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LECTURES ON

REPRESENTATION THEORY FOR BANACH ALGEBRAS AND LOCALLY COMPACT GROUPS

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REPRESENTATION THEORY FOR BANACH ALGEBRAS AND LOCALLY COMPACT TOPOLOGICAL GROUPS.

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P R E F A C E

The aim of these lectures is to provide an introduction to some of the basic theorems of representation theory. They are purely expository ; there is little or nothing that cannot be found in standard treatises such as M.A.Naimark's Normed rings,(revised edition: Noordhoff, Groningen 1964) or Vol.1 of Abstract harmonic analysis by E.Hewitt and K.A.Ross (Springer, Berlin 1963). The background assumed is (a) the elementary theory of Banach algebras, in particular of commutative algebras, up to Gelfand's representation theorem ; (b) the elementary theory of Haar measure on a (not necessarily abelian) locally compact group ; (c) some standard results from linear analysis, such as the spectral theorem, the Banach-Steinhaus theorem and the Krein-Milman theorem. This material is readily available in several excellent texts, and it would perhaps be superfluous to make specific recommendations.

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CONTENTS

Page No

Page No.

CHAPTER 1

GENERALITIES

Most abstract mathematical systems have at least one reasonably well understood concrete realisation, which may have served as the starting-point for the abstract theory. One may think of groups (finite groups at least) envisaged as groups of permutations. Such a concrete realisation may also have been found in the course of, or subsequent to, the development of the general theory. In any case we assume a class $\mathcal{S}$ of abstract systems S and a class $\mathcal{S}_0$ of concrete realisations S_0. A structure preserving map $S \longrightarrow S_0$ is a representation of S as a system of $\mathcal{S}_0$. In general many systems $S_0 \in \mathcal{S}_0$ can contain images of a given $S \in \mathcal{S}$; think of permutations and finite groups. The choice of the class $\mathcal{S}_0$ of "well known, "concrete" systems is to some extent arbitrary ; and in most cases no entirely satisfactory reason can be put forward for selecting one such class $\mathcal{S}_0$ rather than another. However in most instances there are certain conventional choices for $\mathcal{S}_0$ which are clearly in some sense reasonable, and we follow the traditions. In the cases in which we are interested $\mathcal{S}_0$ will be some class of algebras or of groups of linear operators on linear spaces, which are regarded as reasonably familiar objects.

An important general notion is that of 'irreducibility' or the equivalent. We shall be particularly interested in representations that are 'simple' in some suitably defined technical sense ; we demand that they cannot be made up out of simpler pieces. In this kind of context the terms 'simple', 'minimal', 'irreducible', 'indecomposable' are all going to mean much the same thing. To take an elementary case : suppose $\mathscr{S}$ is the class of linear spaces (real say) and $\mathscr{S}_0$ is the class of finite dimensional real Euclidean spaces R^n . Among the various possible maps $S \longrightarrow R^n$. Those for which $n=1$, i.e. the linear functionals on S, evidently have this 'minimal' or 'simple' property.

Another general idea is that of a '<u>faithful</u>' representation : this simply means that the map is 1-1 (from S to S_0) . A set $\{T_i\}$ of representations is <u>complete</u> if whenever $x \neq y$ in S there is a T_i such that $T_i(x) \neq T_i(y)$ in $S_0 \in \mathscr{S}_0$. We would like complete set of irreducible representations in general. Two representations $T_1 : S \longrightarrow S_1$ and $T_2 : S \longrightarrow S_2$ are <u>equivalent</u> if there is a 1-1 map W of S_1 on to S_2 such that W and W^{-1} are both structure-preserving and $T_2(x) = WT_1(x)$ for all $x \in S$. We are usually interested in representations only up to equivalence.

We now turn more particularly to Banach algebras and topological groups. We shall always be concerned with representations by bounded linear operators on a Banach space (which will usually be a Hilbert space) : we assume $\mathcal{S}_0$ to be the class $\{\mathcal{L}_0(E)\}_{E\in\mathcal{E}}$ of all algebras of linear operators on the Banach space E, for $E\in\mathcal{E}$ (the class of all Banach spaces). In each case there is a trivial representation : if B is a Banach algebra then $T(x) = 0$ for all $x\in B$: if G is a group then $T(t) = I$ for all $t\in G$.

Take now B to be a Banach algebra : we shall consider only complex algebras (the real case is similar but slightly more complicated) and we do not assume a unit. If however there is a unit e we shall always demand $T(e) = I$ (the identity operator). A representation $T : B \longrightarrow \mathcal{L}(E)$ is <u>bounded</u> (or continuous) if $\|T(x)\| \leq k\|x\|$ for all $x\in B$ and some real k.

There are atleast two obvious representations for any Banach algebra B. If we consider B as acting on itself (as a linear space) by left multiplication and write

$$T(x)y = xy$$

then it is clear that $x \longrightarrow T(x)$ is a representation of B in $\mathcal{L}(B)$. Also $\|T(x)\| \leq \|x\|$ for all $x\in B$. We shall call it the <u>left obvious representation</u>* of B. The left obvious representation is faithful if and only if the left annihilator of B , that is, $\{x : xy = 0 \text{ for all } y\in B\}$ is zero.

* Sometimes it is also called left regular representation, but this term is also applied in a rather similar but distinct sense, so we avoid it here.

More generally, let J be a closed left ideal in B and take the quotient B/J, which will be a Banach space (but not a Banach algebra unless J is two sided). Let Φ be the canonical map $B \longrightarrow B/J$, then the representation T where $T(x)\Phi(y) = \Phi(xy)$ is the <u>left obvious representation modulo J</u>. Similarly for 'right'.

The <u>kernel</u> of a representation $T : \{x : T(x) = 0\}$ is a two sided ideal in B, closed if T is bounded, and conversely. This fact seems however not to be so useful in the non-commutative case as in the commutative case, where it is of fundamental importance.

A subspace E_1 of E is said to be <u>invariant</u> under the representation T if $T(x)(E_1) \subset E_1$ for all $x \in B$. A representation T is <u>reducible</u> if there exists a nontrivial closed subspace $E_1 \subsetneq E$ which is invariant under T ; otherwise <u>irreducible</u>. If T is irreducible, then the vectors $T(x)\xi$, $x \in B$, $\xi \in E$ are dense in E, otherwise the closure would be a non-trivial subspace E_1 with the required properties.

If there is a single vector $\zeta \in E$ such that the vectors $\{T(x)\zeta : x \in B\}$ are dense in E then ζ is said to be a <u>cyclic</u> vector for T and E is cyclic under T. If for each non-zero $\xi \in E$ there is a $x \in B$ with $T(x)\xi \neq 0$ then T is <u>essential</u> : in general if we write

$$N = \{\xi : T(x) = 0 \text{ for all } x \in B\}$$

then N is a closed subsapce of E. We may take the quotient E/N and then the induced representation T_1 given by $T_1(x)\Phi(y) = \Phi(xy)$ is essential. So in fact we can confine ourselves to a great extent to essential representations.

PROPOSITION 1.1: <u>A non-zero representation is irreducible if and only if every non-zero vector is cyclic for it</u>.

PROOF: If $\xi_1 \neq 0$ is not cyclic then the closure of $\{T(x)\xi_1 : x \in B\}$ would be a proper closed subspace of E invariant under T : so T is reducible. Conversely, if T is reducible, then clearly no vector in a closed invariant proper subspace can be cyclic. ‖

Now we shall take the case of a locally compact topological group. Here we are assured of the existence of an essentially unique invariant measure on the group.

Let G be a locally compact topological group and let dt denote the left invariant Haar measure on G. We have then,

$$\int_G f(t)dt = \int_G f(s^{-1}t)dt \qquad \text{for all } s \in G.$$

Denote $f(s^{-1}t)$ as a function of t by $_sf(t)$, the <u>left translate</u> of by s. If $C_{oo}(G)$ is the linear space of complex valued continuous functions on G with compact support then $\int_G f(t)dt$ makes sense for $f \in C_{oo}(G)$. We can introduce various norms into $C_{oo}(G)$: for example

$$\|f\|_\infty = \sup_{t \in G} |f(t)|$$

$$\|f\|_p = \left(\int_G |f(t)|^p dt\right)^{1/p} \qquad 1 < p < \infty$$

On completion, we have the spaces $C_o(G)$, $L_p(G)$. If $p=2$ we have a Hilbert space with inner product

$$(f,g) = \int_G f(t)\, \overline{g(t)} dt$$

It is clear that we have a wide variety of representations of G as (isometric) linear operators $T(s)$ on one of these spaces. In view of the left invariance, if we write

$$T(s) \quad = T(s)f = {}_s f$$

we have

$$\|T(s)f\| = \|{}_s f\| = \left\{ \begin{array}{l} \left(\int_G |f(s^{-1}t)|^p dt\right)^{1/p} \\ \sup |f(s^{-1}t)| \end{array} \right\} = \|f\|$$

So $\|T(s)\| = 1$. Further $T(s_1 s_2)\, f(t) = f((s_1 s_2)^{-1} t)) = f(s_2^{-1} s_1^{-1} t) = (T(s_2)\, f)\, (s_1^{-1} t) = T(s_1)\, T(s_2)\, f(t)$ so that $T(s_1 s_2) = T(s_1)\, T(s_2)$. Thus we have a representation ; $s \longrightarrow T(s)$ is a homomorphism. It turns out that the norm topology on the operators is not the appropriate one here, but rather the strong topology. We have, in fact, the following

PROPOSITION: 1.2: *If G is a locally compact topological group, it has a faithful representation by isometric operators on* $L_p(G)$. ($1 \leq p < \infty$) or on $C_o(G)$. *This is bicontinuous if operators have strong topology, where the basic neighborhoods of* T_o *are,*

$$\left\{ T : \|T\xi_r - T_o\xi_r\|_p < \varepsilon \ , \ r = 1, 2, 3 \ldots, n \right\}$$

In case $p=2$, *the operators are unitary.*

PROOF: If $s_1 \neq s_2$, then from the local compactness of G, we can find a function $f \in C_{oo}(G)$ such that $f(s_1) \neq f(s_2)$, so that $T(s_1)f \neq T(s_2)f$. So T is 1-1.

Since each $f \in C_{oo}(G)$ is uniformly continuous, given $\varepsilon > 0$ there exists a neighbourhood $N(e)$ of the identity such that $\|T(s)f - f\| < \varepsilon$ for $s \in N$. Since $C_{oo}(G)$ is dense in each $L_p(G)$ and in $C_o(G)$, there exists a neighbourhood $N'(e)$ with $\|T(s)\xi_r - \xi_r\| < \varepsilon$ for $s \in N'$, $\xi_1, \ldots, \xi_n \in L_p$ or C_o. This proves that the map $s \longrightarrow T(s)$ is continuous.

For the converse let $N(e)$ be a given neighbourhood of e. Then there exists a symmetric neighbourhood $N'(e)$

such that $N'N'\subset N$. Let $f \in C_{oo}$ such that $f \geq 0$, the support of $f \subset N'$ and $\|f\|_p = 1$. If $s \notin N$, then f, $T(s)f$ will have disjoint supports and then

$$\|f - T(s)f\|_p \geq \|f\|_p = 1$$

whenever $s \notin N$ and so the strong neighbourhood

$$\{ s : \|T(s)f - f\|_p < 1 \}$$

is contained in the given N as required.

If $p=2$ we have

$$\left(T(s)\xi , T(s)\eta\right) = \int_G \xi(s^{-1}t)\overline{\eta(s^{-1}t)}dt = \int_G \xi(t)\overline{\eta(t)}dt = $$
$$= (\xi, \eta)$$

This completes the proof. ‖

The functions in $C_{oo}(G)$ have a rather rich algebraic structure : in addition to their linear space properties we may introduce an operation of multiplication ; if $f, g \in C_{oo}(G)$ then the function

$$f * g(s) = \int_G f(st)g(t^{-1})dt$$

is easily verified to be again in $C_{oo}(G)$. It is called the <u>convolution product</u> of f and g. With this as multiplication, $C_{oo}(G)$ becomes a linear associative algebra (not commutative in general). We have in general the inequality

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

so that in particular $L_1(G)$ is a Banach algebra. Also regarding f as an operator on C_{00} with $\| \|_p$, the operator norm satisfies $\|f\|_{op} \leq \|f\|_1$, and completing C_{00} under $\| \|_{op}$ we get various operator algebras. The case p=1 gives $L_1(G)$ again and p=2 gives $\bigwedge(G)$, which is in fact a B^*-algebra.

CHAPTER 2

ALGEBRAS WITH INVOLUTION

Let B be a complex Banach algebra, not necessarily with a unit ; we shall denote by B_1 the algebra B with a unit adjoined, in case B lacks a unit. B_1 may be normed in the obvious way : $\|\lambda e + x\| = |\lambda| + \|x\|$ but there also other ways of norming B which are more appropriate in certain cases, in particular if B is a B^*-algebra.

DEFINITION : An <u>involution</u> on B is a map $x \longrightarrow x^*$ of B to itself satisfying at least the following conditions

$$\text{(i) } x^{**} = x$$

$$\text{(ii) } (\lambda x+\mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$$

$$\text{(iii) } (xy)^* = y^* x^*$$

The involution may be related to the norm in various ways

(1) $x \longrightarrow x^*$ is a continuous map ;

(2) $x \longrightarrow x^*$ is an isometric map ;

(3) $\|x\, x^*\| = \|x\|^2$ for all x ;

(3') $\|x^* x\| = \|x\|^2$ for all x .

It is not hard to see (3') $\Longleftrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). A Banach algebra with a involution satisfying (2) will be called a <u>Banach $*$-algebra</u> and we shall always assume this condition from now on. If the stronger condition (3) holds, we have a <u>B^*-algebra</u>.

Examples: (1) B = C , with complex conjugation as involution, is a B^*- algebra. This is the simplest example of a B^*-algebra.

(2) $B = C_0(x)$, the continuous functions on the locally compact Hausdorff space X vanishing at infinity, with $\|x\| = \sup|x(t)|$, $t \in x$ and conjugation as involution. This is the typical commutative B^*-algebra.

(3) B = algebra of all complex $n \times n$ matrices, with x^* = transposed complex conjugate of x and norm $\|x\| = \left(\sum_{i,j=1}^{n} |x_{ij}|^2\right)^{1/2}$; this is a Banach$*$ algebra but not a B^*-algebra.

(4) $B = \mathcal{L}(H)$, the algebra of all bounded linear operators on a Hilbert space H, with the involution $x \longrightarrow x^*$ the natural Hilbert space adjoint : $(x^*\xi, \eta) = (\xi , x\eta)$ for $x \in \mathcal{L}(H)$, $\xi, \eta \in$ H. It is easy to see that with the natural operator norm $\|x\| = \sup_{\|\xi\|=1} \|x\xi\|$, $\mathcal{L}(H)$ becomes a B^*-algebra.

(5) Any closed $*$-sub algebra of $\mathcal{L}(H)$ (known as a <u>C^*-algebra</u>) ; this is the standard model for not necessarily commutative B^*-algebra, as will be proved later (Theorem 7-10).

DEFINITION : If in a $*$-algebra an element x satisfies $x^*=x$, it will be called <u>self-adjoint</u> or <u>Hermitian</u>. If B has a unit e then x is said to be <u>unitary</u> if $xx^* = x^*x = e$. If $x^*x = xx^*$ then x is said to be <u>normal</u>. For any x the elements xx^*, x^*x, $\frac{1}{2}(x+x^*)$, $\frac{1}{2i}(x-x^*)$ are always self-adjoint and x can always be written as x_1+ix_2 where x_1 and x_2 are self-adjoint : $x_1=\frac{1}{2}(x+x^*)$, $x_2=\frac{1}{2i}(x-x^*)$ and x is normal if and only if $x_1x_2=x_2x_1$. If B is a $*$-algebra without a unit then B_1 becomes a $*$-algebra if we define $(\alpha e+x)^*=\bar{\alpha}e+x^*$. Further if we have a Banach $*$-algebra $(\|x\|=\|x^*\|)B_1$, on extension in this way will be a Banach $*$-algebra with the usual norm

$$\|\alpha e+x\| = |\alpha| + \|x\|$$

However if B is a B^*-algebra then with this norm B_1 is not B^*-algebra.

We shall define a norm under which it is so :

PROPOSITION 2.1: <u>If</u> B <u>is a B^*-algebra without a unit then B_1 becomes a B^*-algebra under the norm</u>

$$\|\alpha e+x\|' = \sup_{y\neq 0} \|\alpha y+xy\| / \|y\|$$

<u>and further</u> $\| \; \|'$ <u>induces</u> $\| \; \|$ <u>on</u> B.
(<u>i.e.</u> $\| \; \|'$ <u>is the norm as an algebra of left multiplication operators on B</u>).

PROOF: Since $\alpha y+xy=0$ for all $y\in B$ can only hold for $\alpha=x=0$ (if $\alpha\neq 0$ then $y=-\frac{x}{\alpha}$ all y and $-\frac{y}{\alpha}$ is a unit in B, if $\alpha=0$ then $xy=0$ for all $y\in B$ and this cannot happen in a B^*-algebra), it follows that a nonzero element in B_1 gives a nonzero operator on B . Since the expression on the right is certainly a norm on the operators on B, by general Banach space theory it is also a norm on B_1.

