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MATSCIENCE REPORT 49

SELECTED TOPICS ON GRAVITATION

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Visiting Scientist, MATSCIENCE.

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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20. (INDIA.)

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Lecture 1

The first to discuss Gravitation as a Natural force was Newton and he explained gravitational phenomenon by means of his law of universal gravitation. Though nothing had happened to necessitate a revision of this law of Nature but the basic concepts of Newtonian Mechanics had to be revised.

The special theory of Relativity which came into picture as a result of this revision was based on the following two postulates:

1. Invariance of all physical laws under a set of uniformly moving observers.

2. Homogeneous Euclidean background pervailing the entire space-time.

The theory under the above postulates is just a "restricted theory" for no force field was infused into the system. There is just a general background and the particles, charges, forces were all extraneous entities which are to be superposed on this background. Interactions have to be further postulated.

How to generalise this theory? The simplest way was to consider invariance under a set of observers in arbitrary motion. This indeed made possible for the acceleration of the observers to be an integral part of the theory. Therefore now the theory would be closed with respect to some natural force fields. For this generalization, Einstein made two postulates:

1. Invariance of physical laws under arbitrary transformation of coordinates. This is often referred to as the principle of covariance.

and 2. A non Euclidean background of space-time.
In particular if the non-Euclidean background is taken to be Riemanian, the theory gives "Gravitation" as the infused force field. If instead, the background is non-Euclidean, and non-Riemanian, one may get other forces of Nature. This explains the quest for various unified field theories.

In developing this general theory of relativity Einstein postulated the non-Euclidean background as Riemanian and so the general theory of relativity is essentially Einstein's theory of gravitation.

Description of Einstein's theory

We start with the most general form of the line element

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1, 2, 3, 4)$$

and we get the following field equations

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij}$$

inside matter, and in empty space we have.

$$R_{ij} = 0$$

where R_{ij} is the Ricci tensor and $R = g^{ij} R_{ij}$ the scalar curvature. T_{ij} is the energy momentum tensor.

Under the assumption of weak fields and small velocities $(v << c)$ Einstein's theory reduces to the Newtonian theory, if we choose

$$g_{44} = 1 + \frac{2\varphi}{c^2}.$$

where φ is the Newtonian potential.

One rigorous solution of the field equations for empty space is due to Schwarzschild and is given by

$$g_{ij} = \mathrm{dia}\left\{-\left(1-\frac{2m}{r}\right)^{-1}, -r^2, -r^2\sin^2\theta, \left(1-\frac{2m}{r}\right)\right\}$$

where the field is considered to be spherically symmetric.

A fundamental property of Riemanian space is that at any point one can choose coordinates in such a manner that g_{ij} become stationary at the point. This property is reflected in General Relativity by the Principle of Equivalence which in essence says that in the immediate vicinity of a space time event, one can choose an inertial reference frame in which special relativity works. Thus the fundamental result of the special relativity that no physical effect can be propagated with a velocity $v > c$ gets incorporated in the general theory. Therefore unlike Newton's theory, in Einstein's theory, gravitation must be propagated with a finite velocity.

Now how is gravitation propagated? and with what velocity?

If gravitation propagates in the form of waves, regarding the existence of gravitational waves we have to satisfy the following three conditions.

1. There must exist regular wave like solutions of the field equations .

2. At large distances these waves should asymptotically go over to Spherical wave forms, and

3. Near the source the solution must be joined smoothly with a regular solution of

$$R_{ij} - \frac{1}{2} g_{ij} R = - \frac{8\pi G}{c^4} T_{ij}$$

and there should not be any singularities, in the neighbourhood of the sources.

In fact this programme is far from being fulfilled.

Approximate solutions:

In 1916 Einstein[+](1) studied the weak field solutions of the field equations

$$R_{ij} - \frac{1}{2} g_{ij} R = - \frac{8\pi G}{c^4} T_{ij}$$

obtained by assuming

$$g_{ij} = \eta_{ij} + h_{ij}$$

where η_{ij} is the Minkowskian metric and h_{ij} a first order

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+ Figures within brackets indicate the number at which the full reference is listed at the end of the chapter.

quantity. This substitution gives R_{ij} to be equal to

$$R_{ij} = -\frac{1}{2}\delta^{lm}h_{ij,lm} - \frac{1}{2}\left(h_{,ij} - h^{p}_{i,jp} - h^{p}_{j,ip}\right)$$

where ηs are replaced by δs .

But the terms in the bracket may be written in the form $\left(\frac{1}{2}\delta^{p}_{i}h - h^{p}_{i}\right)_{,pj} + \left(\frac{1}{2}\delta^{p}_{j}h - h^{p}_{j}\right)_{,pi}$ and this may be made to vanish by choosing the coordinate condition such that

$$\left(h^{p}_{i} - \tfrac{1}{2}\delta^{p}_{i}h\right)_{,p} = 0$$

Hence

$$R_{ij} = -\tfrac{1}{2}\delta^{lm}h_{ij,lm}$$

and the field equations turn out to be

$$\Box\varphi^{j}_{i} = \frac{-16\pi G}{C^4}T^{j}_{i}$$

$\varphi^{j}_{i,j} = 0$ are thetcoordinate conditions where we have chosen

$$\varphi^{j}_{i} = h^{j}_{i} - \tfrac{1}{2}\delta^{j}_{i}h^{k}_{k}$$

Thus gravitational fields, like electromagnetic fields propogate in vacuum with fundamental velocity.

In 1939, Fierz and Pauli (2) considered the problem of constructing relativistic wave equation for particles of Spin

2 and rest mass zero, and they were led to the equations

$$\Box \varphi_i^j = 0$$

$$\varphi_{i\ ,j}^{j} = 0$$

Since these are analogous to the equations obtained in the above linearized theory, it was suggested that gravitons may be particles of spin two. Also since the gravitational forces have infinite range, it follows that the rest mass of the gravitons must be zero.

In the linearized theory of Einstein considered above we have chosen quantities upto the first order only. If the second order approximation is made, we get,

$$\Box h_2 = Q(h_1)$$

Q being a quadratic function.

If $h = O(\frac{1}{r})$

we get,

$$\Box h_2 \sim \frac{1}{r^2}$$

The solution of which is given by

$$h_2 = \frac{\log(t+r)}{r} F(t-r)$$

Keeping $(t-r)$ a constant if we let $r \to \infty$, we see that h_2 tends to zero as $\frac{\log r}{r}$, whereas h_1 tends to zero as $1/r$. That is h_2 goes to zero slower than h_1 which indicates that there must be something wrong with this approximation method.

To overcome the above-mentioned difficulty several improvements were introduced in the approximation method. For example Fock introduced Sommerfield's radiation condition namely,

$$g_{ik\,l}\, k^l = O(1/r^2) \qquad \text{where} \quad k^l k_l = 0$$

... and thus avoided the logarithmic in the second order approximation.

Trautman has criticized the validity of the linear approximation so far as energy momentum pseudo-tensor is concerned. He has used Synge's argument for this purpose. According to Synge, the field equations within matter, may be read either from left to right or right to left, in the sense that given the source function T_{ij} one can get the metric tensor which yields it, or starting with a given metric tensor one can get the source distribution. A metric tensor which is just approximate for a given source distribution may be considered to be the exact metric for some other distribution, namely the one which is obtained from it by direct calculation of the energy tensor. With this in mind Trautman claims that the outflow of energy and momentum from a radiating system, calculated from the linearized energy pseudo-tensor $_1t^{\alpha}_{k}$, is exactly cancelled by the (negative) outflow of the ordinary energy tensor $_1T_i^{\alpha}$ which one obtains if one recognizes that the metric is only an approximate one.

Lichnerowicz cleared, many of the doubts and his work established the propogation of gravitational shock waves with fundamental velocity. By a gravitational shock wave is meant a 3 space Σ across which g_{ij} and $g_{ij,k}$ are continuous, but there are essential discontinuities in some of the second derivatives $g_{ij,km}$. Now if the coordinates are chosen in such a way that the surface Σ has the equation

$$x^4 = 0 \quad \text{then over } \Sigma,$$

$$g_{ij}, \quad g_{ij\,k}, \quad g_{ij\,\alpha\beta} \quad g_{ij\,\alpha 4}$$

are continuous. The discontinuity can occur only for $g_{ij\,44}$

Now

$$R_{\alpha\beta} = \tfrac{1}{2} g^{44} g_{\alpha\beta\,44} + F_{\alpha\beta}$$

$$R_{\alpha 4} = -\tfrac{1}{2} g^{4\beta} g_{\alpha\beta,44} + F_{\alpha 4}$$

$$R_{44} = \tfrac{1}{2} g^{\alpha\beta} g_{\alpha\beta,44} + F_{44}$$

where the F's involve the continuous quantities. We see that the field equations $R_{ij} = 0$ do not determine $g_{ij\,kl}$, because $g_{i4\,44}$ do not enter at all. Thus discontinuities in second derivatives may occur across any 3-space. Since we are interested in essential discontinuities, a transformation to Gaussian coordinates[+] will lead to $g_{i4\,44} = 0$ and out of

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+Synge: Relativity, The General Theory.

the ten second derivatives we have only to consider the six $g_{\alpha\beta,44}$. Now in order not to determine $g_{\alpha\beta,44}$, we must have

$$g^{44} = 0$$

If Σ has the equation $f(x^1, x^2, x^3, x^4) = 0$ we have the condition

$$g^{ij} f_{,i} f_{,j} = 0$$

i.e. we find that Σ is a <u>null surface</u>. Hence the gravitational shock waves are identified with null surfaces and therefore gravitation propogates with the fundamental velocity.

Lecture 2

To recapitulate:

A. Reasonably acceptable approximate solutions lead to two doubts.

(i) Linearized theory cannot be trusted because General Relativity is essentially non linear.

(ii) Do approximate solutions really correspond to some exact solutions or are they spurious?

B. Lichenerowicz showed beyond doubt that gravitational shocks are propogated with the fundamental velocity.

Exact Solutions:

1. With spherical symmetry Schwarzschild's solution is the only solution of vacuum field equations and because it is static there are no strictly spherical waves.

