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# THEORY OF FUNCTIONAL EQUATIONS



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PL.Kannappan*

At the outset I express my sincere thanks to the authorities of Matscience for having given me an opportunity to speak in this Institute. The aim of these lectures to start with was to give a brief survey of the theory of functional equations touching all aspects of the functional equations and giving examples for each and also the functional inequalities. But as the lectures progressed, it was felt that within this short time limit, what was originally planned cannot be accomplished. So, in these lectures more concentration was made on Cauchy's functional equations. Its generalizations, related equations, some trigonometric equations etc. were also considered. Finally some applications and unsolved problems were treated.

## C O N T E N T S



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## 1. INTRODUCTION.

Theory of funstional equations is one of the oldest as well as relatively young topics of mathematical analysis which is growing very rapidly. Oldest in the sense D'Alembert [16] , [17] , [18] , was the first to apply and solve functional equations in the sense of modern terminology in these three papers. In many respects they are typical; they are in connection with the study of vibrations of strings and the equation considered is $f(x+y) + f(x-y) = g(x)\ h(y)$. Functional equations have fascinated many mathematicians. Even though such eminent mathematicians like Abel, Cauchy, Darboux, Euler, Gauss, Hilbert and Weierstrass among others contributed to the growth and development of this branch, no systematic presentation of this branch was attempted as late as 1918. Applications of functional equations were found much earlier than any systematic presentation could develop. Hence results found in earlier decades have often been presented anew. Young in the sense that the literature has grown markedly during the past fifty years. Further, an attempt to give a unified theory was first tried by A.R.Schweitzer [78] in 1918. Monographs on functional equations have been written by Aczel - Golab [13] , M.Ghermanescu [34] , J.Anastassiadis [21] and M.Kuczma [57] (who is also preparing a monograph on functional equations in a single variable). An excellent first systematic presentation of this subject ever written is by an expert in this field, Hungarian methematician J. Aczel [5] in 1961. This book also gives a survey of the theory of functional equations and contains a good collection of references at the end (more than 100 pages, from 1747 to the present). In his numerous papers as in his book, he treats the whole class of functional equations, gives general method of solving them and criteria of the existence and uniqueness of solutions. He also indicates many new applications of functional equations. A new edition (English) almost twice its original size, containing the many new contributions since 1960 to the present day has come out [6]. After this publication, we hope (like the author), the growth of this field will be accelerated, more people will take up this study and new applications will be found.

In studying Mathematics and its applications to other branches, the type of equations (algebraic) one first comes across are $ax + b = c$, $ax^2 + bx + c = 0$ etc. or the system of equations $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $(i=1,\ldots,n)$. The problem in all these cases, is to determine particular values of a known function or functions. Only in calculus, for the first time, one encounters the question of determining an unknown function. Functional equations generally deal with this.

The large number of papers appearing in various journals, since 1747, is an index of the interest, the mathematicians and others, have for this field. The first significant by reducing it to the partial differential equation

$$(5) \qquad x\, f_x(x,y) + y\, f_y(x,y) = k\, f(x,y)$$

as $f(x,y) = x^k \ \varphi\,({}^{y}/x)$.

The general continuous solutions of the equation (1.1) known as <u>Cosine Equation</u> or <u>D'Alembert Equation</u> or <u>Poisson Equation</u> were found by Cauchy [25] . Equation (1.1) was solved by Andrade [22] by using the technique of integration and differentiation and reducing it to the form $f''(x) = c\, f(x)$. Equation (1.1) in abstract spaces (Banch, Hilbert, Banach algebra, groups etc.) was also treated in considerable detail. (1.1) is one of the equations extensively studied among others by Aczel [7] , T.M.Flett [31] , D.V.Ionescu [46] , Kaczmarz [49] , Kannappan [50],[51] , [52] , S.Kurepa [62] , [63] , [64] , [65] , G.Maltese [72] , Van der Lyn [82] , L.Vietoris [84] , Wilson [93] , [94] and F.Vajzonic.

Under the hypothesis of continuity, Cauchy [25] , solved the following four equations, widely known in general as <u>Cauchy equations</u>

$$(6) \qquad f(x+y) = f(x) + f(y),$$

$$(7) \qquad f(x+y) = f(x)\ f(y),$$

$$(8) \qquad f(xy) = f(x) + f(y),$$

$$(9) \qquad f(xy) = f(x)\ f(y).$$

(1.6) finds applications almost in every branch of mathematics. (1.6) appears in the problems of the measurement of areas, in projective geometry, in mechanics, in the problem of the parallelogram of forces, in the theory of probability, in the non-Euclidean geometry etc. Cauchy's equations are used in mathematics of finances, in the probability theory and in many other topics. Equation (1.6) is one of the equations which has been extensively studied and was solved among numerous others by Aczel, Alexievicz and Orlicz, Banach, Darboux, Frechet, Gauss, Hamel, Kuczma, S.Kurepa, A.Kuwagaki, Legendre, Satz, Sirpenski and Vincze, A. Ostrowski, H.Kestelman, I.Halperin, P.Erdöss, F.B.Jones etc. under various hypothesis of the function, domain and range. We will deal with them in detail later. The existence of discontinuous solutions of (1.6) was proved by Hamel, using axiom of choice. In case where the domain and range of $f$ are abstract sets (groups etc.) (1.6) and (1.8) play an important role in algebra as the equations of homomorphism, endomorphism, isomorphism etc.

One of the striking features of functional equations is the fact that, unlike differential equations, a single equation can determine more than one function. The generalizations of Cauchy's functional equations, known as Pexider equations

(10) $$f(x+y) = h(x) + g(y)$$

(11) $$f(x+y) = h(x)\, g(y)$$

(12) $$f(xy) = h(x) + g(y)$$

(13) $$f(xy) = h(x)\, g(y)$$

is one such example and they were solved in an elementary way by Pexider [75], for the three unknown functions f, g and h. Generalization of Pexider equations was considered by Aczel [8]. These equations and some generalizations will be considered later.

The Jensen equation [47],

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}, \tag{14}$$

has many properties analogous to those of the equation (1.6). Aczel and Fenyo [11], have applied (1.14) to define the centre of gravity of field of forces. Generalizations of the Cauchy equations and Jensen equations of the type

$$f(x+y) = F\left[f(x), f(y)\right] \tag{15}$$

(Known as addition formula)

$$f\left(\frac{x+y}{2}\right) = F\left[f(x), f(y)\right] \tag{16}$$

and

$$\varphi\left[f(x,y)\right] = F\left[\varphi(x), \varphi(y)\right] \tag{17}$$

were treated by Aczel, Dunford-Hille, Alt, Kuwagaki, Montel, Monroe, I., etc.

Abel published four important papers on this subject [1], [2], [3], [4].

The first gives a general method of solving functional equations by differentiation. The second deals with the system of functional equations

$$(18) \qquad F(x,F(y,z)) = F(z,F(x,y)) = F(y,F(z,x)) = F(x,F(z,y)) = F(z,F(y,x)) = F(y,F(x,z)),$$

for the function $F$. In the third he solves the equation

$$(19) \qquad g(x)+g(y) = h\left[xf(y)+yf(x)\right] ,$$

for the three unknown functions $f,g$ and $h$. The technique employed in these three papers is to reduce the functional equations to differential equations and then solve them. In the last paper Cauchy's equations generalized for complex variable were solved.

Development of the theory of functional equations is closely related to its applications to various branches, namely, mechanics, the theory of continuous groups, the theory of geometrical aspects, vector analysis, Euclidean and non-Euclidean geometry, the theory of probability, characterization of means, characterization of various functions such as Euler's function, exponential and logarithmic functions, trigonometric functions, polynomials, characterization of determinants etc. Characterization of determinants has led to the study of matrix equations.

Matrix equations find applications in invariant theory, theory of geometric objects etc. Systems of functional equations were used by Stokes [81], to determine the intensities of reflected and absorbed light. Weierstrass [95], has used the equation $F(f(x),f(y),f(x+y)) = 0$ for the development of elliptic functions. Functional equations with several unknown functions were considered among others by Sinzov, Stephanos, Suto, Schweitzer and Wilson. Few international conferences on functional equations were held since 1961, at Blatonvilagos, Sarospatak, Oberwolfach and Waterloo.

Of course one may ask what is the reason of this interest taken in functional equations by the mathematicians of all the world. This may be connected with the fact that in many branches of mathematics analytic methods are already exhausted to some extent. A use of elementary methods often allows one to obtain much deeper and more general results than it was possible with a use of classical methods of mathematical analysis. On the other hand, more and more problems of physics and technics require making weaker assumptions regarding the occurring functions.

There is no general method of solving functional equations. This itself could have been one of the reasons that might not have attracted many persons to this field. It used to be said that every functional equation requires its own mode of attack. In recent years the situation has improved. Gradually more general results are available, the classically known results are shown to be valid under less severe restrictions, existence proofs applicable to a wide range of equations are being found etc.

The works of Aczel in recent times had considerably advanced the discipline of this field. The techniques employed are varied, but special mention can be made of the method of specialization of variables, iteration and inverse iteration, method of determinants; reducing functional equations to differential equations, reducing functional equations to integral equations etc.

The most important range of problems in this field is however the developing of a qualitative theory of functional equations - existence, uniqueness, extension, characterization etc.

## 2. Definition and classification

First we shall start with the following questions. What is a functional equation ? How to classify them ? The answer to these seemingly simple questions is not easy. It is not answered in a satisfactory manner and finding a suitable answer is one of the problems in this field. But the present day view eliminates wide class of equations: differential, integral, integro-differential, operator equations etc. However what remains is so vast that it needs further compartmentalization and specialization. Here we give the definition found in [6], [57]. As the definition of functional equation involves the notion of a term, we begin with the definition of a term.

<u>Definition of a term.</u> 1) The independent variables $x_1, x_2, \ldots x_n$ are terms. 2) Given that $y_1, y_2, \ldots y_n$ are terms and a function $f$ of n-variables, then $f(y_1, y_2, \ldots y_n)$ is also a term 3) There are no other terms.

A given term thus contains a definite number of variables and a definite number of functions.

<u>Definition of a functional equation.</u> A functional equation is an equation $f=g$ between two terms $f$ and $g$, which contain $n$ independent variables $x_1, x_2, \ldots x_n$ and $p(\geq 1)$ unknown functions $f_1, f_2, \ldots f_p$ of $i_1, i_2, \ldots i_p$ variables respectively, as well as a finite number of known functions.

<u>Definition of a system of functional equations.</u>

A system of functional equations consists $n(\geq 2)$ functional equations which contain $m(\geq 1)$ unknown functions altogether.

The functional equations or systems must be identically satisfied for certain values of the variables occurring in them in a certain set of any sort, i.e. in a domain which may be real or complex numbers, a vector space or an n-dimensional space (real or complex) or a set of matrices or any abstract algebraic system. The range of the unknown functions may be real or complex numbers, vectors, matrices, conjugate space etc.

The number and behavior of solutions of a functional equation may depend very largely on the domain and a function class, known as the class of admissible functions, which are defined by the analytic properties like analyticity, measurability, continuity, differentiability, integrability etc. It is one of the important differences between differential and integral equations. For example, (1) the only solution of

$$f(x) + f(y) = f(xy - \sqrt{1-x^2}\sqrt{1-y^2}) \quad \text{for all } x,y \text{ in } [-1,1]$$

is $f(x) \equiv 0$, where as the general measurable solution in suitably restricted sets is $f(x) = k \arccos x$, (2) The only solution of (1.8) in $]-\infty,\infty[$ is $f(x) \equiv 0$, whereas in $R - \{0\}$, the continuous solution of (1.8) is $f(x) = c \log |x|$ (we will see this later). Behavior of solutions depends on the function class also. For example, (1.8) has also non-measurable solutions in $R - \{0\}$. This is one of the characteristic features of functional equations. Here we make note of the observation made by Abel that one functional equation can contain several unknown functions in such a way that all the unknown functions can be determined from it.

Classification. Rough but useful classification into four types: functional equations for functions of one or several variables for one function or several functions was made by J. Aczel [6]. Here we follow the monograph of Kuczma [57].

Definition. A functional equation in which all the unknown functions are functions of one variable is called an ordinary functional equation. A functional equation in which at least one of the unknown functions is a function of several variable is called a partial functional equation.

The classification of ordinary functional equations is based on the concept of rank, order and implication index kno as type.

Definition of rank. The number of independent variables occur ing in a functional equation is called the rank of the equatio.

Definition of Order. The smallest number of additional equatic which are necessary in order to reduce a functional equation to a form where under the sign of the unknown function, only singl variable occur, is called the order of the equation.

Definition of implication index. Suppose that a functional equa tion has been reduced to a system of equations as described above. The number of additional equations containing the unkno function is called the implication index of the equation.

These definitions concern only the ordinary functional equations. The above definition of order has some shortcomings. It may be due to the fact that, at times, it is hard to tell whether the number of additional equations is really the smallest. We shall illustrate this and the definitions by the following examples.

(a) The equation $f(x+y) = f(x) + f(y) + f(x)f(y)$ may be written as $f(z) = f(x) + f(y) + f(x)\ f(y)$ where $z = x+y$.

(b) The equation $f(x+y) + f(x-y) = 2\ f(x) + 2\ f(y)$ can be written as $f(z) + f(w) = 2\ f(x) + 2\ f(y)$ where $z = x+y$, $w = x-y$.

(c) The Babbage equation $f^n(x) = x$ (power denotes iteration) can be written as

$$f(z_{n-1}) = x$$

where $z_{n-1} = f(z_{n-2})$, $z_{n-2} = f(z_{n-3}), \ldots, z_1 = f(x)$.

(d) The equation $f(x+f(x)) = f(x)g(x) + h(x)$ may be written as

$$f(z) = f(x)g(x) + h(x)$$

where $z = x + f(x)$.

(e) The functional equation

$$F\left[x_1,x_2,\dots x_n,g(x_1),\dots,g(x_n),\ g\left\{f\left(x_1,\dots,x_n,g(x_1),\dots,g(x_n)\right)\right\}\right]=o$$

with $g$ unknown, may be written as

$$F\left[x_1,x_2,\dots,x_n,g(x_1),\dots,g(x_n),\ g(y)\right]=o$$

where $y=f\left(x_1,\dots,x_n,g(x_1),\dots,g(x_n)\right)$.

The ranks of (a), (b), (c), (d) and (e) are 2,2,1,1 and n. The orders of (a), (b), (c), (d) and (e) are 1,2,(n-1), 1 and 1. The implication indicies of (a), (b), (c), (d) and (e) are 0,0, (n-1), 1 and 1. One can unify the rank $n$, the order $o$ and implication index $i$ of a functional equation into one symbol $[n,o,i]$ called the type of the equation. The types of (a), (b), (c), (d) and (e) are $[2,1,0]$ $[2,2,0]$, $[1,(n-1),(n-1)]$, $[1,1,1]$ and $[n,1,1]$.

(f) Now consider the following equation:

$$f(x+y)=f(x)+y.$$

This has apparently order 1 : $f(z)=f(x)+y$ where $z=x+y$. But in fact it is of order zero, for it can be written as $f(z)=f(x)+z-x$, where $x$ and $z$ are not connected by any relation.

Some results regarding the reduction of the rank have been obtained by Aczel and Kiesewelter [15] . It is evident from their results that rank 2 plays a particular role, in the sense that equations of higher rank usually can be replaced by equivalent equations of rank 2 [for example, the families of solutions of (1.6) $f(x+y) = f(x) + f(y)$ and $f(x_1 + .. + x_n) = f(x_1) + .. + f(x_n)$ are identical] , while similar replacing an equation of rank 2 by an equation of rank 1, is in general not possible. In case of rank $\geq 2$, the most frequently used method is that of a specialization of variables. For example, putting $x = 0$ in the above example (f), we obtain $f(x) = x + c$. In most cases, however, the solution cannot be obtained in such a simple way and the process of specialization must be repeated several times in a rather ingenious manner. The method of specialization of variables cannot be used in the case of equation of rank 1. The reduction of the order has been investigated by Kuczma [58].

All these attempts do not prove satisfactory. Two functional equations with the same characteristics may differ by the structure of their solutions. The Cauchy equation (1.6) and the Jensen equation (1.14), both have the same type [2,1,0] . Nevertheless the former has a one parameter family of continuous solutions $f(x) = cx$, while the latter possesses a two parameter family of continuous solutions $f(x) = cx+d$.

3. The Cauchy Equations.

One of the most important and very widely studied functional equations, is the Cauchy equation

$$f(x+y) = f(x) + f(y). \tag{1.6}$$

This equation has application in many branches of mathematics. Cauchy has found the general continuous solutions of (1.6) as given in theorem (3.1). The same equation (1.6) was treated by Legendre [67] and Gauss [33] before Cauchy.

THEOREM 3.1. Let $f$ be a real valued function of real variables satisfying (1.6). Then if $f$ is continuous, $f$ has the form

$$f(x) = cx, \quad \text{for all real } x, \tag{1}$$

where $c$ is a real constant. Further, if $f$ is defined only for positive or non-negative $x, y$, then also $f$ has the form (3.1) for all positive or non-negative $x$, provided $f$ is continuous.

Proof. First setting $x = o$, $y = o$ in (1.6), we obtain

$$f(o) = o. \tag{2}$$

Now, replacing $y$ by $-x$ in (1.6) and using (3.2), we get

$$f(-x) = -f(x). \tag{3}$$

Now, we will show that $f$ is rational homogeneous, i.e. if $x$ is any real number and $r$ is any rational, then

(4) $$f(rx) = rf(x).$$

From (1.6), it follows by finite induction, that

$$f(x_1+x_2+ \ldots +x_n) = f(x_1) + \ldots + f(x_n).$$

Letting $x_k = x$ $(k = 1,2,\ldots,n)$ in the above, we have

(5) $$f(nx) = n\,f(x).$$

That is, (3.4) is true for any positive integer $n$. Let $n$ be any negative integer. Then using (3.3) and (3.5), we get

$$f(nx) = - f(-nx)$$
$$= -(-n)f(x)$$
(6) $$= nf(x).$$

Hence (3.4) is true for all integers $n$. Let $r$ be any rational and $r = \frac{m}{n}$, i.e. $m = nr$. Then from (3.5) and (3.6), we get

$$f(nrx) = f(mx), \text{ x real}$$
that is $$nf(rx) = mf(x)$$
hence $$f(rx) = \frac{m}{n} f(x)$$
$$= rf(x),$$ so (3.4) is valid for all rational $r$ and real $x$.

Thus taking $f(1) = c$ and $x = 1$ in (3.3), we see that

(7) $f(r) = cr$, for all rational $r$.

So far only the condition that $f$ satisfies (1.6) is used. Now using the hypothesis that $f$ is continuous, it is easy to see from (3.7), that $f(x) = cx$, for all real $x$, $c$ being an arbitrary real constant. It is evident from the above arguments that (3.1) is valid for all non-negative or positive $x$.

There are as many conditions known for the solution of (1.6) to be (3.1) and thus continuous. The hypothesis of continuity of $f$ in (1.6) can be considerably weakened, to obtain the same conclusion. In this connection, first we consider the following results due to Darboux [26].

THEOREM (3.2). If $f$ satisfies (1.6) for all real $x$ and $y$, then the following conditions are equivalent:

(i) $f$ is continuous at a point $x_0$.

(ii) $f$ is non-negative for sufficiently small positive $x$'s

(iii) $f$ is bounded on an arbitrarily small interval.

(iv) $f(x) = cx$, for all real $x$.

Proof. First (i) $\Rightarrow$ (iv). Given that $f$ is continuous at $x_0$. That is,

$$\lim_{t \to x_0} f(t) = f(x_0).$$

Then for every $x$, we have

$$\lim_{t \to x} f(t) = \lim_{t - x + x_o \to x_o} f(t-x+x_o+x-x_o)$$

$$= \lim_{t-x+x_o \to x_o} f(t-x+x_o) + f(x-x_o)$$

$$= f(x_o) + f(x-x_o) = f(x).$$

Hence $f$ is continuous everywhere and so (iv) holds.

Second. (ii) $\Rightarrow$ (iv).

From (1.6) and the hypothesis that $f(x) \geq o$, for sufficiently small $x > o$, it follows that

$$f(x+y) = f(x) + f(y) \geqslant f(y),$$

so that $f$ is monotonically increasing. Choose $\{r_n\}$ and $\{R_n\}$ as increasing and decreasing sequences of rationals respectively, both having the same limit $x$. Then for every $n$, we have $r_n < x < R_n$. Now using (3.7), we obtain

$cr_n = f(r_n) \leq f(x) \leq f(R_n) = cR_n$, from which we can conclude that $f(x) = cx$, for all real $x$. Hence (iv) is true.

Third. (iii) $\Rightarrow$ (iv).

Let $f$ be bounded on $(a,b)$. Let us suppose that

(8) $$\varphi(x) = f(x) - xf(1), \text{ for all real } x.$$

Then by virtue of (1.6) and (3.8) $\varphi$ also satisfies (1.6) for all real $x$ and $y$, and is bounded on $(a,b)$ and

$$\varphi(r) = r\,\varphi(1), \text{ for all rational } r. \text{ But } \varphi(1) = o.$$

Hence for any rational $r$,

(9) $$\varphi(r) = o.$$

Thus we have, $\varphi(x+r) = \varphi(x)$, $x$ real and $r$ rational. Since for any real $x$, we can find a rational $r$ such that $x + r$ is in $(a,b)$, we conclude from (3.8) and (3.9) that $\varphi$ is bounded everywhere. Now we will show that $\varphi \equiv o$. If not, suppose there is an $x_o$ such that $\varphi(x_o) = k \neq o$. Then it is true by (3.6) that $\varphi(nx_o) = n\,\varphi(x_o) = nk$. So, for arbitrarily large $n$, $\varphi$ can take arbitrarily large values, contradicting the boundedness of $\varphi$. Thus, $\varphi(x) \equiv o$. This enables us to deduce that $f(x) = cx$. Hence (iv) holds. Other cases can be easily deduced from the above. The proof of this theorem is thus complete.