We show first that when $\alpha=0$, $\|x\|' = \|x\|$ which means that the norm on B_1 induces the original norm on B. In general $\|xy\| \leq \|x\| \ \|y\|$, so that $\|x\|' \leq \|x\|$. But in a B -algebra, taking $y=x$, we get $\|xx^*\| = \|x\| \ \|x^*\|$ and so $\sup \dfrac{\|xy\|}{\|y\|} \geq \|x\|$ and hence $\|x\|' = \|x\|$. Suppose next δ is a real number > 0. Then there exists y with $\|y\|=1$ and $\|\alpha y+xy\| > (1-\delta) \ \|\alpha e+x\|'$. Then

$$(1-\delta)^2 \ (\|\alpha e+x\|')^2 < \|\alpha y+xy\|^2$$

$$= \|(\alpha y+xy)^* (\alpha y+xy)\|$$

$$= \|y^* (\alpha e+x)^* (\alpha e+x)y\|$$

$$\leq \|(\alpha e+x)^* (\alpha e+x)y\|$$

$$\leq \|(\alpha e+x)^* (\alpha e+x)\|'$$

Since δ could be arbitrarily small (α and x fixed) we get

$$(\|\alpha e+x\|')^2 \leq \|(\alpha e+x)^* (\alpha e+x)\|'$$

$$\leq \|(\alpha e+x)^*\|' \ \|\alpha e+x\|'$$

so that $\quad \|\alpha e+x\|' \leq \|(\alpha e+x)^*\|'$;

similarly $\quad \|(\alpha e+x)^*\|' \leq \|(\alpha e+x)\|'$,

so that $\quad \|(\alpha e+x)^*\|' = \|(\alpha e+x)\|'$.

But then $\quad (\|(\alpha e+x)\|')^2 \leq \|(\alpha e+x)^* (\alpha e+x)\|'$

$$\leq \|(\alpha e+x)^*\|' \|(\alpha e+x)\|' = \|\alpha e+x\|^2$$

i.e. $\|(\alpha e+x)^*(\alpha e+x)\|' = \|\alpha e+x\|^2$ which is what we want.

We still have to verify that B_1 is complete in $\| \|'$. Let $(\alpha_n e+x_n)$ be a Cauchy sequence in B_1. If (α_n) is not bounded then there exists a subsequence $\{\alpha_{n_k}\}$ with $|\alpha_{n_k}| \longrightarrow \infty$; then

$$\alpha_{n_k}^{-1}(\alpha_{n_k}e+x_{n_k}) \longrightarrow 0,$$

i.e.

$$\alpha_{n_k}^{-1}x_{n_k} \longrightarrow -e$$

But then $\{\alpha_{n_k}^{-1}x_{n_k}\}$ is a Cauchy sequence in B_1 hence in B and since B is complete there is a limit in B so that B has a unit e which is not so. Hence α_n is bounded. This being so there is a convergent subsequence $\{\alpha_{n_k}\}$; $\{\alpha_{n_k}e\}$ is a Cauchy sequence hence $x_{n_k} = (\alpha_{n_k}e + x_{n_k}) - \alpha_{n_k}e$ is a Cauchy sequence in B_1 and hence in B; therefore there is a limit, x say, in B. If $\alpha_{n_k} \longrightarrow \alpha$ then $\alpha_{n_k}e + x_{n_k} \longrightarrow \alpha e+x$ and hence $\alpha_n e + x_n \longrightarrow \alpha e+x$ also and B_1 is complete. $\|$

PROPOSITION 2.2: <u>If x is a normal element of a B^*-algebra then $\|x\| = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$: (spectral radius of x) if x is unitary then $\|x\| = 1$.</u>

PROOF: $\|x^*x\|^2 = \|x\|^4 = \|x^*x\|^2 = \|x^2\|^2$. (since $(x^*x)^2 = x^2(x^*)^2$; x being normal) and hence $\|x^2\| = \|x\|^2$. Hence $\|x^{2^n}\| = \|x\|^{2^n}$ for all n and so $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \lim_{m\to\infty} \|x^{2^m}\|^{2^{-m}} = \|x\|$, if $x^*x=e$ then clearly $\|x\| = 1$. $\|$

COROLLARY: In a commutative B^*-algebra $\|x\| = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ for all x.

In general in any $*$-algebra it is clear that either $\lambda e - x$ and $\bar{\lambda} e - x^*$ both have inverses or both fail to have inverses. Hence $\sigma(x^*) = \overline{\sigma(x)}$ (the bar denoting complex conjugation and not closure). If x is self adjoint then $\sigma(x) = \overline{\sigma(x)}$ so that $\sigma(x)$ is symmetric about the real axis. In general this is as far as we can go: $\sigma(x)$ need not be real. However we have the following

PROPOSITION 2.3: If B is a B^*-algebra (with a unit) and $x \in B$ is self-adjoint then $\sigma(x)$ is real.

PROOF: Suppose not, let $\alpha + i\beta \in \sigma(x)$, $\beta \neq 0$. Write $y = x + i\gamma e$ where γ is real; then $\alpha + i(\beta+\gamma) \in \sigma(y)$. Hence

$$\alpha^2 + (\beta+\gamma)^2 \leq [\lim \|y^n\|^{\frac{1}{n}}]^2 \leq \|y\|^2 = \|y^* y\|$$

and $y^* = x - i\gamma e$ so that $y^* y = x^2 + \gamma^2 e$
and $\|y^* y\| \leq \|x\|^2 + \gamma^2$ thus

$$\alpha^2 + \beta^2 + 2\beta\gamma \leq \|x\|^2 \qquad \text{for all real } \gamma \text{ which}$$

is clearly impossible if $\beta \neq 0$. ‖

We now turn to representations of $*$-algebras. By a representation we shall always mean in what follows a representation of B in $\mathcal{L}(H)$ in which the involution in B is mapped onto the natural involution in $\mathcal{L}(H)$; that is T is to be a $*$-representation, in which $T(x^*) = (T(x))^*$ for all $x \in B$. We do not assume the boundedness of T; in fact this will follow automatically (Proposition 3.5).

LEMMA 2.4: <u>If</u> $H_1 \subset H$ <u>is invariant under T then so is</u> $H_1^\perp$.

PROOF: If $\xi \in H_1^\perp$, $\eta \in H_1$, $x \in B$ then

$$\left(T(x)\xi \ , \eta\right) = \left(\xi, T(x)^*\eta\right) = \left(\xi, T(x^*)\eta\right) = 0$$

since $T(x^*)\eta \in H_1$ from this it follows that $T(x)\xi \perp \eta$ for all $\eta \in H_1$ and so $T(x)\xi \in H_1^\perp$. ‖

The next result is a substantial one and will be used essentially in what follows:-

PROPOSITION 2.5: T <u>is irreducible if and only if the only operators on</u> H <u>that commute with all the operators</u> $T(x)$, $x \in B$ <u>are scalar multiples of the identity</u>.

PROOF: If T is reducible let H_1 be a nontrivial invariant subspace and let P be the projection on H_1. If $\xi = \xi_1 + \xi_2$ with $\xi_1 \in H_1$, $\xi_2 \in H_1^\perp$, then

$$T(x)\xi = T(x)\xi_1 + T(x)\xi_2$$

and in view of Lemma 2.4, $T(x)\xi_2 \in H_1^\perp$ so that this must be the unique decomposition of $T(x)\xi$ as the sum of a vector in H_1 and a vector in $H_1^\perp$; that is

$$PT(x)\xi = T(x)\xi_1 = T(x)P\xi$$

and since ξ was arbitrary it follows that $PT(x) = T(x)P$, for all x, as required. We have a non-trivial operator commuting with all the $T(x)$.

If there is a non-trivial projection operator P that commutes with all $T(x)$ then $T(x)\cdot(P\xi) = P\left(T(x)\xi\right)$ so that T leaves invariant the non-trivial subspace $H_1 = P(H)$ and T is reducible.

More generally, suppose that T_0 is a bounued self-adjoint (Hermitian) operator that commutes with all the $T(x)$. Recalling the spectral theorem for self-adjoint operators, we see that there exists a spectral family $P(\lambda)$ of projection operators associated with T_0 that commute with all operators that commute with T_0, in particular $P(\lambda)T(x) = T(x)P(x)$ for all $x \in B$, $\lambda \in R$. If then T is irreducible the only projection operators that commute with $T(x)$ for all x are of the form αI, so that each $P(\lambda)$ is either zero or the identity operator. Since $P(\lambda)\,P(\mu) = P(\min(\lambda,\mu))$ it follows that for some λ_0, $P(\lambda) = 0$ for $\lambda < \lambda_0$ and $P(\lambda) = I$ for $\lambda > \lambda_0$; then

$$T_0 = \int \lambda \, d\,P(\lambda) = \lambda_0 I$$

Finally if T_0 is bounded but not necessarily self adjoint, commuting with all the $T(x)$, write $T_0 = \frac{1}{2}(T_0 + T_0^*) + \frac{1}{2i}(T_0 - T_0^*)$. The operators $\frac{1}{2}(T_0 + T_0^*)$, $\frac{1}{2i}(T_0 - T_0^*)$ are self adjoint. They also commute with $T(x)$ for all x for we have $T_o^* T(x) = ((T(x))^* T_o)^* = (T(x^*)T_o)^* = (T_oT(x^*))^* = (T_o(T(x))^*)^* = T(x)T_o^*$, T_o^* commutates with $T(x)$ hence $\frac{1}{2}(T_o + T_o^*)$ and $\frac{1}{2i}(T_o - T_o^*)$ do so also: by what has just been proved they are respectively $\alpha_1 I$ and $\alpha_2 I$ whence T_o is $(\alpha_1 + i\alpha_2)I$ as required.

Thus T irreducible implies $T_o = \alpha I$. ||

COROLLARY: _If B is commutative, then T is irreducible if and only if_ H _is one-dimensional_.

PROOF: Clearly H one-dimensional implies T irreducible If T is irreducible then for a fixed $x \in B$ we have

$$T(x)T(y) = T(xy) = T(yx) = T(y)T(x) \quad \text{for all } x \in B,$$

from which it follows that

$$T(x) = f(x)I$$

from the proposition. Since every subspace of H is then invariant under T clearly T is irreducible H must be one dimensional. ||

Thus the homomorphisms $B \longrightarrow C$ are the only irreducible $*$-representations in the commutative case.

THEOREM 2.6: _Let_ T _be any representation on_ H. _Then we can write_ H _as a direct sum of mutually orthogonal closed subspaces_

$$H = H_o \oplus \bigoplus_{i \in I} H_i$$

<u>such that</u> T <u>restricted to</u> H_0 <u>is zero and each</u> H_i <u>is cyclic for</u> T (<u>hence invariant under</u> T).

PROOF: Write $H_0 = \{ \xi : T(x)\xi = 0, \text{ for all } x \in B\}$; then evidently H_0 has the properties asserted and T is essential on $H_0^\perp$.

If $\xi' \in H_0^\perp$ then $H' = \mathrm{Cl}\,\{T(x)\xi' : x \in B\}$ is closed and clearly invariant under T. In fact it is cyclic, with ξ' as cyclic vector. This is clear if B has a unit, for then

$$\xi' = T(e)\xi' \quad \text{and} \quad \xi' \in H'.$$

In general, write H'' for $\mathrm{Cl}\,\{\alpha\xi' + T(x)\xi';\ \alpha \in C,\ x \in B\}$; we show $H'' = H'$. Suppose not : let $\xi \in H''$, $\xi \perp T(x)\xi'$ for every x. Then

$$\begin{aligned} 0 = (\xi, T(\alpha y^* + y^* x)\xi') &= (\xi, T(y^*)\cdot(\alpha I + T(x))\xi') \\ &= (T(y)\xi, \alpha\xi' + T(x)\xi') \end{aligned}.$$

Since $\xi \in H''$, then $T(y)\xi' \in H''$ also, and since vectors of the form $\alpha\xi' + T(x)\xi'$ are dense in H'' it follows that $T(y)\xi = 0$ for all y. But now observe that $H'' \subset H_0^\perp$ and T is essential on $H_0^\perp$ so if $\xi \neq 0$ there exists y with $T(y)\xi \neq 0$. Hence $\xi = 0$ and $H' = H''$ as required. Thus there are vectors $T(x)\xi'$ arbitrarily close to ξ' and so $\xi' \in H'$: it is then clearly a cyclic vector for H'.

So there exist systems of mutually orthogonal subspaces H_i , each cyclic for T but possibly not spanning $H_0^{\perp}$. Partially order such systems by inclusion apply Zorn's principle : it follows that maximal systems exist. If such a maximal system did not span $H_0^{\perp}$ then we could extend it by taking any vector in $(\oplus H_i)^{\perp}$ and, starting again, obtaining a new subspace with a cyclic vector, so that the system (H_i) would not be maximal.

What theorem 2.6 shows is that we may confine ourselves to cyclic representation if this is convenient, as any representation can be built up out of cyclic representations as in the Theorem. We write T_i for the restriction of T to H_i. ||

P O S I T I V E   F U N C T I O N A L S

Let B be any algebra with an involution; the norm is really irrelevant to begin with.

DEFINITION : A linear functional p on B is said to be <u>positive</u> if $p(x^*x) \geq 0$ for all $x \in B$. The positive functionals play a part in the non-commutative theory somewhat similar to that of the multiplicative linear functionals in the commutative theory.

PROPOSITION 3.1 : <u>If</u> p <u>is a positive functional then</u>

$$\text{(i)}\ p(y^*x) = \overline{p(x^*y)} \quad (\underline{\text{all}}\ x,\ y \in B)$$

$$\text{(ii)}\ |p(y^*x)|^2 = |p(x^*y)|^2 \leq p(x^*x) \cdot p(y^*y)$$

PROOF : $0 \leq p\,[(x+\alpha y)^*(x+\alpha y)]$

$$= p(x^*x) + \bar{\alpha}p(y^*x) + \alpha p(x^*y) + |\alpha|^2 \cdot p(y^*y)$$

Since the first and fourth term are real (and ≥ 0) the sum $\bar{\alpha}p(y^*x) + \alpha p(x^*y)$ is real. Put $\alpha = 1$ and we get $\mathfrak{Im}\,p(y^*x) + \mathfrak{Im}\,p(x^*y)$ and putting $\alpha = i$ we get $\mathrm{Re}\,p(y^*x) = \mathrm{Re}\,p(x^*y)$ so that we have (i) moreover we have then

$$0 \leq p(x^*x) + 2\,\mathrm{Re}\,[\alpha p(x^*y)] + |\alpha|^2\,p(y^*y)$$

If $p(x^*y) = 0$ then (ii) is obvious. Otherwise, take $\alpha = -\,p(x^*x)/p(x^*y)$ and then

$$0 \leq p(x^*x) - 2p(x^*x) + (p(x^*x))^2\,p(y^*y)/|p(x^*y)|^2$$

i.e. $\quad p(x^*x)\,|p(x^*y)|^2 \leq [p(x^*x)]^2\,p(y^*y)$

from which the required result follows if $p(x^*x) \neq 0$. If $p(x^*x) = 0$ but $p(y^*y) \neq 0$ we can repeat the argument with x and y interchanged. If both $p(x^*x)$ and $p(y^*y)$ are 0 we have

$$\mathrm{Re}\big(\alpha\, p(x^*y)\big) \geq 0$$

for all α which is impossible unless $p(x^*y) = 0$. So (ii) holds in all cases. ||

We now define something like a norm for the positive functionals (which do not assume bounded even when B is a Banach algebra). Write

$M(p) = 0$ if $p=0$, $M(p) = \infty$ if $p \neq 0$ but $p(x^*x)=0$ for all x and in general $M(p) = \sup\limits_{x \in B} \dfrac{|p(x)|^2}{p(x^*x)}$. Thus $|p(x)|^2 \leq M(p)\, p(x^*x)$, with the appropriate conventions about ∞, and $M(p)$ is the least number with this property. We have evidently $M(\alpha p) = \alpha M(p)$ if $\alpha \geq 0$; $M(p) = 0$ if and only if there exists $k < \infty$ with $|p(x)|^2 \leq k\, p(x^*x)$ all $x \in B$.

If B is a $*$-algebra without a unit let B_1 be the algebra with a unit e adjoined ; it is also a $*$-algebra. A positive functional p on B is <u>extendable</u> if there exists a positive functional p' on B_1 which when restricted to B coincides with p.

PROPOSITION 3.2: p <u>is extendable if and only if</u>

(i) $p(x^*) = \overline{p(x)}$ <u>for all</u> $x \in B$

(ii) $M(p) < \infty$.

<u>If</u> p <u>is extendable then for each</u> $\alpha \geq M(p)$ <u>there is an extension</u> p' <u>with</u> $p'(e) = \alpha$.

