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LECTURES ON NON-RELATIVISTIC SCATTERING THEORY

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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20 (INDIA)

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Content

1.- Preliminaries and the 2-body problem.

It is assumed that the reader is familiar with the basic concepts of functional analysis [1].

Let $\mathcal{H} = L^2(\mathbb{R}^{3N})$ resp. $\mathfrak{h} = L^2(\mathbb{R}_0^{3N})$ be the Hilbert space of measurable and square-integrable functions of N three-dimensional vectors $\underline{p} = (\underline{p}_1, \dots, \underline{p}_N)$, resp. satisfying $\sum_{i=1}^{N} \underline{p}_i = 0$. The L^2-norm will be denoted by $\|\cdot\|$ and the scalar product of 2 elements by

$$\int f^*(\underline{p})\, g(\underline{p})\, d^{3N}p = \langle f | g \rangle =: (f, g)$$

The starting point for the investigation of many-particle systems is the corresponding Schrödinger equation. We shall therefore consider Hamiltonians on $\mathcal{H}$ or by $\mathfrak{h}$ of the form

$$\begin{aligned} H'' &= -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{\underline{x}_i} + \sum_{1 \le i < j \le N} V_{ij}(\underline{x}_i - \underline{x}_j) \\ &= H''^{0} + V \end{aligned} \tag{1.1}$$

We shall always assume that the (local) 2-body potentials are real.

As it stands, the unbounded operator H'' does not make much sense, because there does not seem to be a satisfactory criterion for determining the set of L^2-functions on which a differential operator can be allowed to act.

Moreover, the hermitian operator H'' may have several self-adjcint extensions. The way out of these difficulties was indicated by Kato[2]. It was shown first that H''^{o} is essentially self-adjoint (i.e. possesses a unique self-adjoint extension) on the domain which consists of all L^2-functions f whose Fourier transforms $\tilde{f}(p)$ are such that $\int |(1+\sum_{i=1}^{N} |p_i|^2) \tilde{f}(p)|^2 d^{3N}p < \infty$. This set of functions is dense in L^2. This self adjoint extension of H''^{o} is denoted by H'^{o}. Next it was shown that for $H' = H'^{o} + V$ to be essentially self-adjoint, it is sufficient that for any positive constant $\alpha > 0$, there exists a constant $\beta < \infty$ such that for each i, j, $1 \leq i < j \leq N$ and every f in the domain $\Delta(H'^{o})$ of H'^{o}, the "perturbations" V_{ij} are small in the following sense:

$$\| V_{ij} f \| \leq \alpha \| H'^{o} f \| + \beta \| f \| \tag{1.2}$$

Then we call V_{ij} a Kato potential.

If this condition is fulfilled, the domain of the (unique) self-adjoint extension of H', here after denoted by H, is $\Delta(H'^{o})$. We shall always assume that all extensions are already done and consider the self-adjoint Hamiltonian

$$H = H^{o} + V = -\sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{\underline{x}_i} + V \tag{1.3}$$

on $\Delta(H^0)$.

It is further shown that (1.3) is bounded from below. Notice that Kato's criterion is formulated in configuration space. The class of Kato potentials is large. We list some examples:

1) $V_{ij}(\bullet) \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$

2) in $\mathbb{R}^\ell$, $\ell = 4$, any real $V \in L^q + L^\infty, q > 2$ is a Kato potential

3) in $\mathbb{R}^\ell$, $\ell \geq 5$, any real $V \in L^{\ell/2} + L^\infty$ is a Kato potential.

In $\mathbb{R}^3$, the potential $\frac{1}{r^2}$ is just the exceptional limit between "good" and "bad" potentials.

The Hamiltonian (1.3) can also be generalised to a free self-adjoint part H^0 and an interaction V being the sum of static, 2-particle, 3-particle, ..., potentials:

$$V = \sum_1^N V_i^{(1)}(\underline{x}_i) + \sum_{i,j} V_{ij}^{(2)}(\underline{x}_i - \underline{x}_j) + \sum_{i,j,k} V_{ijk}^{(3)}(\underline{x}_i - \underline{x}_j, \underline{x}_j - \underline{x}_k) + \cdots\cdots + \cdots V^{(N)}$$

If V is real, if the static and 2-body potentials are in

$L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and if all $V^{(\alpha)}$ are in $L^{\frac{3\alpha-3}{2}} + L^\infty$ on $\mathbb{R}^{3\alpha-3}$, $\alpha = 3, 4, \ldots, N$, then the Hamiltonian is again essentially self-adjoint on $\Delta(H^0)$. At any rate square-integrable 2-body potentials on $\mathbb{R}^3$ lead to a self-adjoint Hamiltonian. We shall assume this to be the case. The importance of the Hamiltonian H being self-adjoint stems from the fact that this implies the existence of a resolvent and of a spectral resolution. Obviously the spectrum $\sigma(H)$ is contained in the real line.

The resolvent $R(\lambda)$ of H is defined as

$$R(\lambda) = (\lambda - H)^{-1} \tag{1.4}$$

where λ is a complex number. For $\lambda \notin \sigma(H)$ (e.g. $\mathrm{Im}\,\lambda \neq 0$) the operator $R(\lambda)$ exists and is bounded. It satisfies the identity:

$$\begin{aligned} R(\lambda) - R(\lambda') &= (\lambda - \lambda')\, R(\lambda) R(\lambda') \\ &= (\lambda - \lambda')\, R(\lambda') R(\lambda). \end{aligned} \tag{1,5}$$

$$R^\dagger(\lambda) = R(\lambda^*)$$

where $\dagger$ denotes the adjoint of an operator.

For $V = \Sigma V_{ij}$ and f belonging to L^2, it follows that $R(\lambda) f \in \Delta(H^0)$ and $H^0 R(\lambda)$, $V R(\lambda)$ are bounded operators. We may therefore write:

$$V R(\lambda) = (\lambda - H^0) R(\lambda) - \mathbb{1}$$

Applying the free resolvent $R_0(\lambda) = (\lambda - H^0)^{-1}$ to both sides:

$$R_0(\lambda) V R(\lambda) = R(\lambda) - R_0(\lambda)$$

i.e. we obtain the resolvent equation:

$$R(\lambda) = R_0(\lambda) + R_0(\lambda) V R(\lambda) \tag{1.6}$$

which uniquely characterises $R(\lambda)$. [3]

In the first part of this section we shall work in configuration space.

We interpret (1.6) as an integral equation on Hilbert space and are therefore interested in the determination and properties of the kernel $R(\underline{x}, \underline{y}; \lambda)$ of $R(\lambda)$ (Green function):

$$(R(\lambda) f)(\underline{x}) = \int R(\underline{x}, \underline{y}; \lambda) f(\underline{y}) \, d\underline{y}$$

In the case of no interaction, the N-body Green-function $R_0(\underline{x}, \underline{y}; \lambda)$ is known to exist and reads:

$$R_0(\underline{x},\underline{y};\lambda) = \frac{i}{4}\left[\frac{\sqrt{\lambda}}{2\pi|\underline{x}-\underline{y}|}\right]^{\frac{3}{2}N-\frac{5}{2}} H^{(1)}_{\frac{3N}{2}-\frac{5}{2}}\left(\sqrt{\lambda}\,|\underline{x}-\underline{y}|\right) \qquad (1.7)$$

Here the center-of-mass (CM) motion of the N-particle system has been separated off and $\underline{x},\underline{y}$ belong to $\mathbb{R}_0^{3N} = \{\underline{x}=(\underline{x}_1,\ldots,\underline{x}_N)\in\mathbb{R}^{3N}, \sum_1^N \underline{x}_i = 0\}$. $H^{(1)}$ is the Hankel function of the 1st kind.

For N=2, (1.7) reduces to the well-known:

$$R_0(\underline{x},\underline{y};\lambda) = \frac{1}{4\pi|\underline{x}-\underline{y}|}\, e^{i\sqrt{\lambda}\,|\underline{x}-\underline{y}|} \qquad (1.8)$$

In the following we shall always assume $\operatorname{Im}\sqrt{\lambda} > 0$, $0 < \arg\lambda < 2\pi$ unless explicitly stated.

We restrict now to a 2-particle system (N=2) in its CM frame. If $f \in L^2(\mathbb{R}^3)$, then also

$$h(\underline{x};\lambda) = (R(\lambda)f)(x) \quad \text{and} \quad h_0(\underline{x};\lambda) = (R_0(\lambda)f)(\underline{x})$$

belong to L^2. It follows that in configuration space

$$h(\underline{x};\lambda) = h_0(\underline{x};\lambda) + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\underline{x}-\underline{y}|}\, e^{i\sqrt{\lambda}|\underline{x}-\underline{y}|}\, V(\underline{y})\, h(\underline{y};\lambda)\, d^3y \qquad (1.9)$$

This is an integral equation of the 2nd kind for $h(\underline{x};\lambda)$.

Denoting the kernel of this equation by K

$$K(\underline{x},\underline{y};\lambda) = \frac{1}{4\pi}\frac{1}{|\underline{x}-\underline{y}|} e^{i\sqrt{\lambda}|\underline{x}-\underline{y}|} V(\underline{y}) \tag{1.9'}$$

We have for $\mathrm{Im}\sqrt{\lambda} > 0$:

$$\begin{aligned}&\int |K(\underline{x},\underline{y};\lambda)|^2 d^3x\, d^3y \\ &= \frac{1}{16\pi^2}\int |V(y)|^2 d^3y \int \frac{1}{|\underline{x}-\underline{y}|^2} |e^{i\sqrt{\lambda}|\underline{x}-\underline{y}|}|^2 d^3x \\ &= \frac{1}{8\pi\,\mathrm{Im}\sqrt{\lambda}} \int |V(y)|^2 d^3y < \infty\end{aligned} \tag{1.10}$$

Kernels which are square-integrable on Hilbert space in the above sense are known as Hilbert Schmidt (HS) kernels, sometimes also loosely called completely continuous or even compact kernels. Their importance derives from the fact that the solution of integral equations with HS kernels is explicitly known [4].

Define

$$\Delta(\lambda) = \sum_{n=0}^{\infty} \Delta_n(\lambda) \quad , D(\underline{x},y;\lambda) = \sum_{n=0}^{\infty} D_n(\underline{x},y;\lambda) \tag{1.11}$$

where

$$\Delta_0(\lambda) = 1, \quad \Delta_1(\lambda) = 0, \quad D_0(\underline{x},\underline{y};\lambda) = K(\underline{x},\underline{y};\lambda)$$

$$\Delta_n(\lambda) = -\frac{1}{n}\int K(\underline{x},\underline{y};\lambda)\, D_{n-2}(\underline{y},\underline{x};\lambda)\, d^3x\, d^3y, \; (n \geq 2) \tag{1.11'}$$

$$\begin{aligned}D_n(\underline{x},\underline{y};\lambda) = {} & \Delta_n(\lambda)\, K(\underline{x},\underline{y};\lambda) \\ & + \int K(\underline{x},\underline{\xi};\lambda)\, D_{n-1}(\xi,y;\lambda)\, d^3\xi \quad (n \geq 1)\end{aligned}$$

Then it is shown that if $h_0 \in L^2$ and λ is such that $\Delta(\lambda) \neq 0$, the equation (1.9) with the kernel (1.9') has one and only one L^2-solution $h(\underline{x};\lambda)$ of the form:

$$h(\underline{x};\lambda) = h_0(\underline{x};\lambda) + \frac{1}{\Delta(\lambda)} \int D(\underline{x},\underline{y};\lambda)\, h_0(\underline{y};\lambda)\, d^3y \qquad (1.12)$$

This is known as (a part of) the Fredholm alternative. We shall state the remaining part in the sequel.

Furthermore:

$$|\Delta(\lambda)| \leq \left(\frac{e}{n}\right)^{\frac{n}{2}} \| K(\lambda)\|^n$$

so that the series for $\Delta(\lambda)$ is convergent for all values of λ for which $\|K(\lambda)\|$ is finite, i.e. for $\mathrm{Im}\sqrt{\lambda} > 0$.

Also each D_n belongs to L^2:

$$\| D_n(\lambda)\| \leq \sqrt{e}\left(\frac{e}{n}\right)^{\frac{n}{2}} \| K(\lambda)\|^{n+1} \qquad (n \geq 1)$$

so that $D \in L^2$ and the series for D converges uniformly in the L^2-topology.

Using (1.12) and by an allowed interchange of integration orders (Fubini's theorem), the following explicit representation of the 2-body Green function is obtained:

$$R(\underline{x},\underline{y};\lambda) = R_0(\underline{x},\underline{y};\lambda) + \frac{1}{\Delta(\lambda)} \int D(\underline{x},\underline{\xi};\lambda)\, R_0(\underline{\xi},\underline{y};\lambda)\, d^3\xi \qquad (1.13)$$

i.e. essentially as the ratio of 2 convergent series.

