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LECTURES ON
RELATIVISTIC GENERALIZATION OF SU (6)

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Visiting Scientist, Matscience

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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20, (INDIA)

MATSCIENCE REPORT 44.

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PREFACE

These notes are intended to be an informal report of the lectures given at Matscience in December 1965. The lecturer recognizes that they are biased and apologises to all authors who have not been duly quoted. Free use was made of standard works on the Poincare group, especially those of Shirokov and Joos. The lecturer thanks Professor A. Ramakrishnan for his invitation to Madras and for the extremely speedy publication of the notes, and to Srinivasa Rao for writing them up.

H.R.

RELATIVISTIC GENERALIZATION OF SU_6

Lecture 1

Introduction:

There are two distinct categories of symmetries which are used in elementary particle physics.

(a) Space-time symmetries:

Translations (P_μ) } Inhomogeneous Lorentz group or Poincare' group.
Lorentz group ($L_{\mu\nu}$) }
Parity, time reversal etc.,

(b) Internal symmetries:

Charge conservation (Q)

Baryon number conservation, (N)

Isospin conservation imbedded in SU_2 symmetry (which is less fundamental than Q or N) and is generalized to SU_3 symmetry etc.,

These are called Internal Symmetries, because usually they commute with all space-time symmetries. There have been various attempts made in various directions to unify space-time and internal symmetries. We shall discuss only a specific case.

As early as in 1937, Wigner[1)] introduced the super-multiplet model for nuclei. This was a non-relativistic phenomenological model of nuclear forces which assumed:

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1) E.P.Wigner, Phys. Rev. 51, 106 (1937).

Isospin invariance (SU_2)

Conservation of spin (SU_2)

Combined spin-isospin invariance (SU_4).

Combined invariance means here the following:

Consider the lowest Irreducible Representation of SU_2 for which the generators are the Pauli matries (σ_i). Let the generators of isospin (SU_2) be τ_i (i = 1,2,3). Consider the following set of 4 x 4 matrices:

$$\begin{array}{ll} \text{Spin:} & \mathbb{1} \otimes \sigma_i \\ \text{Isospin:} & \tau_j \otimes \mathbb{1} \\ & \tau_i \otimes \sigma_j \end{array} \qquad (1.1)$$

where $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and i, j = 1,2,3. We now have a set of 15 hermitian, traceless 4x4 matrices, which obey the algebra of the group SU_4. This phenomenological model of Wigner provided a good approximation in a limited number of cases.

In 1964, Gursey and Radicati[1] and Sakita[2] extended Wigner's supermultiplet theory for nuclear states to elementary particle physics by replacing the isospin invariance group SU_2 by the larger SU_3 group of Gell-Mann[3] and Ne'eman[4]. So, they get:

$$\begin{array}{ll} \text{Spin:} & \mathbb{1} \otimes \sigma_i \\ \text{Internal symmetry:} & \\ \left(\begin{array}{l}\text{Isospin and hyper-} \\ \text{charge conservation}\end{array}\right) & \lambda_\alpha \otimes \mathbb{1} \\ & \text{and } \lambda_\alpha \otimes \sigma_i \end{array} \qquad (1.2)$$

1) F.Gursey and L.A.Radicati, Phys. Rev. Letts. 13, 173 (1964)
2) B.Sakita, Phys. Rev. 136, B1756 (1964)
3) M.Gell-Mann, Phys. Rev. 125, 1067 (1964)
4) Y.Ne'eman, Nucl. Phys. 26, 222 (1961).

where $\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $i = 1,2,3$ and $\alpha = 1, \ldots, 8$, λ_α being the 3x3 matrices of Gell-Mann. These are the set of 35 generators of SU_6. This generalization is rather trivial but gives rise to nice results as far as the classification is concerned. The 35-dimensional (meson) irreducible representation has the following $SU(3) \otimes SU(2)$ contents:

$$35 = (8, 1) \oplus (9, 3)$$

where (8,1) is the pseudoscalar octet(0^-) and (9,3) the vector nonet (1^-) which can be identified with the observed (π, K, η) and (ρ, ω, K*, φ) multiplets. The fact that ω and φ are in the same multiplet provides a natural explanation of the degeneracy of the vector octet and the vector singlet in the nonet. The 56 dimensional (baryon) irreducible representation has the $SU(3) \otimes SU(2)$ content:

$$56 = (8, 2) \oplus (10, 4)$$

where (8, 2) is the baryon octet ($\frac{1}{2}^+$) which accommodates (N, Σ, Λ, Ξ) and (10, 4) is the decuplet of baryon resonances ($\frac{3}{2}^+$) which accommodates (N*, Y*, Ξ^*, Ω^-). Thus all the known low lying baryons and baryon resonances are in the same multiplet.

