Proceedings of the Conference on

CLIFFORD ALGEBRA, ITS GENERALIZATION AND APPLICATIONS

(30th January -- 1st February 1971)

MATSCIENCE
The Institute of Mathematical Sciences
MADRAS 20
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CLIFFORD ALGEBRA, ITS GENERALIZATION
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Edited by
Alladi Ramakrishnan
Director, MATSCIENCE, Madras 20

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PREFACE

A CHARMED ENTRANCE

It is considered a worn-out cliche to emphasize that Pauli and Dirac matrices are the representations of a Clifford algebra of anticommuting elements. However, the exact mathematical transition from Pauli to Dirac matrices has never been studied in rigorous detail. Recently, by a series of fortuitous circumstances it was demonstrated at MATSCIENCE that the transition from Pauli to Dirac matrices revealed a charmed entrance into a fascinating structure called L-matrices with a definite relationship between the number of elements and the dimensions of the matrix representations of a Clifford algebra. Even more surprising was the possibility of entering the unexplored domain of $\omega$-commutation relations where $\omega$ is the general root of unity.

We later found that these extensions have claimed the attention of mathematicians some years ago but their contributions had not gained general currency since they were published in journals not easily accessible. Nono and Morinaga in Japan had arrived at the Generalized Clifford Algebra based on $\omega$-commutation relations

$$AB = \omega BA$$
and this was independently studied by Yamazaki followed ten years later by A. O. Morris. It was therefore found worthwhile to arrange a conference when the creators of the mathematical structure could meet those who had for the first time applied these ideas to elementary particle physics. They met at Ootacamund, that lovely hill resort of colourful South India, where sunsets swathe the verdant valleys and misty mountains with the hues of heaven and forest pines sway to the 'gentle wind whose breath can teach the wilds to love tranquillity'.

It is our earnest hope that at a time when the theoretical physicist of this Gell-Mannic era is satiate with the triumphs of unitary symmetry in the mood 

"unarm, my friends - the long days' work is done"

we are able to discern the dawn of a new algebra which may further enlighten elementary particle theory in the years to come.

Alladi Ramakrishnan
(Alladi Ramakrishnan)
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GENERALIZED CLIFFORD ALGEBRA AND LINEARIZATIONS OF

A PARTIAL DIFFERENTIAL EQUATION:

\[ q^{i_1 i_2 \cdots i_m} \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_m} \psi = c^m \psi \]

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****

About twenty years ago, the purpose of our work was to generalize the theory of spinors as a whole, and to find its applications for physics. As well known, the theory of spinors is established on the combined basic concepts: Clifford algebra with \((-1)\)-commutation rule, orthogonal group (or Lorentz group), linearization of quadratic form:

\[ (x_i \gamma_i)^2 = c_0 \sum_{i=1}^{n} (x_i)^2 \]

and linearization of quadratic wave equation:

\[ \Box \psi = (cm)^2 \psi \].

First of all, our task was to construct an associative algebra generalizing the ordinary Clifford algebra, and to investigate some properties related to the algebra, and the results obtained at that time were published in a paper \[ 1 \]. Thereafter, the theory of \( \omega \)-Clifford algebra with \( \omega \)-commutation rule, being a special one of our generalized Clifford algebra, were well cultivated independently by many authors: Professor K.Yamanaki \[ 2 \] and A.O.Morris \[ 3 \], and members of Madras school \[ 4, 6, 8 \]. It is a great pleasure and also surprise for me to see that this mathematical theory is now being applied beautifully to the theory of elementary particles by Madras school under the direction of Professor A.Ramakrishnan \[ 7, 8, 9, 10 \].

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In Part I, we shall state the outlines of our paper [11] and, in Part II, generalizing some results of Professor K. Morinaga on the linearization of wave equation $\Box \psi = m c \psi$ [11], we shall have some results on the linearizations of a partial differential equation: $g^{i_{1}i_{2}\ldots i_{m}} \nabla_{i_{1}} \nabla_{i_{2}} \ldots \nabla_{i_{m}} \psi = c^{m} \psi$ [12].

I. Generalized Clifford Algebra.

1. Generalized Clifford Algebra:

We shall define a generalized Clifford algebra (briefly G.C. algebra), by extending the concept of the ordinary Clifford algebra.

Let $G$ be the direct product of $n$ groups $G_{1}, G_{2}, \ldots, G_{n}$, and let a symbol $e_{A}$ correspond to each element $A \equiv (\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}), (\alpha_{i} \in G_{i})$ of $G$. We define the following linear associative algebra $\mathcal{V}$ with the basic elements $e_{A}$, $A \in G$ on a field of characteristic zero.

i) $\mathcal{V}$ is a left and right linear space on a field $K$.

ii) There is a mapping $\alpha \rightarrow \alpha^{A}$ in $K$, such that $e_{A} \cdot \alpha = \alpha^{A} \cdot e_{A}$ for any $e_{A}$, $A \in G$ and $\alpha \in K$.

iii) There is defined an associative multiplication in $\mathcal{V}$, such that, for any two basic elements $e_{A}$ and $e_{B}$:

$A \equiv (\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n})$ and $B \equiv (\beta_{1}, \beta_{2}, \ldots, \beta_{n})$.

$e_{A} e_{B} = \mathcal{V}(A, B), AB \equiv (\alpha_{1}\beta_{1}, \alpha_{2}\beta_{2}, \ldots, \alpha_{n}\beta_{n})$,
where 

\[ \Psi (A, B) = \prod_{k > i} \mathfrak{g} (\alpha_k, \beta_i) \cdot (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_i \beta_i, \ldots, \alpha_k \beta_i) \]

\[ \equiv \mathfrak{g} (\alpha_n, \beta_1) \cdot \mathfrak{g} (\alpha_{n-1}, \beta_1) \cdots \mathfrak{g} (\alpha_1, \beta_1) \]

\[ (\alpha_i \beta_1, \alpha_2 \beta_2, \ldots, \alpha_n \beta_n) (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_{n-1} \beta_{n-1}) \]

in terms \( \mathfrak{g} (\alpha_k, \beta_i) \in K, (k > i), \) and

\[ \mathfrak{g} (\alpha_k, \varepsilon_1) = \mathfrak{g} (\varepsilon_k, \beta_1) = 1 \ (k > i), \]

for the unit element \( \varepsilon_1 \) of \( G_1 \) and the unit element \( \varepsilon_1 \) of \( K \).

iv) For any basic elements \( e_A, e_B \) and any element \( \alpha \in K \)

\[ e_A \cdot (\alpha e_B) = (e_A \alpha) \cdot e_B. \]

Such a linear associative algebra \( \mathfrak{g} \) shall be called a generalized Clifford algebra (G.C. algebra), and its elements generalized Clifford numbers. And moreover we shall call \( G_1, \ldots, G_n \) the basic groups of \( \mathfrak{g} \), and \( \mathfrak{g} (\alpha_i, \beta_j) \) the structure numbers of \( \mathfrak{g} \).
\begin{equation}
(\alpha_1 \beta_1, \ldots, \alpha_{i-1} \beta_{i-1}, \alpha_i, \ldots, \alpha_{k-1})
\end{equation}

Here \( \varphi (\alpha_k, \beta_i) \) is lacking the upper indices \( \alpha_k, \ldots, \alpha_n, \beta_i, \ldots, \beta_n \) in (1.3), means \( \varphi^A(\alpha_k, \beta_i) \) for \( A = (\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_{i-1} \beta_{i-1}, \ldots, \alpha_{k-1}, \varepsilon_k, \ldots, \varepsilon_n) \).

We have a theorem derived directly from this definition.

**Theorem 1.** Let \( G_i \) and \( G_j \) \((i > j)\) be the two basic groups of a G.C. algebra \( \mathcal{O} \), and let \( \varphi (\alpha_i, \beta_j) \) be the structure numbers of \( \mathcal{O} \). Then we have the relations

\[
\begin{align*}
\varphi (\alpha_i, \beta_j, \beta_j) &= \varphi (\alpha_i, \beta_j) \varphi (\alpha_i, \beta_j) \text{ for any } \alpha_i \in G_i, \beta_j \in G_j, \\
\varphi (\alpha_i \alpha_i, \beta_j) &= \varphi (\alpha_i, \beta_j) \varphi (\alpha_i, \beta_j) \text{ for any } \alpha_i \in G_i, \beta_j \in G_j.
\end{align*}
\]

The set \( N_j (\alpha_i) = \{ \beta_j : \beta_j \in G_j, \varphi (\alpha_i, \beta_j) = 1 \} \) is a subgroup of \( G_j \). The necessary and sufficient condition for \( N_j (\alpha_i) \) to be a normal subgroup of \( G_j \) is that \( \varphi (\beta_j)(\alpha_i, \theta_j) = \varphi (\alpha_i, \theta_j) \) for any \( \theta_j \in N_j (\alpha_i), \theta_j \in G_j \). The necessary and sufficient condition for \( N_j (\alpha_i) \) to contain the commutator group \( Q_j \) of \( G_j \) is that \( \varphi (\alpha_i, \beta_j, \beta_j) = \varphi (\alpha_i, \beta_j, \beta_j) \) for any \( \beta_j, \beta_j \in G_j \).

As for \( N_i (\beta_j) = \{ \alpha_i, \alpha_i \in G_i, \varphi (\alpha_i, \beta_j) = 1 \} \), the similar results are obtained.

Here \( \varphi (\beta_j)(\alpha_i, \beta_j) \) means \( \varphi^B (\alpha_i, \beta_j) \) for \( B = (\varepsilon_1, \ldots, \varepsilon_{j-1}, \beta_j, \varepsilon_{j+1}, \ldots, \varepsilon_n) \).

2. G.C. algebra reducible to G.C. algebra each whose group is commutative.

Let \( \mathcal{O}' \) be a G.C. algebra which has the basic groups \( G'_i \) \((i = 1, 2, \ldots, n)\) and has the structure numbers \( \varphi' (\alpha'_i, \beta'_j) \).
Then we shall define that the G.C. algebra \( \mathcal{D} \) is homomorphic to \( \mathcal{D}' \), when the basic groups \( G_i \) are homomorphic onto \( G'_i \) respectively, and in the homomorphisms: \( \alpha_i \rightarrow \alpha'_i \) it holds
\[ f(\alpha_i, \beta_j) = f'(\alpha'_i, \beta'_j). \]
By the correspondence from the basic element \( e_A, A \in G = G_1 \times G_2 \times \cdots \times G_n \) to \( e'_{A'} \),
\[ A' \in G' = G'_1 \times G'_2 \times \cdots \times G'_n, \]
the G.C. algebra \( \mathcal{D}' \) is homomorphic to \( \mathcal{D} \) in the ordinary sense. In this terminology we have

**Theorem 2.** Let G.C. algebra \( \mathcal{D} \) be defined by the basic groups \( G_i \) and the structure numbers \( f(\alpha_i, \beta_j) \), 
\( (i > j, i, j = 1, 2, \ldots, n) \). Then the necessary and sufficient condition in order that the G.C. algebra \( \mathcal{D} \) is homomorphic to a G.C. algebra \( \mathcal{D}' \) each of whose basic groups is commutative is that
\[ f(\alpha_i, \beta_j) = f(\alpha_i, \beta_j) \text{ for any } \alpha_i \in G_i, \beta_j \in G_j, \]
\[ f(\alpha_i \beta_j, \beta_j) = f(\alpha_i \beta_j, \beta_j) \text{ for any } \alpha_i, \beta_j \in G_i, \beta_j \in G_j. \]

3. G.C. algebra whose basic elements are commutative with its structure numbers

Now we shall consider the G.C. algebra whose basic elements \( e_A, A \in G \) are commutative with its structure numbers \( f(\alpha_i, \beta_j) \), \( (i > j, i, j = 1, 2, \ldots, n) \). In this case we have
Theorem 3. Let $E_A$ be a symbol corresponding to an element $A \in G = G_1 \times G_2 \times \cdots \times G_n$, and let

$\Lambda \equiv \{ \varphi(\alpha_i, \beta_j) : \alpha_i \in G_i, \beta_j \in G_j, i > j, i, j = 1, 2, \ldots, n \}$

be a set of numbers which are commutative with $E_A$, $A \in G$. Then the necessary and sufficient condition that there is a G.C. algebra having $G_i$ as the basic groups and $\Lambda$ as the set of the structure numbers is that

1. $\Lambda$ is a commutative set,

and

2. $\varphi(\alpha_i, \beta_j) \varphi(\beta_j) = \varphi(\alpha_i, \beta_j) \varphi(\beta_j)$ for any $\alpha_i \in G_i, \beta_j \in G_j$,

$\varphi(\alpha_i \beta_i, \beta_j) = \varphi(\alpha_i, \beta_j) \varphi(\beta_i, \beta_j)$ for any $\alpha_i \beta_i \in G_i, \beta_j \in G_j$.

As we shall see from this theorem, in the G.C. algebra whose basic elements are commutative with its structure numbers, it is evident that

$\varphi(\alpha_i, \beta_j) \varphi(\beta_j) = \varphi(\alpha_i, \beta_j)$ for any $\alpha_i \in G_i, \beta_j \in G_j$,

$\varphi(\alpha_i \beta_i, \beta_j) = \varphi(\alpha_i \beta_i, \beta_j)$ for any $\alpha_i \beta_i \in G_i, \beta_j \in G_j$.

So by Theorem 2, such a G.C. algebra $\mathcal{G}$ is homomorphic to the G.C. algebra $\mathcal{G}'$ each of whose basic groups is commutative.

In particular, let us suppose that the basic groups $G_1, G_2, \ldots, G_n$ are the cyclic groups of order $m_1, m_2, \ldots, m_n$ respectively, i.e. $G_i \subseteq [\xi_i, \alpha_i, \ldots, \alpha_i^{m_i - 1}]$, $\alpha_i = \xi_i$. By Theorem 3, the set $\Lambda \equiv \{ \varphi(\alpha_i^e, \alpha_j^f) : \alpha_i \in G_i \}$, $\lambda_i = 0, 1, \ldots, m_i - 1$, $i, j = 1, 2, \ldots, n$ of the structure numbers of the G.C. algebra with the basic groups $G_1, \ldots, G_n$ is
commutative and is determined by the conditions:

\[ \varphi (\alpha_i^i, \alpha_j^{\lambda_j^+ \mu_j}) = \varphi (\alpha_i^i, \alpha_j^\lambda) \varphi (\alpha_i^i, \alpha_j^\mu), \]
\[ \varphi (\alpha_i^{\lambda_i^j + \mu_i}, \alpha_j^\lambda) = \varphi (\alpha_i^{\lambda_i^j}, \alpha_j^\lambda) \varphi (\alpha_i^{\mu_i}, \alpha_j^\lambda), \quad (i > j) \]

from which we have

\[ \varphi (\alpha_i^{\lambda_i^j}, \alpha_j^{\mu_j}) = (\varphi (\alpha_i^{\lambda_i^j}, \alpha_j^{\mu_j}))^{\lambda_j^j}, \]
\[ (\mu_j = 0, 1, \ldots, m_j - 1) \]

since \( \alpha_i^m = \varepsilon_i \), we have

\[ \varphi (\alpha_i^i, \alpha_j^m) = \varphi (\alpha_i^i, \alpha_j^m) = \varphi (\varepsilon_i, \alpha_j^m) = 1, \]

and

\[ \varphi (\alpha_i^i, \alpha_j^m) = \varphi (\alpha_i^m, \alpha_j^m) = \varphi (\alpha_i^m, \alpha_j^m) = 1, \]

and therefore \( \varphi (\alpha_i^i, \alpha_j^m) \) must be a primitive \( l_{ij} \)-th root \( \omega_{ij} \) of unity, where \( l_{ij} \) is a factor of \( m_i \) and \( m_j \).

In this case we have

**Theorem 4.** For the G.C. algebra with the basic groups

\[ G_i = [\varepsilon_i, \alpha_i^1, \ldots, \alpha_i^{m_i-1}], \quad \alpha_i^{m_i} = \varepsilon_i, \]

the structure numbers \( \varphi (\alpha_i^{\lambda_i^j}, \alpha_j^{\mu_j}) \) are equal to \( \omega_{ij}^{\lambda_{ij} \mu_{ij}} \) where \( \omega_{ij} \) is a primitive \( l_{ij} \)-th root of unity and \( l_{ij} \) is a factor of \( m_i \) and \( m_j \).

4. **Linearization of** \( \sum_{i=1}^{n} (\alpha_i^i)^m \)

Now, in particular, we shall consider the G.C. algebra

whose basic groups, are the same cyclic group of order \( m \):

\[ G_i = [\varepsilon, \alpha, \alpha^2, \ldots, \alpha^{n-1}], \quad \alpha^m = \varepsilon, \]

and whose basic elements are commutative with its structure numbers. Suppose that the field \( K \) is commutative and contains a primitive \( m \)-th root
of unity. If we take $m = 2$ and $K$ as the field of real
or complex numbers, then we obtain the theory of ordinary
spinors. As a special case of Theorem 4 we have

Theorem 4. For the G.C. algebra with the basic groups
$G_i \equiv [\varepsilon, \alpha, \alpha^2, \ldots, \alpha^{m-1}]$, $\alpha^m = \varepsilon$, the structure
numbers $\vartheta(\alpha^\lambda, \alpha^\mu)$ are equal to $\omega^{\lambda \mu}$ where $\omega$ is
any $m$-th root of unity in $K$.

So if we write $e_i^\lambda = e_{(i)}^{(e, \ldots, e, \alpha^\lambda, e, \ldots, e)}$, then we
have

$$e_i^\lambda e_i^\mu = \omega^{\lambda \mu} e_j e_i^{\lambda \mu}, \quad (i > j),$$

in particular,

$$e_i^\lambda e_j = \omega^{\lambda \mu} e_j e_i^{\lambda \mu}, \quad (i > j).$$

We can prove the following theorem.

Theorem 5. Let $e_i^\lambda$ be the basic element $e_{(e, \ldots, e, \alpha^\lambda, e, \ldots, e)}$
of the G.C. algebra whose basic groups are the same cyclic
group of order $m$ and whose structure numbers are $\vartheta(\alpha^\lambda, \alpha^\mu) = \omega^{\lambda \mu}$
($\omega$ is a primitive $m$-th root of unity in $K$). Then we have
an identity

$$\left( \sum_{i=1}^{n} x_i e_i^\lambda \right)^\ell = e_0 \sum_{i=1}^{n} (x_i)^\ell, \quad (e_0 \equiv e_{(e, \ldots, e)},$$

for $\ell = m/(\lambda, m)$, if and only if $(m/(\lambda, m), (\lambda, m)) = 1$. 
5. Structure of the G.C. algebra associated with the linearization of \( \sum_{i=1}^{n} (x^i)^m \)

Here we write \( P_o \) and \( P_i \) for \( C_0 \) and \( C_i \) in 4, respectively. So we have

\[
P_i P_j = \omega P_j P_i , \quad (i > j) ,
\]

where \( \omega \) is a primitive \( m \)-th root of unity. The G.C. algebra generated by \( P_o \) and \( P_i \) \((i = 1, 2, \ldots, n)\) will be called the G.C. algebra associated with the linearization:

\[
(\sum_{i=1}^{n} x^i P_1)^m = P_o \sum_{i=1}^{n} (x^i)^m .
\]

This G.C. algebra might be called \( \omega \)-Clifford algebra, (G.C. algebra), emphasising \( \omega \)-comutation rule.

Now we shall investigate the center and ideal of W.C. algebra, in each case of the followings:

- **Case (I)**: \( n \) is even,
- **Case (II)\_1**: \( n \) is odd and \( m \) is also odd,
- **Case (II)\_2**: \( n \) is odd, \( n \equiv 1 \pmod{4} \) and \( m \) is even,
- **Case (II)\_3**: \( n \) is odd, \( n \equiv 3 \pmod{4} \), \( m \) is even, and there exists an element \( k \) in \( K \) such that \( \omega = k^2 \),
- **Case (II)\_4**: \( n \) is odd, \( n \equiv 3 \pmod{4} \), \( m \) is even \( m = 2m_0 \), there does not exist an element \( k \) in \( K \) such that \( \omega = k^2 \).

