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LECTURES ON RELATIVISTIC PHYSICS IN ONE SPACE AND ONE TIME DIMENSION

By

DANIEL A. DUBIN

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20. (INDIA).

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by
Daniel A. Dubin,
Visiting Scientist, MATSCIENCE(+)

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LECTURES ON RELATIVISTIC PHYSICS IN ONE SPACE AND ONE TIME DIMENSION

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Introduction:-

These lectures cover certain aspects of two dimensional physics. They are by no means exhaustive and must be supplemented by the comprehensive treatment of Wightman's Cargese lecture notes, 1964.

We do two dimensional physics for various reasons. It seems difficult to accept that the four dimensional case won't be significantly harder, so this is a start. One finds that these models already solved are trivial, in that there is no scattering theory, i.e., $S \equiv 1$. The exception to $S \equiv 1$ is the Federbush model, but here the cross sections are energy independent. The mathematical structure of the class of models we solve have a certain intrinsic interest in themselves. The structure is within the Wightman framework up to questions of positive definiteness. We hope to gain a certain proficiency in field theory, a knowledge of phenomena hidden or obscured by perturbation theory, that will be helpful in realistic cases. The final reason is subjective: the theories are there and can be solved exactly, so do them.

In particular we cover the fairly trivial Lorentz and Poincare groups and then classify the relevant free fields in terms of scalar fields. Then we consider our models. Details and proofs are often omitted, references are often not cited and there might well be mistakes, for which the author takes full responsibility.

PART I: THE REAL LORENTZ GROUP

A. INTRODUCTION

DEF. 1. The real Lorentz group, $\mathcal{L}(\phi)$, is the one parameter matrix group that leaves invariant the quadratic form with G:

$$L \in \mathcal{L}(\phi) \text{ iff } L^T G L = G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} g^{00} & g^{01} \\ g^{10} & g^{11} \end{pmatrix} \tag{1}$$

where $-\infty < \phi = \tanh^{-1}(v/c) < +\infty$ is the velocity parameter we use to parametrize $\mathcal{L}$. Let $\mathcal{L}_+^\uparrow$ be the identity connected component of $\mathcal{L}$. Then $\mathcal{L}_+^\uparrow$ must map the co-ordinate vector, x, into x_ϕ:

$$\phi: x \to x_\phi = (e^{-\phi} x^+ , e^{+\phi} x^-) ; \forall \phi \in \mathcal{L}_+^\uparrow \tag{2}$$

Here $x = (x^+, x^-)$ and $x^\pm = x^0 \pm x^1$, the most useful basis.

DEF. 2. Let $P: x \to (x^-, x^+)$ and $T: x \to -(x^-, x^+)$. Then $\mathcal{L}$ decomposes into

$$\mathcal{L} = (\mathcal{L}_+^\uparrow) \cup (P\mathcal{L}_+^\uparrow) \cup (T\mathcal{L}_+^\uparrow) \cup (PT\mathcal{L}_+^\uparrow) \tag{3}$$

Lemma 1: Let $\mathbb{R}$ be the real line with its usual topology. Then as a group, $\mathcal{L}_+^\uparrow$ is isomorphic to the additive group of integers. With the parametrization of Equation (2), as a topological group

$$\mathcal{L}_+^\uparrow \simeq \mathbb{R} \tag{4}$$

The proof should be obvious. We note that as $\mathbb{R}$ is its own universal covering group we shall not have to deal with multi-valued representations as in higher dimensions.

B. Representations of $\mathcal{L}_+^\uparrow$.

We consider only a certain class of representations of $\mathcal{L}_+^\uparrow$. As all the representations of $\mathcal{L}_+^\uparrow$ are well known(1) we shall prove none of the assertions in this section.

DEF.3: For brevity we shall mean by representation the faithful, irreducible, continuous representations. Further, we shall restrict ourselves to representations with real-valued indices. Then, our representation spaces, $\{V_\lambda : \lambda \in R,\ v_\lambda$ spans $V_\lambda\}$ are all copies of a one dimensional vector space V, over C. $\mathcal{L}_+^\uparrow$ is to act on V_λ through the relation

$$\phi : v_\lambda \rightarrow (v_\lambda)_\phi = e^{\lambda\phi} v_\lambda \,, \quad \forall \phi \in \mathcal{L}_+^\uparrow \tag{5}$$

The concepts of tensor and spinor fields are artificial in two dimensions. We shall first define them and then motivate the definition.

DEF.4: A λ-vector field is a field taking values in $V_\lambda \oplus V_{-\lambda}$. A λ-spinor field is a field taking values in $V_\lambda \oplus V_{\lambda+1}$.

The motivation for λ-vector fields is simple. Firstly, the co-ordinate vector is a 1-vector and secondly. the tensor product of λ-vectors reduces to the direct sum of σ-vectors, $0 \leq \sigma \leq \lambda$, The motivation for λ -spinors is tied up to the classical Dirac equation, as no "spin" exists in two dimensions, i.e., contains no rotation subgroup. The solutions of the Dirac equation are λ -spinors and their tensor product reduces to the direct sum of σ -spinors, $0 \leq \sigma \leq \lambda$. The Dirac equation is defined just below in definition 5.

DEF.5: Let γ^0, γ^1, M be non-singular $n \times n$ matrices and let f be a field taking values in $\sum_{j=1}^{n} \oplus V_{\lambda_j}$. The Dirac equation is the first order partial differential equation

$$\sum_{b=1}^{n} \left[i \left(\gamma^0 \frac{\partial}{\partial x_0} - \gamma^1 \frac{\partial}{\partial x_1} \right) + M \right]_{ab} f_b(x) = 0 \tag{6}$$

which implies the second order Klein-Gordon equation

$$(\Box - m^2) f_b = \left[\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - m^2 \right] f_b = 0 \tag{7}$$

Here m is a parameter relating to the orbits of the Poincare group below and has the physical significance of particle mass.

Lemma 2: The matrices $\gamma^0, \gamma^1, M \equiv me$ and γ^5 satisfy the multiplication table

	e	γ^0	γ^1	γ^5
e	e	γ^0	γ^1	γ^5
γ^0	γ^0	e	γ^5	γ^1
γ^1	γ^1	$-\gamma^5$	$-e$	γ^0
γ^5	γ^5	$-\gamma^1$	$-\gamma^0$	e

and generate a Clifford algebra. The lowest rank irreducible representation of this algebra is for n=2. A particular matrix representation for n=2 is

$$e=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \gamma^1=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\ \gamma^5=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad (8)$$

The proof of this lemma follows the well known analogous calculation in four dimension and is omitted.