PROOF : Suppose p is extendable ; let p' be an extension. Take $y=e$ in (i) of Proposition 3.1 ;
then $p(x^*) = p'(x^*) = \overline{p'(x)} = \overline{p(x)}$. Similarly take $y=e$ in (ii) of Proposition 3.1 and

$$|p(x)|^2 = |p'(x)|^2 \leq p'(x^*x)\, p'(e) = p(x^*x)\, p'(e)$$

so that (ii) holds, and $M(p) \leq p'(e)$.
If (i) and (ii) hold let α be any real number $\geq M(p)$ and write $p'(\lambda e+x) = \lambda\alpha + p(x)$; we then have p' a linear functional which we shall prove is positive

$$\begin{aligned} p'\big((\lambda e+x)^*(\lambda e+x)\big) &= |\lambda|^2\alpha + \bar{\lambda}p(x) + \lambda p(x^*) + p(x^*x) \\ &= |\lambda|^2\alpha + 2\,\mathrm{Re}\big(\lambda\, p(x^*)\big) + p(x^*x) \\ &\geq |\lambda|^2\alpha - 2|\lambda|\cdot|p(x)| + p(x^*x) \\ &\geq |\lambda|^2\alpha - 2|\lambda|\alpha^{\frac{1}{2}}p(x^*x)^{\frac{1}{2}} + p(x^*x) \\ &\geq \left[|\lambda|\alpha^{\frac{1}{2}} - (p(\lambda^*x))^{\frac{1}{2}}\right]^2 \\ &\geq 0 \end{aligned}$$

as required.‖

If (i) and (ii) hold for p_1 and p_2 let p_1', p_2' be extensions with $p_1'(e) = M(p_1)$, $p_2'(e) = M(p_2)$ then $p_1' + p_2'$ is also positive and $M(p_1 + p_2) \leq p_1'(e) + p_2'(e) = M(p_1) + M(p_2)$.

An example of a nonextendable functional:-

Let B consists of (bounded) continuous complex functions on $[0, 1]$ with the usual involution and linear space structure, with all products equal to zero. Then for any fixed $t_o \in [0, 1]$, $p(x) = x(t_o)$ is a positive functional with moreover $p(x^*) = \overline{p(x)}$. However it is not extendable since $M(p) = \infty$.

From now on we usé essentially the relation between the norm in B and the involution, that is, $\|x^*\| = \|x\|$ for all $x \in B$. One or two results hold under weaker conditions also.

PROPOSITION 3.3: <u>If</u> B <u>has a unit</u> e <u>and</u> p <u>is a positive functional then</u>

$$|p(x)| \leq p(e)\|x\| \quad \text{for all } x \in B.$$

PROOF: If $\|x\| < 1$ then the series for $(e - x)^{\frac{1}{2}}$, that is

$$e - \frac{1}{2}x - \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2!}\,x^2 - \frac{1}{2}\,\frac{1}{2}\,\frac{3}{2}\,\frac{1}{3!}\,x^3 - \text{----}$$

converges absolutely to an element $y \in B$ with $y^2 = e - x$. If x is self-adjoint, so is y, from the series. Then we get

$$p(e-x) = p(y^2) = p(y^*y) \geq 0$$

and so $\qquad p(x) \leq p(e) \qquad$ if $\|x\| < 1$

But we can take $\|x\|$ as near to 1 as we please

so that $\qquad p(x) \leq p(e) \qquad$ if $\|x\| \leq 1.$

Thus in general, by the linearity of p ,

$$p(x) \leq p(e)\, \|x\| \quad \text{if } x \text{ is self adjoint,}$$

but, $$p(-x) = -p(x) \leq p(e)\|x\|$$

so $$|p(x)| \leq p(e)\|x\| \quad \text{for all self-adjoint } x.$$

If x is not self-adjoint take x^*x, which is self-adjoint:-

$$p(x^*x) \leq p(e) \cdot \|x^*x\| \leq p(e)\, \|x\|^2$$

By Proposition 3.1(ii) with y=e

$$|p(x)|^2 \leq p(e)\; p(x^*x)$$

so that $$|p(x)|^2 \leq (p(e))^2\, \|x\|^2$$

and the required result follows on taking the square root. ||

COROLLARY 1. *Every positive functional on an algebra with a unit is continuous.*

COROLLARY 2: *Every extendable positive functional on an algebra without a unit is continuous.*

We now turn to the relation between positive functionals and $*$-representations.

THEOREM 3.4: *Let* T *be a representation of* B *on the Hilbert space* H. *If* $\zeta \in H$ *then*

$$p(x) = (T(x)\zeta, \zeta)$$

is an extendable positive functional and $M(p) \leq \|\zeta\|^2$. *If T is cyclic and* ζ *is a cyclic vector then* $M(p) \leq \|\zeta\|^2$.

PROOF: p is evidently linear. Since

$$p(x^*x) = \left(T(x^*x)\zeta, \zeta\right) = \left(T(x^*)T(x)\zeta \;, \zeta\right)$$
$$= \left(T(x)^*T(x)\zeta, \zeta\right) = \left(T(x)\zeta \;, \; T(x)\zeta\right) \geq 0$$

p is clearly positive. We have also

$$p(x^*) = \left(T(x^*)\zeta, \zeta\right) = \left(T(x)^*\zeta, \zeta\right)$$
$$= \left(\zeta \;, \; T(x)\zeta\right) = \overline{\left(T(x)\zeta, \zeta\right)} = \overline{p(x)}$$

and $|p(x)|^2 = \left(T(x)\zeta \;, \zeta\right) \leq \|T(x)\zeta\|^2 \; \|\zeta\|^2$

$$= \|\zeta\|^2 \; p(x^* x).$$

so that $M(p) \leq \|\zeta\|^2 < \infty$. p is thus extendable.

If T is cyclic and ζ is a cyclic vector, then given $\varepsilon > 0$ we can find $x_o \in B$ with $\|T(x_o)\zeta - \zeta\| < \varepsilon$, since $\zeta \in H$ and the vectors $T(x)\zeta$ are dense in H. Then

$$|p(x_o)|^2 = |(T(x_o)\zeta, \zeta)|^2$$

is arbitrary close to $|(\zeta \;, \zeta)|^2 = \|\zeta\|^2$ & $p(x_o^* \; x_o) \quad \|T(x_o)\zeta\|^2$ is arbitrarily close to $\|\zeta\|^4 / \|\zeta\|^2 = \|\zeta\|^2$. But since M(p) is always between $\|\zeta\|^2$ and $|p(x_o)|^2 / p(x_o^* x_o)$ it folllows that M(p) is actually equal to $\|\zeta\|^2$ as asserted. ||

PROPOSITION 3.5: <u>Every</u> $*$-<u>representation of a Banach</u> $*$-<u>algebra is continuous</u> : <u>more precisely</u> $\|T(x)\| \leq \|x\|$ <u>for all</u> x.

PROOF: We may assume B has a unit : if not we could clearly extend any representation from B to B_1 by writing $T(\alpha e+x) = \alpha I+T(x)$. Then if $\xi \in H$

$$p(x) = \big(T(x)\xi, \xi\big) \quad \text{is a positive functional.}$$

Apply Proposition 3.3 and we get $|(T(x)\xi, \xi)| \leq \|x\|(\xi, \xi)$. Replace x by x^*x and we have $\|T(x)\xi\|^2 = \big(T(x^*x)\xi, \xi\big) \leq \|x^*x\| \, \|\xi\|^2 \leq \|x\|^2 \, \|\xi\|^2$ and so $\|T(x)\xi\| \leq \|x\| \, \|\xi\|$. Since $\xi \in H$ was arbitrary $\|T(x)\| \leq \|x\|$ as asserted. ||

We will now go from functionals to representations ; this is much more difficult.

THEOREM 3.6: <u>If p is an extendable positive functional on the Banach *-algebra then there is a cyclic *-representation T of B with cyclic vector ζ such that for all</u> $x \in B$.

$$p(x) = \big(T(x)\zeta, \zeta\big)$$

PROOF: Suppose first that B has a unit e. Write

$$N = \left\{ x \; : \; p(x^*x) = 0 \right\} ;$$

we show first that N is a left ideal in B. If $x \in N$ $y \in B$ then by Proposition 3.1 (ii)

$$|p(yx)|^2 \leq p(x^*x) \; p(yy^*) = 0 \quad \text{so } p(yx) = 0.$$

Then

$$p\big((yx)^* yx \big) = p\big((x^* y^* y) \, x \big) = 0$$

so that $x \in N$ implies $yx \in N$. Also if $x_1, x_2 \in N$ then

$$p\left((x_1+x_2)^*(x_1+x_2)\right)=p(x_1^*x_1)\ +p(x_2^*x_1)\ +p(x_1^*x_2)\ +p(x_2^*x_2)=0$$

and $p\left((\alpha x)^*\alpha x\right) = |\alpha|^2 \cdot p(x^*x) = 0$ so N is indeed a left ideal in B. In fact it is a closed ideal, but we do not require this.

Now take the quotient B/N ; this is a linear space. Denote its elements by ξ, η , ... We shall show that this can be made into a Hilbert space using the functional p. Suppose $x_1-x_2 \in N$ and $y_1-y_2 \in N$; then

$$\begin{aligned} p(y_1^*x_1) - p(y_2^*x_2) &= p\left(y_1^*(x_1-x_2)\right) + \overline{p\left((y_1-y_2)^* x_2\right)} \\ &= p\left(y_1^*(x_1-x_2)\right) + p\left(x_2^*(y_1-y_2)\right) \\ &= 0 + 0 = 0 \end{aligned}$$

It follows that the function $(\xi, \eta) = p(y^*x)$ is well defined on B/N : it does not depend on the choice of x,y in the equivalence classes ξ, η respectively. We can show easily that (ξ, η) has all the proporties of inner product:-

$$(\xi, \eta) = \overline{(\eta, \xi)}\ ,\ (\xi_1+\xi_2, \eta) = (\xi_1, \eta) + (\xi_2, \eta)\ ,$$

$$(\alpha\xi, \eta) = \alpha(\xi, \eta)\ ,\ (\xi, \xi) > 0 \text{ for } \xi \neq 0.$$

B/N is thus a pre-Hilbert space : it is in general not complete under the norm $\|\xi\| = (\xi, \xi)^{\frac{1}{2}}$. Let H be its completion. Now define the operators $T(x)$ on B/N as follows: Suppose $\phi(y) = \eta$ (ϕ the canonical map $B \longrightarrow B/N$) ; define $T(x)\eta$ to be $\phi(xy)$. This is independent of y (subject to

$\phi(y) = \eta$) since N is a left ideal and is easily verified to be a linear operator on B/N. Further $x \longrightarrow T(x)$ is clearly a homomorphism. We now examine the boundedness of $T(x)$: fix, for the moment, $y \in B$ and write

$$q_y(x) = p(y^* xy).$$

Then it is easy to see that q_y is a positive functional on B, and $q_y(e) = p(y^* y)$. By Proposition 3.3,

$$|q_y(x)| \leq p(y^* y) \, \|x\|.$$

Then if $\phi(y) = \eta$ we have

$$\begin{aligned}(T(x)\eta\,,\ T(x)\eta) &= p\left((xy)^* \, xy\right) = q_y(x^* x) \\ &\leq p(y^* y) \, \|x^* x\| = (\eta\,,\ \eta) \cdot \|x\|^2.\end{aligned}$$

Thus $\|T(x)\eta\| \leq \|x\| \; \|\eta\|$: so $T(x)$ is bounded and indeed $\|T(x)\| \leq \|x\|$. Moreover if we have

$$(T(x)\eta\,,\ \zeta) = p(z^* xy) = p((x^* z)^* y) = (\eta\,,\ T(x^*)\zeta)$$

so that $T(x^*) = T(x)^*$. Now take the (unique) extension by continuity of T from B/N to H and we have the required $*$-representation T.

This representation is cyclic : a cyclic vector is given by $\zeta = \phi(e)$. We have $(T(x)\zeta\,,\ \zeta) = p(e^* xe) = p(x)$, and any $\xi \in B/N$ is $\phi(x)$ for some $x \in B$, so as x runs through B, $T(x)\zeta = \phi(xe) = \phi(x)$ runs through the whole of B/N.

We now turn to the case where B has no unit. Take B_1, extend p, and proceed as above, obtaining H and the representation $x \longrightarrow T(x)$. Let H_1 be the subspace of H :

$$\{ \eta : T(x)\eta = 0 \text{ for all } x \in B \}$$

and $$H_2 = H_1^{\perp}.$$

Write $\zeta = \zeta_1 + \zeta_2$, where $\zeta_1 \in H_1$, $\zeta_2 \in H_2$. We show that T restricted to H_2 is the required representation, with ζ_2 as cyclic vector. We have, for $x \in B$.

$$\begin{aligned} p(x) &= (T(x)\zeta, \zeta) = (T(x)\zeta_1 + T(x)\zeta_2,\ \zeta_1 + \zeta_2) \\ &= (T(x)\zeta_2, \zeta_1) + (T(x)\zeta_2, \zeta_2) \end{aligned}$$

Now since H_1 is invariant under T so is H_2, and so

$$T(x)\zeta_2 \in H_2,\quad (T(x)\zeta_2, \zeta_1) = 0 \text{ giving}$$

$$p(x) = (T(x)\zeta_2, \zeta_2) .$$

Now vectors of the form $\alpha\zeta_2 + T(x)\zeta_2$ are dense in H_2, since for any $\eta \in H$ we have

$$\| \alpha\zeta_2 + T(x)\zeta_2 - \eta \| \leq \| \alpha\zeta + T(x)\zeta - \eta \|$$

and B/N is dense in H. Suppose $\xi_2 \in H_2$ is orthogonal to all $T(x)\zeta$, $x \in B$; then for all x, $y \in B$ we get

$$\begin{aligned} 0 &= (\xi_2,\ T(\alpha x^* + x^* y)\zeta_2) = (T(x)\xi_2,\ T(\alpha e + y)\zeta_2) \\ &= (T(x)\xi_2,\ \alpha\zeta_2 + T(y)\zeta_2) \end{aligned}$$

and it follows that $T(x)\xi_2 = 0$ for all $x \in B$ which implies, since $\xi_2 \in H_1^{\perp}$, that $\xi_2 = 0$. Therefore the vectors $\{T(x)\zeta_2\}$ are dense in H_2. Thus the theorem holds, with the Hilbert space H_2 and cyclic vector ζ_2. ||

CHAPTER 4

INDECOMPOSABLE FUNCTIONALS AND IRREDUCIBLE REPRESENTATIONS

We say that the (positive) functional p <u>dominates</u> the (positive) functional q and write $p > q$ or $q < p$ if there exists a positive real α such that $\alpha p-q$ is positive. Note that $p>q$ $q>p$ do not imply together $p=q$: any functional p dominates any positive multiple of itself and is dominated by any strictly positive multiple of itself. We clearly have that $p>q$, $q>r$ implies $p>r$. If p dominates only positive multiples of itself, it is called <u>indecomposable</u>.

In the following theorems p, T, H and ζ are as in theorem 3.6.

THEOREM 4.1: <u>If S is a positive self-adjoint operator on H commuting with all the T(x) then</u>

$$q(x) = (ST(x)\zeta , \zeta)$$

<u>is a positive extendable functional, with</u> $q<p$. <u>Conversely if q is positive extendable functional with</u> $q<p$ <u>there exists a positive self-adjoint S such that</u> $q(x)$ <u>is given by the above formula</u>.

PROOF: If S is positive and self-adjoint it has a (unique) positive self-adjoint square root $S^{1/2}$ which commutes with everything that commutes with S , in particular with all the T(x) . Writing $q(x) = (ST(x)\zeta , \zeta)$, it is clear that q is a linear functional on B. Also

$$q(x^*x) = (ST(x^*x)\zeta, \zeta) = (S^{\frac{1}{2}}T(x)\zeta, S^{\frac{1}{2}}T(x)\zeta) \geq 0;$$

also $q(x^*) = (ST(x^*)\zeta,\zeta) = (\zeta, ST(x)\zeta) = \overline{(ST(x)\zeta,\zeta)} = \overline{q(x)}$

and $|q(x)|^2 = |(ST(x)\zeta,\zeta)|^2 = |(T(x)S^{\frac{1}{2}}\zeta, S^{\frac{1}{2}}\zeta)|^2$

$$\begin{aligned} &\leq \|T(x)S^{\frac{1}{2}}\zeta\|^2 \|S^{\frac{1}{2}}\zeta\|^2 \\ &= (T(x)S^{\frac{1}{2}}\zeta, T(x)S^{\frac{1}{2}}\zeta)\|S^{\frac{1}{2}}\zeta\|^2 \\ &= (ST(x^*x)\zeta,\zeta)\|S^{\frac{1}{2}}\zeta\|^2 \\ &= q(x^*x)\|S^{\frac{1}{2}}\zeta\|^2 \end{aligned}$$

so that (i) and (ii) of proposition 3.2 hold with $M(q) \leq \|S^{\frac{1}{2}}\zeta\|^2$, so q is a positive extendable functional.

Finally if $\alpha \geq \|S\|$ then $\alpha p - q$ is positive; for $q(x^*x) = \|S^{\frac{1}{2}}T(x)\zeta\|^2 \leq \|S^{\frac{1}{2}}\|^2 \|T(x)\zeta\|^2 = \|S^{\frac{1}{2}}\|^2 \cdot p(x^*x) = \|S\| \cdot p(x^*x)$: so if $\alpha \geq \|S\|$ then $\alpha p(x^*x) - q(x^*x) \geq 0$ as required.