We have the following results:

**Theorem 6.** If \( n \) is even (i.e. Case (I)), then the center \( \mathcal{Z} \) of \( \mathcal{V} \) is \( \{ P_0 \} \) and the ideals of \( \mathcal{V} \) are only \( \{ 0 \} \) and the whole algebra \( \mathcal{V} \). If \( n \) is odd, then the center \( \mathcal{Z} \) of \( \mathcal{V} \) is \( \{ P, \frac{1}{2}, \frac{3}{2}, \ldots, P^{m-1} \} \), where

\[
P = P_1 P_2 P_3 \ldots P_{n-1} P_n .
\]
And in the cases (II₁), (II₂) and (II₃), \( \mathcal{V} \) is decomposed into the direct sum \( \sum_{\lambda=0}^{m-1} \mathcal{V}^\lambda \) of the simple ideals of \( \mathcal{V} \)

where

\[
\mathcal{V}^\lambda = \frac{1}{m} \sum_{\mu=0}^{m-1} \omega^{\lambda \mu} \cdot \mathcal{P}^{\frac{1}{4}(\mu-1)(\mu-1)} \cdot \mathcal{P}
\]

for the cases (II₁) and (II₂),

\[
\mathcal{V}^\lambda = \frac{1}{m} \sum_{\mu=0}^{m-1} \omega^{\lambda \mu} \cdot \mathcal{P}^{\frac{1}{4}(\mu-1)^2} \cdot \mathcal{P}
\]

for the case (II₃);

moreover, as for any ideal \( \mathcal{V} \) of \( \mathcal{V}^n \), \( \mathcal{V} \cap \mathcal{V} \) is an ideal \( \sum \Theta \{ \mathcal{V}^\lambda \} \) of \( \mathcal{V} \) and \( \mathcal{V} = \sum \Theta \mathcal{V}^\lambda \). Similarly, in the case (II₄) we have \( \mathcal{V} = \sum_{\alpha=0}^{m^2} \mathcal{V}^\alpha \) where \( \mathcal{V}^\alpha = \mathcal{V}^\alpha \cup \mathcal{V}^\alpha^+ \)

\[
\mathcal{V} \cap \mathcal{V} = \sum \Theta \mathcal{V}^\alpha \text{ and } \mathcal{V} = \sum \Theta \mathcal{V}^\alpha \text{ where } \mathcal{V}^\alpha = \{ u, u^+ \},
\]

\[
\mathcal{V}^\alpha \cup \mathcal{V}^\alpha^+ = \frac{1}{m} \sum_{\mu=0}^{m-2} \omega^{\alpha \mu} \cdot \mathcal{P}^{\frac{1}{4}(\mu-1)^2} \cdot \mathcal{P},
\]

and

\[
\mathcal{V}^\alpha \mathcal{V}^\alpha^+ = \frac{1}{m} \sum_{\nu=0}^{m-1} \omega^{\alpha \nu} \cdot \mathcal{P}^{\frac{1}{4}(\nu-1)^2} \cdot \mathcal{P}.
\]

Moreover, \( \mathcal{V}^\alpha \) (\( \lambda = 0, 1, \ldots, m-1 \)) are mutually isomorphic and have \( m^{m-1} \) linearly independent elements respectively, and \( \mathcal{V}^\alpha \) (\( \alpha = 0, 1, \ldots, m-1 \)) are mutually isomorphic and have \( 2m^{m-1} \) linearly independent elements respectively.

Here \{ \mathcal{V}_{i_1}, \mathcal{V}_{i_2}, \ldots, \mathcal{V}_{i_s} \} denotes the linear space over \( K \) spanned by \( \mathcal{V}_{i_1}, \mathcal{V}_{i_2}, \ldots, \mathcal{V}_{i_s} \).
6. **Matrix representation of the G.C. algebra associated with the linearization of** \( \sum_{i=1}^{n} (z^i)^m \).

We can investigate the representation of the G.C. algebra by means of the structure of \( \mathcal{O} \)-discussed in 5, and the general theory of the representations of algebra. But, in this section we shall consider directly the actual representation of the G.C. algebra \( \mathcal{O} \). We must suppose that \( K \) contains a primitive \( m \)-th root \( \omega \) of unity, and the square root \( \omega^{1/2} \) of \( \omega \).

We have the following results.

**Theorem 7.** Let \( \omega \) be a primitive \( m \)-th root of unity if \( K \) contains \( \omega^{1/2} \) then the general system of matrices

\[ P_i \quad (i=1, 2, \ldots, n) \]

such that

\[ P_i P_j = \omega P_j P_i \quad (i \neq j) \quad \text{and} \quad P_i^m = E \]

are written as follows:

(I) If \( n \) is even: \( n = 2\gamma \), then

\[
\begin{align*}
P_{2\gamma-1} &= \mathcal{W}^{-1} \left[ \sum_{\gamma} \Omega_3 \times \cdots \times \Omega_3 \times \Omega_4 \times E_m \times \cdots \times E_m \times E_L \right] \mathcal{W}, \\
P_{2\gamma} &= \mathcal{W}^{-1} \left[ \sum_{\gamma} \Omega_3 \times \cdots \times \Omega_3 \times \Omega_2 \times E_m \times \cdots \times E_m \times E_L \right] \mathcal{W}
\end{align*}
\]

\((\lambda = 1, 2, \ldots, \gamma)\)

where \( \mathcal{W} \) is an arbitrary regular matrix of order \( \gamma \times \gamma \), \( \gamma = 1, 2, \ldots \).
(II) If \( n \) is odd, \( n = 2\tau + 1 \), then
\[
\mathcal{P}_{2\tau-1} = W^{-1} \left[ \Omega_3 \times \cdots \times \Omega_3 \times \Omega_1 \times E_m \times \cdots \times E_m \times E_\tau \right] W,
\]
\[
\mathcal{P}_{2\tau} = W^{-1} \left[ \Omega_3 \times \cdots \times \Omega_3 \times \Omega_2 \times E_m \times \cdots \times E_m \times E_\tau \right] W,
\]
\[
\mathcal{P}_n = \mathcal{P}_{2\tau+1} = W^{-1} \left[ \Omega_3 \times \cdots \times \Omega_3 \times R \right] W,
\]
\((\tau = 1, 2, \ldots, \tau),\)

where \( R \) is any matrix of order \( \tau \) such that \( R^m = E_\tau \), we may take
\[
R = \begin{bmatrix}
\omega^{\alpha_1} & 0 \\
\omega^{\alpha_2} & 0 \\
\vdots & \vdots \\
0 & \omega^{\alpha_\tau}
\end{bmatrix}, \quad (\alpha_1, \ldots, \alpha_\tau \text{ any integers}, \quad \tau = 1, 2, \ldots)
\]
and \( W \) is an arbitrary regular matrix of order \( m \times \ell \). Here
\[
\Omega_1 = \begin{bmatrix}
1 & 0 \\
\omega & 1 \\
0 & \omega & \ddots & \ddots \\
0 & 0 & \ddots & \ddots & 0
\end{bmatrix}, \quad \Omega_2 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]
and \( \Omega_3 = \omega^{\frac{m-1}{2}} \Omega_1^{-1} \Omega_2 \).

Now if we make correspond the matrix \( \mathcal{P}_1^\lambda \mathcal{P}_2^\lambda \cdots \mathcal{P}_n^\lambda \), \( \mathcal{P}_1 \) being the matrices in Theorem 7, to the basic element
\( \mathcal{P}_1^\lambda \mathcal{P}_2^\lambda \cdots \mathcal{P}_n^\lambda \) of the G.C. algebra \( \mathcal{A} \), then we obtain the general representation of \( \mathcal{A}' \). Then we have the following theorem.
Theorem 6. The general representation of the G.C. algebra \( \mathcal{V} \)
associated with the linearization of \( \sum_{i=1}^{n} (x_i)^m \) is generated
by the system of matrices \( P_i \ (i=1,2,\ldots,n) \) in Theorem 7.
If \( n \) is even: \( n=2r \), then the faithful representation
of minimal order yields the complete matric algebra in \( m^r \)
dimensions, and therefore the representation is irreducible.
If \( n \) is odd: \( n=2r+1 \), the faithful representation of
minimal order is the direct sum of \( m \) complete matric algebras
in \( m^r \) dimensions, and therefore it is completely reducible.
The above results were obtained independently by
Professor A.O. Morris [3]. By Theorem 7, there exist only
three matrices \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) in the lowest dimension
\( m \), satisfying the two generalized Clifford conditions, from
which the general system of matrices \( P_i \ (i=1,2,\ldots,n) \)
can be constructed. The method of construction seems to be
essentially the same as the \( \mathcal{G} \)-operation by Professor
Ramakrishnan [4].

7. Linear transformation leaving invariant \( \sum_{i=1}^{n} (x_i)^m \), \( (m>2) \).

In this last section, we shall determine the linear
transformation

\[ x_i = \sum_{j=1}^{n} k_{ij} x_j \]

leaving invariant \( \sum_{i=1}^{n} (x_i)^m \), \( (m>2) \). The result obtained
is the following:
Theorem 2. The set of all linear transformations leaving invariant \( \sum_{i=1}^{n} (\alpha^i)^m \), \((m > 2)\) is a finite group of order \( m^n \cdot n! \). These linear transformations are written as follows:

\[
\alpha^i = \omega^{i \cdot j(i)}
\]

where \( j(i), i = 1, 2, \ldots, n \) is any permutation of \( 1, 2, \ldots, n \), \( \omega \) is a primitive \( m \)-th root of unity, and \( \lambda_1 \) is any integer mod. \( m \).

II. **Linearization of a partial differential equation**:

\[
g^{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi.
\]

1. **Preliminaries**

The two following theorems were proved by Professor K. Morinaga [11].

**Theorem I.** A partial differential equation

\[
\alpha^i \nabla_{i_1} \psi = mc \psi
\]

is a linearization of the wave equation

\[
\Box \psi = (mc)^2 \psi,
\]

if and only if

\[
\alpha^{(i j)} = g^{i j} E_N,
\]

i.e., \( \alpha^i \) are matrices of Dirac type.

**Theorem II.** A partial differential equation

\[
\alpha^i \nabla_{i_1} \psi = mc \psi, \quad (mc \neq 0)
\]
is a linearization of the wave equation
\[ \Box \psi = (mc)^2 \psi \]

if and only if there exists an integer \( P (\geq 0) \) such that
\[ \alpha_{i_1 i_2 \ldots i_{p+2}} = \hat{g}_{i_1} \alpha_{i_3 \ldots i_{p+2}}. \]

The condition for \( p = 1 \) becomes to:
\[ \alpha_{i_1 i_2 i_3} = \hat{g}_{i_4} \alpha_{i_2 i_3}, \]

which is satisfied by each of the following conditions:
\[ \alpha_{i_1 i_2} = \hat{g}_{i_4 i_2}, \]

and
\[ \alpha_{i_1 i_2 i_3} = \hat{g}_{i_4 i_2 i_3}. \]

The latter is the Duffin-Kemmer relation.

We shall generalize these results for a partial differential equation:
\[ \hat{g}_{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi. \]

Let \( \alpha^i \) be \( n \) variables, \( \nabla_i = \frac{\partial}{\partial \alpha_i} \) \( (i=1,2,\ldots,n) \), \( \alpha^i \) \( N \times N \) matrices, and \( \psi \) an \( N \)-vector, then a partial differential equation:
\[ \alpha^i \nabla_i \psi = c \psi \quad (1) \]

is called a linearization of a given partial differential equation:
\[ \hat{g}_{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi, \quad (2) \]
if and only if any solution of (1) is always a solution of (2).

Here \( g_{i_1i_2...i_m} \) is symmetric with respect to \( i_1, i_2, ..., i_m \).

If we take

\[
g_{i_1i_2...i_m} = \delta_{i_1i_2...i_m} = \begin{cases} 1 & \text{for } i_1 = i_2 = ... = i_m, \\ 0 & \text{otherwise}, \end{cases}
\]

then we have

\[
g_{i_1i_2...i_m} \nabla_{i_1} \nabla_{i_2} ... \nabla_{i_m} \psi = \sum_{i=1}^{n} \frac{\partial^m}{\partial x_{i_m}} \psi.
\]

Let \( \nu = \alpha^i \nu_i \)

then we define \( \| \nu \| \) as follows:

\[
\| \nu \| = ( g_{i_1i_2...i_m} \nu_{i_1} \nu_{i_2} ... \nu_{i_m} )^{1/m},
\]

since

\[
\nu^m = (\alpha^i \nu_i)^m = \alpha^{i_1} \alpha^{i_2} ... \alpha^{i_m} \nu_{i_1} \nu_{i_2} ... \nu_{i_m}
\]

\[
= \alpha^{i_1} \alpha^{i_2} ... \alpha^{i_m} \nu_{(i_1} \nu_{i_2} ... \nu_{i_m)}
\]

\[
= \alpha^{(i_1 \alpha^{i_2} ... \alpha^{i_m})} \nu_{i_1} \nu_{i_2} ... \nu_{i_m},
\]

We see that

\[
\nu^m = \| \nu \|^m E_N,
\]

if and only if

\[
\alpha^{(i_1 \alpha^{i_2} ... \alpha^{i_m})} = g_{i_1i_2...i_m} E_N.
\]
Here \((\ )\) means the symmetric part, \(E_N\) is the unit matrix of order \(N\). From (3) it follows:
\[
(\det \nu)^m = \|\nu\|^{mN},
\]
so that \(\det \nu = 0\) is equivalent to \(\|\nu\| = 0\).

Remark. If we assume that, for a primitive \(m\)-th root \(\omega\) of unity,
\[
\alpha^i \alpha^j = \omega \alpha^j \alpha^i, \quad (i > j)
\]
then we have
\[
(\alpha^i)^m = E_N,
\]
then we have
\[
(\alpha_1 \alpha_2 \ldots \alpha_m)^i = \delta_{i1} \delta_{i2} \ldots \delta_{im} E_N.
\]
similarly we see that
\[
(\nu^m + \nu^p) = \alpha^{i_1} \alpha^{i_2} \ldots \alpha^{i_m} \alpha^{j_1} \ldots \alpha^{j_p} \nu_{i_1} \nu_{i_2} \ldots \nu_{i_m} \nu_{j_1} \ldots \nu_{j_p},
\]
and
\[
\|\nu\|^m \nu^p = (\delta_{i_1} \delta_{i_2} \ldots \delta_{i_m} \nu_{1_1} \nu_{1_2} \ldots \nu_{1_m}) (\alpha^{i_1} \alpha^{i_2} \ldots \alpha^{i_p} \nu_{i_1} \nu_{i_2} \ldots \nu_{i_m} \nu_{j_1} \ldots \nu_{j_p})
\]
\[
= \delta_{i_1} \delta_{i_2} \ldots \delta_{i_m} \alpha^{j_1} \ldots \alpha^{j_p} \nu_{i_1} \nu_{i_2} \ldots \nu_{i_m} \nu_{j_1} \ldots \nu_{j_p}
\]
\[
= \delta_{i_1} \delta_{i_2} \ldots \delta_{i_m} \alpha^{j_1} \ldots \alpha^{j_p} \nu_{i_1} \nu_{i_2} \ldots \nu_{i_m} \nu_{j_1} \ldots \nu_{j_p}.
\]
so we have that
\[ \nu^{m+p} = \| \nu \|^m \nu^p , \] (4)

if and only if
\[ \alpha^{(i_1 \alpha i_2 \ldots \alpha^i \ldots \alpha^{j_1} \ldots \alpha^{j_p})} = g^{(i_1 \alpha i_2 \ldots \alpha^{j_1} \alpha^{i_2} \ldots \alpha^{j_p})} . \] (5)

If there exists an integer \( p \geq 0 \) such that
\[ \nu^{m+p} = \| \nu \|^m \nu^p , \]
then a partial differential equation:
\[ \alpha^i \nabla_i \psi = c \psi \quad (c \neq 0) \] (1)
is a linearization of the partial differential equation:
\[ g^{i_1 \alpha i_2 \ldots \alpha^{i_m}} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi . \] (2)

In fact, operating \( (\alpha^i \nabla_i)^{P+m-1} \) on the both sides of (1), we have
\[ \alpha^{i_1} \alpha^{i_2} \ldots \alpha^{i_{P+m}} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_{m}} \psi = c (\alpha^i \nabla_i)^{P+m-1} \psi \] (6)
since clearly it holds
\[ \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_{P+m}} = \nabla_{(i_1} \nabla_{i_2} \ldots \nabla_{i_{P+m})} , \]
from (6) it follows
\[ \alpha^{(i_1 \alpha i_2 \ldots \alpha^{i_{P+m}})} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_{P+m}} \psi = c^{P+m} \psi . \]
By using (5) we have
\[ q^{(i_1 i_2 \ldots i_m, i_{m+1} \ldots i_{m+p})} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \nabla_{i_{m+1}} \ldots \nabla_{i_{m+p}} \psi = c^{m+p} \psi, \]
i.e.,
\[ q^{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} (\alpha^i \nabla_i)^p \psi = c^{m+p} \psi. \]

By using (1) again we have
\[ c^p q^{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^{m+p} \psi, \]
so that, since \( c \neq 0 \), it holds
\[ q^{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi. \]

Thus, (1) is a linearization of (2).

2. **Main Theorems**

**Theorem 1.** A partial differential equation
\[ \alpha^i \nabla_i \psi = c \psi \] (1)
is a linearization of the partial differential equation:
\[ q^{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi, \] (2)
if and only if
\[ v^m = (\alpha^i \nabla_i)^m = \|v\|^m E_N \text{ for any } v_i, \]
i.e.,
\[ \alpha^{(i_1 i_2 \ldots i_m)} = q^{i_1 i_2 \ldots i_m} E_N. \] (3)
Proof: Let \( \mathbf{v}_1 \) be an arbitrary vector such that 
\[
\mathbf{v}_1 \in \mathbb{C}^{i_1 i_2 \ldots i_m}, \quad \mathbf{v}_{i_1} \circ \mathbf{v}_{i_2} \circ \ldots \circ \mathbf{v}_{i_m} = \mathbf{i},
\]
and then we transform \( \alpha^i \mathbf{v}_1 \) into the Jordan canonical form:
\[
U (\alpha^i \mathbf{v}_1) U^{-1} = \sum \bigoplus \begin{bmatrix} \lambda & \mu & \cdots & 0 \\ \lambda & \cdots & \cdots & \mu \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}, 
\]
\((\mu = 0 \text{ or } 1)\).

Under the assumption that (1) is a linearization of (2), we can prove that
\[
U (\alpha^i \mathbf{v}_1) U^{-1} = \begin{bmatrix} \omega_{n_1} & 0 \\ \omega_{n_2} & \ddots \\ 0 & \cdots & \omega_{n_N} \end{bmatrix},
\]
where \( \omega \) is an \( m \)-th root of unity. Therefore we see that
\[
(U (\alpha^i \mathbf{v}_1) U^{-1})^m = E_N,
\]
i.e.,
\[
(\alpha^i \mathbf{v}_1)^m = E_N.
\]

For an arbitrary \( \mathbf{v}_1 \), if we take
\[
\mathbf{v}_1 \overset{\circ}{=} \frac{\mathbf{v}_1}{\|\mathbf{v}\|},
\]
then clearly \( \|\mathbf{v}_1\| = 1 \), so that we have
\[
(\alpha^i \frac{\mathbf{v}_i}{\|\mathbf{v}\|})^m = E_N,
\]
which is equivalent to:

\[ \nu^m = (\alpha^i \nu_1)^m = \|\nu\|^m E_N, \]

i.e.,

\[ \alpha(i_1 \alpha i_2 \ldots \alpha i_m) = g_{i_1i_2 \ldots i_m} E_N. \quad (3) \]

Conversely, if \( \alpha^i \) satisfy condition (3), then by operating \( (\alpha^i \nabla_1)^m \) on both sides of (1) we have

\[ \alpha(i_1 \alpha i_2 \ldots \alpha i_m) \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c (\alpha^i \nabla_1)^m \psi \]

i.e.

\[ \alpha(i_1 \alpha i_2 \ldots \alpha i_m) \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi, \]

by means of (3) we have

\[ g_{i_1i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi. \]

Therefore, (1) is a linearization of (2).

**Theorem 2.** A partial differential equation

\[ \alpha^i \nabla_i \psi = c \psi \quad (\psi \neq 0) \quad (1') \]

is a linearization of the partial differential equation:

\[ g_{i_1i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi, \quad (2') \]

if and only if there exists an integer \( P (\geq 0) \) such that

\[ \nu^{m+P} = \|\nu\|^m \nu^P, \]

i.e.,

\[ \alpha(i_1 \alpha i_2 \ldots \alpha i_m \alpha j_1 \ldots \alpha j_P) = g_{i_1i_2 \ldots i_m j_1 \ldots j_P}. \quad (4) \]
Proof: Let $\mathbf{v}_1$ be an arbitrary vector such that
$$g_{i_1i_2\ldots i_m} \mathbf{v}_{i_1} \mathbf{v}_{i_2} \cdots \mathbf{v}_{i_m} = 1,$$
and then we transform $\alpha^i \mathbf{v}_1$ into the Jordan canonical form:
$$X = U(\alpha^i \mathbf{v}_1)U^{-1} = \sum \bigoplus \begin{bmatrix} \lambda & 0 \\ \lambda & \ddots \\ & & \mu \\ 0 & & & \lambda \end{bmatrix}, \quad (\mu = 0 \text{ or } 1).$$

Under the assumption that (1') is a linearization of (2''), we can prove that
$$X = U(\alpha^i \mathbf{v}_1)U^{-1} = \begin{bmatrix} \omega^{n_1} & 0 \\ \omega^{n_2} & \ddots \\ & & \omega^n \\ 0 & \cdots & \omega^n \end{bmatrix} \oplus \sum \bigoplus \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots \\ & & \ddots & \ddots \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Let $P_0$ be the maximal value of orders of blocks belonging to zero roots, then we have
$$(X^m - E_N)X^{P_0} = 0_N,$$
so that
$$(\alpha^i \mathbf{v}_1)^m - E_N(\alpha^i \mathbf{v}_1)^{P_0} = 0_N.$$

Let $p$ be the maximal value of $P_0$ for changing $\mathbf{v}_1$ under the restriction $\|\mathbf{v}\|^m = 1$, then clearly we have
$$(\alpha^i \mathbf{v}_1)^m - E_N(\alpha^i \mathbf{v}_1)^{P} = 0_N.$$

For any $\mathbf{v}_1$, if we take $\mathbf{v}_1 = \frac{\mathbf{v}_i}{\|\mathbf{v}\|^m}$, then we know $\|\mathbf{v}_i\|^m = 1$, and hence it holds
$$(\alpha^i \mathbf{v}_1)^m - \|\mathbf{v}\|^m E_N(\alpha^i \mathbf{v}_1)^P = 0_N.$$
that is,

\[ u^m p = \|u\|^m u^p. \]

The converse was already proved in 1.