Since a representation of SU_6 has particles with different spin and fifferent SU_3 representations, one can say that space-time (spin) and internal (SU_3) properties have been unified. The main problem is, however, to find a relativistic theory which, in some sense, incoorporates SU_6.

At this point, one should make a logical distinction between two aspects of a symmetry:

a) It provides a classification of particles

b) It assumes the invariance of the Hamiltonian, providing thus conservation laws and selection rules.

Although these two aspects are not independent, the link may be complicated.

For the usual internal symmetries, the link is immedi[ate]. For example, particles are classified according to their isospin, which is a conserved quantity.

On the contrary, in the case of the Inhomogeneous Lorentz group, particles are classified according to their spin which is not conserved. For example:

$$\begin{array}{lcccc} & N^* & \longrightarrow & N & + \quad \pi \\ \text{spin:} & 3/2 & & 1/2 & \quad 0. \end{array} \qquad (1.3)$$

From (3), it is evident that spin is not conserved. Only a combination of spin and orbital angular momentum gives a Lorentz transformation. The main reason for this pecularity is due to the fact that the Poincare group is not a semi-simple group while the isospin group is semi-simple. The Poincare group is not semi-simple since the translations form an invariant abelian sub-group.

If we want to have a relativistic theory which incorporates SU_6, we have to decide whether we should keep both aspects: classification according to spin and conservation of spin. Although it is mathematically simpler to keep both aspects, one knows that spin conservation is empirically wrong, one example amongst many being reaction (3). Therefore, we shall exclude this aspect of spin conservation.

Before explaining the difficulties of the problem of relativistic generalization of SU_6, it is necessary to introduce some mathematical definitions.

The inhomogeneous Lorentz group or the Poincare group P, is an extension of the homogeneous Lorentz group L by the translation group T_4.

Definition: G is an extension of G_1 by G_2 if $G_1 = G/G_2$

Therefore, G_2 is an invariant subgroup of G and G_1 is a factor group.

Definition: G_2 is an invariant subgroup of G, if for every element $g \in G$ and for any element $h \in G_2$ we have:

$$g^{-1} h g = h'$$

where $h' \in G_2$.

Some people have the inverse notation: viz. G_2 is extended by G_1.

In the case of the Poincare' group:

$$L = P/T_4.$$

The factor group G_1 can be:

1) a subgroup of G

2) not a subgroup of G. (Schreier extension)

L is a subgroup of P, so we deal with case 1. In each of the above two cases, there are again two possibilities:

a) G_1 commutes with G_2

b) G_1 does not commute with G_2.

One may consider a) as a special case of b).

1a) In this case G is obtained by the direct product of G_1 and G_2 and one writes:

$$G = G_1 \otimes G_2 .$$

Note that $G_2 = G/G_1$; so that the extension is symmetrical in G_1 and G_2.

1b) In this case G is obtained by the semidirect product of G_1 and G_2 and one writes:

$$G = G_1 \boxed{\sigma} G_2 .$$

In the case of the Poincare group, since L is a subgroup of P but not an invariant subgroup of P and furthermore since $[L, T_4] \neq 0$, we are in case (1b).

We have seen that H is an invariant subgroup of G, if for every element $g \in G$, and for any element $h \in H$,

$$g^{-1}hg \in H .$$

Therefore, the elements of G transform the elements of H into the elements of H: G induces on automorphism of H.

In particular if $G_1 \in G$ and $G_1 = G/G_2$, G_1 induces an automorphism on G_2.