To prove the converse : let $H' = \{T(x)\zeta : x \in B\}$. Then since ζ is cyclic H' is dense in H (and a linear subspace). For $x, y \in B$ write

$$Q(T(x)\zeta, T(y)\zeta) = q(y^*x).$$

We show first that this depends only on $T(x)\zeta$ and $T(y)\zeta$, not on the particular choice of x and y. Suppose $T(x')\zeta = T(x)\zeta$, $T(y')\zeta = T(y)\zeta$; then $T(x'-x)\zeta = T(y-y')\zeta = 0$ so that

$$p((x-x')^*(x-x')) = p(y-y')^*(y-y') = 0$$

(being $(T(x-x')\zeta,\ T(x-x')\zeta)$, $(T(y-y')\zeta,\ T(y-y')\zeta)$ respectively) and it follows since $q<p$ that

$$q\big((x-x')^*(x-x')\big)=q\big((y-y^*)(y-y')\big)=0.$$

Now use Proposition 3.1 (ii) ; we have

$$\begin{aligned}
|q(y^*x) - q(y'^*x')| &= |q(y^*-y'^*)x + q(y'^*(x-x'))| \\
&\leq |q(y^*-y'^*)x| + |q(y'^*(x-x'))| \\
&\leq [q(y^*-y'^*)(y-y')q(x^*x)]^{1/2} \\
&+ [q((y'^*y')q(x^*-x'^*))(x-x')]^{1/2} \\
&= 0.
\end{aligned}$$

It is clear Q is linear in $T(x)\zeta$ and conjugate-linear in $T(y)\zeta$; moreover if $\alpha p-q$ is positive then

$$\begin{aligned}
|Q\ T(x)\zeta,\ T(y)\zeta| = |q(y^*x)| &\leq \left(q(x^*x)^{\frac{1}{2}}\ q(y^*y)^{\frac{1}{2}}\right) \\
&\leq \alpha\left[p(x^*x)^{\frac{1}{2}}\,(p(y^*y)^{\frac{1}{2}}\right] \\
&= \alpha\|T(x)\zeta\|\ \|T(y)\zeta\|.
\end{aligned}$$

Thus Q is continuous on $H'\times H'$ and hence there is a unique extension by continuity to the whole of $H\times H$, also linear in one variable and conjugate-linear in the other. Now any such function must be of the form

$$Q(\xi,\eta) = (S\xi,\eta) \quad (\text{all } \xi,\eta\in H)$$

where S is some bounded linear operator on H.

We proceed to verify the properties asserted for S. First, $(ST(x)\zeta,\ T(y)\zeta)=q(y^*x) = \overline{q(x^*y)} = \overline{(ST(y)\zeta,T(x)\zeta)}$

$(ST(x)\zeta,\ T(x)\zeta) = q(x^*x) \geq 0$, so that S is positive (="non-negative definite"). For $x,y,z \in B$,

$$(ST(x)T(y)\zeta\ ,\ T(z)\zeta) = q(z^*xy) \text{ and}$$

$$(T(x)ST(y)\zeta\ , T(z)\zeta) = (ST(y)\zeta\ , T(x^*z)\zeta)$$
$$= q((x^*z)^*y) = q(z^*xy)$$

and it follows (H' being dense in H ; all these arguments depend on this fact) that

$$ST(x) = T(x)S \qquad \text{for all } x.$$

Then

$$q(y^*x) = (ST(x)\zeta\ ,\ T(y)\zeta)$$
$$= (ST(y^*x)\zeta\ ,\ \zeta)$$

for all $x,y \in B$.

We wish to show that

$$q(x) = (ST(x)\zeta\ ,\ \zeta) \qquad \text{for all } x.$$

If B has a unit e then simply put y=e in the formula for $q(y^*x)$. In general, write

$$q'(x) = (ST(x)\zeta, \zeta)\ ;$$

by the first part of this theorem and Theorem 3.6 there exists a Hilbert space H', a cyclic representation T' and a cyclic vector $\zeta' \in H'$ with $q'(x) = (T'(x)\zeta', \zeta')$. Also there exist H", T", ζ'' such that $q(x) = (T''(x)\zeta'', \zeta'')$ Now define a map $\mathcal{U}$ as follows

$$\mathcal{U}T'(x)\zeta' = T''(x)\zeta''.$$

the map is well defined: for we have

$T'(x_1-x_2)\zeta'=0 \Longleftrightarrow (T'(x_1-x_2)^*(x_1-x_2)\zeta',\zeta')=0 \Longleftrightarrow$
$q'((x_1-x_2)^*(x_1-x_2))=0 \Longleftrightarrow q((x_1-x_2)^*(x_1-x_2))=0 \Longleftrightarrow$
$T''((x_1-x_2)^*(x_1-x_2)\zeta'',\zeta'')=0 \Longleftrightarrow \|T''(x_1-x_2)\zeta''\|=0 \Longleftrightarrow$
$T''(x_1-x_2)\zeta''=0$; so if $T'(x_1)\zeta'=T'(x_2)\zeta'$ then $T''(x_1)\zeta''=T''(x_2)\zeta''$ and conversely : so $T''(x)\zeta''$ is genuinely a function of $T'(x)\zeta'$. U is then evidently a linear map of a dense subspace of H' onto a dense subspace of H''. It is unitary : for

$$(T''(x)\zeta'', T''(y)\zeta'')=(T''(y^*x)\zeta'',\zeta'')=q(y^*x)$$
$$= q'(y^*x) =(T'(y^*x)\zeta',\zeta')=(T'(x)\zeta',T'(y)\zeta')$$

and so since in particular U is continuous it can be extended uniquely to a unitary transformation of the whole of H' onto the whole of H''.

Then $UT'(xy)\zeta'=T''(xy)\zeta''$ so that

$$UT'(x)T'(y)\zeta'= T''(x)\ T''(y)\zeta''=T''(x)UT'(y)\zeta';$$

since the vectors $T'(y)\zeta'$ are dense in H' it follows that $UT'(x)\zeta'= T''(x)U$ and

$$T''(x)\zeta''=UT'(x)\zeta' = T''(x)U\zeta'.$$

Hence $(U\zeta', T''(x)\zeta'') = (T''(x^*)U\zeta', \zeta'')$

$$= (T''(x^*)\zeta'',\zeta'') = (\zeta'', T''(x)\zeta'') \quad ,$$

and vectors $T''(x)\zeta''$ are dense in H'', so that $U\zeta'=\zeta''$. Then finally

$$q(x) =(T''(x)\zeta,\zeta'')= (T''(x)U\zeta', U\zeta')$$
$$=(UT'(x)\zeta,U\zeta')= (T'(x)\zeta',\zeta') = q'(x)$$
$$= (ST(x)\zeta, \zeta) \qquad \text{as required. } \|$$

THEOREM 4.2: <u>T is irreducible if and only if p is indecomposable</u>.

PROOF: Suppose p is decomposable, $p > p_1$, say, where p_1 is not zero and not a multiple of p. Then $p_1(x) = (ST(x)\zeta, \zeta)$, where S commutes with all the T(x), by Theorem 4.1. This S cannot be of the form αI, otherwise p_1 would be αp. So by Proposition 2.5, T is reducible.

Suppose T reducible ; let P be the projection on a non trivial invariant subspace H_1 say, of H then $PT(x) = T(x)P$ for all x writing

$$p_1(x) = (PT(x)\zeta \ .\zeta) \quad ,$$

p_1 is a positive functional dominated by p (in fact $p(x^*x) - p_1(x^*x) \geq 0$ for all x). This cannot be a multiple of p: for we can find x with $T(x)\zeta$ arbitrarily close to $(I-P)\zeta = \zeta_2$ say. If then $P\zeta = \zeta_1$, we have

$$p(x) = (T(x)(\zeta_1 + \zeta_2), (\zeta_1 + \zeta_2))$$

$$= (\zeta_2, \zeta_2) + \eta \text{ say}$$

and $|p_1(x)| = |(PT(x)\zeta, \zeta)| = |(0, \zeta)| + \eta_1$

so that $\left| p_1(x) / p(x) \right| = \eta_1 / (\zeta_1, \zeta_2) + \eta$

which can be arbitrarily small : this contradicts $p_1 = \alpha p$ for fixed finite real α so p is decomposable.‖

In general if the extendable positive functional p is decomposable and we write $p=p_1+p_2$, where p_1 and p_2 are also extendable positive functionals, then $M(p) \leq M(p_1) + M(p_2)$; for if p', p_1', p_2' are the appropriate extensions then $M(p_1)$ and $M(p_2)$ can be taken for $p_1'(e)$, $p_2'(e)$, respectively ; and we have $M(p) = M(p_1+p_2) \leq p'(e) = p_1'(e) + p_2'(e) = M(p_1) + M(p_2)$.

It will be useful to have the following result, which sharpens this inequality to an equality.

PROPOSITION 4.3: _If p is an extendable decomposable positive functional then there exist positive functionals_ p_1 _and_ p_2, _neither of them a multiple of_ p, _with_ $p = p_1 + p_2$ _and_ $M(p) = M(p_1) + M(p_2)$

PROOF: If p is decomposable, the associated cyclic representation T is reducible by Theorem 4.2. Let P be the projection on a non-trivial invariant subspace H_1 of H and write

$$p_1(x) = \left(PT(x)\zeta, \zeta\right) , \ p_2(x) = \left((I-P)T(x)\zeta, \zeta\right)$$

Then evidently p_1 and p_2 are extendable positive functionals and $p = p_1 + p_2$. By the argument already used at the end of the proof of Theorem 4.2, neither p_1 and p_2 can be a multiple of p.

Writing $P\zeta = \zeta_1$, $(I-P)\zeta = \zeta_2$, we have $p_1(x) = (T(x)\zeta_1, \zeta_1)$, $p_2(x) = (T(x)\zeta_2, \zeta_2)$ and so, by Theorem 3.4

$$M(p_1) + M(p_2) \leq (\zeta_1, \zeta_1) + (\zeta_2, \zeta_2) = (\zeta, \zeta) = M(p)$$

since T is cyclic with cyclic vector ζ (in fact of course $M(p) = (\zeta_1, \zeta_1)$ and $M(p_2) = (\zeta_2, \zeta_2)$ since both ζ_1 and ζ_2 are cyclic vectors in PH and (I-P)H respectively but we do not need this). In any case the required result follows from the general inequality noted immediately before the theorem and the reverse inequality established in the proof of the theorem. ‖

CHAPTER 5

THE SELF-ADJOINT ELEMENTS OF B AS A BANACH SPACE

If B is a Banach $*$-algebra then since any real multiple of a self-adjoint element is again self-adjoint, and any sum of self-adjoint element is self-adjoint, it follows that the self-adjoint elements of B form a real linear subspace. Denote this by B_s. It is evidently normed (as a subspace of B) and if $x_n = x_n^*$ for all n, $x_n \longrightarrow x$ then $\lim x_n^* = (\lim x_n)^* = x^* = x$, so that B_s is closed in B, hence complete, hence a Banach space in its own right. If p is an extendable, positive functional on B then its restriction to B_s is a real linear functional, since $p(x^*) = p(x) = \overline{p(x)}$. As a functional on B it is continuous, by Proposition 3.3, Corollary. Write $\|p\|$ for the norm of p as an element of the dual of B and $\|p\|_s$ for the norm of (the restriction of) p as an element of the dual of B_s. It is immediate that $\|p\|_s \leq \|p\|$.

PROPOSITION 5.1: $\|p\|_s = \|p\| \leq M(p)$: <u>if</u> B <u>has a unit then</u> $\|p\|_s = \|p\| = M(p)$.

PROOF: Suppose $x_0 \in B$, $\|x_0\| = 1$ and $|p(x_0)| > \|p\| - \varepsilon$. Multiply x_0 by $e^{i\theta}$ if necessary : we get an x with $p(x) > \|p\| - \varepsilon$. Then also $p(x^*) = \overline{p(x)} > \|p\| - \varepsilon$, and so $p(\frac{1}{2}(x+x^*)) > \|p\| - \varepsilon$. But $\frac{1}{2}(x+x^*)$ is self-adjoint and $\|\frac{1}{2}(x^*+x)\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|x^*\| = \frac{1}{2} + \frac{1}{2} = 1$ so that

$$\|p\|_s = \sup_{y=y^*} \frac{|p(y)|}{\|y\|} > \|p\| - \varepsilon \quad .$$

Since ε was arbitrary $\|p\|_s \geq \|p\|$ and so $\|p\|_s = \|p\|$.

By Proposition 3.2 there is an extension p' of p with $p'(e) = M(p)$. By Proposition 3.3 $|p'(x)| \leq p'(e)\|x\|$ and so

$$\|p'\| \leq p'(e) \quad \text{(actually equal, of course).}$$

Evidently $\|p\| \leq \|p'\|$ and so finally

$$\|p\| \leq \|p'\| \leq p'(e) = M(p)$$

If B has a unit e then by Proposition 3.1 (ii) we have, taking $y=e$, $|p(x)|^2 \leq p(e)(p(x^*x))$ and so $M(p) \leq p(e) \leq \|p\|$ (since $\|e\|=1$). Hence $\|p\|=M(p)$. $\|$

From now on we shall drop the suffix from $\|p\|_s$ in view of Proposition 5.1. We also note the corollary that if p is non-zero on B then its restriction to B_s must also be non-zero, since if the restriction were zero then $\|p\|_s = 0$, $\|p\|=0$ and so $p=0$ by the basic properties of a norm.

Now denote the set of extendable positive functionals p on B with $M(p) \leq 1$ by P. This is never empty : it contains at least the zero functional. Recall the weak*-topology of the dual of a linear space E ; the basic neighbourhoods of $f_o \in E'$ are

$$\{ f : |f(x_r) - f_o(x_r)| < \varepsilon \quad r=1,2,\ldots\ldots,n \}$$

for $\varepsilon > 0$ and $x_1, x_2, \ldots\ldots x_n \in E$. Then we have the

BOURBAKI-ALAOGLU THEOREM : <u>The unit ball in the dual of a Banach space is compact in the weak*-topology.</u>

PROPOSITION 5.2 : <u>P is a weak*-closed (and hence compact) convex subset of the unit ball of the dual of</u> B_s.

PROOF: Since $\|p\|_s = \|p\| \leq M(p) \leq 1$, evidently P is a subset of the unit ball. If $p_1, p_2 \in P$ then if p_1', p_2' are extensions it is clear that $\alpha p_1' + (1-\alpha) p_2'$ $(0 \leq \alpha \leq 1)$ is an extension of $\alpha p_1' + (1-\alpha)p_2'$ and also $M(\alpha p_1 + (1-\alpha)p_2) \leq \alpha M(p_1) + (1-\alpha)M(p_2) \leq 1$ if $M(p_1) \leq 1$, $M(p_2) \leq 1$; so $\alpha p_1 + (1-\alpha)p_2 \in P$, that is, P is convex.

Suppose $p_o \in \mathrm{Cl}\, P$ (that is, the closure in the dual of B_s). For $x \in B$ write $x = x_1 + ix_2$, where $x_1, x_2 \in B_s$ and extend p_o to B by writing $p_o(x) = p_o(x_1) + ip_o(x_2)$. Then given $\varepsilon > 0$ and $x_1, x_2, x^*x \in B_s$ there exists $p \in P$ with

$$|p(x_1) - p_o(x_1) < \varepsilon, |p(x_2) - p_o(x_2)| < \varepsilon, |p(x^*x) - p_o(x^*x)| < \varepsilon$$

In particular

$$0 \leq p(x^*x) \leq p_o(x^*x) + \varepsilon ;$$

since ε is arbitrary $p_o(x^*x) \geq 0$ and p_o is positive. Also

$$|p(x) - p_o(x)| \leq |p(x_1) - p_o(x_1)| + |p(x) - p_o(x_2)| < 2\varepsilon \quad ,$$

so that we get

$$|p_o(x)|^2 \leq |p(x)|^2 + |\,[p(x)]^2 - [p_o(x)]^2|$$
$$\leq |p(x)|^2 + 2\|x\|\,|p(x) - p_o(x)|$$
$$\leq p(x^*x) + 4\|x\|\varepsilon$$
$$\leq p_o(x^*x) + (4\|x\|+1)\varepsilon \quad .$$

and since ε is arbitrary,

$$|p_o(x)|^2 \leq p_o(x^*x)$$

so that $M(p_o) \leq 1$. Thus $p_o \in P$ and P is closed. $\|$

A point $p \in P$ is <u>extreme</u> if it is not of the form $\alpha p_1 + (1-\alpha)p_2$ where $0 < \alpha < 1$, $p_1, p_2 \in P$ and $p \neq p_1$, $p \neq p_2$. The zero functional is always an extreme point of P.

PROPOSITION 5.3: <u>A non-zero functional $p \in P$ is extreme if and only if</u>

(i) $M(p) = 1$ <u>and</u>

(ii) p <u>is indecomposable</u>.

PROOF: Suppose $0 < M(p) < 1$. Then we can write

$$p = M(p)\,\frac{p}{M(p)} + \bigl(1-M(p)\bigr)\,0$$

and both $p/M(p)$ and 0 distinct from p, so p cannot be extreme. If p is decomposable then by Proposition 4.3, we can write $p=p_1+p_2$ where neither p_1 nor p_2 is a multiple of p (and neither is zero) : we have

$$M(p) = M(p_1) + M(p_2)$$

Thus $p = \frac{M(p_1)}{M(p)} \frac{p_1}{M(p_1)} + \frac{M(p_2)}{M(p)} \frac{p_2}{M(p_2)}$ and now

$\frac{p_1}{M(p_1)} \in P$, $\frac{p_2}{M(p_2)} \in P$, and neither is equal to p so that p cannot be extreme. Thus (i) and (ii) are necessary conditions for p to be extreme.