References


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Report of Research Committee of the Japan Science Council, 1941.
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\[
g^{i_1 i_2 \ldots i_m} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_m} \psi = c^m \psi
\]
(to be published).
STRUCTURE OF ALGEBRA EXTENSION OF FINITE ABELIAN GROUP

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Introduction

During the preparations of [2] and [3], the author has not been aware of [1]. Now it should be referred and its logical connection with [2], [3] should be clarified. The concept of the 'generalized Clifford algebra' given in [1] may be regarded as a special kind of 'ring extension' given in [2] when the basic field is commutative (Cf. 2).

An algebra extension is a ring extension whose elements commute with the elements of the basic field. As a special case, the structure and representation of 'usual' generalized Clifford algebra which is written $C_n^m$ were studied in [1] originally. [2] and [3] contain some clarifications of the structure of more general algebra extensions.

All concepts and theories given in this lecture will be taken only from [2], [3], except some illustrations. For the reader's convenience, some prerequisites will be added.

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1. **Prerequisites**

To avoid possible confusion, we shall recall some elementary concepts from the standard algebra.

By a *monoid*, we mean a set $M$ with an associative binary operation

$$M \times M \rightarrow M \quad (x, y) \rightarrow xy$$

such that there exists the neutral element $1$. A group is a monoid whose elements $x$ have their inverses $x^{-1}$.

**Note.** If the binary operation is written additively: $(x, y) \rightarrow x + y$, the neutral element is written $0$ and the inverse of $x$ is written $-x$.

Let $M, M'$ be monoids. A *(monoid) morphism* from $M$ to $M'$ is a mapping $h : M \rightarrow M'$ satisfying the conditions

$$h(xy) = h(x)h(y) \quad (x, y \in M),$$

$$h(1_M) = 1_{M'},$$

where $1_M$ and $1_{M'}$ mean the neutral elements of $M$ and $M'$ respectively.

**Note.** If $M'$ is a group, the first condition implies the second one.

Let $M_1, M_2$ be monoids. A *pairing* (= bimorphism) from $M_1, M_2$ to $M'$ is a mapping $p : M_1 \times M_2 \rightarrow M'$ such that, for arbitrarily fixed elements $a_1 \in M_1, a_2 \in M_2$, the partial mappings

$$p(\cdot, a_2) : M_1 \rightarrow M' \quad (x_1 \mapsto p(x_1, a_2))$$

$$p(a_1, \cdot) : M_2 \rightarrow M' \quad (x_2 \mapsto p(a_1, x_2))$$
are morphisms.

By a **ring** we mean a set \( R \) with two binary operations - addition and multiplication - such that it is an abelian group under the addition, it is a monoid under the multiplication, and the multiplication

\[
R \times R \rightarrow R \quad (x, y) \mapsto xy
\]

is a pairing under the addition. A **field** is a ring \( K \) such that \( K^* = K - \{0\} \) forms an abelian group under the multiplication, we shall fix a field \( K \).

Let \( R, R' \) be rings. A **(ring) morphism** from \( R \) to \( R' \) is a mapping \( h : R \rightarrow R' \) which is a monoid morphism under the addition and multiplication.

By a **ring over** \( K \), we mean a ring \( R \) with a ring morphism \( K \rightarrow R \). If \( R \) is non-trivial, namely \( R \neq \{0\} \), \( K \) may be regarded to be contained in \( R \). An **algebra over** \( K \) is a ring over \( K \) whose elements commute with the elements of \( K \).

Let \( R, R' \) be rings over \( K \). A **(ring) morphism over** \( K \) from \( R \) to \( R' \) is a (ring) morphism \( h : R \rightarrow R' \) making the following diagram commutative:

\[
\begin{array}{ccc}
R & \xrightarrow{h} & R' \\
\downarrow{K} & & \downarrow{K} \\
& & 
\end{array}
\]
Which means \( h(\lambda a) = \lambda h(a) \), \( h(a \lambda) = h(a) \lambda \) \((\lambda \in K, a \in R)\);

an algebra morphism means always a ring morphism over \( K \).

A ring \( R \) over \( K \) is regarded as a two-sided vector space over \( K \) whose scalar multiplications are naturally defined. A subset of \( R \) which is closed under the addition and scalar multiplication, is called a two-sided \( K \)-subspace. If \( R \) is an algebra over \( K \), the word 'two-sided' may be replaced.

For a bijective (monoid or ring) morphism, its inverse mapping is also a morphism. Such a morphism is called a (monoid or ring) isomorphism. Let \( M \) be a monoid or ring. An isomorphism from \( M \) to itself is called an automorphism of \( M \).

Let \( G \) be a (multiplicative) group acting on an abelian (multiplicative) group \( \Omega \) as automorphism, we denote by \( \sigma \) the action. A mapping

\[
f : G \times G \to \Omega
\]

is called a 2-cocycle of \( G \) in \( \Omega \), if it satisfies

\[
(\sigma(x)f(y, z) f(xy, z)^{-1} f(xyz) f(x, y)^{-1} = 1
\]

\((x, y, z \in G)\).

All cocycles form an abelian group \( Z^2_c(G, \Omega) \).

For any mapping

\[
g : G \to \Omega,
\]

the mapping \( f : G \times G \to \Omega \) defined by

\[
f(x, y) = (\sigma(x)g(y)) g(xy)^{-1}g(x)
\]

\((x, y \in G)\) is a 2-cocycle, such a 2-cocycle is called a 2-coboundary.
All 2-coboundaries form a subgroup $B^2_c(G, \Omega)$ of $Z^2_c(G, \Omega)$.

The quotient group $H^2_c(G, \Omega) = Z^2_c(G, \Omega)/B^2_c(G, \Omega)$ is called the 2-cohomology group of $G$ in $\Omega$, its element is called a 2-cohomology class.

For the trivial action $\sigma$, $H^2_c(G, \Omega)$ will be denoted simply by $H^2(G, \Omega)$.

2. The concept of ring and algebra extension

**DEFINITION.** Let $K$ be a field and $G$ a (multiplicative) group. A **ring extension of $G$ over $K$** is a ring $R$ over $K$ with a direct sum decomposition

$$R = \sum_{x \in G} R_x$$

of $R$ into a family of two-sided $K$-subspaces $R_x$, indexed by $G$, such that

1) $R_x = K e_x = e_x K$ for some $e_x \neq 0$ ($x \in G$);

2) $R_x R_y = R_{xy}$ ($x, y \in G$).

An algebra extension of $G$ over $K$ is a ring extension of $G$ over $K$ which is an algebra over $K$.

**Note.** For an algebra $R$ over $K$, 1) is equivalent to the following:

1') Every $R_x$ is of one dimension over $K$.

Two ring extensions $R, R'$ of $G$ over $K$ are said to be isomorphic if there exists a ring isomorphism $h : R \rightarrow R'$ over $K$ satisfying

$$h(R_x) = R'_x$$

($x \in G$).
Let a ring extension $R$ of $G$ over $K$ be given. Then $R_1$ is identified with $K$, and $G$ acts naturally on $K$ as ring automorphisms we denote it by $\sigma_R$:

$$(\sigma_R(x))(\lambda) = e_x \lambda e_x^{-1} \quad (\lambda \in K, \; x \in G),$$

where the right side depends only on $\lambda, x$. $\sigma_R$ may be considered as an action of $G$ on the multiplicative group $K^*$. 

Note. If $\sigma_R$ is faithful, $R$ is nothing but a so-called 'crossed product' of $G$ and $K$ where $G$ is regarded as the Galois group of $K$ over the fixed subfield. If and only if $\sigma_R$ is trivial, $R$ is an algebra extension.

Taking non-zero elements $e_x \in R_x \; (x \in G)$, we define a mapping $f : G \times G \rightarrow K^*$ by the equalities

$$e_x e_y = f(x, y) e_{xy} \quad (x, y \in G).$$

Then $f$ is a 2-cocycle of $G$ in $K^*$ whose cohomology class $c_R \in H^2(\sigma_R)$ depends only on $R$. In this way, we have a one-to-one correspondence between all isomorphic classes of ring extensions of $G$ over $K$ and all 2-cohomology classes in $H^2(\sigma_R(G, K^*))$, where $\sigma$ runs over all actions of $G$ on the field $K$.

In particular, we have a one-to-one correspondence between all isomorphic classes of algebra extensions of $G$ over $K$ and all 2-cohomology classes in $H^2(G, K^*)$. 


Now suppose that $G$ is decomposed into the direct product of normal subgroups $G_1, \ldots, G_n$. Then we have the natural morphisms

$$H^2_\sigma(G, K^*) \rightarrow H^2_{\sigma_1}(G_1, K^*) \quad (c \mapsto c_1) \quad (1 \leq i \leq n),$$

where $\sigma_1$ is the action of $G_1$ on $K$ induced by $\sigma$. If a ring extension $R$ of $G$ over $K$ satisfies the condition

$$(c_R)_1 = 1 \quad (1 \leq i \leq n),$$

which means $R_1 = \sum_{x \in G_1} R_x$ is essentially the group algebra of $G_1$ over $K$, then $R$ is almost nothing but the 'generalized Clifford algebra' given in [1] *.

2. Some basic considerations

Hereafter we shall always assume the following

**ASSUMPTION.** $G$ is a finite abelian group with exponent $e$ (= the greatest order of elements) and $K$ is a field containing a primitive $e$-th root of unity.

$H^2_{ab}(G, K^*)$ denotes the subgroup of $H^2(G, K^*)$ consisting of all cohomology classes containing 'abelian cocycle' $f$

$$f(x, y) = f(y, x) \quad (x, y \in G).$$

Let $\Omega$ be an arbitrary abelian group. $\text{P.a.s.}(G, \Omega)$ denotes

---

*The author would like to call such a ring extension with direct product decomposition 'of $M$-$N$ type'. However, exactly saying, this is not the same as given in [1] , because of the difference of formulations. To get the concept given in [1] , we should add a little more restrictions.*
the group consisting of all 'anti-symmetric' pairings
\[ \varphi : G \times G \rightarrow \Omega; \]
\[ \varphi(x,y) = \varphi(y,x)^{-1} \quad (x, y \in G). \]

For any \( c \in H^2(G, \Omega), \quad \varphi_c \in \text{P}_{a.s.}(G, \Omega) \) is defined by
\[ \varphi_c(x,y) = f(x,y) f(y,x)^{-1} \quad (x,y \in G) \]
where \( f \) belongs to \( c \), \( \varphi_c \) depends only on \( c \).

**Lemma 1.** We have the following natural splitting exact sequence.
\[ 1 \rightarrow H^2_{ab}(G, \Omega) \rightarrow H^2(G, \Omega) \rightarrow \text{P}_{a.s.}(G, \Omega) \rightarrow 1 \]
\[ (c \mapsto \varphi_c) \]


**Note.** Let \( \mathbb{Z}_m \) denote the cyclic group of order \( m \). For
\[ G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}, \quad H^2_{ab}(G, \Omega) \]
has the following structure.
\[ \cong \Omega^{m_1} \times \cdots \times \Omega^{m_n} \]
where \( \Omega^m \) is consisting of all \( m \)-th powers of elements in \( \Omega \).
(see [2] p. 159). In particular, \( H^2_{ab}(G, \mathbb{C}) \) is trivial, where \( \mathbb{C} \) is the field of complex numbers. Hence we have

**Corollary.** \( H^2(G, \mathbb{C}) \cong \text{P}_{a.s.}(G, \mathbb{C}) \).

\( \text{Hom}(G, \Omega) \) denotes the group consisting of all morphisms from \( G \) to \( \Omega \). Let \( \Omega \) be the quotient group \( \overline{K}/K^* \) where \( \overline{K} \) is the algebraic closure of \( K \). Then to any \( \psi \in \text{Hom}(G, \Omega) \), taking a
section $\lambda : 0 \rightarrow \mathbb{K}^*$ of $\Psi$ and 2-cocycle

$$f(x, y) = \lambda(y) \lambda(xy)^{-1} \lambda(x),$$

we assign a cohomology class $c \in H^2_{ab}(G, K^*)$ containing $f$, $c$
depends only on $\Psi$, which will be denoted by $\Psi_c$.

**Lemma 2.** We have the following isomorphism

$$H^2_{ab}(G, K^*) \cong \text{Hom}(G, K^*/K^*).$$

$$(c \mapsto \Psi_c)$$


Now, for any $c \in H^2(G, K^*)$, we define two subgroups $N_c$
and $M_c$ of $G$ as follows. $N_c$ denotes the annihilator of $\Psi_c$,
namely the subgroup consisting of all $y \in G$ such that

$$\Psi_c(x, y) = 1 \text{ for all } x \in G.$$  

Applying Lemma 1 to the group $N_c$ and $K^*$, we see the restriction
c' of $c$ to $N_c$ belongs to $H^2_{ab}(N_c, K^*)$. So, by the correspondence
in Lemma 2, we have $\Psi_{c'} \mapsto \text{Hom}(N_c, K^*/K^*)$. $M_c$ denotes the kernel
of $\Psi_{c'}$, this is a subgroup of $N_c \subset G$.

4. **Structure theorem.**

Any semi-simple algebra $A$ over a field $K$ is decomposed into
the direct sum of all minimal two-sided ideals which are simple
algebras:

$$A = S_1 + \ldots + S_\mu.$$
Note. $\mu$ depends only on $A$ and may be written $\mu_A$. This is equal to the number of all non-similar irreducible representations of $A$ over $K$.

According to this decomposition, we have the following expressions

$$1 = \varepsilon_1 + \ldots + \varepsilon_\mu \quad (\varepsilon_1 \in S_1).$$

Then we define two subalgebras $B(A)$ and $C(A)$ of $A$ as follows. $B(A)$ denotes the subalgebra generated by $\varepsilon_1, \ldots, \varepsilon_\mu$: $B(A) = K \varepsilon_1 + \ldots + K \varepsilon_\mu$. $C(A)$ denotes the center of $A$ which is the sum of centres of $S_i(1 \leq i \leq \mu)$:

$$C(A) = C(S_1) + \ldots + C(S_\mu).$$

**Theorem 1.** Let $G$ be a finite abelian group of exponent $e$ and $K$ a field containing a primitive $e$-th root of unity. Then any algebra extension $A$ of $G$ over $K$ has the following structure:

1) $A$ is a semi-simple algebra with mutually isomorphic simple components.

2) $B(A) = \sum_{x \in N_c} A_x$ \quad (A corresponds to $c \in H^2(G, K^*)$.)

3) $C(A) = \sum_{x \in N_c} A_x$

Remark. 1) follows from the fact that the group \( \hat{G} = \text{Hom}(G,K^*) \) acts on \( A \) and induces some transitive permutations on all simple components. [3] contains a simplified proof for this fact which means that there exists only one non-similar irreducible 'projective' representation of \( G \) over \( K \) corresponding to a given cohomology class \( c \), \( M_c \) is the kernel of this representation.

Moreover the centres \( C(S_1) \) are isomorphic Kummer fields over \( K \) whose automorphisms are induced by \( G \). See [2] pp.180-182 in detail.

Note. If \( K \) is the field \( \mathbb{C} \) of complex numbers, we have \( M_{\mathbb{C}} = N_{\mathbb{C}} \) and

\[ A \cong M_4(\mathbb{C}) \oplus \ldots \oplus M_d(\mathbb{C}) \quad (\mu_A \text{ copies}) \]

where \( M_d(\mathbb{C}) \) is the total matric algebra over \( K \) of degree \( d \).

We note that \( \mu_A d^2 \) is equal to the order of \( G \).

5. Illustrations-1

Taking some simple cases, let us illustrate the theory given in the sections 3, 4. In this section, we assume that

\[ K = \mathbb{C} \quad \text{(the field of complex numbers)} \]

\[ G = \mathbb{Z}_n \times \ldots \times \mathbb{Z}_n \quad \text{(n copies).} \]

Then the anti-symmetric pairings \( \varphi \in \text{P}_{a,s}(G,K^*) \) are
given by the tables

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( \ldots )</th>
<th>( s_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>( \omega_{11} )</td>
<td>( \omega_{12} )</td>
<td>( \omega_{1n} )</td>
<td></td>
</tr>
<tr>
<td>( s_2 )</td>
<td>( \omega_{21} )</td>
<td>( \omega_{22} )</td>
<td>( \omega_{2n} )</td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_n )</td>
<td>( \omega_{n1} )</td>
<td>( \omega_{n2} )</td>
<td>( \omega_{nn} )</td>
<td></td>
</tr>
</tbody>
</table>

where the \( s_i \) generate \( G \) and the \( \omega_{ij} \in K \) satisfy the conditions

\[
\omega_{ij}^m = 1, \quad \omega_{ij} = \omega_{ji}^{-1} \quad (1 \leq i, j \leq n).
\]

We have such \( m^{n(n-1)/2} \) tables. Hence, by Corollary to Lemma 1, there are \( m^{n(n-1)/2} \) non-isomorphic algebra extensions of \( G \) over \( K \).

**Note.** The above number is not in the sense of algebra but 'algebra extension'. The number of non-isomorphic algebra extensions in the sense of 'algebra' can be also calculated using Theorem 2 in the next section. (Cf.1.)

For each anti-symmetric pairing \( \varphi \), we take the pairing \( f \) given by the table

<table>
<thead>
<tr>
<th>( f )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( \ldots )</th>
<th>( s_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( s_2 )</td>
<td>( \omega_{21} )</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( s_n )</td>
<td>( \omega_{n1} )</td>
<td>( \omega_{n2} )</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
where all entries in the upper half triangle are 1.

**Note.** Any pairing \( f : G \times G \rightarrow K^* \) is a 2-cocycle of \( G \) in \( K^* \).

We see that the cohomology class \( c \) containing \( f \) determines \( \varphi = \varphi_c \). Using the above \( f \), the corresponding algebra extension \( A \) is given by the generators \( e_1, \ldots, e_n \) with the fundamental relations

\[
e_i^m = 1, \quad e_i e_j = \omega^{i j} e_j e_i \quad (i > j).
\]

\( e_i = e_{s_i} \in A_{s_i} \quad (1 \leq i \leq n) \) according to the previous notation.

Now let us find the structure of \( A \). The subgroups \( N_c \) and \( N_\varphi \) of \( G \) are the same and calculated as follows:

Let \( \omega \) be a primitive \( m \)-th root of unity and \( \alpha_{ij} \) integers such that

\[
\omega^{i j} = \alpha_{ij} \quad (1 \leq i, j \leq n).
\]

Since \( N_c \) is the annihilator of \( \varphi_c \), we have

\[
N_c = N_\varphi = \{ s_1^{\beta_1} \ldots s_n^{\beta_n} ; \sum_{j=1}^{n} \alpha_{ij} \beta_j \equiv 0 \pmod{m} \ (1 \leq i \leq n) \}.
\]

**Note.** This subgroup is reduced to the unit group which means \( A \) is simple, if and only if the determinant of the matrix \( (\alpha_{ij}) \) is coprime with \( m \).

**Example 1.** ("usual" generalised Clifford algebras). The algebra \( C_n^m \) may be defined by the generators \( e_1, \ldots, e_n \) with the relations

\[
e_i^m = 1 \quad (1 \leq i \leq n), \quad e_i e_j = \omega e_j e_i \quad (i > j),
\]
where $\omega$ is a primitive $m$-th root of unity.

In this case, we have

$$\alpha_{ij} = \begin{cases} 1 & (i > j) \\ 0 & (i = j) \\ -1 & (i < j). \end{cases}$$

If $n$ is even, then $\det(\alpha_{ij}) = 1$ and $C_n^m$ is simple.

If $n$ is odd, then solving the equation

$$\begin{align*}
-\beta_2 - \beta_3 - & \cdots - \beta_n \equiv 0 \\
\beta_1 - \beta_3 - & \cdots - \beta_n \equiv 0 \\
\beta_1 + \beta_2 - & \cdots - \beta_n \equiv 0 \\
\cdots & \\
\beta_1 + \beta_2 + \beta_3 + & \cdots + \beta_{n-1} \equiv 0
\end{align*}$$

(mod. m)

we see that $M_c = N_c$ is generated by the single element

$$s_1 s_2^{-1} s_3 s_4^{-1} \cdots,$$

whose order is $m$. Hence we have

$$C_n^m = \begin{cases} M_{2n}(\mathbb{C}) & (n = 2\nu) \\ M_{2\nu}(\mathbb{C}) \oplus \cdots \oplus M_{2\nu}(\mathbb{C}) & (n = 2\nu + 1) \text{ (m copies)} \end{cases}$$

**Example 2.** The algebra generated by $e_1, \ldots, e_{2\nu}$ with the relations
\[ e_i^n = 1 \quad (1 \leq i \leq 2\nu), \quad e_i \circ_j e_j = \begin{cases} e_{i+j} & (i = j + 1 \text{ is even}) \\ e_j \circ_i e_i & (\text{otherwise}) \end{cases} \]

is isomorphic to \( M_{\nu}(\mathbb{C}) \). The proof is left to the readers as an exercise.


According to the notation in Theorem 1, we shall determine all possible subgroups \( M_c \) and \( N_c \) of \( G \).

**Theorem 2.** Let \( G \) be a finite abelian group of exponent \( a \) and \( K \) a field containing a primitive \( a \)-th root of unity. Then for given two subgroups \( H \) and \( N \) of \( G \) there exists an algebra extension \( A \) such that \( M_c = H \) and \( N_c = N \), if and only if the following hold:

1) The quotient group \( G/H \) is of symmetric type.

2) The quotient group \( \mathbb{K}^*/\mathbb{K}^+ \) contains a subgroup isomorphic to \( N/H \), where \( N \) is supposed to contain \( H \).

