45

LECTURES ON
TRANSFINITE DIAMETER AND ITS APPLICATIONS

BY

PROFESSOR W. K. HAYMAN, F. R. S.

Visiting Professor, Matscience

NOTES BY

K. R. UNNI

45

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20, (INDIA).

MATSCIENCE REPORT NO.45

THE INSTITUTE OF MATHEMATICAL SCIENCES

MADRAS-20 (India)

TRANSFINITE DIAMETER AND ITS APPLICATIONS

by

W.K. Hayman, F.R.S.*
(Visiting Professor, MATSCIENCE)

Notes by
Dr. K.R. Unni

*Permanent address:

Professor of Mathematics,
Imperial College of Science and Technology,
London, England.

CONTENTS

1. Introduction

Let E be a compact subset in the complex plane. Let n be a positive integer greater than 1 and suppose $z_1, z_2, \ldots, z_n$ are distinct points of E. We set

$$\left(d_n(E)\right)^{\frac{n(n-1)}{2}} = \max\left\{ \prod_{1\leq i<j\leq n} |z_i - z_j| \;\middle|\; z_j \in E, j = 1,2,\ldots n \right\} \tag{1}$$

Then $d_n(E)$ is called the <u>diameter of order n</u> of E. Since every continuous function on a compact set attains its maximum, we are justified in taking maximum in (1) and the existence of $d_n(E)$ is obvious. Notice that when $n=2$, $d_2(E)$ is the ordinary diameter of E.

DEFINITION 1: If $d(E) = \lim_{n\to\infty} d_n(E)$ exists, then $d(E)$ is called the <u>transfinite diameter of</u> E.

We shall now prove that the transfinite diameter really exists.

THEOREM 1: (Fekete [3]). $d_n(E)$ <u>decreases with</u> n <u>and the transfinite diameter exists.</u>

PROOF: Let $z_1, z_2, \ldots\ldots, z_n$ be any points of E satisfying (1). Now, let

$$\prod_{k=1}^{n} \left\{ \prod_{\substack{1 \le i < j \le n \\ i,j \ne k}} |z_i - z_j| \right\} = \prod_{k=1}^{n} \Pi_k \text{ (say).}$$

Each distance $|z_i - z_j|$ occurs in the above product $k-2$ times, so that we have

$$\prod_{k=1}^{n} \left\{ \prod_{\substack{1 \le i < j \le n \\ i,j \ne k}} |z_i - z_j| \right\} = \prod_{k=1}^{n} \Pi_k = \left(d_n(E)\right)^{\frac{n(n-1)(n-2)}{2}}. \quad (2)$$

On the other hand, by definition, we have

$$\Pi_k \le \left(d_{n-1}(E)\right)^{\frac{(n-1)(n-2)}{2}}$$

so that the L.H.S. of (2) is at most $\left(d_{n-1}(E)\right)^{\frac{n(n-1)(n-2)}{2}}$.

Hence

$$\left(d_n(E)\right)^{\frac{n(n-1)(n-2)}{2}} \le \left(d_{n-1}(E)\right)^{\frac{n(n-1)(n-2)}{2}}$$

so that

$$d_n(E) \leq d_{n-1}(E).$$

This proves Theorem 1.

Remark: $d(E) \geq 0$ and $d(E) \leq d_2(E)$.

We shall now look at a different approach to the same. We need the following definition.

DEFINITION 2: Let $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ be a monic polynomial of degree n. Then the set $\{z \mid |P(z)| \leq \lambda^n\}$ where $\lambda > 0$ is called a Lemniscate of order n, with radius λ and centres $z_1, z_2, \ldots, z_n$. If E is compact, we define $r_n(E)$ to be the greatest lower bound of the radii of lemniscates of order n containing E and $r_n(E)$ is called the radius of order n.

THEOREM 2: As $n \to \infty$, $r_n(E) \to d(E)$ where $d(E)$ is the transfinite diameter of E.

We first prove a lemma.

LEMMA 1: For a fixed k, we have

$$\overline{\lim_{n\to\infty}}\, r_n(E) \le r_k(E)$$

PROOF: Let $n = ak+b$ where a, b are integers such that $0 \le b \le k-1$. Suppose $\{z \mid |P_k(z)| \le \lambda^k\}$ be a lemniscate of order k containing E. Suppose further that $|z| \le M$ on E. Set

$$Q_n(z) = z^b \{P_k(z)\}^a$$

Then $Q_n(z)$ is obviously a polynomial of degree n, and

$$|Q_n(z)| = |z^b \{P_k(z)\}^a| \le M^b \cdot \lambda^{ak} \quad \text{for } z \in E.$$

$$= \lambda^n \left(\frac{M}{\lambda}\right)^b$$

so that

$$|Q_n(z)|^{\frac{1}{n}} \le \lambda \left(\frac{M}{\lambda}\right)^{b/n} \quad \text{for } z \in E$$

which implies

$$r_n(E) \le \lambda \left(\frac{M}{\lambda}\right)^{b/n}$$

We have not assumed anything about λ . We now let $\lambda \to r_k(E)$ and obtain

$$r_n(E) \leq r_k(E) \left(\frac{M}{r_k(E)} \right)^{b/n}$$

The lemma now follows by letting $n \to \infty$.

Proof of Theorem 2: Let $\alpha = \underline{\lim}_{n\to\infty} r_n(E)$ and choose k such that $r_k(E) < \alpha + \epsilon$. Then, from lemma 1, it follows that

$$\beta = \overline{\lim_{n\to\infty}} \; r_n(E) \leq \alpha + \epsilon$$

ϵ being arbitrary, we now have $\alpha = \beta$ and $r(E) = \lim_{n\to\infty} r_n(E)$ exists.

To complete the proof we have to show $r(E) = d(E)$. First, we show that $r(E) \leq d(E)$.

Let $z_1, z_2, \ldots\ldots, z_n$ be points in E such that

$$\left(d_n(E)\right)^{\frac{n(n-1)}{2}} = \prod_{1\leq i<j\leq n} |z_i - z_j| \doteq \Pi_n \qquad (3)$$

and let $P(z) = \prod_{i=1}^{n} (z - z_i)$.

Consider $\prod_n |P(z)|$ for $z \in E$. This is the product of the distances between $z_1, z_2, \ldots\ldots z_n, z$. For $z \in E$, this is at most $\left(d_{n+1}(E)\right)^{\frac{(n+1)n}{2}}$. Then we have the inequality

$$|P(z)| \leq \frac{\left(d_{n+1}(E)\right)^{\frac{(n+1)n}{2}}}{\left(d_n(E)\right)^{\frac{n(n-1)}{2}}} \leq \frac{\left(d_{n+1}(E)\right)^{\frac{n(n+1)}{2}}}{\left(d_{n+1}(E)\right)^{\frac{n(n-1)}{2}}} = \left(d_{n+1}(E)\right)^n$$

This shows that $r_n(E) \leq d_{n+1}$. Now letting $n \to \infty$, we get

$$r(E) \leq d(E) \tag{4}$$

This completes the proof if $d(E) = 0$. We now assume $d(E) > 0$. We now prove in this case $d(E) \leq r(E)$.

Choose again $z_1, z_2, \ldots\ldots z_n$ satisfying (3). We write

$$V(z_1, z_2, \ldots, z_n) = \begin{vmatrix} z_1^{n-1} & z_2^{n-1} & . & z_n^{n-1} \\ z_1^{n-2} & z_2^{n-2} & . & z_n^{n-2} \\ . & . & . & . \\ 1 & 1 & . & 1 \end{vmatrix}$$

Then

$$|V(z_1, z_2, \ldots z_n)| = \prod_n$$

Suppose $Q(z) = z^{n-1} + a_1 a^{n-2} + \ldots\ldots + a_{n-1}$ to be a monic polynomial of degree n-1.

Now multiplying the j^{th} row by a_{j-1} for $j = 2,3,\ldots n$ and adding to the first row, we obtain

$$V(z_1, z_2, \ldots, z_n) = \begin{vmatrix} Q(z_1) & Q(z_2) & \cdot & Q(z_n) \\ z_1^{n-2} & z_2^{n-2} & \cdot & z_n^{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & 1 \end{vmatrix}$$

Expand by elements of the first row and obtain

$$\left|\prod_n\right| \leq \prod_{j=1}^{n} |Q(z_j)| \left(d_{n-1}(E)\right)^{\frac{(n-1)(n-2)}{2}}$$

$$\leq \left(\max_{j=1,2,\ldots,n} |Q(z_j)|\right) n \left((d_{n-1}(E)\right)^{\frac{(n-1)(n-2)}{2}}$$

Hence

$$\max_{j=1,2,..,n} |Q(z_j)| \geq \frac{|\prod_n|}{n\left((d_{n-1}(E)\right)^{\frac{(n-1)(n-2)}{2}}}$$

$$= \frac{\left(d_n(E)\right)^{\frac{n(n-1)}{2}}}{n\left((d_{n-1}(E)\right)^{\frac{(n-1)(n-2)}{2}}}$$

Since Q is arbitrary, we have

$$r_{n-1}(E) \geq \frac{(d_n(E))^{\frac{n}{2}}}{(d_{n-1}(E))^{\frac{n-2}{2}}} n^{-\frac{1}{n-1}} \tag{5}$$