If p is not extreme we can write $p=\alpha p_1 + (1-\alpha)p_2$ with $0 < \alpha < 1$ $p \neq p_1$, $p \neq p_2$. There are two possibilities; if neither p_1 nor p_2 is a multiple of p then p is clearly decomposable, since then $p-\alpha p_1$ is a positive functional $\left(= (1-\alpha)p_2\right)$. If on the other hand one (and hence both) of p_1, p_2 is a multiple of p, say $p_1 = \alpha p$, $p_2 = \beta p$, then one of α, β must be >1; say $\alpha > 1$. Then $M(p_1) \leq 1$ and $M(p) = M(p_1/\alpha) \leq \frac{1}{\alpha} < 1$. So (i) and (ii) together are sufficient for p to be extreme. ||

COROLLARY: <u>If</u> p <u>is nonzero and extreme then the associated representation is irreducible.</u>

We next recall the

KREIN-MILMAN THEOREM: <u>Let K be a compact convex subset of a real locally convex linear topological space E and let</u> K_1 <u>be the set of convex combinations of extreme points of</u> K. <u>Then</u> $K = Cl\ K_1^{*}$

[By a convex combination of extreme points we mean a finite sum $\sum \alpha_\gamma e_\gamma$ where the e_γ are extreme and the α_γ are positive real scalars with $\sum \alpha_\gamma = 1$].

[In fact we do not need the full force of the Krein-Milman theorem in order to prove Proposition 5.4, but only the partial results that given any hyperplane in E there exists a supporting hyperplane of K that is parallel to it, and that every supporting hyperplane of K contains an extreme point of K : however we shall not go into this refinement.]

PROPOSITION 5.4: _If for_ $x \in B$ _we have_ $p(x) \neq 0$ _for some_ $p \in P$ _then we have_ $q(x) \neq 0$ _for some extreme point_ $q \in P$.

PROOF: Suppose first x is self-adjoint and $p(x) \neq 0$. Then by the Krein-Milman theorem we can find extreme points $q_1, q_2, \ldots\ldots q_n$ and positive scalars $\alpha_1, \cdots, \alpha_n$ with

$$|p(x) - \sum \alpha_r q_r(x)| < |p(x)|$$

Hence for atleast one value of r, we must have $q_r(x) \neq 0$. In general, if $x = x_1 + ix_2$ where x_1 and x_2 are self-adjoint and $p(x) \neq 0$ then not both $p(x_1)$ and $p(x_2)$ are zero. If (say) $p(x_1) \neq 0$ then there exists an extreme point q with $q(x_1) \neq 0$ thus Re $q(x) \neq 0$ and so $q(x) \neq 0$. ||

COROLLARY: _If_ $p(x^* x) > 0$ _for some_ $p \in P$ _then also_ $q(x^* x) > 0$ _for some extreme point_ $q \in P$.

We can now state one of our main theorems:

THEOREM 5.5: _Let_ B _be a Banach_ $*$_-algebra and_ $x \in B$. _Then the following are equivalent_.

(i) $\exists$ $p \in P$ with $p(x) \neq 0$

(ii) $\exists$ $p \in P$ with $p(x^*x) > 0$

(iii) $\exists$ extreme $p \in P$ with $p(x) \neq 0$

(iv) $\exists$ extreme $p \in P$ with $p(x) > 0$

(v) $\exists$ $*$-representation T with $T(x) \neq 0$

(vi) $\exists$ irreducible $*$-representation T with $T(x) \neq 0$.

PROOF: The implications (iii) $\Longrightarrow$ (i), (iv) $\Longrightarrow$ (ii). (vi) $\Longrightarrow$ (v) are trivial. The implications (i) $\Longrightarrow$ (iii), (ii) $\Longrightarrow$ (iv) have just been established (Proposition 5.4). We now prove (i) $\Longrightarrow$ (ii), (iv) $\Longrightarrow$ (vi), (v) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii): $|p(x)|^2 \leq M(p)\ p(x^*x) \leq p(x^*x)$ so if $p(x) \neq 0$ then $p(x^*x) \neq 0$.

(iv) $\Longrightarrow$ (vi): by Theorem 4.2 if p is indecomposable, T is irreducible: and if $p(x^*x) > 0$ then $T(x) = 0$ since $p(x^*x) \neq 0$ $(T(x)\zeta, T(x)\zeta) = \|T(x)\zeta\|^2$: if $p(x^*x) > 0$ then $T(x)\zeta \neq 0$ and so $T(x) \neq 0$.

(v) $\Longrightarrow$ (i): we note first that T_0 is any linear operator in a complex Hilbert space then $(T_0\xi, \xi) = 0$ for all ξ implies $T_0 = 0$. This follows from the identity

$$4(T_0\xi, \eta) = (T_0(\xi+\eta), (\xi+\eta)) - (T_0(\xi-\eta), (\xi-\eta))$$
$$+ i(T_0(\xi+i\eta), (\xi+i\eta)) - i(T_0(\xi-i\eta), (\xi-i\eta)) \;;$$

if each term on the right is zero then $(T_0\xi, \eta)$ is zero for all ξ, η hence $T_0\xi$ is zero for all ξ hence T_0 is zero. If then there is a representation T(not cyclic in general) with $T(x) \neq 0$ then there exists $\xi \in H$ with $(T(x)\xi, \xi) \neq 0$ and $\|\xi\| = 1$. Then if $p(x) = (T(x)\xi, \xi)$ p is evidently a positive extendable functional; by Proposition 3.4 we have $M(p) \leq \|\xi\|^2 = 1$ so $p \in P$ as required. $\|$

CCROLLARY: _If_ B _is a_ C^*-_algebra and_ $x \in B$ _is non-zero then there is an irreducible representation_ T _with_ $T(x) \neq 0$.

We have at this stage reached the point where we can assert that if representations of a certain kind exist (separating, in particular) then also irreducible representations of the same kind exist. However, we cannot assert that for a general Banach $*$-algebra there are enough representations to separate points. The fact that this is so for $L_1(G)$ is vital for the theory of group representations and is quite easy to prove - we return to this later. In the meantime we specialise our algebras further to the case of a B^*-algebra.

CHAPTER 6

THE ELEMENTS x^*x AS A CONE IN B_s

Throughout this section and the next let B be a B^*-algebra ; some results are valid in more general situations. The results proved in earlier sections are all applicable. The main result proved in the next section is the celebrated Gelfand-Naimark theorem, that B is isometric and isomorphic to a closed sub-algebra of $\mathcal{L}(H)$ for some Hilbert space H. This, it might be emphasised, is for the complex case: the real case is not so easy to discuss. It is clear that in general B cannot be isometrically isomorphic to the whole of $\mathcal{L}(H)$; consider the case where B is commutative and of dimension > 1.

We begin with the remark that there is a no loss of generality in assuming that B has a unit. For, if not, consider B_1 with the norm described in Proposition 2.1. If B is isometrically $*$-isomorphic to a closed subalgebra of $\mathcal{L}(H)$, the same must be true of B, since B is a closed subalgebra of B_1 .

LEMMA 6.1: *If $x,y \in B$, α is a scalar, and one of $(e+\alpha xy)^{-1}$, $(e+\alpha yx)^{-1}$ exists, then so does the other.*

PROOF: Suppose $(e+\alpha xy)^{-1}$ exists. Then

$$(e+\alpha yx)\,[e-\alpha y(e+\alpha xy)^{-1}x] = [e-\alpha y(e+\alpha xy)^{-1}x]\times$$
$$\times(e+\alpha yx) = e$$

so that $(e+\alpha yx)^{-1}$ exists : similarly if $(e+\alpha yx)^{-1}$ exists, so does $(e+\alpha xy)^{-1}$. ||

COROLLARY: *If* $x,y \in B$ *then* $\sigma(xy)$ *and* $\sigma(yx)$ *are the same, except that possibly* 0 *may be in one set but not in the other.*

PROOF: If $\lambda \neq 0$, take $\alpha = -1\,\lambda^{-1}$ in Lemma 6.1 and it follows that if one of $(\lambda e - xy)^{-1}$, $(\lambda e - yx)^{-1}$ exists, so does the other. ||

To see that the sets $\sigma(xy)$ and $\sigma(yx)$ may indeed be different, take for example $B = \mathcal{L}(\ell_2)$, and the infinite matrices

$$x = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . \\ 0 & 0 & 1 & 0 & . & . \\ 0 & 0 & 0 & 1 & . & . \\ & & & & & \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 & 0 & . & . \\ 1 & 0 & 0 & . & . \\ 0 & 1 & 0 & . & . \\ & & & & \end{bmatrix}$$

then $xy = e$ and

$$yx = \begin{bmatrix} 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & . & . & . \\ & & & & & \end{bmatrix}$$

so that $\sigma(xy) = \{1\}$ while $\sigma(yx) = \{0,1\}$. It can be proved that if B is a finite-dimensional algebra then $\sigma(xy) = \sigma(yx)$.

PROPOSITION 6.2: *If* $x \in B$ *is self-adjoint, the following are equivalent.*

(i) $\sigma(x)$ $[0,\infty[$;

(ii) $x = y^2$ *for some self-adjoint* $y \in B$;

(iii) $\|e-\alpha x\| \leq$ <u>for some strictly positive real</u> α ;

(iv) $\|e-\alpha x\| \leq 1$ for all α with $0 \leq \alpha \leq \dfrac{2}{\|x\|}$;

(v) $\big\| \|x\| e-x \big\| \leq \|x\|$.

PROOF: Suppose (i), and take any suitable closed commutative sub-algebra of B containing x in which the spectrum of x is the same as the spectrum in B. [A suitable subalgebra would be B"(x) . all elements that commute with everything that commutes with x ; if $(\lambda e-x)^{-1}$ exists in B it must be in B"(x) and so the spectrum $\sigma''(x)$ of x in B"(x) is exactly $\sigma(x)$]. Now use the representation theorem for commutative B^*-algebra as algebras C(X) (we have X compact here, since B has a unit, although this is really irrelevant). Since the function $\hat{x}$ corresponding to x is non-negative (its values are precisely the points of $\sigma(x)$), it has a (unique) non-negative square root $\hat{y}$: let y be the element of B corresponding to this. Since the correspondence between the algebra and the function algebra C(X) is a $*$-isomorphism, y must be self-adjoint and $x=y^2$.

Conversely, if $x=y^2$ with y self-adjoint take a suitable commutative sub-algebra of B containing y in which the spectrum of y is exactly $\sigma(y)$ say, B"(y). Let $\hat{y}$ be the function corresponding to y in the representation of this algebra as a C(X) : since y is self-adjoint $\hat{y}$ is a real function (Proposition 2.3) and so $\hat{x}=\hat{y}^2$ is non-negative. Since $\sigma(x)$ cannot contain any point that is not a value taken by $\hat{x}$ it follows that $\sigma(x) \subset [0,\infty[$. We have thus proved (i) $\Longrightarrow$ (ii).

To prove (i) $\Longrightarrow$ (iv) : take again B"(x) or some other suitable sub-algebra and consider $\hat{x}$: this is ≥ 0. So if $\alpha \leq \frac{2}{\|x\|} = \frac{2}{\|\hat{x}\|}$ we have $0 \leq \alpha\hat{x} \leq 2$, so that $-1 \leq 1-\alpha\hat{x} \leq 1$, giving $\|1-\alpha\hat{x}\| \leq 1$ and hence (since the correspondence between the algebra and the corresponding $\mathbb{C}(\mathfrak{X})$ is isometric) $\|e-\alpha x\| \leq 1$.

The implication (iv) $\Longrightarrow$ (iii) is of course trivial. To show (iii) $\Longrightarrow$ (i) : suppose $\hat{x}(M) < 0$ for some $m \in \mathfrak{X}$. Then if $\alpha > 0$, $1-\alpha\hat{x}(M) > 1$ and so $\|1-\alpha\hat{x}\| = \|e-\alpha x\| > 1$.

If $x=0$ both (i) and (v) hold. If $x \neq 0$ (v) is just $\|e-\frac{1}{\|x\|}x\| \leq 1$, which implies (iii) and is implied by (iv). This completes the proof. ||

The proposition we have just proved is useful because it enables us to convert a statement about the spectrum into a statement involving only norms of elements in the algebra. We use this in the following proposition.

We denote by Q the set of self-adjoint elements of B that satisfy one (and hence all) of the conditions of Proposition 6.2. It is a subset of B_s: we shall now prove that it is a <u>cone</u>; that is, a set K such that $x,y \in K, \alpha \geq 0 \Longrightarrow x+y$, $\alpha x \in K$ (so that $x_r \in K, \alpha_r \geq 0 \Longrightarrow \sum \alpha_r x_r \in K$) and also $x, -x \in K \Longrightarrow x=0$.

PROPOSITION 6.3: <u>Q is a closed cone in B_s with a non-empty interior ; more precisely e is an interior point of</u> Q.

PROOF: Suppose $x \in Q$, $\alpha \geq 0$ then $\alpha x \in Q$ by condition (i) of Proposition 6.2, since $\sigma(\alpha x) = \alpha\sigma(x)$.

If $x, y \in Q$ choose $\alpha > 0$ with $\alpha \leq \frac{2}{\|x\|}$, $\alpha \leq \frac{2}{\|y\|}$.

Then $\|e-\alpha x\| \leq 1$, $\|e-\alpha y\| \leq 1$ and so $\|e-\frac{1}{2}\alpha(x+y)\| =$
$\|\frac{1}{2}(e-\alpha x) + \frac{1}{2}(e-\alpha y) \leq \frac{1}{2}\|e-\alpha x\| + \frac{1}{2}\|(e-\alpha y)\| \leq \frac{1}{2}+\frac{1}{2} = 1$
so that $x+y \in Q$, by (iii) of Proposition 6.2

If $x \in Q$, $-x \in Q$ then $\sigma(x) = \{0\}$ and so, since $\|x\| = \|\hat{x}\| = 0$ we have $x=0$. Thus Q is certainly a cone.

To show that Q is closed, use condition (v) of Proposition 6.2. If $x \notin Q$ then

$$0 < \big\| \|x\|e - x \big\| = \varepsilon \qquad \text{say:}$$

since
$$\big\| \|x\|e - x\big\| \leq \big\| \|y\|e - y\big\| + \|x-y\| + \big| \|x\| - \|y\| \big|$$
$$\leq \big\| \|y\|e - y\big\| + 2\|x-y\|,$$

it follows that

$$\big\| \|y\|e - y\big\| \geq \big\| \|x\|e - x\big\| - 2\|x-y\|$$

and

$$\big\| \|y\|e - y\big\| - \|y\| \geq \big\| \|x\|e - x\big\| - \|x\| - 3\|x-y\|$$

so that if $\|x-y\| < \varepsilon/3$ we have $\big\| \|y\|e - y\big\| > 0$ and so $y \notin Q$.

To show that Q has a non-empty interior, recall the device used in the proof of Proposition 3.3 ; if $\|x\| < 1$ and x is self-adjoint then there is a self-adjoint y such that $y^2 = e-x$, given by the usual power series for $(e-x)^{\frac{1}{2}}$. That is, the elements of B_s that lie in the open ball $\{x : \|x-e\| < 1\}$ are all in Q and e is certainly an interior point of Q. ‖

It is clear that if $x \in Q$ then x is of the form $x^* y$ (indeed with y self adjoint). Our next result shows that the converse result also holds.

PROPOSITION 6.4: $x^* x \in Q$ <u>for all</u> $x \in B$.

PROOF: We first show that if $-x^* x \in Q$, $x \in B$ then $x=0$. Writing $x=x_1+ix_2$, where x_1 and x_2 are self-adjoint, we have $x^* = x_1 - ix_2$ and

$$x^* x + xx^* = 2x_1^2 + 2x_2^2 ,$$

so that
$$x^* x = 2x_1^2 + 2x_1^2 + (-xx^*).$$

Now if $-x^* x \in Q$ then $-xx^* \in Q$ also, by the Corollary to Lemma 6.1. Since Q is a cone, and the three terms on the right are in Q, $x^* x \in Q$: this with $-x^* x \in Q$ implies $x^* x = 0$ and since we are in a B^*-algebra $x=0$ since $\|x^* x\| = \|x\|^2$.

Now take a general $x \in B$: we wish to show $x^* x \in Q$. Certainly $x^* x$ is self adjoint, so we can write it as a difference of positive elements:

$$x^* x = y-z$$

where y,z are positive, self-adjoint and commute with everything that commutes with x^*x. [In $B''(x^*x)$ take $\hat{y}=(x^*x)^+$ and $\hat{z} = (x^*x)^-$]. We also have $yz=0$, since $\hat{y}\hat{z}=0$. Then $(xz)^*xz = zx^*xz = z(y-z)z = -z^3$ and since $z^3 \in Q$ we get

$$(-xz)^*xz \in Q$$

which implies $xz=0$ by the argument given above. Then $z^3=0$ and so $z=0$ since $\|z^4\| = \|z^2\|^2 = \|z\|^4$ and if $z^3=0$ then $z^4=0$ and so $\|z\|=0$. Thus $x^*x=y \in Q$ as required. $\|$

COROLLARY : $(e+x^*x)^{-1}$ <u>exists for all</u> $x \in B$.