(\( \nu \) corresponds to \( a \)).


**Note.** A group \( H \) is called 'of symmetric type' if it is isomorphic to the direct product of two isomorphic groups: \( H = H' \times H' \). If \( K \) is the field \( \mathbb{C} \) of complex numbers, then the condition 2) is trivial and we have:

**Corollary.** Let \( G \) be a finite abelian group and \( d, \mu \) natural numbers. Then there exists an algebra extension \( A \) isomorphic to

\[ M_d(\mathbb{C}) \oplus \ldots \oplus M_d(\mathbb{C}) \quad (\mu \text{ copies}), \]
If and only if $G$ contains a subgroup $H$ such that

1) $G/H$ is of symmetric type
2) The order of $H$ is equal to $\mu$
3) The order of $G$ is equal to $\mu d^2$

7. Illustrations-2

Taking the same group as in the section 5, let us illustrate the theory given in the section 6. Namely we assume

$$G = Z_m \times \ldots \times Z_m$$

(n copies).

Example 1. As already shown, any algebra extension $A$ of $G$ over the field $\mathbb{C}$ has the following structure:

$$A = M_d(\mathbb{C}) \oplus \ldots \oplus M_d(\mathbb{C})$$

($\mu$ copies)

with the trivial condition

$$m^n = \mu d^2.$$

Now applying Corollary to Theorem 2, we can know what numbers $d, \mu$ are really possible:

There exists an algebra extension $A$ of $G = Z_m \times \ldots \times Z_m$

(n copies) over $\mathbb{C}$ such that

$$A \cong M_d(\mathbb{C}) \oplus \ldots \oplus M_d(\mathbb{C})$$

($\mu$ copies)

if and only if $d, \mu$ satisfy the conditions:

1) $m^n = \mu d^2$.
2) If $n$ is odd, then $n$ divides $\mu$. 
Proof. Let $N$ be any subgroup of $G$. Then there are $n$ positive integers $m_1, \ldots, m_n$ such that

$$G/N \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$$

and $m_i$ divides $m_{i+1}$ ($1 \leq i < n$), since $G/N$ has $n$ generators. Considering the orders of its elements, the $m_i$ must be divisors of $m$. Conversely, for any divisors $m_i$ of $m$, there clearly exists such a subgroup $N$. On the other hand, we note that, for a finite abelian group of order $h$, any divisor of $h$ is possible as an order of some subgroup.

Hence, by Corollary to Theorem 2, we easily see the existence of an algebra extension of given type under the conditions 1), 2). Let us see the converse. For an even number $n$, the condition 2) is trivial. For an odd number $n$, if $G/N$ is of symmetric type then $m_1 = 1$ which implies that $m$ divides $\mu$, namely we have 2).

Example 2. We take $K = \mathbb{Q}$ (the field of rational numbers) and $m = 2$ for the simplicity. Then we can determine all 'commutative' algebra extensions of $G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ over $\mathbb{Q}$:

$$A \cong L \oplus \cdots \oplus L \ (\mu \text{ copies}),$$

where $L$ is a Kummer field over $\mathbb{Q}$. By Theorem 2, we easily see that any divisor $\mu$ of $2^n$ is really possible. We note that any Kummer field with a Galois group homomorphic to $G$ is possible as $L$. Such a fact holds also for the general case. (See [2] pp. 190-192.)
References


PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

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Notation
Let $G$ be a finite group, $K$ an algebraically closed field of characteristic $p 
\not| \text{dim}$ (for example the field of complex numbers)
$K^* = K - \{0\}$, $V$ a finite dimensional vector space over $K$ with
$(V; K) = n$, $GL(V)$ the group of non-singular linear transformations
on $V$ (or the group of invertible $n \times n$ matrices over $K$).

1. Factor Sets

Definition 1.1. A mapping $\alpha : G \times G \to K^*$ is called a factor set
of $G$ in $K$ if

$$\alpha(x, y) \alpha(x', y') = \alpha(x, y') \alpha(x', y), \text{ for all } x, y, z \in G \quad (1.1)$$

and

$$\alpha(x, e) = \alpha(e, x) = \alpha(e, e) = 1, \text{ for all } x \in G, \quad (1.2)$$

where $e$ denotes the identity element of $G$. Two factor sets $\alpha$ and
$\beta$ of $G$ in $K$ are said to be equivalent if there exists a mapping
$\mu : G \to K^*$ such that $\mu(e) = 1$ and

$$\beta(x, y) = \frac{\mu(x) \mu(y)}{\mu(x y)} \alpha(x, y) \quad \text{ for all } x, y \in G.$$
Factor sets satisfy various elementary properties which are obtained in the following series of lemmas. Let $M(G,K^*)$ denote the set of all factor sets of $G$ in $K$.

**Lemma 1.2.** $M(G,K^*)$ is a multiplicative abelian group.

**Proof.** Exercise.

**Lemma 1.3.** If $\alpha \in M(G,K^*)$, then there exists $\beta \in M(G,K^*)$ such that $\beta$ is equivalent to $\alpha$ and

$$\beta(x,x^{-1}) = 1 \quad \text{for all } x \in G. \quad (1.3)$$

**Proof.** Define

$$\mu(x) = \frac{1}{\sqrt{\alpha(x,x^{-1})}} \quad \text{for all } x \in G.$$ 

Then, the factor set $\beta$ defined by

$$\beta(x,y) = \frac{\mu(x) \mu(y)}{\mu(xy)} \alpha(x,y)$$

has the required property, since

$$\beta(x,x^{-1}) = \left\{ \frac{\alpha(x,e^{-1})}{\alpha(x,x^{-1}) \alpha(x^{-1}x)} \right\}^{\frac{1}{2}} \alpha(x,x^{-1}) = 1.$$

**Lemma 1.4.** (Schur [40]) If $\alpha \in M(G,K^*)$, then there exists $\beta \in M(G,K^*)$ such that $\beta$ is equivalent to $\alpha$ and

$$\beta(x,y) = 1 \quad \text{for all } x,y \in G. \quad (1.4)$$

**Proof.** Define

$$\mu(x) = \prod_{g \in G} \alpha(x,g),$$

then
\[ \frac{\mu(x) \mu(y)}{\mu(x, y)} = \prod_{g \in G} \alpha(x, g) \prod_{g \in G} \alpha(y, g) \prod_{g \in G} \alpha(xy, g) = \alpha(x, y) \] using (1.1). If we now let
\[ \nu(x) = \mu(x)^{-\frac{1}{|G|}}, \]
then the factor set \( \beta \) defined by
\[ \beta(x, y) = \frac{\nu(x) \nu(y)}{\nu(xy)} \alpha(x, y) \] for all \( x, y \in G \)
has the required property.

**Definition 1.6.** A factor set \( \alpha \in \mathcal{M}(G, x^+) \) is called a normalized factor set if
\[ \alpha(x, x^{-1}) = 1 \] for all \( x \in G \). (1.5)

**Lemma 1.6.** If \( \alpha \in \mathcal{M}(G, x^+) \) is a normalized factor set, then
\[ \alpha^{-1}(x, y) = \alpha(y^{-1}, x^{-1}) \] for all \( x, y \in G \). (1.6)

**Proof.** By (1.1)
\[ \alpha(x, y) \alpha(x, y^{-1}) = \alpha(x, e) \alpha(y, y^{-1}) = 1, \]
and
\[ \alpha(xy, y^{-1}) \alpha(x, x^{-1}) = \alpha((xy), (xy)^{-1}) \alpha(y^{-1}, x^{-1}). \]
That is, we have
\[ \alpha(x, y^{-1}) = \alpha(xy, y^{-1}) = \alpha(y^{-1}, x^{-1}) , \]
as required.

**Definition 1.7.** An element \( s \in G \) is called a \( \alpha \)-**regular element** if
\[ \alpha(s, x) = \alpha(x, s) \quad \text{for all } x \in C_G(s) = \{ x \in G \mid xs = sx \} . \]
(\( C_G(s) \) is called the centralizer of \( s \) in \( G \), where \( \alpha \in \mathbb{H}(G, K^*) \).
We shall consider in particular for each \( \alpha \in \mathbb{H}(G, K^*) \) and for each \( \alpha \)-regular element \( s \in G \)
\[ f_\alpha(x, s) = \alpha(x, s)^{\alpha^{-1}}(xsx^{-1}, x) \quad \text{for all } x \in G. \quad (1.7) \]
We note in particular that \( f_\alpha(x, s) = 1 \) if \( x \in C_G(s) \). Further, if \( \alpha \) is a normalized factor set
\[ f_\alpha(x, s) = \alpha(x, s) \alpha(xs, x^{-1}) . \quad (1.9) \]
Then, the following can be proved.

**Lemma 1.3.** If \( \alpha \in \mathbb{H}(G, K^*) \) is a normalized factor set and \( s \in G \)
is a \( \alpha \)-regular element, then

(i) \[ f_\alpha(x, s) = f_\alpha^{-1}(x, s^{-1}) \quad \text{for all } x \in G \quad (1.9) \]

(ii) if \( \alpha xsx^{-1} = ysy^{-1} \) with \( x, y \in G \), then
\[ f_\alpha(x, s) = f_\alpha(y, s) . \quad (1.10) \]

**Proof.** (i) \[ f_\alpha(x, s)f_\alpha(x, s^{-1}) = \alpha(x, s) \alpha(xs, x^{-1}) \alpha(x, s^{-1}) \alpha(xs^{-1}, x^{-1}) \]
\[ = \alpha(x, s) \alpha(xs, x^{-1}) \alpha(x, s^{-1}x^{-1}) \alpha(s^{-1}x^{-1}) \]
\[ = 1 \text{ by Lemma 1.5.} \]
\[ (\alpha_\alpha)(x, s) \alpha_\alpha^{-1}(y, s) = \alpha(x, s) \alpha(x s, x^{-1}) \alpha(x s x^{-1}, y) \alpha^{-1}(y, s) \]
\[ = \alpha(x, s) \alpha(x s, x^{-1} y) \alpha(x^{-1}, y) \alpha^{-1}(y, s) \]
\[ = \alpha(x, s x^{-1} y) \alpha(s, x^{-1} y) \alpha(x^{-1}, y s) \alpha^{-1}(x^{-1}, y s) \]

But \( x s x^{-1} = y s y^{-1} \) and \( \alpha^{-1} y \in C_G(s) \) and since \( s \) is \( \alpha \)-regular,
\[ \alpha(s, x^{-1} y) \alpha^{-1}(x^{-1} y, s) = 1 \]

Thus
\[ \alpha_\alpha(x, s) \alpha_\alpha^{-1}(y, s) = \alpha(x, s x^{-1} y) \alpha(x^{-1}, y s) \]
\[ = \alpha(x, x^{-1} y s) \alpha(x^{-1}, y s) \]
\[ = 1 \]

These results are now used to prove

**Theorem 1.2.** If \( \alpha \in M(G, K^*) \) is normalized and \( s \) is a \( \alpha \)-regular element of \( G \), then

(i) \( s^{-1} \) is a \( \alpha \)-regular element,

(ii) every element conjugate to \( s \) is \( \alpha \)-regular.

**Proof.** (i) It is easily verified that \( C_G(s) = C_G(s^{-1}) \). Further
\[ \alpha_\alpha(x, s) = 1 \] for all \( x \in C_G(s) \)

and by Lemma 1.5 (i)
\[ \alpha_\alpha(x, s^{-1}) = 1 \] for all \( x \in C_G(s^{-1}) \),

and so \( s^{-1} \) is a \( \alpha \)-regular element.

(ii) Again, it is easily verified that \( C_G(t s t^{-1}) = t C_G(s) \)

We must show that
\[ f_\alpha (y, t s t^{-1}) = 1 \quad \text{for all } y \in C_\alpha (t s t^{-1}) \]

But
\[ f_\alpha (y, t s t^{-1}) = \alpha (y, t s t^{-1}) \alpha^{-1} (t s t^{-1}, y) \]

Since \( y \in C_\alpha (t s t^{-1}) \), it follows that
\[ (y t)^{-1} (y t)^{-1} = t s t^{-1}, \]

and thus by Lemma 1.3(11)
\[ f_\alpha (y t, s) = f_\alpha (t, s) \]

and
\[ f_\alpha (y t, s) = \alpha (y t, s) \alpha (y t s, t^{-1} y^{-1}) \]
\[ = \alpha (y t, s) \alpha (y t s, t^{-1}) \alpha (t s t^{-1} y, y^{-1}) \alpha^{-1} (t, y^{-1}) \]
\[ = \alpha (y, t s) \alpha (y t s, t^{-1}) \alpha (t s t^{-1} y, y^{-1}) \alpha (t, s) \]
\[ = \alpha (y, t s t^{-1}) \alpha (t s t^{-1} y, y^{-1}) \alpha (t s, t^{-1}) \alpha (t, s) \]
\[ = f_\alpha (y, t s t^{-1}) f_\alpha (t, s) \]

That is
\[ f_\alpha (y, t s t^{-1}) = 1 \quad \text{for all } y \in C_\alpha (t s t^{-1}) \]

As a result of this theorem, we have

**Definition 1.10.** A class of conjugate elements of \( G \) is called a \( \alpha \)-regular class if all its elements are \( \alpha \)-regular. (In the physical literature such a class has been called a ray class (Harter [33]).)
We note in particular that if a class $\mathcal{K}$ of $G$ is $\alpha$-regular, then the class $\mathcal{K}^*$ of inverses of elements of $\mathcal{K}$ is $\alpha$-regular.

Before proceeding to our main theorem, we note the following.

**Lemma 1.11.** If an element $s \in G$ is $\alpha$-regular, then it is $\beta$-regular for all $\beta \in M(G, K^*)$ which are equivalent to $\alpha$.

**Proof.** Exercise.

**Theorem 1.12.** (Conlon [10]). If $\alpha \in M(G, K^*)$, then there exists a $\gamma \in M(G, K^*)$ equivalent to $\alpha$ such that both the following conditions are satisfied

(i) $\gamma(x, x^{-1}) = 1$ for all $x \in G$ \hspace{1cm} (1.11)

(ii) $\ell_\gamma(x, s) = 1$ for all $\gamma$-regular $s \in G$ \hspace{1cm} (1.12)

and for all $x \in G$.

**Proof.** If $\alpha \in M(G, K^*)$, then by Lemma 1.3 there exists a $\beta \in M(G, K^*)$ equivalent to $\alpha$ which satisfies (1.11).

Let $\{a_i, \ldots, a_n\}$ be an arbitrary $\beta$-regular class of $G$, then $\alpha_i = x_i a_i x_i^{-1}$ where $G = \bigcup_{i=1}^{n} C_G(a_i) a_i$. Now, define

$\mu(a_i) = \ell_\beta(x_i, a_i), \hspace{1cm} (i = 1, \ldots, n)$.

Similarly, define $\mu(a)$ for all $\beta$-regular elements $a \in G$. If $a$ is not $\beta$-regular, let

$\mu(a) = 1$.

Then by (1.10), $\mu$ is well-defined and $\mu(a_i) = 1$ (in particular $\mu(e) = 1$). Now, let

$\gamma(x, y) = \frac{\mu(x) \mu(y)}{\mu(xy)} \beta(x, y)$ for all $x, y \in G$. 

Then
\[ \gamma(a_i, a_i^{-1}) = \frac{\mu(a_i) \mu(a_i^{-1})}{\mu(\varepsilon)} \beta(a_i, a_i^{-1}) \]
\[ = \frac{f_\beta(x_i, a_i)}{f_\beta(x_i, a_i)} \]
\[ = \frac{f_\beta(x_i, a_i)}{f_\beta(x_i, a_i)} \beta^{-1}(x_i, a_i) \]
\[ = 1 \]
by (1.9) and so \( \gamma \) satisfies (1.11). In addition
\[ f_\gamma(x, a_i) = \gamma(x, a_i) \gamma^{-1}(x a_i x^{-1} \varepsilon) \]
\[ = \frac{\mu(a_i)}{\mu(x a_i x^{-1})} \beta(x, a_i) \beta^{-1}(x a_i x^{-1}, x) \]
\[ = \beta(x_i, a_i) \beta(x, a_i x_i) \beta(x^{-1} x x_i) \beta^{-1}(x x_i, a_i) \]
\[ = \beta(x_i, a_i) \beta(x, a_i x_i) \beta(x^{-1}, x x_i) \beta(x x_i, a_i) \]
\[ = 1 \]
and so \( \gamma \) also satisfies (1.12).

Remark. Unfortunately, the factor set \( \gamma \) defined in the above theorem does not in general simultaneously satisfy (1.4). Conlon [10] has in fact proved that
\[ \gamma(x, y)^{x_1y_1} = 1 \]
for all \( x, y \in G \).

A factor set \( \alpha \in \gamma(G, K^*) \) which satisfies (1.11) and (1.12) is
called a \textit{\alpha-normalised factor set}.

2. \textbf{Projective Representations and Twisted Group Algebras}

\textbf{Definition 2.1} A mapping $T: G \rightarrow GL(V)$ is called a \textit{projective representation} with representation space $V$ and factor set $\alpha$ if
\begin{equation}
T(x)T(y) = \alpha(x, y)T(xy) \quad \text{for all } x, y \in G \tag{2.1}
\end{equation}
\begin{equation}
T(e) = I \quad \text{.} \tag{2.2}
\end{equation}

where $\alpha(x, y) \in K$. (Alternatively, it may be considered as a mapping $T: G \rightarrow GL(n, K)$ satisfying (2.1) and $T(e) = I_n$, when it is referred to as a \textit{projective representation of degree $r$ with factor set $\alpha$}. These two definitions are taken to be equivalent).

It is easily verified that the $\alpha: G \times G \rightarrow K^*$ is a factor set of $G$ in $K$, that is, $\alpha \in \mathcal{U}(G, K^*)$. If $\alpha(x, y) = 1$, for all $x, y \in G$, then $T$ is an \textit{ordinary representation} of $G$, which we shall call a \textit{linear} representation.

\textbf{Definition 2.2.} If $S$ and $T$ are projective representations of $G$ with representation spaces $V_1$ and $V_2$ respectively and factor sets $\alpha$ and $\beta$ respectively, then $S$ and $T$ are \textit{projectively equivalent} if there exists an isomorphism $\Phi: V_1 \rightarrow V_2$ and a mapping $\mu: G \rightarrow K^*$ with $\mu(e) = 1$ such that
\begin{equation}
\Phi(S(x)) = \mu(x)T(x) \Phi \quad \text{for all } x \in G. \tag{2.3}
\end{equation}
If $\mu(x) = 1$, for all $x \in G$, then $S$ and $T$ are said to be \textit{linearly equivalent}.

\textbf{Lemma 2.3.} The factor sets of projectively equivalent projective representations are equivalent. Linearly equivalent projective
representations have the same factor set.

Proof. Exercise.

Lemma 2.4. For any \( \alpha \in M(G, K^*) \), there exists a projective representation \( T \) of \( G \) with factor set \( \alpha \).

Proof. Let \( V \) be a vector space with \( K \)-basis \( \{ \nu(x) | x \in G \} \).

Define \( T: G \rightarrow GL(V) \) by

\[
T(x) \nu(y) = \alpha(x, y) \nu(xy)
\]

for all \( x, y \in G \).

Then we have

\[
T(x) T(y) \nu(z) = T(x) \alpha(y, z) \nu(yz) = \alpha(x, yz) \nu(yz) = \alpha(x, y) \alpha(xy, z) \nu(xy) = \alpha(x, y) \alpha(x, y) \nu(z) = \alpha(xy, z) \nu(z)
\]

for all \( x, y, z \in G \). That is

\[
T(x) T(y) = \alpha(x, y) \alpha(xy, z) \nu(xy) \nu(z)
\]

for all \( x, y \in G \).

Further,

\[
T(e) \nu(x) = \alpha(e, x) \nu(x) = \nu(x)
\]

that is

\[
T(e) = 1
\]

Thus, \( T \) is the required projective representation with factor set \( \alpha \).

Definition 2.5. Conlon [10], Yamazaki [60], Tasawa [55].

Let \( \alpha \in M(G, K) \) and

\[
(KG)_\alpha = \{ \sum_{x \in G} \xi x \nu(x) | \xi x \in K \}
\]

where \( \{ \nu(x) | x \in G \} \) is a set of elements in one-one correspondence.
with the elements of \( G \). Then, addition on \((KG)_\alpha\) is defined by

\[
\sum_{x \in G} \xi_x \upsilon(x) + \sum_{x \in G} \eta_x \upsilon(x) = \sum_{x \in G} (\xi_x + \eta_x) \upsilon(x),
\]

scalar multiplication of elements of \((KG)_\alpha\) by elements of \( K \) by

\[
\xi \left( \sum_{x \in G} \xi_x \upsilon(x) \right) = \sum_{x \in G} \xi \xi_x \upsilon(x),
\]

and multiplication on \((KG)_\alpha\) by

\[
\left( \sum_{x \in G} \xi_x \upsilon(x) \right) \left( \sum_{x \in G} \eta_x \upsilon(x) \right) = \sum_{x, y \in G} \xi_x \eta_y \alpha(x, y) \upsilon(xy).
\]

Then, the following lemma is easily proved.

**Lemma 2.6.** \((KG)_\alpha\) is an associative algebra with identity \( \upsilon(e) \), called the **twisted group algebra** associated to \( \alpha \in M(G, K^*) \).

This is a natural generalization of the familiar group algebra which plays such an important role in the ordinary representation theory of \( G \) (see e.g. [11]). Our aim now is to show that all of the usual theorems on group algebras can be generalized to twisted group algebras.