<u>THEOREM (3.3).</u> Every measurable (in the Lebesgue sense) function $f$ satisfying (1.6) for all real $x$ and $y$ is continuous (so after Cauchy is of the form $cx$). Number of proofs of this theorem are known. We give below some of them.

<u>Proof 1.</u> (Due to Sierpinski [70] ). Here he uses the fact that if $P$ and $Q$ are 2 linear measurable sets of positive measure, then there exist points $p \in P$ and $q \in Q$ such that $p-z$ is a rational. Let us define

$$(8) \qquad \varphi(x) = f(x) - xf(1), \quad \text{for all real } x.$$

Then we know that $\varphi(r) = 0$, $r$ any rational and

$$\varphi(x+r) = \varphi(x), \quad x \text{ real}, r, \text{ rational}.$$

From the definition of $\varphi$, it follows that $\varphi$ is also measurable (since $f$ is). Now we will prove that, for all real $x$,

$$\varphi(x) = o.$$

Suppose, in fact, there is a real 'a' such that

$$(10) \qquad \varphi(a) \neq o.$$

Let $$E_1 = \{x \in R : \varphi(x) > o\}$$

and $$E_2 = \{x \in R : \varphi(x) < o\}.$$

Since $\varphi(-x) = -\varphi(x)$, $E_1$ and $E_2$ are symmetric to each other.

Further, $E_1$ and $E_2$ are measurable (since the function $\varphi$ is) and therefore of the same measure. Suppose the measure of these sets is positive. Then there exist $x_1 \in E_1$ and $x_2 \in E_2$, such that $x_1 - x_2 = r$, $r$ rational. Then we have $\varphi(x_1) = \varphi(x_2+r) = \varphi(x_2)$, which is impossible, since $x_1 \in E_1$ and $x_2 \in E_2$. The sets $E_1$ and $E_2$, are therefore of measure zero. Let $E = E_1 \cup E_2$. So, measure of $E$ is also zero and $E$ is the set of all points $x$, for which $\varphi(x) \neq o$. Then the set $G = \{x \in R\colon \varphi(x) = o\}$ is of positive measure.

Let $H = \{x \in R\colon \varphi(x+a) = o\}$. Then $H$ has positive measure (for $H \supset G-a$, the translate of $G$).

Let $x \in H$. Then $\varphi(x+a) = o$. Hence

$\varphi(x) + \varphi(a) = o$. Since by (3.10) $\varphi(a) \neq o$, we have $\varphi(x) \neq o$. Hence $H \subset E$, that is, a set of positive measure is contained in a set of null measure, which is impossible. Hence our assumption of existence of 'a' such that $\varphi(a) \neq o$ is false. Therefore, $\varphi(x) = o$ for all real $x$ and $f(x) = xf(1)$.

From this, one can conclude that every discontinuous solution of (1.6) is non-measurable.

Proof 2. (due to Banach [23] ). Let $x_o$ be any real number, $\varepsilon$ any positive number and $(a,b)$ an arbitrary interval. By the theorem of Lusin, there exists for every measurable function $f$ and for every $\sigma > o$ (in particular $\sigma = \frac{b-a}{3}$ ), a continuous function $F$ (for all reals $x$) such that

$$(11) \qquad f(x) = F(x)$$

is true for all $x \in (a,b)$, except perhaps for $x$'s forming a set $E$ of measure $< \sigma$. The function $F$ being continuous, for every $\varepsilon > o$, there is a $\delta(< \sigma)$ such that, for all $x \in (a,b)$

$$(12) \qquad |F(x+h) - F(x)| < \varepsilon$$

whenever $|h| < \delta$. Let $h$ be such a real number satisfying $|h| < \delta$. (3.11) being true for all $x \in (a,b)$ except over a set $E$ of measure $< \sigma$, we can conclude that

$$(13) \qquad f(x+h) = F(x+h)$$

is satisfied for all $x \in (a,b)$ except over a set $G$ of measure $< \sigma + |h| < \sigma + \delta$. The set of $x \in (a,b)$ for which either (3.11) or (3.13) is not satisfied, is therefore of measure $\leq m(E \cup G) < 2\sigma + \delta < 3\sigma < b-a$. Hence there is a point $x \in (a,b)$ (dependent on $h$) for which (3.11), (3.12) and (3.13) are valid. So we have

$$|f(x+h) - f(x)| < \varepsilon \quad . \tag{14}$$

Using (1.6) and (3.14), we have $f(x+h) = f(x) + f(h)$ and $f(x_o+h) = f(x_o) + f(h)$ and so $f(x+h) - f(x) = f(x_o+h) - f(x_o)$ and consequently for any real $x_o$,

$$|f(x_o+h) - f(x_o)| < \varepsilon$$

Hence $f$ is continuous.

<u>Proof 3.</u> (due to Alexawicz and Orlicz [20] ). Let $x \neq o$. Suppose

$$\varphi(t) = f(t) - \frac{f(x)}{x}\, t$$

and

$$\psi(t) = \frac{1}{1 + |\varphi(t)|} \quad .$$

It is evident that $\varphi(t+x) = \varphi(t) + \varphi(x) = \varphi(t)$, since $\varphi(x) = o$. Hence $\varphi$ and so $\psi$ are of period $x$. So,

$$\int_o^x \frac{dt}{1 + |\varphi(t)|} = \int_o^x \psi(t)\,dt$$

$$= \int_o^x \psi(2t)\,dt$$

$$= \int_o^x \frac{dt}{1 + 2|\varphi(t)|} \quad .$$

so

$$\int_0^x \frac{|\varphi(t)|\,dt}{(1+|\varphi(t)|)(1+2|\varphi(t)|)} = 0.$$

It follows that $\varphi(t) = o$ almost everywhere. That is to say that, $f(t) = \frac{f(x)}{x} t$ for almost all $t$, in particular for $x = 1$, $f(t) = f(1)t$ for almost all $t$. Hence for every $x \neq 0$, there is a $t_o \neq o$ such that

$$f(t_o) = \frac{f(x)}{x} t_o$$

and $$f(t_o) = f(1)\, t_o.$$

Therefore $f(x) = f(1)\,x$ for all $x \neq o$. Evidently this equality is also true for $x = o$.

THEOREM 3.3. For Cauchy's equation (1.6), continuity and measurability are equivalent.

Proof. Let $f$ be measurable and satisfy (1.6). Then $f$ is bounded on every bounded interval. Indeed, suppose there is an interval $I = (-A, A)$ on which $f$ is not bounded. Choose a sequence $y_k \in I$ such that $f(y_k) > 2n + f(y_{k-1})$, for fixed $n$. Let $E_m = \{x \in I : |f(x)| \leq m\}$, $m$ any integer. Then $E_1 \subset E_2 \subset \cdots$ and $\bigcup E_m = I$. Therefore, there is an $n$ such that $\mu(E_n)$ is positive ( $\mu$ is the Lebesgue measure). Define

$$F_k = y_k + E_n = \{z : z = y_k + x : x \in E_n\}$$

$$= \{z : |z - y_k| < A, |f(z-y_k)| \leq n\}.$$

Also define

$$G_k = \{z : |z| < 2A, f(y_k) - n \leq f(z) \leq n+f(y_k)\}.$$

Then we see that $F_k \subset G_k$. Now choose an integer $j > k$. Then we have $-f(y_j) + 2n + f(y_k) < o$. If $z_j \in G_j$, then $f(y_j) - n \leq f(z_j)$ by the definition of $G_j$. Adding these two inequalities we have $n + f(y_k) < f(z_j)$. So $z_j \notin G_k$, $k \neq j$, and $G_j \cap G_k = \phi$. Hence $F_j \cap F_k = \phi$ for $j \neq k$. Therefore, $\mu(\bigcup_1^\infty F_k) = \sum_1^\infty \mu(F_k) \leq 4A$, $F_k \subset G_k \subset (-2A, 2A)$. From this we conclude that $\mu(F_k) = o$ for every $k$. So, $\mu(E_n) = \mu(F_k)$ (for every $k$) $= o$, which is a contradiction. Therefore, $f$ is bounded on every finite interval and hence is continuous. Thus for Cauchy's equation, continuity and measurability are equivalent.

<u>THEOREM 3.4</u>. Suppose $f$ is a real additive function i.e. $f$ satisfies (1.6) and is bounded on a set $E$ of positive measure. Then (3.1) holds, i.e. $f$ is continuous.

<u>Proof</u>. (Due to Kestelman [53] ). By a theorem of Steinhaus [80] , there is a positive number $\delta$, such that, every real number $\theta$, satisfying $|\theta| < \delta$, may be expressed as $x-y$ for suitable $x, y$ in $E$. If $M$ is the upper bound of $f$ on $E$, then by using (1.6), we obtain

$$
\begin{aligned}
|f(\theta)| &= |f(x-y)| \\
&= |f(x) - f(y)| \\
&\leq 2M. \qquad (15)
\end{aligned}
$$

Hence, using (1.6) and (3.15), we get, for $|\sigma| < \frac{\delta}{n}$ , that

$$|f(\sigma)| \leq \frac{2M}{n}, \quad (n = 1,2,\ldots). \qquad (16)$$

Let $\alpha$ be a real number. if $r_n$ is a rational such that $|\alpha - r_n| < \frac{\delta}{n}$ , using (1.6) and (3.16), we have

$$
\begin{aligned}
|f(\alpha) - \alpha f(1)| &= |f(\alpha - r_n) + (r_n - \alpha) f(1)| \\
&\leq \frac{2M}{n} + \frac{|f(1)|\delta}{n}, \quad (n = 1,2,\ldots)
\end{aligned}
$$

which means that $f(\alpha) = \alpha f(1)$, which is wanted to be proved.

<u>COROLLARY 3.5</u>. Every discontinuous solution of (1.6) is unbounded on every set of positive interior measure.

<u>COROLLARY 3.6</u>. If $f$ satisfies (1.6) and is measurable in some set of positive measure, then $f$ is continuous, because,

the set of $x$ for which $|f(x)| < N$ has positive measure, if $N$ is large enough.

THEOREM 3.7. Let $f$ be real additive and be bounded from above on some interval $[a,b]$. Then $f$ has the form (3.1).

Proof. [41]. We first show that $f$ is bounded in a neighborhood of the origin. If this were not so, there would exist a sequence $x_n \to o+$ such that $|f(x_n)| \to \infty$. Hence $|f(a+x_n)| = |f(a) + f(x_n)| \to \infty$. Since $f$ is bounded above in $[a,b]$, this means $f(x_n) \to -\infty$ and hence $f(b-x_n) = f(b) - f(x_n) \to \infty$ which is impossible. [So, $f$ is bounded in a neighborhood of the origin which certainly is of positive measure]. So, from theorem (3.4) $f$ has the form (3.1). [We give another proof here]. Now claim that $f(x) \to o$ as $x \to o+$. If the contrary were true, then there would exist a sequence $x_n \to o+$ such that $f(x_n) \geq \varepsilon > o$ (or $f(x_n) \leq -\varepsilon < o$) for some $\varepsilon$. But then $f(\sum_{i=k}^{k+n} x_i) \geq n\varepsilon$ and

$$\lim_{k\to\infty} \sum_{i=k}^{k+n} x_i = o$$

for arbitrary $n$, which again is impossible. So, $f$ is right continuous at the origin. Not only $f$ is right continuous at the origin, but because of the additivity, it is clearly right continuous for all $x \geq o$. From (3.4) we have $f(r) = r\,f(1)$ for any rational number $r$. Making use of the right continuity, we finally obtain $f(x) = f(1)x$, for all $x \geq o$. Hence $f$ has the form (3.1).

<u>THEOREM 3.8</u>. All solutions of the equation (1.6) which are bounded from below (or from above) on an interval are of the form $f(x)=cx$, $c$, a constant.

<u>Proof</u>. Let $f$ satisfy (1.6) and be bounded below in $[a,b]$, that is, there is an $M$ such that $f(x) \geq M$, for all $x \in [a,b]$. Now, first we will show that $f$ is bounded below in $[o,b-a]$.

If $x \in [o,b-a]$, then $(x+a) \in [a,b]$. Hence

$$f(x) = f(x+a) - f(a)$$

(17) $\geq M - f(a)$, that is $f$ is bounded below in $[o,b-a]$.

Consider the function

(18) $$g(x) = f(x) - \frac{f(d)}{d}\,x, \quad \text{where } d = b-a \neq o,$$

$$= f(x) - cx, \text{ where } cd = f(d).$$

Evidently $g$ also satisfies (1.6). It is enough to show that $g(x) = o$ for all real $x$. For all $x \in [o,d]$, $c\,x \leq \max(o,cd) = e$.

From (3.17), (3.18) and the above we see that

$$g(x) \geq M-f(a) -e, \quad \text{for all } x \in [o,d] .$$

That is, $g$ is bounded below in $[o,d]$, say

(19) $$g(x) \geq N, \ x \in [o,d] .$$

From (3.18) we have $g(d) = o$. Thus $g(x+d) = g(x)$, for all real $x$. That is, $g$ is periodic, with period $d$. Since $g$ is bounded in $[o,d]$ and $g$ is periodic with period $d$, we conclude that $g$ is bounded below everywhere by $N$.

Suppose there is an $x_o$ such that $g(x_o) = o$.

If $g(x_o) > o$, then by (3.3), $g(-x_o) < o$.

By (3.6), we can find an integer $n$ sufficiently large such that $g(-nx_o) < N$. If $g(x_o) < o$, as before by (3.6) we can find an integer $n$ sufficiently large such that $g(nx_o) < N$. In either case we get a contradiction to the fact that $g$ is bounded below everywhere by $N$ (3.19). Hence $g(x) = o$, for all real $x$.

Then by (3.18), the result follows.

<u>Existence of a discontinuous solution of (1.6)</u>

Hamel [39], has proved by using the axion of choice that (1.6) has a discontinuous solution.

<u>THEOREM 3.9</u> There exists an $f$ satisfying (1.6) but is <u>not</u> of the form $f(x) = cx$.

<u>Proof</u>. We shall need the idea of a Hamel basis. A set $H$ with the following properties is called a Hamel basis:

(i) every real number $x$ can be represented as a finite linear combination

$$x = r_1 x_1 + r_2 x_2 + - + r_n x_n ,$$

with $x_i \in H$ and $r_i$ rationals for $i = 1,2,\ldots n$.

(ii) No proper subset of $H$ has the property described by (i).

Such a basis can be shown to exist by making use of transfinite induction [42] , or equivalently by Zorn's lemma etc. Note that $H$ is nondenumerable and the representation for $x$ given by (i) is unique, because of (ii).

Now let $H$ be a Hamel basis and for each $b \in H$, choose $f(b)$ as an arbitrary real number. Then for any real $x$ (of the form in (i) define

$$f(x) = \sum_{i=1}^{n} r_i f(x_i). \tag{20}$$

Then $f$ constructed by (3.20) always satisfies the functional equation (1.6). Indeed, if

$$x = \sum_{i=1}^{n} r_i x_i \quad \text{and} \quad y = \sum_{k=1}^{n} q_k x_k, \ (x_i \in H)$$

(some of the coefficients $r_i$ and $q_k$ may be zero, but we use $n$ terms in both cases), then

$$x + y = \sum_{i=1}^{n} (r_i + q_i)\, x_i .$$

From (3.20), we obtain

$$f(x+y) = \sum_{i=1}^{n} (r_i + q_i) f(x_i)$$

$$= f(x) + f(y).$$

Such a solution $f$ is continuous, if and only if, there is a constant $c$ such that

$$f(x_i) = cx_i \text{ for all } x_i \in H.$$

Now to exhibit an $f$ that is discontinuous. For a particular $x_i \in H$, let $f(x_i) = 1$ and $f(x_j) = o$ for $j \neq i$, $x_j \in H$. If $f$ is to be continuous, then we know that $f(x) = cx$.

Hence $\dfrac{f(x_j)}{f(x_i)} = \dfrac{x_j}{x_i}$ . But the left side is zero for all $x_j \neq x_i$ for the above definition of $f$, while the right side can never be zero (since a basis does not contain the zero element). This contradiction shows that $f$ is not continuous.

<u>Reduction to differential and integral equations</u>.

The method employed here is to reduce functional equations to differential or integral equations and thereby solve the functional equation. Here we illustrate these methods by the example of Cauchy functional equation (1.6).

Reduction to differential equation.

Differentiating (1.6) with respect to $x$, we have

$$f'(x+y) = f'(x).$$

Hence $f'$ is periodic with arbitrary period $y$ and consequently we have $f'(x) = c$, where $c$ is a constant.

Hence $f(x) = cx+d$.

Since $f(o) = o$, we get $f(x) = cx$, that is $f$ is of the form (3.1).

Reduction to integral equation.

Let $f$ satisfy (1.6) and $f$ be integrable. Consider the double integral in the region $x \geq o$, $y \geq o$, $x+y < t$.

Then

$$\int_o^t \int_o^{t-y} f(x+y)\, dxdy = \int_o^t \int_o^{t-y} f(x)\, dxdy + \int_o^t \int_o^{t-y} f(y)\, dxdy.$$

Hence

$$\int_o^t \int_y^t f(v)\, dv\, dy = \int_o^t \int_o^{t-y} f(x)\, dx\, dy + \int_o^t (t-Y) f(y)\, dy.$$

Let $F(t) = \int_o^t (t-y)\, f(y)\, dy$, then $F''(t) = f(t)$ and $F'(o) = o = f(o)$. Now, we have

$$\int_0^t [f'(t) - F'(y)]\,dy = \int_0^t \dot{F}'(t-y)\,dy + F(t).$$

that is,

$$tF'(t) - F(t) = F(t) + \dot{F}(t).$$

Hence, $$\frac{F'(t)}{F(t)} = \frac{3}{t} \quad \text{or} \quad F(t) = k\,t^3.$$

Thus $f(t) = 6\,kt$ and hence is of the form (3.1).

Deduction of differentiability from integrability.

As in the above case, let $f$ satisfy (1.6) and be integrable, say in the interval $[0.1]$ .

Integrating with respect to $y$, we have

$$f(x) = \int_0^1 f(x+y)\,dy - \int_0^1 f(y)\,dy = \int_x^{x+1} f(t)\,dt - c, \text{ where}$$

$$\int_0^1 f(y)\,dy = c.$$

Since the right side is continuous, so is $f$. Since $f$ is continuous, again we have the right side is differentiable and thus $f$ is differentiable. Hence $f$ has the form (3.1). But differentiating the above equation, we have, by using (1.6)

$$\begin{aligned} f'(x) &= f(x+1) - f(x) \\ &= f(1), \end{aligned}$$

Thus $$f(x) = f(1)x.$$

Solution of the Cauchy equation (1.6) for complex values.

Theorem 3.10. Let $f$ be a complex-valued function of the complex variables satisfying (1.6). Then the most general solution is given by

$$f(x) = f_1(x_1) + i\, f_2(x_1) + g_1(x_2) + i\, g_2(x_2)$$

where $x = x_1 + i\, x_2$ and $f_1$, $f_2$, $g_1$ and $g_2$ are solutions of (1.6), $x_1$, $x_2$ reals.

Proof. Let $x = x_1+ix_2$, $y = y_1+iy_2$. Further, let

$$f(x) = F(x_1,x_2) + i\, G(x_1,x_2). \tag{21}$$

Then, it is easy to see that

$$F(x_1+y_1,\ x_2+y_2) = F(x_1,x_2) + F(y_1,y_2) \tag{22}$$

and

$$G(x_1+y_1,\ x_2+y_2) = G(x_1,x_2) + G(y_1,y_2). \tag{23}$$

Set $x_2 = o, y_2 = o$ in (3.22). Then, with

$$f_1(x) = F(x,o),\ x \text{ real, we get} \tag{24}$$

$$f_1(x_1+y_1) = f_1(x_1) + f_1(y_1),$$

that is, $f_1$ satisfies (1.6). Similarly, we obtain by defining

(25) $g_1(x) = F(0,x)$ for x real, that

$g_1(x_2+y_2) = g_1(x_2) + g_1(y_2)$, that is, $g_1$ is also a solution of (1.6).

From (3.22), (3.24) and (3.25), we have

$$F(x_1,x_2) = F(x_1,0) + F(0,x_2), \text{ and}$$

(26) $F(x_1,x_2) = f_1(x_1) + g_1(x_2)$ where $f_1$ and $g_1$ satisfy (1.6).

Similarly, we obtain

(27) $G(x_1,x_2) = f_2(x_1) + g_2(x_2)$ where $g_2$ and $f_2$ satisfy (1.6).

From (3.21), (3.26) and (3.27), we have the desired result

$$f(x) = f_1(x_1) + i\, f_2(x_1) + g_1(x_2) + i\, g_2(x_2).$$

<u>Theorem 3.11</u>. The general continuous complex solution of (1.6) is $f(z) = cz + d\bar{z}$ where c and d are arbitrary complex numbers.

<u>Proof</u>. From theorem (3.10), we see that $f_1$, $f_2$, $g_1$ and $g_2$ are continuous solutions of (1.6). Hence by Theorem (3.1), we have, for $z = x + i\,y$,

$$f_1(x) = c_1 x, \; c_1 \text{ real}$$

$$f_2(x) = c_2 x, \; c_2 \text{ real}$$

$$g_1(x) = c_3 x, \; c_3 \text{ real,}$$

and $$g_2(x) = c_4 x, \; c_4 \text{ real.}$$

Hence

$$f(z) = c_1 x + i\, c_2\, x + c_3\, y + i\, c_4\, y \tag{28}$$

$$= c.z + d\,\bar{z} \text{ where } c = \frac{1}{2}\; (c_1+c_4)+i\;(c_2-c_3)$$

and $d = \frac{1}{2} \quad (c_1-c_4) + i(c_2+c_3)$ .

<u>Remark.</u> The general differentiable solution of (1.6) in the complex case is $f(z) = cz$, where $c$ is any complex number, since $\bar{z}$ is not differentiable.

<u>Solution of Cauchy equation for functions of several variables.</u>

<u>Theorem 3.12.</u> Let $f$ be with domain $R^n$, taking real values and satisfy (1.6), Then the most general solution of (1.6) is

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$$

where $f_i^{'}s$ satisfy (1.6). Thus the general continuous solution is $f(x_1,\ldots,x_n) = c_1 x_1 + \ldots + c_n x_n$, where $c_i^{'}$ s are reals.