PROOF: Taking - say - $B''(x^*x)$, we have $\widehat{(e+x^*x)} \geq 1$ and so $(e+x^*x)$ certainly has an inverse in the sub-algebra hence in B. $\|$

An algebra satisfying this condition is called by Naimark <u>completely symmetric</u> : th condition is thus implied by the B^*-condition $\|x^*x\| = \|x\|^2$ (but not $\|x^*\| = \|x\|$).

In general, if K is a cone in a real locally convex topological vector space E. We call a functional f <u>positive with respect to K</u> if $p(x) \geq 0$ for all $x \in K$.

In the present case taking $E=B_s$, $K=Q$, a functional (on B_s) is positive with respect to Q if and only if $f(x^*x) \geq 0$ for all $x \in B$. Hence the extension of f to B by linearity $\left(f(x_1+ix_2) \doteq f(x_1) + if(x_2)\right)$ is precisely what we have already called a positive functional p.

For positive functional we have the following variation of the Hahn-Banach theorem:

KREIN'S EXTENSION THEOREM: *Let* E *be a real locally convex topological vector space and* K *a cone with a non-empty interior. If* E_1 *is a linear subspace of* E *containing an interior point of* K, *and* f_1 *is a linear functional on* E_1 *that is positive with respect to* $K_1 = K \cap E_1$ *then there is an extension* f *of* f_1 *to the whole of* E *that is positive with respect to* K.

We proceed in the next section to apply this result to the case of q as a cone in B_s.

CHAPTER 7

THE REALISATION OF B^*-ALGEBRAS AS C^*-ALGEBRAS

We begin with one or two results relating to ideal theory in B:

PROPOSITION 7.1: <u>If</u> J <u>is a proper left ideal in</u> B <u>there is a positive functional</u> p <u>on B with</u> p(e)=1 <u>and</u> p(x)=0 <u>for all</u> $x \in J$.

PROOF: Consider $F=B_s$; let E_1 be the subset of B_s consisting of elements $(\lambda e+x)$, $(\lambda \in R,\ x \in J \cap B_s)$. This is a linear subspace of B_s and contains an interior point of Q namely e. Write

$$p_1(\lambda e+x) = \lambda ;$$

It is positive with respect to $Q_1=Q \cap E_1$, for if $x \in Q_1$ and $x = \lambda e+y$ then $y \in J$ and $-y^{-1} = (\lambda e-x)^{-1}$ fails to exist so $\lambda \in \sigma(x)$, $\lambda \geq 0$ and thus $p(x) \geq 0$.

We can now apply Krein's extension theorem to p_1: there is a functional p on B_s that is positive with respect to Q and the extension by linearity of this to the whole of B is the required functional. We have p(x)=0 for all $x \in J$ and so in particular $p(x^*x) = 0$ for all $x \in J$. ‖

We have used in the above proposition the fact that if $\lambda e-x$ is in some proper left ideal then $(\lambda e-x)^{-1}$ fails to exist. We would like to use next a converse of this : unfortunately a direct converse would be false, as is seen by considering the elements x and y in $\mathcal{L}(l_2)$ described just after Lemma 6.1 : here y^{-1} fails to exist but the left principal ideal generated by y is the whole of $\mathcal{L}(l_2)$. Clearly if we demand that no left-inverse y_l^{-1} exists then the left principal ideal generated by y will be proper : so we deal for the moment with one-sided inverses.

We begin by defining the (left) radical of B to be the set of all $x \in B$ such that a left-inverse $(e+yx)_l^{-1}$ exists for all $y \in B$. (It will appear later that we get exactly the same set of elements if we start with 'right' rather than 'left'). This evidently reduces to the usual definition of "radical" in a commutative Banach algebra ; that is, all elements whose spectrum is $\{0\}$.

PROPOSITION 7.2: The radical of B is the intersection of all the maximal left ideals of B.

PROOF: Suppose $(e+yx)_l^{-1}$ fails to exist for some $y \in B$. Then the set of elements of the form $z(e+yx)$ is a proper left ideal and hence is contained in a maximal left ideal. If now x is in the intersection of all maximal left ideals then x and hence yx belong to this ideal and hence so does e, being

$e+yx-yx$. This is a contradiction : so the radical contains the intersection of all maximal left ideals.

Suppose $x \notin J$ for some maximal left ideal J. Then the set of all elements of the form $z+yx (z \in J,\ y \in B)$ is again a left ideal and properly contains J (since it contains the element x). Since J was maximal this ideal must be the whole of B. But then $e = z+yx$ for some y,z so $z = e-yx$. But z we have no left inverse, since J is proper : and so $x \notin$ radical thus the radical is contained in each maximal left ideal J, hence in their intersection. ||

COROLLARY: <u>The radical is a closed left ideal of</u> B.

PROPOSITION 7.3: <u>If</u> x <u>belongs to the radical then a two sided inverse</u> $(e+yx)^{-1}$ (<u>necessarily unique</u>) <u>exists for all</u> $y \in B$.

PROOF: If $x \in$ radical and $y \in B$ some left inverse $(e+yx)_{\ell}^{-1}$ exists: say $(e+z)$, so that $(e+z)\cdot(e+yz) = e$. Thus

$$z = -zyx - yx = -(zy+y)x$$

Since the radial is a left ideal $z \in$ radical. Thus $e+z$ has a left inverse w say: $w(e+z) = e$. So $w(e+z).(e+yx)=w$ $= e+yx$ and $e+yx$ has $e+z$ as a two-sided inverse. ||

PROPOSITION 7.4: <u>A</u> B^{*}<u>-algebra is semi-simple</u>.

PROOF: This follows from the above proposition : If $x \in$ radical then necessarily $\sigma(x) = \{0\}$ and then $x=0$. ||

We now show (although it is not really required in what follows) that we could have taken "right" instead of 'left' in the definition of the radical.

PROPOSITION 7.5: <u>The radical is a two sided ideal of B : it is the intersection of all maximal left ideals and also the intersection of all maximal right ideals.</u>

PROOF: This depends on Lemma 6.1 ; taking $\alpha = 1$, if either $(e+yx)^{-1}$ or $(e+xy)^{-1}$ exists so does the other. So $x \in$ (left) radical $\Longleftrightarrow (e+yx)^{-1}$ exists for all $y \Longleftrightarrow (e+xy)^{-1}$ exists for all $y \Longleftrightarrow x \in$ right radical. Hence the left and right radicals coincide. ||

Note that in general the radical is not the intersection of all maximal two-sided ideals of B ; take $B = \mathcal{L}(H)$ for some countably infinite dimensional H. Then there exists a unique proper two-sided ideal, the compact operators : the intersection of all maximal two-sided ideals is therefore precisely compact operators, therefore not zero. But the radical is $\{0\}$ and is not the intersection of maximal two-sided ideals of B.

PROPOSITION 7.6: <u>If</u> $p(x^*x) = 0$ <u>for all positive functionals</u> p <u>on</u> B <u>then</u> $x = 0$.

PROOF: Suppose J is a maximal left ideal. By Proposition 7.1 there is a positive functional p such that $p(x^*x) = 0$ for all $x \in J$ and $p(e) = 1$. But the set of elements

$\{z : p(z^*z) = 0\}$ is a left ideal (this was proved during the course of proving Theorem 3.6) and this ideal must be proper since $p(e) = 1$. Then it coincides with J since J is maximal. Hence if $p(x^*x) = 0$ for all p then $x \in J$ for each maximal J and so x is in the radical which is $\{0\}$. (Propositi n 7.4). ||

PROPOSITION 7.7 : <u>A</u> B^*<u>-algebra has a complete set of</u> $*$<u>-representations and hence a complete set of irreducible</u> $*$<u>-representations</u>.

PROOF: What we want is to show that if $T(x) = 0$ for all $*$-representations T then $x = 0$. But if p is the functional associated with T then $p(x^*x) = (T(x^*x)\zeta, \zeta) = \|T(x)\zeta\|^2 = 0$ and conversely if p is given then there is a T associated with it : so $T(x) = 0$ for all T is equivalent to $p(x^*x) = 0$ for all p which implies $x=0$ by Proposition 7.6. ||

We now proceed to construct a Hilbert space H such that B is isometrically $*$-isomorphic to a closed $*$-subalgebra of $\mathcal{L}(H)$. First we prove two propositions.

PROPOSITION 7.8: <u>Let</u> X <u>be compact</u>, $C(X)$ <u>the usual space of continuous functions with</u> $\|x\| = \sup_{t \in X} |x(t)|$.
<u>Let</u> $\|x\|'$ <u>be any other norm on</u> $C(X)$ <u>in which it is a normed (not necessarily complete) algebra. Then,</u>

$$\|x\| \leq \|x\|' \quad \underline{\text{for all}} \ x.$$

PROOF: Suppose B is the completion of $C(X)$ under $\| \|'$ and let M be its maximal ideal space. For any $m \in M$ let f_m be the associated functional then f_m restricted to $C(X)$ is a non-zero functional on $C(X)$: so there exists $t_m \in X$ with

$$f_m(x) = x(t_m), \qquad \text{(all } x \in C(X)\text{)}$$

Write $X_1 = \{t : t = t_m \text{ for some } m \in M\}$. Then $\mathrm{Cl}\, X_1 = X$. For, if not, there exists an open subset V with $\mathrm{Cl}\, V$ compact and $\mathrm{Cl}\, V \subset X \setminus \mathrm{Cl}\, X_1$. Now choose $x, y \in C(X)$ with $y(t) = 1$ for $y \in \mathrm{Cl}\, V$, $y(t) = 0$ for $t \in \mathrm{Cl}\, X_1$ and $x \neq 0$, $x(t) = 0$ for $t \in X \setminus \mathrm{Cl}\, V$. Then given $m \in M$, $f_m(y) = y(t_m) = 0$. But then $(e-y)^{-1}$ exists and $x = xy$ so that $x(e-y)(e-y)^{-1} = 0$, i.e. $x = 0$ a contradiction. So $\mathrm{Cl}\, X_1 = X$. Then evidently

$$\|x\|' \geq \sup_{m \in M} |f_m(x)| = \sup_{t \in X} |(x(t))| = \|x\|$$

as required. $\|$

COROLLARY : _If_ B _is semi-simple then_ $\| \|$ _and_ $\| \|'$ _are metrically equivalent and_ $B = C(X)$.

PROOF: This follows at once from the Banach inversion theorem if we note that M which we have identified with a subset of X must be all of X. $\|$

We next have the key result which enables us to prove that B is _isometric_ with a closed sub-algebra of $\mathcal{L}(H)$.

PROPOSITION 7.9: Suppose B_1 and B_2 are B^*-algebras and φ is a $*$-isomorphism (no continuity assumed) from B_1 to a dense sub-algebra of B_2. Then φ is necessarily an isometry and hence $\varphi(B_1) = B_2$.

PROOF: Let $x \in B_1$ and let B_3 be the closed subalgebra of B_1 generated by x^*x (and e) : B_3 is commutative. Define for $y \in \varphi(B_3)$,

$$\|y\|' = \|\varphi^{-1}(y)\|_1, \quad \|y\|'' = \|y\|_2$$

where the suffix indicates the norm in B_1 or B_2, respectively. Under $\|\ \|'$, $\varphi(B_3)$ is a commutative B^*-algebra, hence (isometrically isomorphic to) $C(X)$ for some X. Under $\|\ \|''$, $\varphi(B_3)$ is a commutative normed algebra and its completion B_4 is a closed sub-algebra of B_2. Since B_2, being a B^*-algebra, is semi-simple, B_4 is also semi-simple. By the preceding proposition and its corollary $\|y\|' \leq \|y\|''$ for all $y \in \varphi(B_3)$ and $\varphi(B_3) = B_4$.

Now B_4 will also be of the form $C(X')$ for some X'; the result of Proposition 7.8 now yields

$$\|y\|'' \leq \|y\|',$$

and so $\|\varphi(z)\|_2 = \|z\|_1$ if $z \in B_3$. In particular $\|x_1\|^2 = \|x^*x\| = \|\varphi(x^*x)\| = \|\varphi(x^*)\varphi(x)\|_2 = \|\varphi(x)\|_2^2$

and so φ is an isometry on B_1 to B_2 (since evidently the range of φ must then be closed, it is the whole of B_2). ‖

If $\{H_i\}_{i\in I}$ is any collection of Hilbert spaces their Hilbert direct sum $H = \oplus H_i$ is the Hilbert space whose elements are "vectors" $\xi = (\xi_i)_{i\in I}$ such that $\sum_{i\in I} \|\xi_i\|^2 < \infty$ (so that $\xi_i = 0$ for all but a countable set of indices i). We can introduce the inner product $(\xi, \eta) = \sum_{i\in I}(\xi_i, \eta_i)$ and thus the norm $\|\xi\| = (\xi, \xi)^{\frac{1}{2}}$; note that $\|\xi\|^2 = \sum_{i\in I}\|\xi_i\|^2$. It is easy to verify all the Hilbert space axioms (including completeness). If we have a corresponding collection of bounded operators $\{T_i\}_{i\in I}$ then their direct sum is the operator T on H defined by $T\xi = (T_i\xi_i)_{i\in I}$. This is bounded if and only if $\sup_{i\in I}\|T_i\| < \infty$ and then $\|T\| = \sup_{i\in I}\|T_i\|$. To see this, note that $|(T\xi, \eta)| = \left|\sum(T_i\xi_i, \eta_i)\right| \leq \sum|(T_i\xi_i, \eta_i)| \leq \sup\|T_i\|\sum\|\xi_i\|\cdot\|\eta_i\| \leq \sup\|T_i\| \ \|\xi\| \ \|\eta\|$. So if, $\sup\|T_i\| < \infty$, then T is bounded and $\|T\| \leq \sup\|T_i\|$. On the other hand, choose an i with $\|T_i\| > \sup\|T_i\| - \varepsilon$; there exist $\xi_i, \eta_i \in H_i$ with $|(T_i\xi_i, \eta_i)| > (\sup\|T_i\| - 2\varepsilon) \ \|\xi_i\| \ \|\eta_i\|$. Now take ξ to be the vector with only one non-zero component ξ_i and similarly for η: we then have

$$|(T\xi, \eta)| = |(T_i\xi_i, \eta_i)| \geq (\sup\|T_i\| - 2\varepsilon) \ \|\xi\| \ \|\eta\|$$

so that $\|T\| \geq \sup \|T_i\| - 2\varepsilon$ and since ε was arbitrary

$$\|T\| \geq \sup \|T_i\| \text{ giving } \|T\| = \sup \|T_i\| \text{ as}$$

asserted. If T_i is now a collection of representations of B, their direct sum T, where $T(x)\xi = \left(T_i(x)\xi_i \right)_{i \in I}$ is again a representation as is immediately verified.

THEOREM 7.10: <u>A B^*-algebra is isometrically $*$-isomorphic to a closed sub-algebra of $\mathcal{L}(H)$ for some Hilbert space</u> H.

PROOF: Let B be the algebra and let (T_i) be any complete set of representations (not necessarily irreducible) on Hilbert spaces H_i. Taking the direct sum of these we evidently have a faithful $*$-representation $x \longrightarrow T(x)$ of B on the Hilbert space $H = \bigoplus_{i \in I} H_i$. (Since $\|T_i(x)\| \leq \|x\|$ for all i (Proposition 3.5) it follows that $\|T(x)\| \leq \|x\|$, but we do not in fact require this). Taking the closure of the set of operators $\left\{ T(x) \right\}_{x \in B}$ we have a closed $*$-subalgebra of $\mathcal{L}(H)$ and we are in the situation of Proposition 7.5; applying the result of that proposition, it follows that the map $x \longrightarrow T(x)$ is an isometry on to a closed sub-algebra of $\mathcal{L}(H)$, which is what we wanted. ‖

It may be useful to indicate what this representation may be like in a particular case : starting with $B = C[0, 1]$

CHAPTER 8

REPRESENTATIONS OF LOCALLY COMPACT GROUPS

Let G be a locally compact group, not in general abelian. By a representation of G we mean a map $s \longrightarrow V(s)$ where $V(s)$ is an invertible linear operator on a Banach space, with $V(s_1 s_2) = V(s_1) V(s_2)$. It will be called (strongly) continuous if the map is continuous when the operators are given the strong topology. If the Banach space is in fact a Hilbert space and the operators are all unitary (in which case we shall usually write $U(s)$ rather than $V(s)$) we have a unitary representation ; in this case $U(s^{-1}) = [U(s)]^*$. We have already seen in Ch.1 that there always exists a continuous faithful unitary representation of G. We now wish to examine the existence of irreducible representations.

First we define irreducibility : this is exactly the same for groups as for Banach algebras. The representation V is reducible if there is a non-trivial closed subspace E_1 of E with $V(s) E_1 \subset E_1$ for all $s \in G$, otherwise irreducible. For unitary representations on a Hilbert space H we have exactly the same criterion for irreducibility as we had previously for algebras :

PROPOSITION 8.1: <u>The unitary representation</u> U <u>of</u> G <u>on the Hilbert space H is irreducible if and only if the only operators that commute with all the</u> $U(s)$ <u>are scalar multiples of the identity operator</u>.

Also, we can introduce the notion of cyclic representation, cyclic vector etc., for representations of groups in exactly the same way as for representations of Banach algebras: we have a result analogous to Theorem 2.6.