Suppose now that

$\dfrac{d_n(E)}{d_{n-1}(E)} < e^{-\epsilon/n}$, $n \geq n_0$ where $\epsilon > 0$. Then

$$\frac{d_n(E)}{d_{n_0}(E)} = \frac{d_n(E)}{d_{n-1}(E)} \cdot \frac{d_{n-1}(E)}{d_{n-2}(E)} \cdots \frac{d_{n_0+1}(E)}{d_{n_0}(E)} < e^{-\epsilon\left[\frac{1}{n}+\ldots+\frac{1}{n_0+1}\right]}$$

Since $\sum_{k=n_0+1}^{n} \frac{1}{k}$ is unbounded as $n \to \infty$, we deduce that $d_n \to 0$ as $n \to \infty$. But we have assumed that $d > 0$. This gives a contradiction. Hence we have for infinitely many n

$$\frac{d_n(E)}{d_{n-1}(E)} \geq e^{-\varepsilon/n} . \tag{6}$$

We choose such a value of n and apply (5) to get

$$r_{n-1} \geq \left(\frac{d_n(E)}{d_{n-1}(E)}\right)^{n/2} \cdot d_{n-1}(E) \cdot n^{-\frac{1}{n-1}}$$

$$\geq e^{-\frac{\varepsilon}{2}} \; d_{n-1}(E) \; n^{-\frac{1}{n-1}}$$

This inequality holds for infinitely many n. We make n tend to ∞ through this sequence and obtain

$$r(E) \geq e^{-\varepsilon/2} d(E)$$

as $n^{-\frac{1}{n-1}} \to 1$ as $n \to \infty$. ε being arbitrary, we have

$$r(E) \geq d(E)$$

This completes the proof of Theorem 2.

<u>Remark</u>: By virtue of Theorem 2, the transfinite diameter is also called transfinite radius.

THEOREM 3: <u>The transfinite diameter of a Lemniscate is equal to its radius. In particular, the transfinite diameter of a circle is its radius.</u>

First we prove a lemma.

LEMMA 2: <u>If</u> E_1, E_2 <u>are lemniscates of radii</u> r_1, r_2 <u>respectively and</u> $E_1 \subset E_2$, <u>then</u> $r_1 \leq r_2$ <u>with equality if and only if</u> $E_1 = E_2$.

PROOF: Let the two lemniscates E_1, E_2 be given by

$$E_1 : \quad |P_n(z)| \leq r_1^n$$

$$E_2 : \quad |Q_m(z)| \leq r_2^m$$

where $E_1 \subset E_2$. Consider the function

$$\phi(z) = \frac{\left(Q_m(z)\right)^n}{\left(P_n(z)\right)^m}$$

Since the zeros of $P_n(z)$ lie in E_1, $\phi(z)$ is regular outside E_2. Near ∞, $\phi(z) \sim \frac{z^{mn}}{z^{mn}} = 1$.

Hence $\phi(z) \to 1$ as $z \to \infty$ and is regular at ∞. On the boundary of E_2,

$$|Q_m(z)| = r_2^m \quad \text{and} \quad |P_n(z)| \geq r_1^n$$

Hence

$$|\phi(z)| \leq \frac{r_2^{mn}}{r_1^{mn}} = \left(\frac{r_2}{r_1}\right)^{mn}$$

This would contradict the maximum principle if $\frac{r_2}{r_1} < 1$ since $\phi(\infty) = 1$ and $\phi(z)$ is regular outside E_2 including ∞. Thus $r_2 \geq r_1$.

By the maximum principle again, equality is possible only if $\phi(z) \equiv 1$. This means $|Q_m(z)^n| = |(P_n(z)^m|$ or $|Q_m(z)|^{\frac{1}{m}} = |P_n(z)|^{\frac{1}{n}}$ so that E_1 and E_2 are identical.

Proof of Theorem 3: Let E be a lemniscate and r its radius. Let $r(E)$ be its transfinite radius and $r_n(E)$ its radius of order n. We shall prove that $r = r(E)$. Suppose E is given by

$$|P_k(z)| \leq r^k \tag{7}$$

Then, if $n = k$, every lemniscate of degree k containing E must have radius at least r by lemma 2. But, since E is contained in itself, r is precisely the lower bound of the radii of lemniscates of degree k containing E. This means $r_k(E) = r$. If n is a multiple of k, say $n = ck$, the same result holds, since (7) can be written as $|P_k(z)^c| \leq r^{ck}$.

Hence $r_n(E) = r$ for any n which is a multiple of k. Making n tend to infinity through the sequence $k, 2k, 3k \ldots \ldots$, we obtain

$$r = \lim_{n\to\infty} r_n(E) = r(E)$$

In particular, the transfinite diameter of a circle of radius r is precisely r.

2. Upper Semi Continuity.

DEFINITION 3: Let E_n and E be sets of points. We say that $E_n \downarrow E$ if

$$\text{(i)} \quad E_1 \supset E_2 \supset \cdots\cdots \supset E_n \supset \cdots\cdots \supset E$$

and (ii) given $\epsilon > 0$, then E_n is contained in an ϵ-neighbourhood of E for $n > n_o(\epsilon)$. i.e., if $z \in E_n$, there exists $z' \in E$, such that $|z-z'| < \epsilon$.

THEOREM 4: <u>If</u> E <u>and</u> E_n <u>are compact sets such that</u> $E_n \downarrow E$, <u>then</u> $d(E_n) \to d(E)$.

PROOF: Let $d_k(E)$ denote the diameter of E of order k.

It is evident that if $E_2 \subset E_1$, then $d_k(E_2) \leq d_k(E_1)$ for any k and so also $d(E_2) \leq d(E_1)$. Hence, we have

$$d(E_1) \geq d(E_2) \geq \cdots\cdots \geq d(E_n) \geq \cdots \geq d(E)$$

Hence

$$\delta = \lim_{n \to \infty} d(E_n) \text{ exists and } \delta \geq d(E)$$

Now, given $\epsilon > 0$, choose k so large that

$$d_k = d_k(E) < d(E) + \epsilon.$$

Now we wish to show that for large n, $d_k(E_n) < d_k(E) + \epsilon$. i.e. we wish to show that for each fixed k, $d_k(E_n) \to d_k(E)$.

Let $z_{1,n}, \ldots\ldots\ldots, z_{k,n}$ be k points of E_n such that

$$\prod_{1 \leq i < j \leq k} |z_{i,n} - z_{j,n}| = \left(d_k(E_n)\right)^{\frac{k(k-1)}{2}}$$

We vary n and assume that for a suitable subsequence $n = n_p$(say), $z_{j,n_p} \to z_j$, $j = 1, 2, \ldots, k$.

Since for large p, z_{j,n_p} lies within η of some point of E; so any neighbourhood of z_j contains points of E. Therefore z_j lies in the closure of E and so in E, since E is closed.

Thus, as $p \to \infty$,

$$\prod_{1 \leq i < j \leq k} |z_{i,n_p} - z_{j,n_p}| \to \prod_{1 \leq i < j \leq k} |z_i - z_j| \leq (d_k(E))^{\frac{k(k-1)}{2}}$$

The last inequality follows by definition.

Thus $\lim\limits_{n \to \infty} d_k(E_n) \leq d_k(E)$ and so

$$\lim_{n \to \infty} d_k(E_n) = d_k(E)$$

For large n, $d_k(E_n) < d_k(E) + \epsilon < d(E) + 2\epsilon$.

But $d_k(E_n) \geq d(E_n)$.

So $\lim_{n \to \infty} d(E_n) \leq d(E)$ since ϵ is arbitrary

Hence $d(E_n) \to d(E)$.

3. Green'x function and the transfinite diameter.

LEMMA 3: Suppose that Γ is a finite set of mutually disjoint analytic curves forming the frontier of an unbounded domain D_o. Suppose further that $g(z)$ is harmonic in D and remains analytic on Γ (as a function of x and y, $z = x+iy$).

$$g(z) = o \quad \text{on } \Gamma$$

$$g(z) = \log|z| + \gamma + o(1) \quad \text{as} \quad z \to \infty$$

where γ is a constant. Then the transfinite diameter of Γ is $e^{-\gamma}$. (Szego [9])

Here g(z) is called the (classical) Green's function and γ is Robin's constant.

PROOF: We apply Green's formula to the part of D_o outside a small circle $|z-z_o| > \rho$ and inside a large circle $|z| < R$.

Denote $|z-z_o| = \rho$ by C_ρ and $|z|=R$ by C_R. Set $v(z) = \log|z-z_o|$. Then

$$\int_{C_R}\left(v\frac{\partial g}{\partial n} - g\frac{\partial v}{\partial n}\right) ds = \int_{C_\rho}\left(v\frac{\partial g}{\partial n} - g\frac{\partial v}{\partial n}\right) ds + \int_{\Gamma}\left(v\frac{\partial g}{\partial n} - g\frac{\partial v}{\partial n}\right) ds$$

since v and g are harmonic.

We now make $R \to \infty$ and $\rho \to o$. We first make few estimates.