THE POINCARE GROUP

A. The two dimensional translation group, T_2 .

DEF.6: $T_2 = \{a = (a^+, a^-) \in V_1 \oplus V_{-1}\}$. As a topological group, $T_2 \simeq \mathbb{R}_2$.

The representations of interest for the Poincare group are given by the characters of T_2 ,continuous unimodular maps from $\mathbb{R}_2 \times \mathbb{R}_2 \to R$:

$$\chi(p,a) = \exp[\, i \langle p,a\rangle] \quad (9)$$

Here

$$\langle\, ,\rangle : \mathbb{R}_2 \times \mathbb{R}_2 \to R$$
$$\langle p,a\rangle = \tfrac{1}{2}(p^+a^- + p^-a^+) \quad (10)$$

and has the property of being Lorentz invariant, i.e., invariant under $\mathcal{L}_+^{\uparrow}$ if p takes values in $V_1 \oplus V_{-1}$.

B. The Poincare group $\mathcal{P}$.

DEF.7: $\mathcal{P} = \mathcal{L} \circledS T_2 \quad , \quad \mathcal{P}_+^{\uparrow} = \mathcal{L}_+^{\uparrow} \circledS T_2$ (11)

Then semidirect composition law is

$$(a,\phi)(b,\psi) = (a+(b)_\phi, \phi+\psi), \quad a,b \in T_2, \quad \phi,\psi \in \mathcal{L} \tag{12}$$

Equation (12) is well defined, as $(b)_\phi$ is known.

Lemma 3: The orbits[2] of $\mathcal{P}_+^{\uparrow}$ are each contained in one of the solutions of

$$(p^+)(p^-) = c \quad , \quad c \in R, \; p = (p^+, p^-) \in \mathbb{R}_2 \tag{13}$$

of course p is a 1-vector field. The orbits are nine in number and fall into four classes. We list the orbits below by giving a convenient arc-length parametrization for the values of p for each orbit. Let $\lambda \in R$, $\rho \in R^*$. Then

CLASS 1. $O_1 = \{\sqrt{c}\,(e^{\lambda}, e^{-\lambda})\}$ (14)

$O_2 = \{-\sqrt{c}\,(e^{\lambda}, e^{-\lambda})\}$

2. $O_3 = \{+\sqrt{-c}\,(e^{\lambda}, -e^{-\lambda})\}$

$O_4 = \{+\sqrt{-c}\,(-e^{+\lambda}, e^{-\lambda})\}$

3. $O_5 = \{(\rho, 0)\}, \; O_6 = \{(-\rho, 0)\}, \; O_7 = \{(0, \rho)\}, \; O_8 = \{(0, -\rho)\}$

4. $O_9 = \phi = \cancel{\&}\,\{(0,0)\}$.

This lemma is the two dimensional analogue of the four dimensional lemma proven by Mackey[2]. Its proof is therefore omitted, as is the proof of Lemma 4 for the same reason.

Lemma 4: The homogeneous little groups (stability groups) corresponding to the nine orbits are

$$H_j = \Phi \; ; \; j = 1, 2, \cdots, 8 \; ; \; H_9 = \mathcal{L}_+^{\uparrow} \tag{15}$$

The orbital diagram is given in Figure 1:

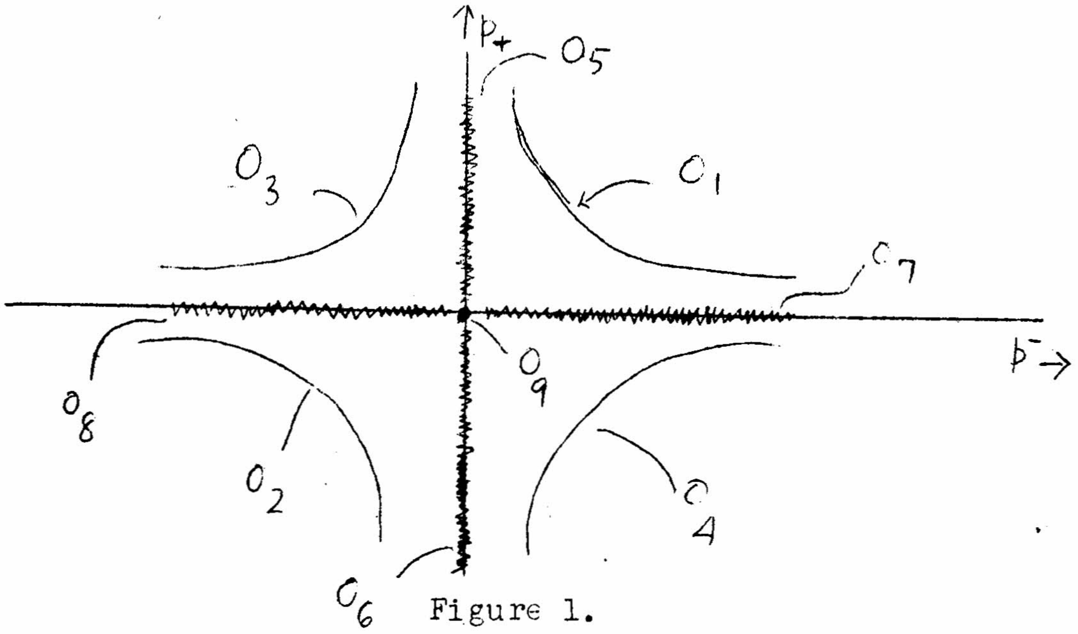

Figure 1.

In inducing from the characters of T_2 to the unitary representations of $\mathcal{P}_+^{\uparrow}$, the covariance condition is automatically trivially satisfied, since H_j is either Φ or $\mathcal{L}_+^{\uparrow}$.