*Proof*. It is similar to the proof of theorem (3.11).

Number of questions were raised of (1.6), some of them were solved very recently and some of them still await answer.

(I.Halperin). Does the continuity of $f$ (real) follow from (1.6) and from $f(\frac{1}{x}) = \frac{1}{x^2} f(x)$, (all $x \neq o$) ?

The answer turns out to be true. Here we give proofs due to Kurepa and Jurkat. The following theorem in this direction is due to Kurepa [66] .

THEOREM 3.13. Let $f$ and $g \neq o$ be two solutions of the Cauchy functional equation (1.6). If $g(t) = P(t)\, f(\frac{1}{t})$ holds for all $t \neq o$, where $P$ is a continuous function such that $P(1) = 1$, then $P(t) = t^2$ and $f(t) + g(t) = 2tg(1)$. Furthermore, the function $F(t) = f(t) - t\, f(1)$ satisfies (1.6) and the equation $F(ts) = tF(s) + sF(t)$ for all real $t$ and $s$($F$ is called a derivative).

*Proof*. Let $t \neq o$ and $r$ a rational number different from zero. Then

$$g(rt) = P(rt)\, f\left(\frac{1}{rt}\right). \tag{29}$$

Since $f$ and $g$ satisfy (1.6), using (3.4) we get from (3.29),

$$rg(t) = P(rt)\, \frac{1}{r}\, f\left(\frac{1}{t}\right).$$

Hence, $g(t) = \frac{P(rt)}{r^2} f(\frac{1}{t})$ so that,

(30) $$\left[\frac{P(rt)}{r^2} - P(t)\right] f(\frac{1}{t}) = o, \text{ for all } r \neq o.$$

If $s \neq o$ is any real number, using the continuity of P and (3.30), we obtain

(31) $$\left[\frac{P(st)}{s^2} - P(t)\right] f(\frac{1}{t}) = o.$$

Since $f \neq o$, from (3.31), we have

(32) $P(st) = s^2 P(t)$, for all real $s \neq o$ and for at least one $t$ say $t_o \neq o$.

In (3.2), setting $s = \frac{1}{t_o}$, we get

(33) $$P(t_o) = t_o^2 P(1) = t_o^2.$$

From (3.32) and (3.33), we have

$$P(st_o) = t_o^2 s^2.$$

In the above replacing $s$ by $\frac{s}{t_o}$ we find that

(34) $$P(s) = s^2, \text{ for all } s \neq o.$$

Thus

(35) $$g(t) = t^2 f(\frac{1}{t}).$$

Evidently $g(1) = f(1)$. Now, define

(36) and $$\begin{cases} F(t) = f(t) - tf(1) \\ G(t) = g(t) - tg(1). \end{cases}$$

Obviously $F$ and $G$ satisfy (1.6). Further from (3.35) and (3.36), we have

(37) $$G(t) = t^2 F(\frac{1}{t}).$$

We have from (3.36) and (3.4), that for any rational $r$,

(38) and $$\begin{cases} F(r) = o \\ G(r) = o. \end{cases}$$

Now, from (3.37), (3.38) and (1.6), we have

$$\begin{aligned} G(t) = G(t+1) &= (t+1)^2 F(\frac{1}{1+t}) \\ &= (t+1)^2 F(1 - \frac{t}{1+t}) \\ &= -(t+1)^2 F(\frac{t}{1+t}) \\ &= -(t+1)^2 . (\frac{1}{1+t})^2 . G(\frac{1+t}{t}) \end{aligned}$$

$$= - t^2 G(\frac{1}{t})$$

$$(39) \qquad = - F(t).$$

(3.37) and (3.39) yield

$$(40) \qquad F(t) = - t^2 F(\frac{1}{t}).$$

From (3.36) and (3.39), we have

$$f(t) + t f(1) = - g(t) + t g(1).$$

that is,

$$(41) \qquad f(t) + g(t) = 2 t f(1).$$

Now using (3.4) and (3.40), we have

$$F(t) + \frac{1}{t^2} F(t) = F(t - \frac{1}{t})$$

$$= F(\frac{t^2-1}{t})$$

$$= -(\frac{t^2-1}{t})^2 . F(\frac{t}{t^2-1})$$

$$= -(\frac{t^2-1}{t^2})^2 \; F(\frac{1}{t-1} - \frac{1}{t^2-1})$$

$$= \frac{(t^2-1)^2}{t^2} \left[ \frac{-1}{(t-1)^2} F(t-1) + \right.$$

$$\left. + \frac{1}{(t^2-1)^2} F(t^2-1) \right]$$

$$(42) \qquad = \frac{(t+1)^2}{t^2} \; F(t) - \frac{1}{t^2} F(t^2).$$

Simplifying we obtain

$$(43) \qquad F(t^2) = 2\, t\, F(t).$$

In (3.42), replace $t$ by $(t+s)$ and use (3.42), we have

$$(44) \qquad F(ts) = tF(s) + sF(t).$$

This completes the proof of the theorem.

Corollary 3.14. If a function $f: R \to R$ satisfies (1.6) and $f(t) = t^2 f(\frac{1}{t})$ holds for $t \neq 0$, then $f(t) = t\, f(1)$.

Proof. It is evident from the proof of the theorem (3.12) and (3.41).

THEOREM 3.15. Let $f$ be additive and satisfy $f(\frac{1}{x}) = \frac{1}{x^2} f(x)$, for all $x \neq 0$. Then $f$ is continuous.

Proof. (Due to Jurkat [48]).

For $x \neq 0$ and 1, we have

$$\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x} .$$

Since

$$(45) \qquad f(\frac{1}{x}) = \frac{1}{x^2} f(x), \quad x \neq 0$$

we have

$$\frac{1}{x^2(x-1)^2} f[x(x-1)] = \frac{1}{(x-1)^2} f(x-1) - \frac{1}{x^2} f(x)$$

That is, $f(x^2)-f(x) = x^2[f(x) - f(1)] - (x-1)^2 f(x)$,

(46) $$f(x^2) = 2xf(x) - x^2 f(1).$$

(46) is also true for $x = o$ and 1.

Using

(47) $$4xy = (x+y)^2 - (x-y)^2$$

and (3.46), we have

$$f(4\,x\,y) = 2(x+y)f(x+y) - (x+y)^2 f(1) - \{2(x-y)f(x-y) - (x-y)^2 f(1)\}$$
$$= 4\,x\,f(y) + 4\,yf(x) - 4\,xy\,f(1).$$

Hence

(48) $$f(xy) = xf(y) + yf(x) - xyf(1).$$

Putting $y = \frac{1}{x}$, $x \neq o$ in (3.48), and using (3.45), we have

$$f(1) = xf(\tfrac{1}{x}) + \frac{1}{x}\,f(x) - f(1)$$

$$= \frac{2}{x}\,f(x) - f(1),$$ from which follows

$$f(x) = x\,f(1), \text{ for } x \neq o.$$

The last equation is also true for $x = o$. Hence $f$ is continuous.

(P. Erdos). A question regarding the domain of (1.6) was raised by P. Erdős. Let $f$ be a real valued function satisfying (1.6) for almost all pairs $(x,y)$. Is it true that $f$ is then equal almost everywhere to a function which satisfies (1.6) for all $x, y$ ? Here again the answer is yes. The result proved under is due to Jurkat [48].

THEOREM 3.16. Let $f$ be real valued, defined for almost all real $x$ and suppose that (1.6) holds for almost all pairs $(x,y)$ in the sense of plane measure (Lebesgue). Then there exists a real-valued function $F$, defined for all $x$ and satisfying (1.6) for all $x$ and $y$, which coincides with $f$ for almost all $x$ in the sense of linear measure (Lebesgue). These requirements determine $F$ uniquely.

Proof. By Fubini's theorem, there are null sets $N$ and $N_x$ such that (1.6) is true, if $X \notin N$, $Y \notin N_x$. Let $M$ be the complement of $N$ and notice that $f$ is defined on $M$. First we will show that (1.6) holds, whenever $x, y, x+y \in M$. Fix $x$ and $y$ for the moment and pick $Z$ such that $Z \notin N_{x+y}$, $Y + Z \notin N_x$ and $Z \notin N_y$. This is possible by avoiding 3 null sets for $z$. But then we have

$$f(x+y+z) = f(x+y) + f(z)$$

$$f(x+y+z) = f(x) + f(y+z)$$

$$f(y+z) = f(y) + f(z),$$

and the result follows. Next, we will prove that

$$f(x_1) + f(x_2) = f(y_1) + f(y_2) \tag{49}$$

whenever $x_1 + x_2 = y_1 + y_2$, $x_1, x_2, y_1, y_2$ all belong to M. Pick $z \in M$ such that $x_2' = x_2 - z \in M$, $y_2' = y_2 - z \in M$, $x_1 + x_2' = y_1 + y_2' = x_1 + x_2 - z = y_1 + y_2 - z \in M$. This is possible by avoiding four null-sets for z. Now, we have

$$f(x_2) = f(x_2') + f(z), \qquad f(y_2) = f(y_2') + f(z)$$
$$f(x_1 + x_2') = f(x_1) + f(x_2'), f(y_1 + y_2') = f(y_1) + f(y_2').$$

Hence

$$\begin{aligned} f(x_1)+f(x_2) &= f(x_1+x_2') + f(z) \\ &= f(y_1+y_2') + f(z) \\ &= f(y_1) + f(y_2). \end{aligned}$$

Finally, we show that given $x_1, x_2, x_3 \in M$, there exists $y_1, y_2 \in M$, such that

$$\text{and} \quad \begin{cases} x_1 + x_2 + x_3 = y_1 + y_2 \\ f(x_1+x_2+x_3) = f(y_1+y_2). \end{cases} \tag{50}$$

This is done by picking $z \in M$ such that $z' = x_3 - z \in M$, $y_1 = x_1 + z \in M$, $y_2 = x_2 + z' = -x_2 + x_3 - z \in M$ (avoid four null sets). Then we have

$$f(x_3) = f(z') + f(z)$$

$$f(y_1) = f(x_1) + f(z)$$

$$f(y_2) = f(x_2) + f(z').$$

Thus (3.50) is obtained. In order to define $F$, we notice that every real number $z$ is of the form $x+y$ with $x \in M$, $y \in M$ (simply pick $x \in M$ such that $y = z - x \in M$). Define $F(z) = f(x) + f(y)$, which is single-valued, because of (3.49). For $z \in M$, (1.6) implies $F(z) = f(z)$. Now take two arbitrary real numbers $z_1$, $z_2$ of the forms $x_1 + y_1$, and $x_2 + y_2$ where $x_1, x_2, y_1, y_2 \in M$. By applying (3.50) twice we obtain, two numbers $z_1'$, $z_2' \in M$ such that

(51) and $$\begin{cases} x_1+y_1+x_2+y_2 = z_1'+z_2' \\ f(x_1)+f(y_1)+f(x_2)+f(y_2) = f(z_1') + f(z_2'). \end{cases}$$

L.H.S. of (3.51) is equal to $F(z_1)+F(z_2)$ by definition, while R.H.S. of (3.51) equals $F(Z_1'+Z_2') = F(Z_1+Z_2)$, thus proving (1.6) for $F$ with unrestricted variables.

It remains to show the uniqueness of $F$. Let $F_1$ and $F_2$ be 2 functions satisfying (1.6) for all $x, y$ and coincide on a set which includes almost all $x$. Let $F = F_1 - F_2$. Then $F$ also satisfies (1.6) and vanishes on $M$. As every real number $z$ is of the form $x+y$ with $x, y \in M$, we see that $F(z) = F(x) + F(y) = o$ holds generally. This completes the proof of this theorem.

<u>The other Cauchy Equations.</u>

Now let us consider the following equations:

$$(1.7) \qquad f(x+y) = f(x)\ f(y)$$

$$(1.8) \qquad f(xy) = f(x) + f(y)$$

$$(1.9) \qquad f(xy) = f(x)\ f(x).$$

One can find solutions of these equations either by a method adopted similar to that employed for (1.6) or by other means. But one can also find more promptly the solutions by putting them in a form analogous to that of (1.6). First let us consider (1.7).

THEOREM 2.17. Let $f$ be a real valued function of the real variable satisfying (1.7). Then the most general solutions of (1.7) are $f(x) \equiv o$ and $f(x) = e^{g(x)}$ where $g$ is an arbitrary solution of (1.6).

Proof. Suppose $f(x_o) = o$ for some $x_o$. Then from (1.7) we have

$$f(x) = f(x - x_o + x_o)$$

(52) $$= f(x - x_o)\, f(x_o)$$

$$= o, \quad \text{for all real } x.$$

Hence $f(x) = o$ everywhere or nowhere. In case (1.7) holds only for positive $x, y$, then also the above condition holds. For what we have from (3.52) is that $f(x) = o$, for all $x \geq x_o$. Let $x \in \left]o, x_o\right[$. Then there is an integer $n$ such that $n\, x \geq x_o$. Now from (1.7), we have

$$f(nx) = f(x)^n.$$

Since $f(nx) = o$, this gives $f(x) = o$ for all positive $x$. So, without loss of generality, we can assume that $f(x) \neq o$ for all real $x$. Replacing $x$ and $y$ in (1.7) by $x/2$, we obtain

$$f(x) = f(x/2)^2 > o,$$

from which follows that, any nontrivial solution of (1.7) is always positive. Now taking logarithm on both sides of (1.7) and

(53) $$g(x) = \log f(x),$$

we have

$$g(x+y) = g(x) + g(y).$$

Hence $f(x) = e^{g(x)}$ is the most general nontrivial solution of (1.7), with $g$ satisfying (1.6).

COROLLARY 3.18. The most general continuous (continuous at one point, measurable on a set of positive measure etc.) solution of (1.7) is $f(x) = e^{cx}$, where $c$ is any constant.

Proof. From (3.53), it follows that $g$ is continuous, since $f$ is, Hence $g(x) = cx$ (by theorem 3.1) and the result follows.

Remark. In case (1.7) is valid only for nonnegative $x, y$, in addition we have also the solution $f(o) = 1$ and $f(x) = o$, $x > o$.

Deduction of differentiability from integrability.

THEOREM 3.19. The continuous solution of the functional equation (1.7) are $f(x) = o$ and $f(x) = e^{cx}$ ($c$, constant) and only these.

Proof. $f(x_o) = o$ at some point $x_o$ implies $f(x) = o$, for all $x$. So, we assume that $f(x) \neq o$, for all real $x$. Then from (1.7), we have

$$f(o) = f(o)^2 \quad \text{and thus} \quad f(o) = 1.$$

Since $f$ is continuous and $f(o) = 1$, there is an $\varepsilon > o$ such that

$$a = \int_o^{\varepsilon} f(x)\, dx \neq o.$$

Integrating (1.7) with respect to $x$ between $o$ and $\varepsilon$, we get

$$\int_o^{\varepsilon} f(x+y)\, dx = \int_o^{\varepsilon} f(x)\, f(y)\, dx$$

$$= af(y).$$

Hence

$$f(y) = \frac{1}{a} \int_o^{\varepsilon} f(x+y)\, dx \text{ for all } y.$$

Replacing $x+y$ by $u$ in the right side, we get

$$f(y) = \frac{1}{a} \int_y^{\varepsilon+y} f(u)\, du$$

$$= \frac{1}{a} \left[ \int_o^{\varepsilon+y} f(u)\, du - \int_o^{y} f(u)\, du \right].$$

The continuity of $f$ gives that the right side is differentiable and hence $f$ is differentiable.

Differentiating (1.7) with respect to $x$, we obtain

$$f'(x+y) = f'(x)\, f(y), \quad \text{for all } x \text{ and } y.$$

Putting $x = o$ and taking $f'(o) = c$, we have

$$f'(y) = cf(y) \text{ for all } y.$$

Thus $f(y) = a\, e^{cx}$, where $a$ is a constant.
Using the fact that $f(o) = 1$, we get $a = 1$. Thus

$$f(x) = e^{cx}, \text{ for all } x.$$

THEOREM 3.20. Let $\nu$ be a complex-valued function satisfying (1.7) non-trivially for $x,y \geq o$ with $\nu(o) = 1$ and $|\nu(x)|$ be bounded in some interval $[a,b]$. Then $|\nu(x)| = \exp \alpha x$ for some real number $\alpha$.

Proof. [41] Let $f(x) = \log |\nu(x)|$, Then $f$ is well defined on $[o,\infty[$. Further $f(x+y) = f(x) + f(y)$, that is, $f$ satisfies (1.6). Also $f(o) = o$ and $f$ is bounded from above on $[a,b]$. Then we know that (Th.3.7), $f$ is continuous, that is $f(x) = f(1)\, x$, for $x \geq o$. Taking $\alpha = f(1)$, we get our desired result.

DEFINITION. (Also refer [40]). Set $\chi(x) = \dfrac{\nu(x)}{|\nu(x)|}$ where $\nu$ satisfies (1.7) with $\nu(o) = 1$. Then it is clear that $\chi$ also satisfies (1.7) for $x,y \geq o$ and $|\chi(x)| = 1$. For negative $x$, we may set $\chi(x) = [\chi(-x)]^{-1}$. Then $\chi$ satisfies (1.7) for all real $x$. Such a function is called a <u>character</u> of the real line.

THEOREM 3.21. If $\chi$ is a measurable character of the real line, then $\chi(x) = \exp(i\beta x)$ for some real $\beta$.

<u>Proof</u>. For $|\delta| > 0$ and $\nu > 0$, we have

$$\left[\chi(x+\delta) - \chi(x)\right]\nu = \int_0^{\nu} \left[\chi(x+\delta-y) - \chi(x-y)\right]\chi(y)\,dy.$$

Hence $|\chi(x+\delta) - \chi(x)| \leq \frac{1}{\nu}\int_0^{\nu} |\chi(x+\delta-y) - \chi(x-y)|\,dy$ which tends to zero with $|\delta|$. Thus $\chi$ is continuous. Further

$$\frac{\chi(\delta)-1}{\delta}\int_0^{Y}\chi(x)\,dx = \frac{1}{x}\int_0^{Y}\left[\chi(x+\delta) - \chi(x)\right]dx$$

$$= \frac{1}{\delta}\int_{Y}^{Y+\delta}\chi\, x(dx) - \frac{1}{\delta}\int_0^{\delta}\chi(x)\,dx.$$

Choose $Y$ so that $\int_0^{Y}\chi(x)\,dx \neq 0$. Since the limit as $|\delta| \to 0$ exists for the terms on the right hand side of this equation, it follows that the derivative of $\chi(x)$ at $x = 0$ exists. Let $\left.\frac{d\,\chi(x)}{dx}\right|_{x=0} = i\,\beta$. Then we will show that the derivative of $\chi$ exists at all $x \in\, ]-\infty,\infty[$ and $\frac{d\,\chi(x)}{dx} = i\,\beta\,\chi(x)$. In fact,

$$\frac{\chi(x+\delta) - \chi(x)}{\delta} = \chi(x).\,\frac{\chi(\delta) - 1}{\delta}\,.$$

Limit as $|\delta| \to o$ exists in the right side. Hence the derivative of $\chi$ exists at all $x$. Taking the limit as $|\delta| \to o$, we get

$$\frac{d\chi(x)}{dx} = \chi(x)\, i\,\beta. \quad \text{Hence } \chi(x) = c \exp(i\,\beta x).$$

Since $\chi(o) = 1$, $c = 1$. As $|\chi(x)| = 1$, we see that $\beta$ is real.

THEOREM 3.22. If $\nu$ is a measurable function (complex-valued) and satisfies (1.7) non-trivially for $x,y \geq o$ with $\nu(o) = 1$, then $\nu(x) = \exp[(\alpha + i\,\beta)\,x]$ for some real numbers $\alpha$ and $\beta$.

Proof. Let $f(x) = \log|\nu(x)|$. Then $f$ is measurable and additive on $[o,\infty[$. Hence $f$ is continuous (Th.3.3). So, by theorem (3.20) $|\nu(x)| = \exp \alpha x$, for some real $\alpha$. Further $\chi(x) = \dfrac{\nu(x)}{|\nu(x)|}$ is measurable. So, by theorem (3.21),

$$\chi(x) = i\,\beta x \quad \text{for some real } \beta.$$

The result now is immediate.

Now we will take up the equation (1.8).

THEOREM 3.23. If $f$ is a solution of (1.8) for all real $x,y \neq o$, then the most general form of $f$ is $f(x) = g(\log|x|)$, where $g$ satisfies (1.6).

Proof. First note that $f$ is even. For, replace $y$ by $x$ and then $x$ and $y$ by $-x$ respectively in (1.7), then we obtain

$$f(x^2) = 2\, f(x)$$
$$\text{also} = 2\, f(-x).$$

Thus $$f(-x) = f(x), \quad \text{for} \quad x \neq o.$$

Now let $x$ and $y$ be positive. There exists $u$ and $v$ such that $x = e^u$ and $y = e^v$. By defining

$$g(u) = f(e^u),$$

and using (1.8), we obtain, $g(u+v) = g(u)+g(v)$.

Thus for $x > o$, $f(x) = g(\log x)$, where $g$ satisfies (1.6). Then the result follows from the fact that $f$ is even.

COROLLARY 3.24. The continuous solution of (1.8) which is defined for all $x, y \neq o$ is $f(x) = c \log x$ .
The proof is immediate from the above theorem (3.23).

Remark. If (1.8) is valid for all real $x$, then $f(x) \equiv o$. For, putting $x = o$ in (1.8), we have $f(o) = f(o) + f(y)$, from which it is easy to see that $f(x) \equiv o$.

Characterization of exponential and logarithmic functions.

The functions $e^{cx}$ and $\log cx$ can be characterized by means of the equations (1.7) and (1.8) respectively (vide above theorems), in two variables. But these functions can also be characterized with the aid of the following equations in a single variable:

$$f(2x) = f(x)^2 \quad (54)$$

and

$$f(x^2) = 2f(x) \quad (55)$$

and some additional conditions. The following theorems will be of interest in that direction (for proof see [59]).

THEOREM 3.25. The function $f(x) = e^x$ is the only function which is differentiable in $[0,\infty[$, satisfies (3.54) and the conditions $f(0) = f'(0) = 1$.