2. Hence we shall now consider cylindrical symmetry we shall start with a coordinate system $(x^1\ x^2\ x^3\ x^4)$. Let $x^1 = 0$ represent the axis of rotation, $(x^1\ x^2)$ the meridian plane, x^3 the angular coordinate, and $x^4 = t$. With these coordinates, the symmetry requirements that, the gravitational field be symmetrical about every plane x_2=const , and about the meridian plane x_3=const. give

$$g_{12} = g_{23} = g_{31} = g_{34} = g_{24} = 0$$

One can choose the two coordinates (x^1, x^4) in such a way that, one gets

$$g_{11} = -g_{44} \quad \text{and} \quad g_{14} = 0$$

We are left with g_{11}, g_{22} and g_{33} as the unknown potentials. Putting $\sigma = (g_{22}g_{33})^{1/2}$, we find that the field equation

$$R_{ij} = 0 \qquad \text{imply} \quad \sigma_{,11} - \sigma_{,44} = 0, \quad \sigma_{,1} = \frac{\partial \sigma}{\partial x^1} \text{ etc}$$

Under what transformation of $(x^1 x^4)$ the conditions $g_{11} = -g_{44}$ and $g_{14} = 0$ are invariant? If $\bar{x}^1$, $\bar{x}^4$ are transformed coordinates for which these conditions also hold true, we find that

$$\frac{\partial \bar{x}^1}{\partial x^1} = \frac{\partial \bar{x}^4}{\partial x^4} \qquad \frac{\partial \bar{x}^1}{\partial x^4} = \frac{\partial \bar{x}^4}{\partial x^1}$$

i.e.

$$\frac{\partial^2 \bar{x}^1}{\partial (x^1)^2} - \frac{\partial^2 \bar{x}^1}{\partial (x^4)^2} = 0$$

This relation is of the same form as that satisfied by σ. Hence we can take

$$\sigma = a x^1$$

where a is any arbitrary constant.

Finally the equations $R_{ij} = 0$, lead to

$$\beta_{,11} - \beta_{,44} + \frac{\beta_{,1}}{x^1} = 0$$

$$\alpha_1 = \tfrac{1}{2} x^1 (\beta_1^2 + \beta_4^2) - \frac{1}{2x^1}$$

$$\alpha_4 = x^1 \beta_1 \beta_4$$

where α and β are so chosen that,

$$-e^{\alpha} = g_{11} \qquad \text{and} \qquad e^{2\beta} = \frac{g_{22}}{g_{33}}$$

This is the Einstein-Rosen (3) solution.

Remarks: No unambiguous way of loss of mass can be ascertained because the source of the field will be an infinite rod and it looks as if energy is being smuggled in from infinity.

3. Plane Waves:

Rosen in 1937 proved that plane gravitational waves do not exist. But what do we mean by plane waves? Bondi, Pirani and Robinson (4) defined plane symmetry as the symmetry defined by plane electromagnetic waves. Space-time traversed by plane waves admit a 5 parametric group of motion. If x-direction is the direction of propagation of waves, they consist of 3 parametric groups of translation along OY OZ and the plane $t-x$ =const , and two parameter group of rotations (null rotations) which leave plane $t-x$ =const. invariant. Such null rotations were studied by Sibata (5).

Several exact solutions of $R_{ij} = 0$ are known which exhibit this type of symmetry. Takeno (6) has given a number of such solutions.

One conclusion from plane wave solutions is that these waves are transverse. Because it is found that a wave passing two test particles at relative rest endows them with relative acceleration perpendicular to the direction of wave.

4. <u>Waves from bounded sources</u>:

Cylindrical and plane waves cannot display important physical characteristics of physically significant waves. General relativity is a "particularly closed" theory and so open waves are likely to be misleading. The simplest field due to a finite source is spherically symmetrical, but Birkhoff's theorem does not allow any possible wave like solutions. Bondi, Van-der-Burg and Metzer (7) have worked out the case of an axial-symmetric bounded system. Pandya (8) gave a scheme for an isolated system without any symmetry. Sachs (9) gave a more complete scheme than Pandya's.

<u>Deviation of the metric</u>:

We assume that space-time is pervaded by a congruence of null vectors which are everywhere hypersurface-orthogonal. Through any point choose a null vector of this congruence as one of the coordinate vectors

$$v^{\mu} = (v^1, 0, 0, 0)$$

and take the orthogonal hypersurface as x^4=const . The normal to this is

$$v_{\mu} = (0, 0, 0, v_4)$$

Since $v_\mu = g_{\alpha\mu} v^\alpha$, $g_{11} = 0$, $g_{12} = 0$, $g_{13} = 0$. If the space is pervaded by gravitational waves, then the rays will be along null vectors.

Coordinate conditions:- Choose x^2, x^3 coordinates in such a way that

$$g_{22} = g_{33}$$
$$g_{23} = 0$$

Hence the line element can be written as,

$$ds^2 = -B\left[(dx^2)^2 + (dx^3)^2\right] + C(dx^4)^2 + 2H\,dx^1\,dx^4 + 2\lambda\,dx^2\,dx^4 + 2\mu\,dx^3\,dx^4$$

where B, C, H, λ and μ are functions of all the coordinates x^1, x^2, x^3, x^4.

Equations $R_{\alpha\beta} = 0$ $(\alpha, \beta = 1, 2, 3)$ along with a rotation in $(x^1\ x^4)$ plane lead to $H=1$. Bondi and coworkers proved this result. Pandya had assumed it.

<u>News functions</u>

It turns out that the equations $R_{\alpha\beta} = 0$ hold on the hypersurface $x^4 =$ const. Therefore they determine B, C, λ, μ on this hypersurface. Their march from one hypersurface to the next is given by

$$R_{14} = 0$$

$R_{\alpha\beta} = 0$ lead to several arbitrary functions of x^4. They bring to us information from this hypersurface. Bondi calls them news function.

In Bondi's axi-symmetric case, there was one news function. In the general case without spatial symmetry there are two news functions. The mass of the isolated system can be calculated and connected with the news function.

Robinson and Trautman (10) gave a particular solution which in a sense described spherical waves but the sources show no loss of mass because there are singularities in the solution which pipe in energy from infinity to the source.

Lecture 3.

To recapitulate:

So far we have considered attempts to solve $R_{ij} = 0$, to obtain solutions which would

(1) correspond to wave propagation

(2) imply a loss of mass of the source, and

(3) lead to a relative acceleration of two test particles.

Equations $R_{ij} = 0$ are too complicated to yield any general results giving general properties of gravitational waves.

A different approach leading to such general properties is needed. Presently we consider the geometrical approach.

Petrov classification:

In 1954 Petrov (11) classified all solutions of $R_{ij} = 0$ into three distinct classes. For this purpose he used the values of R_{abcd} at a point. In this process he had to associate certain null vectors with R_{abcd}.

Pirani (12) attempted to link up this classification with gravitational radiation because gravitational radiation implies certain relationship of R_{abcd} with the null come.

Properties of R_{abcd}:

The Riemann tensor R_{abcd} is given by

$$R_{abcd} = \tfrac{1}{2}\left(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}\right) + g^{mn}\left([ad,m][bc,n] - [ac,m][bd,n]\right)$$

It has the following symmetry properties

$$R_{[ab][cd]} = R_{abcd} \quad , \quad R_{a[bcd]} = 0$$

By contraction of R_{abcd} , we get the Ricci tensor and the scalar curvature.

Ricci tensor $\quad R_{bc} = g^{ad} R_{abcd}$

Scalar Curvature $\quad R = g^{bc} R_{bc}$

<u>Weyl tensor</u>:

Weyl tensor is obtained from Riemann tensor and is given by

$$W_{abcd} = R_{abcd} - \tfrac{1}{2}\left(g_{ad}R_{bc} + g_{bc}R_{ad} - g_{ac}R_{bd} - g_{bd}R_{ac}\right) + \tfrac{1}{6}\left(g_{ad}g_{bc} - g_{ac}g_{bd}\right)$$

It has the same symmetry properties as the Riemann tensor. But unlike Riemann tensor, Weyl tensor vanishes identically on contraction of its two indices

$$g^{ad} W_{abcd} = 0$$

In vacuum (i.e. $R_{bc} = 0$) Weyl tensor reduces to Riemann tensor.

Petrov's method of classification is also known as Matrix method of classification. Synge (Commn. Dublin Inst. of Adv. Studies A. No.15, 1964) gives an excellent account of this method. In this method one takes the advantage of the symmetry properties of the Weyl tensor to write it as a matrix and classifies it in terms of the eigenvalues and eigenvectors of this matrix.

Let $g_{abcd} = -g_{ad}g_{bc} + g_{ac}g_{bd}$

Consider an eigenvalue problem

$$W_{abcd}F^{cd} = \lambda g_{abcd}F^{cd}, \qquad F^{cd} = F^{[cd]}$$

Petrov simplified this to a three dimensional complex valued problem. The Weyl tensor which is skew symmetric in each of two pairs of indices, may be written as a 6×6 matrix. Considering a 6 dimensional structure we get W_{abcd} to be a second rank tensor. Using Petrov notation, we have that, if $K_{\alpha\beta}$ are the physical components of an antisymmetric tensor, the corresponding six vector is obtained by relabeling the suffixes in accordance with the rule

$$23 \to 1, \quad 31 \to 2, \quad 12 \to 3, \quad 14 \to 4, \quad 24 \to 5, \quad 34 \to 6$$

Letting A, B range over the values 1.......6, we can write our problem as

$$W_{AB}F^{B} = \lambda g_{AB}F^{B}$$

At an event we can always choose coordinates such that $g_{ab}=\delta_{ab}$ if we put $x^4 = it$.

Then

$$g_{AB} = \text{dia}\{1, 1, 1, 1, 1, 1\}$$

$$F^B \equiv \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_6 \end{pmatrix} = \begin{pmatrix} G \\ H \end{pmatrix}$$

where G is real and H imaginary.

The matrix W_{AB} , because of the symmetry properties can be written in the form

$$W_{AB} = W = \begin{pmatrix} M & N \\ N & M \end{pmatrix}$$

where M and N are 3x3 matrices with $M = \tilde{M}$, $N = \tilde{N}$ and $\operatorname{tr} M = 0$, $\operatorname{tr} N = 0$. Again M is a real matrix and N is purely imaginary. Our equation

$$WF = \Lambda F \qquad \text{gives,}$$

$$\begin{pmatrix} M & N \\ N & M \end{pmatrix} \begin{pmatrix} G \\ H \end{pmatrix} = \Lambda \begin{pmatrix} G \\ H \end{pmatrix}$$

i.e.

$$MG + NH = \Lambda G$$
$$NG + MH = \Lambda H$$

or $KJ = \Lambda J$ where $K = M + N$ and $J \equiv G + H$ are both complex matrices. The characteristic equati is given by

$$\det \left| K - \lambda I \right| = 0$$

This is a cubic with complex coefficients and hence has three roots. The classification of Weyl tensor is done according to the nature of the roots and the corresponding eigenvectors.

Let λ', λ'' and λ''' denote the eigenvalues and J', J'' and J''' the corresponding eigenvectors. Then we have the following possibilities and classification is made accordingly.