In the case of the Poincare' group $G_1 = L$, and $G_2 = T_4$. Consider the four generators P_μ of the infinitesimal transformations of T_4. Under the automorphism induced by L, they transform amongst themselves.

$$L^{-1} P_\mu L = L_\mu{}^\nu P_\nu$$

This defines a linear mapping of the 4 dimensional vector space spanned by the P_μ which therefore provide a basis for a four dimensional real representation of L. Note also that $P_\mu P^\mu = m^2$ is an invariant of L. We shall see later on how the fact that T_4 is an invariant (abelian) subgroup of $\mathcal{P}$ allows one to construct in an easy way the unitary representations of $\mathcal{P}$. In brief, the procedure is the following:

$P_\mu P^\mu$ commutes with all the generators of $\mathcal{P}$. The four operators P_μ commute together and so they may be simultaneously diagonalized. Their eigen values are K_μ. If K_μ is the time-like vector of a free particle, i.e. $K_\mu K^\mu = m^2 > 0$, there exists a Lorentz transformation which brings it to the form $K_\mu^R = (m,0,0,0)$. Consider the subgroup of L which leaves K_μ^R invariant. It is isomorphic to a rotation group and is called the little group. Its covering group is SU_2. It will be shown that the irredicible representations of the little group together with the four values of K_μ determine all irreducible unitary transformations of $\mathcal{P}$. The little group defines precisely the <u>spin</u> of the particle. That is the place where the classification according to spin comes in. However, for a multiparticle system, one does not have invariance, because the little group is defined with respect to a <u>fixed</u> momentum K_μ^R . In the multiparticle system, the particles will have, in general, different momenta and therefore different little groups.

If one wants to keep this picture when one generalizes SU_6 in a relativistic way, one would consider SU_6 as the little group of some large group G with the following properties:

1) $G \supset P$ and S.

$S \supset SU_3$.

i.e. G contains the Poincare group and the internal symmetry group.

2) T_4 is an invariant subgroup of G.

3) $P_\mu P^\mu$ is invariant

Consider $G/T_4 = L$

Since T_4 is abelian, L induces on T_4 the same automorphism as G. Hence P_μ is again a basis for a real representation of L . If $P_\mu P^\mu = m^2$ is invariant, this is the same representation as that of L. But this is possible only if $L'/S = L$. From this it follows that the little group is the extension of SU_2 (the little group of Poincare group) by the symmetry group. Therefore SU_6 cannot be the little group of G, since SU_6 is a simple group. This argument has been given by L.Michel using more sophisticated methods.

LECTURE 2.

One way of achieving a relativistic generalization of SU_6 is to introduce unphysical momenta: $G' = S \boxed{\sigma} T_n$ where S contains the full Lorentz group and SU_3 and others and the translation group has $n > 4$. Here, T_4 is no longer an invariant subgroup, since one has more than four translations. We shall pursue this point of view and show that inspite of the introduction of unphysical momenta one may arrive at a theory which has physical sense (although many problems remain unsolved). (Coleman) We shall also describe an alternative theory which apparently avoids the introduction of unphysical momenta, but which in fact is equivalent to introducing them. For example Gell-Mann, Lipkin and Meshkov talk of $(SU_6)_W$ or Hybrid or collinear groups and what they do is equivalent to the above approach.

Remark: May be it is worthwhile to give examples of the extension problem.

Example 1: One of the simplest examples of a Schreier extension is the following:

Let G be the additive group of integers, and G_2 the additive group of integers. Then, $G_1 = \frac{G}{G_2} = C_2$.

C_2, the cyclid group of two elements, is obviously not a subgroup of G, since G has no finite subgroup.

<u>Example 2</u>: Let us discuss the four possible extensions of the cyclic group C_4 by C_2 such that

$$G/C_4 = C_2$$

The elements of C_4 are:

$$C_4 : \quad E,\ A,\ A^2\ A^3. \quad \text{with } A^4 = E.$$

The involutary automorphisms of C_4 are:

a) $A \longrightarrow A$

b) $A \longrightarrow A^{-1}$

a) is an automorphism because C_4 is abelian. In the group G there will be an element R which induces the automorphism. So,

$$\text{a)} \quad R\ A\ R^{-1} = A$$
$$\text{b)} \quad R\ A\ R^{-1} = A^{-1}.$$

G is divided into two sets: the invariant subgroups C_4 and its coset RC_4. The square of each element of RC_4 is in C_4. Hence R^2 is in C_4. Since R induces an invountary automorphism, R^2 induces the identity automorphism, so that R^2 commutes with A. Therefore we must choose R^2 to be in the centre of C_4.