<u>On</u> C_R : Clearly

$$g = \log R + \gamma + \frac{O(1)}{R}$$

$$v = \log R + \log \left| 1 - \frac{z_o}{R} \right| = \log R + \frac{O(1)}{R}$$

so that

$$\frac{\partial g}{\partial n} = \frac{1}{R} + \frac{O(1)}{R^2}$$

and

$$\frac{\partial v}{\partial n} = \frac{1}{R} + \frac{O(1)}{R^2}$$

Then

$$g\frac{\partial v}{\partial n} - v\frac{\partial g}{\partial n} = \left(\log R + \gamma + \frac{O(1)}{R}\right)\left(\frac{1}{R} + \frac{O(1)}{R^2}\right)$$

$$- \left(\log R + \frac{O(1)}{R}\right)\left(\frac{1}{R} + \frac{O(1)}{R^2}\right)$$

$$= \frac{\gamma}{R} + \frac{O(\log R)}{R^2}$$

Now the length of Γ is $2\pi R$. Hence

$$\int_{C_R} \left(g\frac{\partial v}{\partial n} - v\frac{\partial g}{\partial n} \right) ds = 2\pi R \left(\frac{\gamma}{R} + \frac{O(\log R)}{R^2} \right) \to 2\pi\gamma \quad \text{as } R \to \infty.$$

<u>On C_ρ :</u> We have

$$g(z) = g(z_0) + O(\rho)$$

$$\frac{\partial g}{\partial n} = O(1)$$

$$v(z) = \log \rho$$

and

$$\frac{\partial v}{\partial n} = \frac{1}{\rho} .$$

Then

$$g\frac{\partial v}{\partial n} - v\frac{\partial g}{\partial n} = \frac{g(z_0)}{\rho} + O\left(\log \frac{1}{\rho}\right),$$

and the length of C_ρ is $2\pi\rho$. Then

$$\int_{C_\rho} \left(g\frac{\partial v}{\partial n} - v\frac{\partial g}{\partial n} \right) ds = 2\pi g(z_0) + O(\rho \log \frac{1}{\rho}) \to 2\pi g(z_0)$$

<u>On Γ :</u>

We have $g = 0$, $\frac{\partial g}{\partial n}$ is analytic and so is bounded.

$v = \log|z-z_0|$, $\frac{\partial v}{\partial n}$ is bounded.

Hence

$$\int_\Gamma \left(g\frac{\partial v}{\partial n} - v\frac{\partial g}{\partial n}\right)ds = -\int_\Gamma \log|z-z_0|\frac{\partial g}{\partial n}ds$$

Thus we obtain the representation formula

$$g(z_0) = \gamma + \frac{1}{2\pi}\int_\Gamma \log|z-z_0|\ \frac{\partial g}{\partial n}\cdot ds \qquad (8)$$

Since g is harmonic in D_0, 0 on Γ and $+\infty$ at ∞, $g > 0$ in D_0 by the maximum principle.

Thus $\frac{\partial g}{\partial n} \geq 0$ on Γ.

If we write $d_\mu(\zeta) = \frac{1}{2\pi}\ \frac{\partial g}{\partial n}\ ds$, then $d_\mu(\zeta)$ is a positive mass distribution along Γ. We now assert that the total mass of Γ is 1.

To see this make $z_0 \to \infty$. Write $|z_0| = r$.

Then

$$g(z_0) = \log r + \gamma + O(1)/r \quad \text{and} \quad \log|z-z_0| = \log r + O\left(\frac{1}{r}\right)$$

uniformly in z.

Substituting in (8), we get

$$\log r + \gamma + \frac{O(1)}{r} = M\left[\log r + \frac{O(1)}{r}\right] + \gamma$$

where M is the total mass $\int_{\Gamma} d\mu(\zeta)$. . Thus we must have $M = 1$ and our assertion is proved.

Before we prove Lemma 3 completely, we need the following.

LEMMA 4: <u>If E is the set consisting of the complement of D_0, i.e., the curves Γ together with their interiors, then E can be approximated from above by Lemniscates</u>

<u>Note:</u> By the maximum principle if a polynomial $P(z)$ satisfies $|P(z)| \leq \lambda^n$ on the closed curve F, then it also holds in the interior.

Thus any lemniscate containing F contains the interior of F and so the transfinite diameter of any set is equal to that of its frontier. So the transfinite diameter of Γ is equal to that of E.

<u>Proof of Lemma 4</u>: We now divide Γ into arcs Γ_ν such that their union makes up Γ, and $\int_{\Gamma_\nu} d\mu(\zeta) = \frac{1}{N}$ except possibly for one arc Γ_ν' on each of the curves constituting Γ. The last arc on each curve of Γ will have a measure between 0 and $\frac{1}{N}$ and we shall ignore this arc. There will be in general p of these extra arcs Γ_ν' if there are p closed curves in Γ. The total measures of these arcs Γ_ν' will be between 0 and $\frac{p}{N}$. On each arc Γ_ν, we choose one point ζ_ν and replace $\int \log|z_0 - \zeta| d\mu(\zeta)$ by $\frac{1}{N}\log|z_0 - \zeta_\nu|$. The difference is

$$\int_{\Gamma_\nu} \log\left|\frac{z_0 - \zeta}{z_0 - \zeta_\nu}\right| d\mu(\zeta) .$$

Now choose z_0 to be distant at least δ from Γ. As $N \to \infty$ the length of Γ_ν' will tend to 0. This is not all trivial and needs a little proof. In fact $\frac{\partial g}{\partial n} > 0$ everywhere on Γ and so bounded below in Γ by m (say) and thus

$$\int_{\Gamma_\nu} \frac{\partial g}{\partial n} ds \geq m s_\nu$$

where s_ν is the arc length of Γ_ν. To see that $\frac{\partial g}{\partial n} > 0$ suppose $\frac{\partial g}{\partial n} = 0$ at ζ_0 in Γ. We write $f = g+ih$ then since the derivative of g vanishes along Γ and perpendicular to Γ, $f'(\zeta_0) = 0$. Also f is not a constant. Thus $f(z) \sim c(z-\zeta_0)^k$, $k \geq 2$, near ζ_0. Hence the regions where $g > 0$ are k in number bounded by curves making angles $\frac{\pi}{k}$ with each other. This is impossible since $g > 0$ along one side of Γ. Thus $k = 1$, $\frac{\partial g}{\partial n} \neq 0$.

Now, on Γ_ν, we have

$$\log\left|\frac{z_0-\zeta}{z_0-\zeta_\nu}\right| = \log\left|1+\frac{\zeta_\nu-\zeta}{z_0-\zeta_\nu}\right| \leq \left|\frac{\zeta_\nu-\zeta}{z_0-\zeta}\right| = \frac{O(1)}{N\delta},$$

since the length of Γ_ν is $\frac{O(1)}{N}$, $|z_0-\zeta_\nu| \geq \delta$.
Now, adding for all the curves, we obtain,

$$\sum_\nu \left\{ \int_{\Gamma_\nu} \log|z_0-\zeta|\, d\mu(\zeta) - \frac{1}{N}\log|z_0-\zeta_\nu| \right\}$$

$$= \sum\left(\frac{O(1)}{N\delta}\,\frac{1}{N}\right) = \frac{O(1)}{N\delta}$$

Also

$$\sum_{\nu}' \int_{\Gamma_\nu'} \log|z_0 - \zeta|\, d\mu(\zeta) = \frac{O(1)}{N} \log \frac{1}{\delta} \qquad \text{if } z_0 \text{ is bounded.}$$

Finally,

$$\int_{\Gamma} \log|\zeta - z_0|\, d\mu(\zeta) - \frac{1}{N}\sum_{\nu} \log|z_0 - \zeta_\nu| = \frac{O(1)}{N\delta} \qquad (9)$$

In the sum $\sum$ in (9), there may be M terms where $N - p \leq M \leq N$. If we replace $\frac{1}{N}$ by $\frac{1}{M}$, the error is again $\frac{O(1)}{N}$. So finally, we get

$$\int_{\Gamma} \log|\zeta - z_0|\, d\mu(\zeta) = \frac{1}{M}\sum_{\nu=1}^{M} \log|z_0 - \zeta_\nu| + \frac{O(1)}{M}$$

provided that z_0 is bounded and at least at a fixed distance from Γ. Consider now the curves Γ_ϵ on which $g = \epsilon$. These will be a finite number of closed analytic curves, unless $f'(\zeta) = 0$ somewhere in Γ_ϵ where $f = g + ih$ as before. This will happen only for isolated values of ϵ. These curves separate Γ from ∞ and as $\epsilon \to \infty$, they tend to Γ. Consider ϵ small (fixed) so that $\Gamma_\epsilon, \Gamma_{2\epsilon}, \Gamma_{3\epsilon}, \ldots,$. are all of this kind.

Make N tend to infinity in the above analysis. On $\Gamma_{2\epsilon}$ we have $g = 2\epsilon$ and so

$$\int_{\Gamma} \log|z_0 - \zeta| \, d\mu(\zeta) = -\gamma + 2\epsilon$$

Hence, if M is large,

$$\frac{1}{M} \sum_{\nu=1}^{M} \log|z_0 - \zeta_\nu|$$

lies between $-\gamma + \epsilon$ and $-\gamma + 3\epsilon$. Thus the lemniscate $\frac{1}{M} \sum \log|z_0 - \zeta_\nu| < -\gamma + 3\epsilon$ contains $\Gamma_{2\epsilon}$ and so also Γ. By the same argument on the other hand, this lemniscate contains no point outside $\Gamma_{4\epsilon}$. Thus this lemniscate approximates Γ in our sense. This proves lemma 4.