Classification table

Class	Eigen Values	Eigen Vectors	type
1	$\lambda' \neq \lambda'' \neq \lambda'''$	J' J'' J''' all distinct and non-null.	I
2 (a)	$\lambda' \neq \lambda'' = \lambda'''$	J' J'' all distinct and non-null.	D degenerate
(b)	$\lambda' \neq \lambda'' = \lambda''$	J' non-null, J" null and two distinct vectors.	II
3 (a)	$\lambda' = \lambda'' = \lambda''' = 0$ $(\because \lambda' + \lambda'' + \lambda''' = 0)$	J' null and others non-null with two distinct vectors.	N
(b)	$\lambda' = \lambda'' = \lambda''' = 0$	One null vector others non-null and 1 distinct vector.	III

Pirani (12) suggested that all solutions except I and D may be taken as containing gravitational radiation, the reason obviously being that the null vectors are associated in II, N, and III type solutions.

At present we are far away from accepting Pirani's suggestion. We have from electromagnetic fields that for a null tensor F_{ab},

$$F_{ab} F^{ab} = 0 \qquad \text{and} \qquad F^{*}_{ab} F^{*ab} = 0$$

and we know that there exists a null vector k^a, such that

$$k^a F_{ab} = 0 \quad , \quad k_a k^a = 0$$

Debever has established a similar result for the tensor R_{abcd} On this basis Sachs (13) made the classification of the Weyl tensor. This method is known as Tensor method.

<u>Tensor method</u>: In the words of Sachs-Debever's theorem states that,

"In every empty space-time $R_{ij} = 0$, at an event, there exist at least one and atmost four directions k_a such that

$$k_a k^a = 0 \qquad k_{[a} R_{b]ij[c} k_{d]} k^i k^j = 0$$

Therefore we have the following consequences:

1. In every solution of $R_{ij} = 0$, R_{abcd} is in some relation to the null cone. Therefore, basis for Pirani's earlier criterion of gravitational radiation is lost.

2. One can classify solutions on the basis of the number of null directions at a point.

Sachs did this classification and the results are as follows:-

A. If we can choose vectors k^{μ} such that

$$R_{abcd}\, k^{d} = 0,$$

(4+0)

then the solutions are of type N (Petrov) and we have all the 4 rays coincident at a point.

B. If $R_{abcd}\,k^{d} \neq 0$, we shall try $R_{abc[d}\,k^{c}k_{e]}$ and if this is zero the solutions are of type III (Petrov). In this case we will have three coincident rays and one distinct ray at a point.

(3+1)

C. If (A) and (B) are not possible we shall consider

$$R_{abc[d}k_{e]}\,k^{a}k^{c}$$

If we can choose two vectors k^{μ} and m^{ν} such that

$$R_{abc[d}\,k_{e]}\,k^{a}k^{c} = 0$$

and

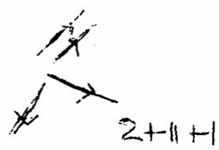
$$R_{abc[d}\,m_{e]}\,m^{a}m^{c} = 0$$

then the solutions belong to type D (Petrov) and we will have two rays disjoint, each being a double ray.

(2+2)

D. If we can have only one vector such that

$R_{abc[d}k_{e]}k^a k^c = 0$ then the solutions belong to class II (Petrov), and the vectors will be 3 rays all disjoint with one of them being a double ray.



E. Finally if none of the above (A to D) cases hold, we shall definitely have,

$$k_{[a}R_{b]ij[c}k_{d]}k^i k^j = 0$$

by the theorem and the solutions belong to class I(Petrov). In this case we will have all the four distinct rays.

If any of these equations (A) to (E), is satisfied then those that follow are also satisfied.

<u>Spinor method</u>

Another important type of classification is the Spinor method and it is due to Penrose (14). Though this method is equivalent to tensor method of Debever, it employs quite a different formalism. This method depends on the following relationship between spinors, null vectors and tensors at a point of space time.

1. A real null vector ξ^a corresponds to the product $\xi^A \xi^{\dot{X}}$. By "corresponds" we mean;

$$\xi^A \xi^{\dot{X}} = \sigma_a{}^{A\dot{X}} \xi^a$$

where $\sigma_a^{\ A\dot{X}}$ are 4 spinors, $(a=1,\cdots 4)$ which can be transformed to the Pauli matrices in a locally cartesian coordinate frame

$$\sigma_1^{\ A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2^{\ A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3^{\ A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_4^{\ A\dot{X}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. A real skew symmetric tensor F_{ab} corresponds to

$$\varphi_{AB}\epsilon_{\dot{W}\dot{X}} + \varphi_{\dot{W}\dot{X}}\epsilon_{AB}$$

where φ_{AB} is a symmetric 2-spinor and $\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. If W_{abcd} is a real Weyl tensor, it corresponds to

$$\Psi_{ABCD}\epsilon_{\dot{W}\dot{X}}\epsilon_{\dot{Y}\dot{Z}} + \Psi_{\dot{W}\dot{X}\dot{Y}\dot{Z}}\epsilon_{AB}\epsilon_{CD}$$

Ψ_{ABCD} being a symmetric 4 spinor.

Now we make use of the property that a spinor can be decomposed into symmetric product of linear factors, and express the quartic $\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D$, where ξ^A is an arbitrary spinor,

$$\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = (\alpha_A\xi^A)(\beta_B\xi^B)(\gamma_C\xi^C)(\delta_D\xi^D)$$

we can at once write,

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$$

Lecture 4

Radiation through matter:

By classification of Weyl tensor we have associated a "ray" (at least one) with every solution of Einstein's field equations

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij}$$

Therefore one can talk of "gravitational radiation within matter".

In order to get unique solutions of the partial differential equations a condition at infinity was introduced namely that at large distances the field will get weaker and weaker so that $R_{abcd} \to 0$ at ∞. This is against the spirit of "completeness" of general relativity, for a flat space time condition assumes that there is a flat background over which gravitational fields are superposed.

This led to the search for various cosmological models. Whatever be the theory of gravitation the line element to describe the uniformity of gravitational field at large distances is the Robertson-Walker line element

$$ds^2 = dt^2 - e^{g(t)} (dx^2 + dy^2 + dz^2) / \{1 + k(x^2 + y^2 + z^2)\}^2$$

where $k = +1$, -1 or 0 according to the curvature of the space t = constant. For this space-time $W_{abcd}=0$ but $R_{abcd}\neq 0$ the condition at infinity is that $W_{abcd}=O(1/r)$ and $R_{abcd}\neq O(1/r)$ Since cosmic background is not empty, one needs to consider gravitational radiation in space which has $R_{ij}\neq 0$.

For the solution within the source region, we must not only have matter but also radiation.

Completeness argument would suggest that we may have gravitational radiation as the source of gravitational radiation. We shall understand this statement by looking at the linearized theory.

We have,

$$\Box\, u_{\mu\nu}=0 \qquad , \qquad u^{\mu\nu}{}_{,\nu}=0$$

If interactions are to be included,

$$\Box\, u_{\mu\nu}=k\,\theta_{\mu\nu} \qquad\qquad k \text{ a coupling constant}$$

$$\theta^{\mu\nu}{}_{,\nu}=0$$

In a linearized theory we have flat background, so the interactions are not generated from within the theory, but they are superimposed on the theory. Since the matter stress tensor $T_{\mu\nu}$ satisfies,

$$T^{\mu\nu}{}_{,\nu}=0$$

we might associate $T^{\mu\nu}$ with $\theta^{\mu\nu}$. Appearence of stress tensor is to be expected in developing a theory of gravitation because stress and energy are sources of the gravitational fields. The gravitational field itself is expected to contribute some stress and energy as well. Hence

$$\Box u_{\mu\nu} = K\,(T_{\mu\nu} + t_{\mu\nu})$$

Now if there is no matter, $T_{\mu\nu} = 0$, we have

$$\Box u_{\mu\nu} = K\, t_{\mu\nu}$$

S.N.Gupta (15) worked out the Lagrangian from which the above equation could be derived. He then considered stress and energy of this gravitational field to be itself the source of and thus added a term $t_{1\mu\nu}$ obtaining $\Box u_{\mu\nu} = K\,(t_{\mu\nu} + t_{1\mu\nu})$ and worked out the corresponding Lagrangian. Proceeding in this manner he obtained a series for the Lagrangian function which summed upto $L = R\sqrt{-g}$. This shows that stress and energy of a gravitational field would itself contribute to the source of the field.

<u>Stress Energy tensor due to flowing energy:</u>

The general macroscopic expression for the energy-momentum tensor corresponding to a flow of radiation in the x^1 direction is given by[+]

$$T_0^{11} = T_0^{44} = T_0^{14} = \rho \;, \quad T_0^{22} = T_0^{33} = 0$$

[+] cf: Tolman: Relativity, Thermodynamics and Cosmology page 272-273 (for the electromagnetic radiation).

where ρ is the density of radiant energy at the point of interest and the suffix 'o' stands for the "natural coordinates" considered at that point. If the radiation is flowing with fundamental velocity, then,

$$d\underset{o}{x}^1 = d\underset{o}{x}^4 = d\tau .$$

Transforming to a general coordinate system, we find that,

$$T^{\mu\nu} = \frac{\partial x^\mu}{\partial \underset{o}{x}^\alpha} \frac{\partial x^\nu}{\partial x^\beta} \underset{o}{T}^{\alpha\beta}$$

Considering the surviving component, we get

$$T^{\mu\nu} = \left(\frac{\partial x^\mu}{\partial \underset{o}{x}^1} + \frac{\partial x^\mu}{\partial \underset{o}{x}^4}\right)\left(\frac{\partial x^\nu}{\partial \underset{o}{x}^1} + \frac{\partial x^\nu}{\partial \underset{o}{x}^4}\right) \rho$$

Substituting $v^\mu = \dfrac{dx^\mu}{d\tau}$, we get

$$T^{\mu\nu} = \rho v^\mu v^\nu \quad \text{and} \quad v_\mu v^\mu = 0$$

Lichnerowicz proved that for fields of type N,

$$k_{[a} R_{bc]de} = 0 \qquad \text{where} \quad k^a k_a = 0$$

$$k^a R_{abcd} = 0$$

From this we have

$$R_{bc} = \tau k_b k_c$$

This agrees with our result.

Hence we have the result that a solution of type N can be generated from the gravitational field produced by flowing energy.

