic moments of the proton and the neutron. The magnetic moments of the free nucleons are not usually the same as the magnetic moments within the nuclei and hence numerical results are separately presented using

- a) free nucleon magnetic moments and
- b) magnetic moments within the nuclei.^{3), 4), 5).}

-
- 3) J.P. Elliot and A.M. Lane, Handbuch der Physik, Vol. 39, Springer-Verlag, 248, 264 (1957).
 - 4) E. Feenberg, Shell Theory of the Nucleus, Princeton University Press, Princeton, p. 45 (1955).
 - 5) H. Miyazawa, Prog. Theor. Phys., 6, 253 (1961)
A. de-Shalit, Helv. Phys. Acta, 24, 296 (1951)
F. Blockh, Phys. Rev., 83, 839 (1951)
S. D. Drell and J.D. Walecka, Phys. Rev., 120, 1069 (1960).

It is reasonable to expect the latter to give correct results. This point has not been observed by earlier authors who have used without any modification the free nucleon cross-sections.

2. General considerations

The matrix element* for the transition from the initial to the final ground state with the photoproduction of charged pion can be written as

$$Q = \left\langle f_{n_f}^{L_f} (r) ; L_f \frac{1}{2} J_f M_f \right| t e^{i \vec{k} \cdot \vec{r}} \left| f_{n_i}^{L_i} (r) ; L_i \frac{1}{2} J_i M_i \right\rangle \quad (1)$$

where \vec{k} represents the momentum transfer to the outer-shell nucleon whose position coordinate is denoted by \vec{r} . The transition operator t has the general structure $\vec{\sigma} \cdot \vec{K} + L$ where \vec{K} represents the spin dependent part and L , the non-spin dependent part. The values of \vec{K} and L can be taken from the complete amplitude given by Chew et. al. for the photo-production of charged pions from free nucleons.

$$\begin{aligned} t(\gamma + n \rightarrow \pi^- + p) = & \frac{2\sqrt{2}\pi}{\sqrt{\mu_0} 2\mu_0} e f \left[\frac{1}{1 + \frac{\mu_0}{M}} \left\{ i \vec{\sigma} \cdot \vec{E} \right. \right. \\ & \left. \left. + 2i \frac{\vec{\sigma} \cdot \vec{k} \vec{\mu} \cdot \vec{E}}{k^2 + 1} \right\} + i \vec{\sigma} \cdot \vec{\mu} \times (\vec{\nu} \times \vec{E}) (\lambda h^- + \alpha) \right. \\ & \left. + \vec{\mu} \cdot \vec{\nu} \times \vec{E} \lambda h^{+-} - 2i \vec{\sigma} \cdot \vec{\mu} \vec{\mu} \cdot \vec{E} \frac{1}{2M\mu_0} \right. \\ & \left. + i \vec{\sigma} \cdot \vec{E} \alpha \mu_0^2 \right] \end{aligned}$$

(2)

where

* We use natural units in which $\hbar = c = 1$ and the pion mass is taken as unity.

where

$$\lambda = \frac{g_p - g_n}{4 M_f^2}, \quad \alpha = \frac{g_p + g_n}{2 M \mu_0}, \quad \vec{k} = \vec{\omega} - \vec{\mu}$$

$$h^{--} = \frac{e^{i\delta_{33}} \sin \delta_{33}}{3 \mu^3}, \quad h^{+-} = -\frac{2}{3 \mu^3} e^{i\delta_{33}} \sin \delta_{33}$$

taking into account only the dominant (3, 3) phase shifts.

g_p and g_n are the magnetic moments of the proton and neutron respectively, ω_0 and μ_0 the energy of the incident photon and the outgoing meson and $\vec{\omega}$ and $\vec{\mu}$, their momenta. $\vec{\sigma}$ is the spin operator for the nucleon, $\vec{\epsilon}$ the polarization vector for the photon and M , the mass of the nucleon.

Expanding $e^{i\vec{k} \cdot \vec{r}}$ into definite angular momentum states and separating the radial and angular integrals, expression (1) can be written as

$$Q = \sum_{l=0}^{\infty} \left\{ (i)^l (2l+1) \langle L_f \frac{1}{2} J_f M_f | t P_l(\hat{k} \cdot \hat{r}) | L_i \frac{1}{2} J_i M_i \rangle \int f_{m_f l_f}^*(r) j_l(kr) f_{m_i l_i}(r) r^2 dr \right\} \quad (3)$$

Following the notation adopted in earlier chapters, t can be written in the form

$$t = A \sigma_+ + B \sigma_- + C \sigma_z + D \quad (4)$$

Operating them on the angular momentum state expanded in terms of orbital and spin wave functions

$$\begin{aligned}
 |L \frac{1}{2} J M\rangle &= C(L \frac{1}{2} J; M - \frac{1}{2}, \frac{1}{2}) |L, M - \frac{1}{2}\rangle | \frac{1}{2}, +\frac{1}{2}\rangle \\
 &+ C(L \frac{1}{2} J; M + \frac{1}{2}, -\frac{1}{2}) |L, M + \frac{1}{2}\rangle | \frac{1}{2}, -\frac{1}{2}\rangle
 \end{aligned}
 \tag{5}$$

we obtain

$$\begin{aligned}
 \langle L_f \frac{1}{2} J_f M_f | t P_l(\hat{k}, \hat{n}) | L_i \frac{1}{2} J_i M_i \rangle \\
 &= \frac{4\pi}{2l+1} \sum_l (-1)^l Y_l^{-m}(\hat{k}) \left[C(L_f \frac{1}{2} J_f; M_f - \frac{1}{2}, \frac{1}{2}) \right. \\
 &\quad C(L_i \frac{1}{2} J_i; M_i + \frac{1}{2}, -\frac{1}{2}) \langle L_f, M_f - \frac{1}{2} | Y_l^m(\hat{n}) | L_i, M_i + \frac{1}{2} \rangle A \\
 &\quad + C(L_f \frac{1}{2} J_f; M_f + \frac{1}{2}, \frac{1}{2}) C(L_i \frac{1}{2} J_i; M_i - \frac{1}{2}, \frac{1}{2}) \\
 &\quad \left. \langle L_f, M_f + \frac{1}{2} | Y_l^m(\hat{n}) | L_i, M_i - \frac{1}{2} \rangle B \right. \\
 &\quad + C(L_f \frac{1}{2} J_f; M_f - \frac{1}{2}, \frac{1}{2}) C(L_i \frac{1}{2} J_i; M_i - \frac{1}{2}, \frac{1}{2}) \\
 &\quad \left. \langle L_f, M_f - \frac{1}{2} | Y_l^m(\hat{n}) | L_i, M_i - \frac{1}{2} \rangle (D+C) \right. \\
 &\quad \left. + C(L_f \frac{1}{2} J_f; M_f + \frac{1}{2}, \frac{1}{2}) C(L_i \frac{1}{2} J_i; M_i + \frac{1}{2}, \frac{1}{2}) \right. \\
 &\quad \left. \langle L_f, M_f + \frac{1}{2} | Y_l^m(\hat{n}) | L_i, M_i + \frac{1}{2} \rangle (D-C) \right]
 \end{aligned}$$

wherein we have used the expansion of $P_l(\hat{k}, \hat{n})$ in terms of spherical harmonics

$$P_l(\hat{k}, \hat{n}) = \frac{4\pi}{2l+1} \sum_m (-1)^m Y_l^{-m}(\hat{k}) Y_l^m(\hat{n}) \tag{7}$$

After summing and averaging over the nuclear spin states, the square of the matrix element is finally obtained as

$$\begin{aligned}
 |Q|^2 = & \frac{1}{2J_f+1} \sum_{M_i} \sum_{M_f} \left| 4\pi \sum_{\ell} (i)^{\ell} F_{\ell} \sum_m (-1)^m Y_{\ell}^{-m}(\hat{k}) \right. \\
 & \left\{ C(L_f \frac{1}{2} J_f; M_f - \frac{1}{2}, \frac{1}{2}) C(L_i \frac{1}{2} J_i; M_i + \frac{1}{2}, -\frac{1}{2}) \right. \\
 & \quad \left. \langle L_f, M_f - \frac{1}{2} | Y_{\ell}^m(\hat{n}) | L_i, M_i + \frac{1}{2} \rangle A \right. \\
 & + C(L_f \frac{1}{2} J_f; M_f + \frac{1}{2}, -\frac{1}{2}) C(L_i \frac{1}{2} J_i; M_i - \frac{1}{2}, \frac{1}{2}) \\
 & \quad \left. \langle L_f, M_f + \frac{1}{2} | Y_{\ell}^m(\hat{n}) | L_i, M_i - \frac{1}{2} \rangle B \right. \\
 & + C(L_f \frac{1}{2} J_f; M_f - \frac{1}{2}, -\frac{1}{2}) C(L_i \frac{1}{2} J_i; M_i - \frac{1}{2}, \frac{1}{2}) \\
 & \quad \left. \langle L_f, M_f - \frac{1}{2} | Y_{\ell}^m(\hat{n}) | L_i, M_i - \frac{1}{2} \rangle (D+C) \right. \\
 & + C(L_f \frac{1}{2} J_f; M_f + \frac{1}{2}, -\frac{1}{2}) C(L_i \frac{1}{2} J_i; M_i + \frac{1}{2}, \frac{1}{2}) \\
 & \quad \left. \langle L_f, M_f + \frac{1}{2} | Y_{\ell}^m(\hat{n}) | L_i, M_i + \frac{1}{2} \rangle (D-C) \right\}^2 \quad (8)
 \end{aligned}$$

with $F_{\ell} = \int f_{m_f L_f}^*(r) j_{\ell}(kr) f_{m_i L_i}(r) r^2 dr$ (9)

where $j_{\ell}(kr)$ denotes the spherical Bessel function. The matrix elements involving the spherical harmonics Y_{ℓ}^m can be easily evaluated using the Wigner-Eckart theorem

$$\langle l_f m_f | Y_l^m | l_i m_i \rangle = c(l_i l_f; m_i m) \\ c(l_i l_f; 00) \left[\frac{(2l_i+1)(2l+1)}{4\pi(2l_f+1)} \right]^{1/2} \quad (10)$$

A more elegant form for the matrix element can be obtained using Racah algebra as shown in the appendix.

The differential cross-section for the process is given by

$$\frac{d\sigma}{d\vec{u} d\vec{h}} = (2\pi)^{-2} |Q|^2 \delta(\vec{u} - \vec{u} - \vec{h}) \delta(\omega_0 - \mu_0)$$

or

$$\frac{d\sigma}{d\omega} = (2\pi)^{-2} \mu \mu_0 |Q|^2 \quad (11)$$

with $\mu_0 = \omega_0$

3. Particular case

We shall consider for the purpose of illustration, the two reactions (a) $^{17}\text{O}_8(\gamma, \pi^-)^{17}\text{F}_9$ and (b) $^{29}\text{Si}_{14}(\gamma, \pi^-)^{29}\text{P}_{15}$. In these two cases, there is only one nucleon in the outer shell and the residual nucleus is a positron emitter. In the former case, the outer nucleon is in $1d_{5/2}$ state and in the latter, it is in $2s_{1/2}$ state. Since there are no experimental results to compare with, we have chosen for numerical calculation the second case which is simpler.

For the reaction (b), the square of the matrix element is obtained after averaging over photon polarisation, as

$$|Q|^2 = \frac{1}{2} |F_0|^2 (|A|^2 + |B|^2 + 2|C|^2 + 2|D|^2)$$

where

(12)

$$\begin{aligned}
 |A|^2 + |B|^2 + 2|C|^2 + 2|D|^2 &= \frac{8\pi^2 e^2 f^2}{\mu_0 \omega_0} \left[\frac{2}{(1 + \frac{\mu_0}{M})^2} \left(1 - \frac{2\mu^2 \sin^2 \theta}{(k^2 + 1)^2} \right) \right. \\
 &+ \frac{1}{1 + \frac{\mu_0}{M}} \left\{ \frac{4\omega\lambda}{3\mu} \cos \delta_{33} \sin \delta_{33} \left(\frac{2\mu \sin^2 \theta}{k^2 + 1} - \frac{\cos \theta}{\mu} \right) \right. \\
 &+ \frac{4}{k^2 + 1} \mu^2 \sin^2 \theta \left(\frac{1}{M\mu_0} (\mu^2 - \mu\omega \cos \theta) + 4\alpha (\mu_0^2 - 2\mu \cos \theta) \right. \\
 &\left. \left. - \frac{2\mu^2 \sin^2 \theta}{M\mu_0} \right) \right\} \\
 &+ \left(\frac{\lambda\omega}{3\mu^2} \right)^2 \sin^2 \delta_{33} (2 + 3 \sin^2 \theta) \\
 &+ \frac{2\omega\lambda}{3} \cos \delta_{33} \sin \delta_{33} \left\{ \frac{\alpha\omega}{\mu} (1 + \cos^2 \theta) - \frac{2\alpha\mu_0^2 \cos \theta}{\mu^2} \right\} \\
 &+ 2\alpha^2 \mu_0^2 (\mu_0^2 - 2\omega\mu \cos \theta) \\
 &+ \alpha^2 \mu^2 \omega^2 (1 + \cos^2 \theta) \\
 &\left. - \frac{2\alpha\mu_0^2 \mu^2}{M\mu_0} \sin^2 \theta + \frac{\mu^4}{M^2 \mu_0^2} \sin^2 \theta \right]
 \end{aligned}$$

(13)

and

$$F_0 = \int_0^\infty f_{2,0}^*(r) j_0(kr) f_{2,0}(r) r^2 dr \quad (14)$$

For evaluating the radial integral we use the oscillator wave function³⁾ (properly normalised)

$$f_{2,0}(r) = \left(\frac{8}{3\sqrt{\pi}}\right)^{1/2} \left(\beta r^2 - \frac{3}{2}\right) \exp\left(-\frac{1}{2}\beta r^2\right) \quad (15)$$

substituting (10) in (9), we get

$$F_0 = \left(1 - \frac{1}{3} \frac{k^2}{\beta} + \frac{1}{24} \frac{k^4}{\beta^2}\right) \exp\left(-\frac{k^2}{4\beta}\right) \quad (16)$$

where $\beta = 0.3708$ obtained from the relation

$$\langle r^2 \rangle = \frac{7}{2\beta} = (2.9)^{2/3} \quad (17)$$

The differential cross-sections are evaluated at photon energies 1.5, 1.75, and 2.0 pion mass using

a) the free nucleon magnetic moments ($g_n = -1.913$,

$$g_p = 2.793$$

and b) the quenched values^{3), 4), 5)} ($g_n = -0.555$

in $^{29}\text{Si}_{14}$, $g_p = 1.435$ in $^{29}\text{P}_{15}$)

and they are presented in Table 1. The cross-sections in general show a forward peaking as we are considering an "elastic" process but this tendency is accentuated in case (b). At large angles, the cross-sections for the case (b) falls below that of (a) which however is small. The total cross-sections evaluated using these two different sets of magnetic moments are presented in Table 2 and they are found to be nearly equal. Hence it is rather difficult to decide in favour of any particular set of magnetic moments by measuring the total cross-sections alone.

Table 1

The differential cross-section $\frac{d\sigma}{d\Omega}$ for the process $^{29}\text{Si}_{14}(\gamma, \pi^-)^{29}\text{P}_{15}$ in units of $\mu\text{b}/\text{sr}$ as a function of photon energy in units of pion mass and laboratory angle of meson emission.

θ	0°	10°	20°	30°	60°	90°	120°	150°	180°	
1.5	(a)	5.027	3.978	2.246	1.044	0.1184	0.3326	0.3789	0.2186	0.1779
	(b)	6.572	5.034	2.758	1.209	0.1120	0.2913	0.3232	0.1852	0.1494
1.75	(a)	3.124	2.160	1.014	0.3659	0.2074	.3818	0.1270	0.0295	0.0023
	(b)	4.495	3.087	1.329	0.4183	0.1766	.3035	0.1001	0.0235	0.0019
2.0	(a)	5.832	3.368	1.175	0.3370	0.3453	0.1688	0.0132	0.0013	0.0005
	(b)	7.930	4.626	1.382	0.3180	0.2621	0.1234	0.0098	0.0010	0.0004

However, the angular distribution offers a check. The predominance of the differential cross-section evaluated with quenched magnetic moments over that evaluated with free magnetic moment at low angles and the subsequent off-set at large angles may be taken advantage of, in deciding the issue.

Table 2

The total cross-section for the process $^{29}\text{Si}_{14}(\gamma, \pi^-)^{29}\text{P}_{15}$ in units of μb as a function of photon energy in units of pion mass. The column (a) corresponds to the cross-section evaluated with free nucleon magnetic moments and the column (b) to the values obtained with quenched magnetic moments.

ω_0	a	b
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1.5	5.453	5.738
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1.75	3.442	3.359
------	-------	-------

2.00	3.063	2.972
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However, the angular distribution offers a chance. The predominance of the differential cross-section evaluated with quenched magnetic moments over that evaluated with free magnetic moments at low angles and the subsequent off-set at large angles may be taken advantage of, in deciding the issue. At incident photon energies 1.75 and 2.0 in units of pion mass, the differential cross-section in case (b) is about 1.5 times that in case (a) at forward angle and hence susceptible to experimental detection. Although we have assumed the harmonic oscillator wave function for the bound nucleon, it may be pointed out that at small angles $j_0(kr)$ is a slowly varying function within the nuclear radius and hence the result obtained does not very much depend on the nature of the wave function.

4. Conclusion

The good agreement obtained with experiment in the earlier calculations on photo production of pions from deuterons using the Chew-Low amplitude encourages one to believe in the validity of the use of this amplitude for the photo-production from bound nucleons. Since the photoproduction process involves the magnetic moment interaction, there occurs in the Chew-Low amplitude the terms with $\lambda = \frac{g_p - g_n}{4 M f^2}$ and $\alpha = \frac{g_p + g_n}{2 M \mu_0}$. The magnetic moment of the bound nucleon is not usually the same as the free nucleon magnetic moment and hence the ambiguity

arises as to the values that have to be fed in the exp. (2). It is hoped that this difficulty will be overcome by considering the "elastic" photoproduction of charged pions from nuclei with one nucleon in the outershell, since it is a single nucleon process with no uncertainty in the final state and hence amenable to reliable calculation. Although in the particular case we have considered viz., $^{29}\text{Si}_{14}(\gamma, \pi^-)^{29}\text{P}_{15}$, the total cross-sections obtained for the two sets of values of magnetic moments were almost the same and only the angular distributions showed deviations susceptible to experimental observation, it is expected that this difference will be more enhanced in the case of a ^{fa-}four-able nucleus. Conversely, the photoproduction of pions from nuclei can be used as a method of determining the magnetic moments of bound nucleons and hence as a test on the idea of quenching⁵⁾.

Appendix

The matrix element Q can be obtained in an alternative form using Racah algebra. For this we introduce a complete set of intermediate states and the matrix element is written as

$$Q = \int r^2 dr f_{n_f L_f}^*(r) \sum_{J' M'} \langle L_f \frac{1}{2} J_f M_f | t | L_f \frac{1}{2} J' M' \rangle \langle L_f \frac{1}{2} J' M' | e^{i \vec{k} \cdot \vec{r}} | L_i \frac{1}{2} J_i M_i \rangle f_{n_i L_i}(r) \quad (A-1)$$

Expressing $\exp(i \vec{k} \cdot \vec{r})$ in terms of spherical harmonics as before and writing t in the form

$$t = \vec{\sigma} \cdot \vec{K} + L \quad (A-2)$$

with

$$\vec{\sigma} \cdot \vec{K} = 2 \sum_{\mu} (-1)^{\mu} K_{-\mu} S_{\mu} \quad (A-3)$$

where S_{μ} are the components of the spin angular momentum operator in spherical tensor form, Q can be obtained in the following form using Racah technique

$$Q = \sum_l (4\pi)^{1/2} F_l(x)^l (-1)^{L_f - L_i} \left[\frac{(2L_i + 1)(2l + 1)}{(2L_f + 1)} \right]^{1/2}$$

$$C(L_i l L_f; 00) \sum_{m=-l}^{+l} (-1)^m Y_l^{-m}(\hat{k})$$

$$\sum_{J' = l + \frac{1}{2}, l - \frac{1}{2}} C(J_i l J'; M', m) U(L_i l \frac{1}{2} J'; L_f J_i)$$

$$\left\{ \sum_{\mu} (-1)^{\mu} K_{-\mu} C(J' l J_f; M' \mu) U(\frac{1}{2} l L_f J_f; \frac{1}{2} J') \sqrt{3} + L \delta_{J_f J'} \delta_{M_f M'} \right\}$$

(A-4)

where U is related to the Racah coefficient W by

$$U(a b c d; e f) = [(2e + 1)(2f + 1)]^{1/2} W(a b c d; e f)$$

(A-5)

and

$$K_{+1} = -\frac{B}{\sqrt{2}}, \quad K_{-1} = \frac{A}{\sqrt{2}}, \quad K_0 = c, \quad L = D$$

(A-6)

(A-6) gives the relationship between the two sets of coefficients, one used here and the other used in the text.