(<u>centre</u>: The set of elements of C_4 which commute with all elements of C_4.) Since C_4 is abelian, it is identical with its centre. The only possibilities for case a) are:

1a) $R^2 = E$

2a) $R^2 = A$

$R^2 = A^2, A^3$ give no new extensions. For case b), since $R^2 \in C_4$, one has

$$R\, R^2\, R^{-1} = R^{-2}$$
$$\text{or} \quad R^4 = E$$

The only two possibilities which satisfy this condition also are:

1b) $R^2 = E$

2b) $R^2 = A^2$ (since $A^4 = E$)

<u>Case 1a)</u>:

$$R\, A\, R^{-1} = A, \qquad R^2 = E$$

Since R commutes with C_4 and $R^2 = E$, we have for the extension of C_4 by C_2, the direct product of C_4 by C_2:

$$G = C_4 \otimes C_2$$

This is confirmal if one uses the non-trivial representation of C_4:

$$\begin{array}{cccc} E, & A, & A^2, & A^3 \\ 1, & i, & -1, & -i \end{array}$$

$R^2 = 1 \Rightarrow R = \pm 1$ and G contains C_4 and its cosets:

$$\begin{array}{lcccccccc}
G: & E, & A, & A^2, & A^3, & R, & RA, & RA^2, & RA^3 \\
C_4 & \multicolumn{4}{c}{\underbrace{1, \quad i, \quad -1, \quad -i,}_{}} & \multicolumn{4}{c}{\underbrace{-1, \quad -i, \quad 1, \quad i}_{}} \\
C_2: & \multicolumn{4}{c}{1} & \multicolumn{4}{c}{-1}
\end{array}$$

In G there exists the subgroup (E,R) which is isomorphic to C_2 in such a way that the homomorphic mapping is not changed.

<u>Case 2a)</u>: In this case $R^2 = A$. For the non-trivial representation under consideration $A = i$, so that $R = \frac{1+i}{\sqrt{2}}$.
The group G now is:

$$\begin{array}{lcccccccc}
G: & E, & A, & A^2, & A^3, & R, & RA, & RA^2, & RA^3 \\
 & 1, & i, & -1, & -i, & \frac{1+i}{\sqrt{2}}, & \frac{i-1}{\sqrt{2}}, & -\frac{1+i}{\sqrt{2}}, & \frac{-i+1}{\sqrt{2}}
\end{array}$$

Since $\left(\frac{1+i}{\sqrt{2}}\right)^8 = 1, \quad G = C_8.$

Therefore $C_8 / C_4 = C_2.$

Here (E, R) is not a subgroup of C_8.

In case b) the automorphisms are no longer trivial.

<u>Case 1b)</u>: $RAR^{-1} = A^{-1}, \quad R^2 = E.$

Obviously, there must be a non-tirival representation of G which is not one dimensional. Since G has eight elements, the representation can be only two dimensional. This representation will be reducible for the subgroup C_4:

$$\begin{array}{lcccc}
C_4: & E, & A, & A^2, & A^3 \\
 & 1, & i, & -1, & -i \\
 & 1, & -i, & -1, & i
\end{array}$$

Since $R^2 = E$, we may choose $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So, we get for G:

$$G: \quad \underset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{E}, \quad \underset{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}{A}, \quad \underset{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}{A^2}, \quad \underset{\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}}{A^3}, \quad \underset{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}{R}, \quad \underset{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}{AR}, \quad \underset{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}{A^2R}, \quad \underset{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}{A^3R}.$$

This is the dihedral group. (E, R) is a subgroup and C_2 is isomorphic to it.

$$D/C_4 = C_2$$

but C_4 does not commute with C_2 so that the extension of C_4 by C_2 is the semidirect product of C_4 by C_2:

$$D = C_2 \boxed{\sigma} C_4.$$

<u>Case 2b)</u>:

$$R\,A\,R^{-1} = A^{-1} \text{ and } R^2 = A^2$$

Again, one needs a two dimensional representation. For C_4 one may retain the set in case 1b). Since $R^2 = A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, one may choose $R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. So we have for G:

$$G: \quad \underset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{E}, \quad \underset{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}{A}, \quad \underset{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}{A^2}, \quad \underset{\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}}{A^3}, \quad \underset{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}{R}, \quad \underset{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}{AR}, \quad \underset{\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}}{A^2R}, \quad \underset{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}{A^3R}$$

(E,R) is not a subgroup. G is the <u>quaternion</u> group.