**Lemma 2.7.** There is a one-one correspondence between the projective representations of \( G \) with factor set \( \alpha \) and the representations of \((KG)_\alpha\). Furthermore, there is a one-one correspondence between the representations of \((KG)_\alpha\) and finite dimensional left \((KG)_\alpha\)-modules.

**Proof.** This is left as an exercise, the proof being identical with that for the corresponding result for ordinary representations (See e.g. [11], p.44-48).
Thus, the problem of determining all the projective representations of $G$ with factor set $\alpha$ is equivalent to the determination of all left $(KG)_\alpha$-modules $V$ which are finite dimensional as vector spaces. If $V$ is a left $(KG)_\alpha$-module, then define $T:G \rightarrow GL(V)$ by

$$T(x) \cdot v = \nu(x) \cdot v$$

for all $x \in G, v \in V$.

Then $T(x) \in GL(V)$, since $V$ is a left $(KG)_\alpha$-module, i.e.

$$T(x)(\alpha \cdot v_1 + v_2) = \nu(x)(\alpha \cdot v_1 + v_2) = \alpha \cdot \nu(x) \cdot v_1 + \nu(x) \cdot v_2$$

$$= \alpha T(x) \cdot v_1 + T(x) \cdot v_2$$

and $T(x) \cdot 0 = 0 \Rightarrow \nu(x) \cdot 0 = 0$ which is impossible. Further

$$T(x)T(y) \cdot v = \nu(x) \nu(y) \cdot v$$

$$= \alpha(x,y) \nu(xy) \cdot v$$

$$= \alpha(x,y) T(xy) \cdot v$$

for all $x, y \in G, v \in V$,

and $T(e)v = \nu(e)v = v$. This representation constructed from the left $(KG)_\alpha$-module $V$ is called the projective representation afforded by $V$. In particular $(KG)_\alpha$ itself may be regarded as a left $(KG)_\alpha$ module. The projective representation afforded by $(KG)_\alpha$ is called the **left regular projective representation with factor set** $\alpha$ of $G$.

That is $\{\nu(g_i) | i = 1, \ldots, |G|\}$ may be regarded as a $K$-basis for the vector space $(KG)_\alpha$, if $g$ is an arbitrary element of $G$, define

$$R(x) \cdot \nu(g_i) = \nu(x) \nu(g_i)$$

for all $g \in G$,

$$= \alpha(x, g_i) \nu(xg_i) = \alpha(xg_i)$$

and

$$R(x) R(y) \cdot \nu(g) = \alpha(x, y) R(x) R(y) \cdot \nu(g)$$

for all $g \in G$. 


and so

\[ R(x) R(y) = \alpha(x, y) R(xy) \]

The matrix representation \( R(x) \) is the matrix whose only non-zero element in the \( i \)'th column is \( \alpha(x, g_i) \) which appears in the \( j \)'th row if \( x g_i = g_j \). For the sake of completeness, we include the following definitions.

**Definition 2.8.** A left \((KG)_\alpha\)-module \( V \) is **irreducible** if its only submodules are the trivial submodules \( \{0\} \) and \( V \). A left \((KG)_\alpha\)-module \( V \) is **completely reducible** module if for every submodule \( V_1 \) of \( V \) there exists a submodule \( V_2 \) of \( V \) such that

\[ V = V_1 \oplus V_2. \]

The following generalization of Maschke's Theorem is now proved.

**Theorem 2.9.** Every left \((KG)_\alpha\)-module \( V \) is completely reducible.

**Proof.** Let \( V_1 \) be a non-trivial submodule of \( V \). Then \( V_1 \) is a subspace of \( V \) and there exists a subspace \( W \) of \( V \) such that

\[ V = V_1 \oplus W. \]

Thus, there exists an \( E \in \text{Hom}_K(V, V_1) \) such that if

\[ V = V_1 + w, \quad v \in V, \quad v_i \in V_1, \quad w \in W, \]

then \( E_v = V_1 \). Let \( F : V \to V \) be defined by

\[ F_v = \frac{1}{|G|} \sum_{x \in G} \nu(x) E \nu(x)^{-1} v \quad \text{for all } v \in V, \]

(which is meaningful since \( |G| \neq 0 \) in \( K \)). Then \( E \in \text{Hom}_{(KG)_\alpha}(V, V) \),
since for all $v \in V$,

$$F \nu(y) v = \frac{1}{|G|} \sum_{x \in G} \nu(x) E \nu(x^{-1}) \nu(y) v$$

$$= \frac{1}{|G|} \nu(y) \sum_{y \in G} \frac{\alpha(g^{-1} y^{-1}, y) \alpha(y_2 z^{-1}) \nu(y_2) E \nu(z^{-1}) v}{\alpha(y_2 y, (y_2 y^{-1}) \alpha(y_2, y_2)}$$

and by (1.1)

$$\alpha(g^{-1} y^{-1}, y) \alpha(y_2 z^{-1}) = \alpha(g^{-1} y^{-1}, y_2) \alpha(y_2, y_2),$$

that is

$$F \nu(y) v = \nu(y) F v$$

for all $y \in G$, $v \in V$.

Furthermore, $F V \subseteq V_1$ since

$$\frac{1}{|G|} \sum_{x \in G} \nu(x) E \nu(x^{-1}) v \in \frac{1}{|G|} \sum_{x \in G} \nu(x) E v \in V_1,$$

since $V_1$ is a submodule of $V$. Also if $v_1 \in V_1$,

$$F v_1 = \frac{1}{|G|} \sum_{x \in G} \nu(x) E \nu(x^{-1}) v_1$$

$$= \frac{1}{|G|} \sum_{x \in G} \nu(x) \nu(x^{-1}) v_1$$

$$= v_1,$$

since $V_1$ is a submodule of $V$ and $F V_1 = V_1$. Thus $F V = V_1$. Now, let

$$V_2 = \{ v - F v \mid v \in V \},$$

then $V_2$ is a submodule of $V$ and

$$V = V_1 \oplus V_2.$$
That is, \( V \) is completely reducible.

As a consequence of this theorem, using well-known results (see e.g. \([11]\)) we have

**Corollary 2.10.** \((KG)\alpha\) is a semi-simple algebra.

**Corollary 2.11.** If \( T:G \to O(n,K) \) is a projective representation of \( G \) with factor set \( \alpha \), then \( T \) is linearly equivalent to a projective representation \( S(\text{with factor set } \alpha) \) such that

\[
S(x) = \begin{bmatrix}
S_1(x) \\
\vdots \\
S_n(x)
\end{bmatrix}
\]

where \( S_i(\alpha)(i=1,\ldots,n) \) are irreducible projective representations with factor set \( \alpha \).

**Corollary 2.12.** \((\{11\}, p.186) (KG)\alpha \cong M_{\nu_1}(K) + \cdots + M_{\nu_s}(K) \)

where \( M_{\nu_i}(K)(i=1,\ldots,s) \) is the full matrix algebra of \( \nu_1 \times \nu_2 \) matrices over \( K \). Further, there are \( s \) distinct irreducible inequivalent \((KG)\alpha\) modules \( M_1, \ldots, M_s \). (In fact, \( \text{Hom}_K(M_1, M_1) \cong M_{\nu_1}(K) \) and \( (M_i : K) = \nu_i \); each \( M_{\nu_i}(K) \) is isomorphic to a direct sum of \( \nu_i \) copies of \( M_i \). This in turn implies that the irreducible projective representation of degree \( \nu_i \) afforded by \( M_i \) appears \( \nu_i \) times as an irreducible component of the regular projective representation with factor set \( \alpha \) afforded by \((KG)\alpha\).)

We now consider \( Z = Z((KG)\alpha) \) the centre of \((KG)\alpha\), that is

\[
Z = \{ \lambda \in (KG)\alpha \mid \lambda \alpha = \alpha \lambda \quad \text{for all } \alpha \in (KG)\alpha \}
\]

It is easily verified (see \([11]\), p.187) that \( Z \) is itself an
algebra over $K$ and that

$$Z \cong Z \left( M_n(K) \right) \oplus \cdots \oplus Z \left( M_m(K) \right) .$$

Further $Z \left( M_n(K) \right)^\alpha \sim K$ (i.e. the only matrices which commute with all matrices in $M_n(K)$ are scalar multiples of the identity matrix). Thus $(Z:K) = s$, the number of inequivalent irreducible projective representations with factor set $\alpha = (Z:K)$.

The following theorem gives another method for computing the dimension of $Z$ over $K$.

**Theorem 2.13.** Let $\alpha \in \pi(G, K^*)$ be a $c$-normalized factor set. Let $K_1, \ldots, K_t$ be the $\alpha$-regular classes of $G$ and let

$$K_\alpha = \sum_{g \in K_\alpha} \nu(g) \quad (i = 1, \ldots, t)$$

Then $\{ K_1, \ldots, K_t \}$ is a $K$-basis for $\mathbb{Z} = \mathbb{Z}(K_\alpha)$. 

*Proof.* If $x \in G$, then

$$\nu(x^{-1} K_j \nu(x) = \sum_{g \in K_j} \nu(x^{-1} g) \nu(x) = \sum_{g \in K_j} \nu(xg) = K_j ,$$

and so $K_j (j = 1, \ldots, t) \in Z$. The set $\{ K_1, \ldots, K_t \}$ is clearly linearly independent over $K$ since each consists of a sum of disjoint sets of group elements. Let $y = \sum_{g \in G} \chi(g) \in Z$. We first show that if $\chi \neq 0$, then $g$ is a $\alpha$-regular element. That is, we must show that if $\alpha \in C_\alpha (g)$, then $F_\alpha(x, g) = 1$. Since $y \in Z$, $\nu(x^{-1} y \nu(x) = y$ for all $x \in G$, or
\[ \nu(x)^{-1} \nu(g) \nu(x) + \sum_{g \neq g'} \nu(g) \nu(x)^{-1} \nu(g') \nu(x) = \nu(x) \nu(g) + \sum_{g' \neq g} \nu(g') \nu(g') \]

This means that

\[ \nu(x, g) \nu(g) + \sum_{g' \neq g} \nu(x, g') \nu(x^{-1} g' x) = \nu(g) \nu(g) + \sum_{g' \neq g} \nu(g') \nu(g') \]

and as \( \nu(g) \neq 0 \) it follows that \( f_{\alpha}(x, g) = 1 \) since \( x^{-1} g' x = g \) would contradict \( g' \neq g \). Thus \( g \) is a \( \alpha \)-regular element. Hence, if \( y = \sum_{g \in G} \nu(g) \in Z \), the summation is over \( \alpha \)-regular elements \( x \) of \( G \). Further now

\[ \nu(x)^{-1} y \nu(x) = y \]

implies that

\[ \sum_{g \in G} \nu(g) f_{\alpha}(x, g) \nu(x^{-1} g x) = \sum_{g \in G} \nu(g) \nu(g) \]

and since \( f_{\alpha}(x, g) = 1 \) for all \( \alpha \)-regular \( g \in G \), and \( x \in G \), then

\[ \nu(g) = \nu_{xg x^{-1}} \]

Thus \( y \) is a linear combination of \( \{ K_1, \ldots, K_n \} \) as required, and the proof is complete.

It now follows directly that

**Corollary 2.14.** The number of non-equivalent distinct irreducible projective representations of \( G \) with factor set \( \alpha \) is equal to the number of \( \alpha \)-regular classes of \( G \).
Let \( P_1 \) and \( P_2 \) be projective representations of \( G \) with factor sets \( \alpha \) and \( \beta \) respectively and representation spaces \( V_1 \) and \( V_2 \) respectively. Then, define

\[
(P_1 \otimes P_2)(x)(v_1 \otimes v_2) = (P_1(x) \otimes P_2(x))(v_1 \otimes v_2)
\]

for all \( x \in G, v_1 \in V_1, v_2 \in V_2 \),

then

\[
(P_1 \otimes P_2)(x)(P_1 \otimes P_2)(y) = \alpha(x, y) \beta(x, y) (P_1 \otimes P_2)(xy)
\]

for all \( x, y \in G \).

That is, \( P_1 \otimes P_2 \) (the direct product of \( P_1 \) and \( P_2 \)) is a projective representation of \( G \) with factor set \( \alpha \beta \) where \( \alpha(\beta(x, y)) = \alpha(x, y) \beta(x, y) \) and representation space \( V_1 \otimes V_2 \). (This indicates an error in Rudra's [46] treatment of the direct product of projective representations. In his paper an incorrect construction for obtaining the direct product of two projective representations with factor set \( \alpha \) to give a projective representation with factor set \( \alpha \) is given.

The above shows that the resulting projective representation will have factor set \( \alpha \beta \). Rudra's construction is only valid for projective representations projectively equivalent to an ordinary representation.

Remark. The following theorem due to Clifford [9] (see also [11], p.351) shows that projective representations arise naturally in the study of ordinary representations.

**Theorem 2.15.** If \( H \) is a normal subgroup of \( G \) and \( V \) is an irreducible \((KG)\)-module, then \( V \) restricted to \( H \) is a direct sum of \( S \) isomorphic irreducible \((KH)\)-modules. If \( T \) is the representation of \( G \) afforded by \( V \), then
\[ T(x) = P_1(x) \otimes P_2(x) \quad \text{for all } x \in G \]

where \( P_1 \) and \( P_2 \) are irreducible projective representations of \( G \) and \( P_1 \) is a projective representation of \( G/H \) of degree \( s \).

For further work of this type, see Tucker [56], [57], Conlon [10], and Mackey [31].

3. Schur Multiplier and Representation Groups

If \( \alpha, \beta \in M(G, K^*) \), we have called \( \alpha \) and \( \beta \) equivalent factor sets if there exists \( \mu : G \rightarrow K^* \) such that

\[ \beta(x, y) = \frac{\mu(x) \mu(y)}{\mu(xy)} \alpha(x, y) \]

This is easily verified to be an equivalence relation on \( M(G, K^*) \).

Let \( \{ \alpha \} \) denote the equivalence class which contains \( \alpha \in M(G, K^*) \).

Let \( H^2(G, K^*) \) denote the set of equivalence classes of factor sets.

If \( \{ \alpha \}, \{ \beta \} \), we define multiplication on \( H^2(G, K^*) \) by

\[ \{ \alpha \} \{ \beta \} = \{ \alpha \beta \} \]

Then, we obtain the following

**Theorem 3.1.** \( H^2(G, K^*) \) is a finite abelian group. The order of every element of \( H^2(G, K^*) \) is a divisor of the order of \( G \).

**Proof.** We have already proved in Lemma 1.4 that if \( \alpha \in M(G, K^*) \), and we define \( \mu : G \rightarrow K^* \) by \( \mu(x) = \prod_{g \in G} \alpha(x, g) \), then

\[ \alpha(x, y) = \frac{\mu(x) \mu(y)}{\mu(xy)} \]
or in other words

\[ \{ \alpha \}^{|G|} = \{ 1 \} \]

which proves the second statement. Furthermore, in Lemma 1.4, we showed that there exists a factor set \( \beta \) equivalent to \( \alpha \) which is a \(|G|\)-th root of unity. Thus, every equivalence class \( \{ \alpha \} \) contains a representative \( \{ \beta \} \) which is a \(|G|\)-th root of unity. Thus, there are at most a finite number of equivalence classes of factor sets and the first statement is proved.

Definition 2.2. \( H^2(G, K^*) \) is called the Schur Multiplier of \( G \) in \( K \).

Before proceeding to give further crucial results on Schur Multipliers, we demonstrate its importance in the theory of projective representations of \( G \).

A pair \((G^*, \pi)\), where \( G^* \) is a group and \( \pi \) a homomorphism of \( G^* \) onto \( G \) is called a group extension of \( G \). \((G^*, \pi)\) is called a central group extension if the kernel \( A \) of \( \pi \) is contained in the centre of \( G^* \). Thus \( G^*/A \cong G \) and \( A \subseteq Z(G^*) \). Let \( \{ \omega(\alpha) | \alpha \in G \} \) be a set of coset representatives of \( A \) in \( G^* \) which are in one-one correspondence with the elements of \( G \). Then, we have

\[ \alpha(x, y) = \omega(x) \omega(y) \omega(xy)^{-1} \in A \quad \text{for all } x, y \in G \]

Let \( T \) be a linear representation of \( G^* \) with representation space \( V \) such that

\[ T(\alpha(x, y)) = \alpha(x, y) \mathbf{1}_V \]

where \( \alpha(x, y) \in \mathbb{K}^* \). Let \( T \) be an irreducible linear representation of \( G^* \), then since \( A \subseteq Z(G^*) \) it follows by Schur's lemma that
$T(a) = \alpha(x \cdot y)$, for all $a \in A$. In particular, let $T(a(x,y)) = \alpha(x,y)$.

If we define $P : G \to \text{GL}(V)$ by $P(x) = T(\omega(x))$ for all $x \in G$, then

$$P(x) P(y) = T(\omega(x)) T(\omega(y)) = T(\alpha(x,y) \omega(x \cdot y))$$

$$= \alpha(x,y) P(x \cdot y) \quad \text{for all } x, y \in G.$$ 

that is $P$ is a projective representation of $G$ with factor set $\alpha$.

We say that the projective representation $P$ is linearized by the linear representation $T$ of $G^*.$

**Definition 3.3.** A representation group $G^*$ of $G$ is a finite group $G^*$ of lowest possible order which is a central group extension of $G$ such that every projective representation of $G$ is equivalent to a projective representation of $G$ which is linearized by a linear representation of $G^*$.

The following theorem due to Schur [48] proves the existence of a representation group. The proof given here is due to Asano and Shoda [4].

**Theorem 3.4.** $G$ has at least one representation group $G^*$ of order $|H^2(G, K^*)| |G|$. Furthermore, the kernel $A$ of the extension is isomorphic to $H^2(G, K^*)$.

**Proof.** $H^2(G, K^*)$ is a finite abelian group and so is a direct product of cyclic groups

$$H^2(G, K^*) = \langle \alpha^{(1)} \rangle \times \cdots \times \langle \alpha^{(n)} \rangle,$$

where $\langle \alpha^{(i)} \rangle$ denotes the cyclic group generated by $\alpha^{(i)}$ of order $e_1$ (say) $(i = 1, \ldots, n)$. By theorem 3.1, $\{ \alpha^{(i)} \}$ may be chosen as an
the root of unity. Let \( \zeta \) denote a primitive \( \omega \) th root of unity, then

\[
\alpha^{(i)}(x, y) = \zeta^{a_{x,y}} \quad \text{where} \quad 0 \leq a_{x,y} \leq \omega - 1
\]

By (1.1) it follows that

\[
\alpha^{(i)}(x, y) + \alpha^{(i)}(x, y) \equiv \alpha^{(i)}(x, y) + \alpha^{(i)}(y, y) \pmod{\omega} \quad \text{(3.1)}
\]

Then if \( \{\alpha\} \in H^2(G, K^*) \), then \( \alpha \) is equivalent to a factor set such that

\[
\beta(x, y) = (\alpha^{(i)(x, y)})^{(\omega)} \cdot \cdots \cdot (\alpha^{(i)(x, y)})^{(\omega)} \quad 0 \leq \omega \leq \omega - 1
\]

\[
= (\zeta^{\omega})^{(x, y)} \cdots (\zeta^{\omega})^{(y, y)} \quad \text{(3.2)}
\]

Let \( A \) be an arbitrary finite abelian group such that

\( A \cong H^2(G, K^*) \) and let \( \alpha_i \in A \) correspond to the element

\( \alpha^{(i)} \in H^2(G, K^*) \). Let

\( \alpha(x, y) = \alpha_i \cdots \alpha_r \)

then it follows from (3.1) that

\( \alpha(x, y) \alpha(xy, z) = \alpha(x, yz) \alpha(y, z) \), \( x, y, z \in G \).

Let \( \chi \in \text{Hom}(A, K^*) \), define

\( \psi(\chi, y) = \chi(\alpha(x, y)) = \chi(a_i) \cdots \chi(a_r) \)

then \( \psi \chi \in H^2(G, K^*) \) and \( \chi(\alpha_i) \) is an \( \omega \) th root of unity.
Then as \( \chi \) runs through the characters of \( A \), then by (3.2) the \( \Psi_{\chi} \) run through all the factor systems of \( H^2(G, K^*) \).

Let \( G^* = \{ (\chi, a) \mid \chi \in G, a \in A \} \) and define
\[
(\chi, a)(\gamma, b) = (\chi \gamma, \alpha(\chi, \gamma) a \beta)
\]

Then, it is easily verified that \( G^* \) is a group such that
\[
\{ (1, a) \mid a \in A \} \cong A \text{ is contained in } Z(G^*) \text{. Further, let }
\nu(x) = x \text{ if } x \in G \text{ then } \{ \nu(x) \mid x \in G \} \text{ is a set of coset representation of } A \text{ in } G^* \text{ and }
\nu(x) \nu(y) = (\chi(x)) (y, 1) = (\chi, 1, \alpha(\chi, y)) \alpha(1, \alpha(x, y)) (x, y, 1)
\]
it follows that \( \frac{G^*}{A} \cong G \) and \( G^* \) is a central extension of \( G \).