1) J. Steinberger and A. Weiss, Phys. Rev., **75**, 451 (1954);

R.P. Taylor, Phys. Rev., **75**, 453 (1954);

George, Phys. Rev., **75**, 454 (1954);

W. Riegler and G. Stenlund, Phys. Rev., **75**, 456 (1954);

2) J.M. Blatt and L. Teller, Phys. Rev., **75**, 304 (1954);

75, 310 (1954).

3) G. Racah, Phys. Rev., **62**, 205 (1947);

CHAPTER VII.PHOTO PRODUCTION OF CHARGED PIONS FROM NUCLEI - II.*GENERAL THEORY1. Introduction

The early experiments^{1) 2)} on the photo production of charged pions from nuclei have been devoted to the measurement of the total cross-section, mostly accompanied by the nucleon emission. The most exhaustive of them is that of Litt^u and Walker²⁾ who have used a bremsstrahlung spectrum of maximum energy 310 Mev and measured the yield of charged photo mesons produced at an angle of 135° in the energy range 65 ± 15 Mev from a wide variety of targets totalling seventeen in number. These experiments do not distinguish between reactions in which the meson is the only particle emitted and those in which meson emission is accompanied by the emission of one or more nucleons. The following broad features are observed.

(i) The yield is considerably less than ^{that} expected from the corresponding number of free nucleons.

(ii) The sum of the π^+ and π^- cross-section exhibits an $A^{2/3}$ dependence.

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- 1) J. Steinberger and A. Bishop, Phys. Rev., 78, 494 (1950);
 R.F. Mozley, Phys. Rev., 80, 493 (1950);
 George, Proc. Phys. Soc., A 69, 110 (1956);
 W.R. Hogg and D. Sinclair, Phil. Mag., Series 8, 1, 466 (1956)
- 2) R.M. Litt^u and D. Walker, Phys. Rev., 83, 206 (1956);
86, 838 (1952).

* V. Devanathan and G. Ramachandran, Nuclear Physics (in press)

In order to explain these gross features, Butler³⁾ has proposed the model of surface production according to which the pion production from the core is suppressed and only the outer nucleons i.e., the nucleons beyond the boundary r_0 , the radius of the central core of constant density, are effective in the process. The mechanism of suppression inside the core has been explained by Wilson⁴⁾ as due to the competing process of photo-disintegration of the nucleons by the capture of produced pions by the other strongly interacting nucleons in the core.

Although this model has met with some early successes, certain discrepancies have also been reported by Laing and Moorhouse⁵⁾ and March and Walker⁶⁾. It is to be observed that Butler's model is meant only to explain the broad features of total cross sections and any attempt to extract the finer details by specifying the final state of the nucleus may yield poor results.

We propose to work the other way : from finer details to gross features. We consider the transition of the nucleus from the given initial state to a well defined final state with

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- 3) S.T.Butler, Phys. Rev., 87, 1117 (1952)
 4) R.R.Wilson, Phys. Rev., 86, 125 (1952)
 5) E.W.Laing and R.G.Moorhouse, Proc. Phys. Soc., A 70, 629 (1957).
 6) I.S.Hughes and P.V.March, Proc. Phys. Soc., A 72, 259 (1958)

the production of a charged pion by the incident photon. Such experiments^{6) 7)} have been reported recently and they are of considerable interest to us since a rigorous theoretical treatment is possible in these cases. Laing and Moorhouse⁵⁾ have attempted earlier to treat such cases using the independent particle model but they have digressed ^{into} ~~to~~ a discussion on the surface and volume production. We hold the view that the elements of the surface production model of Butler^{1) 3)} ^{are} ~~is~~ ingrained in any treatment based on shell-model and hence a complete and rigorous treatment of the photoproduction process based on the shell model description of the nucleus should precede any discussion of the surface and volume production.

In the earlier chapter, we have investigated in impulse approximation the "elastic" photo production of charged pions from nuclei wherein it was assumed that closed shells do not contribute and the matrix elements were evaluated between initial and final states with specified magnetic quantum numbers, the summation over the magnetic quantum numbers having been done as the last step. This restricted the scope of application of the method to a bound nucleon in s -shell or p -shell. For bound nucleons in shells of higher ^{l} values, the method will prove almost formidable due to the summation over magnetic quantum ^{m} numbers. Laing and Moorhouse⁵⁾ have overcome this difficulty by an approximate method. In this ^{chapter}, we outline a rigorous

7) P.V. March and T.G. Walker, Proc. Phys. Soc., 77, 293 (1961)

method of evaluating the cross-section by treating both the spin-flip and non-spin-flip terms alike and summing over the magnetic quantum numbers. In section 2, we evaluate the square of the matrix element between the single particle states and the resulting expression is free from the summation over magnetic quantum numbers. For the purpose of calculation, the independent particle shell model is used and the initial and final states are represented by the harmonic oscillator wave functions. The transition operator which has the general structure $\vec{\sigma} \cdot \vec{K} + L$ is taken to correspond to the Chew-Low amplitude for the photo-production of pions from free nucleons.

In section 3, extension is made to the complex nuclei with N nucleons in the outer shell and cross sections are obtained in terms of single particle matrix element using the concept of fractional parentage coefficients. ^{The j-j Coupling} ~~Various~~ shell model descriptions ^{is} ~~are~~ used in the study.

The assumption that closed shell nucleons do not contribute in impulse approximation is quite reasonable since we cannot convert, for example, a neutron into a proton in a closed shell and still keep it there as it would need a two particle operator to bring this proton to outer shells and substitute one of the outer shell nucleons instead. But the question will arise whether it is sufficient in impulse approximation to consider the nuclear wave function as a product of closed shell wave function and the wave function of the outer nucleons and consider only those single

particle operators operating on the outer shell nucleons to calculate the amplitude since this is what the result of our assumption would be. Section 4 is devoted to this question and starting with completely antisymmetrised nuclear wave functions, we conclude that such an assumption holds either if the single particle operators are spherical tensor operators of non-zero rank or if the nucleus goes into an excited state. This result shows for example that closed shells do not contribute for neutral meson production when the nucleus goes into an excited state but make a non-zero contribution when the process is elastic.

2. Evaluation of the square of the single particle matrix element

The matrix element for the photo-production of pion from a bound nucleon is given by

$$Q_{M_i \rightarrow M_f} = \left\langle f_{m_f l_f}^{(n)} ; L_f \frac{1}{2} J_f M_f \left| (\vec{\sigma} \cdot \vec{K} + L) e^{i \vec{k} \cdot \vec{r}} \right| L_i \frac{1}{2} J_i M_i ; f_{m_i l_i}^{(n)} \right\rangle \quad (1)$$

The initial and final states are explicitly represented as the product of the radial part $f_{n l}^{(n)}$ (Table I) and the angular and spin part $|L \frac{1}{2} J M\rangle$. The transition operator has the general structure $\vec{\sigma} \cdot \vec{K} + L$ where \vec{K} and L are functions of the momenta and energy of the incident photon and the outgoing pion, the polarisation of the photon and the angle of pion emission. They also depend on the magnetic moment of the

proton and the neutron and their complete structure can be obtained from the Chew-Low amplitude. The momentum transfer to the bound nucleon is taken into account by the factor

$e^{i\vec{k}\cdot\vec{r}}$ where \vec{k} represents the difference between the momentum of the incident photon $\vec{\omega}$ and the outgoing pion $\vec{\mu}$ and \vec{r} the position coordinate of the bound nucleon.

The differential cross-section* is given by

$$\frac{d\sigma}{d\vec{\mu} d\vec{k}} = (2\pi)^{-2} |Q|^2 \delta(\vec{\omega} - \vec{\mu} - \vec{k}) \delta(\omega_0 - \mu_0) \quad (2a)$$

Integrating over $d\vec{k}$, we obtain

$$\frac{d\sigma}{d\Omega} = (2\pi)^{-2} \mu \mu_0 |Q|^2 \quad (2b)$$

with $\mu_0 = \omega_0$, making the assumption that the entire energy of the photon ω_0 is given to the outgoing pion. The quantity $|Q|^2$ is the square of the matrix element (1) after performing the sum over the final $(2J_f + 1)$ spin states and averaging over the initial $(2J_i + 1)$ states. This sum can be performed with ease if we treat the spin-dependent and spin-independent parts alike by defining a unit operator, σ_0 in spin space. Writing $\vec{\sigma} \cdot \vec{K} + L$ as $\sum_{n=0,1} \vec{\sigma}_n \cdot \vec{K}_n$ and expanding $\exp(i\vec{k}\cdot\vec{r})$ into spherical harmonics, the product may be written as

* We use natural units in which $\hbar = c = \text{pion mass} = 1$.

$$\begin{aligned}
 (\vec{\sigma} \cdot \vec{k} + L) \exp(i\vec{k} \cdot \vec{r}) &= 4\pi \sum_n \sum_{l,m} (i)^l j_l(kr) \\
 &\quad \cdot (-1)^m Y_l^m(\hat{r}) Y_l^{-m}(\hat{k}) \\
 &\quad \cdot \sum_{\mu} (-1)^{\mu} \sigma_n^{\mu} K_n^{-\mu}
 \end{aligned}
 \tag{3}$$

Separating the angular and radial parts and forming the tensor products of the operators in pairs, we obtain

$$\begin{aligned}
 Q_{M_i \rightarrow M_f} &= 4\pi \sum_{l,n} \sum_{\lambda, m_{\lambda}} (i)^l F_l (-1)^{l+n-\lambda} (-1)^{m_{\lambda}} \\
 &\quad (Y_l(\hat{k}) \times K_n)_{\lambda}^{m_{\lambda}} \langle L_f \frac{1}{2} J_f M_f | (Y_l(\hat{r}) \times \sigma_n)_{\lambda}^{m_{\lambda}} | L_i \frac{1}{2} J_i M_i \rangle
 \end{aligned}
 \tag{4}$$

wherein the orthogonality property of the Clebsch-Gordan coefficients has been used. The quantity F_l represents the radial integral

$$F_l = \int_0^{\infty} f_{n_f l_f}^*(r) j_l(kr) f_{n_i l_i}(r) r^2 dr
 \tag{5}$$

Applying the Wigner-Eckart theorem,

$$\begin{aligned}
& \langle L_f \frac{1}{2} J_f M_f | (Y_l(\hat{n}) \times \sigma_n)_\lambda^{m_\lambda} | L_i \frac{1}{2} J_i M_i \rangle \\
& = C(J_i \lambda J_f; M_i m_\lambda) \\
& \quad \langle L_f \frac{1}{2} J_f || (Y_l(\hat{n}) \times \sigma_n)_\lambda || L_i \frac{1}{2} J_i \rangle
\end{aligned} \tag{6}$$

and squaring, averaging and summing over the initial and final states, we obtain

$$\begin{aligned}
|Q|^2 &= \frac{1}{2^{J_i+1}} \sum_{M_i, M_f} \sum_{l, n} \sum_{l', n'} \sum_{\lambda, m_\lambda} \sum_{\lambda', m'_\lambda} 16 \pi^2 F_l F_{l'}^* \\
& \quad (-1)^{l-l'} (-1)^{l+n-\lambda} (-1)^{l'+n'-\lambda'} (-1)^{m_\lambda+m'_\lambda} \\
& \quad (Y_l(\hat{k}) \times K_n)_{\lambda}^{-m_\lambda} (Y_{l'}(\hat{k}) \times K_{n'})_{\lambda'}^{-m'_\lambda} \\
& \quad C(J_i \lambda J_f; M_i m_\lambda) C(J_i \lambda' J_f; M_i m'_\lambda) \\
& \quad \delta_{m_\lambda, M_f - M_i} \delta_{m'_\lambda, M_f - M_i} \\
& \quad \langle L_f \frac{1}{2} J_f || (Y_l(\hat{n}) \times \sigma_n)_\lambda || L_i \frac{1}{2} J_i \rangle \\
& \quad \langle L_f \frac{1}{2} J_f || (Y_{l'}(\hat{n}) \times \sigma_{n'})_{\lambda'} || L_i \frac{1}{2} J_i \rangle^*
\end{aligned} \tag{7}$$

Using the symmetry properties of the Clebsch-Gordon coefficients and summing over M_i , we finally obtain after omitting the redundant summation over M_f ,

$$\begin{aligned}
 |Q|^2 &= \frac{1}{[J_i]^2} \sum_{l, l'} \sum_{n, n'} \sum_{\lambda, m_\lambda} 16 \pi^2 F_l F_{l'}^* \\
 &\quad (i)^{l-l'} (-1)^{l+l'+n+n'} \frac{[J_f]^2}{[\lambda]^2} \\
 &\quad (Y_l(\hat{k}) \times K_n)^{-m_\lambda} (Y_{l'}(\hat{k}) \times K_{n'})^{-m_\lambda^*} \\
 &\quad \langle L_f \frac{1}{2} J_f \| (Y_l(\hat{k}) \times \sigma_n)_\lambda \| L_i \frac{1}{2} J_i \rangle \\
 &\quad \langle L_f \frac{1}{2} J_f \| (Y_{l'}(\hat{k}) \times \sigma_{n'})_\lambda \| L_i \frac{1}{2} J_i \rangle^*
 \end{aligned} \tag{8}$$

where $[J] = (2J+1)^{1/2}$

Making use of the following standard relations

$$\begin{aligned}
 &\langle L_f \frac{1}{2} J_f \| (Y_l(\hat{k}) \times \sigma_n)_\lambda \| L_i \frac{1}{2} J_i \rangle \\
 &= [L_f] \left[\frac{1}{2} \right] [J_i] [\lambda] \left\{ \begin{matrix} L_i & l & L_f \\ \frac{1}{2} & n & \frac{1}{2} \\ J_i & \lambda & J_f \end{matrix} \right\} \\
 &\quad \langle L_f \| Y_l(\hat{k}) \| L_i \rangle \langle \frac{1}{2} \| \sigma_n \| \frac{1}{2} \rangle
 \end{aligned} \tag{9}$$

$$\langle L_f \| Y_l(\hat{k}) \| L_i \rangle = C(L_i \ l \ L_f ; 0 \ 0)$$

$$\frac{[L_i][l]}{\sqrt{4\pi} [L_f]}$$

(10)

$$\left\langle \frac{1}{2} \parallel \sigma_n \parallel \frac{1}{2} \right\rangle = [n] \quad (11)$$

the square of the matrix element can be finally written as

$$\begin{aligned}
 |Q|^2 = & \frac{1}{[J_i]^2} \sum_{l, l'} \sum_{n, n'} \sum_{\lambda, m_\lambda} 4\pi (i)^{l-l'} \\
 & F_l F_{l'}^* (-1)^{l+l'+n+n'} \\
 & (Y_l(\hat{k}) \times K_n)_{\lambda}^{-m_\lambda} (Y_{l'}(\hat{k}) \times K_{n'})_{\lambda}^{-m_\lambda^*} \\
 & [J_f]^2 [J_i]^2 [L_i]^2 \left[\frac{1}{2}\right]^2 [l] [l'] [n] [n'] \\
 & c(L_i, l, L_f; 00) c(L_i, l', L_f; 00) \\
 & \left\{ \begin{matrix} L_i & l & L_f \\ \frac{1}{2} & n & \frac{1}{2} \\ J_i & \lambda & J_f \end{matrix} \right\} \left\{ \begin{matrix} L_i & l' & L_f \\ \frac{1}{2} & n' & \frac{1}{2} \\ J_i & \lambda & J_f \end{matrix} \right\} \quad (12)
 \end{aligned}$$

In the above summation n and n' can take only two values 0 and 1 and the values of l and l' are highly restricted by the parity Glebsch-Gordon coefficients. The Wigner 9-j symbols $\left\{ \begin{matrix} \end{matrix} \right\}$ also restrict the value of λ and so the calculation can be performed with ease.

In exp. (12), the only term that depends on m_λ is

$$(Y_l(\hat{k}) \times K_n)_{\lambda}^{-m_\lambda} (Y_{l'}(\hat{k}) \times K_{n'})_{\lambda}^{-m_\lambda^*}$$

and the summation over m_λ can be performed to give the square and interference terms involving the quantities \vec{K} which are related to the coefficients A, B, C and D of ref. 7) 8) by *chapters 5 and 6*

$$K_1^1 = -\frac{B}{\sqrt{2}}, \quad K_1^{-1} = \frac{A}{\sqrt{2}}, \quad K_1^0 = C, \quad K_0^0 = D \quad (13)$$

The coefficients, A, B, C and D are determined from the Chew-Low expression for the photoproduction of pions from free nucleons. It is to be observed that the argument \hat{k} of $Y_\ell(\hat{k})$ does not represent the direction of pion emission but that of momentum transfer.

The square of the matrix element is evaluated between the various single particle states of the shell model and they are presented in Tables II, III and IV. The quantities $M_{\ell, \ell'}$ tabulated (Tables II and III) are defined by

$$|Q|^2 = \frac{1}{2J_i+1} \sum_{\ell, \ell'} |Q|_{\ell, \ell'}^2$$

$$|Q|_{\ell, \ell'}^2 = \text{Re} (F_\ell F_{\ell'}^* M_{\ell, \ell'}) \quad (14)$$

where F_ℓ represents the radial integral given by expression (5) and evaluated in Table II. Re denotes the real part.

3. Transition probability between any two nuclear states.

Since the transition operator with which we are concerned is essentially a single particle one, the transition operator between any two nuclear states can be expanded in the form⁸⁾

$$T = \sum_i t(r_i, \sigma_i, \tau_i) \quad (15)$$

8) A.M.Lane and D.H.Wilkinson, Phys. Rev., 97, 1199 (1955)

where each term in the sum operates on only one particle i represented by the usual space, spin and isotopic spin coordinates h_i , σ_i and τ_i . To find the transition probability the matrix element of the corresponding operator taken between the initial and final states must be evaluated. If there are present in the system N outer shell nucleons that are concerned in the transition*, then since the wave function of these initial and final states are antisymmetric in all particles, the matrix element for a transition can be written in terms of the matrix element of just one particular particle

$$\langle f | T | i \rangle = N \langle f | t(h_i, \sigma_i, \tau_i) | i \rangle \quad (16)$$

If now we assume that the fractional parentage expansions of the initial and final states are known, the evaluation of this quantity is straight forward. The total matrix element is expressible as a product of three factors

$$\langle f | T | i \rangle = N \times (\text{Single particle } \times \text{ parentage overlap} \\ \text{overlap matrix element}) \quad (17)$$

The parentage overlap can in no case be greater than one and in some cases be less than $\frac{1}{N}$ so that the transition probability is reduced to less than the single particle value. A detailed study of this may throw some light on the gross features.

* It is shown in section 4 that closed shell nucleons do not contribute for the photo production of charged pions.