Lecture 3

The Poincare' group: In order to give a Lorentz invariant definition of spin, we shall discuss the irreducible representations of the inhomogneous Lorentz group, or Poincare group $\mathcal{P}$. The basis for these representations will be one-particle states and conservation of probability will require the representation to be unitary. Since the Poincare' group is non-compact (the domain of variation of its parameters is infinite), the unitary representations will be infinite dimensional. These representations have first been formed by Wigner[1]. We shall follow the works of Shirokov[2] and of Joos[3].

Notation:

The Poincare group acts on a real four-vector x_μ in the following way:

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu \qquad (\mu = 0,1,2,3) \qquad (3.1)$$

where summation over repeated indices is always implied. Eq. (3.1) is an Inhomogeneous Lorentz transformation (Λ ,a) if

$$g_{\rho\sigma} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma = g_{\mu\nu} \qquad (3.2)$$

where $g_{\mu\nu}$ is the metric tensor with:

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \text{ and } g_{\mu\nu} = 0 \text{ for } \mu \neq \nu .$$

1) E.P.Wigner, Ann. Math. 40, 149 (1939).
2) J.M.Shirokov, JETP. 6, 664, 918, 929 (1958).
3) H.Joos, Fortschritte der Physik. 10, 65 (1962)
See also A.S.Wightman, Los Houches lecture notes 1960.

$g_{\mu\nu}$ is used to relate covariant and contravariant quantities as far as L is concerned. In (3.2) for $a_\mu = 0$ we get the homogeneous Lorentz group L and for $\Lambda = 1$, the translation group T which is an invariant sub-group. Using

$$x_\mu = g_{\mu\nu} x^\nu$$

we get

$$x'_\mu = (\Lambda^{-1})^\mu_\nu \, x^\nu \tag{3.3}$$

Eq. (3.2) ensures the invariance of the metric tensor $g_{\mu\nu}$ and hence the invariance of the scalar product ;

$$x_\mu \, x^\mu = g_{\mu\nu} \, x^\nu \, x^\mu \tag{3.4}$$

which is the invariant length of x , written as:

$$x_\mu \, x^\mu = x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 = x_0^2 - \vec{x}^2 \tag{3.5}$$

If translations are included, the corresponding invariant is $(x_\mu - y_\mu)(x^\mu - y^\mu)$. From eq. (3.2) it follows on taking the determinant on both sides:

$$(\det \Lambda)^2 = 1$$

$$\text{or } \det \Lambda = \pm 1 \tag{3.6}$$

Thus the Lorentz group has two topologically disjoint "pieces", the elements of which are characterized by det Λ = +1 or -1. The Lorentz transformations which have det Λ = +1 are called "Proper Lorentz transformations". Each one of these two pieces contain two further disjoint pieces. The Poincare' group falls into four pieces:

$$\begin{array}{lll} \text{I)} & \det \Lambda = 1, & L^0_0 > 0 \\ \text{II)} & \det \Lambda = -1, & L^0_0 > 0 \\ \text{III)} & \det \Lambda = +1, & L^0_0 < 0 \\ \text{IV)} & \det \Lambda = -1, & L^0_0 < 0 \end{array} \tag{3.7}$$

We shall mainly deal with piece I in our lectures here. Elements of pieces I and III do not change the time component of a time like vector. For this reason the transformations which have this property are called "orthochronous Lorentz transformations". In view of the property:

$$\det \Lambda \, \varepsilon_{\mu\nu\rho\sigma} = \Lambda_\mu{}^{\mu'} \Lambda_\nu{}^{\nu'} \Lambda_\rho{}^{\rho'} \Lambda_\sigma{}^{\sigma'} \varepsilon_{\mu'\nu'\rho'\sigma'} \tag{3.8}$$

with $\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} \pm 1 \text{ if } \mu\nu\rho\sigma \text{ is even or odd} \\ \quad\text{permutation of } 0123 \end{cases}$

it follows that $\varepsilon_{\mu\nu\rho\sigma}$, the Levi-civita tensor, is also an invariant tensor for 'proper' Lorentz transformations.