Finally it may be shown that each function $\Delta_n(\lambda)$ is regular in $\{\operatorname{Im}\sqrt{\lambda} > 0\}$ and also $(g, R(\lambda)f)$ is analytic in ~~the~~ any region of $\operatorname{Im}\sqrt{\lambda} > 0$ where $\Delta(\lambda)$ does not vanish. This last result is equivalent with the analyticity of the operators $R(\lambda)$ in the given region.

Now, the zeros of $\Delta(\lambda)$ are isolated, since $R(\lambda)$ is regular off the real axis, it follows that the poles of the resolvent are located on the negative real λ-axis and also have no accumulation point at infinity. It may be easily seen that these poles are simple, We have:

$$(R(\lambda)f)(\underline{x}) = \int R_0(\underline{x},\underline{\xi};\lambda) f(\underline{\xi}) d^3\xi + \frac{1}{\Delta(\lambda)} \int f(\underline{\xi}) D(\underline{x},\underline{y};\lambda) R_0(\underline{y},\underline{\xi};\lambda) d^3y\, d^3\xi \tag{1.14}$$

and also from (1.5)

$$\frac{d}{d\lambda}(g, R(\lambda)f) = (g, R(\lambda) R(\lambda) f) \tag{1.15}$$

Let λ_α be a zero of $\Delta(\lambda)$ and assume its order being q. Then in the neighbourhood of λ_α, $(g, R(\lambda)f)$ may be expanded in a Laurent series, the most singular term of which is proportional to $(\lambda - \lambda_\alpha)^{-q}$. Hence in the expansion of $\frac{d}{d\lambda}(g, R(\lambda)f)$ the most singular term is proportional to $(\lambda - \lambda_\alpha)^{-q-1}$. But according to (1.15), it is also proportional to $(\lambda-\lambda_\alpha)^{-2q}$ so that $q = 1$.

In order to relate the bound states of the Schrödinger equation to the poles of the resolvent, consider the function:

$$F(\underline{x},\underline{\xi};\lambda_\alpha) = \lim_{\lambda\to\lambda_\alpha} \frac{\lambda-\lambda_\alpha}{\Delta(\lambda)} \int D(\underline{x},\underline{y};\lambda_\alpha)\, R_0(\underline{y},\underline{\xi};\lambda_\alpha)\, d^3y \tag{1.16}$$

which is the "residue" of $R(\underline{x},\underline{\xi};\lambda)$ at $\lambda = \lambda_\alpha$ (cf.1.13).

Due to $R^*(\underline{x},\underline{y};\lambda) = R(\underline{x},\underline{y};\lambda^*)$ and $R(\underline{x},\underline{y};\lambda) = R(\underline{y},\underline{x};\lambda)$ (easy to prove), it follows that F is real and satisfies

$$F(\underline{x},\underline{\xi};\lambda_\alpha) = F(\underline{\xi},\underline{x};\lambda_\alpha).$$

Furthermore $F(\lambda_\alpha)$ is of HS type and can therefore be expanded in a series of orthonormal eigenfunctions in L^2:

$$F(\underline{x},\underline{\xi};\lambda_\alpha) = \sum_{n=1}^{\infty} \mu_{\alpha n}\, \varphi_{\alpha n}(\underline{x})\, \varphi^*_{\alpha n}(\underline{\xi})$$

$$\| F(\lambda_\alpha) \| = \left[\sum_{n=1}^{\infty} \mu^2_{\alpha n}\right]^{1/2} \tag{1.17}$$

the $\mu_{\alpha n}$ being real (since F is hermitian). We may always assume that the eigenfunctions φ are also all real (since φ and φ^* belong to the same eigenvalue).

Consider again the resolvent equation

$$R(\lambda)f = R_0(\lambda)f + R_0(\lambda)VR(\lambda)f$$

Substituting the Green function in both sides, expanding in a Laurent series in powers of $\lambda - \lambda_\alpha$ and equating principal parts yields:

$$\int F(\underline{x}, \underline{\xi}; \lambda_\alpha) f(\underline{\xi}) d^3\xi$$

$$= \int R_0(\underline{x}, \underline{y}; \lambda_\alpha) V(y) d^3y \int F(\underline{y}, \underline{\xi}; \lambda_\alpha) f(\underline{\xi}) d^3\xi \tag{1.18}$$

and in particular with $f = \varphi_{\alpha n}$:

$$\varphi_{\alpha_n}(\underline{x}) = \int R_0(\underline{x}, \underline{y}; \lambda_\alpha) V(\underline{y}) \varphi_{\alpha_n}(\underline{y}) d^3y \tag{1.19}$$

which means that the eigenfunctions φ_{α_n} are solutions of the homogeneous integral equation (1.9).

Recalling that $D(\underline{x}, \underline{y}; \lambda_\alpha)$ being square-integrable is by Fubini's theorem also square-integrable in $\underline{y}$ for almost every $\underline{x}$ and that $V R_0(\lambda_\alpha)$ is a bounded operator, we have from (1.16):

$$\int |V(\underline{\xi}) F(\underline{x}, \underline{\xi}; \lambda_\alpha)|^2 d^3\xi \le \text{const} \, \| V R_0(\lambda_\alpha) \|^2 \int | D(\underline{x}, \underline{y}; \lambda_\alpha|^2 d^3y$$

$$\int |V(\underline{\xi}) F(\underline{x}, \underline{\xi}; \lambda_\alpha)|^2 d^3\xi \, d^3x \le \text{Const} \, \| V R_0(\lambda_\alpha) \|^2 \, \| D(\lambda_\alpha) \|^2$$

By Schwarz' inequality it follows that

$$\int |V(\underline{\xi})\varphi_{\alpha_n}(\underline{\xi})|^2 d^3\xi = \int \left| V(\underline{\xi}) \int F(\underline{x},\underline{\xi};\lambda_\alpha)\varphi_{\alpha_n}(\underline{x})\, d^3x \right|^2 d^3\xi \leq$$

$$\leq \int |V(\underline{\xi})\, F(\underline{x},\underline{\xi};\lambda_\alpha)|^2 d^3x\, d^3\xi \int |\varphi_{\alpha_n}(\underline{y})|^2 d^3y < \infty$$

Hence $V\varphi_{\alpha_n} \in L^2$. It is now permitted to apply the operator $\lambda_\alpha - H^0$ to both sides of (1.19) yielding:

$$(\lambda_\alpha - H^0)\,\varphi_{\alpha_n}(\underline{x}) = V(\underline{x})\,\varphi_{\alpha_n}(\underline{x})$$

i.e.

$$(H - \lambda_\alpha)\,\varphi_{\alpha_n}(\underline{x}) = 0$$

In other words each φ satisfies also the Schrödinger equation. Since it belongs to L^2, it is the eigenfunction of a bound state with energy λ_α. This is the second part of the Fredholm alternative.

The preceding argument can be easily extended to show that the Green function can be written in the form

$$R(\underline{x},\underline{y};\lambda) = \sum_{\alpha,n} \frac{\varphi_{\alpha_n}(\underline{x})\,\varphi^*_{\alpha_n}(\underline{y})}{z - \lambda_\alpha} + R_B(\underline{x},\underline{y};\lambda) \quad (1.20)$$

where $R_B(\underline{x},\underline{y};\lambda)$ is for $\mathrm{Im}\sqrt{\lambda} > 0$ the kernel of a bounded operator. Notice that nothing has been said about the behaviour of the resolvent in the neighbourhood of the positive real axis. In this limit the resolvent equation is certainly singular on Hilbert space.

It is customary to introduce a T -operator by defining

$$T(\lambda) = V + V R(\lambda) V \tag{1.21}$$

Then the explicit solution for $R(\lambda)$ and the previous discussion essentially carries over to $T(\lambda)$. Using the resolvent equation we obtain the Lippmann-Schwinger eqn. (LS) for :

$$T(\lambda) = V + V R_0(\lambda) T(\lambda) \tag{1.22}$$

From now on we shall always work in momentum space and introduce a new complex number $z = \sqrt{\lambda}$. Equation (1.22) is then again an integral equation of the 2nd kind on $L^2(\mathbb{R}^3)$ (in the CM frame):

$$T(p, q; z) = v(p - q) + \int d^3k\, v(p - k) \frac{1}{z - \mu k^2} T(k, q; z) \tag{1.23}$$

where $v(\cdot)$ is the Fourier transform of $V(\cdot)$, $\mu = \frac{1}{2m}$, m reduced mass of the two particles and where p resp. q are interpreted as the incoming resp. outgoing relative momenta.

The kernel $v(p-k)\dfrac{1}{z-\mu k^2}$ is again for $\operatorname{Im} z \neq 0$ of HS type. Therefore the Fredholm alternative applies:

A non-trivial solution $g \in L^2$ of the homogeneous equation $g = V R_0(z)\, g$ leads to an eigenstate $R_0(z)\, g$ of the Hamiltonian with eigenvalue z which is a zero of the Fredholm determinant $\Delta(z)$, or the inhomogeneous integral equation (1.23) has a unique solution $T(\cdot, q; z) \in L^2$ for z not contained in the spectrum of H. This off-energy-shell solution can also be exhibited as a ratio of 2 Fredholm series, similar to (1.13).

It is well known that the LS equation is related to the Schrödinger equation by invertible operations.

We notice that $T(z) R_0(z) = [\mathbb{1} - V R_0(z)]^{-1} V R_0(z)$ is, as product of a bounded operator and a HS operator, again of HS type, with $\| T(z) R_0(z) \| \leq \dfrac{c}{(1+|z|)^{\delta}}$, $\delta > 0$ for $-\operatorname{Re} z$ sufficiently large. It is instructive to sketch a proof of analyticity properties of the on-energy-shell 2-body amplitude.

We shall assume for the sake of simplicity that the potential V does not lead to bound states, so that the spectrum of H is contained in $[0,\infty)$ and that $v(\cdot)$ has some mild regularity properties, namely that $v(p)$ is analytic in $\{p \in \mathbb{C}^3 ; |\mathrm{Im}\, p| < \rho\}$ for some $\rho > 0$ (for instance Yukawa potentials).

Finally some suitable decrease at infinity is required for the potential, say $|v(p)| \leq C(1+|p|)^{-\theta}$, $\theta > \frac{3}{2}$. The following is an illustration of the method of rotation of integration contours [12,13].

It is easy to check that for $z \in \mathbb{C} - \{0\}$, $\arg z \neq 2\varphi$ and with $p^{\varphi} = e^{i\varphi} p$, $q^{\varphi} = e^{i\varphi} q$, $p, q \in \mathbb{R}^3$, the kernel

$$v(p^{\varphi} - q^{\varphi})(z - \mu q^2 e^{2i\varphi})^{-1} \tag{1.24}$$

is a HS operator on the "rotated" Hilbert space, with its norm going to 0 for $\mathrm{Re}\, z \to -\infty$ uniformly in φ . Therefore, the equation

$$T^{\varphi}(p^{\varphi}, q^{\varphi}; z)(z - \mu q^2 e^{2i\varphi})^{-1} = v(p^{\varphi} - q^{\varphi})(z - \mu q^2 e^{2i\varphi})^{-1} +$$

$$+ \int_{\mathbb{R}^3} d^3\xi \, e^{3i\varphi} v(p^{\varphi} - \xi^{\varphi}) \frac{1}{z - \mu \xi^2 e^{2i\varphi}} T^{\varphi}(\xi^{\varphi}, q^{\varphi}; z) \frac{1}{z - \mu q^2 e^{2i\varphi}} \tag{1.25}$$

has as solution a HS operator whose kernel $T^{\varphi}(p^{\varphi}, q^{\varphi}; z) \frac{1}{z - \mu q^2 e^{2i\varphi}}$

is holomorphic in $z \in \{z \neq 0 \quad , \arg z \neq 2\varphi\}$ provided that there are no non-trivial L^2-solutions f^φ of the corresponding homogeneous equation

$$f^\varphi(p^\varphi) = \int_{\mathbb{R}^3} d^3\xi \, e^{3i\varphi} v(p^\varphi - \xi^\varphi) \frac{1}{z - \mu\xi^2 e^{2i\varphi}} f^\varphi(\xi^\varphi) \tag{1.26}$$

Now the necessary and sufficient condition for the existence of a square-integrable solution of (1.26) is the vanishing of the "rotated" Fredholm determinant $\Delta^\varphi(z) = 1 + \sum_{n=1}^{\infty} \Delta_n^\varphi(z)$ and we have [4]:

$$\Delta_n^\varphi(z) = \frac{(-1)^n}{n!} \begin{vmatrix} 0 & n-1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sigma_2^\varphi & 0 & n-2 & 0 & \cdots & 0 & 0 & 0 \\ \sigma_3^\varphi & \sigma_2^\varphi & 0 & n-1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \sigma_{n-1}^\varphi & \sigma_{n-2}^\varphi & \sigma_{n-3}^\varphi & \sigma_{n-4}^\varphi & \cdots & \sigma_2^\varphi & 0 & 1 \\ \sigma_n^\varphi & \sigma_{n-1}^\varphi & \sigma_{n-2}^\varphi & \sigma_{n-3}^\varphi & \cdots & \sigma_3^\varphi & \sigma_2^\varphi & 0 \end{vmatrix} \tag{1.27}$$

where σ_i^φ is the trace of the ith iteration of the kernel (1.24). We notice that for $2\varphi < \arg z < 2\pi$ and $0 \leq \varphi' \leq \varphi$, these iterated traces

$$\sigma_n^{\varphi'}(z) = \int_{\mathbb{R}^{3n}} v(p_1^\varphi - p_2^\varphi) v(p_2^\varphi - p_3^\varphi) \cdots v(p_n^\varphi - p_1^\varphi) \cdot \prod_{j=1}^{n} \frac{e^{3i\varphi}}{z - \mu p_j^2 e^{2i\varphi}} d^3 p_j \tag{1.28}$$

are independent of φ' :

$$\sigma_n^{\varphi'}(z) = \sigma_n^{0}(z)$$

Furthermore, due to the uniform decrease of the potentials and free Green functions at infinity, the deformation of the integration contour does not lead to a contribution at infinity. Finally, for $0 < 2\varphi < \arg z < 2\pi$, we have $\Delta^{\varphi}(z) = \Delta^{0}(z) \neq 0$, since a zero of the Fredholm determinant $\Delta^{0}(z)$ in the region

$$Z^{\varphi} = \{ z \in \mathbb{C} - \{0\}, 2\varphi < \arg z < 2\pi \}, \quad \varphi \geq 0$$

would lead to a non-trivial L^2-solution f of (1.26) and thus to an eigenfunction of the Hamiltonian $H^{\varphi = 0}$ with eigenvalue $z \in Z^{\varphi}$. This is however in contradiction with the assumed spectrum $[0, \infty)$ for $H^{\varphi = 0}$.