We now review briefly one or two aspects of integration on G. As usual dt will denote left invariant Haar measure, with some fixed normalisation. We then have, for $f \in C_{oo}(G)$ at least,

$$\int_G {}_s f(t)dt = \int f(s^{-1}t)dt = \int f(t)dt$$

but

$$\int_G {}_s f(t)dt = \int_G f(ts^{-1})dt \neq \int_G f(t)dt, \quad \text{in general.}$$

However it is clear that

$$\int_G f(uts^{-1})dt = \int_G f_s(ut)dt \quad = \quad \int_G {}_s f(t)dt = \int_G f(ts^{-1})dt$$

so that $f \longrightarrow \int_G f_s(t)dt$ is a left invariant integral on $C_{oo}(G)$; and so by the uniqueness theorem for Haar measure it must be a constant multiple of $\int_G f(t)dt$: the constant depends on s but not on f and we write

$$\int_G f_s(t)dt = \Delta(s) \int_G f(t)dt$$

This function $\Delta(s)$ is the <u>modular function of</u> G. It is by definition real and non-negative. If $\Delta(s) \equiv 1$ then G is called <u>unimodular</u> (the term unimodular is also applied to certain groups of matrices with determinant 1, but we do not use it in the same sense here). $\Delta(s) \equiv 1$ is evidently a necessary and sufficient condition for left and right Haar measures to coincide.

PROPOSITION 8.2: $s \longrightarrow \Delta(s)$ *is a continuous homomorphism of G into the multiplicative group of strictly positive real numbers*.

PROOF: If $f \in C_{oo}$ then $f(ts^{-1})$ is uniformly continuous function ; given ε we can certainly find $N(s_o)$ so that for $s \in N$ we have $|f(ts^{-1}) - f(ts_o^{-1})| < \varepsilon$ throughout some fixed compact set hence $|\int {}_s f(t)dt - \int {}_{s_o} f(t)dt| < k\varepsilon$ if $s \in N$, hence continuity at s_o

$$\Delta(s_1 \cdot s_2) \int f(t)dt = \int f(ts_2^{-1}s_1^{-1})dt = \int f_{s_1}(ts_2^{-1})dt$$

$$= \Delta(s_2) \int_{s_1} f(t)dt$$

The homomorphism property is immediate:

$$\Delta(s_1 s_2) \int f(t)dt = \int f(ts_2^{-1}s_1^{-1})dt = \int f_{s_1}(ts_2^{-1})dt$$

$$= \Delta(s) \int f_{s_1}(t)dt = \Delta(s_2)\ \Delta(s_1) \int f(t)dt$$

and the result follows on choosing f with $\int f(t)dt \neq 0$. ||

PROPOSITION 8.3: $\Delta(s) \equiv 1$ *if* G *is abelian or discrete or compact*.

PROOF: This is immediate if G is abelian or discrete. If G is compact note that the function $f(t) \equiv 1$ is in $C_{00}(G)$ and apply the formula for $\Delta(s)$ with this f. ||

There are of course unimodular groups of other kinds also.

We have $\int f(t)dt = \int f(st)dt = \int f(s^{-1}t)dt = \int f(ts^{-1})\Delta(s^{-1})dt = \int f(ts)\,\Delta(s)dt$. We do not have $\int f(t)dt = \int f(t^{-1})dt$ in general; the appropriate formula is

$$\int f(t)dt = \int f(t^{-1})\,\Delta(t^{-1})dt.$$

To see this, look at $\int f(t^{-1})\Delta(t^{-1})dt$. Using the formula $\int \varphi(t)dt = \Delta(s^{-1})\int \varphi(ts^{-1})dt$, with $\varphi(t) = f(s^{-1}t^{-1})\,\Delta(t^{-1})$, we have $\int {}_s f(t^{-1})\,\Delta(t^{-1})dt = \int f(s^{-1}t^{-1})\Delta(t^{-1})dt = \Delta(s^{-1})\int f(s^{-1}st^{-1})\cdot\Delta(st^{-1})dt = \int f(t^{-1})\,\Delta(t^{-1})dt$, and so this is a left-invariant integral. It must therefore be of the form $f(t^{-1})\int \Delta(t^{-1})dt = c\int f(t)dt$ for some constant c, by the uniqueness of Haar measure.

To see that c must be 1, choose a neighbourhood of e so that $\Delta(s)$ is nearly equal to 1 throughout this neighbourhood. Then choose f to be a non-negative symmetric function $\left(f(t^{-1}) = f(t) \text{ for all } t\right)$ with support in the neighbourhood. It will follow that $\int f(t)dt$ and $\int f(t^{-1})\,\Delta(t^{-1})dt$ are arbitrarily close, and hence that c=1.

As a corollary, $\int f(t)dt = \int f(t^{-1})dt$ if and only if $\Delta(t) \equiv 1$.

We may introduce an involution in C_{oo} by writing $f^*(t) = \overline{f(t^{-1})}\,\Delta(t^{-1})$. It is clear that this has all the linear space properties required : to show that it has the appropriate property relative to convolution we note

$$(f * g)^*(t) = \overline{f * g\ (t^{-1})} \cdot \Delta(t^{-1}) = \int \overline{f(s)g(s^{-1}t^{-1})}ds\ \Delta(t^{-1})$$

$$= \int \overline{g(s^{-1}t^{-1})}\ \Delta(s^{-1}t^{-1})\ \overline{f(s)}\ \Delta(s)ds$$

$$= \int g^*(ts) f^*(s^{-1})ds = g^* * f^*(t)$$

We have also immediately the fact that $f \longrightarrow f^*$ is an isometry for the L_1 norm :

$$\|f^*\|_1 = \int |\overline{f(t^{-1})}\ \Delta(t^{-1})|dt = \int |f(t)|dt = \|f\|_1$$

(on the other hand, it is not an isometric map in any other L_p norm, unless G is unimodular). So we can extend the involution uniquely by continuity from C_{oo} to L_1 and L_1 then becomes a Banach $*$-algebra.

We should note also the fact that if we write T_f for the operator on C_{oo} obtained by left convolution by f: $T_f(g) = f * g$ then we have $(T_f g, h) = (g, T_{f^*} h)$, so that the Hilbert space adjoint of T_f is exactly T_{f^*}. The verification is not difficult. It is thus clear that the natural involution on Λ (the completion of C_{oo} in the operator norm on $L_2(G)$) coincides with the involution on $C_{oo}(G)$ (This is of course a strong argument in favour of defining f^* as we did).

There is one formula that we shall require later : if we translate a convolution product $f*g$ either on the right or the left, we get $(f*g)_s = f*(g_s)$ and $_s(f*g) = (_sf)*g$. If we take the special case of a product $f^* * f$ then if we translate f by s we get exactly the same result : $(_sf)^* * {_sf} = f^* * f$. To see this,

$$\int {_sf}^*(t) \;\; {_sf}(t^{-1}u)dt = \int \overline{f(s^{-1}t^{-1})} \; \Delta(t^{-1})f(s^{-1}t^{-1}u)dt$$

$$= \int \overline{f(s^{-1}t)} \; f(s^{-1}t\,u)dt = \int \overline{f(t)} \; f(tu)dt$$

$$= \int \overline{f(t^{-1})} \; f(t^{-1}u) \; \Delta(t^{-1})dt$$

$$= \int f^*(t)f(t^{-1}u)dt$$

THEOREM 8.4: There is a 1-1 correspondence between continuous unitary representations $U : s \longrightarrow U(s)$ of G and essential $*$-representations $T : x \longrightarrow T(x)$ of $L_1(G)$: in one direction the correspondence is given by

$$\big(T(x)\xi, \eta\big) = \int_G (U(s)\xi, \eta) \; x(s)ds$$

and in the other by

$$U(s) \; T(x)\xi = T(_sx)\xi \quad \text{(for any suitable } x, \xi \text{)}.$$

PROOF: Suppose the representation U given. Consider for $x \in L_1$, $\xi, \eta \in H$, the integral

$$I = \int_G \big(U(s)\xi, \; \eta\big) \; x(s)ds.$$

Evidently $|\ I\ | = \sup \|(U(s)\ \xi\ ,\ \eta)\| \cdot \|x\|$ and since $\|U(s)\| = 1$ we have $|\ I\ | \leq \|x\|\ \|\xi\|\ \|\eta\|$. Evidently the integral is linear in ξ, conjugate-linear in η, so it must be of the form $(T(x)\xi\ ,\ \eta)$ where $\|T(x)\| \leq \|x\|$. It is clear that $T(x)$ is a linear function of x. To complete the verification that it is a representation we have to show that $T(x)^* = T(x^*)$ and $T(x*y) = T(x)T(y)$.

We have

$$\begin{aligned}
(T(x^*)\xi\ ,\eta) &= \int (U(s)\xi\ ,\ \eta)\,\overline{x}(s^{-1})\Delta(s^{-1})ds \\
&= \int (\xi\ ,\ U(s^{-1})\eta)\,x(s^{-1})\ \Delta(s^{-1})ds = \int \overline{(U(s^{-1})\xi\ ,\eta)}\ \overline{x}(s^{-1}) \\
&\qquad\qquad \Delta(s^{-1})ds \\
&= \int (U(s)\xi,\eta\ )\ (x)(s)ds = \overline{(T(x)\eta\ ,\xi)} = (\xi, T(x)\eta\ ) = (T(x)^*\xi,\eta)
\end{aligned}$$

as required ; and

$$(T(x*y)\xi\ ,\eta) = \int (U(s)\xi\ ,\eta) \int x(st)y(t^{-1})dt\ ds.$$

We may interchange the order of integration by Fubini's theorem ; we get

$$\begin{aligned}
&\int\int (U(st)\ U(t^{-1})\xi\ ,\eta\ )\ x(st)\ \Delta(t)ds\ \Delta(t)y(t^{-1})dt \\
&= \int (\ T(x)U(t^{-1})\xi\ ,\eta\ )\ \Delta(t^{-1})y(t^{-1})dt \\
&= \int (\ U(t^{-1})\ \xi\ \ ,\ T(x)^*\eta\ )\ \Delta(t^{-1})\ y(t^{-1})dt \\
&= (\ T(y)\xi\ ,\ T(x)^*\eta\ )\ \ = (T(x)\ T(y)\xi\ ,\eta)
\end{aligned}$$

To see that T is essential : suppose $\xi \neq 0$: then $U(s)\xi$ is nearly equal to ξ for s near e. hence if the support of x is small and x is non-negative with $\int x(s)ds = 1$ then $\int (U(s)\xi, \xi)\, x(s)ds$ is nearly equal to $\int (\xi, \xi)\, x(s)ds = \|\xi\|^2$ and so in particular $(T(x)\xi, \xi) \neq 0$ and $T(x)\xi$ is therefore non-zero.

So, starting from U, we obtain T quite straightforwardly. To go in the reverse direction is somewhat harder. Suppose first to simplify matters that we have a cyclic representation T with cyclic vector ζ : the vectors $T(x)\zeta$ are then dense in H. We first observe that if $T(x)\zeta = 0$ then also $T({}_sx)\zeta = 0$ for all $s \in G$. For, we have

$$\left(T({}_sx)\zeta\,,\ T({}_sx)\zeta\right) = \left(T({}_sx^* * {}_sx)\zeta, \zeta\right) = \left(T(x^* * x)\zeta, \zeta\right)$$
$$= \left(T(x)\zeta\,,\ T(x)\zeta\right)$$

so that the required conclusion follows at once. We now define $U(s)$ by

$$U(s)T(x)\zeta = T({}_sx)\zeta\ .$$

This is well-defined : if $T(x)\zeta = T(y)\zeta$ then $T(x-y)\zeta = 0$, $T({}_sx - {}_sy)\zeta = 0$ and $T({}_sx)\zeta = T({}_sy)\zeta$. We have

$$\left(U(s)T(x)\zeta\,,\ U(s)T(x)\zeta\right) = \left(T({}_sx)\zeta\,,\ T({}_sx)\zeta\right)$$
$$= \left(T({}_sx^* * {}_sx)\zeta, \zeta\right) = \left(T(x^* * x)\zeta, \zeta\right) = \left(T(x)\zeta, T(x)\zeta\right)$$

so that $\|U(s)\xi\| = \|\xi\|$ for all ξ of the form $T(x)\zeta$. Since these are dense in H we can extend U(s) uniquely by continuity to become a unitary operator on H (it is clearly linear, algebraically). It is clear that $U(e) = I$, the identity operator, and

$$U(st)T(x)\zeta = T({}_{st}x)\zeta = T({}_s({}_tx))\zeta = U(s)T({}_t(x))\zeta$$

$$= U(s)U(t)T(x)\zeta \quad ,$$

so that $U(st) = U(s)U(t)$.

The map $s \longrightarrow U(s)$ is continuous : this is proved by essentially the same argument as was used in Proposition 1.2. We have $\|U(s_o)\xi - U(s)\xi\| = \|U(s_o)\,T(x)\zeta - U(s)T(x)\zeta\|$ if $\xi = T(x)\zeta$ and this is $\|T(s_ox - {}_sx)\zeta\| \leq \|({}_{s_o}x - {}_sx)\|\;\|\zeta\|$. So, given $\xi_1, \xi_2 \ldots\ldots\ldots\ldots \xi_n, \varepsilon, s_o$ choose $x_1, \ldots, x_n$ so that $\|(T(x_r)\zeta - \xi_r)\| < \varepsilon/3$ for all r and then $N(s_o)$ so that for $s \in N(s_o)$ $\|{}_{s_o}(x_r) - {}_s(x_r)\| < \dfrac{\varepsilon}{3\|\zeta\|}$ for all r : this is possible since the continuous functions of compact support are dense in L_1 and such functions are uniformly continuous. Then we get

$$\|U(s_o)\xi_r - U(s)\xi_r\| \leq 2\|\xi_r - T(x_r)\zeta\| + \|U(s_o)T(x_r)\zeta - U(s)\cdot T(x_r)\zeta\|$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \;, \quad 1 \leq r \leq n$$

and so the required continuity follows.

We next remark that if ξ is any vector in H and x any element in L_1, then $U(s)T(x)\xi = T({}_sx)\xi$. For, if $\xi = T(y)\zeta$ for some y then we have

$$U(s)T(x)T(y)\zeta = U(s)T(x*y)\zeta = T({}_sx*y)\zeta$$

$$= T\big(({}_sx)*y\big)\zeta = T({}_sx)\cdot T(y)\zeta = T({}_sx)\xi$$

and the general case follows by continuity since vectors of the form $T(y)\zeta$ are dense in H.

If T is not a cyclic representation we can decompose it as a direct sum of cyclic representations, as in Theorem 2.6. For each T_i we form U_i as described above, and then take the direct sum of the U_i. It is easy to verify that U and T are related by

$$U(s)\ T(x)\xi = T({}_sx)\xi$$

for all $\xi \in H$; for if $U_i(s)T_i(x)\xi_i = T({}_sx)\xi_i$ for each i then also $U_i(s)T_i(x)\xi_i = T({}_sx)\xi_i$ (ξ_i the projection of ξ on H_i) and so the required result holds.

We must show that the correspondence indicated is really 1-1. Suppose that T has arisen from U_o and that U has arisen from T by the formula given. Then $\big(T(x)\xi, \eta\big) = \int (U_o(s)\xi, \eta)\, x(s)ds$ and

$$\left(T({}_tx)\xi\,,\eta\right) = \int_G \left(U_o(s)\xi\,,\eta\right) x(t^{-1}s)ds$$

$$= \int_G \left(U_o(t^{-1}s),\ U_o(t^{-1})\eta\right) x(t^{-1}s)ds$$

$$= \left(T(x)\xi\,,\ U_o(t^{-1})\eta\right) = \left(U_o(t)T(x)\xi\,,\eta\right)$$

but this is also $\left(U(t)\,T(x)\xi\,,\eta\right)$ by definition so $U(t) = U_o(t)$ for all t, as required.

Suppose that U has arisen from T_o and T from U For $z \in C_{oo}(G)$, and hence for all $z \in L_1(G)$, the function $\left(T_o(z)\xi\,,\eta\right)$ is a complex integral on C_{oo}; we may write it as

$$\left(T_o(z)\xi\,,\eta\right) = \int_G z(s)d\mu_{t_o\xi\,\eta}(s) = \int_G z(s)d\mu(s) \text{ say}$$

Then $\left(T_o(x*y)\xi\,,\eta\right) = \int_G x(t)y(t^{-1}s)dt\ d\mu(s)$

and we may interchange the order of the integration by Fubini's Theorem : we get

$$\left(T_o(x*y)\xi\,,\eta\right) = \iint_G {}_ty(s)d\mu(s)\ x(t)dt = \int\left(T_o({}_ty)\xi\,,\eta\right)x(t)dt$$

$$= \int\left(U(t)T_o(y)\xi\,,\eta\right)x(t)dt = \left(T(x)\ T_o(y)\xi\,,\eta\right)$$

by the definition of T. But it is also $\left(T_o(x)T_o(y)\xi\,,\eta\right)$ and so $T(x) = T_o(x)$ for all x (since the vectors $\left(T_o(y)\xi\,,\eta\ \right)$ are dense in H as ξ varies throughout H). ||

THEOREM 8.5: If U and T are related as in Theorem 8.4 then U is irreducible if and only if T is irreducible.