The function $f(x) = \log x$ is the only function which is differentiable in $[1,\infty[$, satisfying (3.55) with the condition $f'(1) = 1$.

THEOREM 3.26. The function $f(x) = e^x$ is the only function which is logarithmically convex in $]0,\infty[$, satisfying (3.54) and the condition $f(1) = e$. The function $f(x) = \log x$ is the only function which is concave ($-f$ convex) in $]1,\infty[$, satisfying (3.55) and the condition $f(e) = 1$.

THEOREM 3.27. If $f$ satisfies (1.9) $f(xy) = f(x)f(y)$ for all positive $x,y$ or for all real $x,y$ or for all real $x \neq 0$, $y \neq 0$, then the continuous solutions of (1.9) are

$$(56)\quad \begin{cases} f(x) = x^c \quad \text{or} \quad f(x) = o \\[2ex] f(x) = \begin{cases} |x|^c, & x \neq o, \\ o, & x = o \end{cases} \\[2ex] f(x) = \begin{cases} |x|^c \operatorname{sgn} x, & x \neq o, \\ o, & x = o \end{cases} \end{cases}$$

$$f(x) = 1,\ f(x) = \operatorname{sgn} x$$

$$f(x) = o,\ f(x) = |\operatorname{sgn} x|$$

or

$$f(x) = |x|^c,\ f(x) = |x|^c \operatorname{sgn} x,\ f(x) = o$$

respectively.

<u>Proof</u>. Let $x$ and $y$ be positive. Put $x = e^u$, $y = e^v$, $f(e^u) = g(u)$ in (1.9). Then we have

$$g(u+v) = g(u)\ g(v) \quad \text{which is same as (1.7).}$$

Thus the continuous solutions in this case are

$$f(x) = e^{c \log x} = x^c \quad \text{or} \quad f(x) = o.$$

Now, put $x = o$ in (1.9).. Then we have

$$f(o) = f(x)\ f(o) \quad \text{from which we can conclude}$$

that either

(57) $$f(x) \equiv 1 \quad \text{or} \quad f(o) = o.$$

Now, let $x \neq o$, $y \neq o$. Set first $y = x$ and then replace $x$ and $y$ by $-x$ in (1.9). Then we obtain

$$f(x)^2 = f(x^2) = f(-x)^2.$$

Hence

$$\text{(58)} \qquad \begin{array}{l} f(-x) = f(x) = x^c \\ \text{or} = -f(x) = -x^c \end{array} \quad \text{or } f(-x) = o.$$

Thus since $f$ is continuous, for $x \neq o$, we have

$$f(x) = x^c, \; f(x) = x^c \operatorname{sgn} x, \; f(x) = o,$$

for if there are $x$ and $y$ $(\neq o)$ such that $f(-x) = f(x)$ and $f(-y) = -f(y)$, then from (1.9) we would have $f(x)f(y) = o$. Let $x, y$ be real. Then from (3.57), (3.58) we obtain the continuous solutions as

$$f(x) = \begin{cases} x^c, & x \neq o, \\ o, & x = o \end{cases} \qquad f(x) = \begin{cases} x^c \operatorname{sgn}, & x \neq o, \\ o, & x = o \end{cases}$$

$$f(x) = o, \qquad f(x) = 1 \quad f(x) = \operatorname{sgn} x, \; f(x) = \operatorname{sgn} x \; .$$

THEOREM 3.28. The common solutions (real) of (1.6) and (1.9) are $f(x) = x$ and $f(x) = o$.

Proof. For $x > o$, replace $x$ and $y$ in (1.9) by $x$. Then we have

$$f(x) = f(\sqrt{x})^2 \geq o.$$

Thus $f$ is non-negative for positive $x$. Then by Theorem (3.2) $f$ being a solution of (1.6), we have $f(x) = cx$. This in (1.9) gives the condition that $c\,x\,y = c^2\,x\,y$, from which it follows that either $c = o$ or $c = 1$. Thus $f(x) = x$ or $f(x) = o$.

THEOREM 3.29. The common continuous complex solutions of (1.6) and (1.9) are $f(x) = o$, $f(x) = x$ and $f(x)=\bar{x}$. (here $f$ is a complex function of a complex variable).

Proof. All continuous, complex solutions of (1.6) are given by (theorem 3.11)

(28) $$f(x) = ax+b\bar{x}, \quad \text{for all complex } x; a,b \text{ complex.}$$

Hence for all real $x$, $f(x) = (a+b)x$. This in (1.9) gives either $a+b = o$ or $a+b = 1$. Thus (3.28) takes the form

$$f(x) = a(x-\bar{x})$$

or

$$f(x) = ax+(1-a)\bar{x}\ , \text{ for all complex } x.$$

First let $f(x) = a(x-\bar{x})$.

Putting $x = i, y = 1$ in (1.9), we have

$$f(i) = f(i)\ f(1), \text{ that is,}$$

$$a\ .\ 2i = a\ .\ 2i\ .\ o\ . \quad \text{Thus } a = o.$$

Hence $f(x) = o$.

From (59), next let $f(x) = ax + (1 - a)\,\bar{x}$.

Setting $x = i,\ y = -i$ in (1.9) we have

$$f(1) = f(i)\ f\,(-i).$$

that is,

$$\begin{aligned} 1 &= [ai + (1-a)(-i)]\ [-ai + (1-a)i] \\ &= 1 - 4a + 4a^2. \end{aligned}$$

Hence $a = o$ or $1$. This gives either

(60) $f(x) = \bar{x}$ or $f(x) = x$. Hence the result.

Solution of (1.7) and (1.9) for complex values.

(1.7) $f(x+y) = f(x)\ f(y),\ f : \mathbb{C} \longrightarrow \mathbb{C},\ \mathbb{C}$, complex numbers.

THEOREM 2.30. The continuous, complex solutions (non-vanishing) of (1.7) are $f(x) = e^{ax+b\bar{x}}$, $a, b$ complex constants.

Proof. (due to Abel [4]). Let $p + iq = r\,(\cos\phi + i\sin\phi)$, $p, q$ real. Then $r = \sqrt{p^2 + q^2}$,

$\cos\phi = \frac{p}{r}$ and $\sin\phi = \frac{q}{r}$.

Let $r = h(x,y)$ and $\phi = g(x,y)$, where $p + iq = f(x+iy)$

Then for $x, y, u, v$ real, we have

$$(61)\quad \begin{cases} f(x+iy) = h(x,y)\,[\cos g\,(x,y) + i\sin g(x,y)] \\ f(u+iv) = h(u,v)\,[\cos g\,(u,v) + i\sin g(u,v)] \\ f(x+u+iy+iv) = g(x+u, y+v)\,[\cos g(x+u,y+v) + \\ \qquad\qquad + i\sin g(x+u, y+v)]\,. \end{cases}$$

From (1.7) and (3.61), we get

$$h(x+u,y+v)\left[\cos g(x+u,y+v) + i \sin g(x+u,y+v)\right]$$
$$= h(x,y)\, h(u,v)\left[\cos (g(x,y) + g(u,v)) + i \sin (g(x,y) + g(u,v))\right].$$

Hence

$$(62)\quad \begin{cases} h(x+u,y+v) \cos g(x+u,y+v) = h(x,y)\, h(u,v) \cos (g(x,y) + g(u,v)). \\ h(x+u,y+v) \sin g(x+u,y+v) = h(x,y)\, h(u,v) \sin (g(x,y) + g(u,v)). \end{cases}$$

Squaring and adding (3.62), we get

$$h(x+u,y+v)^2 = h(x,y)^2\, h(u,v)^2 \quad \text{from which follows}$$

$$(63)\qquad h(x+u,y+v) = h(x,y)\, h(u,v) \quad (\text{since } h \text{ is always positive}).$$

From (3.62) and (3.63) results

$$(64)\qquad g(x+u,y+v) = 2\, m\, \pi + g(x,y) + g(u,v).$$

where $m$ is any integer.

Since $f$ is continuous, so are $h$ and $g$.

As $g$ is continuous, $m$ in (3.64) should be a constant.

From Theorem 3.10, it follows that,

$$(65)\qquad g(x,y) = c_1 x + c_2 y - 2\, m\, \pi,\quad c_1, c_2 \text{ real}.$$

But we will prove (3.65) using Abel's argument as follows.

Putting $x = o$ and $u = o$ in (3.65), we get

$$(66) \quad \begin{cases} g(u,y+v) = 2\,m\,\pi + g(o,y) + g(u,v) \\ g(x,y+v) = 2\,m\,\pi + g(x,y) + g(o,v). \end{cases}$$

From (3.64) and (3.66), we get

$$(67) \quad g(x,y+v) + g(u,y+v) = 2\,m\,\pi + g(u,y) + g(o,v) + g(x+u,y+v).$$

Letting

$$(68) \quad \begin{cases} \alpha(x) = g(x,y+v) \\ c = 2\,m\,\pi + g(o,y) + g(o,v), \end{cases}$$

we have from (3.67)

$$(69) \quad \alpha(x) + \alpha(u) = c + \alpha(x+u).$$

$\alpha$ being continuous, we get from (3.69),

$$(70) \quad \alpha(x) = dx + c,$$

where $d = \alpha(1) - c$.

Thus, $\alpha(1)$ being a function of $y$ and $v$, we have

$$(71) \quad g(x,y+v) = d(y,v)\,x + 2\,m\,\pi + g(o,y) + g(o,v).$$

$x = o$ in (3.71) gives

$$g(o,y+v) = 2\,m\,\pi + g(o,y) + g(o,v).$$

Hence, (72) $g(o,y) = ey - 2m\,\pi$, $e$ a real constant.

Therefore from (3.71) and (3.72), we have

$$(73) \quad g(x,y+v) = d(y,v)x + e(y+v) - 2\,m\,\pi.$$

From (3.73) we see that $\phi(x,v)$ is of the form $\phi(y+v)$.

So,

(74) $$g(x,y+v) = \phi(y+v)x + e(y+v) - 2m\pi.$$

Putting $v = o$, in (3.74), we have

(75) $$g(x,y) = \phi(y)\,x + ey - 2m\pi.$$

But from (3.64) we have, putting $u = o$,

(76) $$g(x,y+v) = 2m\pi + g(x,y) + g(o,v).$$

From (3.72), (3.74), (3.75) and (3.76), we get

$$\phi(y+v) = \phi(y) = \text{constant} = \alpha.$$

Thus from (3.75)

(77) $$g(x,y) = \alpha x + ey - 2m\pi,$$ $\alpha$, $e$ real constants which is same as (3.65).

As $h(x,y)$ is positive, with $h(x,y) = e^{H(x,y)}$, (3.63) reduces to

$$H(x+u,y+v) = H(x,y) + H(u,v).$$

Hence

$$H(x,y) = d_1 x + d_2 y,$$ $d_1$ $d_2$ real constants and so,

(78) $$h(x,y) = e^{d_1 x + d_2 y}.$$

The equations (3.61), (3.65) and (3.78) give

$$f(z) = f(x+iy) = e^{d_1x + d_2y}\left[\cos(c_1x + c_2y) + \right.$$

$$\left. + i \sin(c_1x + c_2y)\right]$$

$$= e^{az+b\bar{z}}, \text{ where } a = \frac{1}{2}\left[(d_1+ic_1) + (c_2-i\,d_2)\right]$$

and $b = \frac{1}{2}\left[(d_1 + ic_1) - (c_2 - i\,d_2)\right]$.

This completes the proof of the theorem.

(1.9) $$f(xy) = f(x)\,f(y),\ f : \mathbb{C} \to \mathbb{C}.$$

THEOREM 3.31. Let $f$ be a complex valued function of the complex variable satisfying (1.9). Further let $f$ be a continuous solution of (1.9). Then $f$ is of the form

$$f(z) = \begin{cases} e^{k \log |z|}\, z^n, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

where $k$ is a complex constant and $n$ any integer, provided $f \not\equiv 0,1$.

<u>Proof</u>. Let $T = \{z : |z| = 1\}$. Then $f$ restricted to $T$ is a continuous character [40]. It is known that [40], the character group of $T$ is the group of integers $z$, that is,

$$f(z) = z^n, \text{ for } z \in T,\ n \in Z.$$

By theorem 3.27, for $x$ positive, we have

$$f(x) = x^k,\ x \in R^+.$$

For $z \neq 0$, we have

$$f(z) = f\left(|z| \cdot \frac{z}{|z|}\right)$$

$$= f(|z|) \cdot f\left(\frac{z}{|z|}\right)$$

$$= e^{A \log |z|} \left(\frac{z}{|z|}\right)^n, \quad A \text{ complex constant, } n \text{ any integer}$$

$$= e^{k \log |z|} \cdot z^n, \quad k, \text{ any complex constant.}$$

Thus the theorem is proved.

§ 4. Pexider and Jensen Equations.

Pexider's Equations. The functional equations (1.10), (1.11), (1.12) and (1.13)

(1.10) $$f(x+y) = h(x) + g(y)$$

(1.11) $$f(x+y) = h(x)g(y)$$

(1.12) $$f(xy) = h(x) + g(y)$$

(1.13) $$f(xy) = h(x)g(y)$$

known as Pexider equations are an immediate generalization of the Cauchy Equations to which they can be easily reduced and solved.

THEOREM 4.1. The general solution of (1.10) is $f(x) = a(x) + b + c$, $g(x) = a(x) + c$ and $h(x) = a(x) + b$ where $a(x)$ is an arbitrary solution of (1.6) and $b, c$ are constants. Further if $f$ is a continuous solution of (1.10), then $f(x) = kx + b + c$, $g(x) = kx + c$ and $h(x) = kx + b$, $b, c, k$ are constants.

Proof. Put first $x = o$ in (1.10) and then $y = o$ in (1.10). Then we have

(1) $$f(y) = h(o) + g(y)$$

(2) $$f(x) = h(x) + g(o).$$

From (1.10),(4.1) and (4.2) we have

(3) $$f(x+y) = f(x) + f(y) - b - c \text{ where } h(o) = b$$

and $g(o) = c$.

(4) Setting $a(x) = f(x) - b - c$, from (4.3), we have

$$a(x+y) = a(x) + a(y).$$

From (4.1), (4.2) and (4.4), we obtain the required result.

Remark. As for $g$ and $h$ are concerned, no further assumptions are necessary. Further these functions are continuous, when $f$ is. This follows from (4.1) and (4.2).

THEOREM 4.2. The most general solution of (1.11) is $f(x) = a\, b \exp[c(x)]$, $g(x) = b \exp[c(x)]$, $h(x) = a \exp[c(x)]$, where $c(x)$ is an arbitrary solution of (1.6), $a \neq o$, $b \neq o$ are arbitrary constants excluding the trivial solution $f = o$, $g = o$, $h$ arbitrary, $f = o$, $h = o$, $g$ arbitrary. Further, if $f$ is continuous, then $f(x) = ab \exp(cx)$, $g(x) = b \exp(cx)$, $h(x) = a \exp(cx)$, where $a,b,c$ are non-zero constants (excluding the trivial solution).

**Proof.** If either $h(o) = o$ or $g(o) = o$, we obtain from (1.11), that $f \equiv o$ and either $g \equiv o$ or $h \equiv o$, so that either $h$ or $g$ is arbitrary respectively. Henceforth, we assume $h(o) \neq o$ and $g(o) \neq o$. Putting $x = o$ in (1.11) we obtain,

$$f(y) = h(o)\ g(y),$$

(5) hence $$g(y) = \frac{1}{a} f(y)\ , \text{ where } h(o) = a \neq o.$$

Similarly, putting $y = o$ in (1.11), we have

(6) $$h(x) = \frac{1}{b} f(x), \text{ where } g\ (o) = b \neq o.$$

From (1.11), (4.5) and (4.6), we have

(7) $$f(x+y) = \frac{f(x)\ f(y)}{ab}\ .$$

Setting

(8) $$g(x) = \frac{f(x)}{ab}\ ,$$

from (4.7) and (4.8) we obtain

(1.7) $$g(x+y) = g(x)\ g(y).$$

Thus $g(x) = \exp\ (cw)$. Then from (4.5), (4.6) and (4.8) we obtain the desired result.

**Remark.** When $f$ is continuous (non-trivial solution), then so are $g$ and $h$.

THEOREM 4.3. If $f,g,h$ satisfy (1.12), then $f(x) = \ell(x) + a + b$, $g(x) = \ell(x) + b$ and $h(x) = \ell(x) + a$ where $\ell(x)$ is an arbitrary solution of (1.8), $a,b$ are constants. When $f$ is continuous, $f(x)=\gamma \log (\alpha \beta x)$, $g(x) = \gamma \log (\beta x)$ and $h(x) = \gamma \log (\alpha x)$, $(\alpha, \beta, x > o)$.

<u>Proof</u>. Putting $x = 1$ in (1.12), we have

$$f(y) = g(y) + h(1),$$

or

(9) $$g(y) = f(y) - a, \quad \text{where } h(1) = a.$$

Similarly, putting $y = 1$ in (1.12), we have

(10) $$h(x) = f(x) - b, \text{ where } b = g(1).$$

Setting

$\ell(x) = f(x) - a - b$, we obtain from (1.12), (4.9) and (4.10),

(1.8) $$\ell(xy) = \ell(x) + \ell(y).$$

If $f$ is continuous, $x,y > o$, then $\ell(x) = y \log x$ and taking $a = \gamma \log \alpha$ and $b = \gamma \log \beta$, $\alpha$, $\beta > o$, we have the sought for result.

THEOREM 4.4. If $f,g,h$ satisfy (1.13), then $f = o$, $g = o$, $h$ arbitrary; $f = o$, $h = o$, $g$ arbitrary and $f(x) = abm(x)$, $g(x) = b\, m(x)$ and $h(x) = a\, m(x)$ are the only solutions of (1.13) where $m(x)$ satisfies (1.9) and $a \neq o$, $b \neq o$ constants. If $f$ is continuous, then $f(x) = ab\, x^c$, $g(x) = b\, x^c$ and $h(x) = a\, x^c$ where $x > o$, $a \neq o$, $b \neq o$, $c$ are constants.

Proof. Putting $y = 1$ in (1.13), we get

$$f(x) = h(x)\ g(1)$$

or

$$h(x) = \frac{f(x)}{a}, \text{ where } g(1) = a \neq o. \tag{11}$$

Similarly putting $x = 1$ in (1.13), we obtain

$$g(y) = \frac{f(y)}{b} \quad \text{where} \quad h(1) = b \neq o. \tag{12}$$

If either $g(1) = o$ or $h(1) = o$ we obtain the trivial solutions. Now, setting

$$m(x) = \frac{f(x)}{ab},$$

from (1.13), (4.11) and (4.12), we have

$$m(xy) = m(x)\ m(y). \tag{1.9}$$

Thus, when $f$ is continuous and $x, y > o$, we obtain the desired result.

The functional equations (1.7), $f(x+y) = f(x)\ f(y)$ and (1.11) $f(x+y) = h(x)\ g(y)$ where $f, g, h$ are real valued functions of the real variables have been extensively studied and it is well known that the continuous solutions of (1.7) are given by $f(x) = e^{cx}$ (Theorem 3) and that of (1.11) are of the form $f(x) = ab\ e^{cx}$, $g(x) = b\ e^{cx}$ and $h(x) = a\ e^{cx}$ (Theorem 4.2), where $a, b, c$ are arbitrary constants. Here we consider (1.11) in the following manner.

Let $f : R \to R$ ($R$, real numbers). Then $f$ is said to have property (A) if there exist functions $h, g : R \to R$ such that (1.11)

$$f(x+y) = h(x)\ g(y) \text{ holds, for all } x, y \in R.$$

As pointed out in Theorem 4.2, if either $f$ or $g$ or $h$ is zero at some point, we will have only trivial solutions. In what follows we consider $f, g, h$ to be nowhere zero.

LEMMA 4.5. Let $f : R \to R$. Then the following two conditions are equivalent:

(i) $f$ has property (A)

(ii) $f\,(x+y) = \dfrac{f(x)\ f(y)}{f(o)}$, for every $x, y \in R$.

<u>Proof</u>. Let (i) be true. From (1.11) first with $x = o$ and then with $y = o$, we obtain

$$f(y) = h(o)\ g(y)$$

(13) and

$$f(x) = h(x)\ g(o).$$

From (1.11) and (4.12), we have

(14) $$f(x+y) = \frac{f(x)\,f(y)}{h(o)\,g(o)} = \frac{f(x)\,f(y)}{f(o)}\text{, which is (ii).}$$

Let (ii) be true. Then $f(x+y) = h(x)\,g(y)$ where $h(x) = f(o)$ $f(x)$ and $h(y) = \dfrac{f(y)}{f(o)^2}$. So, (i) holds. This completes the proof of this lemma. Let $f$ have property (A). Then (4.14) holds. Replacing $x$ by $x/2$ and $y$ by $x/2$, we get

$$(15) \qquad f(x) = \frac{f(x/_2)^2}{f(o)}$$

Hence from (4.15), we see that $f$ has always the same sign as that of $f(o)$. From (4.14), it is also evident that $f(o)$ is arbitrary. In the sequence we take $f(o)$ is positive and hence $f$ is always positive. Putting $y = -x$ in (4.14), we obtain

$$f(o) = \frac{f(x)f(-x)}{f(o)} .$$

That is,

$$(16) \qquad f(-x) = \frac{f(o)^2}{f(x)} , \quad \text{for every } x \in R.$$

THEOREM 4.6. Suppose $f : R \rightarrow R$ with property (A). Then for every rational $r$, $f(r) = \dfrac{f(1)^r}{f(o)^{r-1}}$ .

Proof. Put $y = x$ in (4.14) we have

$$(17) \qquad f(2x) = \frac{f(x)^2}{f(o)} .$$

Setting $y = 2x$ (in 4.14) and using (4.17), we obtain

$$f(3x) = \frac{f(x)f(2x)}{f(o)}$$

$$= \frac{f(x)^3}{f(o)^2} .$$

Hence by induction on $n$, we have for any natural $n$,

$$(18) \qquad f(nx) = \frac{f(x)^n}{f(o)^{n-1}} .$$

Let $n = -m$, $m > o$. From (4.16) and (4.18), we have

$$f(nx) = \frac{f(o)^2}{f(mx)}$$

$$= \frac{f(o)^{m+1}}{f(x)^m}$$

$$= \frac{f(x)^n}{f(o)^{n-1}} .$$

Hence (4.18) holds for all integers $n$. Let any rational $r = \frac{p}{q}$. Then $p = q\,r$.