C. <u>The unitary representations of</u> $\mathcal{P}_+^\uparrow$ (2)

The unitary representations are discussed by orbits. First we consider orbit O_1. Let $d\mu_1$ be the $\mathcal{L}_+^\uparrow$ invariant measure concentrated on O_1:

$$d\mu_1(p) = 2\,\delta(p^+p^- - m^2)\,\theta(p^+ + p^-)\,dp^+\,dp^- \tag{16}$$

Here $m^2 > 0$ is the square of the particle mass. Let F,G, $\in \mathbb{L}(d\mu_1,(\,,\,))$. Then

$$(F,G) = \int F(p)\,\overline{G(p)}\,d\mu_1(p) \tag{17}$$

Alternately, let $f_m, g_n \in L_2(d\lambda,(\,,\,))$, so that

$$(f_m, g_n)_1 = \int_{-\infty}^{+\infty} f_m(\lambda)\,\overline{g_n(\lambda)}\,d\lambda \tag{18}$$

Then if $\sqrt{2m}\, f_m(\lambda) = F(me^{\lambda}, me^{-\lambda})$, similarly for g_m and G,

$$(f_m, g_m)_1 = (F,G),\quad \mathbb{L}_2(d\mu_1,(\,,\,)) \simeq \mathbb{L}_2(d\lambda,(\,,\,)_1) \tag{19}$$

Since $H_1 = \phi$, the infinite dimensional unitary representation of $(a,\phi) \in \mathcal{P}_+^\uparrow$ acting on the representation space $\mathbb{L}_2(d\lambda,(\,,\,)_1)$ can be explicitly written

$$(a,\phi) \to U(a,\phi) : f_m(\lambda) \to U(a,\phi)\, f_m(\lambda)$$
$$= \exp\left\{\left(\frac{im}{2}\right)\left[a^+ e^{-(\lambda+\phi)} + a^- e^{+(\lambda+\phi)}\right]\right\} f_m(\lambda+\phi) \tag{20}$$

a similar construction works for $O_2 - O_4$ which correspond to non-positive masses and are thereby excluded on physical grounds.

Next we discuss the orbit O_5. This will lead us to the concept of infraparticles, i.e., an essential indefinite metric will appear. The obvious procedure might be to consider the invariant measure concentrated on O_5, O_6 and O_9,

$$d\Omega_{Pos}(p) = 2\delta(p^0 - |p^1|)\,|p^1|^{-1}\,dp^0 dp^1 \tag{21}$$

or perhaps even $d\Omega_{Pos}(p)\,\Theta(p^+)$. This will introduce a problem because of the point $p=0$. These above measures are positive, but will be too singular, except when restricted to smearing with a set of test functions, smaller than $\mathcal{S}$, e.g., $\mathcal{S}_0 = \{f \in \mathcal{S} : f(p=0) = 0\} \subset \mathcal{S}$. But such a set, the largest allowed within the Wightman framework, is too small to be useful. This can be seen as follows. In the familiar Fock representation for the free massless scalar field, ϕ_{Pos}, represented via Ω_{Pos}, we would have

$$(\Omega_0, \phi_{Pos}(\tilde{f})\,\phi_{Pos}(\tilde{g})\,\Omega_0) \sim \int f(p)\,\overline{g(-p)}\,d\Omega_{Pos}(p) \tag{22}$$

The tilde denotes the fourier transform and $f, g \in \mathcal{S}_0$. This restriction on f, g does not suffice to define the wick ordered cubic fields, as was shown by Tarski.

But it is characteristic of two dimensions that in the massive case, $\sum_{n=0}^{\infty} C_n :\phi^n:(f)$ is a field if $\phi(f)$ is, and if $\sum_{n=0}^{\infty} C_n z^n$ is entire in z. Also characteristic of two dimensions is that the models we can solve, use: $:e^{ig\phi(f)}: = e^{ig\phi^+(f)} e^{ig\phi^-(f)}$, where ϕ is massless, in the solution construct.

Hence we must have massless fields we can exponentiate. We do this by sacrificing positivity. We shall use the word field in the extended non positive sense (i.e., infraparticles) in what follows.

Write

$$
\begin{aligned}
(f,g)_5 &= \int \bar{f}(u)\, g(u)\, d\mu_5(u) \\
&= -\int_0^\infty du \, \ln u \, \frac{d}{du}\left[\bar{f}(u)\, g(u)\right]
\end{aligned}
\tag{23}
$$

where $f, g \in \mathbb{L}_2(\mu_5)$ and we adjoin the conditions $\lim_{u\to+0} f(u)$ exists, $\lim_{u\to 0} f'(u)$ exists and similarly for g. We have the following theorem

<u>Theorem 1</u>: Let f, g satisfy the conditions above, $d\mu_5$ be as in Equation (23). Then $\mathbb{L}_2(d\mu_5, (\,,\,)_5)$ is an incomplete Pontriagin space $\tilde{\Pi}_1$, which can be completed (to Π_1, a complete Pontriagin space)

Briefly, a Pontriagin space Π_K is a vector space of infinite dimension which is equipped with a non-degenerate; Hermitian form. Π_K contains at least one subspace of dimension $K < \infty$ on which the form is negative-definite, but not such $K+1$ dimensional subspace. There also exists at least one orthogonal direct sum decomposition $\Pi_K = \Pi_- \oplus \Pi_+$ for which Π_+ is a Hilbert space. Details and the proof of the theorem can be found in Reference (3).

For orbit O_5, the infinite dimensional unitary representation of $P_+^\uparrow$ we choose is

$$
\begin{aligned}
(a,\phi) \to U(a,\phi) : f(u) \to U(a,\phi) f(u) \\
= \exp\left(\tfrac{i}{2} a^- u\, e^{\phi}\right) f(u e^{\phi})
\end{aligned}
\tag{24}
$$

This normalizes the choice of massless free field we choose to use. as there is a logrithmic scale factor free to choose. In this case we have chosen it so that no contribution of the form $\delta(p)$ appearing. Similar constructs hold for orbits $O_6 - O_8$. For O_9, the action of $P_+^\uparrow$ is the same as that of T_2, since $H_9 = \mathcal{L}_+^\uparrow$.

PART II: FREE QUANTUM FIELDS

A. INTRODUCTION

DEF: 8: We shall mean a field in the sense of Wightman when we say quantum fields. Hence we shall discuss infraparticles only at the end of this section. Therefore fields have strictly positive mass until further notice. The discussion concerning the orbit O_1 is relevant here, as by free we mean that the field satisfy the Klein-Gordon equation. This is the usual viewpoint in Physics. Perhaps it is more revealing to say that a free field leads to a one particle sector whose measure is concentrated on orbit O_1 . We also deal with local fields only.

<u>DEF. 9</u>: An element $W \in \mathcal{S}_2'$ is said to be local if

$$\langle W, f \otimes g \rangle = 0 \quad , \forall f, g \in \mathcal{S} \tag{25a}$$

such that

$$f(x^+) g(x^-) = 0 \qquad x^+ x^- = x^2 \leq 0. \tag{25b}$$

we use the symbol $\langle , \rangle$ for the usual duality pairing from $\mathcal{S}_2' \times \mathcal{S}_2 \to R$.