Now, we may also write:

$$\begin{aligned} T^{\varphi}(\underline{p}^{\varphi}, \underline{q}^{\varphi}; z)(z - \mu \underline{q}^2 e^{2i\varphi})^{-1} &= v(\underline{p}^{\varphi} - \underline{q}^{\varphi})(z - \mu \underline{q}^2 e^{2i\varphi})^{-1} + \\ &+ \int_{\mathbb{R}^3} d^3\xi \, e^{3i\varphi} v(\underline{p}^{\varphi} - \underline{\xi}^{\varphi}) \frac{1}{z - \mu \underline{\xi}^2 e^{2i\varphi}} v(\underline{\xi}^{\varphi} - \underline{q}^{\varphi}) \frac{1}{z - \mu \underline{q}^2 e^{2i\varphi}} + \\ &+ \int_{\mathbb{R}^6} d^3\xi \, d^3\eta \, e^{6i\varphi} v(\underline{p}^{\varphi} - \underline{\xi}^{\varphi}) \frac{1}{z - \mu \underline{\xi}^2 e^{2i\varphi}} T^{\varphi}(\underline{\xi}^{\varphi}, \underline{\eta}^{\varphi}; z) \frac{1}{z - \mu \underline{\eta}^2 e^{2i\varphi}} \\ &\qquad \cdot v(\underline{\eta}^{\varphi} - \underline{q}^{\varphi}) \frac{1}{z - \mu \underline{q}^2 e^{2i\varphi}} \end{aligned} \tag{1.29}$$

From this representation it follows that for all $\underline{\xi}, \underline{\eta} \in \mathbb{R}^3$, $T^\varphi(p^\varphi, q^\varphi; z)$ is analytic in $p^\varphi, q^\varphi, z \in Z^\varphi$ provided that $|\mathrm{Im}\, p| < \rho \cos\varphi$ resp. $|\mathrm{Im}\, q| < \rho\cos\varphi$. $\varphi \geq 0$.

Finally we prove that for $0 \leq \varphi' \leq \varphi$ and $p, q \in \mathbb{R}^3$:

$$T^\varphi(p^{\varphi'}, q^{\varphi'}; z) = T^{\varphi'}(p^{\varphi'}, q^{\varphi'}; z) \tag{1.30}$$

This is shown by using the representation of $T^\varphi(p^\varphi, q^\varphi; z)\dfrac{1}{z - \mu q^2 e^{2i\varphi}}$ as a quotient of a Fredholm series, valid for

$$|\mathrm{Im}\,(e^{i(\varphi-\varphi')}p)| < \rho\cos\varphi, \; |\mathrm{Im}(e^{i(\varphi-\varphi')}q)| < \rho\cos\varphi$$

then inserting this representation into (1.29) and using the Cauchy formula to establish (1.30) in the sense of continuation of analytic functionss

Thereby, the square-integrability of $v(\eta^\varphi - q^\varphi)$ in η for fixed q , uniformly in φ and of $v(p^\varphi - \underline{\xi}^\varphi)\dfrac{1}{z - \mu \xi^2 e^{2i\varphi}}$ in $\underline{\xi}$ are essential for the existence of $T^\varphi(\xi^\varphi, \eta^\varphi; z)$.

Relation (1.30) is then proved by analytic continuation. This establishes the existence of the on-energy-shell 2-body scattering amplitude as boundary value of a function which is analytic in the external momenta and in z , $\mathrm{Im}\, z \longrightarrow 0$. Notice that one is also able to perform analytic continuation across the energy cut into the unphysical sheet.

This method will be used in the investigation of the analyticity properties of many-body scattering amplitudes.

2.- The 3-body problem and the Faddeev equations

In this section, we restrict ourselves to the CM frame, i.e. assume that the global CM-motion of the 3 particle system has been separated off. The Hamiltonians will be denoted by H, H° etc... Hence incoming and outgoing momenta e.g. p_i, q_i $1 \leq i \leq 3$, will satisfy $\sum_1^3 p_i = \sum_1^3 q_i = 0$. We consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_0^9)$ and the Hamiltonian

$$H = H^{\circ} + V = H^{\circ} + \sum_{1 \leq i < j \leq 3} V_{ij}(\underline{x}_i - \underline{x}_j) \tag{2.1}$$

describing the relative motion of the system.

We shall start by assuming that the 3 interparticle potentials $V_{12} = V_3$, $V_{31} = V_2$, and $V_{23} = V_1$, $V = \sum_1^3 V_\alpha$ are of class L^2. Later on, we shall introduce a broader class of 2-body potentials.

As in the previous section, we may consider, for $\operatorname{Im} z \neq 0$, the resolvent $R(z)$ of the self-adjoint operator H and derive the resolvent equation and the corresponding LS equation. Only now these operators act on the 3-particle space $L^2(\mathbb{R}_0^9)$:

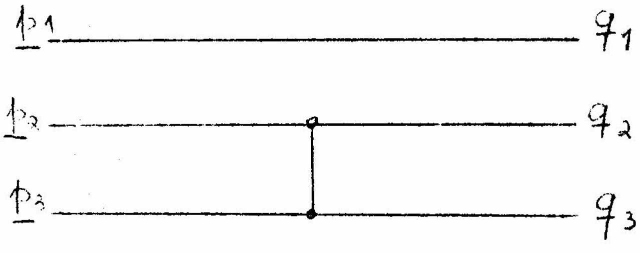
$$T(z) = V + V R_0(z) T(z) \tag{2.2}$$

However this equation is not very useful, since it may be easily seen that its kernel $V R_0(z)$ is not of HS-type. The reason for this is that the operator, say $V_1 R_0(z)$ which appears in (2.2) has the kernel

$$\delta(p_1 - q_1)\, v_1(\hat{p}_2 - \hat{q}_2) \frac{1}{z - \sum_1^3 \mu_i q_i^2} \tag{2.3}$$

where we have set $\mu_i = \frac{1}{2m_i}$, $1 \leq i \leq 3$. $\hat{p}_2 = p_2 - \frac{m_2}{m_2 + m_3}(p_2 + p_3)$ and similarly for $\hat{q}_2$. Remember $\sum_1^3 p_i = \sum_1^3 q_i = 0$.

The corresponding process may be graphically depicted as follows:



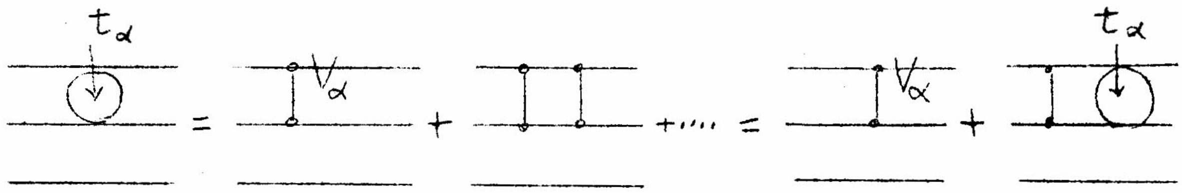
Momentum conservation is manifest at the vertex.

The δ -function in (2.3) expresses the momentum conservation of the 1st particle which is a mere "spectator" in this process. We may now iterate (2.2) as often as we wish: this δ -function will not disappear, due to the fact that the kernel of any $V_1 R_0(z) V_1 R_0(z) \ldots$ contains the same δ -function. It is therefore impossible to reduce (2.2) to an integral equation with a HS kernel in any function space.

However, we note that these δ -functions disappear in the product of the operators $V_1 R_0(z) V_2 R_0^+(z)$ upon iterating (2.2). The same remark applies to any product of the form $V_\alpha R_0(z) V_\beta R_0(z)$, $\alpha \neq \beta$. Now since the LS-equation is not of much use in the 3-body system, we have to replace it by something more appropriate. The central idea of Faddeev [3] in deriving the set of coupled integral equations named after him relies on the above remark. Let us first introduce 2-particle subamplitudes via the definition:

$$t_\alpha(z) = V_\alpha + V_\alpha R_0(z) t_\alpha(z), \quad \alpha = 1,2,3 \tag{2.4}$$

which is to be considered as an operator equation on the 3-particle Hilbert space $L^2(\mathbb{R}_0^9)$. Graphically:



In other words e.g.

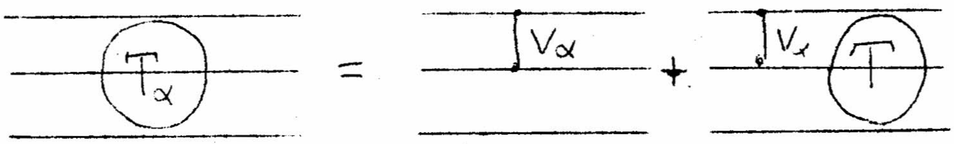
$$\langle p_1 p_2 p_3 | t_3(z) | q_1 q_2 q_3 \rangle$$

$$= \delta(p_3 - q_3) \langle p_1 p_2 | \tau_3\left(z - \frac{1}{2(m_1+m_2)}(p_1+p_2)^2\right) | q_1 q_2 \rangle \tag{2.5}$$

or

$$t_3(p_1,\dots,q_3;z) = \delta(p_3-q_3)\,\tau_3\Big(\hat{p}_1,\hat{q}_1; z - \frac{1}{2(m_1+m_2)}(p_1+p_2)^2\Big),$$
$$\sum_1^3 p_i = \sum_1^3 q_i = 0$$

where $\tau_3(k, k'; z)$ is the off-energy-shell solution of the 2-body LS-equation. With only particles within the subsystem 3=(12) interacting. It is easily verified that $t_\alpha(z)$ is defined on $\Delta(H^0)$. Let now $T_\alpha(z)$ be the sum of all graphs where the 1st interaction is V_α :



or

$$T_\alpha(z) = V_\alpha + V_\alpha R_0(z)\,T(z) \qquad (2.6)$$

Clearly:

$$T(z) = \sum_1^3 T_\alpha(z)$$

From (2.4) and (2.6) and remembering that $V_\alpha R_0(z)$ is a bounded operator for $\mathrm{Im}\, z \neq 0$, we obtain say

$$\big[\mathbb{1} - V_1 R_0(z)\big]\,T_1(z) = V_1 + V_1 R_0(z)\,\big[T_2(z) + T_3(z)\big]$$

and

$$\left[\mathbb{1} - V_1 R_0(z) \right]^{-1} V_1 = t_1(z)$$

so that:

$$T_1(z) = t_1(z) + t_1(z) R_0(z)(T_2 + T_3) \qquad (2.7)$$

Similar equations hold for $T_2(z)$ and $T_3(z)$.

In matrix form these 3 equations may be written as

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} + \begin{pmatrix} 0 & t_1 & t_1 \\ t_2 & 0 & t_2 \\ t_3 & t_3 & 0 \end{pmatrix} R_0(z) \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \qquad (2.8)$$

or

$$T_\alpha(z) = t_\alpha(z) + \sum_{\beta=1}^{3} K_{\alpha\beta}(z) T_\beta(z) \qquad (2.9)$$

These are the Faddeev equations. Their inhomogeneities are solutions of 2-particle subsystems and their kernel is now a 3 x 3 matrix, with elements defined on $\Delta(H^0)$. The kernels of these matrix elements do still contain δ -functions, but it is easily seen that a single iteration of K yields a kernel K^2 which contains only connected graphs.