If two successive Poincare' transformations are applied one gets:

$$\begin{aligned} x''_\mu &= \Lambda'^{\nu}_{\mu} x'_\nu + a'_\mu \\ &= \Lambda'^{\nu}_{\mu} (\Lambda_\nu{}^\rho x_\rho + a_\nu) + a'_\mu \\ &= (\Lambda'\Lambda)_\mu{}^\rho x_\rho + (\Lambda'^{\nu}_{\mu} a_\nu + a'_\mu) \end{aligned}$$

Therefore:

$$(\Lambda' a', \Lambda a) = (\Lambda'\Lambda, \Lambda' a + a') \tag{3.9}$$

Instead of the finite transformations (3.1), it is often useful to consider the infinitesimal transformations. For pure translations:

$$x'_\mu = x_\mu + a_\mu$$

the infinitesimal transformation is

$$\delta x_\mu = x'_\mu - x_\mu = \delta a_\mu$$

which we denote by: $(1, \quad P_\mu \ \delta a_\mu)$. (3.10)

We will show, in the course of this lecture, that a particular representation of the translation group T_4 is: $P_\mu = \dfrac{\partial}{\partial x_\mu} = \partial_\mu$

From this it follows immediately that

$$[P_\mu, P_\nu] = 0 \tag{3.11}$$

We now consider the infinitesimal Lorentz transformation which is written as:

$$x'_\mu = (\delta_\mu^\nu + (M_A)_\mu^\nu \delta\varepsilon^A) x_\nu = \Lambda_\mu^\nu x_\nu \tag{3.12}$$

or

$$\delta x_\mu = x'_\mu - x_\mu = (M_A)_\mu{}^\nu x_\nu \delta\varepsilon^A$$

This we denote by $(1 + M_A \delta\varepsilon^A, \ 0)$ where ε^A are the parameters of infinitesimal Lorentz transformations. Using (3.2), which is the definition for the Lorentz transformations, one gets for infinitesimal Lorentz transformation:

$$g_{\rho\sigma} (\delta_\mu^\rho + (M_A)_\mu^\rho \delta\varepsilon^A)(\delta_\nu^\rho + (M_A)_\nu^\rho \delta\varepsilon^A) = g_{\mu\nu}$$

With the definition

$$g_{\rho\sigma} (M_A)_\mu^\rho = (M_A)_{:\mu\sigma} \text{ and } g_{\rho\sigma}(M_A)_\nu^\sigma = (M_A)_{\rho\nu} \tag{3.13}$$

one gets:

$$g_{\mu\nu} + [(M_A)_{\mu\nu} + (M_A)_{\nu\mu}]\delta\varepsilon^A + O(\delta\varepsilon^A)^2 = g_{\mu\nu}$$

This relation should be true for any $\delta\varepsilon$. Neglecting terms of order $\delta\varepsilon^2$, we have:

$$(M_A)_{\mu\nu} + (M_A)_{\nu\mu} = 0 \tag{3.14}$$

Therefore M_A's are antisymmetric, real 4x4 matrices. There exist six linearly independent matrices of this kind, which shows that the Lorentz group has six real parameters. This is expressed by using two indices $\alpha\beta$ ($\alpha, \beta = 0,1,2,3$) instead of a single index A for the matrices M, in which case (3.14) implies:

$$M_{\alpha\beta} = -M_{\beta\alpha} \tag{3.15}$$

For the moment this is only a covenient notation. Later, we shall see that $M_{\alpha\beta}$'s are anti-symmetric tensors under L.