To be specific, we give below the expansion in terms of the coefficient of fractional parentage^{9), 10)} in the case of photo production of charged pions from a nucleus containing N equivalent equivalent nucleons in the outer shell. The matrix element in this case can be written as

$$Q_{M_i \rightarrow M_f} = \left\langle \psi(J_f M_f T_f M_{T_f}) \left| \sum_i (\vec{\sigma}_i \cdot \vec{K} + L) e^{i\vec{k} \cdot \vec{r}_i} \tau_i^\pm \right| \psi(J_i M_i T_i M_{T_i}) \right\rangle \quad (18)$$

where $\psi(J_i M_i T_i M_{T_i})$ and $\psi(J_f M_f T_f M_{T_f})$ represent the initial and final nuclear states, and the summation \sum is over all the outer shell nucleons. The isotopic spin operator τ^+ corresponds to the photoproduction of negative pion converting a neutron into a proton, and similarly τ^- corresponds to the photoproduction of positive pion. T and M_T denote the total isotopic spin quantum number and its projection. Since the initial and final nuclear states are completely antisymmetric, the matrix element (18) can be written as

$$Q_{M_i \rightarrow M_f} = N \left\langle \psi(J_f M_f T_f M_{T_f}) \left| (\vec{\sigma}_N \cdot \vec{K} + L) e^{i\vec{k} \cdot \vec{r}_N} \tau_N^\pm \right| \psi(J_i M_i T_i M_{T_i}) \right\rangle \quad (19)$$

- 9) G. Racah, Group theory and Spectroscopy, Mimeographed Lecture notes, Princeton (1951)
- 10) G. Racah, Phys. Rev., 76, 1352 (1949)
 H.A. Jahn, Proc. Roy. Soc. (Lond) A, 209, 192 (1951)
 H.A. Jahn and H. van Wieringen, Proc. Roy. Soc. (London), A 209, 502 (1951)
 A.R. Edmonds and B.H. Flowers, Proc. Roy. Soc. (London), A, 214, 515 (1952)

The transition operator in (19) corresponds to that of N^{th} nucleon in the outer shell and this has to be evaluated between the two nuclear states. The procedure is now essentially the same as outlined in section 2. Expanding the transition operator as shown in equation (3), we obtain for the square of the matrix element after summing over the final spin states and averaging over the initial states

$$\begin{aligned}
 |Q|^2 &= \frac{N^2}{[J_i]^2} \sum_{l, l'} \sum_{n, n'} \sum_{\lambda, m_\lambda} 16 \pi^2 (-i)^{l+l'+n+n'} (i)^{l-l'} \\
 &\frac{[J_f]^2}{[\lambda]^2} (Y_l(\hat{k}) \times K_n)_{\lambda}^{-m_\lambda} (Y_{l'}(\hat{k}) \times K_{n'})_{\lambda}^{-m_\lambda} \\
 &\langle \psi(J_f T_f M_{T_f}) \| (Y_l(\hat{r}_N) \times \sigma_n)_{\lambda} j_l(k r_N) \tau_N^{\pm} \| \psi(J_i T_i M_{T_i}) \rangle \\
 &\langle \psi(J_f T_f M_{T_f}) \| (Y_{l'}(\hat{r}_N) \times \sigma_{n'})_{\lambda} j_{l'}(k r_N) \tau_N^{\pm} \| \psi(J_i T_i M_{T_i}) \rangle^*
 \end{aligned} \quad (20)$$

This is essentially the same as expression (8), but the reduced matrix elements now involve the nuclear states. In order to evaluate them, we have to decouple the wave function of the N^{th} nucleon from the nuclear wave function using the fractional parentage expansion. We shall assume j - j coupling shell model for the nuclear wave function

$$|J T M_J M_T\rangle = \sum_{\alpha' J' T'} \langle (j)^{N-1} (\alpha' J' T'), j | j^N (J T) \rangle | j^{N-1} (\alpha' J' T'), j ; J T M_J M_T \rangle \quad (21)$$

where $\langle j^{N-1}(\alpha' J' T') j || j^N(J T) \rangle$ represents the coefficient of fractional parentage and the quantity α' denotes the symmetry and other quantum numbers.

To evaluate the reduced matrix element in (20), first we shall separate the radial and angular parts

$$\begin{aligned} & \langle \psi(J_f T_f M_{T_f}) || (Y_\ell(\hat{r}_N) \times \sigma_n)_\lambda j_\ell(k r_N) \tau_N^\pm || \psi(J_i T_i M_{T_i}) \rangle \\ & = \langle j_\ell(k r_N) \rangle \langle J_f T_f M_{T_f} || (Y_\ell(\hat{r}_N) \times \sigma_n)_\lambda \tau_N^\pm || J_i T_i M_{T_i} \rangle \end{aligned} \quad (22)$$

where $\langle j_\ell(k r) \rangle$ is the expectation value of $j_\ell(k r)$ over the normalised radial wave function $f_{n_L}^{(N)}$ of particle N

$$\langle j_\ell(k r) \rangle = \int_0^\infty |f_{n_L}^{(N)}(r)|^2 j_\ell(k r) r^2 dr \quad (23)$$

The evaluation of the angular and isotopic spin part is straight forward. Using the fractional parentage expansion (21) we finally obtain by the method of Racah's recoupling coefficients that

$$\begin{aligned} & \langle \psi(J_f T_f M_{T_f}) || (Y_\ell(\hat{r}_N) \times \sigma_n)_\lambda j_\ell(k r_N) \tau_N^\pm || \psi(J_i T_i M_{T_i}) \rangle \\ & = \langle j_\ell(k r) \rangle \sum_{\alpha' J' T'} \langle j^{N-1}(\alpha' J' T') j || j^N(J_i T_i) \rangle \\ & \quad \langle j^{N-1}(\alpha' J' T') j || j^N(J_f T_f) \rangle \\ & \quad \left(\pm \sqrt{\frac{3}{2}} \right) C(T_i | T_f ; M_{T_i}, \mp 1) \end{aligned}$$

(Cont.)

$$U(T' \frac{1}{2} T_f 1 ; T_i \frac{1}{2}) U(J' j J_f \lambda ; J_i J)$$

$$\langle j \| (Y_\ell(\hat{r}) \times \sigma_n)_\lambda \| j \rangle$$

$$U(T' \frac{1}{2} T_f 1 ; T_i \frac{1}{2}) U(J' j J_f \lambda ; J_i J) \quad (24)$$

The recoupling coefficient U is related to the Racah coefficient W by

$$U(j_1 j_2 J j_3 ; j_{12} j_{23}) = [j_{12}] [j_{23}] W(j_1 j_2 J j_3 ; j_{12} j_{23}) \quad (25)$$

The matrix element $\langle j \| (Y_\ell(\hat{r}) \times \sigma_n)_\lambda \| j \rangle$

can be evaluated using equations (9), (10) and (11).

Now we can write down the differential cross-section for the process. Neglecting small differences in binding energy and nuclear recoil energy, we have

$$\frac{d\sigma}{d\Omega} = (2\pi)^{-2} \mu \mu_0 |Q|^2 \quad (26)$$

with $\mu_0 = 2\mu$. The square of the matrix element $|Q|^2$ is obtained by substituting equation (24) in equation (20).

$$|Q|^2 = \frac{N^2}{[J_i]^2} \left\{ \sum_{l, l'} \langle j_l(kr) \rangle \langle j_{l'}(kr) \rangle \text{Re } M_{l, l'} \right\} \\ \left\{ \sum_{\alpha' J' T'} \langle j^{N-1}(\alpha' J' T') j \| j^N(J_i T_i) \rangle \right. \\ \left. \sum_{\alpha'' J'' T''} \langle j^{N-1}(\alpha'' J'' T'') j \| j^N(J_i T_i) \rangle \right. \\ \left. \langle j^{N-1}(\alpha' J' T') j \| j^N(J_f T_f) \rangle \right. \\ \left. \langle j^{N-1}(\alpha'' J'' T'') j \| j^N(J_f T_f) \rangle \right\}$$

(Contd.)

$$\left(\frac{3}{2} \right) \left| C(T_i | T_f ; M_{T_i}, \mp 1) \right|^2 \\
 \left. \begin{aligned}
 & U(T' \frac{1}{2} T_f | ; T_i \frac{1}{2}) U(T'' \frac{1}{2} T_f | ; T_i \frac{1}{2}) \\
 & U(T' j T_f \lambda ; T_i j) U(T'' j T_f \lambda ; T_i j)
 \end{aligned} \right\}$$

(27)

Expression (27) represents the reduction of transition probability between any two nuclear systems to that of single particle. The first curly bracket in (27) represents the contribution due to single particle transition and the second curly bracket represents the influence of the presence of other particles. The quantities $M_{l, l'}$ are tabulated (Tables III and IV) and since we are considering only equivalent particles in the outer shell, we need only diagonal terms in the single particle transitions.

4. Discussion on closed shells

In the shell model description, we consider the nucleus to consist of A nucleons placed in n normalized single particle states $\psi_1, \psi_2, \dots, \psi_A$ where the suffixes 1, 2, . . . A denote besides energy, the angular momentum and isotopic spin quantum numbers of single particles. If there are N particles outside closed shells, the remaining $A - N$ particles occupy states, say $\psi_1, \dots, \psi_{A-N}$ the angular momentum and isotopic spin projection quantum numbers of which add to zero and the totally antisymmetric wave function generated out of $\psi_1, \dots, \psi_{A-N}$ is also an eigen

state of total angular momentum and isotopic spin with zero eigen values. The sum of the projection quantum numbers of the N states $\psi_{A-N+1} \dots \psi_A$ are the angular momentum and isotopic spin projection quantum numbers M and M_T associated with the nucleus.

The antisymmetric wave function of the A particle system is the Slater determinant

$$\frac{1}{\sqrt{A!}} \begin{vmatrix} \psi_1(1) & \dots & \psi_1(A) \\ \psi_A(1) & \dots & \psi_A(A) \end{vmatrix} \quad (28)$$

Using the Laplace expansion¹¹⁾ of a determinant (28) can be written as a sum of terms each of which is a product of the

11) A.C. Aitken, Determinants and matrices, Oliver and Boyd, p. 78 (1956)

Consider a determinant $|A|$ of order n . Let us take any m rows; no generality will be lost by taking the first m rows. From these we may form $\frac{n!}{(n-m)!m!}$ minors of order m . Multiplying each minor by its cofactor, a signed minor of order $n-m$, we obtain terms of $|A|$. None of these terms duplicates any other, for they involve different selections of rows and columns, and so each term has a different arrangement of suffixes from every other term. Counting up all the terms of $|A|$, so arising, we have altogether

$$\left(\frac{n!}{m!(n-m)!} \right) m!(n-m)! = n!$$

terms. Hence we have all the terms of $|A|$ without omission or repetition.

antisymmetrised wave function of $A-N$ particles occupying the states $\psi_1, \dots, \psi_{A-N}$ and the antisymmetric wave function of N particles occupying the states $\psi_{A-N+1}, \dots, \psi_A$ multiplet, by the factor $\sqrt{\left(\frac{(A-N)! N!}{A!}\right)}$ since $\frac{1}{\sqrt{(A-N)!}}$

and $\frac{1}{\sqrt{N!}}$ are the normalisation factors associated with the two wave functions in the product and we have in addition $\frac{1}{\sqrt{(A!)}}$ from expression (28). The number of terms in the expansion is

$\frac{A!}{(A-N)! N!}$ corresponding to the number of ways in which $A-N$ out of A particles can occupy the first $A-N$ states of the closed shells; which is the same as every possible set of N out of A particles occupying the N states in the outer shell. The sign of each term is easily found by considering the 'class'¹¹⁾ to which it belongs to.

In general, corresponding to given values of M and M_T we will have more than one Slater determinant with different $\psi_{A-N+1}, \dots, \psi_A$ differing in individual projection quantum numbers but adding up to the same M and M_T . Suitable linear combinations¹²⁾ of these determinants represents different eigen states of total angular momentum, total isotopic spin and seniority. Since all these different determinants have the same quantum numbers n for the first $A-N$ rows, it

12) J.C.Slater, Quantum theory of Atomic Structure, Vol. II, McGraw Hill (1960).
E. Feenberg, Shell Theory of the Nucleus, Princeton University Press (1955)

follows that using the Laplace expansion for each of these, any nuclear state with given seniority α , total angular momentum and isotopic spin quantum numbers J (or $L S J$) and T and their projections M (or $M_L M_S M_T$) and M_T can be written as a sum of terms each of which consists of a product of the antisymmetrised closed shell wave function and the antisymmetrised wave function of the outer N nucleons forming an eigen state of total angular momentum and total isotopic spin multiplied by a factor $\sqrt{\frac{(A-N)! N!}{A!}}$. We have again $\frac{A!}{(A-N)! N!}$ such terms which are orthogonal to each other since between any two terms there will at least be one pair of particles which have changed their occupation between closed shell and outside closed shell orbitals.

Now if we consider the matrix element of a sum

$$T = \sum_{i=1}^A t_i \quad (29)$$

of single particle operators t_i (t_i operating on particle i) between two such states the matrix element can also be written as a sum of $\frac{A!}{(A-N)! N!}$ matrix elements of T since t_i do not connect different particle labels. In each of these matrix elements of T we can group the t_i into two parts one acting on the particles occupying states forming the closed shells and the other on the outer N particles. If t_i is a spherical tensor operator of non-zero rank either in configuration or isotopic spin space the first part contributes zero since a spherical

tensor operator of non-zero rank cannot connect states of zero angular momentum and isotopic spin. The first part contributes zero, also in the case, when the ~~outer~~ ^{outer} N particles form initial and final states which are orthogonal to each other.

Thus, for charged meson production closed shells do not contribute since t_i contains an over all τ^\pm operator or because the values in the initial and final states differ by one. Also if the outer particles are in equivalent orbitals, the desired matrix element is just N times the matrix element of any one single particle operator between the nuclear states (i.e. between the N particle states, since closed shells remain closed shells and contribute one since the states are properly normalised), the factor $\frac{A!}{(A-N)!N!}$ obtained on summation cancelling with the normalisation factors $\sqrt{\left(\frac{(A-N)!N!}{A!}\right)}$ in the initial and final states. However, for the "elastic" photo production of neutral mesons the initial and final N particle states are the same and the t_i are of the form

$$t = \left[(t_P + t_N) + (t_P - t_N) \tau^0 \right] \exp(i \vec{k} \cdot \vec{r}) \quad (30)$$

where only the second terms of t_i 's contain the isobaric spin operator τ^0 and consequently contribute zero to the first part of the matrix element of T . To evaluate the contribution of the terms $(t_P + t_N) \exp(i \vec{k} \cdot \vec{r})$ in t forming the first part of T , we write

$$\begin{aligned}
 & (t_P + t_N) \exp(i \vec{k} \cdot \vec{r}) \\
 & = \left\{ \vec{\sigma} \cdot (\vec{K}_P + \vec{K}_N) + (L_P + L_N) \right\} \exp(i \vec{k} \cdot \vec{r}) \\
 & = \sum_{n=0,1} \sum_{\ell=0}^{\infty} 4\pi(\epsilon)^\ell j_\ell(kr) \sum_{\lambda, m_\lambda} (-1)^{\ell+n-\lambda} (-1)^{m_\lambda} \\
 & \quad (Y_\ell(\hat{k}) \times K_n)_{\lambda}^{m_\lambda} (Y_\ell(\hat{n}) \times \sigma_n)_{\lambda}^{m_\lambda} \quad (31)
 \end{aligned}$$

where $K_n = (K_P + K_N)_n$, treating spin dependent and spin independent terms alike as in section 2. The operative part is $(Y_\ell(\hat{n}) \times \sigma_n)$ and for our purpose only the terms with $\lambda = 0$ need be considered, all other terms contributing zero. The $\lambda = 0$ terms are explicitly

$$K_0 j_0(kr) + 4\pi(\epsilon) j_1(kr) (Y_1(\hat{k}) \times K_1)_0 (Y_1(\hat{n}) \times \sigma_1)_0 \quad (32)$$

of which only the first term contributes since the second term with $\ell = 1$ vanishes due to parity Clebsch-Gordon coefficient. Thus for elastic photoproduction the contribution from the closed shells is to be added to the contribution from the outer shell nucleons which is N times the matrix element of a single particle operator t_i , i referring to one of the outer N nucleons, taken between the N particle states. If however, the final nucleus is left in an excited state (when the process is inelastic) the closed shell

term again gives zero since the scalar product of the initial and final N particle states multiplying the term is zero as the N particle states are now orthogonal.

Orbital	q -dependence	Normalizing factor
$1s$	$e^{-\frac{1}{2}q^2}$	1
$2p$	$q e^{-\frac{1}{2}q^2}$	$2\sqrt{3}$
$2d$	$(\frac{3}{2}q^2 - \frac{3}{2}) e^{-\frac{1}{2}q^2}$	$2\sqrt{3}$
$3d$	$q^2 e^{-\frac{1}{2}q^2}$	$16\sqrt{15}$
$3f$	$(\frac{7}{2}q^3 - \frac{5}{2}q) e^{-\frac{1}{2}q^2}$	$16\sqrt{15}$
$4f$	$q^3 e^{-\frac{1}{2}q^2}$	$24\sqrt{105}$

To obtain the wave-functions $\psi_{nlm}(r)$ that are properly normalized to unity $\int \psi_{nlm}^* \psi_{nlm} d\tau = 1$, the functions of the second column should be multiplied by $(4\pi)^{-1/2} N_{nl}$, where $N_{nl} = \frac{1}{\sqrt{2\pi n! n!}}$ is listed in the third column. The dimensionless quantity q determines the 'size' parameter R by the relation $R = \hbar/q$, and the 'size parameter', in turn, is determined by $\langle R^2 \rangle$, the expectation value of the square of the radius of the nucleus.

*
Table I

Radial integral between various single particle states
Wave-Functions for the lowest states of the Harmonic oscillator in terms of the dimensionless quantity

$$q = \left(\frac{M\omega}{\hbar} \right)^{1/2} r$$

Orbital	q-dependence	Normalising factor N_{nl}
1 s	$e^{-1/2 q^2}$	4
1 p	$q e^{-1/2 q^2}$	8/3
2 s	$(q^2 - \frac{3}{2}) e^{-1/2 q^2}$	8/3
1 d	$q^2 e^{-1/2 q^2}$	16/15
2 p	$(q^3 - \frac{5}{2} q) e^{-1/2 q^2}$	16/15
1 f	$q^3 e^{-1/2 q^2}$	32/105

To obtain the wave-functions $f_{nl}(q)$ that are properly normalised to unity $\int f_{nl}^2(q) q^2 dq = 1$ the functions of the second column should be multiplied by $N_{nl}^{1/2} / \pi^{1/4}$ where

$N_{nl} = \frac{2^{n+l+1}}{(2n+2l-1)!}$ is listed in the third column. The dimensionless quantity q determines the 'size' parameter' b by the relation \hbar/b and the 'size parameter' in turn, is determined by $\langle r^2 \rangle$, the expectation value of the square of the radius of the nucleus.

* J.P.Elliot and A.M.Lane, Handbuch der Physik, Springer Verlag, Vol. 39, p. 248 (1957).