Recalling that for $\mathrm{Im}\, z \neq 0$, $t_\alpha(z)\, R_0(z)$ is a HS operator and that the off-shell amplitudes $t_\alpha(\cdot, q; z)$, $t_\alpha(p, \cdot\,; z)$ are square-integrable in either argument uniformly in z , it is clear that each element of K^2 is of HS type. Hence we may again use the Fredholm method to solve the iterated system and the Fredholm alternative also applies provided that $z \notin \sigma(H)$. It may further be shown [3] that the operators $T_\alpha(z)$, $\alpha = 1,2,3$, satisfying (2.9) uniquely characterise the T -operator. Thus the Faddeev equations are equivalent with the Schrödinger equation and there are no spurious solutions.

The rest of this section is devoted to a presentation of the such called Faddeev programm.

Clearly the Faddeev equations are singular in Hilbert space in the physical region, i.e. for $\mathrm{Im}\, z \to 0$. In order to investigate the physical scattering amplitude several new assumptions and definitions have to be made. We shall always work in momentum space.

The potentials $v_\alpha(\cdot)$, $1 \leq \alpha \leq 3$ are assumed to satisfy the following conditions:

(i) Boundedness and decrease at infinity:

$$|v_\alpha(\underline{k})| \leq C(1+|\underline{k}|)^{-\theta}, \quad \theta > 1$$

(ii) Smoothness (Hőlder continuity (HC)):

$$|v_\alpha(\underline{k}+\underline{h})-v_\alpha(\underline{k})| \le C(1+|\underline{k}|)^{-\theta}|\underline{h}|^\mu, \ |\underline{h}|\le 1, \ \mu>0$$

(iii) Real-valuedness

$$v_\alpha(\underline{k}) = v_\alpha^*(-\underline{k})$$

Functions satisfying (i)-(iii) are said to belong to the class $\mathcal{B}(\theta,\mu)$. It is not difficult to check that if $\theta > \frac{3}{2}$, the potentials V_α are Kato potentials. Hence we shall assume this to be the case. The potential class considered by Faddeev is very broad but still allows for a rigorous treatment of the singular equations. We further introduce a Banach space, also denoted by $\mathcal{B}(\theta,\mu)$ with the norm

$$\|f\|_{\theta,\mu} \doteq \sup_{\underline{h},\underline{k}\in\mathbb{R}^3}(1+|\underline{k}|)^\theta\left[|f(\underline{k})|+\frac{|f(\underline{k}+\underline{h})-f(\underline{k})|}{|\underline{h}|^\mu}\right] \quad (2.10)$$

Or more generally, let $\mathcal{B}_M(\theta,\mu)$, $\theta>0$, $\mu>0$ be the Banach space of all functions $f: \mathbb{R}_0^{3M} \longrightarrow \mathbb{C}$ with

$$\|f\|_{\theta,\mu} = \sup_{\underline{h},\underline{k}\in\mathbb{R}_0^{3M}} N_M^{-1}(\underline{k},\theta)\left[|f(\underline{k})|+\frac{|f(\underline{k}+\underline{h})-f(\underline{k})|}{|\underline{h}|^\mu}\right] \quad (2.11)$$

where $N_M(\underline{k},\theta) = \sum_i \prod_{j=1}^{M-1}(1+|\underline{k}_{ij}|)^{-\theta}$, the summation extending over all (M-1)-tuples $\underline{k}_{ij}$ of partial sums

$$\underline{k}_{ij} = \sum_{\alpha=1}^{M} \sigma_{ij\alpha}\, \underline{k}_{\alpha} \quad , \quad \sigma_{ij\alpha} = 0, \pm 1$$

, which span a configuration $\underline{k}_1, \ldots, \underline{k}_M$.

In the Faddeev program one leaves the Hilbert space and works in the scale of Banach spaces $B(\theta,\mu)$. The essential steps are as follows: one iterates the Faddeev equations, finds suitable estimates for the iterated kernels and a function space $B(\theta,\mu)$ in which these iterations are contained (from a certain order onward). It may be shown that for $\theta > \bar\theta > 0$, $\mu > \bar\mu > 0$, every bounded linear mapping from $B(\bar\theta,\bar\mu)$ onto $B(\theta,\mu)$ is compact in $\mathcal{L}(B(\bar\theta,\bar\mu))$. Then one shows the existence of a unique solution in the singular limit. Also the associated homogeneous equation is investigated. Without going into details, the results obtained for the 2- and 3-particle systems are as follows: [3]

A. Two-particle system:

Assume that the potential belongs to the class $B(\theta_0,\mu_0)$ with $\theta_0 > \frac{3}{2}$, $\mu_0 > \frac{1}{2}$ and that the discrete spectrum of the (relative) Hamiltonian consists of a single negative eigenvalue $-\mathcal{H}^2$, $\mathcal{H} \neq 0$. Then

$$t(\underline{p},\underline{q};z) = v(\underline{p}-\underline{q}) + \int d^3k\, v(\underline{p}-\underline{q}) \frac{1}{z-\mu \underline{k}^2}\, t(\underline{k},\underline{q};z) \quad (2.12)$$

has for any z, except $z = -\mathcal{H}^2$ a unique solution in some $\mathcal{B}(\bar{\theta},\bar{\mu})$ with $\theta > \frac{3}{2}$, $\mu > 0$, uniformly in z. The solutions of the homogeneous equation belong to $\mathcal{B}(\theta_0,\mu_0)$.

We note that the restriction to a single simple negative eigenvalue is not essential and that the same result also holds in case of a finite number of negative eigenvalue with finite multiplicities. However the exclusion of the origin as bound state is not only generally fulfilled, but also necessary for a subsequent treatment of the 3-body system.

For instance $z = 0$ may be a singular (i.e. the homogeneous Faddeev equations have a non-trivial solution for z =0) as well as a regular point. If 0 is singular for the potential v, then it is also a singular point for the potential $(1+\varepsilon)v$, $\varepsilon > 0$. Thus it is possible to remove any zero-energy bound state by a slight change of the potential.

The discrete spectrum of the (relative) 2-body Hamiltonian has been studied by many authors. In particular, if $V(\underline{x})$ is quite arbitrary within a sphere of finite radius, but satisfies the estimate $|V(\underline{x})| \leq C(1+|\underline{x}|)^{-\alpha}$, $\alpha > 1$, outside this sphere, then there is no <u>positive</u> discrete spectrum[5]. Also Birman[6] has given almost necessary conditions for the finiteness of the non-positive spectrum. At any rate, a detailed knowledge of the spectrum of the

Hamiltonian is necessary in order to maintain control over the Fredholm alternative.

B. Three-particle system:

One assumes again that all 3 interparticle potentials are of the class $B(\theta_0, \mu_0)$, $\theta_0 > \frac{3}{2}$, $\mu_0 > \frac{1}{2}$ and that each 2-particle subsystem has only one simple negative eigenvalue $-\mathcal{H}_\alpha^2$, $\alpha = 1,2,3$.

Then the Faddeev equations have a solution in some $B(\theta, \mu)$, $\theta > \frac{3}{2}$, $\mu > 0$ uniformly in z in the complex plane slit from $-\mathcal{H}^2 \equiv -\max_\alpha \mathcal{H}_\alpha^2$ to $+\infty$, outside the discrete spectrum of the 3-body Hamiltonian. This set of singular points lies in a finite interval, is closed, denumerable and may have as limit points only the points $-\mathcal{H}_\alpha^2$, $\alpha = 1,2,3$. Thereby the following general theorem is used [7]:

Let the operator A be defined on a dense domain $\Delta(A)$ in the Banach space B and such that

(i) $\Delta(A)$ is stable under A

(ii) for some fixed $N > 0$ and any $n \geq N$, the operators A^n may be extended over the whole Banach space B into completely continuous operators

(iii) the homogeneous equation $\varphi = A\varphi$ has no non-trivial solution in $\Delta(A)$.

Then, the equation $\psi = \tilde{\psi} + A\psi$, $\tilde{\psi} \in \Delta(A)$ has a unique solution in B. Needless to say, the proof of the existence of the singular limit is intricate in the sense that it requires lengthy and ingenious majorisations. The validity of the Faddeev equations as an equivalent version of the Schrödinger equation is an essential ingredient of this proof.

Finally asymptotic completeness is proved in the following form for N=2,3 (we state the result only for the 3-body case):

Let $\hat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$, where $\mathfrak{h}_0$ respectively $\mathfrak{h}_\alpha$ are L^2-spaces of 2 resp. one vector variable. Let $\hat{H} = \hat{H}_0 \oplus \hat{H}_1 \oplus \hat{H}_2 \oplus \hat{H}_3$, where $\hat{H}_0$ operates on $\mathfrak{h}_0$, $\hat{H}_\alpha$ on $\hat{\mathfrak{h}}_\alpha$ and $\hat{H}_0$ is the free Hamiltonian of the relative motion and $(\hat{H}_\alpha f_\alpha)(\hat{p}) = (\mu_\alpha \hat{p}^2 + \varkappa_\alpha^2) f_\alpha(\hat{p})$, α =1,2,3. If $\mathbb{P}$ denotes the projector onto the subspace $\mathfrak{h}_d$ spanned by the eigenfunctions of the discrete spectrum of H, then it is shown that there exist operators $\Omega_0^{\pm}$, $\Omega_\alpha^{\pm}$, = α =1,2,3, mapping $\mathfrak{h}_0$, $\mathfrak{h}_\alpha$ into $\mathfrak{h}$ such that any $f \in \mathfrak{h}$ is uniquely represented as

$$f = f_d + \sum_{\alpha=0}^{3} \Omega_\alpha^{\pm} f_\alpha^{\pm} \tag{2.13}$$

where

$f_d \in \mathfrak{H}_d \ (\mathfrak{H} = \mathfrak{H}_d \oplus \mathfrak{H}_c)$, $f_\alpha^\pm \in \mathfrak{H}_\alpha$, $f_\alpha^\pm = (\Omega_\alpha^\pm)^\dagger f$, $\alpha = 0, 1, 2, 3$

Moreover:

$$\| f \|^2 = \| f_d \|^2 + \sum_{\alpha=0}^{3} \| f_\alpha \|_\alpha^2$$

with $\| \circ \|_\alpha$ denoting the norm in $\mathfrak{H}_\alpha$, and the operators $\Omega^\pm = \sum_{\alpha=0}^{3} \Omega_\alpha^\pm$ map isometrically $\hat{\mathfrak{H}}$ onto $\mathfrak{H}_c$.

This is the expansion theorem for an arbitrary $f \in \mathfrak{H}$ into eigenfunctions of H.

Consider now the operator $S = (\Omega^+)^\dagger \Omega^-$ defined in $\hat{\mathfrak{H}}$. In particular $S_{\alpha\beta} = (\Omega_\alpha^+)^\dagger \Omega_\beta^-$, $\alpha, \beta = 0, \dots, 3$ maps $\mathfrak{H}_\beta$ into $\mathfrak{H}_\alpha$. Then, the operator S is unitary and the relation $\Omega^- = \Omega^+ S$ i.e. $\Omega_\alpha^- = \sum_\beta \Omega_\beta^+ S_{\beta\alpha}$ holds.

These results will be connected in the next section with the concept of channel. Also the connection of the various Ω -operators with time dependent scattering theory, i.e. asymptotic solutions of $i \frac{\partial}{\partial t} f = H f$, for $|t| \to \infty$ will be exhibited. If $\mathbb{P}_\alpha$ is the projector in $\mathfrak{H}$ on the subspace spanned by the functions of the form

$$f_\alpha(\underline{P}, \hat{\underline{t}}) = \psi_\alpha(\hat{\underline{t}}) f_\alpha(\underline{P}),$$

with $f_\alpha \in L^2(\mathbb{R}^3)$ and $\psi_\alpha(\hat{\underline{r}})$ a discrete eigenfunction of the α^{th} 2-particle system, and if one identifies $\mathbb{P}_\alpha \mathfrak{h}$ with $\mathfrak{h}_\alpha$, $\mathfrak{h}$ with $\mathfrak{h}_0$, then the strong limitts

$$\Omega_0^{\pm} = s-\lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}, \quad \Omega_\alpha^{\pm} = s-\lim_{t\to\pm\infty} e^{iHt} e^{-iH_\alpha t}$$

exist.

3.- The N-body problem

We start with some notations and definitions which are the generalisations of the concepts used in the preceding sections. Again we restrict ourselves to the CM motion. The interparticle potentials are taken to be square-integrable. Then one may expect that the N-particle system can break up into parts which are independent, provided they are far separated from each other.

To formulate this, one has to introduce the concept of channel. [8]. A fragment α is any subset of the N particles. Let $V_\alpha = \sum_{i,j\in\alpha} V_{ij}$ denote the interactions within the fragment α. Then the Hamiltonian of (the relative motion of) this fragment reads: $H_\alpha = H^0_\alpha + V_\alpha$.