<u>Lemma 5</u>: Every N-component free quantum field $\Phi(f)$ can be written

$$\Phi(f) = \sum_{\mu=1}^{N} {}^{\oplus} \phi_\mu(f) \qquad N < \infty , f \in \mathcal{S} \tag{26}$$

and where

(i) $\langle (\Box - m^2) \phi_\mu , f \rangle = 0 \quad \forall f \in \mathcal{S}$

(ii) $\phi_\mu(f)$ takes values in V_{λ_μ}

The proof follows by the defunction of a free quantum field. Point (ii) follows as $\mathcal{L}_+^{\uparrow} \simeq \mathbb{R}$. The condition $N < \infty$ is an ad hoc addition to eliminate infinite towers of fields.

<u>DEF. 10</u>: Using "field" for the fields of lemma 5 we define Boson fields as fields whose commutator is local in the sense of DEF. 9.

We first consider the simplest case, that of the <u>real massive scalar boson field</u> ϕ. It has the frequency decomposition $\phi = \phi^{(+)} + \phi^{(-)}$ and well known two point function

$$(\Omega_0, \phi^{(-)}(x)\,\phi^{(+)}(y)\,\Omega_0) = i^{-1}\,\Delta_m^{(-)}(\xi) \tag{27}$$
$$= -\frac{1}{4}\,H_0^{(1)}\left[j\,m(\xi^2)^{1/2}\right]$$

where $\xi = x - y$ and $j = -1$ for $\xi^2 > 0,\ \xi^0 > 0$; $j = +1$ for $\xi^2 > 0,\ \xi^0 < 0$ and for $\xi^2 < 0$. We are assuming a Fock space, the Hilbert space with cyclic vector Ω_0 having the Fock property $\phi^{(-)}(f)\,\Omega_0 = 0$; the Hilbert space has the inner product $(\ ,\ )$. The commutator function is $\Delta_m(x) = \Delta_m^{(-)}(x) - \Delta_m^{(-)}(-x)$.

It is seen, thereby, that ϕ is local. All free local boson fields are classified in the theorem below.

<u>Theorem 2</u>: Let Φ be an n-component free boson field whose j-th component takes values in V_{α_j}. Then

(i) $\phi_j(f) = (\text{constant})\left(\dfrac{\partial}{\partial x^+}\right)^{\alpha_j}\phi(f)\,,\ f \in \mathcal{S}_2$ (28a)

(ii) If Φ is local, then

$$\alpha_j + \alpha_k \in Z \quad , j,k = 1, \ldots, n \tag{28b}$$

Note that negative power derivatives are to be interpreted as $\left(-m^2 \frac{\partial}{\partial x^-}\right)^{\alpha_j}$ since the Klein-Gordon equation holds.

Proof: Equation (28a) is obvious in the momentum representation. For (28b) we must define our normalization of fractional derivatives. In one dimension, $\phi \in S'$, $f \in S$,

$$\partial^\alpha \phi(f) = e^{i\pi\alpha} \phi(f_\alpha) \tag{29a}$$

If $\mathcal{F}$ denotes the Fourier-Plancherel transformation map $S \to S$, then f_α is defined through

$$\mathcal{F}(f_\alpha) = e^{i\pi\alpha/2} (p)^\alpha \check{f}(p) \tag{29b}$$

We assume known the standard relations between the locality properties of distributions and the analyticity properties of the holomorphic functions which have the distributions as boundary values.

The support of $\Delta_m(x)$ is contained in the closed timelike cone. The support of $R(x) = \theta(x^0)\Delta_m(x)$ is contained in the closed future cone.

Then $\tilde{R}_1^{\alpha}(p) = \mathcal{F}\left[\left(\frac{\partial}{\partial x^1}\right)^{\alpha} R(x)\right](p)$ is the boundary value

$$\tilde{R}_1^{\alpha}(p) = \lim_{\substack{\eta \to 0 \\ \eta \in \bar{V}_+}} \tilde{R}_1^{\alpha}(q) \tag{29c}$$

where $\tilde{R}_1^{\alpha}(q) = (iq^1)^{\alpha}(q^2 - m^2)^{-1}$

$$\tag{29d}$$

and $q = p + i\eta$, $\bar{V}_+ = \{\eta^0 > \eta^1 \geq 0\}$ (29e)

But for $\left(\frac{\partial}{\partial x^1}\right)^{\alpha} \Delta_m(x)$ to be Local, $\tilde{R}^{\alpha}(q)$ must be analytic for all $q = p + i\eta$, $\eta \in \bar{V}_+$. This is not so, as $\tilde{R}^{\alpha}(q)$ has a branch point at $q^1 = 0$ unless $\alpha \in \mathbb{Z}$.

For $\left(\frac{\partial}{\partial x^0}\right)^{\alpha} R(x)$, the corresponding $\tilde{R}_0^{\alpha}(q)$ has a branch point at $q = im$, unless $\alpha \in \mathbb{Z}$. Thus $\left(\frac{\partial}{\partial x^1}\right)^{\alpha} R(x)$ are not local unless $\alpha \in \mathbb{Z}$. This implies equation (28b), and the theorem is proven. q.e.d.

<u>EXAMPLES</u>

1. Vector field: The usual presentation would take the form

$$(\Box - m^2) U^{\sigma} = 0 \quad , \quad \partial_{\sigma} U^{\sigma} = 0 \quad , \quad [U^{\sigma}, \check{U}^{\tau}] = \Pi^{\sigma\tau} c^{-1} \Delta_m \tag{30a-c}$$

where $$\Pi^{\sigma\tau} = -\left(g^{\sigma\tau} + m^{-2} \partial^{\sigma} \partial^{\tau} \right) \tag{30d}$$

In the terms of our classification theorem,

$$u^{\pm} = u^{0} \pm u^{1} = \mp (m^{-1}) \partial^{\mp} \phi \tag{30e}$$

$$u^{0} = \frac{1}{m} \partial^{1} \phi \quad , \quad u^{1} = -\left(\frac{1}{m}\right) \partial^{0} \phi \tag{30f}$$

2. "Curl"-field: In two dimensions we can define the anti-symmetric second rank tensor

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \; ; \; \epsilon_{01} = -\epsilon^{01} = +1 \tag{31a}$$

Then we can define the curl-field B^{μ} ,

$$B^{\mu} = \epsilon^{\mu\nu} \partial_{\nu} \phi \; ; \tag{31b}$$

this is "chiral" to u^{σ} , as can be seen by

$$B^{\pm} = B^{0} \pm B^{1} = \pm \frac{1}{m} \partial^{\pm} \phi \tag{31c}$$

3. As a third example we consider the second rank symmetric tensor field $V^{\sigma\tau}$,

$$(\Box - m^{2}) V^{\sigma\tau} = 0 \, , \; \partial_{\sigma} V^{\sigma\tau} = \partial_{\sigma} V^{\tau\sigma} = 0 \tag{32a,b}$$