TABLE III

Radial integral between various single particle states

Single particle transitions		$\langle J_l(kr) \rangle$
$1s \rightarrow 1s$	$\langle J_0(kr) \rangle$	$e^{-k^2 b^2/4}$
$1p \rightarrow 1p$	$\langle J_0(kr) \rangle$	$(1 - \frac{k^2 b^2}{6}) e^{-k^2 b^2/4}$
	$\langle J_2(kr) \rangle$	$\frac{1}{6} k^2 b^2 e^{-k^2 b^2/4}$
$1d \rightarrow 1d$	$\langle J_0(kr) \rangle$	$\frac{1}{15} b^7 e^{-k^2 b^2/4}$ $\times \left\{ \frac{k^4 b^4}{4} - 5k^2 b^2 + 25 \right\}$
	$\langle J_2(kr) \rangle$	$\frac{2}{15} b^7 e^{-k^2 b^2/4}$ $\times \left\{ -\frac{b^4 k^4}{8} + \frac{7k^2 b^2}{4} - 5 \right\}$
	$\langle J_4(kr) \rangle$	$\frac{b^7}{15} e^{-k^2 b^2/4} \{ 5k^2 b^2 - 14 \}$
$2s \rightarrow 2s$	$\langle J_0(kr) \rangle$	$e^{-b^2 k^2/4} \left\{ 1 - \frac{1}{3} b^2 k^2 + \frac{1}{24} b^4 k^4 \right\}$

TABLE III

Single particle transitions	$M_{0,0}$	$M_{2,2}$	$M_{0,2} = M_{2,0}$
$s_{1/2} \rightarrow s_{1/2}$	$2 \{ k_0 ^2 + \sum_{m_\lambda} k_1^{m_\lambda} ^2 \}$	—	—
$p_{1/2} \rightarrow p_{1/2}$	$2 \{ k_0 ^2 + \frac{1}{9} \sum_{m_\lambda} k_1^{m_\lambda} ^2 \}$	$\frac{64}{9} \pi \sum_{m_\lambda} (Y_2(\hat{k}) \times K_1)^{m_\lambda} ^2$	$-\frac{16\sqrt{2}}{9} \pi \sum_{m_\lambda} (Y_0(\hat{k}) \times K_1)^{m_\lambda} (Y_2(\hat{k}) \times K_1)^{m_\lambda*}$
$p_{3/2} \rightarrow p_{3/2}$	$4 \{ k_0 ^2 + \frac{5}{9} \sum_{m_\lambda} k_1^{m_\lambda} ^2 \}$	$\frac{16}{5} \pi \sum_{m_\lambda} \{ (Y_2(\hat{k}) \times K_0)^{m_\lambda} ^2 + \frac{2}{9} (Y_2(\hat{k}) \times K_1)^{m_\lambda} ^2 + 3 (Y_2(\hat{k}) \times K_1)^{m_\lambda} ^2 \}$	$-\frac{16}{9} \sqrt{2} \pi \sum_{m_\lambda} (Y_0(\hat{k}) \times K_1)^{m_\lambda} (Y_2(\hat{k}) \times K_1)^{m_\lambda*}$
$p_{3/2} \rightarrow p_{1/2}$	$\frac{16}{9} \sum_{m_\lambda} k_1^{m_\lambda} ^2$	$\frac{16}{5} \pi \sum_{m_\lambda} \{ (Y_2(\hat{k}) \times K_0)^{m_\lambda} ^2 + \frac{5}{18} (Y_2(\hat{k}) \times K_1)^{m_\lambda} ^2 + \frac{3}{2} (Y_2(\hat{k}) \times K_1)^{m_\lambda} ^2 + \sqrt{6} (Y_2(\hat{k}) \times K_0)^{m_\lambda} (Y_2(\hat{k}) \times K_1)^{m_\lambda*} \}$	$\frac{32}{9} \pi \sum_{m_\lambda} (Y_0(\hat{k}) \times K_1)^{m_\lambda} (Y_2(\hat{k}) \times K_1)^{m_\lambda*}$

TABLE IV

Single particle transitions	$M_{0,0}$	$M_{2,2}$	$M_{0,2} = M_{2,0}$
$d_{3/2} \rightarrow d_{3/2}$	$4 \left\{ K_0 ^2 + \frac{1}{5} \sum_{m_\lambda} K_1^{m_\lambda} ^2 \right\}$	$\frac{32}{35} \pi \sum_{m_\lambda} \left\{ \frac{7}{2} (Y_2(\hat{k}) \times K_0)_2^{m_\lambda} ^2 + 7 (Y_2(\hat{k}) \times K_1)_1^{m_\lambda} ^2 + \frac{1}{7} (Y_2(\hat{k}) \times K_1)_3^{m_\lambda} ^2 \right\}$	$-\frac{16\sqrt{2}}{5} \pi \sum_{m_\lambda} (Y_0(\hat{k}) \times K_1)_1^{m_\lambda} (Y_2(\hat{k}) \times K_1)_1^{m_\lambda*}$
$d_{5/2} \rightarrow d_{5/2}$	$6 \left\{ K_0 ^2 + \frac{7}{15} \sum_{m_\lambda} K_1^{m_\lambda} ^2 \right\}$	$\frac{192}{35} \sum_{m_\lambda} \left\{ (Y_2(\hat{k}) \times K_0)_2^{m_\lambda} ^2 + \frac{1}{3} (Y_2(\hat{k}) \times K_1)_1^{m_\lambda} ^2 + \frac{9}{7} (Y_2(\hat{k}) \times K_1)_3^{m_\lambda} ^2 \right\}$	$-\frac{16\sqrt{2}}{5} \pi \sum_{m_\lambda} (Y_0(\hat{k}) \times K_1)_1^{m_\lambda} (Y_2(\hat{k}) \times K_1)_1^{m_\lambda*}$
$d_{5/2} \rightarrow s_{1/2}$	-	$16 \pi \sum_{m_\lambda} \left\{ \frac{3}{10} (Y_2(\hat{k}) \times K_0)_2^{m_\lambda} ^2 + \frac{1}{5} (Y_2(\hat{k}) \times K_1)_2^{m_\lambda} ^2 + \frac{1}{2} (Y_2(\hat{k}) \times K_1)_3^{m_\lambda} ^2 - \frac{\sqrt{6}}{5} (Y_2(\hat{k}) \times K_0)_2^{m_\lambda} (Y_2(\hat{k}) \times K_1)_2^{m_\lambda*} \right\}$	-
$d_{5/2} \rightarrow d_{3/2}$	$\frac{16}{5} \sum_{m_\lambda} K_1^{m_\lambda} ^2$	$\frac{160}{7} \pi \sum_{m_\lambda} \left\{ \frac{3}{50} (Y_2(\hat{k}) \times K_0)_2^{m_\lambda} ^2 + \frac{7}{100} (Y_2(\hat{k}) \times K_1)_1^{m_\lambda} ^2 + \frac{1}{4} (Y_2(\hat{k}) \times K_1)_2^{m_\lambda} ^2 + \frac{16}{175} (Y_2(\hat{k}) \times K_1)_3^{m_\lambda} ^2 + \frac{\sqrt{30}}{50} (Y_2(\hat{k}) \times K_0)_2^{m_\lambda} (Y_2(\hat{k}) \times K_1)_2^{m_\lambda*} \right\}$	$\frac{16\sqrt{2}}{5} \pi \sum_{m_\lambda} (Y_0(\hat{k}) \times K_1)_1^{m_\lambda} (Y_2(\hat{k}) \times K_1)_1^{m_\lambda*}$

TABLE IV (contd.)

Single particle transitions	$M_{4,4}$	$M_{2,4} = M_{4,2}$
$d_{3/2} \rightarrow d_{3/2}$	$\frac{2304}{245} \pi \sum_{m\lambda} (Y_4(\hat{k}) \times K_1)_{3}^{m\lambda} ^2$	$-\frac{192\sqrt{3}}{245} \pi \sum_{m\lambda} (Y_2(\hat{k}) \times K_1)_{3}^{m\lambda} (Y_4(\hat{k}) \times K_1)_{3}^{m\lambda*}$
$d_{5/2} \rightarrow d_{5/2}$	$\frac{16}{7} \pi \sum_{m\lambda} \left\{ (Y_4(\hat{k}) \times K_0)_{4}^{m\lambda} ^2 + \frac{4}{85} (Y_4(\hat{k}) \times K_1)_{3}^{m\lambda} ^2 + 5 (Y_4(\hat{k}) \times K_1)_{5}^{m\lambda} ^2 \right\}$	$\frac{96}{35} \sqrt{3} \pi \sum_{m\lambda} (Y_2(\hat{k}) \times K_1)_{3}^{m\lambda} (Y_4(\hat{k}) \times K_1)_{3}^{m\lambda*}$
$d_{5/2} \rightarrow s_{1/2}$	<p style="text-align: center;">—</p>	<p style="text-align: center;">—</p>
$d_{5/2} \rightarrow d_{3/2}$	$\frac{32}{7} \pi \sum_{m\lambda} \left\{ (Y_4(\hat{k}) \times K_0)_{4}^{m\lambda} ^2 + \frac{27}{140} (Y_4(\hat{k}) \times K_1)_{3}^{m\lambda} ^2 + \frac{5}{4} (Y_4(\hat{k}) \times K_1)_{4}^{m\lambda} ^2 + \sqrt{5} (Y_4(\hat{k}) \times K_0)_{4}^{m\lambda} (Y_4(\hat{k}) \times K_1)_{4}^{m\lambda*} \right\}$	$\frac{192}{245} \sqrt{3} \pi \sum_{m\lambda} (Y_2(\hat{k}) \times K_1)_{3}^{m\lambda} (Y_4(\hat{k}) \times K_1)_{3}^{m\lambda*}$

CHAPTER VIIISCATTERING OF PIONS BY COMPLEX NUCLEI1. Introduction

Extensive experimental study of the elastic and inelastic scattering of charged pions by complex nuclei has been made using various devices such as photographic emulsions¹⁾, cloud chambers²⁾ and counter techniques^{3) 4)}. Of these, it is claimed that the counter techniques offer a better energy resolution and hence more reliable for the investigation of elastic scattering of charged pions. Special mention may be made of the experiments⁴⁾ of Baker, Rainwater and Williams who have investigated the elastic scattering of 80 Mev π^- meson by lithium, carbon, aluminium and copper using a π^- detector which gives improved discrimination against low-energy loss inelastic events. The differential

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T.F. Fujii, Phys. Rev., 113, 695 (1959)
- 4) W.F. Baker, J. Rainwater and R. E. Williams, Phys. Rev., 112, 1763 (1958).

cross-sections they have obtained agree well with the previously reported results at small angles but smaller at large angles by a factor varying between two and ten.

Theoretical studies of the elastic scattering of fast particles by atomic nuclei have frequently been made within the framework of the 'optical model'. According to this model, one replaces the interaction between particle and nucleus by a complex potential well having the form

$$V_c(x) = V(\omega) \rho(x)$$

where $\rho(x)$ is the nuclear density and $V(\omega)$ is a complex quantity called the 'well-depth' and is a function of ω , the energy of the incident particle.

The application of the optical model to nuclear scattering has been two-fold. The most frequent application is to obtain the well-depth $V(\omega)$ for some $\rho(x)$ which best fits the scattering data at energy ω . The second application is complementary in character and it attempts to deduce the value of $V(\omega)$ from the knowledge of the elementary two-particle interactions. This has been done by Frank, Gammel and Watson⁵⁾ who have evaluated both the real and imaginary parts of the potential from the knowledge of the pion-nucleon scattering.

The usual optical model does not make use of the angular distribution of the elementary pion-nucleon scattering process but uses only the forward scattering amplitude $f(\theta)$. Thus the predicted angular distribution of the scattering is

5) R.M. Frank, J.L. Gammel and K.M. Watson, Phys. Rev. 101, 891, (1953).

determined by the nuclear density and the forward scattering amplitude $f(\theta)$ for neutrons and protons. But recently a modified optical analysis has been developed by Kisslinger⁶⁾ and this takes better account of $f(\theta)$. The need for such a modification has been felt by Baker, Byfield and Rainwater⁷⁾ who have found it not possible to fit their experimental data with the usual optical potential model having either a square or diffuse edge but found it necessary to use a Kisslinger-type model which takes account of the p -wave nature of the basic π -nucleon scattering process.

We attempt to develop in this chapter the opposite extreme point of view based on the 'impulse approximation' which uses the complete angular dependence of the elementary pion-nucleon scattering amplitude modified by a form factor of the nucleus. Such an attempt has already been made by Williams, Rainwater and Pevsner⁸⁾ who have tried to fit in their theoretical cross-section obtained by combining coherently the scattering amplitude for the pion-nucleon interactions weighted by a form factor with their experimental results of the scattering of charged pions from lithium at 78 Mev. It may be pointed out that Li^7 has spin 3/2 and hence the spin-dependent term of the pion-nucleon scattering amplitude will contribute along with the spin independent term even for the elastic scattering process; but Williams et al have neglected the spin-dependent part without much justification. We outline below

6) L. S. Kisslinger, Phys. Rev., 98, 761 (1955)

7) W. F. Baker, H. Byfield and J. Rainwater, Phys. Rev. 112, 1773 (1958).

8) Williams, Rainwater and Pevsner, Phys. Rev., 101, 412 (1956).

a rigorous method of evaluating the cross section and the procedure is essentially the same as outlined in Chapter VII.

2. Method based on impulse approximation

The pion-nucleon scattering amplitude can be symbolically written as $\vec{\sigma} \cdot \vec{K} + L$ of which the first term is the spin-dependent part and the second, the spin-independent part. The coefficients \vec{K} and L are functions of pion energy and momentum and the direction of the scattered pion and the pion-nucleon scattering phase shifts. We have already evaluated the square of the matrix element for such an operator between any two nuclear states assuming that the outer-shell nucleons alone contribute. That assumption is true only for the photoproduction of charged pions and the charge-exchange scattering of pions from nuclei. In the case of photoproduction of neutral pions and the direct scattering of pions from nuclei, the inner closed shells also contribute besides the outer shell nucleons.

The matrix element for the 'elastic' scattering of pions by a nucleus of mass number A can be written as

$$Q_{M_i \rightarrow M_f} = \left\langle \Psi^A (J M_f T M_T) \right| \sum_{i=1}^A \left(\frac{t_P^{(\omega)} + t_N^{(\omega)}}{2} + \frac{t_P^{(\omega)} - t_N^{(\omega)}}{2} \tau^o^{(\omega)} \right) e^{i\vec{k} \cdot \vec{r}_i} \left| \Psi^A (J M_i T M_T) \right\rangle$$

(1)

where τ^o is the isotopic spin operator. The quantities t_P and t_N are the direct scattering amplitudes of pion by the proton and the neutron respectively.

Writing

$$t_P = \vec{\sigma} \cdot \vec{K}_P + L_P \quad (2)$$

and

$$t_N = \vec{\sigma} \cdot \vec{K}_N + L_N$$

we obtain

$$\begin{aligned} \frac{t_P + t_N}{2} &= \vec{\sigma} \cdot \left(\frac{\vec{K}_P + \vec{K}_N}{2} \right) + \frac{L_P + L_N}{2} \\ &= \sum_{n=0,1} \vec{\sigma}_n \cdot \vec{K}_{nm}, \quad m=0 \end{aligned} \quad (3)$$

and

$$\begin{aligned} \frac{t_P - t_N}{2} &= \vec{\sigma} \cdot \left(\frac{\vec{K}_P - \vec{K}_N}{2} \right) + \frac{L_P - L_N}{2} \\ &= \sum_{n=0,1} \vec{\sigma}_n \cdot \vec{K}_{nm}, \quad m=1 \end{aligned}$$

where

$$\begin{aligned} K_{00} &= \frac{L_P + L_N}{2}, & K_{01} &= \frac{L_P - L_N}{2} \\ K_{10} &= \frac{K_P + K_N}{2}, & K_{11} &= \frac{K_P - K_N}{2} \end{aligned} \quad (4)$$

Now the matrix element (1) can be rewritten as

$$Q_{M_i \rightarrow M_f} = \left\langle \Psi^A (J M_f T M_T) \left| \sum_{l=1}^A \sum_{n=0,1} \sum_{m=0,1} \vec{\sigma}_n \cdot \vec{K}_{nm} \tau_n^{(k)} \right| \Psi^A (J M_i T M_T) \right\rangle \quad (5)$$

The evaluation of the square of such a matrix element has been discussed in great detail in Chapter VII and here we give only the results. After summing over final spin states and averaging over initial spin states, the square of the matrix element is obtained as

$$\begin{aligned}
 |Q|^2 &= \frac{1}{[J]^2} \sum_{l, l'} \sum_{n, n'} \sum_{m, m'} \sum_{\lambda, m_\lambda} 16 \pi^2 (i)^{l-l'} (-1)^{l+l'+n+n'} \\
 &\quad \frac{[J]}{[\lambda]^2} (Y_l(\hat{k}) \times K_{nm})_{\lambda}^{-m_\lambda} (Y_{l'}(\hat{k}) \times K_{n'm'})_{\lambda}^{-m_\lambda} \\
 &\quad \langle J T \parallel \sum_{i=1}^A (Y_l(\hat{r}_i) \times \sigma_n)_{\lambda} j_l(k r_i) \tau_m^{o(\epsilon)} \parallel J T \rangle \\
 &\quad \langle J T \parallel \sum_{i=1}^A (Y_{l'}(\hat{r}_i) \times \sigma_{n'})_{\lambda} j_{l'}(k r_i) \tau_{m'}^{o(\epsilon')} \parallel J T \rangle^*
 \end{aligned} \tag{6}$$

The reduced matrix elements in (6) can easily be evaluated.

If the nucleus contains certain closed shells plus N nucleons in the outer shell,

$$\begin{aligned}
 &\langle J T \parallel \sum_{i=1}^A (Y_l(\hat{r}_i) \times \sigma_n)_{\lambda} j_l(k r_i) \tau_m^{o(\epsilon)} \parallel J T \rangle \\
 &= \langle 00 \parallel \sum_{i=1}^{A-N} (Y_l(\hat{r}_i) \times \sigma_n)_{\lambda} j_l(k r_i) \tau_m^{o(\epsilon)} \parallel 00 \rangle \\
 &\quad + \langle J T \parallel \sum_{i=A-N+1}^A (Y_l(\hat{r}_i) \times \sigma_n)_{\lambda} j_l(k r_i) \tau_m^{o(\epsilon)} \parallel J T \rangle
 \end{aligned} \tag{7}$$

The first term will contribute only for $\lambda = m = 0$ since the matrix element is taken between states of zero angular momentum and zero isotopic spin. Besides, the value of l is restricted to zero by the parity Clebsch-Gordon coefficients. Thus the first term in (7) reduces to

$$\begin{aligned} & \langle 00 \parallel \sum_{i=1}^{A-N} (Y_{\ell}(\hat{r}_i) \times \sigma_n)_{\lambda} j_{\ell}(kr_i) \tau_m^{o(\omega)} \parallel 00 \rangle \\ & = (A-N) \frac{1}{\sqrt{4\pi}} \langle j_{\ell}(kr) \rangle \delta_{\ell,0} \delta_{n,0} \delta_{m,0} \end{aligned} \quad (8)$$

if there is only one closed shell or to a sum of similar terms if there are more shells. $\langle j_{\ell}(kr) \rangle$ is the expectation value of $j_{\ell}(kr)$ and it is different for different shells. The second term in (7)

$$\begin{aligned} & \langle \mathcal{J}T \parallel \sum_{i=A-N+1}^A (Y_{\ell}(\hat{r}_i) \times \sigma_n)_{\lambda} j_{\ell}(kr_i) \tau_m^{o(\omega)} \parallel \mathcal{J}T \rangle \\ & = N \langle \mathcal{J}T \parallel (Y_{\ell}(\hat{r}_i) \times \sigma_n)_{\lambda} j_{\ell}(kr_i) \tau_m^{o(\omega)} \parallel \mathcal{J}T \rangle \end{aligned} \quad (9)$$

can be evaluated as shown earlier in sec.3 of Chapter VII by using the concept of fractional parentage coefficients. The calculation of this term is of course a little tedious.

If we consider the elastic scattering of pions from nuclei for which both the proton and the neutron shells are closed e.g. carbon, then the problem becomes enormously simple. The square of the matrix element in the case of carbon is

$$|Q|^2 = |K_{00}|^2 \left\{ 4 \langle j_0(kr) \rangle_{1s} + 8 \langle j_0(kr) \rangle_{1p} \right\}^2 \quad (10)$$

The suffixes $1s$ and $1p$ denote that the expectation value of $j_0(kr)$ is taken between the corresponding states. K_{00} is the spin-independent term and thus we see that only the spin independent part contributes for the elastic scattering of pions by carbon. Using harmonic oscillator wave functions, exp. (10) can be evaluated.

$$|Q|^2 = 16 |K_{00}|^2 e^{-k^2/3\beta} \left\{ 3 - \frac{k^2}{3\beta} \right\}^2 \quad (11)$$

where β is a parameter which depends on the oscillation potential well depth, the value of which can be determined by finding the expectation value $\langle r^2 \rangle$ and equating it with the experimentally determined value. Alternately, the experimental data of the scattering of pions from nuclei can be used to determine the value of β which gives the best fit. Thus the scattering of pions from nuclei can be used as a tool to probe the structure of nuclei.

3. Discussion

Preliminary calculations have been carried out in the case of carbon and the agreement is good at low angles but the theoretical values obtained are much smaller at large angles. In this connection, attention may be drawn to the experimental findings of Baker, Rainwater and Williams. The differential cross-sections they have obtained agree well with the previously reported results at small angles but smaller at large angles by a factor varying between two and ten. It may be pointed out that at large angles, the inelastic events predominate due to large momentum transfers; and it is very likely that some of these events with low loss of energy may be counted as elastic since in almost all experiments, the elastic and inelastic events are distinguished only by measuring the energy of the scattered pions and the reliability of the experimental result thus largely depends on the energy resolution of the apparatus.

With the improvement in the experimental technique, the experiments register a lower cross section at large angles and with further improvement, it may be expected that the large angle cross-sections may further decrease giving an agreement with our preliminary findings. On the other hand, in order to obtain a fit with the existing experimental results for the elastic scattering at large angles, we have to take into account some of the inelastic events with low energy losses.