The CM momentum of α is $\underline{K}(\alpha) = \sum_{i\in\alpha} \underline{k}_i$ where $\underline{k}_i$ is the momentum of the ith particle and the relative momenta within α will be chosen according to

$$\hat{\underline{k}}_i(\alpha) = \underline{k}_i - \frac{m_i}{m(\alpha)} \underline{K}(\alpha) \tag{3.1}$$

with $m(\alpha) = \sum_{i\in\alpha} m_i$.

We shall denote

$$\mu_i = \frac{1}{2m_i}\ , \quad \mu(\alpha) = \frac{1}{2m(\alpha)} \tag{3.2}$$

Considering a fragment α as a system of closely bound particles, we introduce the fragment wave function ψ_α^r via

$$H_\alpha \psi_\alpha^r = E_\alpha^r \psi_\alpha^r \tag{3.3}$$

The ψ_α^r , $r = 1,2,\cdots$ constitute an orthonormal basis $\mathcal{F}_\alpha$ for the discrete spectrum of H_α .

If $\{\alpha_1, \alpha_2, \cdots \alpha_k\}$ is a partition of the N particles into k fragments, a channel A_k is a set of wave functions

$$A_k = \{\psi_{\alpha_\nu}^{r_\nu} \in \mathcal{F}_{\alpha_\nu}, \nu = 1,2,\ldots,k\} \tag{3.4}$$

Needless to say each fragment within A_k may still move freely relative to the others. The channel wave function will be denoted by $\Psi_{A_k} = \bigotimes_{\nu=1}^{k} \psi_{\alpha_\nu}^{r_\nu}$

To a given channel A_k corresponds a partition a_k of $\{1,2,\ldots,N\}$, $a_k = a_k(A_k)$. Obviously there is only one (scattering) channel A_1 with one fragment and one free channel A_N with N (non interacting) fragments.

Let us define for $\underline{k} = (k_1,\ldots,k_N)$ and $a_i = \{\alpha_1,\ldots,\alpha_i\}$

$$\begin{aligned}
E_{a_i}(\underline{k}) &= \sum_{\nu=1}^{i} \mu(\alpha_\nu)\, \underline{K}^2(\alpha_\nu) \\
E_{A_i}(\underline{k}) &= E_{a_i}(\underline{k}) + \sum_{\nu=1}^{i} E_{\alpha_\nu}^{r_\nu} \qquad (3.5) \\
E(\underline{k}) &= \sum_{i=1}^{N} \mu_i \underline{k}_i^2 = E_{a_i}(\underline{k}) + \sum_{n=1}^{N} \mu_n \hat{\underline{k}}_n^2(\alpha_{\nu(n)})
\end{aligned}$$

with $\nu(n)$ satisfying $n \in \alpha_{\nu(n)}$. Then the channel Hamiltonian $\overset{o}{H}_{A_i} = \sum_{\nu=1}^{i} (\overset{o}{H}_{\alpha_\nu} + E_{\alpha_o}^{r_\nu})$ is in momentum space a multiplication operator by $E_{A_i}(\underline{k})$ and every channel A_i (or equivalently, every partition $a_i(A_i)$) induces a decomposition of $\mathfrak{h}$ into free motion and internal motion Hilbert spaces: $\mathfrak{h} = \mathcal{H}_{A_i} \otimes \mathfrak{h}_{A_i}$.

It may be shown [9] that for $V_{ij}(\cdot) \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, such that the L^∞-component of V_{ij} can be chosen arbitrarily small in the L^∞-norm, the spectrum of the N-particle Hamiltonian (in the CM frame) consists of a continuum $\sigma(H) = [\min_{A_i, i \geq 2} \sum_{\nu=1}^{i} E_{\alpha_\nu}^{r_\nu}, \infty) \equiv [E^c, \infty)$ and in the complement of $\sigma(H)$, of eigenvalues which are of finite multiplicity and can only accumulate at E^c. This theorem is also valid if $V_{ij}(\cdot)$ is locally square integrable and vanishes (arbitrarily slowly) at infinity. In particular, this covers the case of Coulomb-interactions.

For subsequent use, we define a chain α_k as a sequence of partitions

$$\alpha_k = (a_k, a_{k+1}, \ldots, a_{N-1}) = (a_k, \alpha_{k+1}) = \ldots \ldots \tag{3.6}$$

$$= (a_k, a_{k+1}, \ldots, a_{N-2}, \alpha_{N-1})$$

where every a_{n+1} is obtained from a_n by partitioning some of its clusters, i.e. $a_n \supset a_{n+1}$, $k \leq n \leq N-2$, $\alpha_{N-1} \equiv a_{N-1}$.

Let $a_i = (\alpha_1, \alpha_2, \dots, \alpha_i)$ and define

$$V_{a_i} = \sum_{\nu=1}^{i} V_{\alpha_\nu}, \quad H_{a_i} = H^0 + V_{a_i}, \quad R_{a_i}(z) = (z - H_{a_i})^{-1} \tag{3.7}$$

The corresponding T-operator $T_{a_i}(z)$ is defined accordingly. Furthermore, let

$$V(a_i/a_{i+1}) = V_{a_i} - V_{a_{i+1}} \tag{3.8}$$

be the sum of 2-body potentials contained in a_i but not in a_{i+1}. Consider the chain $\alpha_1 = (a_1, a_2, \dots, a_{N-1})$ and suppose that a_{i+1} has been obtained from a_i by breaking its cluster $\alpha \cup \mathcal{B}$ into the 2 fragments α and $\mathcal{B}$.

Then a convenient coupling scheme for Jacobi coordinates is:

$$K(a_1) \equiv k(a_1) = \sum_{i=1}^{N} \underline{k}_i, \; \underline{k}(a_2/a_1), \dots, \underline{k}(a_N/a_{N-1}) \tag{3.9}$$

where $$k(a_{i+1}/a_i) = \frac{m(\alpha)\,\underline{K}(\mathcal{B}) - m(\mathcal{B})\,\underline{K}(\alpha)}{m(\alpha) + m(\mathcal{B})}$$

With the definition $\mu(a_{i+1}/a_i) = \mu(\mathcal{O}) + \mu(\mathcal{B})$, it is easily seen that for any i, $2 \leq i \leq N-1$:

$$E(\underline{k}) = E_{a_i}(\underline{k}) + \sum_{j=i}^{N-1} \mu(a_{j+1}/a_j)\, \underline{k}^2(a_{j+1}/a_j) \quad (3.10)$$

Finally, in a multichannel scattering, momentum conservation is conveniently exhibited by the use of

$$\delta_{a_i}(\underline{k}-\underline{\ell}) = \prod_{j=1}^{i-1} \delta\left[\underline{k}(a_{j+1}/a_j) - \underline{\ell}(a_{j+1}/a_j)\right] = \delta_{A_i}(\underline{k}-\underline{\ell}) \quad (3.11)$$

for $\underline{k}, \underline{\ell} \in \mathbb{R}_0^{3N}$.

We turn now to a brief review of multichannel scattering[8]. Consider a given channel A_s. The very channel concept implies that in the distant past ($t = -\infty$) the system behaved according to some wave function $e^{-iH^0_{A_s}t} f_{A_s} \otimes \Psi_{A_s}$, where $f_{A_s} \in \mathcal{H}_{A_s}$ describes the free motion of the fragments with respect to each other. A multichannel scattering process is equivalent with the existence of the limits $\lim_{t\to\pm\infty} e^{iHt} e^{-iH^0_{A_s}t} f_{A_s} \otimes \Psi_{A_s}$ in the L^2-topology. Now, the result of time-dependent scattering theory is that:

For L^2-potentials $V_{ij}(\cdot)$ and for any channel A_s, the Mölter operators

$$\Omega^{\pm}_{A_s} = \underset{t \to \pm\infty}{s-\lim} e^{iHt} e^{-iH^{0}_{A_s} t} \tag{3.12}$$

exist on $\mathfrak{h}_{A_s} = \mathcal{H}_{A_s} \otimes \Psi_{A_s}$ and that for any 2 channels A_r and A_s, $r \neq s$ the asymptotic Hilbert spaces $\mathfrak{h}^{ex}_{A_s} = \Omega^{ex}_{A_\varrho} \mathfrak{h}_{A_\varrho}$, $\varrho = r, s$, $ex = \pm$, are orthogonal to each other. (Note that $\mathfrak{h}^{+}_{A_r} \perp \mathfrak{h}^{-}_{A_s}$ is not implied, otherwise there would be no transition from one channel into another). Moreover, on $\mathfrak{h}^{ex}_{A_s}$, H is unitarily equivalent to $H^{0}_{A_s}$ and the spectrum $\sigma(H)$ contains $[E^{c}, \infty)$. If at time $t = -\infty$, the system was in the state $e^{-iH^{0}_{A_s} t} f_{A_s} \otimes \Psi_{A_s}$ the probability of finding it in the state $e^{-iH^{0}_{A_r} t} f_{A_r} \otimes \Psi_{A_r}$ at time $t = +\infty$ is given by

$$\lim_{t \to \infty} \left| \left(e^{-iH^{0}_{A_r} t} f_{A_r} \otimes \Psi_{A_r}, e^{-iHt} \Omega^{+}_{A_s} f_{A_s} \otimes \Psi_{A_s} \right) \right|^{2} =$$

$$= \left| \left(f_{A_r} \otimes \Psi_{A_r}, (\Omega^{-}_{A_r})^{\dagger} \Omega^{+}_{A_s} f_{A_s} \otimes \Psi_{A_s} \right) \right|^{2} \tag{3.13}$$

It follows that the transitions from channel A_s to channel A_r are determined by the operator $S_{rs} = (\Omega^{-}_{A_r})^{\dagger} \Omega^{+}_{A_s}$.

The orthogonality of the asymptotic Hilbert spaces is equivalent with the uniqueness of the channel concept: the statement that the system is in channel A_r excludes its being in channel $A_{r'}$, $r' \neq r$.

Since the L^2 -space is separable and the (closed) subspaces $\mathfrak{h}^{\pm}_{A_s}$ are mutually orthogonal, the number of channels is finite or denumerably infinite. In the former case, it is obvious that the operators $\sum_{r=1}^{M} \Omega^{\pm}_{A_r} (\Omega^{\pm}_{A_r})^{\dagger}$ are projectors. In the latter case, it may be shown that $\sum_{r} \Omega^{\pm}_{A_r} (\Omega^{\pm}_{A_r})^{\dagger} = \mathbb{P}^{\pm}$ are also projectors, with ranges $\mathfrak{h}^{\pm} = \bigoplus_{r} \mathfrak{h}^{\pm}_{A_r}$.

Now we expect that if r runs through all channels,

$$\mathbb{P}^{-} \Omega^{+}_{A_s} f_{A_s} \otimes \Psi_{A_s} = \Omega^{+}_{A_s} f_{A_s} \otimes \Psi_{A_s}$$

i.e. that $\Omega^{+}_{A_s} f_{A_s} \otimes \Psi_{A_s}$ has no components which do not correspond to some channel A_r . Therefore, it is necessary that $\mathfrak{h}^{+}_{A_r} \subseteq \mathfrak{h}^{-}$. On the other hand, in order that every function of the form $e^{-i \overset{\circ}{H}_{A_s} t} f_{A_s} \otimes \Psi_{A_s}$ can be realised as the result of a scattering event, we must require $\mathfrak{h}^{+} \supseteq \mathfrak{h}^{-}_{A_s}$. Hence, we expect on physical grounds that $\mathfrak{h}^{+} = \mathfrak{h}^{-}$.

Defining the S-matrix as an isometric mapping from $\mathfrak{h}^+$ onto $\mathfrak{h}^-$, the unitarity of the S-matrix or the statement of asymptotic completeness is equivalent with $\mathfrak{h}^+ = \mathfrak{h}^- = \mathfrak{h}$.

We now turn to the generalisation of the Faddeev-equations to the N-particle case, $N \geq 4$.

The resolvent $R(z)$, free resolvent $R_0(z)$ and T-operator $T(z)$ are defined in the well-known way and the LS-equation holds:

$$T(z) = V + V R_0(z) T(z) \tag{3.14}$$

Needless to say, equation (3.14) shares the same pathologies as the corresponding equation in the 3-body case. Therefore, we shall start by defining N-body amplitudes which have specified first connections. These will be the analogous of the $T_\alpha(z)$, $\alpha = 1, 2, 3$ of the previous section. However, due to the various disconnected graphs (for $N \geq 4$, there may be "spectator fragments"), several subamplitudes shall be defined.

The following remark will be repeatedly used in the sequel:

Given 2 different partitions a_k and b_k, there is at most one common partition a_{k+1} such that $a_{k+1} \subset a_k$

and $a_{k+1} \subset b_k$. If there is such a partition a_{k+1}, then there exists a unique a_{k-1} such that $a_k \subset a_{k-1}$ and $b_k \subset a_{k-1}$.