We shall now consider the above situation for the case of cylindrical symmetry[+], wherein we use the Einstein and Rosen metric. In the scheme of Einstein and Rosen discussed in our second lecture, put $r e^{\beta} = e^{2\psi}$ and $r e^{\beta} e^{\alpha} = e^{2\gamma}$ where γ and ψ are functions of r and t only. With $x^1 = r$, $x^2 = z$, $x^3 = \varphi$ we have the following relations of the non-vanishing components of mixed Ricci tensor

$$R_{ab} = \tau k_a k_b$$

$$R^1_1 + R^4_4 = 0$$

$$R^1_1 + R^1_4 = 0$$

$$R^2_2 = 0 , \quad R^3_3 = 0$$

i.e. we get

$$\psi_{11} - \psi_{44} + \frac{\psi_1}{r} = 0$$

$$\gamma_1 + \gamma_4 - r(\psi_1 + \psi_4)^2 = 0$$

$$\gamma_{11} - \gamma_{44} + \psi_1^2 - \psi_4^2 = 0$$

[+]cf: J. Krishna Rao: Pro. of N. Ins. Sc. India. Vol.30A, No.4, 440.

Since this is a consistent set of 3 equations with only two unknowns, we shall leave out the third of the above set of equations and also we note that in the case of pure gravitational waves the second equation decomposes into

$$\gamma_1 - r(\psi_1^2 + \psi_4^2) = 0$$

$$\gamma_4 - 2r\psi_1\psi_4 = 0$$

If $\overset{0}{\psi}$ and $\overset{0}{\gamma}$ are the solutions corresponding to $R_{ij} = 0$, then

$$\psi = \overset{0}{\psi}$$

$$\gamma = \overset{0}{\gamma} + f(t-r)$$

are the solutions corresponding to

$$R_{ab} = \tau k_a k_b$$

In particular if we consider the trivial solutions $\overset{0}{\psi} = 0$ and $\overset{0}{\gamma} = 0$, we get

$$\psi = 0 \ , \quad \gamma = f(t-r)$$

Then the line element takes the form

$$ds^2 = -e^{2\gamma}(dr^2 - dt^2) - r^2 d\varphi^2 - dz^2$$

This gives the geometry of the region through which gravitational waves are passing and the energy that is flowing is itself serving as the source of the Gravitational field through

$$R_{ab} = \tau k_a k_b$$

<u>Gravitational radiation through the universe:</u>

The cosmic background is

$$ds^2 = dt^2 - e^{g(t)}(dx^2 + dy^2 + dz^2) / [1 + k(x^2+y^2+z^2)]^2$$

Since $W_{abcd} = 0$, for these space times we could write them in a form conformal to flat.

Infeld and Schild (16) found this conformally flat form:

$$ds^2 = e^{2H}[dt^2 - dx^2 - dy^2 - dz^2]$$

where H has different forms under different conditions as $k = +1$, 0 or -1.

Transforming to cylindrical coordinates we have,

$$ds^2 = e^{2H}[dt^2 - dr^2 - r^2 d\varphi^2 - dz^2]$$

Take

$$ds^2 = e^{2H}[e^{2f}(dt^2 - dr^2) - r^2 d\varphi^2 - dz^2]$$

where $f = f(t \pm r)$. This gives a type N solution and therefore a null gravitational field whose source is matter plus flowing radiation.

Thus the effect of flowing radiation through material universe can be discussed[+].

Spherical symmetry:

For the case of $R_{ij} = 0$, we have no other solution except that of Schwarzschild. But there may be solutions of

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij}$$

exhibitting spherical symmetry which may give radiation through matter.

Conclusions:

1. A vast field is open for exploration.

2. Bondi once remarked to the following effect: Any interior solution which is not spherically symmetric will be useful. It does not matter if it has some physically unexpected features. Let us have an interior solution fitted with an exterior solution going to spherical waves at infinity. A rigorous solution will point the way to get physically reliable approximate solutions.

We have the following questions:

(1) Will this classification of Weyl tensor lead to quantisation of the wave field?

(2) Will flowing gravitational energy give a definite contribution to the source of gravitational field?

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Chapter II. Lecture 5.

1. Gravitational Collapse

For more than fifteen years the nature and origin of extragallactic radio sources has been one of the most fascinating problems of modern astronomy. For some time it was believed that such radio sources were due to energy liberated during collision of galaxies. But in the last few years it has turned out that this explanation is untenable in most cases. The estimates indicate that more than 10^{60} ergs of energy are involved in the radio emission. The energy requirement has so far ruled out all possible explanations.

In 1963, Hoyle and Fowler suggested that energies which lead to formation of radio sources could be supplied through the gravitational collapse of certain super stars. Such a super sun if it were to shrink down close to Schwarzschild radius could supply the necessary energy.

The source 3C273 seems to be a superstar and has a diameter of about a light week. It is the brightest known object in the universe about a million million times brighter than the Sun. Various problems arose at that time and the important ones are commuticated as follows+.

1) Astronomers observed some unusual objects connected with radio sources. Are these the debris of a gravitational implosion?

+Preface to Quasistellar sources and Gravitational collapse Chicago Univ. Press, 1965.

2) By what machinery is the gravitational energy converted into radio waves?

3) Does gravitational collapse on our present assumptions lead to indefinite contraction and a singularity of space time?

4) If so how should we change our theoretical assumptions in order to avoid this catastrophe?

As an answer to question 3, we know by general relativity that indefinite contraction is not possible and there is a natural limit for the radius of any spherically symmetric object. This is the Schwarzschild limit and at the limiting valve we have a singularity. The understanding of this Schwarzschild singularity is very important and presently we look into it.

2. Schwarzschild Singularity:

For a spherically symmetrical distribution of matter, outside the matter we have the Schwarzschild's solution of the equation

$$R_{ij} = 0$$

given by

$$ds^2 = \left(1 - \frac{2m}{r}\right) dT^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

As it is apparent, this solution has two singularities, one at

$r = 0$ and other at $r = 2m$. The surface $r = 2m$ is a null surface. Can we transform away this singularity? Choose a time coordinate u such that,

$$\frac{\partial u}{\partial r}\left(1 - \frac{2m}{r}\right)^{1/2} + \frac{\partial u}{\partial t}\left(1 - \frac{2m}{r}\right)^{-1/2} = 0$$

Clearly u is the retarded time. The transformation of T to retarded time u leads to the following line element

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2\,du\,dr - r^2 d\Omega^2$$

Apparently, $r = 2m$ is not a singularity here. To see this more clearly consider the transformation $u = t - r$. This makes,

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{4m}{r}\,dr\,dt - \left(1 + \frac{2m}{r}\right) dr^2 - r^2 d\Omega^2$$

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - \frac{2m}{r}(dt - dr)^2$$

This clearly shows that $r = 0$ is the only singularity of our solution. This line element was first derived by Eddington (1) and later by Finkelstein (2).

Since we know from general relativity that any set of transformations to any different systems do not alter the physical situation (principle of covariance) obviously the singularity $r = 2m$ could not have escaped into nothing. By regarding the transformation from T to t directly we find that it is given by

$$T = t + 2m \log (r - 2m)$$

Apparently we have the singularity in the transformation equation.

Now of the two, the Schwarzschild and Finkkstein solutions, which represent the physical situation? If we had derived Finkelstein's solution first, we would not have obtained the Schwarzschild solution, because the connecting transformation is a singular one and hence not permitted. We find that for Finkelstein's solution too, $r = 2m$ is a null surface. Let us discuss its nature.

For $r > 2m$, the Schwarzschild solution go over to Finkelstein line element by the transformation

$$T = t + 2m \log (r - 2m)$$

Hence in this case we have the t-axis as time like vector and the solution obtained is static.

For $r < 2m$ the transformation will have to be

$$T = t + 2m \log (2m - r)$$

and we find when we go over to the Schwarzschild metric the roles of t and r will have interchanged. This involves t in the coefficients and hence the solution is non-static.

Kruskal (3) introduced the transformations

$$-\frac{w}{v} = e^{T/2m}, \quad -vw = (x-1)e^{x}, \quad x = r/2m$$

and transformed Schwarzschild's line element into the form

$$d\lambda^2 = -16m^2 e^{-x}\, dv\, dw + 4m^2 x^2\, d\Omega^2$$

This does not have any singularity in the finite region. Once again the singularity has been transferred to the transformation. For $r = 2m$, $vw = 0$ and therefore the surface $r = 2m$ now corresponds to $T = +\infty$ or $T = -\infty$.

Let us now fix our attention on the Physical properties of the surface $r = 2m$.

1. If an object with $r < 2m$ emits radiation, we won't be able to observe it, because the light which starts from the object gets zero velocity at $r = 2m$. Hence all light signals stop at this barrier.

2. On the other hand we would have its gravitational field as given by Finkelstein or Kruskal (3). Thus we shall have a causeless gravitational field! This is highly repungant.

3. If an object has a radius $r > 2m$ and then contracts can it withdraw itself into the Schwarzschild's radius and thus disappear before our eyes?

As the object contracts the process appear~~s~~ slower and slower to an external observer and as it nears the Schwarzschild limit the process is extremely slowed down. For, we have,

$$d\lambda^2 = \left(1 - \frac{2m}{r}\right) dt^2$$

Since ds is finite, as $r \to 2m$, $\left(1-\frac{2m}{r}\right) \to 0$ and hence $dt \to \infty$. As viewed by an external observer the object will take infinite time to contract into Schwarzschild singularity. For an observer on the surface of the collapsing star, it may take finite time to reach Schwarzschild radius.

4. One may carry an impression that an object with $r < 2m$, must be very massive. But actually that is not so. For, consider

$$\frac{2m}{r} \sim 1$$

i.e.
$$\frac{8\pi r^2 \rho}{3} \sim 1$$

$\therefore$
$$\rho \sim \frac{3}{8\pi r^2}$$

i.e. ρ varies as $1/r^2$. Hence taking r to be sufficiently large we can maintain a lower density.

<u>Energy Consideration</u>

Can one build up a star of radius $r < 2m$ by bringing in particles from a diffused state at infinity? McCrea (4) has discussed this problem. A particle of proper mass m is brought from infinity in the Schwarzschild field to a radius R and then brought to rest. Then the energy in excess of the rest energy is radiated radially outwards in the form of a photon. The photon reaches infinity as one having energy $h\nu$. McCrea proves that as $R \to 2M$ (M=Schwarszchild mass of the object to which the particle is projected) the energy thus received back at infinity tends to m so that no further rest mass is added to M from infinity. Hence M is the maximum mass in radius R, or $R = 2M$ is the minimum radius for given M. Thus we see that the objects with $R \leq 2M$ is of no physical interest.

Lecture 6.