The preliminary investigation of the elastic scattering of pions by carbon using the impulse approximation and single nucleon scattering amplitudes seem to suggest that all the nucleons participate in the process and the scattering takes place throughout the volume. A detailed study is being made on the scattering by various nuclei and it is expected that this will throw light on the amount of absorption of pions by nuclei and indicate whether the scattering is a surface or a volume process. Any theory that is able to explain the photoproduction process from nuclei should also consistently explain the scattering process and it is very unlikely that it will be a surface process in one case and a volume process in the other case. It is also of great importance to investigate at the resonance region, for we have obtained discordant results in the case of elastic scattering of pions from deuterons in this energy region and it is still a matter of speculation whether the impulse approximation is a good approximation in the resonance region.

APPENDIX*

We briefly discuss in this appendix, the algebra of tensor operators, the use of which greatly simplifies the calculation of matrix elements of physical operators. The concept of tensor operator is due to Racah¹⁾. He defines a tensor operator T_λ of degree λ to have $(2\lambda+1)$ components T_λ^q with $q = \lambda, \lambda-1, \dots, -\lambda$, such that they transform under rotations like the spherical harmonics Y_λ^q or in other words ^{that} they transform according to the representation D_λ of the rotation group R_3 , or that they have the same commutation relations with the total angular momentum operator J as do the spherical harmonic operators Y_λ^q . Like the spherical harmonics, a hermitian tensor operator satisfies the relation

$$(T_\lambda^q)^* = (-1)^q T_\lambda^{-q} \quad (\text{A-1})$$

The tensor product of two tensor operators of rank λ_1 and λ_2 is given by

$$\begin{aligned} T_\lambda^{q(\lambda_1 \lambda_2)} &= [T_{\lambda_1}^{(1)} \times T_{\lambda_2}^{(2)}]_\lambda^q \\ &= \sum_{q_1} C(\lambda_1, \lambda_2, \lambda; q_1, q_2) T_{\lambda_1}^{q_1(1)} T_{\lambda_2}^{q_2(2)} \end{aligned} \quad (\text{A-2})$$

* In this appendix, we list only the formulae, the details regarding the derivation may be found in any standard text-book on Angular Momentum. M.E. Rose, Elementary Theory of Angular Momentum, John Wiley and Sons. A.R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press (1957). Also see, J.P. Elliott and A.M. Lane, Handbuch der Physik, Springer-Verlag (1957), p. 394.

1) G. Racah, Phys. Rev., 62, 438 (1942)

The value of λ is restricted by the Clebsch-Gordon coefficient $|\lambda_1 - \lambda_2| \leq \lambda \leq \lambda_1 + \lambda_2$. The inverse formula can be written as

$$T_{\lambda_1}^{q_1(1)} T_{\lambda_2}^{q_2(2)} = \sum_{\lambda} c(\lambda_1, \lambda_2, \lambda; q_1, q_2) T_{\lambda}^{q(1,2)} \quad (\text{A-3})$$

The scalar product of two tensor operators is defined by

$$(T_{\lambda} \cdot T_{\lambda}) = \sum_q (-1)^q T_{\lambda}^q T_{\lambda}^{-q} \quad (\text{A-4})$$

The reduced matrix element of a tensor operator can be defined by Wigner-Eckart theorem

$$\langle J' m' | T_{\lambda}^{q(1,2)} | J m \rangle = c(J q J'; m q) \langle J' || T_{\lambda} || J \rangle \quad (\text{A-5})$$

The reduced matrix elements of tensor operators that frequently occur are given below:

$$\begin{aligned} & \langle J_1' J_2' J' || T_{\lambda}^{q(1,2)} || J_1 J_2 J \rangle \\ &= [J_1'] [J_2'] [J] [\lambda] \left\{ \begin{array}{ccc} J_1 & \lambda_1 & J_1' \\ J_2 & \lambda_2 & J_2' \\ J & \lambda & J' \end{array} \right\} \\ & \langle J_1' || T_{\lambda_1} || J_1 \rangle \langle J_2' || T_{\lambda_2} || J_2 \rangle \end{aligned} \quad (\text{A-5})$$

$$\begin{aligned} & \langle J_1' J_2' J' || T_{\lambda_1}^{(1)} \cdot T_{\lambda_2}^{(2)} || J_1 J_2 J \rangle \\ &= (-1)^{J_1' + J_2 - J} [J_1'] [J_2'] W(J_1 J_2 J_1' J_2'; J \lambda_1) \end{aligned}$$

$$\langle J_1' || T_{\lambda_1}^{(1)} || J_1 \rangle \langle J_2' || T_{\lambda_2}^{(2)} || J_2 \rangle \quad (\text{A-6})$$

$$\langle J_1' J_2' J' \| T_{\lambda_1}^{(1)} \| J_1 J_2 J \rangle$$

$$= (-1)^{J_1 - J_1' - J + J'} [J_1'] [J] W(J_2 J_1 J' \lambda_1; J J_1')$$

$$\langle J_1' \| T_{\lambda_1} \| J_1 \rangle \delta_{J_2 J_2'} \quad (A-7)$$

$$\langle J_1' J_2' J' \| T_{\lambda_2}^{(2)} \| J_1 J_2 J \rangle$$

$$= [J_2'] [J] W(J_1 J_2 J' \lambda; J J_2')$$

$$\langle J_2' \| T_{\lambda_2} \| J_2 \rangle \delta_{J_1 J_1'} \quad (A-8)$$

where

$$[J] = (2J+1)^{1/2}$$

The 9-j symbol can be expanded as a sum of the products of Racah coefficients

$$\left\{ \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array} \right\} = \sum_x (2x+1) W(a b c; e x) W(d f h; e x) W(h e d a; g x)$$

(A-9)

In the summation, the values of x are restricted by the properties of Racah coefficients. Numerical tables of symbols and Racah coefficients are now available.

- 2) Tables of the Racah coefficients, edited by T. Ishidazu, Pan Pacific Press, Tokyo (1960).
 The 3-j and 6-j symbols, M. Rotenberg et al, The Technology Press, Massachusetts Institute of Technology, Cambridge, Mass (1959).
 Tables of the Wigner 9-j coefficients, J.W. Stephenson and K. Smith, North Carolina State College, Raleigh, N.C. (1957).

The reduced matrix elements of the angular momentum operators are given by

$$\langle j' \| \mathbf{J} \| j \rangle = \sqrt{j(j+1)} \delta_{jj'} \quad (\text{A-10})$$

$$\langle \frac{1}{2} \| \boldsymbol{\sigma} \| \frac{1}{2} \rangle = \sqrt{3} \quad (\text{A-11})$$

The reduced matrix element of a spherical harmonic is

$$\langle L_f \| Y_l \| L_i \rangle = c(L_i, l, L_f, 0, 0) \frac{[L_i][l]}{\sqrt{4\pi} [L_f]} \quad (\text{A-12})$$

CHAPTER IX

ON THE EQUIVALENCE OF FIELD THEORIES AND FORMAL POTENTIALS*

1. Introduction

Recently, Rosenbluth, Zeldovich and Lifshitz¹⁾ (hereinafter

referred to as RZL) **PART III** and the connection between

the field theories and the Feynman description of quantum

electrodynamical collision processes and have arrived at a new and

single word **ELECTRODYNAMIC AND DEUTERON** as follows.

Therefore it seems appropriate to elaborate arguments and

relations with **DISINTEGRATION PROCESSES** through the introduction

of a new procedure and the application of Wick's theorem

for deriving the equivalence which is stated throughout this

and similar to that article. We shall in this chapter make

a comparative study of the well-known method of establishing

the equivalence using Wick's theorem and the method of formal

potentials of RZL and prove that if the theory is "renormalizable"

* A. Rosenbluth, S. Zeldovich and L. P. Lifshitz, *Soviet Phys. Usp.* **10**, 100 (1968)

1) A. Rosenbluth, S. Zeldovich and L. P. Lifshitz, *The Physical Basis of Particle Theory*, J. Math. Anal. and Mech. **1**, 491 (1967)

2) S. D. Drell, *Relativistic Quantum Theory*, Wiley-Interscience, New York, 1967, p. 207, (1967)

S. D. Drell, *Phys. Rev.* **157**, 1318 (1967)

CHAPTER IX

ON THE EQUIVALENCE OF FIELD THEORETIC AND FEYNMAN FORMALISMS*

1. Introduction

Recently, Ramakrishnan, Radha and Thunga¹⁾ (hereinafter referred to as RRT) have re-examined the connection between the field theoretic and the Feynman description of quantum mechanical collision processes and have arrived at a new and simpler proof of the equivalence of the two formalisms. Therefore it seems surprising why elaborate arguments and notations were necessary particularly through the introduction of normal products and the application of Wick's theorem²⁾ for proving the equivalence which is almost transparent without recourse to such methods. We shall in this chapter make a comparative study of the well-known method of establishing the equivalence using Wick's theorem and the method of Ramakrishnan et al and prove that if the concept of 'typical

* A. Ramakrishnan, K. Venkatesan and V. Devanathan
J. Math. Anal. and Applns., (in press)

1) A. Ramakrishnan, T.K.Radha and R. Thunga,
The Physical Basis of Quantum Field Theory,
J. Math. Anal. and Applns., 4, 494 (1962)

2) S.S.Scheweber, H.A.Bethe and P.D.Hoffman,
Mesons and Fields, Vol. 1, (Row, Peterson & Co.,
New York), p. 207, (1956)

G.C.Wick, *Phys. Rev.*, 80, 268 (1950)

sequence of events' ^{*} used by RRT is applied to Wick's procedure, we obtain the same simplification.

2. Comparison of the two methods.

We shall first briefly recall Wick's procedure for reducing the S-matrix which contains the chronological product of operators into normal products. The first step is to replace the chronological operator P which occurs in the S-matrix

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dx_1 \dots dx_n P(\mathcal{H}_I(x_1) \dots \mathcal{H}_I(x_n)) \quad (1)$$

by the time-ordered product T of Wick; the two being the same in the conventional theories since the fermion operators which alone may cause a relative sign between P and T products occur in bilinear combination. The time-ordering fixes the relative positions of the interaction Hamiltonian densities, $\mathcal{H}_I(x)$ in the integrand of the S-matrix, but $\mathcal{H}_I(x)$ which itself is a product of annihilation and creation

* They conceive of the scattering as though it were a stochastic process and of course they deal with complex amplitudes instead of positive definite probabilities. "There seems to be a prejudice against the use of stochastic methods, a prejudice which is as deep rooted as it is unreasonable. As long as we are remembering the principle of superposition of amplitudes for 'typical complexions' in quantum mechanics, there is no reason why we cannot view a collision as a stochastic process."

operators has to be arranged in the normal product form i.e. with creation operators to the left of destruction operators.

The motivation for defining such a normal product stems from the property that its vacuum expectation value is zero. Also the introduction of the normal product in \mathcal{N}_I will remove a difficulty regarding the definition of the current operator in electrodynamics pointed out by Heisenberg viz., the infinity obtained for the vacuum expectation value of the current operator $j_\mu = \bar{\psi}(x) \gamma_\mu \psi(x)$ considered as the limit of

$\bar{\psi}(x) \gamma_\mu \psi(x')$ as $x' \rightarrow x$. This infinity in the vacuum expectation value for the current is attributed to the negative energy sea and it can be overcome by symmetrizing the theory in particle and anti-particle variables and defining the current density four vector as the commutator

$j_\mu = \frac{e}{2} [\bar{\psi}(x) \gamma_\mu \psi(x)]$ by writing j_μ in the normal product form. We shall show that this consideration does not in any way enter the discussion in a scattering process where the reduction will always concern operators belonging to different times (except in the trivial case of lowest order potential scattering) and in such a case the procedure suggested by RRT using the conventional Hamiltonian is identical with that using \mathcal{N}_I defined in normal product form.

The fundamental lemma used by Wick is

$$T(UV) = N(UV) + \dot{U}\dot{V} \quad (2)$$

where U and V are two field operators and T and N represent Wick's chronological and the normal product operators respectively. Writing $\mathcal{Y}_{b_I}(x)$ in the normal product form, we find the integrand of the S -matrix to involve mixed T products which can be expressed by means of Wick's theorem as a sum of normal products with all possible contractions (omitting of course the contractions between factors already in normal product form). For instance, in quantum electrodynamics, the integrand of the second order term in the S -matrix expansion can be written as

$$\begin{aligned} & T\left(N\left(\bar{\psi}(x_1) A(x_1) \psi(x_1)\right) N\left(\bar{\psi}(x_2) A(x_2) \psi(x_2)\right)\right) \\ &= N\left(\bar{\psi}(x_1) A(x_1) \psi(x_1) \bar{\psi}(x_2) A(x_2) \psi(x_2)\right) \\ &+ N\left(\bar{\psi}(x_1) A^a(x_1) \psi(x_1) \bar{\psi}(x_2) A^a(x_2) \psi(x_2)\right) \\ &+ N\left(\bar{\psi}(x_1) A(x_1) \psi^a(x_1) \bar{\psi}^a(x_2) A(x_2) \psi(x_2)\right) \\ &+ N\left(\bar{\psi}^a(x_1) A(x_1) \psi(x_1) \bar{\psi}(x_2) A(x_2) \psi^a(x_2)\right) \\ &+ N\left(\bar{\psi}(x_1) A^l(x_1) \psi^a(x_1) \bar{\psi}^a(x_2) A^l(x_2) \psi(x_2)\right) \\ &+ N\left(\bar{\psi}^a(x_1) A^l(x_1) \psi(x_1) \bar{\psi}(x_2) A^l(x_2) \psi^a(x_2)\right) \\ &+ N\left(\bar{\psi}^a(x_1) A(x_1) \psi^l(x_1) \bar{\psi}^l(x_2) A(x_2) \psi^a(x_2)\right) \\ &+ N\left(\bar{\psi}^a(x_1) A^c(x_1) \psi^l(x_1) \bar{\psi}^l(x_2) A^c(x_2) \psi^a(x_2)\right) \end{aligned} \quad (3)$$

where $A = \delta_{\mu} A_{\mu}$ and the superscripts a, b and c represent contracted pairs which are the Feynman propagators in configuration space. Each normal product N corresponds to a Feynman diagram in configuration space. For a given initial and final state, there is one and only one type of normal product which has non-vanishing matrix element between the given states. This normal product reduces to a product of propagators and the normal product of operators to be matched with the initial and final states, yielding the initial and final state wave functions.

It is to be noted that Wick's method is quite general and the reduction into normal products is made without reference to any specific process with given initial and final state.

RRT have emphasised that all the algebraic techniques involved in the method of Wick can be avoided and the procedure much simplified by specifying the initial and final state from the beginning itself. They do not work with the field operators but with their components, i.e. the annihilation and creation operators with their wave functions and consider a 'typical sequence' which contributes a nonvanishing term to the matrix element and then sum over all momenta to get the respective propagators corresponding to the intermediate states. Of course, the integral over space-time of the non-vanishing term corresponding to a typical sequence will vanish if energy and momenta are not conserved for the process. The procedure consists

therefore in building up the propagator from the non-vanishing constituents of the integrand of the S-matrix rather than picking out the non-vanishing matrix element from a general reduction of an n^{th} order product of field operators. No use, however, is made of the normal product form for \mathcal{H}_I . This does not lead to any discrepancy since we can now demonstrate that it is without consequence whether we take \mathcal{H}_I in the conventional or normal product form.

We shall now proceed to show that the method of matching prescribed in Ref. 1 is identical with the procedure of Wick provided we take only the non-vanishing term in the latter. Following RFT the integrand of the S-matrix taken between the initial and final states can be written as

$$\langle f | [n] \dots [l] \dots [k] \dots [j] \dots [i] | i \rangle \quad (3)$$

for a typical realisation of the time ordered sequence of operators with definite momentum labels. For simplicity, we shall consider the one-particle initial and final states defined by

$$|i\rangle = a_{p_1}^\dagger | \rangle_0 \quad \text{and} \quad |f\rangle = a_{p_2}^\dagger | \rangle_0$$

In (3), each bracket contains

$$\hat{a}_p^\dagger \hat{a}_{p'} + \hat{a}_p^\dagger \hat{b}_{p'}^\dagger + \hat{b}_p \hat{a}_{p'} + \hat{b}_p \hat{b}_{p'}^\dagger \quad (4)$$

which - leaving aside for the present the interaction potential eA which is not relevant for the discussion - is the integrand of ψ_I with definite momentum labels. Explicitly

$$\hat{a}_p = \frac{1}{\sqrt{2E_p}} a_p u_p e^{-i p x} \quad (5)$$

and similarly each creation and annihilation operator is accompanied by its corresponding wave function. The creation operator a_p^\dagger , which operating on the vacuum gives the initial state has to be matched with the corresponding annihilation operator a_p , having the same momentum label which for the particular realisation may occur in the k^{th} bracket.

Looking at the structure of the interaction (4), we find that this annihilation operator a_p may be accompanied either by \hat{a}_p^\dagger or \hat{b}_p , which in turn has to be matched with the corresponding annihilation or creation operator as shown by RRT.

Case I: If it is \hat{a}_p^\dagger , then it has to be matched with the corresponding annihilation operator a_p in l^{th} bracket with $t_l > t_k$.

Case II: On the other hand, if in the k^{th} bracket \hat{a}_p is accompanied by \hat{b}_p , which is the annihilation operator for a positron, we readily see that the corresponding

creation operator $b_{p'}^+$ should occur in the earlier bracket $[j]$ with $t_j < t_k$ since the positron which is destroyed should have been created earlier. From (4), we find that b^+ always occurs to the right of a^+ or b and so in bringing $b_{p'}$ to the left of $b_{p'}^+$, it has necessarily to cross over the left hand member of the bracket containing $b_{p'}^+$ which yields the negative sign, usually attributed to the positron propagator.

The process can be continued until all the creation and annihilation operators are matched. The matched pairs operating on the vacuum leave it undisturbed, for

$$\dots a_{p'} a_{p'}^+ a_{p_1} a_{p_1}^+ | \rangle_c = | \rangle_c$$

or

$$\dots b_{p'} b_{p'}^+ a_{p_1} a_{p_1}^+ | \rangle_c = | \rangle_c$$

Thus the integrand of the S-matrix is stripped off all its creation and annihilation operators and the brackets in (3) will now consist only of wave functions. In case I, we have

$$\langle | [n] \dots [l] \dots [k] \dots [1] | \rangle_c \quad (6)$$

with

$$[k] = \frac{1}{\sqrt{2E_{p'}}} \bar{u}_{p'} e^{-i p' x_k} \frac{1}{\sqrt{2E_{p_1}}} u_{p_1} e^{i p_1 x_k} \quad (7)$$

and

$$[l] = \frac{1}{\sqrt{2E_{p''}}} \bar{u}_{p''} e^{-i p'' x_l} \cdot \frac{1}{\sqrt{2E_{p'}}} u_{p'} e^{i p' x_l} \quad (8)$$

In case II, we have

$$\langle 0 | [n] \dots [k] \dots [j] \dots [1] | 0 \rangle \quad (9)$$

with

$$[k] = \frac{1}{\sqrt{2E_{p'}}} \bar{v}_{p'} e^{i p' x_k} \cdot \frac{1}{\sqrt{2E_{p_1}}} u_{p_1} e^{i p_1 x_k} \quad (10)$$

and

$$[j] = -\frac{1}{\sqrt{2E_{p''}}} \bar{u}_{p''} e^{-i p'' x_j} \cdot \frac{1}{\sqrt{2E_{p'}}} v_{p'} e^{-i p' x_j} \quad (11 a)$$

or alternatively

$$[j] = -\frac{1}{\sqrt{2E_{p''}}} \bar{v}_{p''} e^{i p'' x_j} \cdot \frac{1}{\sqrt{2E_{p'}}} v_{p'} e^{-i p' x_j} \quad (11 b)$$

Now each bracket is a mere number and so it may be rearranged as follows:-

Case I :

$$\langle 0 | \dots [l] [k] | 0 \rangle \quad (12)$$

Case II :

$$\langle 0 | \dots [j] [k] | 0 \rangle \quad (13)$$

It may be observed that the sequence in which the brackets are to be rearranged is the same as the sequence in which the brackets are picked in the process of matching the annihilation and creation operators. This rearrangement directly yields the

Feynman ordering and gives the complete Feynman matrix element in momentum representation if we sum over all momenta and spin.