<u>Completion of proof of Lemma 3</u>. The approximating Lemniscate for Γ which we constructed has radius $e^{-\gamma+3\epsilon}$. If we let $\epsilon \to 0$ through a suitable sequence ϵ_n, the corresponding lemniscate $L_n \downarrow \Gamma$. Also their radii which are equal to their transfinite diameters tend to $e^{-\gamma}$. Hence by Theorem 4, $d(\Gamma) = e^{-\gamma}$ as required, and the proof of Lemma 3 is completed.

THEOREM 5: (Hilbert-Fekete). <u>If E is a compact set with connected complement, then E can be approximated from above by lemniscates</u>.

PROOF: Consider the collection of all discs of radius ϵ with centres in E. These discs will cover E. Then by the Heine-Borel theorem a finite number of these discs will cover E. Let E' be the union of these closed discs and let D_0 be the unbounded component of the complement. We notice that as $\epsilon \to 0$, D_0 will tend to the complement of E. Hence if E_0 is the complement of D_0, then, as $\epsilon \to 0$, E_0 will tend to E since any point outside E will be in D_0 and so outside E_0 if ϵ is small enough.

Next we notice that E_0 is bounded by curves consisting of arcs of circles. Hence we can solve the problem of Dirichlet for D_0 (Ahlfors: Complex Analysis, p.205). This means we can construct a function h(z) harmonic in E_0 (including ∞) and taking given continuous boundary values. We choose the boundary values $-\log|z-z_0|$ where z_0 is in the interior of E_0 and set

$$g(z) = h(z) + \log|z-z_0|$$

Then g(z) is the classical Green's function of D_0. $g(z) = 0$ on the boundary of D_0 and g(z) is harmonic in D_0 except at ∞ where $g(z) - \log|z|$ remains harmonic.

Now let D_δ be the set where $g > \delta$. We can choose δ as small as we please such that the boundary Γ_δ of D_δ consists of a finite number of analytic curves. Then the complement E_δ of D_δ approximates D_o from above. Γ_δ and $g-\delta$ satisfy the hypotheses of Lemma 4 and so Γ_δ can be approximated by lemniscates and hence so can E.

LEMMA 5: (Harnack). <u>Let</u> $\{u_n(z)\}$ <u>be an increasing of harmonic functions in a domain D. Then, either</u> $u_n(z) \to \infty$ <u>at each point of D or</u> $u_n(z)$ <u>converses uniformly on compact subsets of D to a harmonic limit</u> <u>u(z)</u>.

PROOF: Suppose $h(z)$ is harmonic and positive in $|z| \leq R$. Then

$$h(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) \frac{R^2 - r^2}{R - 2Rr\cos(\theta-\phi) + r^2} d\phi, 0 \leq r < R$$

$$\leq \frac{R+r}{R-r} \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\phi}) d\phi$$

$$= \frac{R+r}{R-r} h(0)$$

Now suppose that D contains the disc $|z-z_o| \leq R$ and that $u_n(z_o)$ converges. Then if $m > n > N_o(\epsilon)$, $h(z) = u_m(z) - u_n(z) \geq 0$ in $|z-z_o| \leq R$, and $h(z_o) < \epsilon$ since $u_n(z_o)$ converges. Hence for $|z-z_o| \leq r < R$, we have

$$h(z) \leq \frac{R+r}{R-r} h(z_o),$$

that is

$$0 < u_m(z) - u_n(z) < \frac{R+r}{R-r} \epsilon.$$

Thus $\{u_n(z)\}$ converges uniformly in $|z-z_o| \leq r$ for $r < R$. The argument is completed by a step by step process.

It remains to show that the limit $u(z)$ is harmonic. We assume for simplicity that $|z| \leq r$ lies in D. We show now that $u(z)$ is harmonic in $|z| < r$. If $z = e^{i\theta}$, $0 \leq \rho < r$ then

$$u_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u\,(re^{i\phi})(r^2-\rho^2)}{r^2-2r\rho\cos(\theta-\phi)+\rho^2} d\phi \qquad (10)$$

Making $n \to \infty$, we can replace u_n by u in (10). Being the uniform limit of continuous functions, u is also continuous. Hence we can differentiate under the integral sign and show that u also satisfies Laplace's equation. Hence u is harmonic.

Construction of Green's function: We say that if D is an unbounded domain whose complement E is compact, then $g(z)$ is the Green's function of D if $g(z)$ satisfies the following properties:

(i) $g(z)$ is positive and harmonic in D except at ∞

(ii) $g(z) - \log|z|$ remains harmonic at ∞ or equivalently

(ii) $g(z) - \log|z| \to \gamma$ as $z \to \infty$, where γ is called Robin's constant.

(iii) $g(z)$ is minimal subject to (i) and (ii). That is, if $h(z)$ satisfies (i) and (ii), then $h(z) \geq g(z)$ in D.

In some cases $g(z)$ may satisfy instead of (iii) the stronger property.

(iii') $g(z) \to 0$ as z approaches any frontier point of D.
We see that (iii') implies (iii) and hence (iii') is really stronger for if (iii') holds (iii) follows from the maximum principle applied to $g(z)-h(z)$.

We shall now obtain criterion for the existence of Green's function.

THEOREM 6: Green's function $g(z)$ exists if and only if $d(E) > 0$. In this case, $\gamma = \log d(E)$ or $d(E) = e^{-\gamma}$ (Szegö [9]).

PROOF: Let L_n be a sequence of lemniscates, $|P_{k_n}(z)| \leq r_n^{k_n}$ such that $L_n \downarrow E$. We now set

$$g_n(z) = \frac{1}{k_n} \log \left| \frac{P_{k_n}(z)}{r_n^{k_n}} \right|$$

Then $g_n(z) > 0$ outside L_n, $g_n(z) = 0$ on the boundary of L_n, and at ∞

$$g_n(z) - \frac{1}{k_n} \log \left| \frac{z}{r_n} \right|^{k_n} + O(1)$$

$$= \log|z| - \log r_n + O(1)$$

Clearly, g_n satisfies (i) and (ii) for the outside of L_n with $\gamma = -\log r_n$. Consider $g_{n+1}-g_n$ outside L_n. The function is harmonic including ∞. On the boundary, $g_n=0$ and and $g_{n+1}>0$ since L_{n+1} lies inside L_n. Thus if z is outside E, the sequence $\{g_n\}$ is finally increasing and harmonic near z. Near infinity, $g_n(z) - \log|z|$ is finally increasing and harmonic. Hence by Harnack's lemma, $g_n \to +\infty$ everywhere and $g_n - \log|z| \to \infty$ at ∞ or $g_n(z) \to g(z)$ where $g(z)$ satisfies (i), (ii), and (ii').

Now since $L_n \downarrow E$, $r_n \to d(E)$. By considering the behaviour of $g_n - \log|z|$, which is $\log\left(\frac{1}{r_n}\right)$ at ∞, we see that if $d(E) > 0$, we have the second case and if $d(E) = 0$, we have the first case.

Suppose $d(E) = 0$. We shall show in this case that no function in D satisfies (i) and (ii). Suppose, in fact, that $g(z)$ satisfies (i) and (ii). Then on the frontier of L_n, $g(z) > 0$, $g_n(z) = 0$, and $g-g_n \geq 0$. Also, $g-g_n$ is harmonic outside L_n including ∞. Hence $g-g_m > 0$ outside L_n. Now making $n \to \infty$, we obtain

$$g \geq \lim_{n\to\infty} g_n = +\infty$$

which gives a contradiction. Thus if $d = 0$, no Green's function exists.

Suppose next that $d(E) > 0$. Then we show that $g(z)$ also satisfies (iii). This will then prove that $g(z)$ is Green's function. Suppose that $h(z)$ satisfies (i) and (ii) Then by the same argument as above, $h(z) \geq g_n(z)$ outside L_n. Thus $h(z) \geq \lim g_n(z) = g(z)$ which is (iii).

It remains to prove the $d(E) = e^{-\gamma}$. To see this, we notice that near infinity,

$$g(z) - \log|z| = \lim_{n\to\infty} g_n(z) - \log|z| = \lim \log \frac{1}{r_n}$$

$$= \log \frac{1}{d(F)}$$

so that $\gamma = \log \dfrac{1}{d(E)}$ or $d(E) = e^{-\gamma}$.

4. Applications to Conformal Mapping.

THEOREM 7: Suppose that $f(z)$ is meromorphic in a domain D whose complement E is compact and that $f(z)$ maps D into a domain D' whose complement is E'. Further suppose that $f'(\infty) = 1$ which means that

$$f(z) = z + a_0 + \frac{a_1}{z} + \dots \qquad \text{for large } z.$$

Then $d(E') \leq d(E)$. Equality holds if $f(z)$ maps D in a one to one fashion conformally onto D'.