This is expressible as

$$V^{--} = m^{-2}(\partial^+)^2\phi \, , V^{-+} = \phi \, , V^{++} = m^{-2}(\partial^-)^2\phi \tag{32c}$$

The commutator is then

$$[V^{\sigma\tau}, V^{\mu\nu}] = \frac{1}{2}\left(\Pi^{\sigma\mu}\Pi^{\tau\nu} + \Pi^{\sigma\nu}\Pi^{\tau\mu}\right) i^{-1}\Delta_m \tag{32d}$$

4. This last illustration is a counterexample to a usual spin and statistics theorem. Let

$$\Phi = \begin{bmatrix} i(\partial^+)^{-1/2} \\ m(\partial^-)^{1/2} \end{bmatrix} (\partial^+)^{\sigma}\phi \quad , \sigma \in \mathbb{Z} \tag{33}$$

Then ϕ is a local boson field~~s~~, but takes values in $V_{(\sigma_2-1/2)} \oplus V_{(\sigma_2-1/2)+1}$, i.e., is a $[\sigma-1/2]$-spinor.

<u>DEF.11</u>: A Dirac field is a free quantum field satisfying the Dirac equation; bilinear expressions formed from the field with elements of the Dirac-Clifford algebra are local.

<u>DEF.12</u>: The Dirac scalar field ρ is a non local free field ρ satisfying

$$\{\rho, \rho^*\} = i^{-1}\Delta_m^{(1)} \tag{34a}$$

where $\Delta_m^{(1)} = \Delta_m^{(-)}(x) + \Delta_m^{(-)}(-x)$, the even solution to the Klein-Gordon equation. Note that, being free, $\rho = \rho^{(+)} + \rho^{(-)}$ and

$$\left\{ \rho^{(-)}, \rho^{(+)*} \right\} = i^{-1} \Delta_m^{(-)} \tag{34b}$$

Theorem 3: All Dirac fields, upto chirality, are tensor products of Dirac fields of the type $\psi = \psi^{(+)} + \psi^{(-)}$, where

$$\psi^{\pm} = \left(\frac{i \gamma^{+}}{m} \right) (\partial^{+})^{\sigma} \rho^{(\pm)} \tag{35}$$

so that

$$\left\{ \psi, \bar{\psi} \right\} = (-)^{\sigma} \partial^{2\sigma+1} i^{-1} S_m \tag{36a}$$

where

$$(S_m)_{ab} = \left(i \sum_{0}^{1} \gamma^{\mu} \partial_{\mu} + m e \right)_{ab} \Delta_m , \quad a,b = 1,2 \tag{36b}$$

and

$$2\sigma \in \mathbb{Z} \tag{36c}$$

Proof: Direct computation implies that Eqs. (36a,b) follow. Locality gives (36c) just as in Theorem (2). That this exhausts the possibilities is evident from the independence of γ^{μ}, e and γ^5. Our theorem should be amended, in fact, to include possibilities of γ^5 appearing. This is in practice largely ignored in four dimension so we do so here. That is, if

$\psi \to \gamma^5 \psi$, $\bar{\psi} \to \bar{\psi}$ say, then locality would not be affected.

Theorem 4: Let C have the same meaning as in four dimensions, and define the field mapping $\Theta = PCT$. The connectivity of $L(C)$ is the same as in four dimensions. In particular, $L_+(C,2)$ contains both sheets of $L_+(R,2)$ as real subgroup. In Ref.(4) an explicit geometrical setting for the relevant analytic domains, in two dimensions, due to Ruelle, is given. Then one can follow the proof thereof the PCT-theorem for scalar fields word for word. Certainly it holds for our free fields. The spin-statistics connection seems lost, however[4].

PART III: THE VECTOR MESON MODEL[5]

We deal with a vector meson, A^τ, interacting with a fermion field of mass zero ψ. The coupling is formally

$$L \sim \int g \left(\bar{\psi} \gamma^\mu \psi A_\mu\right)_{ord} d^4x \tag{1}$$

when the mass of A_μ, $0 < m_0 < \infty$ this is the vector meson model. When $m_0 = 0$ we have Schwinger's model; correspondingly, $m_0 \to \infty$ is Thirrings model.

DESCRIPTION OF THE MODEL:

We use what seems like the most natural gauge and write

Field equations

$$i \alpha^\mu \partial_\mu \psi = -g \alpha^\mu :A_\mu \psi: \tag{2a}$$

$$(\Box - m_0^2) A_\mu = g (j_\mu)_{ord} \tag{2b}$$

Divergence condition

our-gauge demands

$$\partial_\mu A^\mu = 0 \tag{3}$$

commutation relations

$$[-A^1(t,x'), (\partial^0 A^1 - \partial^1 A^0)(t,y')] = i^{-1}\delta(\xi')$$

$$\{(\bar\psi \gamma^0)_a(t,x'), \psi_b(t,y')\} = \delta_{ab} Z_2^{-1}\delta(\xi') \tag{4}$$

$$A^0 = -m_0^{-2}[gj^0 + \partial_1(\partial^0 A^1 - \partial^1 A^0)]$$

In this form, $(\partial^0 A^1 - \partial^1 A^0)$ is <u>the</u> field conjugate to A^1, and A^0 is constrained by the above. For ordering we use wick ordering, $: \;:$. The terms above have to be suitably defined.

<u>Current</u>: We can define the current without any divergence as follows. We set

$$j^\mu(x;\epsilon) = \frac{1}{2}[\bar\psi(x+\epsilon)\gamma^\mu\psi(x) + \bar\psi(x-\epsilon)\gamma^\mu\psi(x)] \tag{5}$$

Let ϵ_s be spacelike, ϵ_t timelike and orthogonal:

$$\epsilon_s^\mu \epsilon_{s\mu} < 0 \;;\; \epsilon_s^\mu \epsilon_{t\mu} = 0 \;;\; \epsilon_s^\mu \epsilon_s^\nu / \epsilon_s^2 + \epsilon_t^\mu \epsilon_t^\nu / \epsilon_t^2 = g^{\mu\nu} \tag{6}$$

There are two independent limiting procedures extant in the literature leading to two currents, the Schwinger and the

Johnson limits and currents

$$j_S^{\mu}(x) = \lim_{\epsilon_S \to 0} |\epsilon_S^2|^{K} j^{\mu}(x;\epsilon_S)\, e^{i g \int_x^{x+\epsilon_S} A_\sigma(y)\, dy^\sigma} \tag{7a}$$

$$j_J^{\mu}(x) = \frac{1}{2} \lim_{\substack{\epsilon_S \to 0 \\ \epsilon_t \to 0}} \left\{ |\epsilon_S^2|^{K} j^{\mu}(x;\epsilon_S) + |\epsilon_t^2|^{K} j^{\mu}(x;\epsilon_t) \right\} \tag{7b}$$

Here $K = K(Z_2)$ and picks out the (finite) leading term.