It may be pointed out that the process of matching a $b_{p'}$ in any bracket with $b_{p'}^+$ in an earlier bracket yields a negative sign since we have to cross a b_p or an a_p^+ which is the left hand side partner of $b_{p'}^+$. On the other hand, in Wick's procedure the negative sign is already fed in by writing ψ_I in normal product form; i.e. in a typical term of ψ_I , $b_{p'}^+$ occurs to the left but has a negative sign attached. This process of matching is continued until all the annihilation and creation operators in the integrand of the S-matrix are exhausted except the creation operator $a_{p_2}^+$ which for the particular realisation may occur in the l th bracket. This has to be matched with the annihilation operator a_{p_2} in the final state. The matching of all intermediate operators (leaving aside the operators $a_{p_1}^+$ and a_{p_2} corresponding to initial and final states) corresponds in Wick's procedure, to contracted pairs and the matching of $a_{p_1}^+$ and a_{p_2} actually yields the initial and final state wave functions which are obtained in the method of Wick from the uncontracted field operators in the non-vanishing normal product when taken between the initial and final states.

1) A. Messiah, *Quantum Mechanics*, Vol. 1, 2nd ed., Wiley (1961)
 2) A. Messiah, *Quantum Mechanics*, Vol. 2, Wiley (1961)

Thus we see that the method of RRT is equivalent to taking the non-vanishing 'typical terms' of normal product between the initial and final states in Wick's procedure. All we need to employ in this simplified method is the commutation relations between the annihilation and creation operators without evaluating the commutation relations of the field variables in which case one is forced to consider not the spinors themselves but their elements.

In an earlier paper, Ramakrishnan and Ranganathan^{3), 4)} using stochastic arguments postulated the following interaction Hamiltonian

$$H_I(x) \supset \sum_{P, P'} \hat{a}_{P'}^+ \hat{a}_P + \hat{b}_{P'}^+ \hat{a}_P + \hat{b}_{P'}^+ \hat{b}_P + \hat{b}_{P'} \hat{a}_P \quad (14)$$

where the creation operator b^+ always occurs to the left of either a^+ or b , unlike the ordering obtained from $\Psi\psi$. At any vertex ~~xxx~~ one of the four fundamental processes can occur and the operators should occur in pairs but not necessarily in the order prescribed by $\Psi\psi$. A unique prescription can be given for the choice of the correct order based on the following arguments. The transition at a vertex due to interaction takes place not at a single space-time point but in a small

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- 3) A. Ramakrishnan and N.R. Ranganathan, Stochastic methods in Quantum Mechanics, J. Math. Anal. and Appln. 3, 261(1961)
 4) A. Ramakrishnan, Elementary Particles and Cosmic Rays, Pergamon Press (1962)

interval of time Δ . (This may obviate the difficulty regarding the vacuum expectation of the current operator mentioned earlier*.) The process of pair annihilation at t represents the transition of a positive energy electron at t to a negative energy state at $t + \Delta$, the interaction taking place in the interval Δ so that the electron destruction operator should be placed to the right of the positron destruction operator. In the case of pair creation in the interval between $t - \Delta$ and t , we view the process as though we trace the negative energy state of the electron at $t + \Delta$ back to a positive energy state at t so that in this case b_p^\dagger should be placed to the left of a_p^\dagger . For electron and positron scattering, the creation operators will be to the left of the annihilation operators. Thus the interaction Hamiltonian reads as (14).

Having postulated the interaction Hamiltonian in the above form, the matching of an a in some bracket with a^\dagger of the initial state is done assuming that the partner of a is moved along with it i.e. by bringing the bracket containing a and placing it in juxtaposition to the initial state, ignoring all intermediate brackets. If b is the

* Recently Cainiello has suggested that the limiting operation at $x \rightarrow x'$ need not necessarily make the vacuum expectation value of the current operator infinite but the latter may be merely undefined under such an operator. E.R.Cainiello, Nuovo Cimento, 14, 185 (1959).

partner of a , it will now be to the right of the bracket containing the corresponding b^\dagger which is to the left of its partner therein. In matching it with b , we have to cross that partner and this generates a negative sign and then we set $b^\dagger b = 1$ when it operates on the vacuum by interpreting $b^\dagger b$ as $a^\dagger a$ with opposite energy and momentum. In this procedure, the wave functions can be carried along with the operators while in the method of RRT, the operators have to be moved separately from the spinors and equivalently only the elements of the spinors can be carried along with the operators as in Wick's procedure. The choice between the various procedures is therefore a matter of taste; the only simplification being that we work with typical realisation of the integrand of the S-matrix i.e. with annihilation and creation operators and not with field variables.

Application via., the Doppler effect, electron-electron scattering, electron-positron scattering and the bremsstrahlung have been studied using Lee's procedure. Explicit calculations for the lowest order matrix elements are given and they have been shown to be identical with those obtained by an Feynman method. All the Feynman graphs have been obtained as a consequence without

- 1) Y. Nambu and S. Tomonaga, *Annals of the New York Academy of Sciences*, **13**, 467 (1958).
- 2) P. D. Lee, *Phys. Rev.*, **11**, 1702 (1959).

Chapter X

On the equal time commutator in electrodynamic processes*

1. Introduction

Any new method which has been developed to tackle problems in meson physics for which the perturbation expansion is not valid, should naturally be applicable to problems in electrodynamics and should reproduce the well-established results under suitable approximations. The method of Low¹⁾ which enables one to write the matrix element in the Heisenberg representation by postulating an interaction Hamiltonian is one such attempt to overcome the defects of perturbation expansions; but it should be anticipated that this method should also yield identical results in electrodynamics problems where the perturbation methods have been proved to be of great success. The various problems in quantum electrodynamics viz., the Compton effect, electron-electron scattering, electron-positron scattering and the bremsstrahlung have been studied using Low's procedure. Explicit expressions for the lowest order matrix elements are derived and they have been shown to be identical with those obtained by ~~the~~ Feynman method. All the Feynman graphs have been obtained as a consequence without

* V.Devanathan and K. Venkatesan, Presented at the Summer School in Theoretical Physics, Simla (1962).

1) F.E. Low, Phys. Rev., 97 , 1392 (1955).

the need of separate enumeration. The relative signs of the various terms in the matrix elements are also obtained without any recourse to ad hoc principles.

This investigation is undertaken with a view to assessing the contribution of the equal time commutator term in the various processes in electrodynamics and it is found to give, in the lowest order, either a part or the whole of the matrix element. The equal term commutator term occurs both in Low's theory and in the dispersion theory - the latter being model-independent and hence more general. The difference between the two approaches is that we are interested in the former case the time-ordered product and in the latter the retarded product involving a commutator. Although the name "equal time commutator term" has come into vogue in dispersion theory, we shall use the same nomenclature for the corresponding term in Low's approach.

The equal time commutator term always arises when we are converting into current operator a particle in the state vector, the field variable of which occurs either in bilinear or multilinear combination in the interaction Hamiltonian. This term is usually considered troublesome for it brings in its wake unrenormalized quantities. In dispersion theoretic treatments²⁾,

- 2) S. Gasiorowicz, Fortschritte der Physik, 8, 665 (1960).
 M.L. Goldberger, Phys. Rev., 99 979 (1955)
 J.C. Taylor, Lectures on Dispersion relations, Rochester University, (1960).

the contribution from this term is shown to be a polynomial in the square of the total energy, the dependence on the momentum transfer being arbitrary. The statement is usually made that the equal time commutator which can easily be evaluated once an interaction and the canonical commutation rules are assumed, need not be evaluated explicitly as they can be subtracted out since they do not depend on energy (provided there are no derivative interactions involved). It is interesting to study the role of equal time commutator in the conventional Lagrangian approach for as we shall show below by examples involving the electromagnetic interaction, that this term in fact forms part and in some cases even the whole of the matrix element in the lowest order (which would mean that it is dependent both on energy and momentum transfer).

2. The Compton Effect

First we shall consider the Compton effect viz., the scattering of a photon by an electron. The matrix element* for the process can be written as

$$\langle p' q' | S | p q \rangle = (-i)^2 \iint \frac{dx dy}{\sqrt{4 v_0 v_0'}} e^{-i q x} e^{i q' y} \langle p' | P(j_\mu(y) j_\nu(x)) | p \rangle \quad (1)$$

where p and p' represent the initial and final four

* We use natural units in which $\hbar = c = 1$ and the metric in which the scalar product of two ^{four}vectors a and b are represented by

$$a \cdot b = a_\mu b_\mu = a_0 b_0 - \vec{a} \cdot \vec{b}$$

momenta of the electron and q and q' the initial and final four momenta of the photon. $j_{\nu}(x)$ and $j_{\mu}(y)$ are the Heisenberg photon current operators and they satisfy the equation of motion

$$\square A_{\nu}(x) = j_{\nu}(x) = e \bar{\psi}(x) \gamma_{\nu} \psi(x) \quad (2)$$

Using the principle of translational invariance

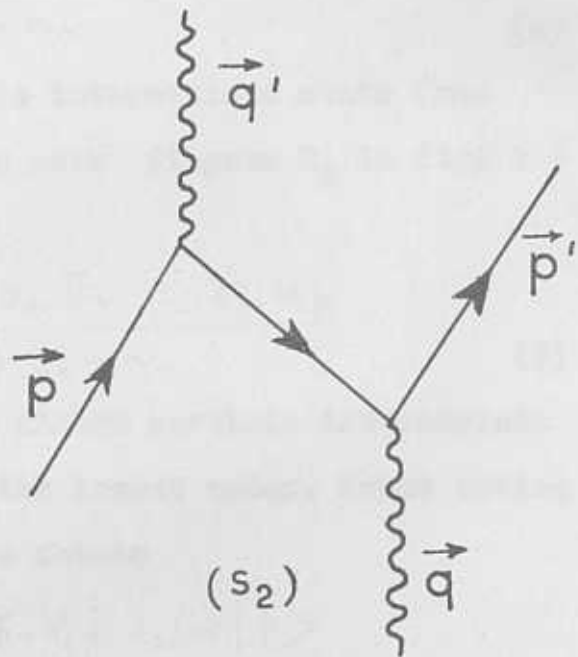
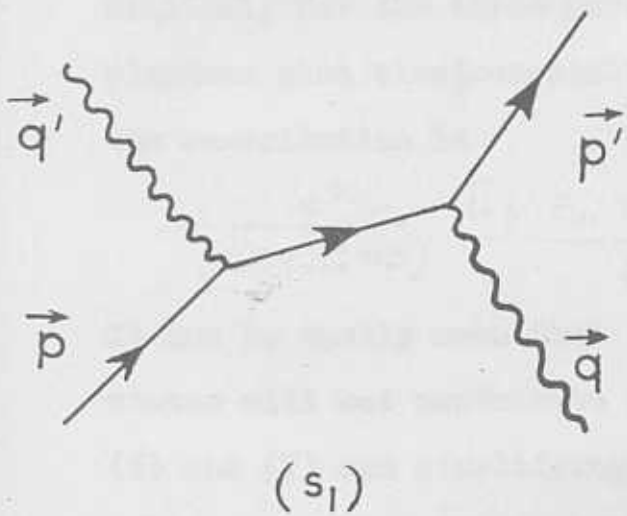
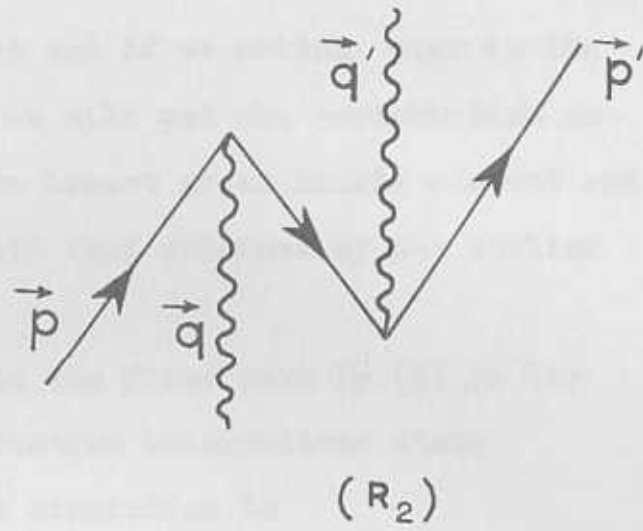
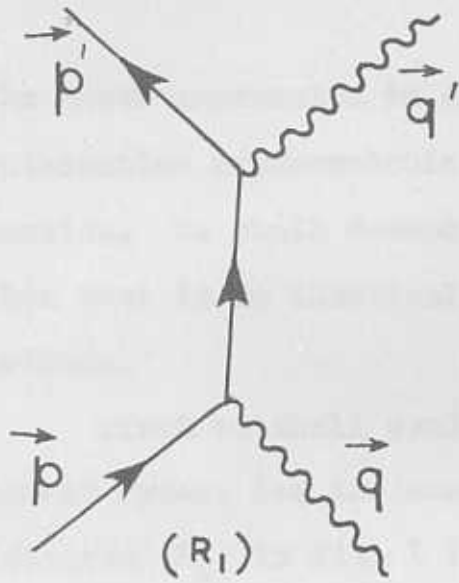
$$j_{\nu}(x) = e^{iPx} j_{\nu}(0) e^{-iPx} \quad (3)$$

where P is the four-momentum operator, the x and y dependence of the operators can be removed. The integration of the four-momentum y yields the energy-momentum conservation and we obtain

$$\langle P'q' | S | Pq \rangle = - \int \frac{dx'}{\sqrt{4q_0q'_0}} e^{-iqx'} \delta(P'+q'-P-q) \langle P' | P(j_{\mu}(0) j_{\nu}(x')) | P \rangle \quad (4)$$

where $x' = x - y$. To remove the x' dependence, a complete set of intermediate states have to be interposed and the integration of the variable x' gives the momentum of the intermediate state and the energy denominators. Finally, we obtain

$$\begin{aligned} \langle P'q' | S | Pq \rangle = & - \frac{i}{\sqrt{4q_0q'_0}} \delta(P'+q'-P-q) \\ & \left[\sum_{\vec{n}=\vec{P}+\vec{q}} \frac{\langle P' | j_{\mu}(0) | n \rangle \langle n | j_{\nu}(0) | P \rangle}{P_0+q_0-n_0+i\epsilon} \right. \\ & \left. + \sum_{\vec{n}=\vec{P}-\vec{q}} \frac{\langle P' | j_{\nu}(0) | n \rangle \langle n | j_{\mu}(0) | P \rangle}{P_0'-q_0-n_0+i\epsilon} \right] \quad (5) \end{aligned}$$



R_1, S_1 : ONE PARTICLE INTERMEDIATE STATE

R_2, S_2 : THREE PARTICLE INTERMEDIATE STATE

FIG. 1. THE COMPTON EFFECT.

The above expression is exact and if we switch over to the interaction representation, we will get the perturbation expansion. We shall deduce the lowest order matrix element and show that it is identical with that obtained by the earlier methods.

First we shall evaluate the first term in (5) in the lowest order. For the one-electron intermediate state (diagram R_1 in fig. 1) it simplifies to

$$\frac{e^2}{\sqrt{16 p_0 p'_0 n^2}} \frac{\bar{u}_{p'} \epsilon_\mu \gamma_\mu u_n \bar{u}_n \epsilon_\nu \gamma_\nu u_p}{p_0 + q_0 - n_0} \quad (6)$$

Similarly for the three-particle intermediate state (one electron plus electron-positron pair diagram R_2 in fig. 1) the contribution is

$$\frac{e^2}{\sqrt{(16 p_0 p'_0 n_0^2)}} \frac{\bar{u}_{p'} \epsilon_\mu \gamma_\mu v_n \bar{v}_n \epsilon_\nu \gamma_\nu u_p}{p_0 + q_0 + n_0} \quad (7)$$

It can be easily seen that the higher particle intermediate states will not contribute in the lowest order. Hence adding (6) and (7) and simplifying, we obtain

$$\begin{aligned} & \frac{\langle p' | j_\mu^{(0)} | n \rangle \langle \vec{n} = \vec{p} + \vec{q} | j_\nu^{(0)} | p \rangle}{p_0 + q_0 - n_0} \\ &= \frac{e^2}{\sqrt{4 p_0 p'_0}} \left[\bar{u}_{p'} \epsilon_\mu \gamma_\mu \frac{1}{p_0 + q_0 - n_0} \epsilon_\nu \gamma_\nu u_p \right] \quad (8) \end{aligned}$$

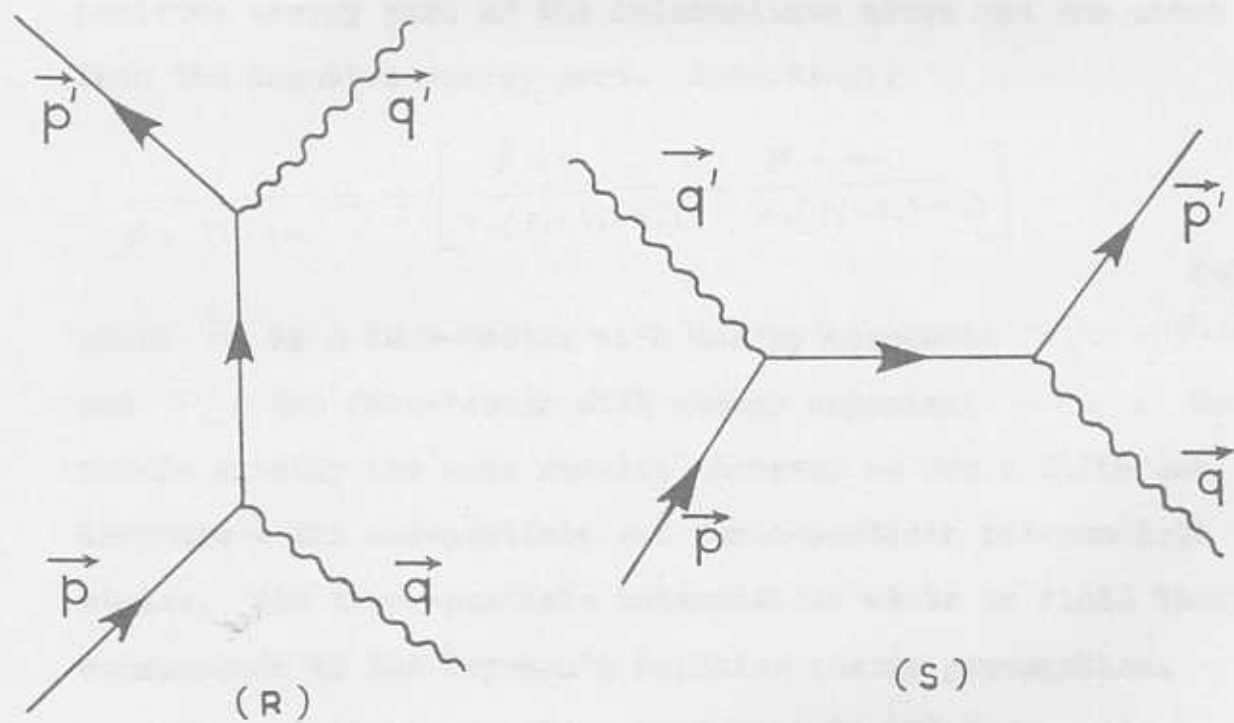


FIG.2 . THE COMPTON EFFECT

Expression (8) is exactly ~~the~~ ^{the} matrix element³⁾ corresponding to one of the Feynman diagrams (Diagram R in fig. 2) . Ramakrishnan et al⁴⁾ have shown from different considerations that the Feynman propagator in momentum representation can be conveniently split up into two parts - one arising from the positive energy part of the intermediate state and the other from the negative energy part. Accordingly

$$\frac{1}{\not{p} + \not{q} - m} = \frac{1}{2} \left[\frac{\not{p} + m}{n_0(p_0 + q_0 - n_0)} - \frac{\bar{\not{p}} + m}{n_0(p_0 + q_0 + n_0)} \right] \quad (9)$$

where \not{p} is a four-vector with energy component $n_0 = +\sqrt{(\vec{p} + \vec{q})^2 + m^2}$ and $\bar{\not{p}}$, the four-vector with energy component $-n_0$. We obtain exactly the same result; however we use a different language - the one-particle and three-particle intermediate states. The three-particle intermediate state in field theory corresponds to the Feynman's negative energy propagation.