For a spherical distribution of given mass M, there is a lower limit to the radius in the scheme of general relativity and we have considered some properties of this limiting radius

$$R_g = \frac{2GM}{c^2}$$

R_g is called the gravitational radius of the distribution. So when a star contracts, distances of various particles from the center become smaller; gravitational forces therefore become stronger, higher pressures are needed to maintain equilibrium . But due to contraction densities also would have increased. Therefore how are the new density and pressure related with respect to themselves as well as with their earlier values?

We shall now consider the case of an equilibrium sphere in an adiabatic state in the Newtonian scheme. We have for such a case the pressure and mass variations given by

$$\frac{dp}{dr} = -\frac{m}{r^2}\rho \qquad \frac{dm}{dr} = 4\pi r^2 \rho \qquad \text{with } G=1$$

As we follow the matter through contraction, m is conserved, and hence let us use m as independent variable.

$$\frac{dp}{dm} = -\frac{m}{4\pi r^4} \qquad \frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$\rho = \frac{1}{4\pi r^2 \frac{dr}{dm}}$$

We shall limit the nature of contraction to the Homologus type, wherein every particle looses the same percentage of radial distance. Therefore as r decreases to λr, $r = f(m)$ decreases to $r = \lambda f(m)$. Then for pressure in density, we have

$$p \sim 1/\lambda^4 \quad \text{and} \quad \rho \sim 1/\lambda^3$$

Hence we have $p \sim \rho^{4/3}$.

Now if p varies as $\rho^{4/3}$ the star is in neutral equilibrium. If p varies as a higher power of ρ than $4/3$, then in order that the star may be in equilibrium in the new situation, heat must be taken away from it. Such a star will be able to contract only when it is radiating and the rate of contraction is determined by the rate of radiation.

If p varies less steeply than $4/3$ power of the density then any contraction leads to catastrophic collapse unless heat is constantly poured into the star to cook it up.

Thus we find that the entire problem of energy release is connected with the thermodynamic property of the material.

We shall consider the collapse with no internal pressure usually referred to as the classical implosion problem; or the free-fall collapse:-

The object is assumed to be spherically symmetric, of uniform density and with internal pressure equal to zero. Outside there is empty space with the flat space-time background at infinity. Our object is taken to implode from rest without any rotation.

The field equations are given by

$$R_{ij} - \frac{1}{2} g_{ij} R = -8\pi T_{ij} = -8\pi\rho\, v_i v_j$$

wherein we have considered the comoving coordinates

$$v^i = (0, 0, 0, v^4)$$

The solution for such a situation was first worked out by Datt (5) and later by Oppenheimer and Snyder (6). Hoyle, Fowler, Burbidge and Burbidge (7) have worked out the problem in detail.

The line-element for the interior field is found to be the same as that of Robertson and Walker given by

$$ds^2 = dt^2 - S^2(t)\left[\frac{1}{1-r^2}\, dr^2 + r^2\, d\Omega^2\right]$$

Einstein's equations give,

$$\frac{\dot{S}^2}{S^2} = -\frac{1}{S^2} + \frac{8\pi\rho}{3}$$

where the proper density ρ satisfies,

$$\rho S^3 = \text{Constant}$$

In equilibrium $\dot{S} = 0$ and hence $S_0 = \left(\frac{8\pi\rho}{3}\right)^{-1/2}$, where S_0 represent the value of S initially. We shall redefine the r coordinate replacing r by $\sqrt{\alpha}\, r$ where $\alpha = \frac{8\pi\rho_0}{3}$.

This makes $S_0 = 1$. Then the line element takes the form

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-r^2} - r^2 d\Omega^2\right]$$

with

$$\frac{\dot{S}^2}{S^2} = \frac{\alpha(1-S)}{S^3}$$

This can be integrated immediately giving

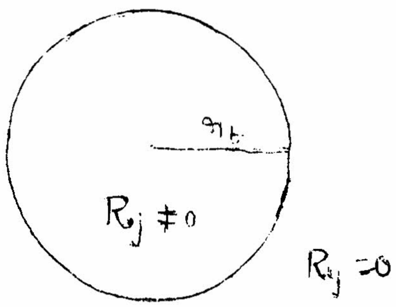
$$\sqrt{\alpha}\, t = \pi/2 - \sin^{-1}\sqrt{S} + \sqrt{S}\,\sqrt{1-S}$$

Unlike in the case of cosmology where $r = 0$ can be any assigned particle, here $r = 0$ represents uniquely the center of the collapsing body.

Now for the exterior solution, i.e. for $R_j = 0$ we may choose the Schwarzschild solution

$$ds^2 = -\left(1-\frac{2M}{R}\right)^{-1} dR^2 - R^2 d\Omega^2 + \left(1-\frac{2M}{R}\right) dT^2$$

where the sphere of radius $R = r_b S(t)$ will have an area $4\pi R^2$ which decreases with increasing t. Here r_b is the constant value of r on the boundary of the collapsing object. One can complete the exterior solution in terms of r and t of Oppenheimer and Snyder and later get the transformation to R and T. For



the exterior solution the line element turns out to be

$$ds^2 = dt^2 - S^2(x)\left[\frac{K^2 dr^2}{1 - \frac{x r_b}{r}} + r^2 d\Omega^2\right]$$

where

$$x = \frac{t\, r_b^{3/2}}{r^{3/2}}, \qquad K = \frac{1}{S}\frac{\partial (rS)}{\partial r}$$

Again S satisfies the same old differential equation, the only change being that the differentiation is with respect to $x = t\, r_b^{3/2}/r^{3/2}$. Similarly one can use the Schwarzschild type coordinates R and T for the interior solution also. The transformation equations from (R,T) to (r,t) are

$$R = rS(t) \qquad \text{and} \qquad T = \phi\left(\int \frac{r\,dr}{1 - \alpha r^2} + \int \frac{dt}{S\dot{S}}\right)$$

where ϕ is any differentiable function.

Then continuity of g_{rr} will lead to

$$\left(1 - \frac{2M}{R}\right)^{-1} = \left(1 - \alpha r^2 - r^2 \dot{S}^2\right)^{-1}$$

$$= \left(1 - \frac{\alpha r^2}{S}\right)^{-1} = \left(1 - \frac{\alpha r^3}{R}\right)^{-1}$$

Remembering the definition of α, we find that at the boundary $r = r_b$,

$$M = \frac{4}{3}\pi r_b^3 \rho$$

and this completes the required solution.

The proper time required for the object to contract from $R = r_b$ to $R = 2M = R_g$. is given by

$$\int_{S=1}^{\alpha r_b^2} \frac{dt}{ds}\, ds = \int_{1}^{\alpha r_b} \frac{ds}{\dot{S}}$$

$$= \frac{1}{\sqrt{\alpha}}\left[\frac{\pi}{2} - \sin^{-1}\sqrt{\alpha}\, r_b + \sqrt{\alpha}\, r_b\sqrt{(1-\alpha r_b^2)}\right]$$

which is finite.

The proper time required for the implosion into a singularity, i.e. for contraction to $r=0$ is

$$\int_{1}^{0} \frac{ds}{\dot{S}} = \frac{\pi}{2\sqrt{\alpha}}$$

of course we use the negative values of $\dot{S}$ in the above calculations.

For an external observer the time dilation factor $\left(1 - \frac{2M}{R}\right)^{-1/2}$ tends to infinity as $R \to R_g$ and so to him star takes infinite time to contract into the Schwarzschild radius.

After the object contracts beyond the gravitational radius R_g , we can send messages to it but can't receive any reply.

We have the exterior field given by

$$ds^2 = dt^2 - S^2(x)\left[\frac{K\, dr^2}{1-\frac{\alpha r_b}{r}} + r^2\, d\Omega^2\right]$$

with $K = \frac{1}{S}\frac{\partial}{\partial r}(r S)$ and $\chi = t\, r_b^{3/2} \cdot r^{-3/2}$

$$\left(\frac{dS}{d\chi}\right)^2 = \frac{\alpha(1-S)}{S}$$

For an imploding star $\left(dS/d\chi\right)$ has to be negative. Now a radial light pulse has

$$d\theta = 0 \quad \text{and} \quad d\Omega = 0$$

$$\therefore \quad \left(1 - \frac{\alpha r_b^3}{r}\right)^{1/2} dt = \pm\, S(x)\, K\, dr$$

The positive sign is chosen for an outward moving ray and negative for the inward moving one. But we have $R = r S$

$\therefore$

$$dR = r\frac{\partial S}{\partial t}\, dt + \frac{\partial}{\partial r}(r S)\, dr$$

$$= \frac{r_b^{3/2}}{r^{1/2}}\frac{dS}{d\chi}\, dt + S K\, dr$$

Using negative values of $\left(dS/d\chi\right)$ and dr for the light ray

$$dR = dt\left[-\frac{\sqrt{\alpha}\, r_b^{3/2}}{r^{1/2}}\left(\frac{1-S}{S}\right)^{1/2} \pm \left(1 - \frac{\alpha r_b}{r}\right)^{1/2}\right]$$

+ sign for outward moving ray, - sign for inward moving ray. It is easy to see that dR could vanish only if + sign is selected i.e. for outward moving light and this would happen when $R = r S = \alpha r_b^3 = R_g$.

Thus for imploding objects light can cross in the inward direction but can't return. For an exploding object light can cross into the outward direction but cannot enter. [cf.8].

Is it possible to halt this implosion before the radius reaches R_g?

Two ways out have been suggested. Hoyle and Narlikar have discussed the effect of their c-field on the imploding object and shown that singularities do not develop. As the body approaches singular state, positive energy flows out producing a repulsive gravitational force which halts the implosion.

Bonnor (9) has worked out an equilibrium configuration which is spherically symmetric, with zero pressure, i.e. consisting of dust which is however charged. The electrostatic repulsion balances the gravitational contraction. Therefore without introducing any extra-relativistic ideas like the c-field he proposes that the repulsive force of electrostatics may be used to halt the contraction.

Lecture 7

Collapse with emission of Radiation

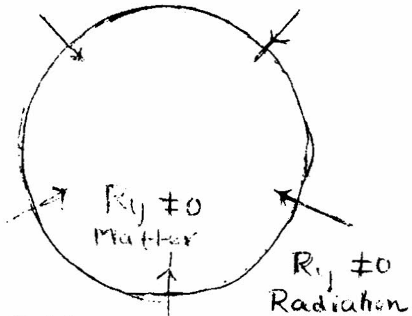
The dependence of internal energy or pressure and density may be such that in the process of contraction there is a copious release of energy which must be radiated away leading to a substantial decrease in m.