From (4.18), we have $f(px) = \dfrac{f(x)^p}{f(o)^{p-1}}$

$$\text{also} \quad = \frac{f(rx)^p}{f(o)^{q-1}} .$$

$$\text{Hence} \quad f(rx)^q = \frac{f(x)^p}{f(o)^{p-q}} .$$

$$\text{That is,} \quad f(rx) = \frac{f(x)^{p/q}}{f(o)^{p/q - 1}}$$

$$(19) \qquad = \frac{f(x)^r}{f(o)^{r-1}} \quad \text{for all} \quad x \in R.$$

Now setting $x = 1$ in (4.19), we have

$$f(Y) = \frac{f(1)^r}{f(o)^r} ,$$

$$= f(o) \cdot \frac{f(1)^r}{f(o)} .$$

$$(20) \qquad = c\,a^r, \quad c = f(o) \text{ and } a = \frac{f(1)}{f(o)} .$$

THEOREM 4.7. Let $f : R \to R$ be continuous at some point. Then

(i) $f$ has property (A)

(ii) $f(x) = c\,a^x$, for every $x \in R$

are equivalent.

Proof. Let (i) hold. First we will prove that $f$ is continuous everywhere. Let $f$ be continuous at $x_o$. Then, from (4.14) and (4.16), we have

$$f(x_o) = \lim_{x_n - x + x_o \to x_o} f(x_n - x + x_o)$$

$$= \lim_{x_n - x + x_o \to x_o} \frac{f(x_n - x) \cdot f(x_o)}{f(o)}$$

$$= \frac{f(x_o)}{f(o)} \cdot \lim_{x_n \to x} \frac{f(x_n) \cdot f(-x)}{f(o)}$$

$$\lim_{x_n \to x} f(x_n) = \frac{f(o)^2}{f(-x)} = f(x).$$

Hence $f$ is continuous everywhere. From theorem 4.6, now it is easy to see that $f(x) = ca^x$ for all $x \in R$. Hence (ii) holds. Suppose (ii) holds. Then $f$ is continuous everywhere. Also

$$f(x+y) = c\,a^{x+y} = \frac{c\,a^x \cdot c\,a^y}{c}$$

$$= \frac{f(x)\,f(y)}{f(o)}, \text{ which is (4.14).}$$

Hence (i) is true. This completes the proof of this theorem.

Jensen's Equation: Now let us consider the equation

$$(1.14) \qquad f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2},$$

known as Jensen's equation. The solution of this equation can be obtained by reducing it to the Cauchy equation (1.6).

THEOREM 4.8. The most general solution of (1.14) is $f(x) = a(x) + b$, where $a(x)$ is a solution of (1.6) and $b$ is an arbitrary constant.

Proof. Put $y = o$ in (1.14). Then we have

$$f\left(\frac{x}{2}\right) = \frac{f(x) + b}{2}, \quad \text{where} \quad b = f(o).$$

Thus

$$\frac{f(x+y) + b}{2} = f\left(\frac{x+y}{2}\right)$$

$$(21) \qquad = \frac{f(x) + f(y)}{2}.$$

Now set $a(x) = f(x) - b$. Then (4.21) becomes

$$a(x+y) = a(x) + a(y).$$

Thus $f(x) = a(x) + b$, where $a$ satisfies (1.6).

Remark. The general continuous solutions of (1.14) are $f(x) = cx + b$, where $b$ and $c$ are constants.

THEOREM 4.9. Let $f$ be defined on an arbitrary interval and satisfy (1.14). If $f$ is continuous, then $f(x) = cx + b$ for all $x$ in the interval.

Proof. Without loss of generality, let us assume the interval to be $[o,1]$. Let $f(o) = b$ and $f(1) = a$. For every $x,y \in [o,1]$, it is evident that $\frac{x+y}{2} \in [o,1]$. The proof is based on induction. First, let us show that $f(x) = cx + b$, for $x$ a dyadic number in $[o,1]$. Then since the dyadic numbers are dense in $[o,1]$ and $f$ continuous, we will have the required result. From (1.14), we have

$$f(\tfrac{1}{2}) = f\left(\frac{0+1}{2}\right) = \frac{b+a}{2} = b + \frac{1}{2}(b - a)$$

$$= b + \frac{1}{2}c, \text{ where } c = b - a.$$

Now,

$$f(\tfrac{1}{4}) = \frac{f(0) + f(+\frac{1}{2})}{2}$$

$$= b + \frac{1}{4}c, \text{ etc.}$$

Suppose $f(x) = cx + b$, for all dyadic s with denominator $2^n$. Then by induction hypothesis, we have

$$f\left(\frac{2k}{2^{n+1}}\right) = f\left(\frac{k}{2^n}\right) = b + c \cdot \frac{k}{2^n}$$

$$= c \cdot \frac{2k}{2^{n+1}} + b$$

and

$$f\left(\frac{2k+1}{2^{n+1}}\right) = \frac{f\left(\frac{k}{2^n}\right) + f\left(\frac{k+1}{2^n}\right)}{2}$$

$$= \frac{1}{2}\left[c \cdot \frac{k}{2^n} + b + c \cdot \frac{k+1}{2^n} + b\right]$$

$$= c \cdot \frac{2k+1}{2^{n+1}} + b, \text{ which proves our assertion}$$

that $f(x) = cx + b$, for all dyadic $x$ in $[o,1]$.

§ 5. <u>Some generalizations of Cauchy, Pexider type equations</u>.

A). (1) $f(ax+by+c) = p\, f(x) + q\, f(y) + r,\ a,b,p,q \neq o,\ f: R \to R.$

The equation (5.1) possesses measurable and non-constant solutions if and only if $a = p$, $b = q$.

The equation (5.1) can be reduced to the equation (1.6) by a sequence of substitutions as follows.

$x = -c/a$, $y = o$ in (5.1) gives

(2) $f(o) = p\, f(-c/a) + q\, f(o) + r.$

$x = \frac{t-c}{a}$, $y = o$ in (5.1) gives

(3) $$f(t) = pf\left(\frac{t-c}{a}\right) + q\, f(o) + r.$$

$x = -c/a$, $y = \frac{z}{b}$ in (5.1) gives

(4) $f(z) = p\, f(-c/a) + q\,(z/b) + r.$

Lastly, $x = (t-c)/a$, $y = z/b$ in (5.1) gives

(5) $f(t+z) = pf((t-c)/a) + q\,(z/b) + r.$

Adding (5.2) and (5.5) and then subtracting (5.3) and (5.4), we get

(1.6) $\quad f_o(t+z) = f_o(t) + f_o(z)$, where

(7) $\quad f_o(t) = f(t) - f(o)$.

Hence if $f$ in (5.1) is continuous, from (1.6) and (5.7), we have,

(8) $\quad f(x) = ex + d,\ d = f(o),\ e$ , constant.

Putting this value of $f$ in (5.1), we obtain

$$a = p,\ b = q,\ ec - r = (a+b-1)d.$$

<u>Remark</u>. For $a = b = 1/2$, $c = o$, $p = q = 1/2$, $r = o$, the equation (5.1) reduces to the Jensen equation (1.14).

As (1.6) possesses discontinuous solution, all solutions of (5.1) are not continuous. As for non-measurable solutions, it has been proved for

(9) $\quad f(ax+y) = pf(x) + y$

(with $b = q = 1$, $c = r = o$) that if $a$ or $p$ is rational, then (5.9) has non-constant solutions only for $a = p$; if $a$ or $p$ is algebraic and (5.9) has non-constant solution, then $a$ and $p$ are algebraic and are roots of the same irreducible monic polynomial [27] .

B.(10) $\quad f(x+y) = g(x)\ k(y) + h(y), \quad f: R \to R.$

Let $f$ be not constant. Putting $y = o$ in (5.10), we get

(11) $\quad f(x) = g(x)\ k(o) + h(o).$

From (5.10) and (5.11), we have

(12) $\quad f(x+y) = \alpha(y)\ f(x) + \beta(y),$ where

(13) $\quad \alpha(x) = \dfrac{k(x)}{k(o)}, \quad \beta(x) = h(x) - \dfrac{h(o)}{k(o)}\ k(x).$

Now to find the solution of (5.12).

Set $x = o$ in (5.12) and subtract the equation

(14) $\quad f(y) = \alpha(y)\ f(o) + \beta(y),$

then we have

(15) $\quad \delta(x+y) = \alpha(y)\ \delta(x) + \delta(y),$ where

(16) $\quad \delta(x) = f(x) - f(o).$

Interchanging $x$ and $y$ in (5.15) and using (5.15), we get

$$\alpha(y)\ \delta(x) + \delta(y) = \alpha(x)\ \delta(y) + \delta(x), \text{ that is}$$

(17) $\quad [\alpha(x) - 1]\ \delta(y) = [\alpha(y) - 1]\ \delta(x).$

If $\alpha(x) \equiv 1$, (5.15) reduces to (1.6) $\delta(x+y) = \delta(x) + \delta(y).$
Hence, from (5.16) and (5.14), we have

(18) $\quad f(x) = \delta(x) + f(o), \ \alpha(x) = 1, \ \beta(x) = \delta(x)$

as a solution of (5.12), where $\delta$ satisfies (1.6).

From (5.13), (5.18) and (5.11), we see that, a solution of (5.10) is of the form

$$f(x) = \delta(x) + f(o)$$

$$g(x) = \frac{\delta(x) + f(o) - h(o)}{b}$$

$$h(x) = \delta(x) + h(o)$$

$$k(x) = \text{constant} = b.$$

Let $\alpha(x) \not\equiv 1$. Then there is an $x_o$ such that $\alpha(x_o) \neq 1$. Putting $y = x_o$ in (5.17), we have

$$(19) \qquad \delta(x) = c\,(\alpha\,(x) - 1), \quad c = \frac{d(y_o)}{\alpha(y_o) - 1}\,.$$

$c = o$ in (5.19) gives $\delta \equiv o$ and hence $f$ is a constant from (5.15) which cannot be.

The equations (5.15) and (5.19) yield

$$(1.7) \qquad \alpha(x+y) = \alpha(x)\,\alpha(y).$$

Hence from (5.14), (5.16), (5.19), we have

$$(20) \qquad \begin{cases} f(x) = c\,\alpha(x) + d, \; d = f(o) - c \\ \beta(x) = d\,(1 - \alpha(x)) \end{cases}$$

as solutions of (5.12), where $\alpha$ is a solution of (1.7). Thus (5.11), (5.13), (5.20), yield, a solution of (5.10) as

$$f(x) = c\,\alpha(x) + d$$

$$g(x) = \frac{c\,\alpha(x) + d - h(o)}{b}$$

$$k(x) = b\,\alpha(x)$$

$$h(x) = d + (d - h(o))\,\alpha(x)$$

where $\alpha$ satisfies (1.7).

In case $f$ is a continuous solution of (5.10), we get

$$f(x) = ex + f(o)$$

$$g(x) = \frac{ex + f(o) - h(o)}{b}$$

$$h(x) = ex + h(o)$$

$$k(x) = \text{const} = b$$

and

$$f(x) = c\, e^{Ax} + d \qquad , A \text{ a constant}$$

$$g(x) = \frac{c\, e^{Ax} + d - h(o)}{b}$$

$$k(x) = b\, e^{Ax}$$

$$h(x) = d + (d - h(o))\, e^{Ax}$$

are the only solutions of (5.10).

C. <u>Equation of the type $f(x+y) = F\,[f(x-y),\ f(x),\ f(y)]$.</u>

(21) $\qquad f(x+y)\, f(x-y) = f(x)^2$, $f : R \to R$, with $f$ differentiable.

Evidently $f \equiv$ constant is a solution of (5.21).

Suppose there is an $x_o$ such that $f(x_o) = o$. To show that $f \equiv o$. Putting $x = o$, $y = x_o$ in (5.21), we see that

$$f(x_o)\, f(-x_o) = f(o)^2 \text{ implies } f(o) = o.$$

Hence putting $x = o$ in (5.21), we see that $f(y)f(-y) = o$, all $y$. Suppose $f(-x_1) = o$,

Putting $x = -x_1$ and $y = 2x_1$, we see that

$$f(x_1)\ f(-3x_1) = 0.$$

If $f(x_1) = o$, there is nothing to prove.

Otherwise $f(-3x_1) = o$.

Put $x = x_1$ and $y = -4x_1$ in (5.21) to obtain

$$f(-3x_1)\ .\ f(+5x_1) = f(x_1)^2 \quad \text{implying} \quad f(x_1) = o.$$

Hence in either case, we have $f(x_1) = o$. So, $f \equiv o$. Let us assume that $f \not\equiv c$, in particular, $f$ vanishes nowhere.

Differentiating (5.21) with respect to $x$ and $y$, we have

$$(22) \qquad \begin{cases} f'(x+y)\ f(x-y) + f(x+y)\ f'(x-y) = 2f(x)\ f'(x), \\ f'(x+y)\ f(x-y) - f(x+y)\ f'(x-y) = o. \end{cases}$$

Thus from (5.22), we have

$$(23) \qquad \begin{aligned} f(x)\ f'(x) &= f(x+y)\ f'(x-y), \text{ for all } y, \\ &= f(2x)\ f'(o), \text{ for } y = x \\ &= \frac{f(x)^2}{f(o)}\ f'(o), \end{aligned}$$

since putting $y = x$ in (5.21) we have $f(2x)\ f(o) = f(x)^2$.

Since $f \not\equiv o$, we have

$$f'(x) = c\ f(x), \text{ all } x \in R, c = \frac{f'(o)}{f(o)}\ .$$

$f'(o) = o$ gives $f(x) \equiv c$, which cannot be.

Thus, $f(x) = A\, e^{cx}$, A, c, Constants.

D. (24) $f(x+y) = \alpha\, f(x) + \beta\, f(y)$, $\alpha$, $\beta$ constants

$x = o$, $y = o$ in (5.24) gives either $\alpha + \beta = 1$ or $f(o)=o$. Suppose $f(o) = o$.

$\therefore$ $y = o$ in (5.24) gives $f(x) = \alpha\, f(x)$

Similarly $f(y) = \beta\, f(y)$

Thus $f(x+y) = f(x) + f(y)$.

Suppose $\alpha + \beta = 1$.

$x = o$ in (5.24) gives

$$f(y) = \alpha\, f(o) + (1 - \alpha)\, f(y)$$

that is, $\alpha\, f(y) = \alpha\, f(o)$ and hence

$$f(x) = \text{constant}.$$

E. Equation of the type $F(x * y) = G(x) + H(y) + K(x)\, L(y)$.

Let F, G, H, K, L : C A $\rightarrow$ C (where A is an arbitrary Abelian semigroup with operation $*$ such that, there is a fixed element $a \in A$ with the property that the equation $a * x = b$ for arbitrary $b \in A$ has at least one solution, C the complex numbers) satisfy

(25) $F(x * y) = G(x) + H(y) + K(x)L(y)$, $x, y \in A$.

It has been proved by Vincze [85], that the following are the only solutions of (5.25).

I. $$F(x) = \varphi(x) + \alpha_1,$$

$$G(x) = \varphi(x) - \alpha_3 K(x) + \alpha_2 + \frac{\alpha_1}{2},$$

$$H(x) = \varphi(x) + \frac{1}{2}\alpha_1 - \alpha_2,$$

$K(x)$ arbitrary

$$L(x) = \alpha_3.$$

II. $$F(x) = \alpha_1\psi(x) + \varphi(x) + \alpha_2,$$

$$G(x) = \alpha_3\psi(x) + \varphi(x) + \alpha_4,$$

$$H(x) = \alpha_5\psi(x) + \varphi(x) + \alpha_6,$$

$$K(x) = \alpha_7\psi(x) + \alpha_8,$$

$$L(x) = \alpha_9\psi(x) + \alpha_{10},$$

$$\alpha_1 = \alpha_7\alpha_9,\ \alpha_3 + \alpha_7\alpha_{10} = 0,\ \alpha_5 + \alpha_8\alpha_9 = 0,\ \alpha_2 = \alpha_4 + \alpha_6 + \alpha_8\alpha_{10};$$

III. $$F(x) = \alpha_1\phi(x)^2 + \alpha_2\phi(x) + \phi_1(x) + \alpha_3$$

$$G(x) = \alpha\,\phi(x)^2 + \phi_1(x) + \alpha_4$$

$$H(x) = \alpha_1\phi(x)^2 + \alpha_5\phi(x) + \phi_1(x) + \alpha_6$$

$$K(x) = 2\alpha_1\phi(x) + \alpha_7$$

$$L(x) = \phi(x) + \alpha_8$$

$$\alpha_2 = 2\alpha_1\alpha_8 = \alpha_5 + \alpha_7,\ \alpha_3 = \alpha_4 + \alpha_6 + \alpha_7\alpha_8;$$

where $\phi$ , $\phi_1$ respectively $\psi$ are the solutions of

$$\phi(x * y) = \phi(x) + \phi(y)$$

$$\psi(x * y) = \psi(x).(\psi(y),$$

$\phi$, $\psi$ being : $A \to C$, $\alpha$'s are arbitrary constants.

F. <u>Equation of the type $f(x+y) = \sum_{i=1}^{n} g_i(x)\, h_i(y)$.</u>

Let $F,G,H$: $A \to C$ satisfy

(26) $$F(x * y) = F(x) + F(y) + G(x)\, H(y) + G(y)\, H(x),$$

for $x,y \in A$, where $A$ is an arbitrary Abelian group with operation $*$ and $C$, the set of complex numbers. Then it is known [87] that the following are the only solutions of (5.26).

I. $$F(x) = \phi(x)$$
$$G(x) \quad \text{arbitrary}$$
$$H \equiv 0.$$

II. $$F(x) = \alpha_1 \phi_1(x)^3 + \phi_1(x)\phi_2(x) + \phi_3(x)$$
$$G(x) = \phi_1(x)$$
$$H(x) = 3\alpha\phi_1(x)^2 + \phi_2(x)$$

III. $$F(x) = 2\alpha\, \beta(\psi(x)-1) + \beta\phi_1(x)\, \psi(x) + \phi_2(x)$$
$$G(x) = \alpha\, (\psi(x)-1) + \phi_1(x)\, \psi(x)$$
$$H(x) = \beta\, (\psi(x)-1).$$

IV. $$F(x) = 2\alpha^3\beta(\psi(x)-1) - \alpha\beta\phi_1(x)^2 + \phi_2(x)$$

$$G(x) = \alpha^2(\psi(x)-1) - \alpha\phi_1(x)$$

$$H(x) = \alpha\beta(\psi(x)-1) + \beta\phi_1(x).$$

and

V. $$F(x) = 2\alpha^2\gamma\left[\psi_1(x)-1\right] - 2\beta^2\gamma\left[\psi_2(x)-1\right] + \phi(x)$$

$$G(x) = \alpha\gamma\left[\psi_1'(x)-1\right] + \beta\gamma\left[\psi_2(x)-1\right]$$

$$H(x) = \alpha\left[\psi_1(x)-1\right] - \beta\left[\psi_2(x)-1\right]$$

where $\phi, \phi_1$ respectively $\psi, \psi_1, \psi_2$ are the solutions of

$$\phi(x * y) = \phi(x) + \phi(y)$$

$$\psi(x * y) = \psi(x)\,\psi(y)$$

$(\phi, \psi, \phi_1, \psi_1, \psi_2 : A \to C)$ and $\alpha, \beta, \gamma$ are arbitrary complex constants.

G. (27) $$\phi(x) = \phi(ax)\,\phi(bx).$$

Let $\phi : R \to C$ be twice differentiable at $x = o$ and satisfy (5.27) with $a > o$, $b > o$, $a^2 + b^2 = 1$. If $\phi$ is non-constant, then $\phi(x) = e^{mx^2}$. (refer [88] ).

H. (28). $$f(xy) = g(y)\,f(x) + h(y)\,x + k(y).$$

The general solution of (5.28) bounded on a set of positive measure are [9] , [37]

$$f(x) = a \log |x| + b + c$$

$$f(x) = a x \log |x| + bx + c$$

$$f(x) = a |x|^{d} + bx + c$$

$$f(x) = a |x|^{d} \operatorname{sgn} x + bx + c.$$

I. (29) $$\phi\left[x + y\,\phi(x)\right] = \phi(x)\,\phi(y).$$

The equation (5.29) has continuous solutions which are not differentiable, also measurable and bounded solutions which are not continuous. The continuous solutions of (5.29) are [38], [28], [35], [89], [77],

$$f \equiv 0$$

$$f(x) = cx + 1$$

$$f(x) = \begin{cases} 1 - \frac{x}{x_1}, & x \leq x_1 \\ 0, & x \geq x_1 \end{cases}$$

$$f(x) = \begin{cases} 0, & x \leq x_1 < 0 \\ 1 - \frac{x}{x_1}, & x \geq x_1. \end{cases}$$

The first two solutions are differentiable.

The function $$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

satisfies (5.29). Here $f$ is not continuous, but bounded and measurable.

§ 6. <u>Miscellaneous equations</u>. In this section, some examples of equations belonging to cyclic equations, iterated equations, trigonometric equations etc. are treated.

A. <u>Solution of equations by simple substitution</u>.

a) (1) $f(x+y) + f(x-y) = 2f(x) \sin y$, where $f: \mathbb{C} \to \mathbb{C}$, $\mathbb{C}$, complex numbers.

Here $f$ is a solution if and only if $f \equiv o$.

$f \equiv o$ is evidently a solution.

Put $x = o$, $y = o$. Then we have $f(o) = o$.

Put $x = o$ and use $f(o) = o$. We have $f(y) = -f(-y)$,

$f$ is odd.

Interchanging $x$ and $y$, we get

$$f(x+y) + f(y-x) = 2f(y) \sin x.$$

Since $f$ is odd,

$$f(x+y) - f(x-y) = 2f(y) \sin x$$

Hence, $$f(x+y) = f(x) \sin y + f(y) \sin x.$$

Putting $y = o$, we obtain, $f(x) = o$.