Let us introduce the symbols

$$\delta(a_k, b_k) = \begin{cases} \mathbb{1} & \text{for } a_k = b_k \\ 0 & \text{otherwise} \end{cases}, \quad \delta(b_{i+1} \subset a_i) = \begin{cases} \mathbb{1} & \text{for } b_{i+1} \subset a_i \\ 0 & \text{otherwise} \end{cases}$$

and the "numerical" matrix $X^k_{a_i}$ of type (k, a_i), $k > i$:

$$X^{\alpha_k \beta_k}_{a_i} = \prod_{j=k}^{N-1} \left[\mathbb{1} - \delta(a_j, b_j) \right] \prod_{j'=k}^{N-2} \delta(b_{j'+1} \subset a_{j'})$$

whose elements are labeled by chains α_k, β_k with $a_k, b_k \subset a_i$. We shall further assume $\mathrm{Im}\, z \neq 0$ and often drop z as a variable. The subamplitudes T_{a_i} with connectivity a_i, $1 \leq i \leq N$ are introduced and we define the α_k-connected components $T^{\alpha_k}_{a_i}$, $a_k \subset a_i$ of T_{a_i} recursively, beginning by:

$$T^{\alpha_{N-1}}_{a_i} = V_{a_{N-1}} + V_{a_{N-1}} R_{a_i} V_{a_i} \tag{3.15}$$

The Faddeev equations for $T^{\alpha_{N-1}}_{a_i}$ read:

$$T^{\alpha_{N-1}}_{a_i} = T^{\alpha_{N-1}}_{a_{N-1}} + T^{\alpha_{N-1}}_{a_{N-1}} R_0 \sum_{b_{N-1} \neq a_{N-1}} T^{\beta_{N-1}}_{a_i} \tag{3.16}$$

$$T_{a_i} = \sum_{\alpha_{N-1}} T^{\alpha_{N-1}}_{a_i}$$

It is easily checked that these equations are related to the LS-equation for T_{a_i} by invertible operations.

We shall consider the different $T_{a_i}^{\alpha_{N-1}}$ as components of a column vector $T_{a_i}^{N-1}$ of type $(N-1, a_i)$.

Graphically, $T_{a_i}^{\alpha_{N-1}}$ is the sum of all graphs with connectivity a_i, with α_{N-1} as first interaction (from the left).

Introducing a 1×1 matrix $Q_{a_i}^{N-1}$ of type $(N-1, a_i)$ via, its elements

$$Q_{a_i}^{\alpha_{N-1}, \beta_{N-1}} = T_{a_{N-1}}^{\alpha_{N-1}} X_{a_i}^{\alpha_{N-1}, \beta_{N-1}} R_0 \tag{3.17}$$

we obtain in matrix form:

$$T_{a_i}^{N-1} = \tilde{T}_{a_i}^{N-1} + Q_{a_i}^{N-1} T_{a_i}^{N-1} \tag{3.18}$$

Here we have set $\tilde{T}_{a_i}^{\alpha_{N-1}} \equiv T_{a_{N-1}}^{\alpha_{N-1}}$.

The successive a_i-kernels with increasing prescribed connectivity are then defined recursively [10] by:

$$\begin{aligned} M_{a_i}^{\alpha_{k-1}, \beta_{k-1}} &= M_{a_k}^{\alpha_{k+1}, \beta_{k+1}} \delta(a_k, b_k) \delta(a_{k-1}, b_{k-1}) \\ &+ \sum_{\substack{d_k \subset a_{k-1} \\ d_k \neq a_k}} Q_{a_i}^{\alpha_k S_k} M_{a_i}^{S_k, \beta_k} \end{aligned} \tag{3.19}$$

We have used the convention over all repeatedly occuring indices, the restrictions being indicated under the summation sign. It is tacitlyy assumed that the upper chain indices α_k, β_k of all matrices of type (k, a_i) satisfy always $a_k, b_k \subset a_i$. Finally, we have set:

$$M^{N}_{a_{N-1}} \equiv \widetilde{T}^{N-1}_{a_i} \tag{3.19'}$$

Clearly, $M^{k-1}_{a_i}$ can be constructed once all $M^{\nu}_{a_i}$, $i < k \leq \nu \leq N-1$ are known and

$$T_{a_i} = \sum_{\alpha_{N-1}, \beta_{N-1}} M^{\alpha_{N-1}, \beta_{N-1}}_{a_i} = \sum_{b_{N-1}} \sum_{\alpha_k} M^{\alpha_k, b_k b_{k+1} \ldots b_{N-2} b_{N-1}}_{a_i} \tag{3.20}$$

Also if $M^{k}_{a_i}$ is known:

$$M^{\alpha_{k+1}, \beta_{k+1}}_{a_i} = \sum_{a_k} M^{\alpha_{k+1} a_k, b_k \beta_{k+1}}_{a_i} \quad \text{for arbitrary } b_k (b_k \subset a_i).$$

Let us define

$$Q^{\alpha_k, \delta_k}_{a_i} = \sum M^{\alpha_{k+1}, \gamma_{k+1}}_{a_k} \delta(a_k, c_k) X^{\gamma_k \delta_k}_{a_i} R_0 \tag{3.21}$$

We introduce then

$$T_{a_i}^{\alpha_{k-1}} = \sum_{b_k < a_{k-1}} Q_{a_i}^{\alpha_k, \beta_k} T_{a_i}^{\beta_k} \tag{3.22}$$

Due to the preceding remark, it is easily recognised that $T_{a_i}^{k-1}$ $(i < k-1)$ is unambiguously defined in terms of the $T_{a_i}^{k}$.

We also set:

$$\widetilde{T}_{a_i}^{\alpha_{k-1}} \equiv T_{a_{k-1}}^{\alpha_{k-1}} \tag{3.23}$$

It may then be shown as in [10] that for $2 \leq k \leq N-1$, $k > i$, the $T_{a_i}^{\alpha_k}$ satisfy the following equations on $\Delta(H^0)$:

$$T_{a_i}^{k} = \widetilde{T}_{a_i}^{k} + Q_{a_i}^{k} T_{a_i}^{k} \tag{3.24}$$

and

$$T_{a_i}^{\alpha_k} = \widetilde{T}_{a_i}^{\alpha_k} + \sum_{a_{k-1}} T_{a_i}^{\alpha_{k-1}}$$

Finally, it is not difficult to prove the cluster decomposition of T_{a_i}:

$$T_{a_i} = \sum_{\alpha_k} T_{a_i}^{\alpha_k} + \sum_{\alpha_{k+1}} T_{a_{k+1}}^{\alpha_{k+1}} + \cdots\cdots + \sum_{\alpha_{N-1}} T_{a_{N-1}}^{\alpha_{N-1}} \tag{3.25}$$

For the full interacting N-particle system $(T(z) \equiv T_{a_1}, i=1)$ we have:

$$T_{a_1}^{\alpha_2} = \tilde{T}_{a_1}^{\alpha_2} + \sum_{\beta_2} Q_{a_1}^{\alpha_2,\beta_2} T_{a_1}^{\beta_2} \qquad (3.26)$$

These coupled integral equations of the 2nd kind are the Faddeev-Yakubovskii (FY) equations for the N-body system. They reduce to the Faddeev equations in the case N=3. It may be shown [10] that the FY-equations are related to the LS-equation and hence to the Schrödinger equation by invertible operations and that the homogeneous equations have no non-trivial solution for $\mathrm{Im}\, z \neq 0$. Thus there are no spurious solutions.

Also it is easily seen that the kernels $Q_{a_1}^{\alpha_2,\beta_2}$ do contain disconnected graphs and therefore cannot be of HS type. We note that there are $\binom{N}{2}\binom{N-1}{2}\cdots\binom{3}{2}$ subamplitudes $T_{a_1}^{\alpha_2}$ of connectivity a_2 and that the solution of (3.26) (whenever possible) requires the knowledge of all possible subamplitudes.

Finally, we quote at this stage a useful formula: Consider an arbitrary channel A_i and let $a_i = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$, $\alpha_\nu = \{(\nu 1), (\nu 2), \ldots, (\nu n_\nu)\} \subset \{1, 2, \ldots, N\}$, be the corresponding partition. Let

$$\tau_{a_i}(\hat{p}_{11}, \ldots, \hat{p}_{1n_1}, \hat{p}_{21}, \ldots, \hat{p}_{2n_2}, \ldots\ldots, \hat{q}_{i1}, \ldots \hat{q}_{in_i}; z)$$

be the kernel of $T_{a_i}(z)$ in the relative motion Hilbert space $\mathfrak{h}_{a_i}$ i.e. with the CM-motions of the clusters factored out:

$$\underline{P}(\alpha_\nu) \equiv \sum_{j=1}^{n_\nu} \hat{\underline{p}}_{\nu_j} = 0, \quad \underline{Q}(\alpha_\nu) \equiv \sum_{k=1}^{n_\nu} \hat{\underline{q}}_{\nu_k} = 0$$

Then the kernel of $T_{a_i}(z)$ in the N-particle Hilbert space reads:

$$\langle \underline{p}_1, \ldots \underline{p}_N | T_{a_i}(z) | q_1, \ldots q_n \rangle = \prod_{\nu=1}^{i} \delta(\underline{P}(\alpha_\nu) - \underline{Q}(\alpha_\nu)) \times$$

$$\times \tau_{oi}\Big(p_{11} - \frac{m_{11}}{m(\alpha_1)} \underline{P}(\alpha_1), \ldots, p_{1n_1} - \frac{m_{1n_1}}{m(\alpha_1)} \underline{P}(\alpha_1), \underline{p}_{21} - \frac{m_{21}}{m(\alpha_2)} \underline{P}(\alpha_2),$$

$$\ldots, \underline{p}_{2n} - \frac{m_{2n_2}}{m(\alpha_2)} \underline{P}(\alpha_2), \ldots, q_{i_1} - \frac{m_{i1}}{m(\alpha_i)} \underline{Q}(\alpha_i); \; z - \sum_{\nu=1}^{i} \mu(\alpha_\nu) \underline{P}^2(\alpha_\nu)\Big)$$

We now turn to a graphical interpretation of the FY-equations. [11] Due to the assumed square-integrability of the potentials $V_{a_{N-1}}$, we have again for $\mathrm{Re}\, z<0$:

$$\| V_{a_{N-1}} R_0(z) \| \leq \frac{c}{|z|} \quad , \quad c>0 \tag{3.27}$$

By choosing $\mathrm{Re}\, z<0$ and sufficiently small, the perturbation series

$$T(z) = \sum_{n=0}^{\infty} [V R_0(z)]^n V \tag{3.28}$$

obtained by iterating the LS equation converges.

Every term $V_{i_1 j_1} R_0 V_{i_2 j_2} R_0 \dots R_0 V_{i_n j_n}$ can be represented by a graph $G(p, q; z)$ where $p, q \in \mathbb{R}_0^{3N}$ denote the external momenta. Every graph G has a unique sequential connectivity $\alpha_i = (a_i, a_{i+1}, \dots a_{N-1})$ from the left. If α_j, $j > i$ is the sequential connectivity of the sequence $V_{i_1 j_1}, \dots V_{i_r j_r}$ and if $V_{i_\nu j_\nu}$ is the next potential which connects two disjoint clusters $\mathfrak{A}$ and $\mathfrak{B}$ in a_j, then $\alpha_{j-1} = (a_{j-1}, \alpha_j)$, where a_{j-1} is obtained from a_j by replacing $\mathfrak{A}$ and $\mathfrak{B}$ by their union. The coarsest partition a_i is called the connectivity of G. Now $T_{a_i}^{\alpha_{N-1}}(z)$ is the analytic continuation from $\{\operatorname{Re} z < c\}$ sufficiently large, into $\{\operatorname{Im} z \neq 0\}$ of the sum of all graphs with connectivity a_i, whose first vertex is $\alpha_{N-1} = (i, j)$. Similarly $T_{a_i}^{\alpha_k}$ is the analytic continuation of the sum of all graphs with connectivity β_n and left-sequential connectivity $\beta_n = (b_n, b_{n+1}, \dots, b_{k-1}, \alpha_k)$, $b_n \subset a_i$ or $b_n = a_i$.

Finally, we investigate the FY-kernels $Q_{a_1}^{\alpha_2, \beta_2} \equiv Q^{\alpha_2, \beta_2}$. We may write:

$$
\begin{aligned}
T_{a_1}^{\alpha_2} &= \tilde{T}_{a_1}^{\alpha_2} + T_{a_1}^{\alpha_1} \\
T_{a_1}^{\alpha_1}(z) &= \sum Q_{a_1}^{\alpha_2, \beta_2}(z)\, T_{a_1}^{\beta_2}(z) = \\
&= \tilde{T}_{a_1}^{\alpha_2} R_0(z) \sum_{c_{N-1} \not\subset a_2} T_{a_1}^{\gamma_{N-1}}
\end{aligned}
\tag{3.29}
$$

Then every graph G in $T^{\alpha_1}_{a_1}(z)$ can be decomposed into 2 graphs G_1 and G_2 where the left most potential in G_2 is the 1st interaction from the left in G belonging to $V(a_1/a_2)$. G_1 contributes to $\widetilde{T}^{\alpha_2}_{a_1}$. If one determines a "critical" potential $V_{C_{N-1}}$ such that G_1 may be splitted as $G_4 R_0(z) G_3$ with G_3 having left sequential connectivity β_3 and $V_{C_{N-1}}$ being the first potential to the right in G_4 with $C_{N-1} \subset a_2$ and $C_{N-1} \not\subset b_3$, then $G_3 G_2$ has left connectivity β_2 with $b_2 \neq a_2$. Therefore $G_3 G_2$ will contribute to $T^{\beta_2}_{a_1}$ and G_4 to $Q^{\alpha_2 \beta_2}(z)$.