This theorem was proved by Hayman [4] for the special case when D is $|z| > 1$ and the general result seems to be due to Fekete, but the proof has not yet appeared

PROOF: Suppose first that D is the complement of a lemniscate E: $|P_n(z)| \leq d^n$ and that D' is the complement of a lemniscate E': $|Q_m(z)| \leq d'^m$. Notice that we are now proving a very special case. We need to show that $d' \leq d$. To prove this, consider

$$\phi(z) = \frac{(P_n(z))^m}{(Q_m(f(z)))^n}$$

Since $Q_m(w)$ is regular and non-zero in D', it follows that $Q_m(f(z)) \neq 0$ in D. Thus $\phi(z)$ is regular in D except possibly at ∞. Further at ∞,

$$\phi(z) \sim \frac{z^{mn}}{(f(z))^{mn}} \to 1$$

Thus $\phi(z)$ is also regular at infinity. As z approaches the boundary of D, $|P_n(z)| \to d^n$, $f(z)$ lies in D' and so $|Q_m(f(z))| \geq d'^m$. This means

$$\overline{\lim}\, |\phi(z)| \leq \left(\frac{d}{d'}\right)^{mn}$$

Then by the maximum principle, since $\phi(\infty) = 1$, we must have $d \geq d'$.

Let us now consider the general case. Suppose L is a lemniscate approximating E from above and consider the closure $\overline{D}_1'$, of the image of the values taken by $f(z)$ outside L. This is a compact set on the Riemann sphere which lies in D' and so is at a positive distance from E'. Hence we can find a lemniscate L' containing D' and not meeting $\overline{D}_1'$. Hence by the first part, $d(L') \leq d(L)$ and so $d(E') \leq d(L)$. Since L is any lemniscate containing E in its interior, we can make $d(L) \to d(E)$ and thus obtain $d(E') \leq d(E)$.

Now if f maps D in a one-to-one manner conformally onto D', we can consider the inverse function and obtain the opposite inequality

$$d(E) \leq d(E')$$

Putting the two together, we get

$$d(E) = d(E')$$

This completes the proof of the theorem.

<u>Remark:</u> In the special case when D is $|z| > 1$, then $d(E') \leq 1$ with equality if f maps $|z| > 1$, in a one-to-one manner conformally onto D'.

THEOREM 7 is the only one that helps to compute the transfinite diameter of a given set.

We shall give below few examples where we actually compute the transfinite diameters making use of Theorem 7.

Example 1. <u>Transfinite diameter of an ellipse.</u> Let E be an ellipse with axes 2a, 2b. Consider the transformation

$$w = z + \frac{\beta}{z}$$

$z = re^{i\theta}$, $r > \sqrt{\beta}$ where β is positive. The circle $|z| = r$ corresponds to the ellipse $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ where $a = r + \frac{\beta}{r}$, $b = r - \frac{\beta}{r}$ and the outside of ellipse corresponds to outside of the circle in a one-to-one manner. So

$$r = \frac{1}{2}(a+b)$$

Given a and b, we can find r and β. And

$$d(E) = r = \frac{1}{2}(a+b)$$

<u>Example 2.</u>* <u>Transfinite diameter of a line segment</u>.

Making $r \to \sqrt{\beta}$ in Example 1, $b \to 0$ and we see that line segment of length $2a = \ell$ has transfinite diameter $(1/2)a = \ell/4$.

<u>Example 3</u>. Suppose that $C(x)$ is a one parameter family of curves, $a \leq x \leq b$, such that

(i) $C(x)$ meets the real axis at the point x.

(ii) if P, P' lie on $C(x)$, $C(x')$ respectively, then the distance PP' is at least $|x-x'|$.

Then if E is any set which meets each of the curves $C(x)$, then

$$d(E) \geq \frac{b-a}{4}$$

For examples of $C(x)$, we may take for instance

(i) $C(x)$ is the line through x perpendicular to the real axis

(ii) $C(x)$ is the circle with centre origin and radius r.

---

* See [4]

PROOF: Suppose that $x_1, x_2, \ldots, x_n$ are points on the segment $s = [a,b]$ of the real axis such that

$$\left\{ \prod_{1 \leq i < j \leq n} |x_i - x_j| \right\}^{\frac{2}{n(n-1)}} = d_n(s)$$

Let z_i be a point of E on $O(x_i)$. Then by hypothesis $|z_i - z_j| \geq |x_i - x_j|$. Thus

$$d_n(E) \geq \left\{ \prod_{1 \leq i < j \leq n} |z_i - z_j| \right\}^{\frac{2}{n(n-1)}} \geq \left\{ \prod_{1 \leq i < j \leq n} |x_i - x_j| \right\}^{\frac{2}{n(n-1)}} = d_n(s)$$

This inequality is true for every n. Thus making n tend to infinity, we obtain

$$d(E) \geq d(s) = \frac{b-a}{4}$$

Example 4. (Fekete). Let E be an arc of length ℓ Then $d(E) \leq \frac{\ell}{4}$.

PROOF: We can parameterize E by $z = z(s)$ $0 \leq s \leq \ell$ (in terms of arc length. Let I be the interval $[0, \ell]$.

If $z_i = z_i(s_i)$ give the diameter of order n, we have

$$|z_i - z_j| \leq |s_i - s_j|$$

so that $d_n(E) \leq d_n(I)$.

Thus making $n \to \infty$, we get $d(E) \leq d(I) = \frac{\ell}{4}$.

THEOREM 8: [4] <u>Suppose that</u>

$$f(z) = z + \ldots\ldots\ldots$$

<u>is meromorphic in $|z| > 1$ and let $C(x)$ be curves satisfying Example 3. Then if $b-a > 4$, $f(z)$ assumes all values on at least one of the curves $C(x)$.</u>

PROOF: Suppose the contrary. Let E' be the set of values not taken by $f(z)$. Then E' meets every one of the curves $C(x)$ and then by Example 3, $d(E') \geq \frac{b-a}{4} > 1$. This contradicts Theorem 7.

COROLLARY 1: <u>With the above hypotheses, $f(z)$ assumes all values on a circle $|w| = R$ with $R < 4+\epsilon$.</u>

Take $C(x)$ as the circle $|w|=x$, $0 \leq x \leq 4+\epsilon$.

COROLLARY 2: If $f(z) = z+a_2z^2+\ldots$ is regular in $|z| < 1$, then $f(z)$ assumes all values on some circle $|w| = R$ with $R > 1/4 - \epsilon$.

This is a sharp form of a Theorem of Landau. [7]

PROOF: We apply corollary 1 to

$$\frac{1}{f(\frac{1}{z})} = \phi(z) = z+\ldots\ldots \qquad |z| > 1$$

$\phi(z)$ assumes all values on $|w| = R$ with $R < 4+\epsilon$ and so $f(\frac{1}{z})$ and $f(z)$ assume in $|z| > 1$ and $|z| < 1$ respectively all values on $|w| = \frac{1}{R} > \frac{1}{4+\epsilon}$.

Example: $f(z) = \frac{2}{(1-z)^2}$ maps $|z| < 1$ into the plane cut from -1/4 to ∞ along the real axis. This proves that the above result is sharp.

5. Transfinite diameter and capacity

Let E be a compact set and let $d\mu$ be a positive mass distribution over E of total mass 1.

Let us consider the integral

$$u(z) = \int_E \log|z-\zeta|\, d\mu_\zeta$$

Then u(z) is harmonic outside E and

$$u(z) = \log|z| + o(1), \text{ as } z \to \infty$$

Since u(z) cannot assume a minimum outside E, either u(z) assumes a minimum value in E(which may be $-\infty$) or tends to a minimum value for a sequence $\{z_n\}$ with a limit in E. We call this minimum m(u). Let m be the upper bound of this minimum m(u), for varying mass distribution μ on E.

Now, for every $d\mu$, we have

$$u(z) \leq \log (d_2(E)) \text{ in } E$$

so that $m \leq \log d_2(E)$; but m may be $-\infty$

DEFINITION: The capacity of the set E is defined by

$$\begin{aligned} \mathrm{Cap}(E) &= e^{-m} \quad \text{if } m > -\infty \\ &= 0 \quad \text{if } m = -\infty \end{aligned}$$

THEOREM 9: For any compact set E, Cap(E) = d(E).

PROOF: Suppose first that E is a lemniscate given by $|P_n(z)| \leq d^n$ and suppose further that $P_n'(z) \neq 0$ for $|P_n(z)| = d^n$ Then this set consists of a finite number of disjoint analytic curves. Further the function

$$g(z) = \frac{1}{n} \cdot \log\left(\frac{|P_n(z)|}{d^n}\right)$$

satisfies the hypotheses of Lemma 3, since

$$g(z) = \log|z| + \log\frac{1}{d} + o(1) \text{ at } \infty$$

Then, outside E we have the representation.

$$g(z) = \log\frac{1}{d} + \int_E \log|z-\zeta|\, d\mu e_\zeta$$

Since the mass distribution is smooth ($\frac{\partial g}{\partial n} ds$)the integral remains continuous on E and so

$$u(z) = \int_\Gamma \log|z-\zeta|\, d\mu = \log d \text{ on } \Gamma.$$

The mass lies on the boundary Γ of E and so the integral is harmonic inside Γ and is log d = constant on Γ . Thus the integral is constant, equal to log d inside, that is, on the whole of E. Thus for this particular u(z), m(u) = log d, Thus m(E) ≥ log d, which means that Cap(E) ≥ d(E).