Let \( T: G \rightarrow GL(V) \) be an arbitrary projective representation of \( G \) with factor set \( \alpha \). Then, by the above considerations, there exists a linear character \( \Psi \) of \( A \) such that
\[
\Psi(\alpha(x, y)) = \alpha(x, y)
\]
Define \( T^*: G^* \rightarrow GL(V) \) by
\[
T^*(\nu(x) a) = T(x) \Psi(\alpha)
\]
then
\( T^* \) is a linear representation of \( G^* \), that is, \( T \) is a linearizable representation. Thus \( G^* \) is a central extension of \( G \) such that every projective representation of \( G \) is linearized in \( G^* \) and 
\[
|G^*| = |H^2(G, K^*)| \| G \|
\]  
But Schur [45] also proved that the order of a representation group is \( \geq |H^2(G, K^*)| \| G \| \) and so \( G^* \) is a representation group of \( G \).
In [49], Schur also proved that the representation group of \( G^* \) is characterized by the following properties:  
1. \( A \subseteq G^* \cap Z(G^*) \)
2. \( \frac{G^*}{A} \cong G \)
3. \( |A| = |H^2(G, K^*)| \)

The representation group is not in general unique. Schur [50] obtained an upper bound for the number of non-isomorphic representation groups.

**Theorem 3.6.** The number of non-isomorphic representation groups of \( G \) is less than or equal to

\[
\prod_{\delta} (\xi_{\delta}, \eta_{\delta}).
\]

where \( \{\xi_{\delta}\} \) and \( \{\eta_{\delta}\} \) are the invariants of \( \frac{G}{G^*} \) and \( H^2(G, K^*) \) respectively. If \( G \) is a perfect group (i.e., \( G' = G \)), the equality sign holds. If \( (\frac{|G|}{|G^*|}, H^2(G, K^*)) = 1 \), then \( G \) has only one representation group.

For an alternative elementary proof of this theorem, see Asano and Shoda [4].

The proof of Theorem 3.4 is a purely existence proof and is of no value in the construction of a representation group. Schur [50] also gave a proof for the existence of a representation group which at the same time gives a general method for the construction of the representation group and Schur multiplier.
Theorem 3.6. Let \( F \) be a free group in \( n \) generators such that \( F/R \cong G \). Let \( H = \frac{F}{[R,F]} \) and \( N = \frac{R}{[R,F]} \). Then \( F \) is a finitely generated abelian group contained in the centre of \( H \). Let \( H_1 \) be the torsion group of \( H \), then \( H_1 = \frac{R \cap F}{[R,F]} \cong H^2(G, K^*) \) and rank \( \sim n \).

If \( F = H_1 \times N_2 \), then \( \frac{H}{N_2} \) is a representation group of \( G \).

For a modern proof of this theorem, see Suctt [24].

Other results were obtained by Schur and others which are useful in determining the Schur multiplier. These results, together with others involving projective representations are now presented without proof.

Theorem 3.7. (Schur [50]). The degree of every irreducible projective representation of \( G \) is a divisor of \( |G| \).

For a proof of this theorem see for example [11].

Theorem 3.8. (Schur [50]). The only primes that occur in \( |H^2(G, K^*)| \) are those which occur to at least the second power in \( |G| \). Thus, groups of square free order have trivial Schur multiplier.

Theorem 3.9. (Sctt [50]). Let \( S \) be a subgroup of \( G \) of index \( n \) and \( K^{(n)} \) denote the elements of \( H^2(G, K^*) \) of order relatively prime to \( n \). Then \( K^{(n)} \) is isomorphic to a subgroup of \( H^2(S, K^*) \).

Theorem 3.10. (Schur [50]). Let \( S \) be a \( p \)-Sylow subgroup of \( G \). Then a \( p \)-Sylow subgroup of \( H^2(G, K^*) \) is isomorphic to a subgroup of \( H^2(S, K^*) \). For an alternative proof and an improvement of this result see Kochendorfer [29].

This result is especially useful in proving that the Schur multiplier of a group is trivial.

Theorem 3.11. (Schur [51]). If \( G \) has \( n \) generators and \( r \) defining relations, then \( H^2(G, K^*) \) has \((r-n)\) generators. Thus if \( r-n \leq 0 \),
$H^2(G,K^*)$ is trivial.

**Definition 3.12.** Let $G$ and $H$ be arbitrary groups, then define

\[ P(G,H; K^*) = \left\{ f : G \times H \to K^* \mid f(g_1 g_2, h) = f(g_1, h) f(g_2, h) \right\} \]

\[ f(g_1, h_1, h_2) = f(g_1, h_1) f(g_1, h_2) \]

for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

If $f_1, f_2 \in P(G,H; K^*)$ define

\[ f_1 f_2 (g, h) = f_1 (g, h) f_2 (g, h). \]

Then $P(G,H; K^*)$ is an abelian group. This may be used to determine the Schur multiplier of a direct product of finite groups.

**Theorem 3.13.** If $G$ and $H$ are finite groups, then

\[ H^2(G \times H, K^*) \cong H^2(G, K^*) \times H^2(H, K^*) \times P(G,H; K^*) \]

This may be extended by induction to prove that if

\[ G = G_1 \times G_2 \times \cdots \times G_r, \]

then

\[ H^2(G, K^*) = \prod_{i=1}^r H^2(G_i, K^*) \times \prod_{1 \leq i < j \leq r} P(G_i, G_j; K^*) \]

4. **Projective Characters**

Throughout this section we shall assume that the factor set $M(G,K^*)$ is $\omega$-normalized, that is

(1) $\alpha(x, x^{-1}) = 1$, for all $x \in G$

(2) $\alpha(x, s) \alpha^{-1}(x s x^{-1}, x) = \alpha(x, s) \alpha(x s, x^{-1})$

= 1 for all $\alpha$-regular $s$ and for
Definition 4.1. Let $T$ be a projective representation of $G$ with factor set $\alpha$. Define

$$\zeta_\alpha(x) = \text{Trace } T(x) \quad \text{for all } x \in G$$

Then $\zeta_\alpha$ is called a projective character of $G$.

Lemma 4.2. $\zeta_\alpha$ is a class function on $G$.

Proof. We shall prove that

$$\zeta_\alpha(xsx^{-1}) = \zeta_\alpha(s) \quad \text{for all } \alpha\text{-regular } s \in G \text{ and for all } x \in G$$

and

$$\zeta_\alpha(s) = 0 \quad \text{if } s \text{ is not } \alpha\text{-regular}.$$ 

It is easily verified that for arbitrary $x \in G$ and $\alpha$-regular $s \in G$ 

$$T(x)T(s)T(x)^{-1} = \alpha(x,s)\alpha^{-4}(xsx^{-1},x)T(xsx^{-1})$$

$$= T(xsx^{-1}),$$

since $\alpha$ satisfies (ii) above. Thus, since equivalent matrices have the same traces, the first result follows.

If $s$ is not $\alpha$-regular, there exists $x \in C_{\alpha}(s)$ such that $\alpha(x,s) \neq \alpha(s,x)$. Since $xsx^{-1} = s$, 

$$T(s) = \alpha(x,s)\alpha^{-4}(s,x)T(s)$$

and taking traces implies that 

$$\zeta_\alpha(s) = \alpha(x,s)\alpha^{-4}(s,x)\zeta_\alpha(s)$$

which means that $\zeta_\alpha(s) = 0$. 
Lemma 4.3. Linearly equivalent projective representations have the same character.

Proof. Exercise.

Projective characters also satisfy the usual orthogonality relations.

Let \( T_1, \ldots, T_t \) form a complete set of irreducible projective representations with factor set \( \alpha \) with degrees \( f_1, \ldots, f_t \) and characters \( \gamma_1, \ldots, \gamma_t \) respectively. Let \( C = [\gamma_{rs}] \) an arbitrary \( f_i \times f_j \) matrix and

\[
P = \sum_{x \in G} Z_i(x) C Z_j(x^{-4})
\]

Then, it follows that

\[
Z_i(y) P = \sum_{x \in G} \alpha(y, x) Z_i(yx) C Z_j(x^{-4})
\]

\[
= \left\{ \sum_{x \in G} \alpha(y, y^{-1}x') \alpha^{-1}(x'^{-4}, y) Z_i(x') C Z_j(x'^{-4}) \right\} Z_j(y)
\]

\[
= P Z_j(y)
\]

if we assume that \( \alpha \) is normalized (i.e. \( \alpha(x, x^4) = 1 \) for all \( x \in G \)). Thus, we may apply Schur's lemma and obtain that \( P = 0 \) if \( i \neq j \) and \( P = \lambda I_{f_i} \) if \( i = j \). For fixed \( l, m \), put \( \gamma_{lm} = \lambda \), and \( \gamma_{rs} = 0 \) if \( \gamma, s \neq (l, m) \), then if \( Z_l(x) = [\omega_{l}(x)] \),

\[
\sum_{x \in G} \omega_{\gamma, k}(x) \omega_{\gamma, m}(x^{-4}) = 0
\]

and
\[
\sum_{x \in G} a^{(i)}_{\tau \ell}(x) a^{(j)}_{\mu \delta}(x^{-1}) = \frac{\delta_{\ell \mu}}{S_{\ell \mu}} \delta_{ij} \cdot 
\]

for all \((r,i,m,s)\). Put \(s=r\) and sum over \(s\), then
\[
\sum_{x \in G} \frac{\delta_{r \ell}}{S_{r \ell}} a^{(i)}_{\tau \ell}(x) a^{(j)}_{\mu \tau}(x^{-1}) = \frac{\delta_{ij}}{S_{ij}} \lambda_{\ell \mu} 
\]
and so
\[
\sum_{x \in G} a^{(i)}_{\tau \ell}(x) a^{(j)}_{\mu \delta}(x^{-1}) = \frac{|G|}{S_{ij}} \delta_{ij} \cdot 
\]
Now, put \(l=r\) and \(m=s\) and sum over \(r\) and \(s\) which gives
\[
\sum_{x \in G} \gamma_{ij}(x) \gamma_{ij}(x^{-1}) = S_{ij} |G| 
\]
which is the first orthogonality relation for projective characters.
Let \(\mathcal{K}_1, \ldots, \mathcal{K}_t\) denote the \(\alpha\)-regular classes of \(G\); since projective characters are class functions, define
\[
\gamma_{ij}^p = \gamma_i(x) 
\]
for any \(x \in \mathcal{K}_p\).
By Theorem 1.2, the class containing the inverses of the elements of \(\mathcal{K}_p\) is also \(\alpha\)-regular; denote this class as \(\mathcal{K}_p^*\). Then (4.1) can be rewritten as
\[
\sum_{p=1}^t \lambda_{ij} \gamma_{ij}^p \gamma_{ij}^{p*} = S_{ij} |G| 
\]
where $\lambda_\rho$ is the number of elements in the class $\mathcal{C}_\rho$. From this, the second orthogonality relation

$$\sum_{i=1}^{t} \zeta_i^\rho \zeta_i^{\sigma^*} = \frac{|G|}{\lambda_\rho} \delta_{\rho\sigma^*} \tag{4.2}$$

is obtained in the usual way. Over the complex field,

$$\zeta_i^{\rho^*} = \overline{\zeta_i^\rho}$$

As a consequence, we can prove

**Theorem 4.4.** Two projective representations $T_1$ and $T_2$ with factor set $\alpha$ are linearly equivalent if and only if they have the same projective character.

**Proof.** Exercise.

Let $H$ be a subgroup of $G$ and let $V$ be a left $(KG)_\alpha$-module, then $V$ may be regarded as a left $(KH)_\alpha$-module by restricting the domain of operators to the subalgebra $(KH)_\alpha$; in this case, we denote the module by $V_H$. If $V$ affords the projective representation $T$ and $V_H$ the projective representation $T_H$, then

$$T_H(h \cdot v) = T(v) \text{ for all } h \in H.$$ 

Let $\zeta$ denote the projective character of $T$ and $\zeta_H$ the projective character of $T_H$. If $\zeta$ is an irreducible character then in general $\zeta_H$ is a reducible character of $H$.

We now consider the inverse problem; let $W$ be a left $(KH)_\alpha$-module. Then

$$(KG)_\alpha \otimes_{KH_\alpha} W = W^G$$
\[ \zeta^G(q) = \sum_{i=1}^{t} \zeta(q^{-1} g \gamma_i) = \frac{1}{|H|} \sum_{t \in G} \zeta(q^{-1} g t), \]

where \( \zeta(x) = \zeta(x) \) if \( x \in H \)

= 0 if \( x \notin H \).

This conforms with the usual formula (see [11], p.266) for obtaining a class function on \( G \) from a class function on \( H \). Therefore, all the usual induced character theorems for linear characters are valid for projective characters without further proof. For example,

**Theorem 4.6.** (Frobenius Reciprocity Theorem). Let \( T_1, \ldots, T_t \) be a complete set of inequivalent projective representations of \( G \) with factor set \( \alpha \) and \( S_1, \ldots, S_s \) a complete set of inequivalent projective representations of \( H \) with factor set \( \alpha \). Let the character of \( T_i \) be \( \phi_i \) \((i=1, \ldots, t)\) and the projective character of \( S_j \) be \( \zeta_j \) \((i=1, \ldots, t)\) then if

\[ \phi_i \zeta_j = \sum_{j=1}^{s} \alpha_{ij} \zeta_j, \quad (i=1, \ldots, t) \]

then

\[ \zeta_j \phi_i = \sum_{i=1}^{t} \alpha_{ij} \phi_i, \quad (j=1, \ldots, s). \]
5. **Schur Multipliers and Projective Representation of Particular Groups.**

We shall concentrate on finite groups which are important in physics. Namely, cyclic groups, dihedral groups and symmetric groups.

**Example 1. Cyclic groups $\mathbb{K}_n$.**

Let $\mathbb{K}_n$ be a cyclic group of order $n$ generated by $x$. Then $x^n = e$ and by Theorem 3.11 it follows immediately that $H^2(\mathbb{K}_n, k^*) = E$. Alternatively, if we let $T$ be a projective representation of $\mathbb{K}_n$ of degree $m$ and factor set $\alpha$, then

$$T(x)^n = \mu(x) I_m,$$

where $\mu(x) = \alpha(x, x)\alpha(x^2, x)\ldots\alpha(x^{n-1}, x)$. Define

$$T'(x) = \frac{1}{(\mu(x))^k} T(x), \text{ for all } x \in G,$$

then

$$(T'(x))^n = I_m,$$

and so $T'$ is a linear representation of $\mathbb{K}_n$ which is projectively equivalent to $T$. That is, every projective representation of $\mathbb{K}_n$ is projectively equivalent to a linear representation of $\mathbb{K}_n$.

**Example 2. Dihedral Groups $D_{2n}$.**

$$D_{2n} = \{ x, y \mid x^n = e, y^2 = e, y^{-1} x y = x^{-1} \}$$

If $n$ is odd, $D_{2n}$ has only cyclic Sylow subgroups and thus $H^2(D_{2n}, k^*) = E$. If $n$ is even, then Schur showed that $D_{4n}$ is a
\[
T(S_{2i}) = \frac{\sqrt{2}}{\sqrt{2}} \left( \otimes_{i=1}^{i-1} (e \otimes \sigma \otimes \cdots \otimes \sigma) \otimes T \otimes \cdots \otimes T \right)
\]

then the relations (5.4) are easily verified. For a systematic construction of projective (spin) representations using Clifford Algebras, see Morris [36]. Schur [51] has completely determined the irreducible projective representations of the symmetric and alternating groups. Projective character tables for small values of \( n \) are available in Morris [36].

**Bibliography**

\[ P(xy) = P(x)P(y). \]

Thus, it follows that
\[
P(xy)P(x) = P(y),
\]
\[
P(xy)P(x)\{P(xy)\}^{-1} = P(y)\{P(xy)\}^{-1} = P(x)^{-1}.
\]

and raising this to the \( n \)th power gives \( \lambda^2 = 1 \) and so \( \lambda = \pm 1 \). If \( \lambda = 1 \) then \( P \) is a linear representation of \( H_{2n} \) and for a non-trivial projective representation \( \lambda \) must be equal to -1. If \( n \) is odd, let \( P'(x) = -P(x) \), \( P'(y) = P(y) \) and
\[
P'(xy) = -P(xy) = -P(x)P(y) = P'(x)P'(y)
\]
and then
\[
(P'(x))^n = (P'(y))^2 = (P'(xy))^2 = I_{2n}.
\]

and so \( P' \) is a linear representation. That is, if \( n \) is odd, \( H_{2n} \) has no non-trivial projective representation. If \( n \) is even, no further such simplification is possible and furthermore, if we put
\[
P(x) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \quad \text{and} \quad P(y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{where} \ \omega \ \text{is a primitive} \ 2n \ \text{th root of unity, then}
\]
\[
[P(x)]^n = -I_{2n}, \ P(y)^2 = I_{2n} \quad \text{and} \quad P(xy)^2 = I_{2n}.
\]

\( P \) is not equivalent to a linear representation as can be verified by comparing characters.
Example 3. Symmetric groups.

Let $S_n$ be the symmetric group of degree $n$. Then $S_n$ is generated by the transpositions $S_{i,i+1} = (i,i+1) \ (i=1,\ldots,n-1)$ with the defining relations

$$S_{i,i+1}^2 = e \quad (i=1,\ldots,n-1)$$

$$(S_i,S_{i+1})^3 = e \quad (i=1,\ldots,n-2)$$

$$S_i S_j = S_j S_i \quad (i+1 < j)$$

Schur [51] proves that $S_n$ has two representation groups $\Gamma_n$ and $\Gamma'_n$ of order $2(n!)$, where $\Gamma_n$ is generated by $f, t_i (i=1,\ldots,n-1)$ with defining relations

$$f^2 = e, \quad t_i^2 = f, \quad (t_i t_{i+1})^3 = f, \quad t_i t_j = f t_j t_i$$

and $\Gamma'_n$ is generated by $f, t'_i (i=1,\ldots,n-1)$ with defining relations

$$f^2 = e, \quad t'_i^2 = e, \quad (t'_i t'_{i+1})^3 = e, \quad t'_i t'_j = f t'_j t'_i$$

The Schur multiplier $\mathbb{Z}^2 (S_n, \mathbb{K}^*) \cong \mathbb{K}$.

Again in this case, we shall obtain the projective representations directly. Let $P$ be a projective representation of $S_n$ of degree $m$. Then

$$P(S_{i,i+1})^2 = \alpha_i I_m \quad (i=1,\ldots,n-1), \quad (5.1)$$

$$(P(S_i) P(S_{i+1}))^3 = \beta_i I_m \quad (i=1,\ldots,n-2), \quad (5.2)$$

$$P(S_i) P(S_j) = \gamma_{i,j} P(S_j) P(S_i) \quad (i < j+1) \quad (5.3)$$
From (5.3) it follows that
\[ P(S_{i,j}) P(S_{j,i}) P(S_{i,j})^{-1} = \gamma_{i,j} P(S_{j,i}), \]
and squaring both sides gives
\[ \gamma_{i,j}^2 = 1. \]

Let \( S_{i,j} = (i, i+1) \) and \( S_{j,i} = (j, j+1) \), where \( j > i+1 \). Then \( 1, i+1, j, j+1 \) are four distinct elements. Let \( S'_{i,j} = (i', i'+1) \), \( S'_{j,i} = (j', j'+1) \) and \( t \in S_n \) be defined by
\[ t = \left( \cdots, i'+1, \cdots, j', j'+1, \cdots \right) \]
then
\[ t^{-1} S_{i,j} t = S'_{i,j} \quad \text{and} \quad t^{-1} S_{j,i} t = S'_{j,i} \quad \text{and so} \]
\[ P(t)^{-1} P(S_{i,j}) P(t) = \lambda P(S'_{i,j}) \quad \text{and} \quad P(t)^{-1} P(S_{j,i}) P(t) = \mu P(S'_{j,i}), \]
where \( \lambda, \mu \in K^* \). From (5.3) it follows that
\[ P(t)^{-1} P(S_{i,j}) P(S'_{j,i}) P(t) = \gamma_{i,j} P(t)^{-1} P(S'_{j,i}) P(S'_{i,j}) P(t), \]
and
\[ \lambda \mu P(S'_{i,j}) P(S'_{j,i}) = \lambda \mu \gamma_{i,j} P(S'_{j,i}) P(S'_{i,j}). \]

Thus \( \gamma_{i,j} = \gamma_{i',j'} \) for all \( i, i', j, j' \in \{1, \ldots, r \} \). Let \( \gamma_{i,j} = c \), where \( c = \pm 1 \). From (5.3) it follows that
\[ T(S_{i,j}) T(S_{i+1}) T(S_{i,j}) = \beta_{i,j} T(S_{i,j})^{-1} T(S_{i+1}) T(S_{i,j})^{-1}, \]
and squaring both sides gives

$$\alpha_i^3 = \alpha_i \alpha_{i+1}.$$  

The $\alpha_i (i = 1, \ldots, n-1)$ in (5.1) can be chosen arbitrarily. Let $\alpha_i \in \{1, \ldots, n-2\}$, then $\beta_i = \pm 1 (i = 1, \ldots, n-2)$. Then define

$$T'(xi) = c \beta_1 \cdots \beta_{i-1} T(xi) \quad \text{if } i \text{ is even},$$

$$= \beta_1 \cdots \beta_{i-1} T(xi) \quad \text{if } i \text{ is odd}.$$  

Then, an elementary calculation shows that

$$T'(xi)^2 = c I_m, \quad [T'(xi)T'(xi+1)]^3 = c I_m,$$

$$T'(xi)T'(xi) = c T'(xi)T'(xi).$$ (5.4)

That is, every projective representation is projectively equivalent to a projective representation satisfying (5.4). If $c=1$, then $T'$ is a linear representation. A non-linear projective representation, if it exists will satisfy (5.4) with $c=-1$. We now give a construction for such a projective representation.