Projection conventions:

$$v = v^{TR} + v^{L} \quad ; \quad v^{L} = \partial(-D^{c}\partial v) \tag{8a}$$

for a vector field, not of zero mass. For zero mass

$$(\partial^\mu \phi)^L = (\partial^\mu \phi) \; ; \; (\epsilon^{\mu\nu}\partial_\nu \phi)^{TR} = (\epsilon^{\mu\nu}\partial_\nu \phi) \tag{8b,c}$$

Solution Ansatz

Let Φ be a linear superposition of free fields, all dynamically independent. We seek a solution in the form

$$\psi = e^{ig\Phi} : \psi(0) \; ; \; \bar{\psi} = \bar{\psi}(0) : e^{-ig\Phi} . \tag{9}$$

we see the necessity of using massless fields that can be exponentiated, even at the expense of not knowing the positive definiteness of the theory.

With this form of ψ, the Dirac field equation gives A^{σ} in terms of Φ

$$\partial_0 \Phi - A_0 = -\alpha^1 (\partial_1 \Phi - A_1) \tag{10}$$

Solubility rests on the commutativity of the α -matrices. We may pick $\alpha^0 = e$, $\alpha^1 = \gamma^5$ with the correct structure. Then Φ must be of the form

$$\Phi = e\,\Phi_a + \alpha^1 \Phi_b \tag{11}$$

where $\Phi_{a,b}$ do not contain matrix properties. Then

$$(\partial_\lambda \Phi_a) = (A_\lambda)^L \quad (\partial_\lambda \Phi_b) = (A_\lambda)^{TR}$$

as illustration, after eliminating $|\epsilon^2|^K$,

$$j_J^\mu(x) = \frac{1}{2} \lim_{\substack{\epsilon_s \to 0 \\ \epsilon_t \to 0}} \left\{ j^{(0)\mu}(x) + \frac{g}{2\pi} \frac{\epsilon_s^\sigma \epsilon_s^\lambda}{\epsilon_s^2} \mathrm{tr}\left[\partial_\lambda \Phi\, \alpha_\sigma \alpha^\mu\right] + O(|\epsilon|) \right. \tag{12a}$$

$$+ (s \to t)$$

$$= J^{(0)\mu}(x) + g/_{2\pi} \left[(A^\mu)^{TR} - (A^\mu)^L\right](x)$$

similarly

$$j_S^\mu(x) = j^{(0)\mu}(x) + \frac{g}{2\pi} (A^\mu)^{TR}(x) \tag{12b}$$

Note that C.Sommerfield sets $I^{(0)}(x) = \lim_{\epsilon_s \to 0} j^0(k;\epsilon_s)$, I'

by covariance. But this gives $I^\mu = j_S^\mu$. We use this process in the Thirring model where no A^σ appears and no line integral is available to define j_S^μ .

Thus two classes of solutions appear, as we use J_S^μ or J_j^μ respectively. There will be found to be different gauges in each class giving a solution table like

gauges	Landau,	Feynman,	Vector-meson,	----
Schwinger	-	-	-	---
Johnson	-	-	-	---

Note also the deep point that well-defining j^μ gives mass renormalization automatically here. For S:

$$(\Box - m_0^2)[(A^\mu)^{TR} + (A^\mu)^L] = g\, j^{(0)\,\mu} + \frac{g^2}{\pi}(A^\mu)^{TR} \tag{13a}$$

or

$$(\Box - m_0^2)(A^\mu)^L + (\Box - m_S^2)(A^\mu)^{TR} = g\, j^{(0)\,\mu} \tag{13b}$$

$$(A^\mu)^{TR} \text{ gets } m_0^2 \to m_S^2 = m_0^2 + g^2/\pi \tag{13c}$$

In all our solutions, to get from S to J class, change

$$m_0 \to (m_J)_-\,,\quad m_S \to (m_J)_+\,,\quad (m_J)_\pm = m_0^2 \pm g^2/2\pi \tag{14}$$

This is for the above reasons. Note also that were "Schwinger terms" to appear in we would throw out the limiting procedure and search for one that was well defined. There is no divine recipe for finding j^μ - it must be well defined only.

The Functional Approach

With neither details nor explanations we write

$$J(j,\eta^*,\eta) = \langle T\, \exp i \int (j^\sigma A_\sigma + \eta^* \psi + \psi^* \eta) \rangle_0 \quad (15)$$

I can be found explicitly through two factors. First, as $[\alpha^\mu, \alpha^\nu] = 0$, an exact expression for $G(u,v;B)\,\alpha(i\partial^\mu - gB^\mu)_u$ $G(u,v;B) = \delta(u-v)$ in closed form is available. Secondly, our ψ ansatz leads to j^μ forms and hence to

$$\langle j^\mu \rangle_B \stackrel{S}{=} \langle j^{\mu(0)} + \frac{g}{\pi} (B^\mu)^{TR} \rangle_0 = \langle j^{\mu(0)} \rangle_0 + \frac{g}{\pi} \{ (P_{nj})^{TR} B \}^\mu \quad (16a)$$

$$= g/\pi \; P_S^{\mu\nu} B_\nu . \quad (16b)$$

Similarly for J -class calculations. With this, it is possible to find J explicitly and hence all the T-product VEV's. I won't write them down as they are boringly long(+). Suffice it to say, compairing $\langle T\, \psi(x_1) \cdots \bar{\psi}(y_n) \rangle_0$ with our ansatz, assuming

(+) See APPENDIX A.

$\Phi_{ab} = \sum C^{j}_{a,b}\, \phi_j(m_j)$ including vectors, the simplest solution is·

$$\Phi = \gamma^5 (m_S^{-2} \epsilon^{\sigma\tau} \partial_\tau U_\sigma + m_S^{-} C) + m_0^{-1} \varphi \tag{17a}$$

$$\omega^\sigma = U^\sigma + m_S^{-1} \epsilon^{\sigma\tau} \partial_\tau C + m_0^{-1} \partial^\sigma \varphi \tag{17b}$$

Here

U^σ	= vector,	mass m_S	,	$[\ ,\ ] = +$
φ	= scalar,	0	,	$[\ ,\ ] = +$
C	= "	, "	,	$[\ ,\ ] = -$ (18)
ψ^0	= spinor,	"	,	$[\ ,\ ]_+ = +$

<u>The Solution Proper</u>

A) We have $H_{un} = \bigotimes_{\text{free fields}} H_j$; we only need $H_{Phys} \subset H_{un}$,

H_{Phys} = closure $\{ P(A, \psi, \bar{\psi}) \Omega_0 \}$ with induced inner product.