We now wish to show that $\mathrm{Cap}(E) \leq d(E)$. Suppose that $u_1(z)$ is potential function

$$u_1(z) = \int_E \log |z-\zeta| \, d\mu_1(\zeta)$$

such that $u_1 \geq \log d_1$ outside E.

Then $u_1 - \log d_1 \geq 0$ outside E. Also, $u_1 - \log d = \log|z| - \log d_1 + o(1)$ as $z \to \infty$. Thus $u_1 - \log d_1$ satisfies hypotheses (i) and (ii) for Green's function. Thus

$$u_1 - \log d_1 \geq g(z) = u(z) - \log d$$

Making $z \to \infty$, we deduce that $-\log d_1 \geq -\log d$ which is the same as $m(u) \leq \log d$. Hence $\mathrm{Cap}(E) \leq d(E)$. Thus $\mathrm{Cap}(E) = d(E)$. in this case.

In the general case, we first notice that $\mathrm{Cap}(E)$ increases with E. Let $\{E_n\}$ be a sequence of lemniscates such that $E_n \downarrow E$. $\mathrm{Cap}(E_n)$ decreases and $d(E) = \lim_{n\to\infty} d(E_n) = \lim_{n\to\infty} \mathrm{Cap}(E_n)$.

Since, on the other hand, $\mathrm{Cap}(E) \leq \mathrm{Cap}(E_n)$ for every fixed n, we deduce that $\mathrm{Cap}(E) \leq d(E)$. If $d(E) = 0$, then the result follows immediately. Thus we assume that $d(E) > 0$. We now assert that $\mathrm{Cap}(E) < d(E)$ is impossible. Let us consider the Green's function $g_n(z)$ on E_n and we write

$$g_n(z) = \int_{\Gamma_n} \log|z-\zeta|\, d\mu_n(\zeta) + \log\frac{1}{d_n} = u_n(z) + \log\frac{1}{d_n} \text{ (say)}$$

As $n \to \infty$, $g_n(z)$ is finally increasing outside E so that $g_n(z) \to g(z)$ and $d_n \to d = d(E)$. Thus at least outside E

$$u_n(z) \to u(z) = g(z) + \log d.$$

On E, $u_n(z)$ is a constant and is equal to $\log d_n$. So on E, $u_n(z) \to \log d$. Thus $u(z) = \lim_{n\to\infty} u_n(z)$ exists in the whole plane and $u(z) \geq \log d$ with equality only on E.

It can be shown that by taking a subsequence $\{n_p\}$, if necessary, the measure μ_{n_p} converges to a limiting measure μ, of total mass 1, distributed over the frontier of E and such that by Fatou's Lemma $u(z) \leq \int \log|z-\zeta|\, d\mu(\zeta)$. Hence $Cap(E) \geq d(E)$. That is, $Cap(E) = d(E)$.

Remark. We also note that from this argument we obtain an integral representation of the Green's function $g(z)$ of the complement of E

$$g(z) = \int_E \log|z-\zeta|\, d\mu(\zeta) + \log\frac{1}{d} \tag{11}$$

outside E.

Also at interior points of E, if any,

$$\log d = \int_E \log|z-\zeta|\, d\mu(\zeta)$$

THEOREM 10: If $d(E) > 0$ it is possible to extend Green's function as a potential function with respect to a measure $d\mu$ on the frontier of E by(11). The resulting function is 0 at interior points of E and ≥ 0 on the frontier of E. It is subharmonic in the whole plane.

We only remark that the required extension is given by (11).

6. Sets of capacity zero and Meromorphic functions [8,pp 260-264]

THEOREM 11: Suppose that $f(z)$ is meromorphic in $|z|<R<+\infty$ and that $T(r,f) \to +\infty$ as $r \to R$. Then $f(z)$ assumes in $|z|<R$ all values except possibly a set of capacity zero.

Remark. A set of capacity zero in the closed plane is a closed set which can be transformed into a set of capacity zero under a bilinear map. We shall now introduce the Ahlfors-Shimuzu characteristic (see e. g [5,p.12]). This has certain advantages over the Nevanlinna characteristic.

We set

$$m_0(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{k(f(re^{i\theta}),a)}\,d\theta$$

where

$$k(w,a) = \frac{|w-a|}{(1+|a|^2)^{\frac{1}{2}}(1+|w|^2)^{\frac{1}{2}}}$$

is the Chordal distance corresponding to a, w on the Riemann sphere.

The first fundamental theorem may now be stated as follows:

For every a including ∞,

$$T_0(r,f) = N(r,a)+m_0(r,a)-m_0(0,a) \qquad (12)$$

where

$$T_0(r,f) = \frac{1}{\pi}\int_0^r \frac{A(t)}{t}\,dt$$

and $A(t)$ is the area of the image $|z| < t$ by $f(z)$ into the Riemann sphere.

We also write that

$$T_0(r,f) = T(r,f)+O(1)$$

Assume now E is bounded. Since the set of values assumed by $f(z)$ is open, E is closed. Thus E is compact. Assume $d(E) > 0$. Then we shall obtain a contradiction if $f(z)$ has unbounded characteristic in $|z| < 1$.

Let $d\mu$ be the associated measure on E, and consider the potential function

$$u(z) = \int_E \log|z-\zeta|\,d\mu e_\zeta$$

Suppose that E is a compact set lying in the disk $|a|<t$, $t\geq\frac{1}{2}$, $d(E)=$ capacity of E. Suppose also that $|w|\leq 2t$.

Then

$$\log\frac{1}{k(w,a)} \leq \log\frac{1}{|w-a|} + 2\log^{+}(2t)+\log 2.$$

For,

$$\log\frac{1}{k(w,a)} = \log\frac{1}{|w-a|} + \frac{1}{2}\left\{\log(1+|a|^2)+\log(1+|w|^2\right\}$$

$$\leq \log\frac{1}{|w-a|} + \log(1+4t^2)$$

$$\leq \log\frac{1}{|w-a|} + 2\log^{+}(2t) + \log 2 + \log 1$$

$$= \log\frac{1}{|w-a|} + 2\log^{+}(2t) + \log 2$$

since $2t \geq 1$.

Let $d\mu$ be a mass distribution associated with E. Now multiplying by $d\mu(a)$ and integrating with respect to a, we obtain

$$\int \log\frac{1}{k(w,a)}\,d\mu(a) < -\int \log|w-a|\,d\mu(a) + 2\log^{+}(2t)+\log 2$$

$$\leq \log\frac{1}{d(E)} + 2\log^{+}(2t) + \log 2$$

Now suppose that $|w|>2t > 1$. Then $|w-a| > \frac{1}{2}|w|$ so that

$$\frac{1}{k(w,a)} < \frac{(1+|a|^2)^{\frac{1}{2}}(1+|w|^2)^{\frac{1}{2}}}{\frac{1}{2}|w|} < 4(1+|a|^2)^{\frac{1}{2}} < 4(1+t^2)^{\frac{1}{2}}$$

and hence

$$\int \log\frac{1}{k(w,a)}\, d\mu(a) \leq \log 4(1+t^2)^{\frac{1}{2}} \leq 3\log 2 + \log^+ t$$

Thus in all cases, we have

$$0 \leq \int \log\frac{1}{k(w,a)}\, d\mu(a) \leq \log^+ \frac{1}{d(E)} + 2\log^+(2t) + 3\log 2.$$

We multiply (12) by $d\mu(a)$ and integrate. This gives

Lemma 6 [8, pp.169-173]. *If E is a compact set in $|a| < t$, where $d(E) > o$ and $d\mu$ an associated extremal mass distribution, then*

$$\left| T_0(r,f) - \int_E N(r,a)\, d\mu(a) \right| \leq \log^+ \frac{1}{d(E)} + 2\log^+(2t) + 3\log 2 .$$

In fact, the left hand side is by (12)

$$\frac{1}{2\pi}\int d\mu(a) \int_0^{2\pi} \log\frac{1}{k(f(re^{i\theta}),a)}\, d\theta - \frac{1}{2\pi}\int d\mu(a) \int_0^{2\pi} \log\frac{1}{k(f(o),a)}\, d\theta$$

We can invert the order of integration. Integrating first with respect to μ, we get the required bound for each fixed θ. Then integrating with respect to θ, the result follows.

If $f(z)$ assumes no value on the set E, we deduce at once that

$$T_0(r,f) \leq \log^+ \frac{1}{d(E)} + \text{constant} \quad .$$

that is, $f(z)$ has bounded characteristic. This proves Theorem 11.