Let

$$\xi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then it is easily verified that

$$\rho^2 = \sigma^2 = \tau^2 = \xi, \quad \rho \sigma = -\sigma \rho, \quad \rho \tau = -\tau \rho, \quad \sigma \tau = -\tau \sigma.$$

Further, if $n = 2 \omega$ or $n = 2 \omega + 1$, set
representation group of $D_{2n}$ and $H^2(D_{2n}, K^*) \cong \mathbb{Z}_2$. Alternatively, the problem may be considered directly as in Frucht [13].

Let $P$ be a projective representation of degree $m$ with factor set $\alpha$ of $D_{2n}$. Then

$$(P(x))^\lambda = \lambda I_m, \quad (P(y))^\mu = \mu I_m \quad \text{and} \quad (P(xy))^\nu = \nu I_m$$

Clearly, $P$ may be replaced by a projectively, equivalent set projective representation $P'$ such that

$$P'(x^r y^s) = P'(x)^r P'(y)^s, \quad 0 \leq r \leq n-1, \quad 0 \leq s \leq 1.$$  

(that is, put $P'(x^r y^s) = \alpha(x^r, x) \cdots \alpha(x^{r-1}, x) \alpha(x^r, y^s) P(x^r y^s)$.)

In particular

$$P'(xy) = P'(x) P'(y).$$

Furthermore, let

$$P''(y) = \frac{1}{\mu} P'(y) \quad \text{and} \quad P''(xy) = \frac{1}{\nu} P'(xy),$$

then

$$(P''(y))^\mu = I_m \quad \text{and} \quad (P''(xy))^\nu = I_m.$$  

Thus, we may assume that our factor set has been chosen such that

$$P(x)^\lambda = \lambda I_m, \quad P(y)^\mu = \mu I_m, \quad P(xy)^\nu = \nu I_m \quad \text{and}$$

$$P(x^r y^s) = P(x)^r P(y)^s \quad \text{where} \quad r = 0, \ldots, n-1; \quad s = 0, 1$$

in particular


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GENERALIZED CLIFFORD ALGEBRA AND ITS APPLICATIONS

OR

A NEW APPROACH TO INTERNAL QUANTUM NUMBERS

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The confluence of relativity and quantum mechanics was achieved when Dirac just wrote down his famous equation in 1928. This achievement was made possible since he was able to construct four mutually anti-commuting matrices so that the Hamiltonian was consistent with the quadratic relativistic relation between energy momento and mass. Earlier, in non-relativistic quantum mechanics three mutually anti-commuting matrices were found sufficient to include the concept of intrinsic spin. It was immediately noticed by Dirac that the quantum mechanical concept of spin was also imbedded in his Hamiltonian. In the years of uninterrupted triumph that followed the birth of relativistic quantum mechanics, the study of the mathematical significance of the transition from Pauli to Dirac matrices was considered quite academic and therefore ignored. But it was obvious that it was still a live and unsolved problem since immediately after Dirac's formulation. Pauli attempted such a study and as late as 1956, Feynman himself raised the question of the relationship between spin and relativity in his famous Caltech lectures even after the total triumph of his graphical formalism in electrodynamics.
We therefore set as our objective the understanding of the mathematical procedure of obtaining Dirac matrices from the basis Pauli set. We thought it was just the right time now to take it up since the spirit of the hour demanded a re-examination of the whole structure from the point of view of mathematical rigour and logical precision. To our surprise we found that the procedure which Dirac used was of such general significance that it could be extended into a grammar of anti-commuting matrices, the ramifications of which give us a better insight into various branches of theoretical physics -- relativity, complementarity, propagator formalism and the fundamental concepts of spin and mass of elementary particles. Even more surprising was the possibility of enlarging the concept of anti-commutation to $\omega$-commutation where $\omega$ is a general root of unity.

This work was presented in a series of papers most of which were published in the Journal of Mathematical Analysis and Applications. We shall now present a rapid survey of this work at this conference.


Our starting point in the fundamental cyclic $n \times n$ matrix obtained by shifting in the $n \times n$ unit matrix $k$th row to the $(k-1)$th row, $k = 2, 3, \ldots, n$ and the first to the $n$th row.

i.e.

\[
C = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
& \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
\] (1)

where \( C \) operates on an \( n \times n \) matrix to the right, it just does this shifting operation.

We immediately note that

\[
C^n = I
\] (2)

i.e. \( C \) is one of the \( n \)-th roots of the unit matrix of dimension \( n \). The eigenvalues of the matrix \( C \) are the \( n \) roots of unity:

\[1, \omega, \omega^2, \ldots, \omega^{n-1}\]

being a primitive \( n \)-th root of unity. The corresponding eigenvectors can be placed together as columns forming a matrix which we call at the U-matrix. Hence

\[
U^{-1} CU = B
\]

where \( B \) is a diagonal matrix.

\[
B = \begin{bmatrix}
1 \\
\omega \\
\omega^2 \\
& \ddots \\
& & & \ddots \\
& & & & \omega^{n-1}
\end{bmatrix}
\] (3)
and

\[ U = \begin{bmatrix}
1 & i & \cdots & i \\
i & \omega & \cdots & \omega^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
i & \omega^{n-1} & \cdots & (\omega^{n-1})^{n-1}
\end{bmatrix} \tag{3a} \]

We immediately observe that \( B \) and \( C \) possess an interesting commutation relation:

\[ \omega B C = \omega BC \tag{4} \]

This we shall call the \( \omega \)-commutation relation which is the basis of the Generalized Clifford Algebra. It is an inexplicable fact in the history of matrix theory that this commutation relation has not been noticed till very recently.

We next observe that any \( n \times n \) matrix \( A \) can be expressed as a linear combination of the \( n^2 \) independent matrices each of the form \( B_k^l \); \( k, l = 0, 1, 2, \ldots, n-1 \), i.e.

\[ A = \sum_{k, l=0}^{n-1} a_{kl} B_k^l C^l. \tag{5} \]

Among these \( n^2 \) matrices there is a matrix \( CB \) such that these matrices \( (C, CB, B) \) have a mutual \( \omega \)-commutation relation. Denoting these as \( \Sigma_x', \Sigma_y', \Sigma_z' \) we find they constitute a generalization of the anticommuting Pauli matrices to the basis set of \( \omega \)-commuting matrices. It is to be noted that the \( \Sigma_x', \Sigma_y', \Sigma_z' \)
obey an ordered commutation relation.

\[\Sigma_x \Sigma_y = \omega \Sigma_y \Sigma_x\]
\[\Sigma_y \Sigma_z = \omega \Sigma_z \Sigma_y\]
\[\Sigma_x \Sigma_z = \omega \Sigma_z \Sigma_x\]  \hspace{1cm} (6)

The order being irrelevant for the case \(\omega = -1\).


The matrices \(B, B^2, \ldots, B^{n-1}\) commute as they are powers of the matrix \(B\). Calling the eigenvalues of \(B, B^2, \ldots, B^{n-1}\) as \(\eta_1, \eta_2, \ldots, \eta_{n-1}\) let us define a set of 'quantum numbers' \(s_1, s_2, \ldots, s_n\) as the following linear combinations of the

\[s_1 = \frac{1}{n} \{ \eta_1 + \eta_2 + \ldots + \eta_{n-1} \}\]
\[s_2 = \frac{1}{n} \{ \omega^{-1} \eta_1^2 + (\omega^{-1})^2 \eta_2 + \ldots + (\omega^{-1})^{n-1} \eta_{n-1} \}\]
\[\vdots\]
\[s_n = \frac{1}{n} \{ \omega \eta_1 + \omega^2 \eta_2 + \ldots + \omega^{n-1} \eta_{n-1} \}\]

These can be written as a vector-matrix equation as

\[\hat{\vec{s}} = \frac{1}{n} \vec{S} \vec{\eta}\]  \hspace{1cm} (7)

where

\[
\hat{\vec{s}} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \quad \text{and} \quad \vec{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-1} \end{bmatrix} \]  \hspace{1cm} (8)
and

\[ S = U \] (9)

Since the first set in \( \eta \) is set equal to zero, the choice of the first column in \( S \) is irrelevant, \( S \) is the well-known Sylvester matrix. We now notice that in the case of the quark, we identify the quantum numbers \( s_1, s_2, \) and \( s_3 \) as

\[
\begin{align*}
  s_1 &= Q \\
  s_2 &= Y - Q \\
  s_3 &= -Y
\end{align*}
\] (9a)

where \( Q \) and \( Y \) are the charge and hypercharge respectively. Moreover, the commuting matrices in Gell-Mann's formulation of SU(3) can be identified as a linear combination of the commuting matrices \( B, B^2 \) in the Generalized Clifford Algebra (G.C.A.) for \( \omega^3 = 1 \). The shift matrices of SU(3) algebra can also be expressed as linear combinations of products of \( B \) and \( C \). In other words, we assert now that the commuting generators of Lie algebra are the linear combinations of the commuting elements of the G.C.A. The same is true in the case of the shift operators.

We now proceed to define eigenvalues \( I_{k\ell} \) which are the differences between the \( s_k \) and \( s_\ell \), \( (k \neq \ell) \)

\[ I_{k\ell} = s_k - s_\ell \quad ; \quad k, \ell = 1, 2, \ldots, n \] (10)

For \( n = 3 \), we can identify \( I_{12}, I_{23} \) and \( I_{31} \) as the z-components of I spin, U spin and V spin in the language of SU(3).
We now also recognise that the equation (10) for 
\[ n = 3 \] as the Gell-Mann-Nishijima relation i.e.

\[ I_{12} = 2 I_z = s_1 - s_2 = 2 Q - Y \]
or \[ Q = I_z + \frac{Y}{2} \].

But eq. (10) can be applied to the SU(3) quarks, and thus constitutes the generalized Gell-Mann-Nishijima relations. What we have described now is the correspondence between the quantum numbers of the quarks in the language of the Lie and Clifford algebras.

If we consider physical particles as composed of quarks with \( \eta^i(j) \) denoting the \( j \)-th quantum number of the \( i \)-th particle then we can define the quantum number of the physical particle \( \eta_j \) as

\[ \eta(j) = \sum_i \eta^i(j) \] (11)

In a similar way, the composite s quantum number of a physical particle can be defined. The vector matrix relation given by eq. (8), still holds between \( s(j) \) and \( \eta(j) \).


The three Pauli matrices are the lowest dimensional matrices which obey anti-commutation relation. It has been shown by the author that to obtain a greater number of mutually anti-commuting matrices, we must increase the dimension of the matrices, more precisely there are \((2m+1)\) mutually anticommuting
matrices of dimension $\mathcal{C}^n$. Denoting them by $\chi_1, \chi_2, \ldots, \chi_{2m+1}$ and their linear combination as

$$L_{2m+1}^2 = \sum_{i=1}^{2m+1} \lambda_i \chi_i$$

(12)

where the $\lambda_i$'s are all real or all imaginary parameters we notice that

$$L_{2m+1}^2 = \Lambda_m^2 I = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{2m+1}^2)I.$$  

(13)

Setting $m = 2, \lambda_1 = p_x, \lambda_2 = p_y, \lambda_3 = p_z, \lambda_4 = 0, \lambda_5 = m$ the rest mass and $\Lambda_2 = E$, energy, and $\Lambda_1^2 = \text{helicity} \cdot |\vec{p}|$ we obtain the familiar features of the Dirac Hamiltonian.

A similar result holds even for the $\omega$-commuting matrices. It can be shown that there are $(2m+1)$ $\omega$-commuting matrices of dimension $\mathcal{C}^n$ where $\omega^n = 1$. It follows therefore that

$$\left( L_{2m+1} \right)^n = \left( \sum_{i=1}^{2m+1} \lambda_i \chi_i \right)^n = (\Lambda_m)^n I$$

(14)

with

$$\left( \Lambda_m \right)^n = \lambda_1^n + \lambda_2^n + \cdots + \lambda_{2m+1}^n$$

This constitutes a generalization of the Dirac procedure to $\omega$-commuting matrices and it is considered worthwhile to unravel the physical meaning of the sequence of eigenvalues.

$\Lambda_m(1), \Lambda_m(2), \ldots, \Lambda_m(n)$
where $\Lambda_m^{(i)} = \omega^i \Lambda_m^{(1)}$, $\omega$ being a primitive $n$-th root of unity and $i = 1, 2, \ldots n$.

The question now may be raised why do we need to invoke the G.C.A. in elementary particle physics. My answer is: the structure is too fundamental to be unnoticed, too consistent to be ignored and much too pretty to be without consequence.

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GENERALIZED SPINOR STRUCTURE

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We present the principal results of [10] and [11] on some grading classes associated to generalized Clifford algebras. We apply these results to construct some special bases of unitary Lie algebras (see also [12] for SU_q) and to define the generalized spinor structures. One establishes a prolongation theorem of structures [6] associated to space-times, by means of metric of Sasakian type [13], [15].

By an algebra over a field F we understand an associative algebra with unity 1 and of finite dimension over F. The field F is identified with F.

Let G be a finite Abelian group, \{e_a\}_{a \in G} a basis of the algebra A over F and \(\theta : G \times G \to F\). If the structure of A is given by

\[ e_a e_b = \theta(a, b) e_{a+b} \quad (a, b \in G), \]

then the triplet \(G = (G, \{e_a\}, \theta)\) is a maximal G-grading (or maximal grading) of A. There results immediately that \(e_0 (e \in G)\) and 1 are collinear and we always suppose \(e_0 = 1\). The equality

\( (e_a e_{-a}) e_a = e_a (e_{-a} e_a) \) implies \(\theta(a, -a) = \theta(-a, a)\).

The maximal grading G is normal if, for any \(a, b \in G\), we have

\[ \theta(a, b) \theta(b, a) = 1. \]

For such a grading

\[ \theta(a, -a) = \theta(-a, a) = \varepsilon_a = \pm 1 \]
for any \( a \in G \ (\xi_0 = 1) \).

Let \( M \) be the set of all maximal \( G \)-gradings of \( A \). Every system \( k = \{ k_a \}_{a \in G} \ (k_0 = 1) \) of non zero scalars defines an operator \( \overline{k} \) of \( M \) by

\[
(1') \quad \overline{k}(G) = (G, \{ k_a e_a \}, \theta') , \quad \theta'(a, b) = \frac{k_a k_b \theta(a, b)}{k_{a+b}} .
\]

Let \( M' \) be the set of all maximal \( G' \)-gradings of \( A \) (\( G \) is isomorphic to \( G' \)). Every isomorphism \( h : G' \rightarrow G \) defines an operator \( \overline{h} : M \rightarrow M' \) by

\[
(1'') \quad \overline{h}(G) = (G', \{ e_{h(a)} \}, \theta') , \quad \theta'(a, b) = \theta(h(a), h(b)) .
\]

Let \( G = (G, \{ e_a \}, \theta) \) and \( G' = (G', \{ e_{b}' \}, \theta') \) be two maximal gradings of \( A \) and \( A' \) respectively. If \( G = G' \) and \( \theta = \theta' \), then \( G \) and \( G' \) are isomorphic. If there exists an operator \( \overline{h} \cdot \overline{k} \) such that \( \overline{h} \cdot \overline{k}(G) \) and \( G' \) are isomorphic, then \( G \) and \( G' \) are equivalent (\( G \sim G' \)). In addition, if \( k_a = \pm 1 \ (a \in G) \), then \( G \) and \( G' \) are \( \xi \)-equivalent (\( G \xi G' \)). If \( G \sim G' \), then the algebras \( A \) and \( A' \) are isomorphic.

It is easily proved that \( \sim \) (resp. \( \xi \)) is an equivalence relation in the set of all maximal gradings (resp. normed maximal gradings) of the algebras over a field \( F \). Assuming \( G \xi G' \), it follows that \( G' \) is normed if and only if \( G \) is normed.

Let \( G_i = (G_i, \{ e_{z_i}^i \}, \theta_i) \) be a maximal grading of the algebras \( A_i \) over \( F \) \((i = 1, 2)\). Then \( A_1 \otimes_F A_2 \) has the maximal grading (the tensor product of \( G_1 \) with \( G_2 \)).
\[ G_1 \otimes G_2 = (G \times G, \{ e_1^a \otimes e_2^b \}, \theta), \]

where

\[ \theta((a_1, a_2), (b_1, b_2)) = \theta_1(a_1, b_1) \theta_2(a_2, b_2). \]

If \( G_1 \) and \( G_2 \) are normed, then \( G_1 \otimes G_2 \) is normed.

Let \( M_n(F) \) be the algebra of all \( n \times n \) matrices over the field \( F \). We suppose that \( F \) contains a primitive \( n \)-th root \( \omega \) of the unity. Then, from the representation theory of generalized Clifford algebras \([7]\), \([8]\), there results that \( M_n(F) \) is the polynomial algebra generated over \( F \) by the matrices \( e_1, e_2 \) subject to the relations

\[(3) \quad e_1^n = e_2^n = 1, \quad e_1 e_2 = \omega e_2 e_1. \]

Let \( Z_n \) be a cyclic group of period \( n \) and \( \alpha_1, \alpha_2 \) two generators of \( Z_n \times Z_n \). The elements of \( Z_n \times Z_n \) have the form

\[ a = a_1 \alpha_1 + a_2 \alpha_2, \quad b = b_1 \alpha_1 + b_2 \alpha_2, \ldots. \]

The matrices

\[(3') \quad e_a = e_1^{a_1} e_2^{a_2} \quad (a \in Z_n \times Z_n) \]

define a basis of \( M_n(F) \) and

\[(3'') \quad e_a e_b = \omega^{a_1 b_1} e_{a+b} \quad (a, b \in Z_n \times Z_n). \]

Hence \( \{ e_a \} \) is the basis of a \( Z_n \times Z_n \)-maximal grading \( A_n \) of \( M_n(F) \) and \( e_1, e_2 \) are called the generators of \( A_n \). It is easily proved that \( A_n \) is independent up to an equivalence of \( e_1, e_2 \) and \( \omega \).
If \( \zeta = \sqrt{\omega} \in \mathbb{F} \) we define an operator \( k \) given by the scalars \( k_a \) \( (a \in \mathbb{Z}_n \times \mathbb{Z}_n) \) as follows. If \( n \) is odd, then we take
\[
\zeta = \omega^{(p+1)/2}, \quad k_a = \zeta^{a_1 a_2}.
\]
If \( n \) is even we associate to every \( a \in \mathbb{Z}_n \times \mathbb{Z}_n \) the supplementary component \( a_3 \) given by \( a_1 + a_2 + a_3 \equiv 0 \pmod{n} \) and we take
\[
k_a = \zeta^{a_1 a_2 + a_3^2} \quad (0 \leq a_4 < n; \quad s, t = 1, 2, 3).
\]
In both cases we have:

\[(4) \quad k_a^2 = \omega^{a_1 a_2}, \quad k_a = k_{-a}.
\]

From relations (1'), (3') and (4) there results that the maximal grading \( A_n^0 = \overline{k}(A_n) \) is normed and the coefficients \( \xi_a \) defined by (2') are all equal to 1. The matrices \( e_1, e_2 \) are also called the generators of \( A_n^0 \). For \( n \) even we consider the operator \( k' \) given by the scalars
\[
k'_a = \zeta^{a_1} \quad (a \in \mathbb{Z}_n \times \mathbb{Z}_n), \quad 0 \leq a_4 < n.
\]
Then the algebra \( M_n(\mathbb{F}) \) has also normed maximal grading \( A_n^1 = \overline{k}'(A_n^0) \) for which

\[(5) \quad \xi_a = \begin{cases} 1 & \text{if } a_1 \equiv 0 \pmod{n} \\ -1 & \text{if } a_1 \not\equiv 0 \pmod{n} \end{cases}.
\]

The matrices \( e'_1 = \xi e_1 \) and \( e'_2 = e_2 \) are called the generators of \( A_n^1 \). The normed maximal gradings \( A_n^0 \) and \( A_n^1 \) are independent up to an \( \xi \)-equivalence of \( e_1, e_2, \omega \) and \( \xi \).
It is known that every Abelian group $G_n$ with $n$ elements ($n > 1$) has a representation of the form

\[(6) \quad G_n \cong \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_x} \quad (q_1 \ldots q_x = n),\]

where

\[(6') \quad q_d = p_i^{m_i} \quad (p_i \text{ prime; } d = 1, 2, \ldots, x).\]

The numbers $x$ and $q_d$ are invariants of $G_n$.

It is clearly that the algebra $M_n(F) (\omega \in F)$ has the maximal $G_n \times G_n$ - grading

\[(7) \quad G_n = A_{q_1} \oplus A_{q_2} \oplus \ldots \oplus A_{q_x}.\]

Two maximal gradings (7) are equivalent if and only if their grading groups are isomorphic.

**THEOREM 1.** Let $G$ be a maximal $G$-grading of a simple algebra $A$ over $F$. If the field $F$ has characteristic zero and contains the roots of all binomial equations over $F$, then $G \cong G_n \times G_n$, $A \cong M_n(F)$ and $G \cong G_n$.

The proof is given in [11] where a more general class of maximal gradings, which are also characterised in terms of generalized Clifford algebras, is constructed.

The algebra $M_n(F) (\omega \in F)$ has the normed maximal $G_n \times G_n$ - grading

\[(7') \quad G_n^o = A_{q_1}^o \oplus A_{q_2}^o \oplus \ldots \oplus A_{q_x}^o \quad (G_n^o \cong G_n).\]
Suppose that in (6), (6') we have

\[ p_1 = p_2 = \ldots = p_s = q, \quad p_{i+2} \quad \text{for } i > s \]

and that the sequence \( q_1, q_2, \ldots, q_s \) contains \( t = t(G_n) \) different terms \( q_1' > q_2' > \ldots > q_t' \). If \( s_j' \) terms are equal to \( q_j' \) (\( j = 1, \ldots, t \)), we may write

\[ q_1 = \ldots = q_{s_1} = q_1' > q_{s_1+1} = \ldots = q_{s_1+s_2} = q_2' > \ldots > q_{s} = q_t' \]

\[ q_{s-s_t+1} = \ldots = q_s = q_t' (s_1 + s_2 + \ldots + s_t = s). \]

The invariant \( t(G_n) \) is zero if and only if \( n \) is odd.