For any two chains α_k, β_m, we define the symbol $a_k \llcorner \beta_m = \gamma_m = (C_m, C_{m+1}, \ldots, C_{k-1}, \alpha_k)$ as the completion of the chain α_k by all new connections arising from β_m in the order $b_{N-1}, b_{N-2}, \ldots, b_m$. Then $Q^{\alpha_2 \beta_2}_{a_1}(z)$ is the sum of all graphs with left connectivity α_k, $2 \leq k \leq N-1$ and with a rightmost potential $V_{C_{N-1}}$, $C_{N-1} \subset a_k$ such that $\alpha_k \llcorner \beta_3 = \alpha_2$, $C_{N-1} \not\subset b_3$.

This graphical interpretation of $Q^{\alpha_2, \beta_2}_{a_1}(z)$ is used to prove that the (N-1)st iteration of the matrix $Q^{2}_{a_1}(z)$ is of HS type with HS norm of the order $O(|z|^{-\delta}), \delta > 0$ as $\mathrm{Re}\, z \longrightarrow -\infty$. [11]

We close this section with the remark that amplitudes with specific sequential connectivities from the left as well as from the right can be constructed in a similar fashion. These amplitudes will be useful for the proof of regularity propoerties.

4.- Perturbation Theory

In this section we study the iterations of the FY equations in the frame work of perturbation theory. Our aim is to establish analyticity properties for the perturbative N-body amplitude and we may already state the outcome of this investigation:

For holomorphic two-body potentials, the Feynman amplitude of any N-body graph with sufficiently high connectivity is holomorphic in the external momenta $p = (p_1, \dots, p_N) \in \mathbb{R}_0^{3N}$, $q = (q_1, \dots, q_N) \in \mathbb{R}_0^{3N}$ for $\mathrm{Im}\, z \to 0$, except on the thresholds

$$z = E_{a_i}(p), \quad z = E_{b_i}(q), \quad 1 \leq i \leq N-1 \qquad (4.1)$$

This amplitude is furthermore HC in $\underline{p}, \underline{q}, z$ and uniformly decreasing in p, q for $\mathrm{Re}\, z \leq E$, $E \geq 0$ arbitrary but fixed, $\mathrm{Im}\, z \geq 0$ or $\mathrm{Im}\, z \leq 0$. We shall only sketch the proof of the analyticity properties. Thereby the method of rotation of integration contours [12] (of Section 1) will be used in order to perform analytic continuations.

It is well-known that to each perturbation graph, there corresponds canonically a Feynman integral over the internal momenta. The integrand is a rational function of internal and external momenta. The potentials contribute to the numerator whereas the free resolvents (propagators) appear only in the denominator. Therefore, we shall consider a class of potentials which is analytic in a neighbourhood of the real axes:

For any i, j, $1 \leq i < j \leq N$, the potentials $v_{ij}(p)$ belong to the class

$$H(\theta, \rho) = \Big\{ v(p) = v^{*}(-p), v(p) \text{ holomorphic for } |\mathrm{Im}\, p| < \rho, \sup_{p \in \mathbb{R}^3} (1+|p|)^{\theta} |D v(p)| < \infty \Big\} \tag{4.2}$$

with $\theta > \frac{3}{2}$, and some $\rho > 0$ and where D is any differential monomial. These potentials decrease exponentially. It is easily checked that for instance suitable superpositions of Yukawa potentials belong to $H(\theta, \rho)$.

The analyticity of these potentials allows us then to restrict to the denominators of the Feynman integrands.

Consider a connected graph in the Born series $\sum_{n=0}^{\infty} [V R_0(z)]^n V$: we call a graph c-connected, if c is the largest integer such that by cutting the graph at c-1 intermediate states one obtains c connected subgraphs. It will turn out that there exists a c_0 depending only on the masses of the particles, such that less than c_0-connected graphs (weakly connected graphs) correspond to rescattering singularities, whose positions depend in a complicated manner on the configuration of both the initial and final momenta. In the 3-particle case, it is still possible to handle these singularities [13], but they rapidly grow out of control with increasing N.

Strongly connected graphs (more than C_0 -connected) have only threshold singularities at loci which do not couple incoming and outgoing momenta. Therefore we shall only consider such graphs.

Finally, we should also mention disconnected graphs which contain momentum conservation δ -functions and are only defined on certain subvarieties in the external momenta.

Now according to the "Feynman rules" [14], the contribution $G(p, q; z)$ of any N-body graph is holomorphic in p, q, z if in its Feynman integral the real contour of loop momenta avoids all singularities of the free propagators and if the integral converges absolutely together with all its derivatives with respect to z and to the external momenta $p, q \in \mathbb{R}_0^{3N}$. Recalling the remark made in the previous section concerning FY-amplitudes with two-sided connectivities, we consider a typical c-connected graph, with $c \geq c_0$, c_0 to be determined, with left-and right sequential connectivities

$$\alpha_1 = (a_1, a_2, \ldots\ldots a_{N-1}), \; \beta_1 = (b_1, b_2, \ldots\ldots b_{N-1}).$$

We may hence decompose G as $G_1 R_0 G_2 R_0 G_3$ where G_1

resp G_3 have the left-resp. right-sequential connectivities α_1 resp. β_1 .

The essential idea may now be sketched as follows:

Let G-consist of $V^1 R_0(z) V^2 R_0(z) \ldots V^{K-1} R_0(z) V^K$, where we have set $V^\lambda \equiv V_{i_\lambda j_\lambda}$. Denote the external momenta by $\underline{k}^0 = \underline{p}$ resp. $\underline{k}^K = \underline{q}$. There are K-1 intermediate states and the internal momenta between $V^\varkappa$ and $V^{\varkappa+1}$ will be denoted by $\underline{k}^\varkappa = (\underline{k}_1^\varkappa, \ldots, k_N^\varkappa), 1 \le \varkappa \le K-1$. To each intermediate state $\varkappa$, there corresponds an energy denominator $D_\varkappa = z - \sum_{\ell=1}^{N} \mu_\ell (\underline{k}_\ell^\varkappa)^2$.

Now, due to momentum conservation at the different vertices, we have for every x, $1 \le \varkappa \le K$:

$$
\begin{aligned}
\underline{k}_\ell^{\varkappa-1} &= \underline{k}_\ell^\varkappa \qquad \text{for } \ell \neq i_\varkappa, j_\varkappa \\
\underline{k}_{i_\varkappa}^{\varkappa-1} + \underline{k}_{j_\varkappa}^{\varkappa-1} &= \underline{k}_{i_\varkappa}^\varkappa + \underline{k}_{j_\varkappa}^\varkappa
\end{aligned}
\tag{4.3}
$$

We shall use a particular choice of internal momenta, say fulfilling (4.3) and such that $\bar{\underline{k}}^\lambda = 0$ for some $1 \le \lambda \le K-1$. Then for $0 \le x < \lambda$ resp. for $\lambda < x \le K$, the internal momenta $\bar{\underline{k}}_\ell^\varkappa$, $1 \le \ell \le N$ will be linear combinations of $\underline{k}_\ell^0 \equiv \underline{p}_\ell$ resp. $\underline{k}_\ell^K = \underline{q}_\ell$, $1 \le \ell \le N$. Necessary for the existence of this solution is the high connectivity of the graph.

On the other hand, every arbitrary choice say $\underline{\tilde{k}}^{x}$ of internal momenta satisfying (4.3) with $\underline{\tilde{k}}_{\ell}^{0} = \underline{\tilde{k}}_{\ell}^{K} = 0$, $1 \leq \ell \leq N$, can be parametrised by L (real) loop momenta $\underline{s}^{1}, \ldots, \underline{s}^{L}$, $(L = K+1-N)$. Then the general solution of (4.3) will be of the form $\underline{\bar{k}}^{x} + \underline{\tilde{k}}^{x}(\underline{s}^{i})$.

We first consider G_1. Pour fixer les idees, let us make a provisory choice of the loop momenta, which might also be useful in other connections. The general prescription for the choice of the internal momenta will be given later on.

It is useful to introduce the "angular" variables $\underline{x}_i, \underline{y}_i$, $1 \leq i \leq N$, on the energy shell via:

$$\underline{p}_i = k\,\underline{x}_i,\; \underline{q}_i = k\,\underline{y}_i,\; 1 \leq i \leq N,\; E = k^2 = \sum_{i=1}^{N} \mu_i \underline{p}_i^2 \qquad (4.4)$$

with

$$\underline{x}_i, \underline{y}_i \in \mathbb{R}^3,\; \sum_1^N \underline{x}_i = \sum_1^N \underline{y}_i = 0,\; \sum_{i=1}^{N} \mu_i \underline{x}_i^2 = 1 \qquad (4.4')$$

We denote by Ω_c^N the following complex submanifold of $\mathbb{C}^{3N}$:

$$\Omega_c^N = \left\{ (\underline{x}_1, \ldots, \underline{x}_N) \in \mathbb{C}^{3N}, \sum_1^N \underline{x}_i = 0, \sum_1^N \mu_i \underline{x}_i^2 = 1 \right\} \qquad (4.5)$$

Finally, let $\alpha_i = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$ be a partition of $\{1,2,\ldots,N\}$ and:

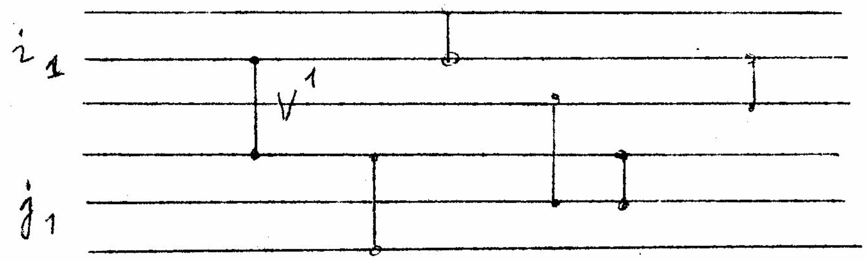
$$\Omega^N = \Omega^N_c \cap \mathbb{R}^{3N} \tag{4.6}$$

$$\Omega^N_{a_i} = \{ \underline{x}_1, \ldots, \underline{x}_N \} \in \Omega^N, \sum_{\nu=1}^{i} \mu(\alpha_\nu)\underline{x}^2(\alpha_\nu) = 1 \}$$

Ω^N is a compact (3N-1)-dimensional surface in $\mathbb{R}^{3N}$ and $\Omega^N_{a_i}$ is a "Landau" subvariety of Ω^N where all the relative momenta within the fragments $\alpha_1, \alpha_2, \ldots, \alpha_i$ vanish. We shall use

$$\Omega^N_0 = \Omega^N - \bigcup_{a_i, i>1} \Omega^N_{a_i} \tag{4.7}$$

Now let $V^1, V^2, \ldots, V^t$ be the sequence of potentials in G_1 from left to right



Define:

$$\underline{k}^1_\ell = \begin{cases} \underline{k}^0_\ell & , \ell \neq i_1, j_1 \\ \mu_{j_1}(\mu_{i_1}+\mu_{j_1})^{-1}(\underline{k}^0_{i_1} + \underline{k}^0_{j_1}), & \ell = i_1 \\ \mu_{i_1}(\mu_{i_1}+\mu_{j_1})^{-1}(\underline{k}^0_{i_1} + \underline{k}^0_{j_1}), & \ell = j_1 \end{cases} \tag{4.8}$$

where we also set $\underline{k}^0_\ell = k\underline{x}_\ell \equiv \underline{p}_\ell \, , \; 1 \leq \ell \leq N$.

Suppose the (i_1)st particle line be not connected to any other line before the (j_1)st line. Then we put the 1st loop momentum $\underline{s}^1$ on this "longest segment", and define:

$$\underline{s}_\ell^1 = \begin{cases} 0 & \ell \neq i_1, j_1 \\ \underline{s}^1 & \ell = i_1 \\ -\underline{s}^1 & \ell = j_1 \end{cases} \tag{4.9}$$

Then the 1st intermediate state carries the momentum $\underline{k}_\ell^1 + \underline{s}_\ell^1$, $1 \leq \ell \leq N$ with $\sum_1^N \underline{k}_\ell^1 = \sum_1^N \underline{s}_\ell^1 = 0$.