<u>Capacities of more general sets.</u>

If G is an open set, we define

$$\mathrm{Cap}(G) = \mathrm{Sup}\left\{ \mathrm{Cap}(F) \,|\, F \subset G,\ F \text{ is compact} \right\}$$

Then if E is an arbitrary bounded set, we define the outer capacity by

$$\mathrm{Cap}(E) = \inf\left\{ \mathrm{Cap}(G) \,|\, E \subset G,\ G \text{ is open} \right\}$$

Further, if E is unbounded, we define

$$\mathrm{Cap}(E) = \lim_{n\to\infty} \mathrm{Cap}(E \cap \left\{ z \,\middle|\, |z| < n \right\})$$

LEMMA 7: Suppose that E_ν, $\nu=1,2,\ldots$ are sets in $|z| < \frac{1}{2}$ and that $E = \bigcup_\nu E_\nu$. Then

$$\left\{\log \frac{1}{\mathrm{Cap}(E)}\right\}^{-1} \leq \sum_\nu \left\{\log \frac{1}{\mathrm{Cap}(E_\nu)}\right\}^{-1}$$

PROOF: First we assume that there are only a finite number of sets E_ν (say N) and that they are compact. Then E is also compact. Let

$$|P_{k_\nu}(z)| < r_\nu^{k_\nu}$$

be lemniscates containing E.

We can assume that the zeros of these lemniscates lie in $|z| < \frac{1}{2}$. Then $|P_{k_\nu}(z)| < 1$ for $|z| < \frac{1}{2}$. Let us now suppose that the δ_ν's be positive rational numbers such that $\Sigma\delta_\nu = 1$. Then

$$\prod_{\nu=1}^{N} |P_{k_\nu}(z)|^{\frac{\delta_\nu}{k_\nu}} \leq \mathrm{Sup}\ r_\nu^{\delta_\nu} = r(\text{say}) \text{ on } E$$

By taking a limit, this result remains true for irrational δ_ν's with sum 1. For δ_ν rational, we obtain a lemniscate by writing $\delta_\nu = \frac{p_{k_\nu}}{Q}$, where Q is a common denominator of

radius r. So $d(E) \le r$. We choose δ_ν (irrational in general) so as to make r as small as we please. That is

$$\delta_\nu \log r_\nu = \text{constant} = \log r \text{ (say)}$$

$$\sum_\nu \delta_\nu = \log r \sum_\nu \frac{1}{\log r_\nu} = 1$$

$$\frac{1}{\log r} = \sum \frac{1}{\log r_\nu}$$

which can also be written as (by multiplying by -1)

$$\frac{1}{\log \frac{1}{r}} = \sum_\nu \frac{1}{\log \frac{1}{r_\nu}}$$

Making $r_\nu \to d(E_\nu)$ and remembering that $d(E) < r$ we deduce our result.

In the general case, suppose that $\mathrm{Cap}(E_\nu) < r_\nu$ and let G_ν be open sets containing E_ν such that $\mathrm{Cap}(G_\nu) < r_\nu$. Let $G = \bigcup_\nu G_\nu$.

Let F be any compact subset of G. Then by the Heine-Borel theorem we could find a finite N such that

$$F \subset \bigcup_{\nu=1}^{N} G_\nu$$

Hence we can find compact subsets F in G_ν such that

$$F \subset \bigcup_{\nu=1}^{N} F_\nu$$

Then, we have

$$\sum_{\nu=1}^{N} \left(\log \frac{1}{d(F_\nu)}\right)^{-1} \geq \left(\log \frac{1}{d(F)}\right)^{-1}$$

But

$$\sum_{\nu=1}^{N} \left(\log \frac{1}{d(F_\nu)}\right)^{-1} \leq \sum_{\nu=1}^{N} \left(\log \frac{1}{d(G_\nu)}\right)^{-1}$$

$$\leq \sum_{\nu=1}^{N} \left(\log \frac{1}{r_\nu}\right)^{-1}$$

This is true for every compact set F in G and hence by definition we replace F by G.

Since $E \subset G$, we may replace G by E. Finally we choose r_ν so that

$$\left(\log \frac{1}{r_\nu}\right)^{-1} < \left(\log \frac{1}{\operatorname{Cap}(E_\nu)}\right)^{-1} + \frac{\epsilon}{2^\nu}$$

Then

$$\left(\log \frac{1}{\operatorname{Cap}(E)}\right)^{-1} \leq \sum_\nu \left((\log \frac{1}{\operatorname{Cap}(E_\nu)})\right)^{-1} + \epsilon$$

Here ϵ may be chosen as small as we please and hence the result is immediate.

COROLLARY: *If* $\operatorname{Cap}(E_\nu) = 0$ *for every* ν, *then* $\operatorname{Cap}(E) = 0$. *In particular if E is countable*, $\operatorname{Cap}(E)=0$

THEOREM 12: [8 p.263]. *Let* $f(z)$ *be meomorphic of unbounded characteristic in* $|z| < R$, $0 < R \leq \infty$. *Suppose* $\epsilon > 0$ *is given. Then if* a *is outside a set of capacity zero, we have*

$$N(r,a) > T(r,f) + T(r,f)^{\frac{1+\epsilon}{2}}, \quad \rho_0(a) < r < R$$

Hence

$$\frac{N(r,a)}{T(r,f)} \to 1$$

PROOF: We take $T_0(r,f)$ instead of $T(r,f)$. Set $\lambda(r) = T_0(r)^{\frac{1+\epsilon}{2}}$ and define a sequence $\{r_n\}$ as follows. Choose r_0 arbitrarily such that $0 < r_0 < R$; r_ν is defined inductively by

$$T_0(r_{\nu+1}) = T_0(r_\nu) + \lambda(r_{\nu+1})$$

Note that

$$T_0(r) - \lambda(r) = T_0(r) - T_0(r)^{\frac{1+\epsilon}{2}}$$

increases with r when r is sufficiently near 1. Thus $r_{\nu+1}$ is uniquely defined, we have $r_{\nu+1} > r_\nu$ and $r_{\nu+1} \to R$ as $\nu \to \infty$. For suppose $r_{\nu+1} \to R' < R$. Then in the limit

$$T_0(R') = T_0(R') + \lambda(R')$$

which gives a contradiction.

Let $|a_0 - a| \leq \frac{1}{2}$ be a fixed disk C with $|a_0| \leq t - \frac{1}{2}$.
Let e_ν be the set of points in C for which $N(r_\nu, a) \leq T_0(r_\nu) - \lambda(r_{\nu+1}) - 2\log^+(2t) - 3\log 2$. Since $N(r_\nu, a)$ is a continuous function of a for fixed r, it follows that e_ν is compact.

Hence by Lemma 6, we obtain

$$\log^+ \frac{1}{d(e_\nu)} \geq \lambda(r_{\nu+1})$$

For, by Lemma 6, we have

$$\left| T_0(r) - \int_{e_\nu} N(r,a)\,d\mu(a) \right| \leq 2 \log^+(2t) + 3\log 2 + \log^+ \frac{1}{d(e_\nu)}$$

or equivalently

$$\left| \int_{e_\nu} \left\{ T_0(r) - N(r,a) \right\} d\mu(a) \right| \leq 2 \log^+(2t) + 3\log 2 + \log^+ \frac{1}{d(e_\nu)}$$

which gives

$$\lambda(r_{\nu+1}) + 2 \log^+(2t) + 3 \log 2 \leq 2 \log^+(2t) + 3\log 2 + \log^+ \frac{1}{d(e_\nu)}$$

We now set

$$E_N = \bigcup_{\nu=N}^{\infty} e_\nu \text{, and } E = \bigcap_{N=1}^{\infty} E_N$$

We want to show that Cap(E) = 0.

By Lemma 7, we have

$$\left(\log\frac{1}{\mathrm{Cap}(E_N)}\right)^{-1} \leq \sum_{\nu=N}^{\infty}\left(\log\frac{1}{d(e_\nu)}\right)^{-1}$$

$$\leq \sum_{\nu=N}^{\infty}\frac{1}{\lambda(r_{\nu+1})}$$

Now

$$\frac{1}{\lambda(r_{\nu+1})} = \frac{1}{\lambda(r_{\nu+1})}\cdot\frac{T_0(r_{\nu+1}) - T_0(r_\nu)}{(r_{\nu+1})}$$

$$= \frac{1}{(\lambda(r_{\nu+1}))^2}\int_{r}^{r_{\nu+1}} dT_0(r)$$

$$\leq \int_{r}^{r_{\nu+1}}\frac{dT_0(r)}{(\lambda(r))^2}$$

Hence

$$\left(\log \frac{1}{\mathrm{Cap}(E_N)}\right)^{-1} \leq \sum_{\nu=N}^{\infty} \int_{r}^{r_{\nu+1}} \frac{dT_o(r)}{(\lambda(r))^2}$$

$$= \int_{r_\nu}^{\infty} \frac{dT_o(r)}{(\lambda(r))^2}$$

$$= \int_{r_\nu}^{\infty} \frac{dT_o(r)}{(T_o(r))^{1+\epsilon}} = \frac{T_o(r)^{-\epsilon}}{\epsilon}$$

Hence $\mathrm{Cap}(E_N) \to 0$ as $N \to \infty$.

Since $E \subset E_N$ for every N, we conclude that $\mathrm{Cap}(E) = 0$. Suppose now that a is a point of $|a_o - a| \leq \frac{1}{2}$ which is outside E. Then a is outside E_N for some N. This means that a is outside e_ν for every $\nu \geq N$.