This point is crucial, as <u>s</u>:

$$(\Box - m_0^2) A_\sigma - g j_\sigma = -[m_0 \partial_\sigma \varphi + m_S \epsilon_{\sigma\tau} \partial^\tau C + g j^{(0)}_\sigma] = R_\sigma \tag{19}$$

is not identically zero, i.e. $(2 | R_\sigma | 1)_{H_{un}} \neq 0$. But

$(2 | R_\sigma | 1)_{H_{Phys}} = 0$. We see this by examining R_σ further:

$$\Box R_\sigma = 0 \implies R_\sigma = R_\sigma^+ + R_\sigma^- \tag{20a}$$

$$[R_\sigma^\pm, A_\tau] = [R_\sigma^\pm, \psi] = [R_\sigma^\pm, \bar{\psi}] = 0 \tag{20b}$$

B) $\langle T(AB)\rangle_0 = AB \qquad x_A^0 > x_B^0$

$= \pm BA \qquad x_A^0 < x_B^0$

we define a T with properties at $x_A^0 = x_B^0$ such that

$$\langle T^* \partial^\mu \phi\, \partial^\nu \lambda\rangle = \partial^\mu_{(\phi)}\, \partial^\nu_{(\lambda)} \langle T^* \phi \lambda\rangle \tag{21}$$

Also T^{**} for second derivatives, etc.

GAUGES

When finding $J(j,\eta,\eta^*)$ there appears the factor $\Delta_c^{\mu\nu}$, the causal vector propagator. This can be written, in k space, as

$$\frac{-1}{m^2-k^2-i\epsilon}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + d_\ell \frac{k^\mu k^\nu}{k^2}\right] \tag{22}$$

Our above gauge, VM, is when $d_\ell = 1 - k^2/m^2$. For the F-gauge, $d_\ell = 1$, and

$$\Phi_F = \Phi_{VM} + m_0^{-1} b \quad , \; b = \text{scalar}, m_0 [\;,\;] = -1 \tag{23}$$

However, $Z_2 = 0$ here ! This is an infinity in the wave function renormalization but we see it is due to the wrong prescription.

For the L-gauge $d_\ell = 0$, and

$$\Phi_L = \gamma_5 \left[m_s^{-2} \epsilon_{\sigma\tau} \partial^\tau u^\sigma + m_s^{-1} c \right]. \tag{24}$$

which looks better than Φ_{VM} by one field; but $\nexists\ H_{Phys}$ on which $(R_\sigma)_L$ vanishes weakly. The above gauges are all fine as regards the Wightman functions, but one can see the advantage of the VM gauge.

We note that these operator solutions might be termed gauge fields and will show no scattering for that reason. Up to non-definition mathematically the various gauges can be interrelated via

$$(A_\mu)_1 \to (A_\mu)_2 = (A_\mu)_1 + \partial_\mu \Lambda_{12}\ ,\ \psi \to :e^{-ig\Lambda_{12}}:\psi_1 = \psi_2 \text{ etc} \tag{25}$$

$$\Lambda_{12} = P_{12}(\square)\, \partial_\nu (A^\nu)_1$$

LIMITS

A) Thirring Model

Theorem : Let $\lambda = g^2/m_0^{\,2}$, $0 \leq \lambda < 2\pi$. Let

$$T(mx) = \exp\left(\pm i\, g^2/m^2\ \Delta_m^{(-)}(x)\right)$$

Then $\lim\limits_{m\to\infty} T(mx) \to 1$ in $\mathcal{S}$

$$\lim_{m_1 \cdots m_N \to \infty} T(m_1 x) \cdots T(m_N x) \to 1 \text{ in } \mathcal{S}$$

provided $g^2 \to \infty$ such that $\lambda \to$. The proof is in the paper.

Then take $m_0 \to \infty$ rather freely to get the Thirring model with operator solutions. For VM we have

$$\psi_{S,J} = :e^{i\Phi_{S,J}}:\ \psi^{(0)} \tag{26a}$$

$$\Phi_S = \sqrt{\lambda}\left[\gamma_5 \frac{1}{\sqrt{1+\lambda/\pi}}\,(\,C + \varphi\,)\right] \tag{26b}$$

$$\Phi_J = \sqrt{\lambda}\left[\gamma_5 \frac{1}{\sqrt{1+\frac{\lambda}{2\pi}}} C + \frac{1}{\sqrt{1-\lambda/2\pi}} \varphi\right] \tag{26c}$$

Again there is an $H_{in} \supset H_{Ph}$. Also note that

$$J^{\mu}_{S} \rightarrow (J^{\mu}_{S})_{Th} \;; \; -g A^{\mu}_{S} \rightarrow \lambda (j^{\mu}_{S}) A_{S} \tag{27a}$$

such that

$$-g^{-1} R^{\mu} \rightarrow \gamma^{\mu} = (j^{\mu}_{S})_{Th} - (j^{\mu}_{S})_{A_S} \tag{27b}$$

and

$$(2 | \gamma^{\mu} | 1)_{H_{in}} = 0.$$

B) Schwinger Model

Here $m_0 \rightarrow 0$. Our solutions show singularities even in the Wightman functions, except in the L gauge. But here we have no true operator solution. It is felt that the operator solution will come through the Coulomb gauge. But the usual relations between the Coulomb and Landau gauges. The two obvious Coulomb Green's functions are

$$Y_0(x) = \frac{1}{2m_0} \sinh[m_0 |x|] \tag{28a}$$

$$Y_1(x) = -\frac{1}{2m_0} e^{-[m_0 |x|]} \tag{28b}$$

or $a_0 Y_0 + a_1 Y_1$. But Y_1 is singular at $m_0 \rightarrow 0$, and Y_0 at ~~∞~~ $m_0 \rightarrow \infty$?!