Hence $f \equiv o$ is the only solution of the above equation (1).

b) (2) $f(x+y) + f(x-y) = 2f(x) \cos x$, $f: \mathbb{C} \to \mathbb{C}$

Here again $f \equiv o$ is the only solution of (6.2)

Putting $x = o$, we get

(3) $$f(y) + f(-y) = 2f(o).$$

Putting $x = \frac{3x}{2}$ and replacing $y$ by $y + \pi$ in (6.2), we have

$$f(y + \frac{5x}{2}) + f(\pi/2 - y) = 0.$$

Putting $x = \pi/2$ in (6.2) we have $f(y+\pi/2) + f(\pi/2-y) = o$.

Hence, $f(y + \frac{5\pi}{2}) = f(y+\pi/2)$. that is

$f(y + 2\pi) = f(y)$, $f$ is periodic with period $2\pi$.

Putting $y = \pi$ in (6.3), we have $f(\pi) + f(-\pi) = 2\,f(o)$.

Hence $f(\pi) = f(o)$, since $f(-\pi) = f(\pi)$.

Set $x = \pi$, $y = \pi$ in (6.2). We have

$$f(2\pi) + f(o) = -2f(\pi) = -2f(o).$$

Therefore $\quad f(o) = o.$

From (6.3), we see that $f(-y) = -f(y)$, $f$ is odd.

$x = \pi$ in (6.2) gives, $f(\pi+y) + f(\pi-y) = o$, that is,

$$f(\pi+x) - f(x-\pi) = o.$$

$y = \pi$ in (6.2) gives $\quad f(x+\pi) + f(x-\pi) = 2\,f(x)\cos x.$

Thus, we have $\quad f(x+\pi) = f(x)\cos x.$

Replacing $x$ by $x + \pi$ and using the periodiarity Af, we get

$$f(x) = -f(x+\pi)\cos x$$

$$= -f(x)\cos^2 x, \text{ for all } x. \tag{4}$$

Putting $y = o$ in (6.2) we get either

$$f(x) = o \text{ or } \cos x = 1.$$

Hence whenever $\cos x \neq 1$, $f(x) = o$.

Now, let $\cos x_o = 1$.

Then from (6.4), we get $f(x_o) =-f(x_o)$, that is $f(x_o) = o$.

Hence $f(x) = o$ for all $x$.

c) (5) $f(x+y) + f(x-y) = 2f(x) \cos y$, $f : \mathbb{C} \to \mathbb{C}$.

$x = o$ in (6.5) gives

$$f(y) + f(-y) = 2 f(o) \cos y.$$

So, $$f(y+\pi/2) + f(-y-\pi/2) = - 2 f(o) \sin y.$$

Put $x = \pi/2$ in (6.5),

$$f(\pi/2 + y) + f(\pi/2 - y) = 2 f(\pi/2) \cos y.$$

Hence $$f(-y-\pi/2) - f(-y+\pi/2) = -2f(o) \sin y - 2f(\pi/2) \cos y$$

or $$f(y-\pi/2) - f(y+\pi/2) = a_1 \sin y + b_1 \cos y.$$

$x = y$, $y = \pi/2$ in (6.5) gives $f(y+\pi/2) + f(y-\pi/2) = o$.

So, $f(y-\pi/2)\ a_2 \sin y + b_2 \cos y$.

Hence $$f(y) = a \sin y + b \cos y.$$

Every solution of (6.5) is $f(x) = a \sin x + b \cos x$.

d) (6) $f(x+y) + f(x-y) = 2f(x) + 2f(y)$, $f: R \to R$.

$x = o, y = o$ in (6.6) gives $f(o) = o$.

$x = o$ in (6.6) gives $f(-y) = f(y)$, $f$ is even.

By induction, let us prove that $f(nx) = n^2 f(x)$, $n$ an integer.

Put $y = nx$ in (6.6)

$$f((n+1)x) + f(-(n-1)x) = 2f(x) + 2f(nx)$$
$$f((n+1)x) = 2f(x) - (n-1)^2 f(x) + 2n^2 f(x)$$
$$= (n+1)^2 f(x).$$

Hence $\qquad f(nx) = n^2 f(x).$

Putting $x = 1$, $\quad f(x) = f(1)n^2 = cn^2.$

Similarly, we have $f(\frac{m}{n}) = c(\frac{m}{n})^2$, that is, for any rational $m$ $\qquad f(r) = cr^2.$

If $f$ is continuous, then $f(x) = cx^2$, for all $x \in R$, is the only solution of (6.6), where $c$ is any constant.

B. Iterated equations.

The equation of the type

(7) $f^n(x) = g\left[f^m(x)\right]$, $m$, $n$ integers ($f^n$ denotes the $n$-th iterate of $f$) belongs to this category. The equation (6.7) for $m < n$, can be reduced to the equation [60] .

(8) $$f^n(x) = g(x),$$

which is a generalization of the well-known Babbage equation.

(9) $$f^n(x) = x.$$

It has been proved [90] , that the general continuous solution of (6.9) for odd $n$ is $f(x) \equiv x$, for even $n$, continous solutions of (6.9) are the continuous solutions of $f^2(x) = x$. Further every continuous solution of (6.9) is strictly monotonic. For $g = f$ in (6.8), the following two classes of functions occur as solutions [32] .

Class 1. i) The function $f$ is continuous for all real $x$

ii) $f(x) = x$ on a connected subset $s$ of the x-axis and

iii) $g$ $L \leq f(x) \leq M$, $L, M$ being the infimum and supremum of $f$ on $s$.

Class 2. iv) The function $f$ is continuous for all real $x$ and either

v) $f^2(x) = x$

or vi) $f^2(x) = x$ on a non-degenerate closed interval $[a,b]$ , $f(a) = b$, $f(b) = a$, $a \leq f(x) \leq b$.

The general solution of (6.8) has been constructed by G.Lojasiewicz [71] and the general continuous solution under the assumption that $g$ is monotonic been constructed by Kuczma [61] .

For $n = 2$ in (6.8) it has been proved by Thron, W.J. [82] , that, when $g$ is an entire function of finite order, which is not a polynomial and which takes on a certain value $P$ only a finite number of times, (6.8) does not have a solution $f$ which is an entire function. A stronger version of the above result was proved by R.Osserman [73] , for $g(z) = e^{z-1}$ and $n = 2$ in (8). Let $z = x+iy$ and let $\Omega$ denote an infinite strip $|y| < b$, for some constant $b > \pi$. Let $f$ be a function defined in some domain $D$ containing $\Omega$. If $f$ satisfies in $D$, then $f$ cannot be analytic.

C. Method of determinants.

The following results are needed to prove the main theorem in this section. For more details and proof refer [86] , [6]

Notation. Let $Q_o$ be an arbitrary Abelian semigroup and $\mathbb{C}$, the field of complex numbers. Let $F_\nu : Q_o \longrightarrow \mathbb{C}$, $\nu = 1,2, \ldots , n$. Let

$$(10) \quad \Delta[F_1(z_1),F_2(z_2),\ldots,F_n(z_n)] = \begin{vmatrix} F_1(z_1) & F_2(z_1) & \cdots & F_n(z_1) \\ F_1(z_2) & F_2(z_2) & \cdots & F_n(z_2) \\ \vdots & \vdots & \cdots & \vdots \\ F_1(z_n) & F_2(z_n) & \cdots & F_n(z_n) \end{vmatrix},$$

$z_i \in Q_o$.

Lemma 1. The functions $F_1,F_2,\ldots,F_n$ are linearly dependent, that is,

$$\alpha_1 F_1(z) + \ldots + \alpha_n F_n(z) = o, \quad \sum_1^n |\alpha_\nu| > o$$

if and only if $\Delta\left[F_1(z_1),\ldots,F_n(z_n)\right] = o$ for all $z_1,z_2,\ldots z_n$.

Lemma 2. If $F_\mu(z) = F_\nu(z)$ holds, then

$$\Delta\left[F_1(z_1),\ldots,F_\mu(z_\mu),\ldots,F_\nu(z_\nu),\ldots,F_n(z_n)\right] = o.$$

Lemma 3. Whenever $\sum_{\nu=o}^{k} \Delta[F_{\nu 1}(x_1),\ldots,F_{\nu n}(x_n)] = o$ is true,

then $\sum_{\nu=o}^{k} \Delta\left[F_{\nu 1}(x_1),\ldots, F_{\nu n}(x_n), F(x_{n+1})\right] = o$ is also true

for arbitrary $F : Q_o \longrightarrow \mathbb{C}$.

The second equation is a result of 'enlarging' the first equation by F. This lemma will play the important role in proving many results in this direction.

<u>Lemma 4.</u> Let $\xi \in Q_o$ be an independent parameter from the variables $z_1, z_2, \ldots, z_n$. Then from

$$\Delta\left[F_1(z_1,\xi), F_2(z_2), \ldots, F_n(z_n)\right] = o$$

follows at least one of the equations

$$F_n(z) \equiv o$$

$$F_\nu(z) = \alpha_{\nu+1} F_{\nu+1}(z) + \ldots + \alpha_n F_n(z), \quad = 2,3,.., n-1.$$

$$F_1(z_1, \zeta) = \alpha_2(\zeta) F_2(z) + \ldots + \alpha_n(\zeta) F_n(z),$$ is true.

The main theorem in this section is the following:

Let $f\colon Q_o \longrightarrow \mathbb{C}$. Then the following two equations are equivalent,

(11) $$f(z_1 * z_2)^n = \left[f(z_1) + f(z_2)\right]^n$$

(12) $$f(z_1 * z_2) = f(z_1) + f(z_2),$$

where $z_1, z_2, z_1 * z_2 \in Q_o$. That is, any solution of (6.11) is also a solution of (6·12) and conversely.

Proof [41] . It is evident that any solution of (6.12) is also a solution of (6.11). So, to prove that any solution of (6.11) is also a solution of (6.12). Further $f \equiv o$ is a solution of both (6.11) and (6.12). So, let us assume that $f \not\equiv o$. Using the associativity of $*$ , we have

$$f\left[(z_1 * \zeta) * z_2\right]^n = f\left[z_1 * (\zeta * z_2)\right]^n .$$

Now from (6.11), we have

$$\sum_{\nu=o}^{n} \binom{n}{\nu} f(z_1)^{n-\nu} f(\zeta)^{\nu} + \sum_{\mu=1}^{n} \binom{n}{\mu} f(z_1 * \zeta)^{n-\mu} f(z_2)^{\mu}$$

$$= \sum_{\nu=o}^{n-1} \binom{n}{\nu} f(z_1)^{n-\nu} f(\zeta * z_2)^{\nu} + \sum_{\mu=o}^{n} \binom{n}{\mu} f(\zeta)^{n-\mu} f(z_2)^{\mu}.$$

Using commutativity of $*$ , we have

$$\sum_{\mu=1}^{n-1} \binom{n}{\mu} \left[f(z_1 * \xi)^{n-\mu} - f(\xi)^{n-\mu}\right] f(z_2)^{\mu} -$$

$$\sum_{\nu=1}^{n-1} \binom{n}{\nu} \left[f(z_2 * \xi)^{n-\nu} - f(\xi)^{n-\nu}\right] f(z_1) = o,$$

that is,

$$(13) \qquad \sum_{\mu=1}^{n-1} \binom{n}{\mu} \Delta \left[f(z_1 * \xi)^{n-\mu} - f(\xi)^{n-\mu}, f(z_2)^{\mu}\right] = o.$$

Enlarging (6.13) with $f(z)$, $f(z)^2$, ... , $f(z)^{n-2}$ and using the lemmas, we obtain, for $n \geq 3$,

$$\sum_{\mu=2}^{n-1} \binom{n}{\mu} \Delta \left[ f(z_1 * \xi)^{n-\mu} - f(\xi)^{n-\mu},\ f(z_2)^{\mu},\ f(z_3) \right] = o$$

$$\sum_{\mu=3}^{n-1} \binom{n}{\mu} \Delta \left[ f(z_1 * \xi)^{n-\mu} - f(\xi)^{n-\mu},\ f(z_2)^{\mu}, f(z_3)^2, f(z_4) \right] = o$$

$$(14) \quad \binom{n}{n-1} \Delta \left[ f(z_1 * \xi) - f(\xi),\ f(z_2)^{n-1},\ f(z_3)^{n-2}, \ldots, f(z_n) \right] = o.$$

The equation (14) holds when either

$$f(z) \equiv o\ ,$$

or (15) $$f(z)^{n-\nu} = \sum_{\mu=1}^{n-\nu-1} \alpha_{\mu}\ f(z)^{\mu}, \quad \nu = 1,2,\ldots, n-1,$$

or

$$(16) \quad f(z_1 * \xi) - f(\xi) = \sum_{\mu=1}^{n-1} \alpha_{\xi}\ (\xi)\ f(z)^{\mu},$$

with $\sum_{\mu=1}^{n-2} |\alpha_{\mu}| > o$ and $\sum_{\mu=1}^{n-1} |\alpha_{\mu}(\xi)| > o$. $f(z)$ cannot be identically zero by assumption. (6.15) implies that the functions $f(z)$, . $f(z)^2$, .. $f(z)^{n-2}$ are linearly dependent and hence $f \equiv o$ (refer [ ]) which cannot be by assumption. Hence only (6.16) is true.

Again, enlarging (6.13) with $f(z)^{n-1}$, $f(z)^{n-2}$, . . $f(z)$, we get

(18) $$f(z_1 * \zeta)^{n-1} - f(\zeta)^{n-1} = \sum_{\nu=1}^{n-1} \beta_\nu(\zeta)\, f(z)^\nu \quad , \text{ with}$$

$$\sum_{\nu=1}^{n+1} \left| \beta_\nu(\zeta) \right| > 0.$$

From (6.16) and (6.18), we have

(19) $$\left[f(\zeta) + \sum_{\mu=1}^{n-1} \alpha_\mu(\zeta)\, f(z)^\mu\right]^{n-1} = f(\zeta)^{n-1} + \sum_{\nu=1}^{n-1} \left[\beta_\nu(\zeta) \cdot f(z)^\nu\right].$$

On account of the linear independence of the powers of $f$, comparing the corresponding coefficients, we have

$$\alpha_\mu(\zeta) = 0, \ \mu \geq 2, \quad \text{thus}$$

(20) $$f(z * \zeta) = \alpha_1(\zeta)\, f(z) + f(\zeta).$$

To determine $\alpha_1(\zeta)$.

Interchanging $z$ and $\zeta$ in (6.20), we obtain

$$\left[\alpha_1(\zeta) - 1\right] f(z) = \left[\alpha_1(z) - 1\right] f(\zeta).$$

Since $f \equiv 0$, we have

$$\alpha_1(\zeta) - 1 = a\, f(\zeta), \ a = \frac{\alpha_1(z_0) - 1}{f(z_0)}, \ f(z_0) \neq 0.$$

Now (6.20) becomes

$$f(z * \zeta) = a\, f(\zeta)\, f(z) + f(z) + f(\zeta)$$

From this equation and (6.11) for $\zeta = z$, we have

$$\left[a\, f(z)^2 + 2f(z)\right]^n = \left[2f(z)\right]^n.$$

As before, comparing the corresponding coefficients, we get $a = o$ and hence $\alpha_1(\zeta) = 1$. Thus from (6.20), we have

$$f((z) * (\zeta)) = f(z) + f(\zeta),$$ which is wanted to be proved.

D. <u>Uniqueness theorem.</u>

There exists at most one continuous function $f$ satisfying the functional equation

$$f,(F(x,y)) = H(f(x),f(y)) \tag{21}$$

for all $x,y \in (A,B)$ and the initial conditions $f(a) = c, f(b) = d$ $(a,b \in (A,B))$, if $F$ is continuous in $(A,B) \times (A,B)$; and $F(x,y)$, $H(u,v)$ are strictly increasing or strictly decreasing in $x,y$ in $(A,B)$, respectively in $u,v$ in $(f(A), f(B))$, when $(A,B)$ is a closed, half-closed, open, finite or infinite interval.

<u>Proof.</u> [14] . Define

$$g(x) = F(x,x). \tag{22}$$

Then $g$ takes every value assumed by $F(x,y)$. Indeed, if $F$ is increasing in both variables, we have

$$g(A) = F(A,A) \leq F(x,y) \leq F(B,B) = g(B).$$

Thus, $g$ being continuous, $g$ assumes the value $F(x,y)$ and also $g$ is strictly increasing.

Hence the inverse $g^{-1}$ exists on $(F(A,A), F(B,B))$ and

(23) $$G(x,y) = g^{-1}(F(x,y)), \quad \text{for } x,y \in (A,B)$$

is well defined, continuous and increases in both variables. Moreover, for $x \leq y$,

$$x = g^{-1}(g(x)) = g^{-1}(F(x,x)) = G(x,x) \leq G(x,y)$$

$$\leq G(y,y) = y, \text{ that is } G \text{ is intern.}$$

Putting $x = y = G(s,t)$ in (6.21), we have

(24) $$f_{,}(F(G(s,t), G(s,t))) = H(f(G,(s,t)), f(G(s,t))).$$

Now, set

(25) $$h(x) = H(x,x).$$

From (6.22), (6.23), (6.24) and (6.25), we obtain

$$f_{,}(F(s,t)) = h(f[G(s,t)]).$$

This by (6.21), becomes

(26) $$H(f(x),f(t)) = h(f[G(s,t)]).$$

Now define

(27) $$K(u,v) = h^{-1}(H(u,v)).$$

$h^{-1}$ exists, since $h$ defined by (25) is strictly monotonic in u. From (6.26) and (6.27), we have

$$(28) \qquad f\left[G(s,t)\right] = K(f(s),f(t)), \text{ where}$$

$K$ is strictly monotonic in $u$ and $v$.

This equation (6.28) is of the form

$$f(F(x,y)) = H(f(s),\ f(y),x,y)$$

and satisfies all conditions of Theorem 1 in [10] . Hence the proof of this theorem is complete.

E. Cyclic functional equation.

An equation of the type

$$(29) \quad F(x_1,x_2,\ldots,x_p) + F(x_2.x_3,\ldots,x_p,x_{p+1}) + \ldots + F(x_{n-p+1},x_{n-p+2},\ldots,x_n) + F(x_{n-p+2},x_{n-p+3}\ldots,x_n,x_1) + \ldots + F(x_n,x_1,\ldots,x_{p-1}) = 0$$

where $p$ and $n(> p)$ are two arbitrary positive integers, is a cyclic functional equation. We shall consider this equation (6.29) later. First we shall consider a particular case of (6.29) for $n = 3$, $p = 2$ known as Sinzov's functional equation.

Sinzov's functional equation. The functional equation

$$(30) \qquad F(x,y) + F(y,z) = F(x,z)$$

is called the Sinzov's functional equation.

THEOREM 1. The general solution of (6.30) is $F(x,y) = g(y) - g(x)$, where $g$ is an arbitrary function.

Proof. Put $x = c$ in (6.30) and define $g(x) = -F(x,c)$. Then we have

$$F(x,y) + F(y,c) = (x,c).$$

Hence (6.31) $$F(x,y) = g(y) - g(x).$$

Obviously $F(x,y)$ defined by (6.31) satisfies (6.30).

Remark. The equation (6.30) can be considered as a generalization of (1.6). For, letting

$$F(x,y) = f(x-y), \tag{32}$$

the equation (6.30) reduces to $f(x-y) + f(y-z) = f(x-z)$. Putting $z = o$ and replacing $x$ by $x + y$ in the above equation, we get

$$f(x) + f(y) = f(x+y) \text{ which is (1.6).}$$

The solution of (1.6) does not follow from that of (6.30). For, from (6.31) and (6.32), we have

$$f(x-y) = g(y) - g(x)$$

which is the Pexider equation (1.10). This illustrates that the solution of a particular functional equation may be more difficult than that of a general one. Now let us take up (6.29). The following theorems are proved under the following assuptions [6] , [29] , [12] .

(i) $x_i \in S$, where $S$ is an arbitrary non-empty set.

(ii) The values of $F$ lie in an additive Abelian group $G$.

(iii) The group $G$ is such that $mx = s$, $(x, s \in G)$ has a unique solution $x = s/m$ for every $m \leq n$, $m$ an integer.

THEOREM 2. For $n \geq 2p-1$, the general solution of the functional equation (6.29), under the hypothesis (i) and (ii), is

$$(33)\quad F(x_1, x_2, \ldots, \ldots, x_n) = f(x_1, x_2, \ldots, x_{p-1}) - f(x_2, x_3, \ldots, x_p) + A$$

where $f$ is an arbitrary function and $A$ an arbitrary element of $G$ such that $nA = 0$.

Proof. Set $x_{p+1} = x_{p+2} = \ldots = x_n = c$ (constant) in (6.29).

Then we have

$$(34)\quad F(x_1, x_2, \ldots, x_p) + F(x_2, x_3, \ldots, x_p, c) \ldots, \ldots\ldots +$$

$$F(x_p, c, \ldots, c) + (n-2p+1)\, F(c, c, \ldots, c) + F(c, c, \ldots, c, x_1)$$

$$+ F(c, \ldots, c, x_1, x_2) + \ldots + F(c, x_1, \ldots, x_{p-1}) = 0.$$

Putting $x_p = c$ in (6.34), we obtain

$$(35)\quad F(x_1, \ldots, x_{p-1}, c) + F(x_2, \ldots, x_{p-1}, c, c) + \ldots\ldots$$

$$F(x_{p-1}, c, \ldots, c) + (n-2p+2)\, F(c, c, \ldots, c) + F(c, c, \ldots, c, x_1)$$

$$+ F(c, c, . c, x_1, x_2) + \ldots + F(c, x_1, \ldots, x_{p-1}) = 0.$$

Subtracting (6.35) from (6.34), we get

$$(36)\quad F(x_1,x_2,\ldots,x_p) = F(x_1,\ldots,x_{p-1},c) - F(x_2,\ldots,x_p,c) + F(x_2,\ldots,x_{p-1},c,c) - F(x_3,\ldots,x_p,c,c) + \ldots + F(x_{p-1},c,\ldots,c) - F(x_p,c,\ldots,c) + F(c,c,\ldots,c)$$

Now let

$$(37)\quad f(x_1,x_2,\ldots,x_{p-1}) = F(x_1,x_2,\ldots,x_{p-1},c) + F(x_2,\ldots,x_{p-1}.c.c) + \ldots + F(x_{p-1},c,c,\ldots,c)$$

and

$$(38)\qquad A = F(c,c,\ldots,c).$$

Then from (6.36), (6.37) and (6.38), we obtain

$$F(x_1,x_2,\ldots,x_p) = f(x_1,x_2,\ldots,x_{p-1}) - f(x_2,\ldots,x_p) + A$$

which is precisely (6.33).