Macvitte[10] has found that the general relativity solution is incomplete when p is uniform (non-zero) and conjectures that this is essentially due to the fact that the process of contraction is non-adiabatic. In the case of Oppenheimer-Snyder model we have free collapse of an object. In order to consider collapse with radiation, we have to have two types of solutions.

1) A solution holding good within the collapsing matter.

2) A solution holding good outside the collapsing matter since in this region there is net flux of energy and so $R_{ij} \neq 0$ and therefore Schwarzschied's exterior solution does not hold.

Radiation solution



For a flow of energy

$$T^{ij} = q\, k^i k^j \qquad , \qquad k^i k_i = 0$$

We can normalize the null vector k^μ in the following way. Let the line element be

$$ds^2 = -e^{\lambda} dR^2 - R^2 d\Omega^2 + e^{\nu} dT^2$$

Let an observer at rest in (R, θ, ϕ, T) measure the radiation density as q. If v^i is the velocity of such an observer then

$$v^1 = v^2 = v^3 = 0, \quad v^4 = e^{-\nu/2}$$ and

$$v^i v^j T_{ij} = q$$

This leads to

$$k^4 = e^{-\nu/2}$$

$k_\mu k^\mu = 0$ now gives

$$k^1 = e^{-\lambda/2}$$

We put

$$k^\mu = \frac{\partial}{\partial x^\mu} = e^{-\lambda/2}\frac{\partial}{\partial R} + e^{-\nu/2}\frac{\partial}{\partial T} = \frac{d}{d\tau}$$

with this normalization of k^μ we find

now $\quad T_1^1 = -q, \quad T_4^4 = q \qquad T_1^4 = -e^{\frac{(\lambda-\nu)}{2}} q$

$$T_2^2 = T_3^3 = 0$$

Field-equations are

$$T_1^1 + T_4^4 = 0 \quad , \quad T_1^4 + e^{\frac{\lambda-\nu}{2}} T_4^4 = 0$$ and $T_2^2 = 0$

putting $\quad e^{-\lambda} = 1 - \frac{2m}{R}, \quad m = m(R,T)$, we get from the second of the above field equations,

$$e^{-\lambda/2}\frac{\partial m}{\partial R} + e^{-\nu/2}\frac{\partial m}{\partial T} = 0$$

which can be written as

$$\frac{dm}{d\tau} = 0$$

Expressing ν in terms of m, we get

$$e^{\nu/2} = - \frac{\dot{m}}{m'} \left(1 - \frac{2m}{R}\right)^{-1/2}$$

where dot and dash stand for differentiations with respect to T and R respectively. The first of the field equations give with the above values of λ and ν,

$$\left(\frac{\dot{m}'}{\dot{m}} - \frac{m''}{m'}\right)\left(1 - \frac{2m}{R}\right) = \frac{2m}{R^2}$$

and the first integral of this differential equation is

$$m'\left(1 - \frac{2m}{R}\right) = f(m)$$

where $f(m)$ is an arbitrary function of m.

Finally we get the radiation metric to be (see 11)

$$ds^2 = -\left(1 - \frac{2m}{R}\right)^{-1} dR^2 - R^2 d\Omega^2 + \frac{\dot{m}^2}{f^2}\left(1 - \frac{2m}{R}\right) dT^2$$

where

$$m'\left(1 - \frac{2m}{R}\right) = f(m).$$

Radiation Coordinates

The above line-element can be put into a very simple form. Let us choose u as a coordinate such that

$$\frac{du}{d\tau} = 0$$

i.e.

$$u' k^1 + \dot{u} k^4 = 0 \qquad \text{or} \qquad u' e^{-\lambda/2} + \dot{u} e^{-\nu/2} = 0$$

The radiation-metric takes the simple form (see 12)

$$ds^2 = \left(1 - \frac{2m}{R}\right) du^2 + 2\, du\, dR - R^2 d\Omega^2 \; ; \; [m = m(u)]$$

Lindquist et al[13] have discussed in detail the properties of this line element.

If m is a constant this go over into Schwarzschild solution.
If m is a function of u, then this gives the solution for a radiating star.

Let us consider radially moving observers. Their velocities are,

$$v^\mu = (v^1, 0, 0, v^4)$$

$$v^1 = \frac{dR}{d\tau} \qquad v^4 = \frac{du}{d\tau}$$

$$g_{\mu\nu} v^\mu v^\nu = 1 \qquad \text{gives}$$

$$2 v^1 v^4 + \left(1 - \frac{2m}{R}\right)(v^4)^2 = 1$$

If $v^1 = U$,

$$v^4 = \frac{-U \pm \sqrt{U^2 + 1 - \frac{2m}{R}}}{1 - \frac{2m}{R}}$$

$$= \left(1 - \frac{2m}{R}\right)^{-1} (\gamma - U) = \frac{1}{\gamma + U}$$

where

$$\gamma = \left(1 + U^2 - \frac{2m}{R}\right)^{1/2}$$

Since,

$$q = v^{\mu} v^{\nu} T_{\mu\nu}$$

field equations are

$$v^{\mu} v^{\nu} R_{\mu\nu} = -8\pi q$$

The only surviving component of $R_{\mu\nu}$ is R_{44} and it has the value $2\frac{dm}{du} / R^2$. The field equations now lead to

$$8\pi q = -(v^4)^2 R_{44} = -(v^4)^2 \frac{2\,dm/du}{R^2}$$

$$q = -\frac{1}{4\pi R^2(\gamma+v)^2}\frac{dm}{du}$$

Consider an observer at rest at a large distance from the object and he measures the luminosity L_∞, then

$$L_\infty = \lim_{\substack{R \to \infty \\ v \to 0}} 4\pi R^2 q = -\frac{dm}{du}$$

Thus we have the following conclusion.

1) $\frac{dm}{du}$ is negative.

2) Luminosity of the object as measured by an observer at rest at a large distance is $-\frac{dm}{du}$

3) If the luminosity at any distance R for an observer moving with velocity v is given by L then

$$L_\infty = L(\gamma+v)^2$$

Radiation solution in matter-co-moving coordinates

The above solution is in coordinates which are co-flowing with the radiation. We have a congruence of null vectors and have chosen one of our coordinate axes at any point to be in the direction of the null vector at the point.

In a contracting material distribution we shall have a congruence of time-like vectors and we would like to choose one of these vectors as a coordinate vector at a point. Consider the general spherically symmetric line element -

$$ds^2 = -e^{\lambda} dr^2 - r^2 e^{\mu} d\Omega^2 + e^{\nu} dt^2 + 2a\, dr\, dt,$$

We can impose two restrictions on the coordinates (r, t). Choose them so as to give

1) the congruence of time-like vectors v^{μ} to be $v^{\mu} = (0, 0, 0, v^4)$ at every point and

2) $g_{14} = 0$

With these restrictions

$$ds^2 = -e^{\lambda} dr^2 - R^2 d\Omega^2 + e^{\nu} dt^2$$

where $\lambda = \lambda(r,t)$, $R = R(r,t)$ and $\nu = \nu(r,t)$

How does the exterior solution for a radiating star look in these coordinates? We shall begin with,

$$ds^2 = \left(1 - \frac{2m}{R}\right)du^2 - R^2 d\Omega^2 + 2\,du\,dR$$

$$8\pi T^1_4 = -R^1_4 = -\frac{2\,dm/du}{R^2}$$

Change R to r. $R = R(r, u)$

$$ds^2 = \left(1 - \frac{2m}{R} + 2R_u\right)du^2 + 2R_r\,du\,dr - R^2 d\Omega^2$$

$$8\pi T^1_4 = -\frac{2\,dm/du}{R^2 R_r}$$

where the suffixes denote differentiation.

Now change u to t, $u = u(r, t)$ such that the term $dr\,dt$ disappears from the resulting line-element. We shall then obtain

$$ds^2 = -\left(1 - \frac{2m}{R} + 2\frac{\dot{R}}{\dot{u}}\right)\left(u'^2 dr^2 - \dot{u}^2 dt^2\right) - R^2 d\Omega^2$$

with

$$u'\left(1 - \frac{2m}{R} + \frac{\dot{R}}{\dot{u}}\right) = -R'$$

and

$$8\pi T^1_4 = \frac{-2\frac{dm}{du}\,\dot{u}}{R^2(-u')\left(1 - \frac{2m}{R} + 2\frac{\dot{R}}{\dot{u}}\right)}$$

But

$$8\pi T^1_4 = 8\pi q k^1 k_4 = 8\pi q e^{\frac{\nu-\lambda}{2}} = 8\pi q \frac{\dot{u}}{-u'},$$

$$8\pi q = \frac{-2\, dm/du}{R^2\left(1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}\right)}$$

We have thus an external solution which can be matched on to an interior solution describing contracting matter which permits a copious flow of energy.

If

$$D_t = v^\mu \frac{\partial}{\partial x^\mu} \qquad \text{with} \qquad v^\mu = (0, 0, 0, v^4)$$

$$D_t = e^{-\nu/2}\, \partial/\partial t$$

Let $U = D_t R$ = rate at which Schwarzschild distance changes.

$$U = e^{-\nu/2} \dot{R} = \frac{\dot{R}}{\dot{u}\left(1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}\right)^{1/2}}$$

$$U^2 = \frac{\dot{R}^2}{\dot{u}^2} \frac{1}{1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}}$$

$$\frac{\dot{R}^2}{\dot{u}^2} = U^2\left(1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}\right)$$

Now

$$e^{\lambda} = u'^2 \left(1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}\right)$$

$$= \frac{R'^2\left(1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}\right)}{\left(1 - \frac{2m}{R} + \frac{\dot{R}}{\dot{u}}\right)^2}$$

$$e^{\lambda} = \frac{R'^2\left(1 - \frac{2m}{R} + 2\frac{\dot{R}}{\dot{u}}\right)}{\left(1 - \frac{2m}{R}\right)^2 + 2\left(1 - \frac{2m}{R}\right)\frac{\dot{R}}{\dot{u}} + U^2\left(1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}\right)} = \frac{R'^2}{\left(1 - \frac{2m}{R} + U^2\right)}$$

$\therefore$ m gives the mass within a sphere of radius R, where

$$2m = R\left(1 + U^2 - R'^2 e^{-\lambda}\right)$$

Lecture 8

Field inside a radiating collapsing sphere

Considering the fluid in such a sphere, we shall describe it by its local thermodynamic properties, such as

the four velocity of a particle v^{μ}

the particle number density n

the energy density ρ and

the pressure p.