In a similar way, the second term in (5) can easily be evaluated. As shown earlier, the one-particle and three-particle intermediate states alone contribute and adding them we obtain

$$\begin{aligned} \sum_n \frac{\langle p' | j_\nu(\omega) | n \rangle \langle \vec{n} = \vec{p}' - \vec{q} | j_\mu(\omega) | p \rangle}{p'_0 - q_0 - n_0 + i\epsilon} \\ = \frac{e^2}{\sqrt{4 p_0 p'_0}} \left[\bar{u}_{p'} \epsilon_\mu \gamma_\mu \frac{1}{\not{p} - \not{q}' - m} \epsilon_\nu \gamma_\nu u_p \right] \quad (10) \end{aligned}$$

3) R.P.Feynman, Quantum Electrodynamics, Lecture Notes, Caltech (1953).

4) A. Ramakrishnan, T.K.Radha and R. Thunga, Proc. Ind. Acad.Sci., 52, 228 (1960)

Expression (10) is identical with the Feynman matrix element corresponding to the other diagram. (Diagram S in fig. 2)

Adding (8) and (10), we get the complex matrix element for the Compton effect in the lowest order (second order)

$$M^{(2)} = - \frac{ie^2}{\sqrt{(16 q_0 q'_0 p_0 p'_0)}} \delta(p' + q' - p - q) \left\{ \bar{u}_{p'} \not{\epsilon} \left(\frac{1}{\not{p} + \not{q} - m} + \frac{1}{\not{p} - \not{q}' - m} \right) \not{\epsilon} u_p \right\} \quad (11)$$

One of the principal features of Low's method is the transcription of some of the particles in the initial or final state into current operators and there is a variety of choice. In the above discussion, we have converted the initial and final photons into current operators and thereby avoided the equal time commutator. Instead, if we convert one of the photons and one of the electrons into current operators, we will obtain the equal time commutator term as shown below

$$\langle p' q' | S | p q \rangle = - \frac{1}{\sqrt{4 q_0 p_0}} \bar{u}_{p'} \iint dx dy e^{-iqx} e^{ip'y} \langle q' | T(j(y), j_\mu(x)) + i\delta(x-y) e e_\mu \gamma_\mu \psi(x) | p \rangle \quad (12)$$

$j(y)$ is the electron current operator given by

$$(i\not{\epsilon} - m) \psi(y) = e \gamma_\mu \psi(y) A_\mu(y) + \delta m \psi(y) = j(y)$$

Removing the x and y dependence of the operators and integrating them out, we obtain

$$\begin{aligned}
 \langle p'q' | S | pq \rangle = & - \frac{i}{\sqrt{4q_0 p_0'}} \delta(p'+q'-p-q) \bar{u}_{p'} \\
 & \left[\sum_n \frac{\langle q' | J(\omega) | n \rangle \langle \vec{n} = \vec{p}' + \vec{q}' | j_{\mu}(\omega) | p \rangle}{p_0' + q_0' - n_0 + i\varepsilon} \right. \\
 & - \sum_n \frac{\langle q' | j_{\mu}(\omega) | n \rangle \langle \vec{n} = \vec{p}' - \vec{p} | J(\omega) | p \rangle}{p_0' + n_0 - p_0 - i\varepsilon} \\
 & \left. + e e_{\mu} \gamma_{\mu} \frac{\langle q' | J(\omega) | p \rangle}{\gamma_{\mu} (p-q')_{\mu} - m} \right] \quad (13)
 \end{aligned}$$

Equation (13) is exact and the operators and the state vectors are in Heisenberg representation. The third term in (13) arises from the equal time commutator and when we rewrite it as

$$- \frac{i}{\sqrt{4p_0' q_0}} \delta(p'+q'-p-q) \sum_n \bar{u}_{p'} e e_{\mu} \gamma_{\mu} u_n(p-q') \frac{1}{(p-q')^2 - m^2} \bar{u}_n(p-q') \langle q' | J(\omega) | p \rangle$$

We find that the first half of the matrix element represents the bare electron-photon vertex and the second half, the physical electron-photon vertex. Thus when we write the former in terms of renormalized quantities, we will be having

* This point has been noticed by Goldberger, Lurie and Yennie while studying the single variable dispersion relations for nucleon-nucleon scattering. *Ann. Phys.* 20, 273 (1967)

the renormalization constants as factors* and they have to be evaluated suitably.

Reverting to the interaction representation and thereby to the perturbation expansion, the first term in (13) is evaluated in the lowest order. The contribution from the one-particle intermediate state is the same as (6). However, the three-particle intermediate state vanishes, in the second order, if we make a direct evaluation. This is, of course, not correct and the three particle intermediate state do contribute as we have seen earlier and hence it seems desirable to convert whenever possible only the bosons into current operators instead of fermions and if we do otherwise, the vertices have to be interpreted suitably. This does not give rise to any ambiguity as long as we work in the Heisenberg representation and identify the matrix element as the product of two matrix elements ~~which~~ which arise when we have a three-particle intermediate state and take the contribution in the lowest order. Now we can go back to the interaction representation and obtain the complementary term (7).

The second term in (13) identically vanishes. The third term directly gives the propagator for the other Feynman diagram and in the lowest order it reduces to the second term in (11).

* This point has been noticed by Goldberger, Oehme and Nambu while studying the single variable dispersion relations for nucleon-nucleon scattering; M.L. Goldberger et al, Ann. Phys. 2, 226 (1957)

As the third alternative, we can pull in both the electrons from the state vector and convert them into current operators. But, as pointed out earlier, it is convenient to keep the fermions in the state vectors and convert only the bosons into current operators.

3. Electron-electron and electron-positron scatterings

The matrix element for electron-electron scattering can be written as products of the known electron-photon vertices by converting two of the electrons, one from the initial and the other from the final state, into current operators. In this case, the equal time commutator term is unavoidable and in fact, it represents the matrix element corresponding to one of the Feynman diagrams (diagram R in fig. 3). The complete matrix element can be written as

$$\begin{aligned}
 \langle P_3 P_4 | S | P_1 P_2 \rangle &= - \frac{i}{\sqrt{4 R_0 P_{30}}} \delta(P_1 + P_2 - P_3 - P_4) \\
 &\left[\bar{u}(P_3) \left\{ \frac{\langle P_4 | J^{(\omega)} | n \rangle \langle \vec{n} = \vec{P}_1 + \vec{P}_2 | J^{(\omega)} | P_2 \rangle}{(P_1)_0 + (P_2)_0 - n_0 + i\epsilon} \right. \right. \\
 &+ \left. \frac{\langle P_4 | J^{(\omega)} | n \rangle \langle \vec{n} = \vec{P}_4 - \vec{P}_1 | J^{(\omega)} | P_2 \rangle}{(P_4)_0 - (P_1)_0 - (P_n)_0 + i\epsilon} \right\} \\
 &\left. - e \bar{u}(P_3) \gamma_\mu u(P_1) \frac{\langle P_4 | j_\mu^{(\omega)} | P_2 \rangle}{(P_4 - P_2)^2} \right] \quad (14)
 \end{aligned}$$

Equation (14) is an integral equation, the first term being the homogeneous term involving the electron-electron scattering matrix element. The second term is the product of electron-photon vertices which can be easily evaluated. The one-photon intermediate state in the lowest order gives the following contribution

$$\frac{1}{2\omega_n} \frac{\bar{u}(p_3) \gamma_\mu u(p_2) \bar{u}(p_4) \gamma_\mu u(p_1)}{\{(p_4)_0 - (p_1)_0\} - \omega_n}$$

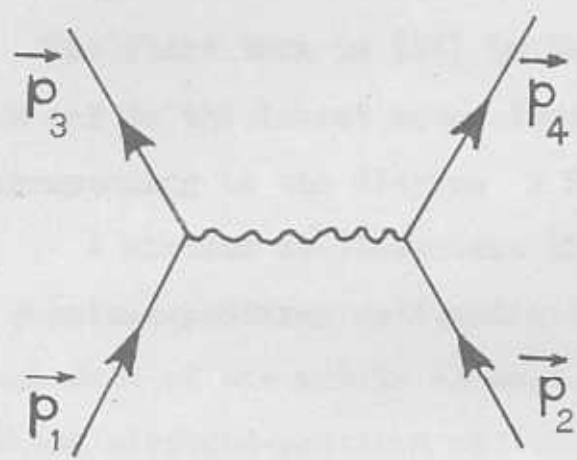
where ω_n represents the energy of the photon. The above corresponds to the emission of a photon by the electron of momentum p_2 and its absorption by the electron of momentum p_1 . To this the complementary term representing the emission of the photon by the electron of momentum p_2 and its subsequent absorption by the electron of momentum p_1 is to be added. This yields the complete matrix element corresponding to one of the Feynman diagrams (diagram S in fig.3)

$$\frac{(\bar{u}(p_3) \gamma_\mu u(p_2)) (\bar{u}(p_4) \gamma_\mu u(p_1))}{(p_4 - p_1)^2}$$

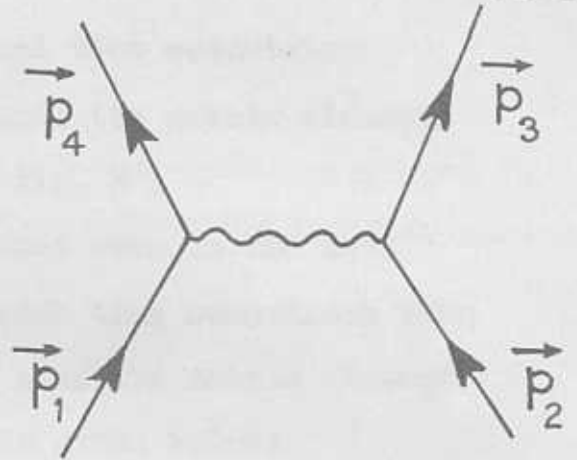
since

$$\frac{1}{2\omega_n} \left[\frac{1}{(p_4)_0 - (p_1)_0 - \omega_n} - \frac{1}{(p_4)_0 - (p_1)_0 + \omega_n} \right] = \frac{1}{(p_4 - p_1)^2}$$

The next higher intermediate state viz., one photon plus electron-positron pair gives disconnected Feynman diagrams in the second order.

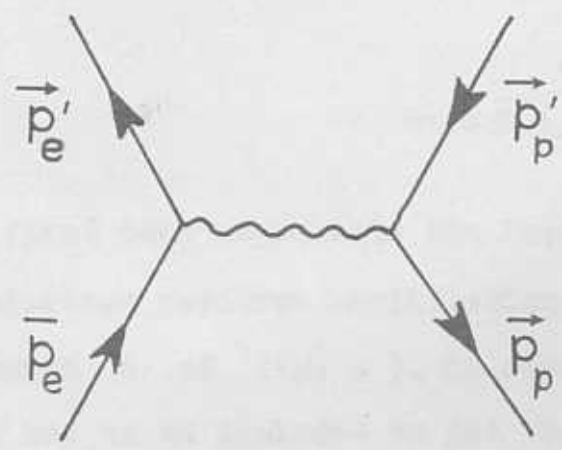


(R)

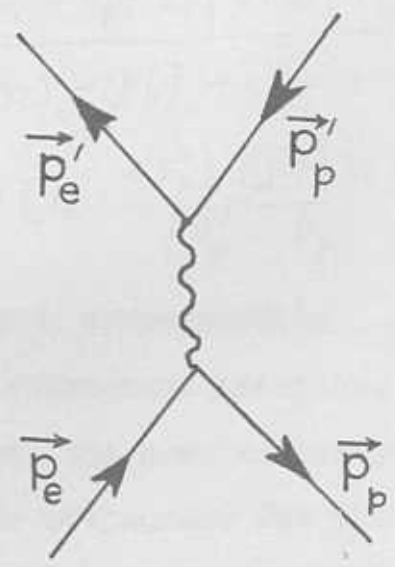


(S)

FIG. 3. ELECTRON - ELECTRON SCATTERING



(R)



(S)

FIG. 4. ELECTRON - POSITRON SCATTERING

The third term in (14) is the equal time commutator term and in the lowest order it represents the matrix element corresponding to the diagram Σ R of fig. 3.

A similar consideration shows that even in the case of electron-positron scattering the equal time commutator term forms part of the matrix element. The complete matrix element for the electron-positron scattering is given below:

$$\begin{aligned} \langle P'_e P'_p | S | P_e P_p \rangle &= - \frac{i}{\sqrt{4(P_e)_0 (P'_e)_0}} \delta(P_e + P_p - P_{e'} - P_{p'}) \\ &\left[\bar{u}(P'_e) \left\{ \frac{\langle P'_p | J^T(\omega) | n \rangle \langle \vec{n} = \vec{P}_p + \vec{P}_e | J(\omega) | P_p \rangle}{(P_p)_0 + (P_e)_0 - (n)_0 + i\varepsilon} \right. \right. \\ &\quad \left. \left. - \frac{\langle P'_p | J(\omega) | n \rangle \langle \vec{n} = \vec{P}'_p - \vec{P}_e | J^T(\omega) | P_p \rangle}{(P_e)_0 + (n)_0 - (P'_p)_0 - i\varepsilon} \right\} \right. \\ &\quad \left. - e \bar{u}(P'_e) \gamma_\mu u(P_e) \frac{\langle P'_p | j_\mu(\omega) | P_p \rangle}{(P'_p - P_p)^2} \right] \end{aligned}$$

The first term represents the Feynman diagram corresponding to electron positron annihilation and its subsequent creation. (diagram S of fig. 4). As shown earlier, the complementary term has to be included to get the complete propagator for the virtual photon. The second term vanishes. The third term is the equal time commutator term and it represents the matrix element corresponding to the other diagram (diagram R of fig. 4).

4. Other Processes

Some further examples where the equal time commutator term gives the whole matrix element or a part thereof, are as follows:

a) Electron-proton scattering where in the lowest order of the electromagnetic coupling constant, the complete matrix element is given by the equal time commutator term.

b) Bremsstrahlung which may be considered as the limiting case⁵⁾ of radiative scattering of electrons by a proton as the proton mass is made infinite.

c) Electro-pion production⁶⁾ where in the lowest order, the equal time commutator term gives the complete matrix element for the process, making it equivalent to pion production by a virtual photon.

d) Electro-disintegration of the deuteron which again can be equated to disintegration by a virtual photon.

e) The contribution to the matrix element for the photo-production of pions from nucleons arising from the photon-pion interaction term, viz.,

$$\left(\varphi^* \frac{\partial \varphi}{\partial x_\mu} - \varphi \frac{\partial \varphi^*}{\partial x_\mu} \right) A_\mu(x)$$

5) A. Ramakrishnan, G. Bhanathi and S. Induathi, Proc. Ind. Acad. Sci., 53, 306 (1961)

6) S. Fubini et al, Phys. Rev., 111, 329 (1958). This result can be obtained in a much easier way by using Low's method and commuting both the operators of the initial and final electrons through the S-matrix.

which is bilinear in the pion field variable ϕ and hence can give rise to an equal time commutator term which is the "pion-current" term.

In all these cases, we see that the equal time commutator term forms the complete matrix element (processes (a), (b), (c) and (d) or a part of it in the other cases considered. Thus we are forced to the conclusion that in the conventional approach using an interaction Hamiltonian we have to include the contribution from the equal time commutator in the calculation. Another notable feature that emerges from our consideration is that there is no need for separate symmetrization or antisymmetrization of the matrix element as this is already imbedded in this method of deriving the matrix element.

are studied in a unified way using Eqs. (1) and (2) which bring out the similarity and interconnection between the matrix elements for these processes.

1) Considered process I under the regular approximation. They note the observation that to obtain the observed angular distribution, it is necessary to include the

2) ρ -state wave function of the deuteron. Later they have calculated the contribution of the pion absorption process to the total scattering cross section of the pion on deuteron. To avoid the label for pion absorption due to deuteron and various other terms the absorption is a co-operative process

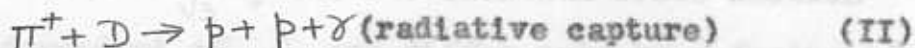
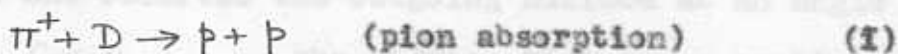
1) G. S. Gatto, E. L. Gell-Mann, M. L. Goldberger and S. M. Dan, Phys. Rev., 125, 201 (1961)
 2) E. S. Gatto, Phys. Rev., 125, 205 (1961)

CHAPTER XI

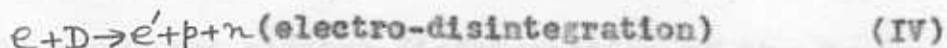
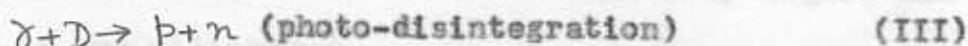
DISINTEGRATION PROCESSES IN DEUTERIUM

1. Introduction

In Chapter IV, we have studied the elastic, inelastic and charge exchange scattering of pions by deuterons. In the present chapter, we discuss the absorption processes



and also the disintegration processes initiated by a photon and an electron, namely



These processes are studied in a unified way using Low's method which brings out the similarity and interconnection between the matrix elements for these processes.

1) Chew et al considered process I under the impulse approximation. They made the observation that to explain the observed angular distribution, it is necessary to include the

2) D -state wave function of the deuteron. Later Rockmore calculated the contribution of the pion absorption process to the total scattering cross section of the pion on deuterium. He used the model for pion absorption due to Brueckner and Watson which treats the absorption as a co-operative process

1) G. F. Chew, M. L. Goldberger, Steinberger and C. N. Yang, Phys. Rev., 84, 581 (1951)

2) R. M. Rockmore, Phys. Rev., 105, 256 (1957)

between two nucleons. He obtained an order of magnitude accuracy for the cross section which, according to the experiment of Rogers and Lederman³⁾ is about 7 millibarns at pion kinetic energy of 25 Mev. The differential cross section is of the form

$$\frac{d\sigma}{d\Omega} = \frac{A + \cos^2\theta}{A + \frac{1}{3}} \cdot \frac{\sigma}{4\pi}$$

so that if one observes the outgoing nucleon at an angle θ for which $\cos\theta = \frac{1}{\sqrt{3}}$, the differential cross-section will be independent of A and differ only by a factor of 4π from the total cross section. The total cross section for the radiative capture at 85 Mev as given by Rogers and Lederman is one millibarn.

The photo and electro-disintegration of the deuteron have been studied by different approaches by various workers. Pearlstein and Klein⁴⁾ have suggested that it is convenient to divide the analysis of the photo-disintegration problem into two parts corresponding to two energy regions. In one, the region below 100 Mev, the experimental results can be completely understood within the framework of conventional quantum mechanics which gives the result that the process essentially proceeds through a dipole transition in this

3) K.C. Rogers and L.M. Lederman, Phys. Rev., 105, 247 (1957)

4) L.D. Pearlstein and A. Klein, Phys. Rev., 108, 193 (1960)

energy region. In the region above 100 Mev, however, there is evidence for the presence of virtual meson currents which make themselves felt with increasing photon energy. Mixing a covariant formalism for deriving a formally exact expression for the matrix element for the process with a phenomenological procedure, they show that such a division has a theoretical basis and that contributions from the virtual mesons are negligible below 100 Mev. Retaining only the one-meson exchange effect they connect the matrix element for dipole transition with the p -wave meson-nucleon coupling constant and the amplitude for the photo-production off the energy shell.

As regards the electro-disintegration process it has been most useful in studying the form factors of the nucleon. The basic non-relativistic theory of the process was given by Jankus⁵⁾ who calculated the differential cross section using a ~~Maxwell~~ Hulthenian wave function of the outgoing nucleons. The final state interaction between the outgoing nucleons was treated approximately using a central force model for the interaction. Durand⁶⁾ made a more complete formulation of the problem of calculating final state effects and studying the influence of the D-state component of the deuteron wave function on the process. He has also discussed the relativistic corrections using the methods of

5) V.Z. Jankus, Phys. Rev., 102, 1586 (1956)
 6) Durand III, Phys. Rev., 115, 1020 (1959);
123, 1393 (1961)

of dispersion relations and has pointed out that the calculations of the final state interaction would require an examination of the double spectral function in the Mandelstam representation ⁷⁾ for the transition matrix element. The importance of the final state interaction and mesonic corrections near the threshold for deuteron break-up was also pointed out.