Hence, for $\nu \geq N$, we have

$$N(r_\nu, a) \geq T_o(r_\nu) - \lambda(r_{\nu+1}) - 2\log^+(2t) - 3\log 2$$

$$= T_o(r_{\nu+1}) - 2\lambda(r_{\nu+1}) - 2\log^+(2t) - 3\log 2.$$

Now N(r,a) and T(r,a) increase with r and

$$\frac{T(r_{\nu+1})}{T(r_\nu)} \to 1 \quad \text{as} \quad \nu \to \infty .$$

Also $\dfrac{\lambda(r_{\nu+1})}{\lambda(r_\nu)} \to 1$ and $\lambda(r_\nu) \to \infty$ as $\nu \to \infty$

Thus for large ν, we have

$$N(r_\nu, a) \geq T_0(r_{\nu+1}) - 3\lambda(r_\nu)$$

Hence, we obtain

$$N(r,a) \geq T_0(r) - 3\lambda(r) \tag{13}$$

for $r_\nu \leq r \leq r_{\nu+1}$, $\nu \geq N$ so that this inequality holds for $r_N \leq r < R$ and $|a-a_0| \leq \frac{1}{2}$, a outside E.

Thus we take the union F of all the sets E corresponding to a sequence of circles $|a_p - a| \leq \frac{1}{2}$ which cover the whole plane, Cap(F) = 0, and (13) holds finally outside F. This proves the result with $3\lambda(r)$ instead of $\lambda(r)$ and the 3 can be absorbed into $T_0(r)^{\epsilon/2}$. Valiron [10] has constructed examples of integral functions for which

$$\varliminf_{n\to\infty} \frac{N(r,a)}{T(r)} < 1$$

holds on a perfect set E, of values a. Such values are called Valiron-deficient. We are for knowing the general nature of possible sets of this type.

THEOREM 13: [8, p.198]. Suppose that $f(z)$ is meromorphic of bounded characteristic in $|z| < 1$ and let E be a compact set in the plane such that $d(E) = 0$. Then the set of θ for which $\lim_{r \to 1} f(re^{i\theta})$ exists and lies in E has Lebesgue measure zero.

PROOF: We set $r_n = 1 - \frac{1}{n}$ and suppose that

$f(r_n e^{i\theta}) \to f(r_e e^{i\theta}) \subset E$ for a set of θ of positive Lebesgue measure. By a theorem of Egorov, the convergence is uniform in a subset F of θ, which has positive Lebesgue measure and may be taken as closed.

Suppose that $f(0)$ is outside E and is at a distance 2η from E.

$$\text{Let } \Lambda = \left\{ w \,\Big|\, |P_k(w)| < \epsilon^k \right\} \tag{14}$$

be a lemnscate of radius ϵ containing E in its interior and itself contained in the η-neighbourhood N of E; $E \subset \Lambda \subset N$ Note that the centres w_k of Λ lie in Λ and so in N. Set

$$\Phi(z) = P_k \left\{ f(z) \right\}$$

Applying Jensen's formula to $\Phi(z)$, we get

$$\frac{1}{k} T(r, \Phi) = \frac{1}{k} T(r, \frac{1}{\Phi}) + \frac{1}{k} \log \Phi(o) \tag{15}$$

Choose n so large that for $\theta \in F$, $f(r_n e^{i\theta}) \in \Lambda$. We also suppose that $|w| < M$ in N where M is some positive constant.

Then for any w, we have

$$\frac{1}{k} \log^+ |P_k(w)| \leq \frac{1}{k} \sum_{\nu=1}^{k}{}' \log^+ |w - w_\nu|$$

$$\leq \frac{1}{k} \sum \left\{ \log^+ |w| + \log^+ |w_\nu| + \log 2 \right\}$$

$$\leq \log^+ |w| + \log^+ M + \log 2.$$

Thus

$$\frac{1}{k} m(r, \Phi) \leq m(r, f) + \log^+ M + \log 2.$$

Note that the poles of Φ are precisely those of f with k times the multiplicity.

$$\frac{1}{k} N(r, \Phi) = N(r, f)$$

Thus

$$\frac{1}{k} T(r, \Phi) \leq T(r, f) + \log^+ M + \log 2.$$

On the other hand,

$$T(r_n, \frac{1}{\Phi}) \geq m(r_n, \frac{1}{\Phi}) \geq \int_F \log^+ \frac{1}{|\Phi(r_n e^{i\theta})|} d\theta \qquad (16)$$

Since by hypothesis $f(r_n e^{i\theta}) \in \Lambda$, we have on F

$$\left| P_k \left\{ f(r_n e^{i\theta}) \right\} \right| < \epsilon^k$$

Thus
/the right hand side of (16) is at least $k\delta \cdot \log \frac{1}{\epsilon}$ where δ is the measure of F.

Now (15) yields,

$$\frac{1}{k} \log \Phi(o) = \frac{1}{k} T(r, \Phi) - \frac{1}{k} T(r, \frac{1}{\Phi})$$
$$\leq T(r,f) + \log^+ M + \log 2 - \delta \log \frac{1}{\epsilon}$$

Also

$$|\Phi(o)| \geq \prod_{\nu=1}^{k} |f(o) - w_\nu| \geq \eta^k \text{ where } \eta \text{ is fixed.}$$

We now make $\epsilon \to 0$ and obtain a contradiction.

This proves the result if $f(o)$ is outside E (and $f(o) \neq \infty$). If $f(o)$ is in E or ∞, we consider $f\left(\frac{z_o+z}{1+\bar{z}_o z}\right)$ where z_o is suitable. This leaves radial limits unchanged except on a set of measure zero and the result follows as before. (See e.g. [5, Theorem 6.12, p.178])

THEOREM 14: [8, p. 201] Let D be a domain whose complement E is a set of capacity zero. Let f be the function mapping $|z| < 1$ into the infinite covering surface over D (we assume E has at least two points). Then f has an unbounded characteristic. (For the existence of f see e.g. [2, Chapter IV])

PROOF: Assume contrary to this that f has bounded characteristic so that radial limits $f(e^{i\theta})$ exists a.e. These radial limits must lie on E. For, if $f(e^{i\theta}) \in D$, then the closed radius [0,1] corresponds to a curve γ lying entirely in D. Then the inverse function can be analytically continued along Γ with values in $|z| < 1$ so that γ corresponds to a compact subset of $|z| < 1$ giving a contradiction since γ corresponds to a radius. This contradicts Theorem 13 if f has bounded characteristic since $d(E) = o$.

Combining theorems 11 and 14, we see that if E is a compact set, then every function meromorphic in $|z| < 1$ and assuming no value on E has bounded characteristic if and only if $d(E) > 0$.

References

1) L. Ahlfors, Complex Analysis (Mcgraw-Hill 1953).

2) L. Ahlfors, L. Sario, Riemann Surfaces (Princeton 1960).

3) M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzahligen Koeffizienten (Math. Zeitschrift 17 (1923), 228-249).

4) W.K. Hayman, Some applications of the transfinite diameter to the theory of functions (J. Analyse Math. 1 (1951), 155-179).

5) W.K. Hayman, Meromorphic functions (Oxford 1964).

6) D. Hilbert, Über die Entwickelung einer beliebigen analytischen Funktion einer Variablen in eine unendliche nach ganzen rationalen Funktionen fortschreitende Reihe, in Gottinger Nachrichten, 1897. Reprinted in Gesammelte Abhandlungen, Vol.3, pp.3-9, Springer-Verlag, Berlin, 1935.

7) E. Landau, Zum Koebeschen Verzerrungssatz (Rend. Circ. Mat. Palermo 46 (1922), 347-8).

8) R. Nevanlinna, Eindeutige analytische Funktionen (Springer 1936).

9) G. Szego, Bemerkungen zu einer Arbeit von Herrn Fekete: Über die Verteilung der Wurzeln bei gewissen algebraischen mit ganzzahligen Koeffizienten (Mat. Zeitschrift 21 (1924), 203-208).

10) G. Valiron, Sur les zéros des fonctions entières d'ordre fini. (Rend. Circ. Mat. Palermo 43 (1919), 255-268; 44 (1920), 250-260).

+++++++

Proceedings of Matscience Symposia

Edited By

ALLADI RAMAKRISHNAN

There are two scientific meetings conducted by the Institute one in winter, the Anniversary symposium in January, and the other, the Summer School in August. The proceedings of these winter and summer meetings are being published in a series entitled

PROCEEDINGS OF THE MATSCIENCE SYMPOSIA

by the Plenum Press, New York, a division of the Consultants Bureau Enterprises Inc.

List of Matscience Reports (1964-65)

Report No.	*Author*	*Title*
35	K. Symanzik	Lecture on a modified models of Euclidean Quantum Field Theory.
36	K. Venkatesan	Report on Recent Experimental Data (1964).
37	A. Fujii	Lectures on Fermi dynamics.
38	M. Gourdin	Mathematical introduction to unitary symmetries.
39	J. V. Narlikar	Theories of Gravitation.
40	K. Venkatesan	Report on recent experimental data (1965).
41	K. R. Unni	Introduction to Hilbert space.
42	L. Rosenfeld	Theory of nuclear reactions.
43	K. R. Unni	Concepts in Modern Mathematics-I (Algebra)
44	H. Ruegg	Relativistic generalization of SU_6