Then the equation of continuity

$$(n\, v^{\mu})_{;\mu} = 0$$

expresses the matter conservation. Let us consider the stress energy tensor to be made up of

$$T^{\mu\nu} = M^{\mu\nu} + E^{\mu\nu}$$

where $M^{\mu\nu} = (p+\rho)v^{\mu}v^{\nu} - p\, g^{\mu\nu}$ is due to matter part and $E^{\mu\nu} = q k^{\mu}k^{\nu}$ is due to the radiation. k^{μ} is the null vector. The conservation of the total stress-energy tensor requires

$$(M^{\mu\nu})_{;\nu} = -(E^{\mu\nu})_{;\nu} \neq 0$$

If the rate of decrease of internal energy due to emission of radiation per particle is denoted by C the flow rate of energy is

$$n c\, v_{\mu} = E^{\nu}_{\mu;\nu}$$

i.e. $$n c = (E^{\nu}_{\mu;\nu})\, v^{\mu} = -M^{\nu}_{\mu;\nu}\, v^{\mu}$$

$$\therefore \quad -n C = (\rho\, v^{\mu})_{;\mu} + (p\, v^{\mu})_{;\mu}$$

The equations of state are $p = p(n, \rho)$, $c = c(n, \rho)$,

Choose a comoving coordinate system defined by,

$$ds^2 = -e^\lambda dr^2 - r^2 e^\beta d\Omega^2 + e^\nu dt^2$$

with

$$v^k = (0, 0, 0, e^{-\nu/2})$$

$$\therefore \quad T^1_1 = -p - q, \quad T^2_2 = T^3_3 = -p$$

$$T^4_4 = \rho + q \qquad T^4_1 = -q e^{\frac{\lambda-\nu}{2}}$$

Altogether we have nowei ht equations and eight unknowns namely, n, ρ, p, c, λ, β, ν and q. Misner[14] has given a scheme for working with these eight equations.

<u>Analytical Solution</u>

The mathematical problem as formulated above is quite a tough one. However, a rigorous analytical solution will be very useful in understanding the inner structure of the equations. Efforts have been made by Bondi[15] and by Vaidya[16] to get analytical solutions.

We have the equations,

$$T^1_1 = -p - q, \quad T^2_2 = -p, \quad T^4_4 = \rho + q$$

$$T^4_1 = -q e^{\frac{\lambda-\nu}{2}}$$

$$\therefore \quad T^1_1 - T^2_2 = e^{(\nu-\lambda)/2} \, T^4_1$$

This is an equation between potentials $g_{\mu\nu}$ independent of p, ρ, q, r etc. The essence of the method is to solve this equation for λ, β and ν along with two convenient mathematical assumptions for λ, β, ν. The above equation looks like the following:

$$\frac{\beta''}{2} + \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\beta'}{4} - \frac{\beta'\nu'}{4} - \frac{\lambda'}{4} - \frac{1}{r}\left(\frac{\lambda'}{2} + \frac{\nu'}{2}\right)$$

$$-\frac{1}{r^2} + \frac{e^{\lambda-\beta}}{r^2} + e^{\lambda-\nu}\left[\frac{\ddot{\beta}}{2} - \frac{\ddot{\lambda}}{2} - \frac{\dot{\lambda}^2}{4} + \frac{\dot{\beta}^2}{2} - \frac{\dot{\lambda}\dot{\beta}}{4} - \frac{\dot{\beta}\dot{\nu}}{4} + \frac{\dot{\lambda}\dot{\nu}}{4}\right]$$

$$+ e^{\frac{\lambda-\nu}{2}}\left[\dot{\beta}' - \frac{\dot{\beta}\nu'}{2} + \left(\frac{\beta'}{2} + \frac{1}{r}\right)(\dot{\beta} - \dot{\lambda})\right] = 0$$

Putting $e^{\lambda} = s^2 e^{\alpha}$, $e^{\beta} = s^2 e^{\mu}$, $s = s(t)$

$\alpha = \alpha(r,u)$, $\mu = \mu(r,u)$, $\nu = \nu(r,u)$ u being the retarded times, we can make assumptions which will split the above equation into two equations leaving $s(t)$ undetermined. One of the two equations is

$$\nu' = u'(\alpha_u - \mu_u)$$

This therefore is the first of the two assumptions. If we add the assumption,

$$u'\mu_u = \lambda_r - 2\mu_r$$

the equation $T^1_1 - T^2_2 = e^{\frac{\nu-\lambda}{2}} T^4_1$ becomes amenable to integration and we find the final solution in the form

$$ds^2 = -s^2 e^{-\nu}\left[\frac{dr^2}{(ar^2+br+1)} + \frac{r^2 d\Omega^2}{(ar^2+br+1)}\right] + e^{\nu} dt^2$$

$$\nu'(ar^2+br+1) + u'(a_1 r^2 + b_1 r) = 0 \qquad a_1 = \frac{da}{du}$$

$$u'(ar^2+br+1) = f(u) \qquad b_1 = \frac{db}{du}$$

a, b and f being undetermined functions of u. If a and b are chosen as constants, $\nu' = 0$, so ν can be chosen equal to zero and we get

$$ds^2 = -s^2\left[\frac{d\bar{r}^2}{(1-\alpha\bar{r}^2)} + \bar{r}^2 d\Omega^2\right] + dt^2$$

where $\bar{r}^2 = r^2(ar^2+br+1)^{-1}$, $\alpha = a - \frac{b^2}{4}$

which is the Oppenheimer-Snyder solution.

ρ, p and q are given by

$$8\pi\rho = \frac{3}{4}(D_t\beta)^2 + 3\left(a - \frac{b^2}{4}\right)\frac{e^{\nu}}{S^2}$$

$$8\pi p = -\left[D_{tt}\beta + \frac{3}{4}(D_t\beta)^2 + \left(a-\frac{b^2}{4}\right)\frac{e^{\nu}}{S^2} + D_{rt}\beta\right]$$

$$8\pi q_1 = D_{rt}\beta + \frac{rb+2}{rS} D_r\nu$$

$$D_t = e^{-\nu/2}\frac{\partial}{\partial t}, \quad D_r = e^{-\frac{\lambda}{2}}\partial/\partial r$$

This solution can be fitted to our exterior solution

$$ds^2 = \left(1 - \frac{2m}{R} + \frac{2\dot{R}}{\dot{u}}\right)\left(-du'^2 dr^2 + \dot{u}^2 dt^2\right) - R^2 d\Omega^2$$

with $\quad u'\left(1 - \frac{2m}{R} + \frac{\dot{R}}{\dot{u}}\right) = - R'$

over a constant radius $r = r_0$.

At the boundary $p = 0$ and $g_{\mu\nu}$ and their first deviation are continuous. We find

$$(D_t R_q)^2 = \frac{2m_0}{R_0} - \frac{r_0^2 (a_0 - \frac{1}{4} b_0^2)}{a_0 r_0^2 + b_0 r_0 + 1},$$

which can be put in the form

$\frac{\dot{S}^2}{S^2} = \frac{\alpha(1-S)}{S^3}$ in the appropriate coordinate system.

<u>Physical properties of the above solution</u>

Density ρ is always positive. p and ρ together satisfy the following theorem,

$$e^{\nu}(p+\rho) = f(t) \text{ only.}$$

On the boundary

$$e^{\nu}(p+\rho) = e^{\nu_0}\rho_0$$
$$e^{\nu} p = e^{\nu_0}\rho_0 - e^{\nu}\rho$$

If $e^{\nu}\rho$ increases outwards in the distribution, p will always be non-negative. This property holds everywhere in the solution. Thus we have an exact generalization of Oppenheimer-Snyder solution.

Conclusion:

The general features of Oppenheimer-Snyder model are retained even when we introduce a non-adiabatic model. Since the equation for $R(t)$ is not obtained from the interior solution, but only on fitting it with the exterior solution, this case is worked only as viewed by an external observer. The situation as viewed by a comoving observer needs to be studied.

It was suggested that introducing rotation to the collapsing object may avoid the collapse into singularity because of the centrifugal force.

A detailed study of the gravitating distribution with rotation under the scheme of general relativity may lead to beneficial results in gravitational waves, gravitational collapse and certain cosmological problems.

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Lecture 9

Non-isotropic Cosmological Models

People have been speculating about the universe since the dawn of civilization. The scientific basis to the speculations is provided by the principle of uniformity. Another principle which is sometimes used is the Principle of Extrapolation. Some cosmological theories extrapolate from the physical theories which have been tested in an immediate neighbouhood. Newtonian or Einsteinian cosmologies are of this type. Some cosmological theories only use the Principle of Uniformity in one form or another and do not extrapolate from physical theories holding goal in our neighbourhood, Milne's theory and steady state theory are illustrations of this later type.

Robertson(1) deduced the basic geometry of the universe as a whole from an axiomatic point of view independently of any field theory of gravitation. His axioms were,

1) Weyl Postulate:- The world lines of galaxies form a non-intersecting congruence of time-like geodesics, which are hypher-surface orthogonal. Thus the time-element can be put in the form

$$ds^2 = dt^2 + g_{ij}\, dx^i\, dx^j = dt^2 - d\sigma^2$$

where $d\sigma^2$ is the time-element on the 3-space S orthogonal to these geodecies.

2) <u>The Cosmological Principle</u>: The 3-space S is homogeneous and isotropic i.e. it permits a six parameter group of motion which takes S onto itself, and the line element n it takes the form

$$d\sigma^2 = \frac{R^2(t)}{(1+kr^2)^2}(dx^2+dy^2+dz^2)$$

$$k = 0, +1 \text{ or } -1$$
$$r^2 = x^2+y^2+z^2$$

Thus the general line element for the universe known as Robertson-Walker line element is

$$ds^2 = dt^2 - \frac{R^2(t)}{(1+kr^2)^2}(dx^2+dy^2+dz^2)$$

with $k = 0, +1$, or -1 according as the 3-space is flat or has positive or negative curvature. The geometry of the universe is always Riemanian. Any theory of Gravitation can be used to discuss the material physical content of this universe. There are two lines of progress in the study of world models.

1) With Robertson-Walker geometry being fixed, we may consider different gravitational theories.

2) Accepting Einstein's theory of gravitation one may change the geometry.

The theories developed in this manner may be aptly termed cosmological speculations, as Eienstein put it, for at present we have rather few contacts with experiments or observations.