PROOF: Suppose U reducible. Then there is an operator $P \neq \alpha I$ (we can take P to be the projection on a non-trivial invariant subspace) such that $PU(s) = U(s)P$, all $s \in G$. Then

$$\left(PT(x)\xi, \eta\right) = \left(T(x)\xi, P^{*}\eta\right) = \int_G \left(U(s)\xi, P^{*}\eta\right) x(s)ds$$

$$= \int_G (PU(s)\xi, \eta)\, x(s)ds = \int_G (U(s)P\xi, \eta)\, x(s)ds$$

$$= \left(T(x)P\xi, \eta\right),$$

and since this holds for all ξ, η, we have $PT(x) = T(x)P$ and T is reducible by Proposition 2.5.

Suppose T reducible, and let P commute with all the $T(x)$. Then $PU(s)\, T(x)\xi = PT({}_s x)\xi = T({}_s x)\, P\xi = U(s)T(x)P\xi = U(s)PT(x)\xi$, and since the vectors $T(x)\xi$ are dense in H (since T is essential) it follows that $P(U(s)) = U(s)P$ and U is reducible by Proposition 8.1. ||

THEOREM 8.6 (GELFAND-RAIKOV) : A locally compact group G always has enough continuous irreducible unitary representations to separate the points of G : given $s \neq e$ there is a representation of the kind described with $U(s) \neq I$.

PROOF: If $s \neq e$ we can find $x \in C_{oo}(G)$ with ${}_s x \neq x$; suppose $s \notin N(e)$ and take a symmetric N' with $N'N' \subset N$, then if the support of x is in N', the supports of x and of ${}_s x$ are disjoint. Given any non-zero function $y \in C_{oo}$, we can find a function $z \in C_{oo}$ such that the convolution $y * z$ is non-zero: we have only to take z non-negative, with sufficiently small support near e, and with $\int z(t)dt = 1$: then y and $y * z$ will be uniformly close and if $y \not\equiv 0$ then $y * z \not\equiv 0$. Thus if we consider the representation of $L_1(G)$ as left convolution operators on L_2, where

$$T(x)\xi = x * \xi,$$

then $y \neq 0 \Longrightarrow T(y) \neq 0$. Then by Theorem 5.5 there is an irreducible representation of L_1 with $T(y) \neq 0$. Taking $y = x - {}_s x$ we see that if U is the associated unitary representation of G we have $T(x - {}_s x) = T(x) - U(s)T(x) \not\equiv 0$ so that $U(s) \neq I$: and this is what we wanted. ‖

We conclude the section by remarking that in the proof of Theorem 8.4 we did not use the full force of the assumption that $U(s)$ is strongly continuous, in going from U to the associated T. It is clear that weak continuity (that is, the continuity of $(U(s)\xi, \eta)$ for each $\xi, \eta \in H$) would suffice: we could then go to T and back to U which must then necessarily be strongly continuous. So, for unitary representations, weak continuity implies strong continuity.

If we only assume that $U(s)$ is weakly measurable then we can obtain an associated representation T as before, except that now we have no assurance that T is non-zero, to say nothing of being essential. For example, let G=R and let H be the space of functions $\xi(t)$ of the real variable t with $\sum|\xi(t)|^2 < \infty$ (so that $\xi(t) \neq 0$ for a countable set of values of t only). The inner product $(\xi,\eta) = \sum_{t \in R} \xi(t)\overline{\eta(t)}$ is then defined for all $\xi, \eta \in H$. Take the representation of R on H given by

$$U(s)\xi(t) = \xi(t-s) \ ;$$

this is evidently unitary. It is evidently also weakly measurable : indeed for a fixed ξ, η we have $\sum \xi(t-s)\overline{\eta(t)} \neq 0$ for a countable set of values of s only i.e. $(U(s)\xi, \eta) = 0$ for almost all s for fixed ξ and η. But then of course $(T(x)\xi, \eta) = \int (U(s)\xi, \eta)\, x(s)ds = 0$ for all ξ, η and so $T(x) = 0$ for all x.

However, if U is weakly measurable and H is separable, we can conclude that T is essential. For, let η_n be an orthonormal basis for H ; if $U(s)$ is unitary and weakly measurable then $(U(s)\xi, \eta)$ cannot be almost everywhere zero for all n: if it were, then in view of the formulae

$$U(s)\xi = \sum_{n=1}^{\infty} (U(s)\xi \cdot \eta_n)\eta_n$$

$$\|U(s)\xi\|^2 = \sum_{n=1}^{\infty} |(U(s)\xi, \eta_n)|^2$$

it would follow that $\|U(s)\xi\| = 0$ for almost all s; but $\|U(s)\xi\| = \|\xi\|$ for all s since U is unitary. So we can find, for $\xi \neq 0$, an η so that $(U(s)\xi, \eta)$ is not almost everywhere zero. Then there exists $x(s) \in C_{oo}$ such that

$$\int (U(s)\xi, \eta)\, x(s)ds \neq 0$$

and hence $(T(x)\xi, \eta) \neq 0$, $T(x)\xi \neq 0$ as required.

So for unitary representations U on a separable Hilbert space, weak measurability implies strong continuity. This is not true in general : the representation of R described above is not even weakly continuous.

CHAPTER 9

REPRESENTATIONS OF COMPACT GROUPS ETC

In this section we assume that Haar measure on G has been normalised so that $\int_G ds = 1$.

THEOREM 9.1: *If G is compact, every continuous irreducible unitary representation is finite-dimensional*.

PROOF: For $\xi, \eta, \zeta \in H$ consider the integral

$$\int_G (U(s)\zeta, \eta)\overline{(U(s)\zeta, \xi)}ds$$

For fixed ζ this is linear in ξ, conjugate-linear in η: since it is evidently bounded (by $\|\zeta\|^2 \|\xi\| \|\eta\|$) it must be of the form $(A(\zeta)\xi, \eta)$ where $A(\zeta)$ is some bounded linear operator on H. Now

$$\begin{aligned}
\big(A(\zeta)\, U(t)\xi, \eta\big) &= \int_G \big(U(s)\zeta, \eta\big)\overline{\big(U(s)\zeta, U(t)\xi\big)}\, ds \\
&= \int_G \big(U(ts)\zeta, \eta\big)\overline{\big(U(ts)\zeta, U(t)\xi\big)}\, ds \\
&= \int_G \big(U(s)\zeta, U(t')\eta\big)\overline{\big(U(s)\zeta, \eta\big)}\, ds \\
&= \big(A(\zeta)\xi, U(t^{-1})\eta\big) = \big(U(t)A(\zeta)\xi, \eta\big) .
\end{aligned}$$

Since ξ, η were arbitrary $A(\zeta)U(t) = U(t)A(\zeta)$ for all t: since U is assumed irreducible, $A(\zeta) = a(\zeta)$ for some scalar $a(\zeta)$. That is,

$$\int_G (U(s)\zeta,\eta)\overline{(U(s)\zeta,\xi)}\,ds = a(\zeta)\,(\xi,\eta)$$

and in particular, taking $\eta=\xi$,

$$|(U(s)\zeta,\xi)|^2 = a(\zeta)\,\|\xi\|^2$$

for all $\zeta,\xi \in H$.

Also

$$a(\xi)\,\|\zeta\|^2 = \int_G |(U(s)\xi,\zeta)|^2 ds = \int_G |(\xi,\, U(s^{-1})\zeta)|^2 =$$

$$\int_G |(U(s^{-1})\zeta,\xi)|^2 ds ,$$

and in a compact group $\Delta(s) \equiv 1$, so that Haar measure is inverse-invariant : the integral is equal to

$$\int_G |(U(s)\zeta,\xi)|^2 ds = a(\zeta)\,\|\xi\|^2 .$$

It follows that $a(\xi)/\|\xi\|^2 = a(\zeta)/\|\zeta\|^2$ for any ξ,ζ : that is, there is a constant k such that

$$a(\xi) = k\|\xi\|^2 \text{ for all } \xi \in H.$$

Thus $\int_G |(U(s)\xi,\xi)|^2 ds = a(\xi)\,\|\xi\|^2 = k\|\xi\|^4$ and if $\|\xi\| = 1$ then $\int_G |(U(s)\xi,\xi)|^2 ds = k$; this shows that $k\neq 0$ since $|(U(s)\xi,\xi)|$ is a continuous function of s and takes the value 1 at $s=e$.

Now let $\xi_1, \xi_2 \ldots\ldots \xi_n$ be an orthonormal set of vectors in H, and ξ any vector with $\|\xi\| = 1$. Then

$$\int_G |(U(s)\xi_i, \xi)|^2 ds = a(\xi_i)\ \|\xi\|^2 = k,$$

and so

$$\begin{aligned} nk &= \sum_{i=1}^{n} \int |(U(s)\xi_i, \xi)|^2 ds \\ &= \sum_{i=1}^{n} \int | U(s)\xi_i , \xi |^2 ds \\ &= \int_G \sum |(\xi_i, U(s^{-1})\xi)|^2 ds \end{aligned}$$

But we have $\|U(s^{-1})\xi\|^2 \geq \sum_{i=1}^{n} |(\xi_i, U(s^{-1})\xi)|^2$, by Bessel's inequality, (if $\xi_1, \xi_2 \ldots\ldots \xi_n$ is a complete orthonormal set) and so

$$nk \leq \int_G \|U(s^{-1})\xi\|^2 ds = \int_G ds = 1,$$

and it follows that $n \leq k^{-1}$, so that the dimension of H cannot exceed k^{-1} and so in particular is finite. ‖

There follows from this result and from Theorem 8.6 the celebrated Peter-Weyl theorem (1927) : there are enough representations of a compact group by unitary (finite) matrices to separate the points of the group. A direct proof of this would of course avoid many of the complicated considerations necessary to deal with the locally compact case.

It should not be supposed that given a compact group G there is an integer $n=n(G)$ such that every continuous irreducible unitary representations is of dimension $\leq n$. Take for example for each integer m, G_m to be the group of all $m \times m$ unitary matrices, with usual topology as a subset of R^{2m^2}. G_m is compact for each m and if $G = \prod G_m$ is the product of the G_m's with the usual topology then G is compact also. The map $s \longrightarrow s_m$ where s_m is the m^{th} coordinate of s is a unitary representation of G on a space of dimension m and is clearly irreducible.

The next theorem generalises a result that is well known for finite groups : we recall that two representations V_1 on E_1, V_2 on E_2 are <u>equivalent</u> if there is a bounded linear operator W from E_1 to E_2 with a bounded inverse such that $V_2(s) = W V_1(s) W^{-1}$ for all $s \in G$.

THEOREM 9.2: <u>Let G be compact and V a continuous representation, not in general unitary, on a Hilbert space H. Then V is equivalent to a continuous unitary representation.</u>

PROOF: Introduce a new inner product in H by writing

$$(\xi, \eta)_1 = \int_G (V(s)\xi, V(s)\eta)\,ds$$

Since V is continuous and G is compact the integral certainly exists. It is easily verified that all the inner product properties hold : in particular, $(\xi, \xi) = 0$ implies $\xi = 0$ since if $\xi \neq 0$ then the function $\left(U(s)\xi , U(s)\xi \right)$ being continuous, non-negative and equal to (ξ, ξ) at $s=e$ has an integral which is strictly positive.

Then $\left(V(t)\xi , V(t)\eta \right)_1 = \int_G \left(V(s)V(t)\xi, V(s) V(t)\eta \right) ds$

$= \int_G \left(V(st)\xi , V(st)\eta \right) ds$ and since in a compact group left invariant Haar measure is also right invariant, this is $\int_G \left(V(s)\xi , V(s)\eta \right) ds = (\xi, \eta)_1$; so V is unitary with respect to the inner product $(\xi, \eta)_1$.

Now for each $\xi \in H$, $V(s)\xi$ is continuous, hence $\|V(s)\xi\|$ is continuous hence (since G is compact)

$$\sup_{s \in G} \|V(s)\xi\| < \infty .$$

It follows from the Banach-Steinhaus theorem that

$$\sup_{s \in G} \|V(s)\| < \infty$$

Writing k for this supremum we have

$$\|\xi\|_1^2 = (\xi, \xi)_1 = \int_G \|V(s)\xi\|^2 ds \leq k^2 \|\xi\|^2 \int_G 1 ds$$

$$= k^2 \|\xi\|^2 .$$

and on the other hand

$$\|\xi\|^2 = \|V(s^{-1})\ V(s)\xi\|^2 \leq k^2\ \|V(s)\xi\|^2$$

Integrate both sides of this with respect to s and we have

$$\|\xi\|^2 \leq k^2\ \|\xi\|_1^2$$

and hence the norms $\|\ \|$ and $\|\ \|_1$ are equivalent. If now H_1 is simply H with the norm $\|\ \|_1$ instead of $\|\ \|$, and W is the identity map of H onto H_1 then W and W^{-1} are bounded and

$$U(s)\left(= V(s)\right) = W\ V(s)W^{-1}$$

is unitary as an operator on H_1. ‖

We conclude by showing that there are groups that admit no non-trivial finite dimensional unitary representations.

LEMMA 9.3: Let V be a non-singular normal n x n matrix. If for every integer m ≥ 1 there is an integral multiple of m, say k(m), and a non-singular n x n matrix W_m such that $V^{k(m)} = W_m V W_m^{-1}$ then V = I.

PROOF: Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of V; then $V^{k(m)}$ has eigenvalues $\lambda_1^{k(m)}, \ldots\ldots, \lambda_n^{k(m)}$ and $W_m V W_m^{-1}$ has eigenvalues $\lambda_1, \ldots\ldots, \lambda_n$: so $\lambda_1^{k(m)}, \ldots\ldots, \lambda_n^{k(m)}$ are simply a permutation of $\lambda_1, \ldots\ldots, \lambda_n$. Fix attention on λ_i: $\lambda_i^{k(1)}, \lambda_i^{k(2)}, \ldots \ldots\ldots, \lambda_i^{k(r)}, \ldots$ is an infinite sequence selected from the finite set $\lambda_1, \ldots\ldots, \lambda_n$.

Hence for some j, $\lambda_i^{k(m)}$ takes the value λ_j for infinitely many values of m. If m_o is the first of these we can find an m such that $k(m) > k(m_o)$ and $\lambda_i^{k(m)} = \lambda_i^{k(m_o)} = \lambda$ (since k(m) is always a multiple of m). Then $\lambda_i^{k(m) - k(m_o)} = 1$ and λ is a root of unity.

This holds for any i ; we obtain integers $r_1 \ldots\ldots r_n$ such that $\lambda_1^{r_1} = \ldots\ldots = \lambda_n^{r_n} = 1$. But then if m is any integer containing $r_1 \ldots\ldots, r_n$ as factors (e.g., l.c.m$(r_1 \ldots r_n)$) then k(m) also contains $r_1 \ldots\ldots, r_n$ as factors and so

$$\lambda_1^{k(m)} = \ldots\ldots = \lambda_n^{k(m)} = 1,$$

and hence $\lambda_1 = \ldots\ldots = \lambda_n = 1$. This clearly implies that V=I as required, since V is a normal matrix. ||

For any locally compact group G let G_o be the subset : $\{ s : U(s) = I$ if U is a finite dimensional continuous unitary representation$\}$. That is G_o is the set of elements that cannot be separated from e by a finite dimensional continuous unitary representation. It is immediate that G_o is a closed invariant subgroup G. Then we have

PROPOSITION 9.4: Let $s \in G$ be such that for each integer m there exists $t_m \in G$ and an integral multiple k(m) of m such that

$$s^{k(m)} = t_m s t_m^{-1} ;$$

then $s \in G_o$.

PROOF: This follows at once from Lemma 9.3, on going over to a finite dimensional unitary representation. ||

PROPOSITION 9.5: _Let_ G _be the group of_ 2×2 _complex matrices with determinent_ 1 (_the special linear group_ $SL(2, c)$ _or the_ 2×2 _unimodular group) then_ G _has no finite dimensional unitary representations_.

PROOF: We may as well take the discrete topology on G ; if we show that the result holds in this case it is of course true a _fortiori_ for the usual topology. We proceed to show $G_0 = G$ here,

Let $s = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ $t_m = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}$;

then $t_m s t_m^{-1} = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m^{-1} & 0 \\ 0 & m \end{bmatrix}$

$$= \begin{bmatrix} m & ma \\ 0 & m^{-1} \end{bmatrix} \begin{bmatrix} m^{-1} & 0 \\ 0 & m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & m^2 a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{m^2}$$, so the required conditions hold with $k(m) = m^2$; by Proposition 9.4 $s \in G_0$.

Now G_o is invariant and $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so it follows that $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in G_o$. Then if $c \neq 0$ we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{a-1}{c} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{d-1}{c} \\ 0 & 1 \end{bmatrix} \in G_o$$

and if $c=0$ then $d \neq 0$ (since $ad-bc=1$) and we have

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} -b & 0 \\ -d & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in G_o \quad \text{since both}$$

factors are of the form which we have just proved to be in G_o. ||

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45	W. K. Hayman	Transfinite diameter and its applications.
46	Ph. Meyer	Selected topics in Weak Interactions.
47	J. Rzewuski	Gauge transformations in quantum field theory.
48	P. L. Kannappan	Theory of functional equations.
49	P. C. Vaidya	Selected topics in Gravitation.
50	K. Venkatesan	Report on recent experimental data (1966).
51	K. R. Unni	Concepts in modern mathematics II (Topology).
52	K. R. Unni	,, ,, III (Analysis)
53	V. Krishnamurthy	Duality Theory in locally convex spaces.
54	J. H. Williamson	Representation theory for Banach algebras and locally compact groups.