We have studied the above processes in a unified way using Low's technique. The deuteron which is of composite structure is always kept in the state vector and only the other particles are converted into current operators when necessary and so the matrix element in each case will include a term involving the vertex which has a pole, the residue of which is the normalization factor of the deuteron-wave function. Integral equations are derived for the matrix elements for the various processes which are interrelated. A better insight into the physical process can be obtained by representing the various terms of the matrix element by means of diagrams.

7) S. Mandelstam, Phys. Rev., 112, 1344 (1958)

2. The Matrix Elements

Using Low's method, the matrix element for the process

$$\pi^+ + D \rightarrow p + p$$

can be written as

$$\langle p_1 p_2 | T | q d \rangle = - \frac{1}{\sqrt{4(p_2)_0 q_0}} \bar{u}(p_2) \left[\iint dx dy e^{-i q x} e^{i p_2 y} \langle p_1 | (j_\pi(x) J_N(y))_+ + i \delta(y-x) g \gamma_5 \tau_x \psi(x) | d \rangle \right] \quad (1)$$

where q and d represent the four-momenta of the incident pion and the target deuteron respectively, and p_1 and p_2 , the four-momentum of the outgoing protons. The operators in equation (1) are the Heisenberg operators and their dependence on the variables x and y can be removed by means of translational invariance.

$$F(x) = e^{i P x} F(0) e^{-i P x}$$

where P is the four-momentum operator. The integration of one of the variables in equation (1) implies the conservation of momentum and energy and the integration over the other variable, which can be performed on interposing the complete set of intermediate states, yields the energy denominations. Finally, we obtain

$$\langle p_1 p_2 | T | q d \rangle = - \frac{i}{\sqrt{4(p_2)_0 q_0}} \bar{u}(p_2) \delta(p_1 + p_2 - q - d) \left\{ \sum_n \frac{\langle p_1 | J_N(0) | n \rangle \langle n | j_\pi(0) | d \rangle}{(p_2)_0 + (p_1)_0 - n_0 + i \varepsilon} \delta(\vec{n} - \vec{p}_1 + \vec{p}_2) \right. \\ \left. - \sum_n \frac{\langle p_1 | j_\pi(0) | n \rangle \langle n | J_N(0) | d \rangle}{(p_2)_0 + n_0 - d_0 - i \varepsilon} \delta(\vec{n} - \vec{d} + \vec{p}_2) \right.$$

(cont.)

$$+ g \gamma_5 \tau_i \left. \frac{\langle p_i | J_N(\omega) | d \rangle}{\gamma_{\mu} (d - p_{\mu}) - m} \right\}$$

(2)

The lowest intermediate state that contributes to the first term in k is the two nucleon state so that this term will give rise to an integral equation for the matrix element for the process, the kernel being the matrix element for nucleon-nucleon scattering, which, in our procedure, represents the final state interaction. Diagrammatically this term is given by Fig. (2). The second term gives the 'pole' or Born approximation term if we take the single nucleon intermediate state only. The numerator is then the product of the pion-nucleon vertex and the deuteron-proton-neutron (D-p-n) vertex. The latter, on the energy shell of the intermediate nucleon, is just the normalisation of the deuteron wave function, N , which is equal to $\left(\frac{2\alpha}{1-\alpha\rho}\right)^{1/2}$ where $\alpha = M\epsilon$, M being the mass of the nucleon and ϵ the binding energy of the deuteron. ρ is the triplet effective range. In terms of the dispersion graphs, this term would, if we assume it to be written in terms of the propagator (instead of the energy denominator) by addition of a term containing a suitable higher particle intermediate state (see earlier chapter), correspond to Fig. (1 a). Fig. (1 b) corresponds to the third term of (2) if we rewrite it such that the denominator becomes $(d - p_i)^2 + m^2$. However by this procedure, the first factor in the numerator, namely, the pion-nucleon vertex, would then be written in terms of

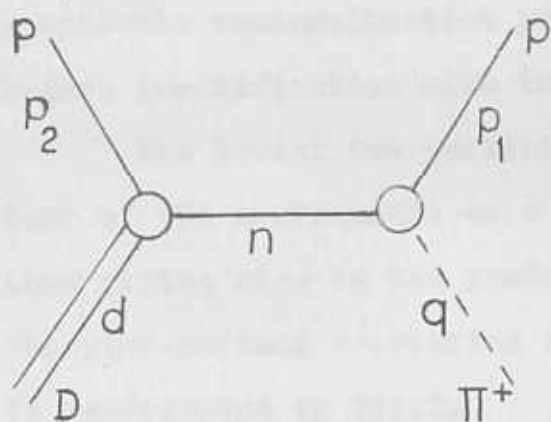


FIG. 1 (a)

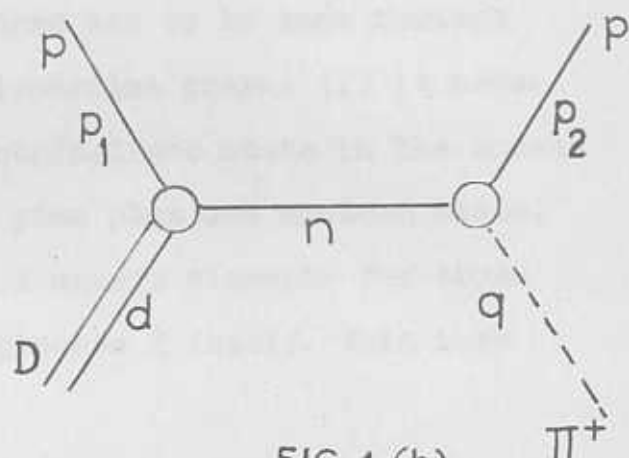


FIG. 1 (b)

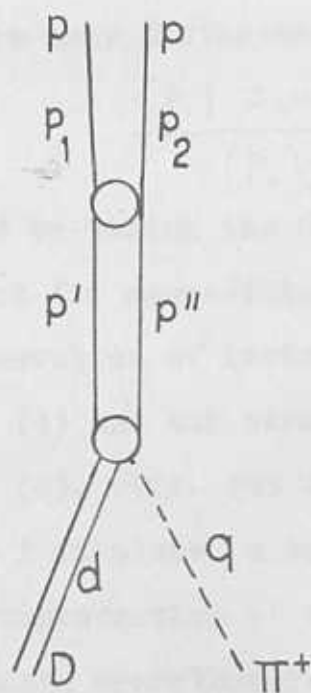


FIG. 2

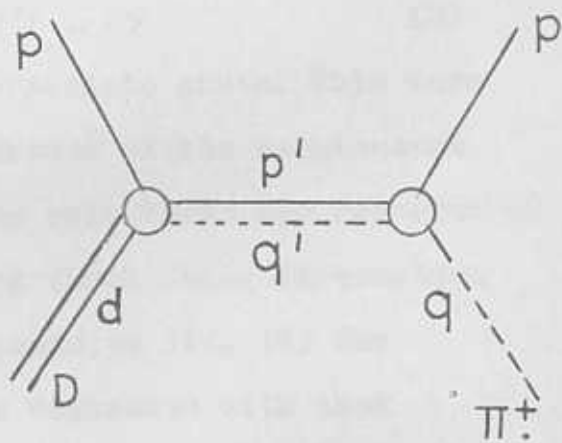


FIG. 3.

bare quantities whereas the second factor which is the d-p-n vertex is given in the Heisenberg representation.

A suitable renormalization procedure has to be gone through before identification with the dispersion graphs (1) is made.

The lowest two particle intermediate state in the second term of (2) corresponds to a one pion plus one nucleon state, thus giving rise to the product of matrix elements for those for pion-nucleon scattering and process I itself. This term is represented by Fig. 3.

The matrix element for the photo-pion disintegration of the deuteron (process III) is similar to (2) if we replace the pion-current operator J_π by the photon current operator J_γ , the only difference being an additional pole term

$$\frac{\langle P_1 | J_N(\omega) | d' \rangle \langle d' | J_\gamma(\omega) | d \rangle}{(P_2)_0 + (P_1)_0 - (d')_0 + i\varepsilon} \quad (3)$$

obtained by taking the deuteron intermediate state. This term is absent for meso-disintegration because of the requirement of conservation of isotopic spin. The pole terms are represented by Fig. (4) and the term representing final state interaction by Fig. (5). Fig. (6) which corresponds to Fig. (3) for process I involves a matrix element connected with that for photoproduction of a pion from a nucleon and represents the mesonic contribution to the process referred to in the introduction.

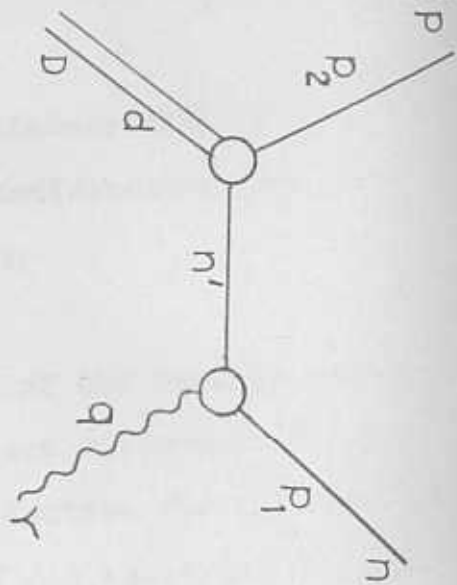


FIG. 4(a)

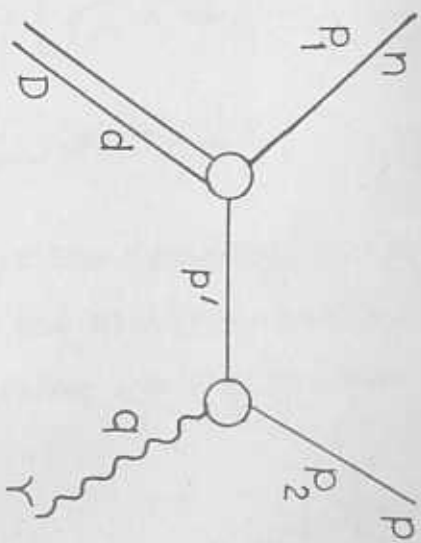


FIG. 4(b)

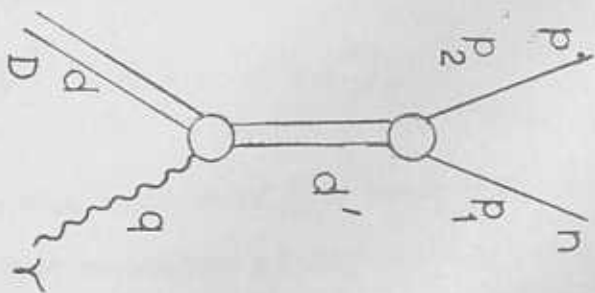


FIG. 4(c)

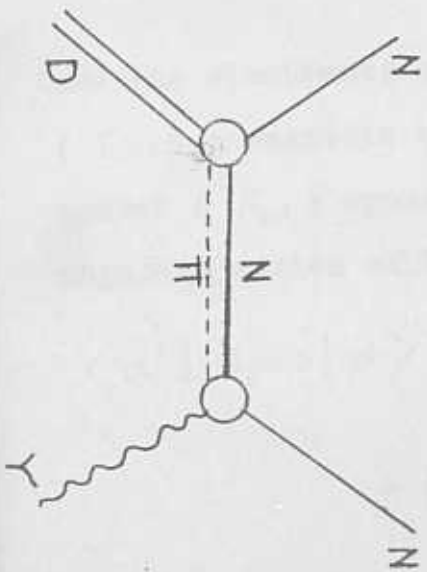
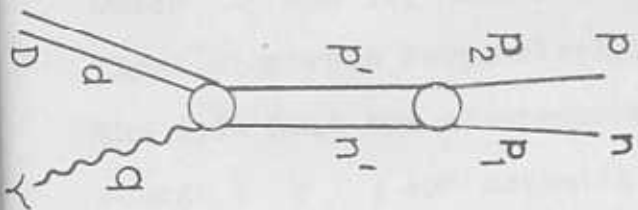


FIG. 6

The matrix element for electro-disintegration of the deuteron can be written as that for photodisintegration by a virtual photon with polarization vector

$$\frac{e \bar{u}(s_2) \gamma_\mu u(s_1)}{(s_1 - s_2)^2}$$

where s_1 and s_2 are the four-momenta of the initial and final electrons respectively. An important difference is in the fact that the electromagnetic form factors for the nucleon charge (F_1) and magnetic moment (F_2) appearing in the expression for the electromagnetic vertex of the nucleon

$$\langle p | J_3(\omega) | n \rangle = \frac{e}{\sqrt{4 p_0 n_0}} \bar{u}(p) \left[\gamma_\mu F_1(q^2) + \frac{\mu}{2m} F_2(q^2) \sigma_{\mu\nu} (p-n)_\nu \right] u(n) \quad (4)$$

and the electromagnetic form factors of the deuteron charge (F_{1d}), magnetic moment (F_{2d}) and the electric quadrupole moment (F_{3d}) appearing in the expression for the electromagnetic vertex of the deuteron

$$\langle d' | J_3(\omega) | d \rangle = \frac{e}{\sqrt{4 d_0 d'_0}} \left[F_{1d}(q^2) \xi'^* \cdot \xi (d'+d)_\mu + F_{2d}(q^2) \left(\xi_\mu \xi'^* \cdot q - \xi'_\mu \xi \cdot q \right) + F_{3d}(q^2) \left(\frac{\xi'^* \cdot q \xi \cdot q}{m^2} \right) (d+d')_\mu \right] \quad (5)$$

are no longer constants (since the square of the momentum transfer q^2 which is the square of the momentum of the virtual photon

need not be zero) but will depend on q^2 . In (5), ξ is a complex vector representing the polarization vector of the deuteron. As in the case of photodisintegration, the mesonic contribution arising from Fig. (6) is likely to become important for energies corresponding to and above the threshold for pion production:

Finally the matrix element for the radiative capture of the pion of momentum q by a deuteron of momentum, d (process II), with emission of a photon of momentum k is given by

$$\begin{aligned}
 \langle p_1 p_2 k | T | q d \rangle = & \frac{1}{\sqrt{4 q_0 k_0}} \left[\frac{\langle p_1 p_2 | j_\pi(\omega) | d \rangle \langle d' | j_\gamma(\omega) | d \rangle}{q_0 + (p_1)_0 + (p_2)_0 - d'_0 + i\varepsilon} \right. \\
 & + \iint \frac{\langle p_1 p_2 | j_\pi(\omega) | p' n' \rangle \langle p' n' | j_\gamma(\omega) | d \rangle}{q_0 + (p_1)_0 + (p_2)_0 - (p')_0 - n'_0 + i\varepsilon} \\
 & \quad \delta(\vec{p}' + \vec{n}' - \vec{p}_1 - \vec{p}_2 - \vec{q}) d\vec{p}' d\vec{n}' \\
 & + \iint \frac{\langle p_1 p_2 | j_\gamma(\omega) | p' p'' \rangle \langle p' p'' | j_\pi(\omega) | d \rangle}{q_0 + (p')_0 + (p'')_0 - d_0 - i\varepsilon} \\
 & \quad \delta(\vec{p}' + \vec{p}'' - \vec{d} - \vec{q}) d\vec{p}' d\vec{p}'' \Big]
 \end{aligned}
 \tag{6}$$

if we retain only one and two-particle intermediate states. In the above, the incoming pion and the outgoing photon are converted into current operators. The first term on the right hand side of (6) is represented in Fig. (7 a). The numerator represents the product of the matrix element for pure absorption in deuterium (process I) and the electromagnetic vertex

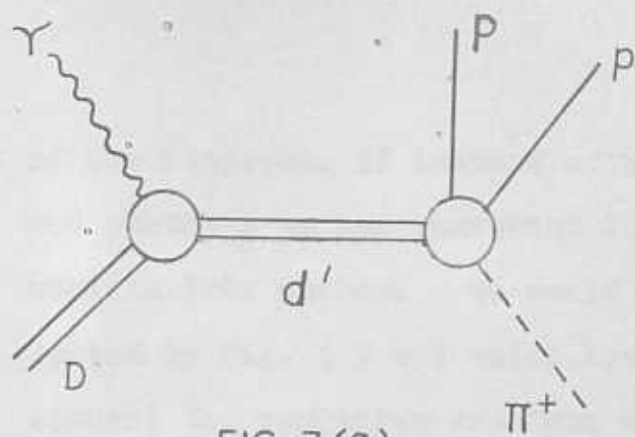


FIG. 7 (a)

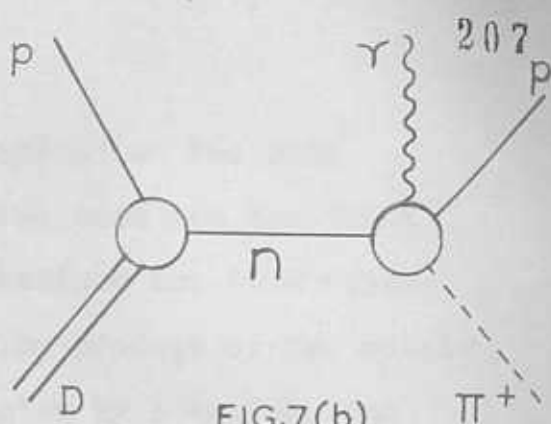


FIG. 7 (b)

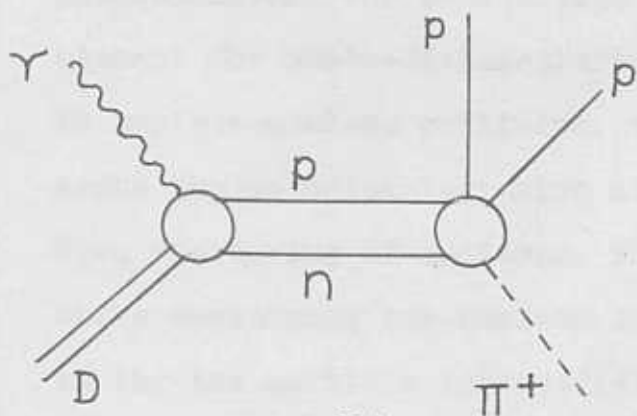


FIG. 8 (a)

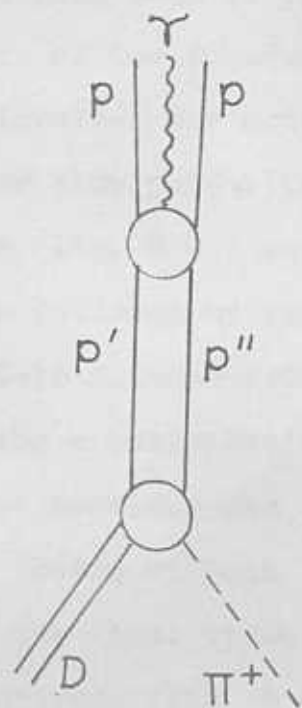


FIG. 8 (b)

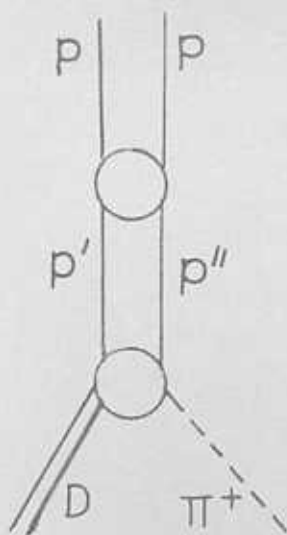


FIG. 9

of the deuteron. If instead of 'contracting' on the pion and photon x we had converted the initial pion and the final nucleon into current we would have obtained the term represented by Fig. (7 b) which involves the product of the matrix element for radiative capture of the pion by a nucleon and the D-p-n vertex. It is interesting to note that this is precisely the term we should expect on the basis of the impulse approximation. The second term (Fig. 3 a) involves the matrix element for photo-disintegration and that for pion production in nucleon-nucleon collision. The third term (Fig. 3 b) represents the mesodisintegration of the deuteron followed by radiative scattering of nucleons. The three particle intermediate state containing two-nucleon and a photon gives a contribution to the two particle intermediate states if we consider the intermediate photon to go off as the final photon without interaction. Then this term will represent the final state interaction (between the nucleons) for the process (Fig. 9).