Putting $x_1 = c = x_2 = . = x_n$ in (6.29) and using (6.38), it is easy to see that $nA = o$.

THEOREM 3. For $n = 2p - 2 > p$ and $m = 2$, the general solution of (6.29) under the hypotheses (i), (ii) and (iii) is

$$(39)\quad F(x_1,x_2,\ldots,x_p) = f(x_1,x_2,\ldots,x_{p-1}) - f(x_2,\ldots,x_p) + G_1(x_1,x_p) - G_1(x_p,x_1) + A$$

with $nA = o$.

<u>Proof.</u> Put $x_{p+1} = x_{p+2} = \dots = x_n = c$ in (6.29). Then we get

$$(40)\quad F(x_1,x_2,\dots,x_p) + F(x_2,\dots,x_p,c) + \dots + F(x_{p-1},x_p,c,\dots,c)$$
$$+ F(x_p,c,\dots,x_1) + F(c,c,\dots,x_1,x_2) + \dots + F(c,x_1,\dots,x_{p-1}) = 0.$$

Putting $x_p = c$ in (6.40), we have

$$(41)\quad F(x_1,\dots,x_{p-1},c) + F(x_2,\dots,x_{p-1},c,c) + \dots + F(x_{p-1},c,\dots c) + F(c,c,\dots,c,x_1) + F(c,\dots,c,x_1,x_2) + \dots + F(c,x_1,x_2,\dots,x_{p-1}) = 0.$$

Subtracting (6.41) from (6.40), we have

$$(42)\quad F(x_1,x_2,\dots,x_p) = \big[F(x_1,\dots,x_{p-1},c) - F(x_2,\dots,x_p,c)$$
$$+ F(x_2,\dots,x_{p-1},c,c) - F(x_3,\dots,x_p,c,c) + \dots$$
$$+ F(x_{p-2},x_{p-1},c,\dots,c) - F(x_{p-1},x_p,c,\dots,c)\big] +$$
$$F(x_{p-1},c,\dots,c) - F(x_p,c,\dots,c,x_1) + F(c,c,\dots,c,x_1).$$

In (6.29), first set $x_2 = x_3 = \dots = x_{p-1} = x_{p+1} = \dots = x_n = c$, then set $x_2 = x_3 = \dots = x_{p-1} = x_p = x_{p+1} = . = x_n = c$ and then $x_1 = x_2 = \dots = x_{p-1} = x_{p+1} = x_{p+2} = \dots = x_n = c$ respectively, we obtain

$$(43)\quad F(x_1,c,\dots,c,x_p) + F(c,\dots,c,x_p,c) + F(c,c,\dots,x_p,c,c) + \dots + F(c,x_p,c,\dots,c) + F(x_p,c,\dots,c,x_1) + F(c,\dots,c,x_1,c) + F(c,c,\dots,x_1,c,c) + \dots + F(c,x_1,c,\dots,c) = o.$$

$$(44)\quad F(x_1,c,\dots,c) + F(c,x_1,c,\dots,c) + \dots + F(c,c\dots,c,x_1) + (p-2)\, F(c,c,\dots,c) = o.$$

and

$$(45)\quad F(x_p,c,\dots,c) + F(c,x_p,c,\dots,c) + \dots + F(c,c,\dots,c,x_p) + (p-2)\, F(c,c,\dots,c) = o.$$

Adding (6.44) and (6.45) and then subtracting it from (6.43) and using $n\,F(c,c,\dots,c) = o$, we have

$$(46)\quad F(x_1,c,\,.,\,c,x_p) + F(x_p,c,\dots,c,x_1) - F(x_1,c,\,.\,,c) - F(c,\dots,c,x_1) - F(x_p,c,\,.\,,c) - F(c,c,.,c,x_p) + 2F(c,c,.,c) = o.$$

From (6.42) and (6.46), we have

$$2F(x_1,x_2,..,x_p) = 2\left[F(x_1,x_2,..,x_{p-1},c) - F(x_2,...,x_p,c) + F(x_2,...,x_{p-1},c,c) - F(x_3,...,x_p,c,c) + ..... F(x_{p-2},x_{p-1},c,...,c) - F(x_{p-1},x_p,c,...,c)\right] + 2F(x_{p-1},c,...,c) - 2F(x_p,c,...,c,x_1) + 2f(c,c,..c,x_1) + F(x_1,c,.,c,x_p) + F(x_p,c,...,c,x_1) - F(x_1,c,...,c,) - F(c,...,c,x_1) - F(x_p,c,...,c) - F(c,...,c,x_p) + 2\,F(c,c,...,c)$$

$$= 2\left[F(x_1,x_2,...,x_{p-1},c) - F(x_2,...,x_p,c) + F(x_2,..,x_{p-1},c,c) - F(x_3,...,x_p,c,c) + ... F(x_{p-1},c,...,c) - F(x_p,c,\,.\,,\,c)\right] + \left[F(x_1,c,...,c,x_p) - F(x_p,c,...,c,x_1) + F(c,c,...,c,x_1) - F(c,c,...,c,x_p) + F(x_p,c,...,c) - F(x_1,c,...,c)\right] + 2F(c,...,c).$$

Dividing by 2 (which is permissible by the hypothesis (iii)), we obtain the required result (6.29). This completes the proof of this theorem.

F. Trigonometric equations.

Sine equation. THEOREM 1. Let $f : R \to R$ and such that

(47) $f(x+y)\ f(x-y) = f(x)^2 - f(y)^2$, holds for all

$x, y \in R$. The general system of continuous solutions of (6.47) is

$$f(x) = cx$$
$$f(x) = A \sin cx$$
$$f(x) = A \sinh cx, \qquad A,\ c \text{ real.}$$

Proof. [69]. Evidently $f \equiv 0$ is a solution of (1). So, we exclude this trivial solution in the following considerations. Since $f \not\equiv 0$, there exists $a$ and $b$ such that

$$k = \int_a^b f(x)\ dx \neq 0. \tag{48}$$

From (6.47) and (6.48), we get

$$k\ f(y) = \int_a^b f(y)\ f(x)\ dx$$

$$= \int_a^b f\left(\frac{x+y}{2}\right)^2 dx - \int_a^b f\left(\frac{x-y}{2}\right)^2 dx$$

$$= \int_{\frac{a+y}{2}}^{\frac{b+y}{2}} f(x)^2\ dx - \int_{\frac{a-y}{2}}^{\frac{b-y}{2}} f(x)^2\ dx,$$

from which follows that, i) $f$ is odd, ii) $f$ is differentiable on $R$, iii) $f'$ is a linear combination of $f(\frac{a+y}{2})^2$ and $f(\frac{b+y}{2})^2$ and hence $f$ has derivatives of all orders.

Now differentiating (6.47) with respect to $y$ twice and then setting $y = o$, we have

$$(48) \qquad f(x)\, f''(x) - f'(x)^2 = c, \text{ where } c = -\frac{1}{2}\left[f(y)\right]^2_{y=o} .$$

Differentiating (6.48) with respect to $x$, we have

$$f(x)\, f'''(x) = f'(x)\, f''(x).$$

Hence

$$f(x) = cx, \; f(x) = A \sin cx, \; f(x) = A \sinh cx, \; A, c \text{ real}.$$

Equation (6.47) can also be solved by reducing it to the well known cosine equation, as follows.

Since $f \not\equiv o$, there is an '$a$' such that $f(a) \neq o$.

Define

$$(49) \qquad g(x) = \frac{f(x+a) - f(x-a)}{2\, f(a)} .$$

Using (6.47) and (6.49), we get

$$g(x+y) + g(x-y) = \frac{1}{2f(a)}\left[f(x+y+a)-f(x+y-a)+f(x-y+a)-f(x-y-a)\right]$$

$$= \frac{1}{2f(a)^2}\Big[f(x+y+a)f(a)+f(x-y+a)f(a) -$$

$$-f(x+y-a)f(a)-f(x-y-a)f(a)\Big]$$

$$= \frac{1}{2f(a)^2}\Big[f(\tfrac{x+y}{2}+a)^2-f(\tfrac{x+y}{2})^2 + f(\tfrac{x-y}{2}+a)^2$$

$$- f(\tfrac{x-y}{2})^2 - f(\tfrac{x+y}{2})^2 + f(\tfrac{x+y}{2}-a)^2 -$$

$$- f(\tfrac{x-y}{2})^2 + f(\tfrac{x-y}{2}-a)^2\Big]$$

$$= \frac{1}{2f(a)^2}\Big[f(\tfrac{x+y}{2}+a)^2 - f(\tfrac{x-y}{2})^2 + f(\tfrac{x-y}{2}+a)^2$$

$$- f(\tfrac{x+y}{2})^2 + f(\tfrac{x+y}{2}-a)^2 - f(\tfrac{x-y}{2})^2 + f(\tfrac{x-y}{2}-a)^2$$

$$- f(\tfrac{x+y}{2})^2\Big]$$

$$= \frac{1}{2f(a)^2}\Big[f(x+a)\ f(y+a) - f(x-a)\ f(y-a)$$

$$+ f(x-a)\ f(y-a) - f(x-a)\ f(y+a)\Big]$$

$$= \frac{1}{2f(a)^2}\left[f(x+a) - f(x-a)\right]\left[f(y+a) - f(y-a)\right]$$

(50) $= 2g(x)\ g(y)$, which is the cosine equation.

The continuous solutions of (6.50) are $f(x) \equiv o$, $f(x) \equiv 1$, $f(x) = \cos cx$, $f(x) = \cosh c\,x$, c real.

The right side of (6.49) is independent of the choice of 'a', [92] . That is, for all $y$ such that $f(y) \neq o$, we have

$$g(x) = \frac{f(x+y) - f(x-y)}{2\,f(y)} .$$

Thus

(51) $$f(x+y) - f(x-y) = 2g(x)\,f(y), \text{ whenever } f(y) \neq o.$$

Putting $x = o$, in (6.51) and using $g(o) = 1$, we have

(52) $$f(-y) = -f(y), \quad f \text{ is odd.}$$

The equation (6.51) now becomes

(53) $$f(y+x) + f(y-x) = 2\,f(y)\,g(x).$$

Now taking $g(x) = \cos c\,x$ and using (6.5), we have

$$f(x) = B\cos cx + A'\sin cx.$$

$f$ being odd, we get $B = o$ and thus $f(x) = A\sin c\,x$.

If $g(x) \equiv 1$, (6.5) reduces to $f(\frac{x+y}{2}) = \frac{f(x) + f(y)}{2}$

and thus $f(x) = cx + d$. Equation (5.2) implies $d = o$ and hence $f(x) = cx$. Similarly from $g(x) = \cosh cx$, $f(x) = A$ in $cx$ can be obtained.

THEOREM 2. Let $f, \phi : R \rightarrow \mathbb{C}$ such that

(53) $$\phi(x+y) = f(x)\,\phi(y) + \phi(x)\,f(y).$$

Further let $f$ and $\phi$ be differentiable.

Then $\phi(x) = A\,(e^{cx} - e^{dx})$ and $f(x) = \frac{1}{2}(e^{cx} + e^{dx})$,

where $A, c, d$ are constants, [1] , [24] .

<u>Proof.</u> Differentiating (6.53) with respect to $x$, we have

$$\phi'(x+y) = f'(x)\,\phi(y) + \phi'(x)\,f(y)$$

also $= f(x)\,\phi'(y) + \phi(x)\,f'(y)$ (differentiating with respect to $y$). Hence

(54) $$f'(x)\,\phi(y) - f(x)\,\phi'(y) - f'(y)\,\phi(x) + f(y)\,\phi'(x) = o.$$

Further let $f(o) = 1$ and $\phi(o) = o$.

Then from (6.54) with $y = o$, we get

$$-\phi'(o)\,f(x) - f'(o)\cdot\phi(x) + \phi'(x) = o$$

or

(55) $$f(x) = k_1\phi(x) + k_2\phi'(x).$$

From (6.54) and (6.55), we obtain

$$\phi''(x) + a\,\phi'(x) + b\,\phi(x) = o.$$

Thus,

(56) $$\phi(x) = A e^{cx} + B e^{dx}.$$

Now (6.55) and (6.56) yield

(57) $$f(x) = D e^{cx} + F e^{dx}.$$

Making use of (6.53, (6.56) and (6.57), we get

$$2 A D = A, \quad 2 B F = B \quad \text{and} \quad AF + BD = o.$$

Supposeing $A$ and $B$ non-zero, we have $D = \frac{1}{2} = F$ and $A = -B$.

Hence

$$f(x) = \frac{1}{2} (e^{cx} + e^{dx})$$

$$\text{and} \quad \phi(x) = A (e^{cx} - e^{dx}).$$

For $c = -d = i$, $A = \frac{1}{2} i$, $f(x) = \cos x$, $\phi(x) = \sin x$.

For $c = -d = 1$, $A = \frac{1}{2}$, we have $f(x) = \cosh x$, $\phi(x) = \sinh x$.

THEOREM 3. Let $f, \phi, \psi : R \to \mathbb{C}$ such that $f, \phi$, twice differentiable and $f, \phi, \psi$ satisfy

(58) $$\psi(x+y) = \phi(x) f'(y) + f(y) \phi'(x), \quad x, y \in R.$$

Then
$$f(x) = a \sin (bx + c)$$
$$\phi(x) = d \sin (bx + e)$$
$$\text{and} \quad \psi(x) = ad\, b \sin (bx + c + e).$$

Proof [1]. Differentiating (6.58) with respect to $x$, we have

$$\psi'(x+y) = \phi'(x)f'(y) + f(y)\ \phi''(x)$$

$$\text{also} = \phi(x)f''(y) + f'(y)\ \phi'(x),$$

(differentiating (6.58) with respect to $y$. Hence

(59) $$\phi(x)\ f''(y) - f(y)\ \phi''(x) = o.$$

In (6.59), making $x$ constant, we get

(60) $$f(x) = a \sin (bx+c).$$

Similarly making $y$ constant in (6.59), we obtain

(61) $$\phi(x) = d \sin (bx+e).$$

From (6.58), (6.60) and (6.61), we have

$$\psi(x+y) = a\ d\ [\sin (bx+e)\ .\ b \cos (by+c) + \sin (by+c)\ .\ b\ .\ \cos (bx+e)]$$

$$= a\ b\ d\ .\ \sin (b(x+y) + c + e).$$

Thus the theorem is proved.

For $a = d = b = 1,\ c = o = e$, we get

$$f(x) = \sin x,\ \phi(x) = \sin x,\ \psi(x) = \sin x,$$

consequently,

$$\sin (x+y) = \sin x.\ \sin' y + \sin y\ \sin' x.$$

THEOREM 4. Let $f$ and $\phi$ be non-constant real functions such that

(62) $$f(x-y) = f(x)\, f(y) + \phi(x)\, \phi(y).$$

Let $f$ and $\phi$ be differentiable. Then $f(x) = \frac{1}{2}(e^{cx} + e^{-cx})$ and $\phi(x) = \pm \frac{1}{2i}(e^{cx} - e^{-cx})$, c a constant.

Proof [74]. By symmetry of $x$ and $y$ in the right side of (1), we get

$$f(-x) = f(x), \text{ all } x \in R, \text{ that is, } f \text{ is even.}$$

Changing $x$ to $-x$ and $y$ to $-y$ in (6.62) and using $f$ even,

$$\phi(x)\, \phi(y) = \phi(-x)\, \phi(-y).$$

From this we see that, $\phi$ cannot be the sum of an even and an odd function. Further, if $\phi$ is also even, by putting $y = x$ first in (6.62) and then $y = -x$ in (6.62), we get

$$f(o) = f(x)^2 + \phi(x)^2$$

also $= f(2x)$, which is a contradiction since $f$ is nonconstant. Hence $\phi$ is odd. Thus

(63) $$\phi(o) = o.$$

Putting $y = o$ in (6.62), using (6.63) and the fact that $f$ is non-zero, we get

(64) $$f(o) = 1.$$

Letting $y = x$ in (6.62), (6.62), reduces to, using (6.64),

(65) $$f(x)^2 + \phi(x)^2 = 1.$$

Changing $y$ into $-y$ in (6.62), we have

(66) $$f(x+y) = f(x)\,f(y) - \phi(x)\,\phi(y).$$

Replacing $x$ by $x+y$ in (6.62) and using (6.65), (6.63), we get

(67) $$\phi(x+y) = \phi(x)\,f(y) + f(x)\,\phi(y).$$

Here changing $y$ into $-y$, we obtain

(68) $$\phi(x-y) = \phi(x)\,f(y) - f(x)\,\phi(y).$$

Solution of (6.67) by theorem 2 is

$$f(x) = \frac{1}{2}(e^{cx} + e^{dx})$$
$$\text{and } \phi(x) = A\,(e^{cx} - e^{dx}).$$

Substituting these values of $f$ and $\phi$ in (6.62), we get

$$\frac{1}{2}\left[e^{c(x+y)} + e^{d(x-y)}\right] = \left(\frac{1}{4}+A^2\right)\left[e^{c(x+y)} + e^{d(x+y)}\right] + \left(\frac{1}{4} - A^2\right)\left[e^{cx+dy} + e^{dx+cy}\right].$$

From this follows $c = -d$ and $\frac{1}{4} + A^2 = 0$ or $A = \pm \frac{1}{2} i$. Hence

$$f(x) = \frac{1}{2}\,(e^{cx} + e^{-cx})$$
$$\text{and} \quad \phi(x) = \pm \frac{1}{2}\, i\,(e^{cx} - e^{-cx}).$$

We will derive some further interesting results from the above equations. The following equations are true:

$$f(2x) = f(x)^2 - \phi(x)^2$$

$$\phi(2x) = 2\phi(x)\, f(x)$$

$$\phi(2x) \pm \phi(2y) = 2\phi(x \pm y) \,.\, f(x \mp y)$$

$$f(2x) + f(2y) = 2f(x+y)\, f(x-y)$$

$$f(2x) - f(2y) = -2\phi(x+y)\,\phi(x-y).$$

Setting $\psi(x) = \dfrac{\phi(x)}{f(x)}$, we get

$$\psi(x+y) = \frac{\psi(x) + \psi(y)}{1-\psi(x)\psi(y)}$$

$$\psi(x-y) = \frac{\psi(x) - \psi(y)}{1+\psi(x)\,\psi(y)} \,.$$

Suppose there is a $t \neq o$ such that $\phi(t) = 1$. Then from (6.65), $f(t) = o$. $f$ and $\phi$ are periodic with period $4t$. Indeed, putting $y = t$ in (6.67) and (6.68), we get

$$\phi(x+t) = f(x)$$

$$\phi(x-t) = -f(x).$$

Hence $f(x) = \phi(x+t) = -f(x+2t) = f(x+4t)$.

Similarly $\phi(x) = \phi(x+4t)$.

Let

(69) $$\lambda(x) = f(x) + i\,\phi(x).$$

The equations (6.66), (6.67) and (6.69) yield

$$\lambda(x+y) = \lambda(x) \cdot \lambda(y).$$

Hence

$$\left[f(x) + i\,\phi(x)\right]^n = \lambda(x)^n = \lambda(nx)$$

$$= f(nx) + i\,\phi(nx),$$ general form of

De Moivre's theorem.

By taking $f(x)$, $\phi(x)$, $\psi(x)$ as $\sin x$, $\cos x$, $\tan x$ respectively the above results proved reduce to the standard formulae in circular functions.

G. Vector and matrix equations.

Instead of taking the domain and the range to be real or complex numbers, the domain and range could be $R^n$ (n-dimensional vector space, $n \geq 1$), $\mathbb{C}^n$ $(n \geq 1)$ (n-dimensional complex vector space), $G[n,n]$, square matrices of order $n$ etc. Here we will consider briefly most of the equations we treated before.

THEOREM 1. Let $f : R^n \longrightarrow R^m$ such that

(70) $$f(x+y) = f(x) + f(y), \quad x,y \in R^n.$$

If $f$ is continuous, then

$$f(x) = A\,x, \text{ where } A = (a_{ij}) \text{ is}$$

a $m \times n$ matrix over $R$.

Proof. Let $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$.

Then each $f_j (j = 1,2,\ldots m)$ satisfies

$$f_j(x+y) = f_j(x) + f_j(y), \quad x,y \in R^n.$$

Hence by Theorem 3.10,

$$f_j(x) = a_{j1} x_1 + \ldots + a_{jn} x_n, \text{ where}$$

$$x = (x_1, \ldots x_m) \text{ and } a_{ji} \ ( i = 1,\ldots,n) \text{ are constants.}$$

Thus

$$f(x) = A\,x, \text{ where } A = (a_{ij}) \quad i = 1,2,\ldots m, \ j = 1,2,\ldots n.$$

THEOREM 2. Let $f,g,h\colon R^n \longrightarrow R^m$ such that

(71) $$f(x+y) = g(x) + h(y), \quad x,y \in R.$$

If $f$ is continuous, then

$$f(x) = A\,x + b + c$$
$$g(x) = A\,x + b$$
$$h(x) = A\,x + c,$$

where $A = (a_{ij})$ is a $m \times n$ matrix, $b,c$ are elements in $R^m$.

Proof. As in Theorem 4.1, equation (6.72) can be reduced to (6.70) be the following substitution

$$\phi(x) = f(x) - b - c, \quad b = h(o), \; c = g(o).$$

$$f(x) = g(x) + b$$

$$f(y) = h(y) + c.$$

Then $\phi$ satisfies (6.70) and hence $\phi(x) = Ax$ and the rest follows.