1. Friedmann's Model

Friedmann applied the Einstein theory of gravitation to the Robertson-Walker geometry and obtained a world model.

$$R_{ij} - \frac{1}{2} g_{ij} R^k{}_k = -8\pi G\, T_{ij} + \Lambda g_{ij}$$

$$T_{ij} = \rho v_i v_j \qquad p = 0, \qquad v^1 = v^2 = v^3 = 0, \quad v^4 = 1$$

$$ds^2 = dt^2 - \frac{R^2(t)}{(1 + kr^2)^2}(dx^2 + dy^2 + dz^2)$$

$$\frac{4}{3}\pi R^3 \rho = M > 0$$

If $\dot{R} \neq 0$ $\qquad \dot{R}^2 = \frac{GM}{4\pi R} + \frac{1}{3}\lambda R^2 - k$

If $\dot{R} = 0$ $\qquad \frac{GM}{4\pi R} + \frac{1}{3}\lambda R^2 - k = 0$

Particular cases:

1. Einstein's static model: $\qquad \dot{R} = 0, \quad \lambda = 1/R^2$

$d\sigma^2$ is a sphere of radius R. We get a 4-dimensional cylindrical model whose section $t = \text{const}$ is a sphere.

2. Desitter Universe:

We have three types of solutions

$0 \leq x \leq \pi \qquad ds^2 = dt^2 - \frac{1}{\alpha^2}\cosh^2\alpha t\,(dx^2 + \sinh^2 x\, d\Omega^2), \quad k = +1$

$\qquad ds^2 = dt^2 - e^{2\alpha t}(dx^2 + dy^2 + dz^2), \quad k = 0$

$0 \leq x < \infty \qquad ds^2 = dt^2 - \frac{1}{\alpha^2}\sinh^2\alpha t\,(dx^2 + \sinh^2 x\, d\Omega^2), \quad k = -1$

according as the time like geodesics never intersect, intersect asymptotically in the part, and intersect at a finite time. Of these, the case $k = 0$ satisfies Weyl's postulate fully.

Criticisms Use of Einstein's theory in Cosmology

1) Introduction of Λ term and later its removal.

2) Freidman's equation can be derived from Newtonian theory also.

3) ... $R_{ij}=0$ supported by three critical tests, but what experimental support is there for the field equation

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -8\pi T_{ij} ?$$

Non-isotropic Universes

In spite of the criticisms, we shall use Einstein's theory because there has been no experimental or observational evidence against that theory. Since world postulates of homogeneity and isotropy are satisfied by Nature only approximately, we could think of more general world models by appropriately changing these postulates.

Taking analogies from classical hydrodynamics we can define for our material medium, the

scalar of expansion: $\theta = \frac{1}{3} v^{\mu}{}_{;\mu}$

tensor of rotation : $w_{\mu\nu} = \frac{1}{2}(v_{\mu;\nu} - v_{\nu;\mu})$ and

tensor of shear : $\sigma_{\mu\nu} = \frac{1}{2}(v_{\mu;\nu} + v_{\nu;\mu}) - \frac{1}{3}(g_{\mu\nu} - v_{\mu}v_{\nu})\, v^{\mu}{}_{;\mu}$

where v_μ is a tangent vector to a momenta of the congruence time-like geodesics. A Raychoudhuri (2) proved an important theorem for inhomogeneous models with incoherent matter; From the field equations, he obtained the result

$$\theta_{;\lambda} v^\lambda + \theta^2 = -\tfrac{1}{3} q_{\mu\nu} q^{\mu\nu} - \frac{8\pi\rho}{6} + \frac{1}{3}\left(\Lambda + W_{\mu\nu} W^{\mu\nu}\right)$$

If Λ be nonpositive $W_{\mu\nu} = 0$ and $\frac{d}{d\tau} \equiv v^\lambda \frac{\partial}{\partial x^\lambda}$
we find $\left(\frac{d\theta}{d\tau} + \theta^2\right)$ is negative say $-\alpha^2$.

If $\theta = 1/\varphi$, $\frac{d\varphi}{d\tau} - 1 = \alpha^2 \varphi^2$ or $\frac{d\varphi}{d\tau} = 1 + \alpha^2\varphi^2 > 0$

Therefore $d\theta/d\tau$ is always negative. So there must appear in the past, an instant where θ would be infinite. A singularity must appear. Hence if one wishes to avoid a singularity he must choose $W_{\mu\nu} \neq 0$.

Godel's rotating Universe

Godel's(3) line element is given by

$$ds^2 = a^2\left[(dx^0 + e^{x^1} dx^2)^2 - (dx^1)^2 - \tfrac{1}{2} e^{2x^1}(dx^2)^2 - (dx^3)^2\right]$$

where a is a constant $\neq 0$, $8\pi G\rho = 1/a^2$, $\Lambda = -\frac{1}{2a^2}$
and $\omega^2 = 4\pi G\rho$, ω being the scalar of rotation.
Godel's type of universe can be obtained from Newtonian theory.

We consider

$$\nabla^2 \varphi = 4\pi G \rho - \lambda$$ and we remove the condition that as $r \to \infty$ $\varphi \to 0$

For $\varphi = 0$ and $v = 0$ one obtains the Einstein static universe. On the other hand one can obtain the Newtonian form of Godel's model with

$$\varphi = \frac{1}{2}\omega^2 \left((x^1)^2 + (x^2)^2\right) \qquad v^2 = \omega x^1, \; v^1 = -\omega x^2, \; v^3 = 0$$

$$2\omega^2 + \lambda = 4\pi G \rho \qquad \text{where} \quad \lambda = -4\pi G \rho < 0$$

Godel's universe is the first illustration of a universe where space sections are homogeneous but not isotropic. Such world models allow a transitive 3 -parameter group of motions in their space sections which has the same constants of structure as the 3-parameter group of orthogonal transformations in Euclidean space. From a theoretical point of view Godel's model is highly interesting in several aspects. It shows that in an infinite space the matter of the universe can rotate absolutely. In fact, this is the first indication that Mach's ideas are not included inthe general theory of relativity.

The general formalisms by which such models as Godels could be constructed is given by A. Taub(4) Also one can refer to Heckmann and Schuking(5) with their formalism. Heckmann and Schuking derived several solutions, giving universes containing incoherent matter, the matter flow having rotation and shear.

Lecture 10

We have carried out a survey of cosmological models and hinted at methods of getting homogeneous but non-isotropic models.

Generalisations:

Homogeneous world models with rotating and expanding incoherent matter lead to rather complicated equations. It can be shown that the matter in these models generally has a non-vanishing shear. It might be possible that in a rotating and expanding universe the singularity could disappear which otherwise is always present in a universe filled with nonrotating incoherent matter. Godel announced a simple stationary model homogeneous in space-time and I.Ozvath constructed the solution in the tetrad formalism as given by

$$
\begin{aligned}
e^1{}_\mu &= (-\sin x^3,\ \sin x^1 \cos x^3,\ 0,\ 0) \\
e^2{}_\mu &= (\cos x^3,\ \sin x^1 \sin x^3,\ 0,\ 0) \\
e^3{}_\mu &= (0,\ \cos x^1,\ 1,\ 0) \\
e^4{}_\mu &= (0,\ 0,\ 0,\ 1)
\end{aligned}
$$

$$
\gamma_{ij} = \begin{pmatrix} a(1-k\cos\tau) & ak\sin\tau, & 0 & 0 \\ ak\sin\tau & a(1+k\cos\tau) & 0 & 0 \\ 0 & 0 & a(1+2k^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

where γ and g are connected by the relation

$$\gamma_{ij} = e^{\mu}_{i}\, g_{\mu\nu}\, e^{\nu}_{j}$$

and a, k are constants with $a<0$, $|k|<\frac{1}{2}$,

$$\lambda = -\frac{1}{4a(1-k^2)} \quad , \quad \frac{4\pi\rho}{\Lambda} = 1-(2k)^2$$

$$\tau = t\sqrt{-\frac{2(1-2k^2)}{a(1+2k^2)}}$$

Putting, $k=0$

we get Einstein's model.

Another generalisation is given by Raval (6) and the solution is

$$ds^2 = a^2\left[(dx^0+e^{x^1}dx^2)^2-(dx^1)^2-(dx^3)^2+2\left(\frac{\beta}{1-\beta}\right)^{1/2}dx^0\,dx^1\right]$$

where a, β are constants with $0 \leq \beta < \frac{1}{2}$

$$8\pi\rho = \frac{1-\beta}{1-2\beta}\,\frac{1}{a^2} \qquad \Lambda = -4\pi\rho$$

$\beta = 0$ gives the Godel's universe.

Heckmann and Schuking's anisotropic models

Bianchi has classified homogeneous 3-dimensional spaces of signature +3 or -3 into 9 different classes*. These spaces allow for a simply transitive three parameter groups of motion. There are nine different types of constants of the structure of this group and the Bianchi classes correspond to these types. Three special solutions are given by Heckmann and Schuking.

(A) Model with Shear but no rotation:-

$$ds^2 = dt^2 - (R_1(t)\,dx^1)^2 - (R_2(t)\,dx^2)^2 - (R_3(t)\,dx^3)^2$$

where R_1, R_2 and R_3 are different functions of t.

The solution contains a parameter a which when put equal to zero leads to Einstein-deSitter model.

E) $$ds^2 = dt^2 - R^2\left[(dx^1)^2 + S^2 e^{2x^1}(dx^2)^2 + S^2 e^{2x^1}(dx^3)^2\right]$$

$$\dot{R}^2 = 1 + \frac{\Lambda}{3}R^2 + \frac{2MG}{R} + \frac{a^2}{3R^4}$$

If $a = 0$ we get isotropic models.

+ cf: E. Cartan: Lecon Sur la geometrie des espaces de Riemann. Paris 1951.

C) This is a model with shear as well as notation

$$ds^2 = dt^2 + 2e^{x^1} dt\, dx^2 - c_{11}(dx^1)^2 - 2c_{12} e^{x^1} dx^1 dx^2 + \alpha c_{11} e^{2x^1}(dx^2)^2 - S^2 (dx^3)^2$$

$$w = \frac{1}{\sqrt{2}\, R} \qquad R^2 = c_{11} - \alpha (c_{11})^2 - (c_{12})^2$$

$$\rho = \frac{\alpha^2}{RS}$$

All models correspond to incoherent matter.

The differential equations for c_{11}, c_{12} and S could not be solved explicitly and in such an investigation approximate integration is hardly of any use.

Solutions for coherent matter

If we forgo the requirement of incoherent matter, but introduce viscous fluid, it is possible to get a mathematically tractable set of equations.

A solution is

$$ds^2 = (dx^0)^2 + 2e^{x^1}(dx^0)(dx^2) + a e^{2x^1}(dx^2)^2 - \alpha a (dx^1)^2 - \frac{1-a}{a}(dx^3)^2$$

where $a = a(x^0)$ satisfies

$$\frac{da}{dx^0} = A(1-a)e^{-1/a} \qquad A = \text{const},$$

This solution represents an expanding Godel's model which ultimately ends up in a physical singularity (7).

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