The algebra \( M_n(F) \) (\( n \) even, \( \mathcal{E} \in F \)) has the normed maximal \( G_n \times G_n \) grading

\[ G_n^j = A_{q_1}^{X_1} \otimes \ldots \otimes A_{q_s}^{X_s} \otimes A_{q_{s+1}}^{X_{s+1}} \otimes \ldots \otimes A_{q_t}^{X_t} \quad (q_n \sim_{G_n} \quad j = 1, \ldots, t). \]

Where

\[ x_i = \begin{cases} 1 & \text{if } i = s_1 + \ldots + s_j + 1 \\ 0 & \text{otherwise.} \end{cases} \]

Hence the relations (7'), (8), (8') associate to every group \( G_n \) the normed maximal gradings \( G_n^j, 0 \leq j \leq t(G_n) \). The gradings \( G_n^j \) and \( G_n^{j'} \) are \( \mathcal{E} \)-equivalent if and only if their grading groups are isomorphic and \( j = j' \).

The results established in [10] together with Theorem 1 imply

**THEOREM 2.** Let \( G \) be a normed maximal \( G \)-grading of the simple algebra \( A \) over the field \( F \) considered in Theorem 1. Then \( G \cong G_n \times G_n \), \( A \cong M_n(F) \) and \( G \cong G_n^j, 0 \leq j \leq t(G_n) \).
Let \( \mathcal{G} = (G, \{E_a\}, \theta) \) be a normed maximal grading of the algebra \( M_n(F) \). If the characteristic of \( F \) is not a divisor of \( n \), the following relation of Pauli type [10] holds

\[
\sum_{a \in \mathcal{G}} E^{i}_{ja} E^{ka}_j = n \delta^i_1 \delta^k_1 \quad (E^a = E_a E_{-a}) ,
\]

where \( E^{i}_{ja} \) and \( E^{ia}_j \) are the elements of \( i \)-th line, \( j \)-th row of matrices \( E^a \) and \( E^a \) respectively, or

\[
\sum_{a \in \mathcal{G}} E^{i}_{ja} E^{ia}_j = n \text{ tr} X \quad (X \text{ arbitrary in } M_n(F)) .
\]

From (9) it results that \( \{E_a\}_{a \neq 0} \) is a basis of \( M_n(F)' \) (subspace of \( M_n(F) \) of all matrices of null trace). Hence

\[
\text{tr}(E^a E_b) = n \delta^a_b \quad (a, b \text{ arbitrary in } G).
\]

Every pair \( (n, p) \) of integers \( (0 \leq p \leq \lfloor n/2 \rfloor) \) defines an antiinvolution \( X \to X^+ \) of the algebra \( M_n(C) \) (\( C \) complex field) by:

\[
X^+ = H_p X^* H_p , \quad H_p = \begin{pmatrix} I_{n-p} & 0 \\ 0 & -I_p \end{pmatrix} \quad (I_p \text{ unity of } M_p(C)),
\]

where \( X^* \) is the adjoint of \( X \).

We consider the real vector spaces

\( \mathcal{J}'(n, p) = \{ X \in M_n(C) : X = X^+ \} \), \( \mathcal{J}'(n, p)' = \mathcal{J}'(n, p) \cap M_n(C)' \)

and the real Lie groups

\( U(n, p) = \{ X \in SL(n, C) : XX^+ = \pm I \} \), \( U_0(n, p) = \{ X \in SL(n, C) : XX^+ = I \} \).

The space \( \mathcal{J}'(n, p) \) has a natural structure of real Jordan algebra. We have \( U(n, o) = U_0(n, o) = SU_n \). If \( n \) is odd, then \( U(n, p) = U_0(n, p) \).
The map $X \rightarrow \frac{1}{n} t \times X^2$ defines a real nondegenerated quadratic form $Q_p$ (resp. $Q_p'$) on $J(n,p)$ (resp. $J(n,p)'$). According to the relation $J(n,p) = J(n,p)' \oplus R \cdot 1$, we have $Q_p = Q_p' \oplus 1$, where $1 : x \rightarrow x^2 \ (x \in R)$. The index of $Q_p'$ is equal to $(n-2p)^2 - 1$. The forms $Q_o$ and $Q_o'$ are positive defined.

Let $\lambda(X)$ be the inner automorphism of $M_n(C)$ given by an arbitrary element $X \in SL(n,C)$ and let $\lambda$ be the $n:1$ representation $X \rightarrow \lambda(X)$. The group $\lambda(SL(n,C))$ is the group of all automorphisms of $M_n(C)$. It is easily proved that $J(n,p)$ is invariant to $\lambda(X)$ if and only if $X \in U(n,p)$. We denote by $\rho_p$ and $\rho_p'$ the $n:1$ representations of $U(n,p)$ induced by $\lambda$ on $J(n,p)$ and $J(n,p)'$, respectively. We have $\rho_p = \rho_p' \oplus 1$, where $1 : X \rightarrow I_n$ and $\rho_p'$ is equivalent with the adjoint representation of $U(n,p)$. On the other hand, $\rho_p(U(n,p)) \subset SO(Q_p)$ and $L(n,p) = \rho_p'(U(n,p)) \subset SO(Q_p')$.

We suppose that the normed maximal grading $G = (Q, [\varepsilon_a], \theta)$ of $M_n(C)$ verifies, for some $p$, the conditions

(11) $E_a^+ = E_a$ (a arbitrary in $G$).

(11') $E_a^- = \varepsilon_a E_a$

These conditions are invariant to every operator $\bar{h} \circ k$ where $k_a^{\pm 1}$ and $h$ is an arbitrary isomorphism of $C$. On the other hand, we can suppose [10] that the generators $e_1, e_2$ (resp. $e_1', e_2'$) of the normed maximal grading $A_n^0$ (resp. $A_n^0'$) verify the relations...
\[ e_i^* = e_i^{-1}, \quad e_i^+ = e_i, \quad (n-2p = \frac{3+(-1)^{n}}{2}) \]
\[ e_i^+ = e_i \quad (p = n/2). \]

Then, from (3), (3'), (4) there results that \( A_n^0 \) and \( A_n^1 \) verify (11), (11'). But according to (5), for \( A_n^1 \) we have \( E_a^2 = -1 \) \((a = \frac{n}{2} \alpha_i)\).

Hence for even \( n \), \( U(n, n/2) \neq U_0(n, n/2) \). We also obtain

**Theorem 3.** Every normed maximal grading \( G \) of \( M_n(C) \) verifies, up to an isomorphism, conditions (11) and (11'). The integer \( p = r(G) \) in (10) is uniquely determined by \( G \) and namely if \( G < G_n^j \), then
\[ p = n/2 \quad \text{for} \quad j < 0 \quad \text{and} \quad n-2p \quad \text{is the number of the elements of period 2 (zero, too) of} \quad G_n \quad \text{for} \quad j = 0. \]

Therefore \( \{E_a^j\}_{a \neq o} \) and \( \{iE_a^j\}_{a \neq o} \) \((i^2 = -1)\) are bases of the space \( J(n, p) \) and of the unitary Lie algebra \( U(n, p) \) respectively. We consider a total ordering relation \( \prec \) in \( G \) and we denote by \( G' = \{a \in G : -a < a, a \neq o\} \). Then
\[ \{E_a^j + E_a^{-j}, i(E_a^j - E_a^{-j})\}_{a \neq o}, \quad \text{and} \quad \{E_a^j, E_{-a}^j, i(E_a^j + E_{-a}^j)\}_{a \neq o}, \quad (E_a^j = E_aE_{-a}) \]
give two bases of the spaces \( J(n, o) \) and \( SU_n \) respectively. But we write the elements of \( J(n, o) \) under the form \( X = \sum_{a \neq o} a E_a \), where \( X_a = E_a X_a^{-1} \quad (a \in G) \).

The quadratic form \( Q_p \) and the isomorphism \( d\varphi_p \) are given in the basis \( \{E_a\} \) by
\[ Q_p(X) = \sum_{a \in G} E_a X_a X_a^{-1}, \quad X \in J(n, p) \]
\[ d\varphi_p(X_a) = [X, E_a], \quad X \in U(n, p)' \]

*) If \( L \) is a Lie group, we denote by \( L' \) its Lie algebra.

**) In [1,2] this result for \( SU_n \) is obtained, using the basis of the maximal grading \( A_n \).
Using relations (9), (9′) and (13), we obtain the explicit form for \( d\rho_p \) and \( d\rho_p^{-1} \):

\[
(13') \quad d\rho^i_p(X)^b_a = \frac{1}{n} \, t X (X \, [E_a^b, E_b^i]) , \\
\quad d\rho^{-1}_p(T) = \frac{1}{n} \sum_{a, b \neq 0} T_a^b E^a E_b
\]

which hold also for \( d\rho^o \) and \( d\rho^{-1}_o \).

We consider the quadratic form \( Q \) on \( V = \mathbb{R}^4 \) given by:

\[
Q(x) = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2
\]

and the following representation of \( O(Q) \):

\[
\mu = \bigoplus_{x = 0} \wedge^x \text{Id}, \quad \text{Id}: X \to X, \ \wedge^o \text{Id}: X \to 1 (X \in O(Q))
\]

having the exterior algebra \( \wedge V \) as representation space.

Choosing a system of generators \( \alpha_1, \ldots, \alpha_4 \) of the group \( \mathbb{Z}_2^4 \), we write its elements under the form \( a = \sum a_i \alpha_i \), \( b = \sum b_i \alpha_i \), \ldots where \( a_i, b_i \) are equal to 0 or 1 (\( i = 1, \ldots, 4 \)).

Let \( \Delta \) be the set given by the zero element of \( \mathbb{Z}_2^4 \) and all sequences \( (i_1, \ldots, i_r) \) where \( 1 \leq i_1 < \ldots < i_r \leq 4, \ r = 1, \ldots, 4 \). The \# \# \# \# map \( \delta: \alpha_1 + \ldots + \alpha_r \to (i_1, \ldots, i_r) \) establishes a 1:1 correspondence \( \mathbb{Z}_2^4 \to \Delta \).

Every orthonormal basis \( \{ e_i \} \) of \( V \) verifies in the Clifford algebra \( C(Q) \) the relations

\[
e_i e_j + e_j e_i = 2 \eta_i \delta_{ij} \quad (\eta_1 = \eta_2 = \eta_3 = -\eta_4 = -1)
\]
and defines a basis \( \{ e_a \} \) of \( C(\mathbb{A}) \), where
\[
e_a = e_{i_1} \cdots e_{i_r} \quad \text{with} \quad \delta(a) = (i_1, \ldots, i_r), \quad e_0 = 1.
\]

There is a vector space isomorphism \( \Lambda V \to C(\mathbb{A}) \) such that
\[
e_{i_1} \wedge \ldots \wedge e_{i_r} \to e_a \quad \text{for any orthonormal basis of} \ V. \quad \text{Using this isomorphism, we suppose that the representation } \mu \text{ acts also on } C(\mathbb{A}).
\]

Let \( h_o \) be the following automorphism of \( \mathbb{Z}_2^4 \)
\[
\alpha_1 \mapsto \alpha_1 + \alpha_2 + \alpha_3, \quad \alpha_2 \mapsto \alpha_4, \quad \alpha_3 \mapsto \alpha_4, \quad \alpha_4 \mapsto \alpha_2
\]
and \( d_o = h_o(A_2 \otimes A_2^\circ) = (\mathbb{Z}_2^4, \{ E_{i_1}^\circ, \theta \}). \) The elements \( e_i = E_{i_i} \)
verify the relations (16), and we can suppose that \( e_1 \) is an orthonormal basis of \( V. \) From Theorem 3 there results that (11) holds
for \( n = 4, p = 2 \) and according to (16) we have:
\[
(17) \quad E_a = n_a e_{i_1} \cdots e_{i_r}, \quad E_a = e(a, -a) = \eta_{i_1} \cdots \eta_{i_r}
\]
where
\[
(17') \quad n_a = \begin{cases} 1 & \text{if } r = 1 \text{ or } r = 4, \\ \sqrt{2} & \text{if } r = 2 \text{ or } r = 3, \end{cases} \quad \delta(a) = (i_1, \ldots, i_r).
\]

Therefore we can suppose that \( \mu \) acts also on the space \( \mathcal{U}(4, 2) \)
and we denote by \( \mu' \) the representation induced by \( \mu \) on \( \mathcal{U}(4, 2)' \)

Any \( A \in \mathcal{O}(\mathbb{A}) \) defines an orthonormal basis \( e'_i = Ae_i \) of \( V \)
and a map
\[
T : E_a \to E'_a = n_a e'_{i_1} \cdots e'_{i_r}, \quad T \in L(4, 2).
\]
Since $E'_a = \mu'(A) E_a$, there results that $\mu'(A) = T$ and we denote by $S$ the subgroup of $U(4, 2)$ given by $\varphi_2(S) = \mu(O(\mathfrak{g}))$. The map $\xi : X \to A$, $X \in \varphi_2^{-1}(\mu(A))$ is a $4 : 1$ representation of the group $S$ on $O(\mathfrak{g})$ and we have:

$$(18) \quad \varphi_2(X) = \mu(\xi \Phi), \quad \varphi'_2(X) = \mu'(\xi'(X)) \quad (X \text{ arbitrary in } S)$$

Hence $\varphi_2$ induces on the space $V$ the representation $\xi$ and we can write:

$$(19) \quad d\xi(X)_{ij} = \frac{1}{4} \operatorname{tr}(X [e_i, e_j]), \quad d\xi^{-1}(A) = \frac{1}{4} A_{ij} e_i^a e_j^a (e_i^a = \eta_i e_i^a).$$

Therefore $\xi^{-1}$ is locally isomorphic to the spinor representation of the Lorentz group $O(\mathfrak{g})$. Similarly, a representation of the group $O(4, \mathbb{R})$ can be obtained, by means of the normed maximal grading $\bar{H}_0(A^0 \otimes A^0)$. From (13') and (19) there results that the representation $d\varphi^{-1}_P$ has, in the basis $\{E_a\}$, the expression as the spinor representation of $O(\mathfrak{g})$.

Let $M$ be a space-time which is, or is not, orientable and let $P = P(M, O(\mathfrak{g}))$ be the principal fibre bundle of orthonormal frames over $M$. A spinor structure of $M$ [6] is a principal fibre bundle homomorphism $f$:

$$(\Sigma, \mathcal{G}) \to P(M, O(\mathfrak{g}))$$

with the corresponding structure group homomorphism $\xi : \mathcal{G} \to O(\mathfrak{g})$ such that the induced map on $M$ is the identity map$^*$. For every principal fibre bundle we suppose that the structure group acts freely on the total space. Hence the above map $f$ is a $4 : 1$ map.
The existence conditions of spinor structures defined by replacing $\mathcal{G}$ by the natural 2:1 representation $\text{Pin}(\mathcal{G}) \to O(\mathcal{G})$ are given in [2] and [3]. The same problem for the covering map of the connected Lorentz group $\text{SL}(2, \mathbb{C}) \to \text{L}_4$ is considered in [4].

Let $G$ be a normed maximal grading of $M_n(\mathbb{C})$ and let $M$ be a Riemannian manifold of dimension $n^2 - 1$ and index $(n-2p)^2 - 1$, such that the principal fibre bundle of orthonormal frames over $M$ has a reduction $P \simeq L(n,p) = \mathcal{P}_p(U(n,p)) \subset O(\mathbb{C}_p)$. If $p \neq 0$ we suppose that $P = P(G)$. A generalized spinor structure of $M$ associated to $G$ is a principal fibre bundle homomorphism $f: \Sigma \to P(M, U(n,p)) \to P(M, L(n,p))$ with the corresponding structure group homomorphism $\rho'_P: U(n,p) \to L(n,p)$ such that the induced map on $M$ is the identity map.

We can study the spinor and spin-tensor fields both for generalized spinor structures and usual spinor structures. The relations (13') and (19) imply similar expressions of the spinor structure covariant derivatives in both cases.

**Theorem 4.** Let $f: \Sigma \to P$ be a spinor structure of the space-time $M$. There exists a Riemannian manifold $M'$ of dimension $15$ and index $-1$ a projection $\Pi$ of $M'$ onto $M$ and a generalized spinor structure $f': \Sigma' \to P'$ of $M'$ associated to the normed maximal grading $G_0$ ($\Sigma'$ reducible to $G$) such that the transition functions $\psi_{\alpha_\beta}$ and $\psi'_{\alpha_\beta}$ of the principal fibre bundles $\Sigma$ and $\Sigma'$ respectively verify the condition

$$\psi_{\alpha_\beta} = \psi'_{\alpha_\beta} \circ \Pi.$$
Proof. Let \( V \) be a \( n \)-dimensional vector space and let \( B \) be a symmetric nondegenerated bilinear form on \( V \). There exists a unique symmetric nondegenerated bilinear form \( B' \) on \( \bigotimes \bigotimes V \) such that:

\[
B' \left( X_1 \otimes \ldots \otimes X_r, Y_1 \otimes \ldots \otimes Y_r \right) = B(X_1, Y_1) \ldots B(X_r, Y_r)
\]

for all \( X_1, \ldots, X_r, Y_1, \ldots, Y_r \in V \). We denote by \( B'_r \) the restriction of \( \frac{1}{r!} B' \) to \( \bigwedge^r V \). If for the basis \( \{e_i\} \) we have \( B(e_i, e_j) = k_i \delta_{ij} \) then

\[
B'_r(e_{i_1} \wedge \ldots \wedge e_{i_r}, e_{j_1} \wedge \ldots \wedge e_{j_r}) = k_{i_1} \ldots k_{i_r} \delta_{i_1i_2} \ldots \delta_{i_rj_r}.
\]

We associate to any point \( x \) of the space-time \( M \) the vector space \( F_x = \bigoplus_{i=2}^4 \bigwedge^i T_x \), where \( T_x = T_x(M) \) is the tangent space of \( M \) at \( x \). The metric \( g \) of \( M \) induces on each space \( F_x \) the bilinear form \( g' = g_2' \oplus g_3' + g_4' \).

Let \( M' \) be the vector bundle over \( M \) of all pairs \((x, y)\) where \( x \in M \), \( y \in F_x \) and let \( \Pi: M' \to M \) be the natural projection \((x, y) \to x\). The Riemannian connection of \( M \) associates to each point \( u \in M' \) the horizontal space \( H_u \) and the vertical space \( V_u = T_u(F_x) \) with \( x = \Pi(u) \) such that \( T_u(M') = H_u \oplus V_u \). Let \( h \) be the isomorphism which carries each vector \( X \in T_x(M) \) in its horizontal lift \( hX \in H_u \) and let \( v \) be the translation \( F_x \to V_u \). The manifold \( M' \) has Saskian metric of index -1

\[
\mathcal{A} = g \circ (h^{-1} x h^{-1}) \oplus g' \circ (v^{-1} x v^{-1})
\]
Let \( \{ U_\alpha \} \) be an open covering of \( M \) such that on every \( U_\alpha \) there exists a field \( \{ \theta_i \} \) of orthonormal frames with respect to \( g \). If \( \{ \theta_i' \} \) is the field of orthonormal frames on \( U_\beta \), then the transition functions of \( P \) are given by:

\[
\theta_i'(x) = \psi_\alpha(x) \theta_i \psi_\beta(x) \quad , \quad x \in U_\alpha \cap U_\beta
\]

and we can suppose that

\[
\psi_\alpha = \theta_0^* \psi_\alpha.
\]

From (12), (17), (17') and (22) there follows that the field of frames \( \{ \psi_a \} \) \( (a \in \mathbb{Z}_2^4 \ , \ a \neq 0) \) on \( V_\alpha = \pi^{-1}(U_\alpha) \) given by:

\[
\psi_a = \begin{cases} 
\theta_i \quad (\delta_a = \theta_i \circ \pi) & \text{if } \delta(a) = i \\
\psi(\beta \_ \_ \_ \_ r) & \text{if } \delta(a) = i_1 \_ \_ \_ i_r, r \geq 2
\end{cases}
\]

and the similar field \( \{ \psi_a' \} \) on \( V_\beta \) are orthonormal with respect to the metric (21) and we have:

\[
\psi_a'(u) = \mu^\prime (\psi_{a \beta}(x)) b \psi_b(u) \quad (u \in V_\alpha \cap V_\beta , x \in \pi(u)).
\]

Hence the principal fibre bundle of orthonormal frames over \( M' \) has a reduction \( P' \) to \( L(4,3) \) whose transition functions are

\[
\psi_a' = \mu^\prime \circ \psi_{a \beta} \circ \pi.
\]
Relations (18), (23) and (23') imply

\[(23') \quad \varphi_{\alpha\beta}^i = \varphi_2^i \circ \psi_{\alpha\beta} \circ \Pi.\]

We consider \( \Sigma' \), the principal fibre bundle whose transition functions are given by (29) and the homomorphism \( f' : \Sigma' \to \mathcal{P}' \) which preserves the conditions \( \varphi_{\alpha\beta}^i = \rho_2^i \circ \varphi_{\alpha\beta}. \)

Q.E.D.

A similar theorem for the normed maximal grading \( \Gamma_0 (A_2^0 \otimes A_2^0) \) and Riemannian manifolds of dimension 4 and index zero holds.

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