Let $f : G[n,n] \to G[m,m]$. Consider the following equations

(72) $\quad f(X + Y) = f(X) + f(Y), \quad X, Y \in G[n,n]$

(73) $\quad f(X + Y) = f(X) \cdot f(Y)$

(74) $\quad f(X \cdot Y) = f(X) + f(Y)$

(75) $\quad f(X \cdot Y) = f(X) \cdot f(Y).$

These equations had been treated extensively and they have many applications. All measurable solutions of (6.72), (6.73), (6.74) and (6.75) are given by A.Kuwagaki [69]. Under the regularity supposition

$$f(V^{-1} X V) = f(X),$$

for all matrices $V$ which are unitary or orthogonal, S.Kurepa [68] has solved these equations.

<u>The equation (6.75)</u>

For $m = 1$, $n = 2$, Golab [36] proved without any condition on $f$, that every solution of (6.75) is of the form

$$f(X) = \phi(\det X),$$

where $\phi$ is an arbitrary scalar-valued function of a single variable, satisfying (1.9).

This result has been generalized to the case $m = 1$ and $n$ arbitrary by M.Kucharzewski [54] and by M.Hosszu [44].

Here we give the proof due to Hosszu.

THEOREM 3. Let $f: G[n\ n] \to K$, such that

$$f(A\ B) = f(A)\ f(B), \quad A,B \in G[n,n] \tag{75}$$

where $G[n,n]$ denotes the multiplicative semi-group of square matrices of order $n$ over the real or complex field $K$. Then

$$f(A) = \phi\ (\det A),$$

where $\phi$ satisfies (1.9).

Proof. $A = H\ V$, where $H$ is hermitian and $V$ is unitary. Also $H$ and $V$ are equivalent to diagonal matrices. But from (6.75), we see that $f$ is the same for equivalent matrices.

$$f(B^{-1}\ AB) = f(B^{-1})\ .\ f(A)\ f(B) = f(B^{-1})\ .\ f(B)\ f(A)$$
$$= f(A).$$

It is enough to prove the theorem for diagonal matrices.

$$D = \begin{bmatrix} d_1 & o & \dots & o \\ o & d_2 & \dots & o \\ o & o & \dots & d_n \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & & & & \mathbf{0} \\ & 1 & & & \\ & & o & & \\ & & & o & \\ \mathbf{0} & & & & o \end{bmatrix} \begin{bmatrix} 1 & & & & \mathbf{0} \\ & d_2 & & & \\ & & o & & \\ & & & o & \\ \mathbf{0} & & & & o \end{bmatrix} \cdots\cdots\cdots$$

$$= \prod_{k=1}^{n} P_k \begin{bmatrix} d_k & & & \mathbf{0} \\ & 1 & & \\ & & o & \\ \mathbf{0} & & & o \end{bmatrix} P_k^{-1} ,$$

where $P_k$ is obtained from the unit matrix by interchanging the first and the $k^{th}$ rows.

Hence

$$f(D) = \prod_{k=1}^{n} f(P_k) \; f\left( \begin{bmatrix} d_k & & & \mathbf{0} \\ & 1 & & \\ & & o & \\ \mathbf{0} & & & o \end{bmatrix} \right) f(P_k^{-1})$$

$$= f\left( \begin{bmatrix} \prod_{k=1}^{n} d_k & & & \mathbf{0} \\ & 1 & & \\ & & o & \\ \mathbf{0} & & & o \end{bmatrix} \right)$$

$$= \phi \; (\det D), \text{ where } \phi \text{ satisfies } (1.9)$$

Hence the result.

For $m = 2$, $n = 2$, Kucharzewski and Kuczma [55] have proved the following result.

Let $f$ satisfy (6.75) for all non-singular matrices $X, Y$ of order 2. Then, we have

either $\quad f(X) = o$

or $\quad f(X) = \phi(\det X) \cdot C\begin{pmatrix} 1 & o \\ o & o \end{pmatrix} C^{-1}$

or $\quad f(X) = \phi(\det X) \cdot C X C^{-1}$

or $\quad f(X) = G(\det X)$,

where $\quad \phi(x) = \begin{bmatrix} \phi(x) & o \\ o & \phi(x) \end{bmatrix}$ and $\phi \not\equiv o$ is a solution (1.9), $G : R \rightarrow GL_2(R)$ is multiplicative with $G(1) = \begin{pmatrix} 1 & o \\ o & 1 \end{pmatrix}$ and $C$, a non-singular matrix.

For $m \leq n$, Kucharzewski and Zajtz [56] proved the following result. Let $GL_n(R)$ denote the multiplicative group of square matrices of order $n$ over $R$.

Let $f : GL_n(R) \rightarrow GL_m(R)$ and satisfy (6.75).

For $m < n$, $f(X) = \phi(\det X)$, where $\phi$ satisfies (1.9).

For $m = n$, either

$$f(X) = G(\det X), \text{ where } G : R \rightarrow GL_n(R)$$

is multiplicative.

or $\quad f(X) = \phi(\Delta)\, C X C^{-1}$

or $\quad f(X) = \phi(\Delta)\, C (X^{\tau})^{-1} C^{-1}$

where $\phi$ satisfies (1.9) and $C$ is an arbitrary non-singular matrix.

§ 7. Applications. Now we will give some applications of functional equations in vector analysis, analysis, statistics etc.

1) Addition of vectors.

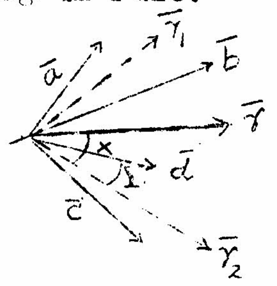
Assumptions. 1) Vectors under addition from an Abelian group.

2) Addition is rotation automorphic; that is, by rotating a pair of vectors, the resultant is rotated through the same amount. This implies that the resultant of two vectors of equal magnitude, lies in the same plane, along the bisector of the angle.

3) The resultant depends continuously upon the magnitude of the vectors and their angle, and

4) parallel vectors are added algebraically.

Conclusion. Conditions 1 to 4 imply the composition of vectors by the parallelogram rule.




Let $\bar{a}$ and $\bar{b}$ and $\bar{c}$ and $\bar{d}$ be two pairs of unit vectors with same included angle $2y$, with resultants $\bar{r}_1$ and $\bar{r}_2$.

By condition 1, since the magnitude of two unit vectors depends only onthe angle between them, let $|\bar{r}_1| = |\bar{r}_2| = 2\, f(y)$. Since the magnitude of the resultant of two vectors of equal magnitude is proportional to the magnitudes of the original vectors, we have,

$$|\bar{r}| = |\bar{r}_1+\bar{r}_2| = 2f(y)\ .\ 2f(x), \quad |\bar{a}+\bar{b}| = 2f(x+y) \quad \text{and}$$

$$|\bar{b}+\bar{d}| = 2f(x-y).$$

By conditions 2 and 4, we obtain

$$|\bar{r}| = |\bar{r}_1+\bar{r}_2| = |\bar{a}+\bar{b}+\bar{c}+\bar{d}| = |\bar{a}+\bar{c}+\bar{b}+\bar{d}|,$$

thus
$$4\, f(x)\, f(y) = 2f(x+y) + 2f(x-y).$$

The only continuous solutions of the above equation, known as, D'Alembert's functional equations or cosine equation or Poisson equation, are

$$f(x) \equiv o$$

$$f(x) = \cos a\, x, \quad a, a \ \text{constant}$$

$$f(x) = \cosh a\, x, \quad a, a \ \text{constant.}$$

Since for two parallel unit vectors, the resultant has magnitude two, we have

$$f(o) = 1.$$

Similarly since the magnitude of the resultant of two antiparallel unit vectors is zero, we have

$$f(\pi/2) = o.$$

Thus $f(x) = o$ and $f(x) = \cosh ax$ cannot be true. Hence $f(x) = \cos ax$, with $a = (2k+1)$, $k = o,1,\ldots$ .

Suppose $k \neq o$. Then $f\left[\frac{\pi}{2(2k+1)}\right] = o$ would imply, two vectors including an angle $\frac{\pi}{2k+1} \neq \pi$ would have the resultant zero, contrary to condition $2(\bar{a} + \bar{b} = \bar{o}$ only if $\bar{a} = -\bar{b})$

Therefore, $f(x) = \cos x$.

Hence , two vectors of equal magnitude $x$ and included angle $2\phi$ , has their resultant along the bisector, with magnitude $2x \cos \phi$.

The general case can be similarly considered.

2. <u>Vector analysis. Definition of scalar (dot) and cross (vector) products</u>.

These products are used to give counterexamples to the well known properties of associativity, commutativity etc. But these products satisfy the distributive laws with regard to addition. With regard to these products, we shall prove the following result. Let us assume that the vectors satisfy the following assumptions:

(1) Products are rotation-automorphic, that is, for a rotation of the space, the scalar product is invariant and the vector product undergoes the same rotation.

(2) $$\left.\begin{aligned}(\bar{A}+\bar{B}) \cdot \bar{C} &= \bar{A} \cdot \bar{C} + \bar{B} \cdot \bar{C},\\ (\bar{A}+\bar{B}) \times \bar{C} &= \bar{A} \times \bar{C} + \bar{B} \times \bar{C}.\end{aligned}\right\} \quad \text{distributivity}$$

(3) $(K\bar{A}) \cdot \bar{B} = \bar{A} \cdot (K\bar{B}) = K(\bar{A} \cdot \bar{B})$,

$(K\bar{A}) \times \bar{B} = \bar{A} \times (K\bar{B}) = K(\bar{A} \times \bar{B})$, k,a scalar.

<u>Conclusion:</u> $\bar{A} \cdot \bar{B}$ and $\bar{A} \times \bar{B}$ are the scalar and vector products to within multiplicative constant.

It is not hard to show from the assumptions that,

$$\text{if } \bar{A} \perp \bar{B}, \text{ then } \bar{A} \cdot \bar{B} = 0$$

$$\text{if } \bar{A} \parallel \bar{B}, \text{ then } \bar{A} \times \bar{B} = 0$$

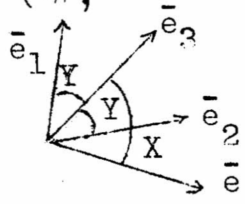
and $\bar{A} \times \bar{B}$ is perpendicular to the plane determined by $\bar{A}$ and $\bar{B}$.

Let $\bar{e}_1, \bar{e}_2, \bar{e}_3$ be unit vectors coplanar with $\bar{e}$, making angles x+y, x-y and x with the direction of $\bar{e}$.

We know that

(i) $$\bar{e}_1 + \bar{e}_2 = 2\bar{e}_3 \cos x.$$

Also, from condition (2),




(ii) $$\begin{cases} (\bar{e}_1 + \bar{e}_2) \cdot \bar{e} = \bar{e}_1 \cdot \bar{e} + \bar{e}_2 \cdot \bar{e} \\ (\bar{e}_1 + \bar{e}_2) \times \bar{e} = \bar{e}_1 \times \bar{e} + \bar{e}_2 \times \bar{e}. \end{cases}$$

(iii) $\begin{cases} (\bar{e}_3 \cdot \bar{e} = f(y) & \text{and} \\ (\bar{e}_3 \times \bar{e} = f(y)\ \bar{i}, & \text{where } \bar{i} \perp \bar{e}_3 \text{ and } \bar{e} \text{ such that} \end{cases}$

$\bar{e}_3, \bar{e}, \bar{i}$

form a right handed system.

Then we have from (i), (ii) and (iii),

$$2f(y) \cos x = f(x+y) + f(x-y).$$

Thus by (6.5) of the miscelleneous equation C, we have

$$f(x) = a \cos x + b \sin x.$$

Since $f(\pi/2) = o$ for the scalar product, we have in this case

$$f(x) = a \cos x.$$

Hence $\bar{A} \cdot \bar{B} = a \left|\bar{A}\right| \left|\bar{B}\right| \cos \theta_1 \quad \theta = \measuredangle(\bar{A},\bar{B})$ .

Since $f(o) = o$ for the vector product, we have in this case

$$f(x) = b \sin x.$$

Hence $\bar{A} \times \bar{B} = b \left|\bar{A}\right| \left|\bar{B}\right| \cos \theta\ \bar{e}$, where $\theta = \measuredangle(\bar{A},\bar{B})$

and $\bar{e} \perp \bar{A}$ and $\bar{B}$ such that $\bar{A},\bar{B},\bar{e}$ form a right handed system.

2) <u>Area of a rectangle</u>. [70] . It is well known that the area of a rectangle of sides $x$ and $y$ is $xy$. Here it is established using Cauchy functional equations.

Let $F: R \times R \longrightarrow R^+$ ($R$, reals; $R^+$, positive reals) be such that $F$ be additive in both variables, that is

(1) $\qquad F(x+u,y) = F(x,y) + F(u,y)$ and

(2) $\qquad F(x,y+v) = F(x,y) + F(x,v)$.

Then $F(x,y) = cxy$, where $c$ is a constant.

For let

$$(3) \qquad C_y(x) = F(x,y).$$

Then by (7.1) and (7.3), $C_y$ satisfies (1.6) and further $C_y$ is positive, since $F$ is. So, by theorem 3.1.

$$(4) \qquad C_y(x) = k(y)x, \text{ where } k(y) \text{ is a constant depending upon } y.$$

From (7.2), (7.3) and (7.4), we see that $k$ satisfies (1.6) and $k$ is positive. Thus,

$$k(y) = cy, \quad \text{where } c \text{ is a constant.}$$

Hence $F(x,y) = cxy$.

The value of $c$ depends on the choice of the area-unit. By choosing the area of the square with unit sides is equal to 1, we obtain $c = 1$.

Remark. $F$ above represents the area of a rectangle of sides $x$ and $y$. The suppositions (7.1) and (7.2) correspond to the area $F$ which depends on the sides $x$ and $y$, is additive in both $x$ and $y$.

4) Analysis. It is well known that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Here a proof based on Cauchy functional equation is given [85].

Here angles are measured in any linear scale, viz degrees etc.

Let

$$(5)\qquad \begin{cases} f_n(x) = 2^n \operatorname{Sin} \dfrac{x}{2^n} \\ g_n(x) = \operatorname{Cos} \dfrac{x}{2} \cdot \operatorname{Cos} \dfrac{x}{2^2} \cdots\cdots \operatorname{Cos} \dfrac{x}{2^n} \end{cases} .$$

We know that

$$(6)\qquad \operatorname{Sin} x = 2 \operatorname{Sin} \frac{x}{2} \cdot \operatorname{Cos} \frac{x}{2} .$$

From (7.5) and (7.6), we have

$$(7)\qquad \operatorname{Sin} x = f_n(x) \cdot g_n(x).$$

$g_n$ is a bounded, decreasing sequence. For, from (7.5),

$$g_n(x) = g(x) \operatorname{Cos} \frac{x}{2^n} < g_{n-1}(x) < 1, \text{ for } o < x < R,$$

(R, numerical value of the right angle).

Hence $\lim_{n\to\infty} g_n(x) = g(x)$ exists and $g(x) < 1$.

Further, let $h_n(x) = g_n(x) \operatorname{Cos} \dfrac{x}{2^n}$ .

Then

$$h_{n-1}(x) = h_n(x) \cdot \frac{\operatorname{Cos} \dfrac{x}{2^{n-1}}}{\operatorname{Cos} \dfrac{x}{2^n}}$$

$$= \frac{h_n(x) \cdot \operatorname{Cos}^2 \dfrac{x}{2^n} - \operatorname{Sin}^2 \dfrac{x}{2^n}}{\operatorname{Cos}^2 \dfrac{x}{2^n}}$$

$$< h_n(x), \text{ for } o < x < R.$$

Thus $\{h_n\}$ is a bounded, increasing sequence and

$$h_n(x) < g_n(x), \quad o < x < R.$$

So,

(8) $$1 > g(x) > h_1(x) = \cos^2 \frac{x}{2} > o.$$

Hence we have from (7.7), that $\lim\limits_{n\to\infty} f_n(x) = f(x) > o$ exists and we have

(9) $$\sin x = f(x)\ g(x).$$

Further,

$$\begin{aligned} f(x+y) &= \lim_{n\to\infty} f_n(x+y) \\ &= \lim_{n\to\infty} 2^n \sin \frac{x+y}{2^n} \\ &= \lim_{n\to\infty} 2^n \left[\sin \frac{x}{2^n} \cos \frac{y}{2^n} + \sin \frac{y}{2^n} \cdot \cos \frac{x}{2^n}\right] \\ &= f(x) + f(y), \text{ since } \cos x \text{ is continuous at zero.} \end{aligned}$$

Since, for $o < x < R$, $f_n(x)$ is increasing and so $f(x)$ is non-decreasing. Hence $f(x) = cx$.

Therefore

(10) $$\sin x = c\ x\ g(x).$$

From (7.8), we have $\lim\limits_{x \to +0} g(x) = 1$

Thus we have, from (7.10),

$$\lim_{x \to 0} \frac{\operatorname{Sin} x}{x} = c, \quad \text{where } c \text{ depends on the scale.}$$

Now, let us introduce the natural angular measure, $t = cx$ and define

$$\sin x = \operatorname{Sin} \frac{x}{c} \quad \text{and} \quad \cos x = \operatorname{Cos} \frac{x}{c}.$$

Then we have

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{1}{c} \cdot \frac{\operatorname{Sin} \frac{x}{c}}{\frac{x}{c}} = 1.$$

5. Statistics. Normal distribution. Let $f$ be continuous and have continuous derivative and such that,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Then $g(x) = f(x_1-x) \cdot f(x_2-x)\, f(x_3-x)\, f(x_4-x)$ has maximum at $x = \frac{x_1+x_2+x_3+x_4}{4}$ if and only if

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp \left[ \frac{x^2}{2\sigma^2} \right].$$

Indeed, let $g$ have a maximum at $x_0 = \frac{x_1+x_2+x_3+x_4}{4}$. Then

$$f'(x_1-x_o)\ f(x_2-x_o)\ f(x_3-x_o)\ f(x_4-x_o) + f(x_1-x_o)\ f'(x_2-x_o)$$

$$f(x_3-x_o)\ f(x_4-x_o) + f(x_1-x_o)\ f(x_2-x_o)\ f'(x_3-x_o)\ f(x_4-x_o) +$$

$$+f(x_1-x_o)\ f(x_2-x_o)\ f(x_3-x_o)\ f'(x_4-x_o) = o.$$

Set $\qquad h(x) = \dfrac{f'(x)}{f(x)}$ .

Then $\sum_{i=1}^{4} h(x_i-x_o) = o$, with $\sum_{i=1}^{4} (x_i-x_o) = o$.

Hence by [6,p.47] ,, h is additive and so, $h(x) = cx$.

Therefore,

$$\frac{f'(x)}{f(x)} = cx, \quad \text{or}$$

$$f(x) = a \exp\left(\frac{-u^2}{2\sigma^2}\right),$$ negative necessary

for the convergence of the integral $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Again by the same integral we get, $a = \dfrac{1}{\sqrt{2\pi}\sigma}$ .

Thus $\qquad f(x) = \dfrac{1}{\sqrt{2\pi}\ \sigma} \exp\left(-\dfrac{u^2}{2\sigma^2}\right)$.

## § 8. Some unsolved problems in functional equations.

1. The function $f(x) = \frac{1}{x}$ can be characterized by the functional equation $f(x+1) = \frac{f(x)}{f(x)+1}$, for $x \in \left]0,\infty\right[$ and some additional condition, namely convexity. The same function $f(x) = \frac{1}{x}$ also satisfies $f^2(x) = x$ (iteration). It will be interesting to charactrize $f$ by the above function and some additional conditions.

2. Babbage equation. The equation $f^n(x) = x$ ($n$ denotes the $n$-th iteration) has been treated well and is known that for continuous $f$, when $n$ odd, $f(x) = x$ and when $n$ even, every solution satisfies $f^2(x) = x$. Also every continous solution is monotonic. Under what conditions on $g$, $f^n(x) = g(x)$ or $f^2(x) = g(x)$ has a convex solution and whether such a solution is unique ? Also, find the general continuous solution of $f^n(x) = g(x)$, without assuming $g$ monotonic.

3. Find the general solution of $f\left[x+yf(x)\right] = f(x)f(y)$.

4. Find all solutions of $f(AB) = f(A)\,f(B)$, where $f$: $GL_n(R) \to GL_m(R)$ $\left[GL_n(R) = \text{all square matrices of order } n\right]$ without any supposition whatsoever on $f$ for arbitrary $m$ and $n$.

5. Find all solutions of $f(AB) = f(A)f(B)$, $g(B) = f(A)g(B) + g(A)$, $f,g\colon GL_n(R) \longrightarrow GL_m(R)$, without any further assumption on $f$ and $g$ for $m,n$ arbitrary.

6. Consider the equation $f(mn) = f(m) + f(n)$, where $m$ and $n$ are integers such that $(m,n) = 1$ ($m,n$ are relatively prime). Suppose there is a constant $C$ such that $|f(n+1) - f(n)| < C$. Do there exist constants $a$ and $M$ such that $f(n) = a \log n + g(n)$ with $|g(n)| < M$ ?

7. Determine all homomorphisms of multiplicative groups of algebras in each other, that is find all solutions of

$$f(xy) = f(x)\, f(y), \text{ where}$$

$f : A_n(F) \longrightarrow A_m(F)$, ($A_n(F)$ an algebra of order $n$ over the field $F$).

8. Find all solutions of $f(xy) = h(x)\, g(y)$, where the domain is a semigroup or quasigroup and the range is in a quasigroup.

9. Find the solutions of the composite equations [45]

$$F\{F[x,F(x,y)], F[F(x,y),y]\} = F(x,y)$$

$$F[F(x,y),x] = F[x,F(y,x)]$$

$$F[F(x,y),x] = y.$$

10. Find all the solutions of

$$x\, f\left[\frac{f(y)}{x}\right] = y\, f\left[\frac{f(x)}{y}\right]$$ (unsolved without continuity)

$$f(x+y) = f(x)\,.\ f\left[\frac{y}{f(x)}\right]$$ (unsolved without continuity)

$$f(x) + f\left[f(y) - f(x)\right] = f\left[\,x + f(y-x)\right]$$ unsolved

without differentiability.

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