Now suppose $\underline{k}_\ell^\varrho$, $\underline{s}_\ell^\varrho$, $1 \leq \ell \leq N$ be chosen for all intermediate states following V^ϱ, $1 \leq \varrho \leq r < t$. After V^{r+1} we introduce

$$\underline{k}_\ell^{r+1} = \begin{cases} \underline{k}_\ell^r & , \ell \neq i_{r+1}, j_{r+1} \\ \mu_{j_{r+1}}(\mu_{i_{r+1}} + \mu_{j_{r+1}})^{-1}(\underline{k}_{i_{r+1}}^r + \underline{k}_{j_{r+1}}^r), & \ell = i_{r+1} \\ \mu_{i_{r+1}}(\mu_{i_{r+1}} + \mu_{j_{r+1}})^{-1}(\underline{k}_{i_{r+1}}^r + \underline{k}_{j_{r+1}}^r), & \ell = j_{r+1} \end{cases} \tag{4.10}$$

and

$$\underline{s}_\ell^{r+1} = \begin{cases} \underline{s}_\ell^r & \ell \neq i_{r+1}, j_{r+1} \\ \underline{s}^{r+1} & \ell = i_{r+1} \text{ ("longest segment")} \\ \underline{s}_{j_r}^r + \underline{s}_{i_r}^r - \underline{s}^{r+1} & \ell = j_{r+1} \end{cases} \tag{4.11}$$

Obviously: $\sum_1^N \underline{k}_\ell^{r+1} = \sum_1^N \underline{s}_\ell^{r+1} = 0$.

We prove now that the rth energy denominator

$$D_r = k^2 - \sum_{\ell=1}^N \mu_\ell (\underline{k}_\ell^r + \underline{s}_\ell^r)^2 \tag{4.12}$$

never vanishes for real loop momenta $\underline{s}^1, \ldots \underline{s}^r$, real $(\underline{x}_1, \ldots, \underline{x}_N) \in \Omega_0^N$ and $\mathrm{Im}\, k \neq 0$:

Setting $k = |k| e^{i\varphi}$, $|k| \neq 0$, $\varphi \neq n\pi$, n integer, we have:

$$D_r = -|k|^2 \left[1 - \sum_1^N \mu_\ell (\underline{x}_\ell^r)^2\right] - \sum_1^\ell \mu_\ell (\underline{s}_\ell^r)^2 + \mathrm{Im}\, D_r \cdot \mathrm{ctg}\, \varphi + i\, \mathrm{Im}\, D_r$$

Now either $\mathrm{Im}\, D_r \neq 0$, in which case there is nothing to prove or $\mathrm{Im}\, D_r = 0$ and then $D_r < 0$ for $\underline{s}^\varkappa \in \mathbb{R}^3$, $1 \leq \varkappa \leq r$, provided that $\sum_1^N \mu_\ell (\underline{x}_\ell^{?r}) < 1$.

Now by construction, for all $r \geq 0$ and $\varrho > r$:

$$\sum_1^N \mu_\ell (\underline{x}_\ell^\varrho)^2 \leq \sum_1^N \mu_\ell (x_\ell^r)^2 \leq 1$$

Therefore either $\sum_1^N \mu_\ell (\underline{x}_\ell^r)^2 < 1$ for some $r \geq 0$ and then the same inequality holds for all $\varrho > r$, or $\sum_1^N \mu_\ell (\underline{x}_\ell^r)^2 = 1$ and then also for all $0 \leq \varrho < r$.

Let now a_i correspond to the connectivity

$$(i_1, j_1) \sqsubset (i_2, j_2) \sqsubset \ldots\ldots \sqsubset (i_r, j_r)$$

Then the equality $\sum_1^N \mu_\ell (\underline{x}_\ell^r)^2 = 1$ holds iff

$$\sum_{\nu=1}^{i} \mu(\alpha_\nu)\, \underline{x}^2(\alpha_\nu) = 1 \tag{4.13}$$

To see this, we note that (4.12) is necessary and sufficient for r=1 since $\sum_1^N \mu_\ell \underline{x}_\ell^2 = 1$ and

$$\sum_{\ell \neq i_1, j_1}^{N} \mu_\ell \underline{x}_\ell^2 + \left[\mu_{i_1} \left(\frac{\mu_{j_1}}{\mu_{i_1} + \mu_{j_1}} \right)^2 + \mu_{j_1} \left(\frac{\mu_{i_1}}{\mu_{i_1} + \mu_{j_1}} \right)^2 \right] (x_{i_1} + x_{j_1})^2 = 1 \tag{4.14}$$

are compatible iff the relative momenta of the (i_1)st and (j_1)st particles vanish, i.e.

$$\mu_{i_1} \underline{x}_{i_1} = \mu_{j_1} \underline{x}_{j_1}$$

which is equivalent to (4.12).

Assume now that for some λ, $1 \leq \lambda < r$, the equality $\sum_1^N \mu_\ell (\underline{x}_\ell^\lambda)^2 = 1$ is equivalent to the vanishing of all relative momenta within the clusters of the partition $b_h = \{ B_1, \ldots, B_h \}$ corresponding to $(i_1, j_1) \sqsubset (i_2, j_2) \sqsubset \ldots \ldots \sqsubset (i_\lambda, j_\lambda)$. If $V^{\lambda+1}$ connects 2 particles within the same fragment say $\mathcal{B}_\nu$, $1 \leq \nu \leq h$, then clearly $\underline{x}_\ell^\lambda = \underline{x}_\ell^{\lambda+1}$, $1 \leq \ell \leq N$. If $V^{\lambda+1}$ connects 2 particles within the different fragments $\mathcal{B}_\nu$ and $\mathcal{B}_{\nu'}$, then a calculation similar to (4.13) shows that

$$\sum_1^N \mu_\ell (\underline{x}_\ell^\lambda)^2 = \sum_1^N \mu_\ell (\underline{x}_\ell^{\lambda+1})^2 \tag{4.15}$$

is equivalent with the vanishing of the relative momentum of the fragments $\mathcal{B}_\nu$ and $\mathcal{B}_{\nu'}$. This finishes the proof.

From this first partial result, one deduces that for real loop momenta $\underline{s}^\lambda$, $1 \leq \lambda \leq r$, real $(\underline{x}_1, \ldots, \underline{x}_N) \in \Omega^N$ and $\mathrm{Im}\, k \neq 0$, the rth energy denominator D_r can only vanish for $\underline{s}_1^r = \underline{s}_2^r = \ldots\ldots = \underline{s}_N^r = 0$. We note also that the subtracted manifold $\bigcup_{a_i; i>1} \Omega_{a_i}^N$ corresponds to the threshold configurations.

We have not yet used the assumed high connectivity of the graph G. This will be necessary in the investigation of the behaviour of the subgraphs of G: by cutting the graph from the left once it is connected, twice-connected...., c-connected and choosing suitable combinations of particle momenta for the loop momenta, then repeating this procedure within the connected subgraphs of G for the highly connected subsystems and so on, it is possible to show [11] that the previous result can be extended to the whole graph $G = G_1 R_0 G_2 R_0 G_3$. Using the method of the rotation of integration contours it may then be shown that the contribution $G(p, q; z)$ of any strongly connected graph is analytic in $(x_1, \dots, y_N) \in \Omega_0^N \times \Omega_0^N$ and in the energy k^2 in a cut plane with cuts running from $-\infty$ to some $-\varsigma^2$ and from 0 to ∞ .

However, we prefer to mention a more compact form of the choice of internal momenta which also leads to the same result[15]. We state this lemma in terms of the internal momenta $\bar{\underline{k}}^\lambda, \tilde{\underline{k}}^\lambda$ previously introduced:

Lemma.- Let G be c-connected, $c \geq c_0$, $z = E + i\varepsilon$, $\varepsilon > 0$, $0 \leq \varphi \leq \varphi_0$, $\varphi_0 \geq 0$. Then, there exists a particular choice $\bar{\underline{k}}^\lambda$ of internal momenta with the required properties (cf.(4.3) and subsequent remarks) which can be chosen holomorphic in E , p , q except on the thresholds (4.1) and HC everywhere. For every real solution $\tilde{\underline{k}}^\lambda$ of (4.3) with $\tilde{\underline{k}}^0 = \tilde{\underline{k}}^K = 0$, all energy denominators

$$D_\lambda = z - E(\bar{\underline{k}}^\lambda + \tilde{\underline{k}}^\lambda (1 - i\varphi))$$ remain different from 0

for all $\varepsilon > 0$ and φ_0 sufficiently small. For $0 < \varphi \leq \varphi_0$ and $\varepsilon \downarrow 0$, D_λ vanishes only if $\tilde{\underline{k}}^\lambda = 0$ and if $z = E$, p, q satisfy (4.1).

In the proof of this lemma, the essential construction of the particular choice $\underline{k}^{\lambda}$ depends upon the left and right connectivity of the graph and is closely related to (4.8)-(4.11). The remaining statements are also proved along lines similar to (4.12)-(4.15).

This lemma allows one to avoid the singularities of the free propagators for $\mathrm{Im}\ z \to 0$ by an infinitesimal deformation of the contour of the loop momenta: $\tilde{\underline{k}}^{\lambda} \to (1-i\varphi)\tilde{\underline{k}}^{\lambda}$ (remember: $\tilde{\underline{k}}^{\lambda} = \underline{k}^{\lambda}(\underline{s}^{i})$) and by subsequent analytic continuation. Thereby the behaviour of the integrand at infinity (convergence of the "rotated" integral) should be taken into account. Also, it is necessary to check the analyticity of the potentials on the rotated contours.

The relevant information is brought about by proving the existence of a suitable integration contour: [12,15]

Let $\tilde{\underline{s}}^{\tau}$ be the combination of loop momenta $\underline{s}^{1}, \ldots, \underline{s}^{L}$ in the τ th intermediate state. Then for every $\lambda > 0$ there exists a constant $C(\lambda, m_1, \ldots, m_N) < \infty$ and a one-parameter $\mathcal{C}^{\infty}$ class $\Gamma(\varphi)$, $0 \leq \varphi \leq \varphi_0$, of distorted contours in the space $\mathbb{C}^{3L}$ of complex loop momenta such that:

(i) $\Gamma(\varphi)$ projects one-to-one onto the original contour $\Gamma(0)$ under the projection which sends every complex loop momentum onto its real part ($\Gamma(\varphi)$ is semiflat)

(ii) if $E(\mathrm{Re}\,\underline{s}) < \lambda$, then $\mathrm{Im}\,\tilde{\underline{s}}^{r} = -\varphi . \mathrm{Re}\,\tilde{\underline{s}}^{r}$

(iii) on $\Gamma(\varphi) : |\mathrm{Im}\,\underline{s}^{r}| \leq C \cdot |\varphi|$ for r = 1,2,...,L.

Due to these properties it is not difficult to see that for any fixed λ ; there is a $\varphi_0 > 0$ such that the argument $\underline{k}$ of any potential remains in $\{|\mathrm{Im}\,\underline{k}| < \varsigma\}$ for $|\varphi| \leq \varphi_0$ and $(\underline{s}^1, \ldots, \underline{s}^L) \in \Gamma(\varphi)$.

Finally, for any $E > 0$, $\mathrm{Re}\, z \leq E$, $\mathrm{Im}\, z \cdot \mathrm{sign}\,\varphi > 0$ and sufficiently small φ_0, we have:

$$G(p, q; z) = \int_{\Gamma(0)} d^{3L}s\; F(p, q, \underline{s}; z) = \int_{\Gamma(\varphi)} d^{3L}s\; F(p, q, \underline{s}; z) \tag{4.16}$$

since by power counting the integrand vanishes at infinity for fixed z and external momenta and since the distortions $\Gamma(\varphi'),\ 0 \leq |\varphi'| \leq |\varphi|$ proceed over semi flat contours.

The analytic continuation of $G(p, q; z)$ for $\mathrm{Im}\, z \to 0$ outside (4.1) is provided by (4.16) and the result stated at the beginning of this section follows. In conclusion of this section we recall that the stated perturbative analyticity properties are obviously a first step towards the proof of a similar result for the exact N-body amplitudes.

If there are only finitely many channels (which would be the case if all subsystems of $M \leq N$ particles had no <u>positive</u> discrete spectrum), then the best one might expect is that the N-particle resolvents are as holomorphic as in perturbation theory (and of similar asymptotic behaviour), i.e. analytic in the physical region, outside the union of finitely many Landau varieties corresponding to physical rescattering and threshold singularities. This would then be the exact meaning of the concept of physical region maximal analyticity. It is possible to obtain maximal analyticity for analytic and purely repulsive potentials (i.e. when only the scattering channel is open) for the connected part of the N-particle scattering amplitude [15]. Then these regularity properties can be used to prove a corresponding version of asymptotic completeness. Thereby the validity of the FY-equations and of the Fredholm alternative are essential ingredients for the rigor of this proof.

Due to this rather unsatisfactory feature of the present state of affairs we shall not try to sketch the proof of the above results.

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