NOTE ADDED:

Hagen proposes a trivial modification in the current, leading to our Schwinger results with

$$m_0^2 \to m_\eta^2 = m_0^2 - \eta\, g^2/\pi$$

$$m_S^2 \to m_S^2 = m_0 - \xi\, g^2/\pi$$

For $(\eta,\xi) = (0,1)$ Schwinger's results; $(\frac{1}{2},\frac{1}{2})$ Johnson's results. Otherwise this trivial class. We can relax his conditions on replaces these, or analytically extend, (ξ,η) so that $\xi+\eta \neq 1$ as he demands. For the Thirring model,

$$1/\sqrt{1+\frac{\lambda\lambda}{\pi}} \quad , \quad 1/\sqrt{1-\frac{\eta\lambda}{\pi}}$$

same expressions with $(\xi,\eta) = (1,0)$ as in our results.

APPENDIX A

The generating functional for time ordered Green's functions is

$$\mathcal{J}(J,\eta^*,\eta) = (\Omega_0, T\exp i\int (J^\sigma A_\sigma + \eta^* \psi + \psi^* \eta)\, dy\, \Omega_0) \tag{A.1}$$

It satisfies the equations

$$\alpha^\mu (i\partial_\mu - g\, i^{-1} \frac{\delta}{\delta J_\mu(x)}) \frac{\delta \mathcal{J}}{\delta \eta^*(x)} = -i\eta(x)\mathcal{J} \tag{A.2a}$$

$$(\Box - m_0^2) \frac{\delta \mathcal{J}}{\delta J_\mu(x)} = (-i J^\mu_{(x)} + i g \sum_{a,b=1}^{2} \delta / \delta\eta_a(x)\, \alpha^\mu_{ab} \frac{\delta}{\delta \eta^*_b(x)}) \mathcal{J} \tag{A.2b}$$

The solution for $\mathcal{J}$ may be verified to be

$$\mathcal{J}(J,\eta^*,\eta) = N^{-1} \exp[\tfrac{1}{i}\int d^2u\, d^2v\, \eta^*(u)\, G(u,v;\tfrac{1}{i}\tfrac{\delta}{\delta J})\eta(v)]$$
$$\times \exp[-L(\tfrac{1}{i}\tfrac{\delta}{\delta J})] \exp[-\tfrac{1}{2i}\int d^2w\, d^2z\, J^\mu(w) \Delta^c_{\mu\nu}(m_0, w-z) J^\nu(z)] \tag{A.3}$$

In Eq. A.3, N is chosen so that $\mathcal{J}(0,0,0) = 1$ The gauge is put in via the choice of $\Delta^c_{\mu\nu}$. G is the fermion Green's function in an external field:

$$\alpha^\mu (i\partial_\mu - gB_\mu)_x G(x,y;B) = \delta(x-y) \tag{A.4}$$

Its solution is

$$G(x,y;B) = G^{(0)C}(x-y)\exp\Big[g\int d^2z\,\big[G^{(0)C}(x-z)-G^{(0)C}(y-z)\big] \times \alpha^{\sigma}B_{\sigma}(z)\Big] \qquad (A.5)$$

Here

$$i\alpha^{\mu}\partial_{\mu}G^{(0)C}(x-y) = \delta(x-y) \qquad (A.6)$$

with causal boundary conditions. L is the closed loop functional and is defined through (for the S-class)

$$\frac{\delta L}{\delta B^{\mu}(x)} = i^{-1}g\langle J_{\mu}(x)\rangle_B = i^{-1}g\langle J_{\mu}^{(0)} + \frac{g}{\pi}B_{\mu}^{TR}\rangle_0 \qquad (A.7)$$
$$= i^{-1}\frac{g^2}{\pi}\int (P_S)_{\mu\nu}(x-y)\,B^{\nu}(y)\,d^2y$$

We have again used prior knowledge for simplification. P_S is the transverse projection operator. For J-class solutions, use the contraction operator P_J :

$$P_S^{\mu\nu}(x-y) = g^{\mu\nu}\delta(x-y) - \partial_x^{\mu}D^C(x-y)\partial_y^{\nu} \qquad (A.8a)$$

$$P_J^{\mu\nu}(x-y) = \frac{1}{2}g^{\mu\nu}\delta(x-y) - \partial_x^{\mu}D^C(x-y)\partial_y^{\nu} \qquad (A.8b)$$

Thus

$$L_{S,J}(B) = \frac{g^2}{2\pi i}\int d^2x\,d^2y\,B_{\mu}(x)\,P_{S,J}^{\mu\nu}(x-y)\,B_{\nu}(y) \qquad (A.9)$$

Following Thirring and Wess: Ann.Phys.(NY)27,331(1964), one is led through the algebra to the Green's functions and thence the Wightman functions. The n-point fermion functions are necessary to find Φ in $:e^{ig\Phi}:\psi(0) = e^{ig\Phi^{(+)}}e^{ig\Phi^{(-)}}\psi(0)$. We have

$$\langle \psi(x_1)\cdots\bar{\psi}(y_n)\rangle_0$$
$$= \sum_{\mathrm{Perm}} (\mathrm{Sgn}\,P) \prod_{j<k} F^{j,P_k}(x_j - y_{P_k})\, F^{P_j,k}(y_{P_j} - x_k) \tag{A.10}$$
$$\times \prod_{\ell} F(x_\ell - y_{P_\ell})\, i^{-1} S_0^{(-)}(x_\ell - y_{P_\ell})$$

The factors F depend upon the current and are given by

$$F_S^{\ell,m}(x_\ell - y_m) = \exp\Big\{\lambda i^{-1} D^{(-)}(x_\ell - y_m) + \gamma_5^{(\ell)}\gamma_5^{(m)} \tag{A.11a}$$
$$\times \lambda\left(1+\frac{\lambda}{\pi}\right)^{-1}\left[\Delta^{(-)}_{m_0}(x_\ell - y_m) - D^{(-)}(x_\ell - y_m)\right]\Big\}$$

$$F_J^{\ell,m}(x_\ell - y_m) = \exp\Big\{\lambda\left(1-\frac{\lambda}{2\pi}\right)^{-1} i^{-1} D^{(-)}(x_\ell - y_m) \tag{A.11b}$$
$$+ \gamma_5^{(\ell)}\gamma_5^{(m)}\lambda\left(1+\frac{\lambda}{2\pi}\right)^{-1}\left[\Delta^{(-)}_{m_{J+}}(x_\ell - y_m) - D^{(-)}(x_\ell - y_m)\right]\Big\}$$

The functions F follow from the corresponding $F^{\ell m}$ by replacing $\gamma_5^{(\ell)}\gamma_5^{(m)}$ by unity. The $\gamma_5^{(\ell)}\gamma_5^{(m)}$ refer to the corresponding spinor indices of $\psi(x_\ell)$ and $\psi(y_m)$ respectively.

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