Using (3.12) and (3.2) one can easily find that (3.14) implies

$$\begin{aligned} (M_{\alpha\beta})_i^{\ j} &= -(M_{\alpha\beta})_j^{\ i} \\ (M_{\alpha\beta})_0^{\ i} &= (M_{\alpha\beta})_i^{\ 0} \end{aligned} \tag{3.16}$$

It is always possible to choose the six independent matrices $M_{\alpha\beta}$ in such a way that the only matrix elements different from zero are those for which $\mu=\alpha$, $\nu=\beta$ or $\mu=\beta$, $\nu=\alpha$. Taking into account the symmetry properties (3.15) and (3.16) one may write:

$$(M_{\alpha\beta})_\mu^{\ \nu} = g_{\alpha\mu}\delta_\beta^\nu - g_{\beta\mu}\delta_\alpha^\nu \tag{3.17}$$

or

$$(M_{ij})_j^{\ i} = -(M_{ij})_i^{\ j} = 1 \ ; \quad (M_{ij})_\mu^{\ \nu} = 0 \text{ for } \mu,\nu \neq i,j$$

$$(M_{oi})_o^{\ i} = (M_{oi})_i^{\ o} = 1 \ ; \quad (M_{oi})_\mu^{\ \nu} = 0 \text{ for } \mu,\nu \neq o,i$$

(3.17) is a 4-dimensional, real, non-unitary representation for the homogeneous Lorentz group. In order to verify that (3.17) forms a Lie algebra, one has to calculate the commutation relations among the M's.

$$\begin{aligned}
([M_{\alpha\beta}, M_{\gamma\delta}])_\mu^{\ \nu} &= (M_{\alpha\beta})_\mu^{\ \rho}(M_{\gamma\delta})_\rho^{\ \nu} - (M_{\gamma\delta})_\mu^{\ \rho}(M_{\alpha\beta})_\rho^{\ \nu} \\
&= (g_{\alpha\mu}\delta_\beta^\rho - g_{\beta\mu}\delta_\alpha^\rho)(g_{\gamma\rho}\delta_\delta^\nu - g_{\delta\rho}\delta_\gamma^\nu) + \\
&\quad -(g_{\gamma\mu}\delta_\delta^\rho - g_{\delta\mu}\delta_\gamma^\rho)(g_{\alpha\rho}\delta_\beta^\nu - g_{\beta\rho}\delta_\alpha^\nu) \\
&= g_{\gamma\beta}(g_{\alpha\mu}\delta_\delta^\nu - g_{\delta\mu}\delta_\alpha^\nu) + \\
&\quad -g_{\delta\beta}(g_{\alpha\mu}\delta_\gamma^\nu - g_{\gamma\mu}\delta_\alpha^\nu) + \\
&\quad -g_{\alpha\gamma}(g_{\beta\mu}\delta_\delta^\nu - g_{\delta\mu}\delta_\beta^\nu) + \\
&\quad + g_{\alpha\delta}(g_{\beta\mu}\delta_\gamma^\mu - g_{\gamma\mu}\delta_\beta^\nu)
\end{aligned}$$

Therefore

$$[M_{\alpha\beta}, M_{\gamma\delta}] = g_{\gamma\beta} M_{\alpha\delta} - g_{\beta\delta} M_{\alpha\gamma} - g \tag{3.18}$$

It can be easily verified that (3.18) satisfies the Jacobi identity:

$$[[M_{\alpha\beta}, M_{\gamma\delta}], M_{\rho\sigma}] + [[M_{\gamma\delta}, M_{\rho\sigma}], M_{\alpha\beta}] + [[M_{\rho\sigma}, M_{\alpha\beta}], M_{\gamma\delta}] = 0 \tag{3.19}$$

The commutation relation (3.18) does not depend upon the representation as long as the representation is faithful. (3.18) may be put into a simpler form by defining:

$$\vec{M} = (M_{23}, M_{31}, M_{12}) \text{ and } \vec{N} = (M_{01}, M_{02}, M_{03})$$

or

$$M_i = \frac{1}{2} \varepsilon_{ijk} M^{jk}$$

$$M_{ij} = \varepsilon_{ijk} M^k \qquad \text{and } N_i = M_{oi} \tag{3.20}$$

These have the following commutation relations:

$$[M_i, M_j] = \varepsilon^{ijk} M_k$$

$$[M_i, N_j] = \varepsilon^{ijk} N_k \tag{3.21}$$

$$[N_i, N_j] = -\varepsilon_{ijk} M^k$$

Therefore the three matrices $M_{ij} = \varepsilon_{ijk} M_k$ form a sub-algbra. From (3.2) and (3.4) it follows that they leave invariant the leng h of a Euclidean three vector. Hence they generate the Euclidean rotation group.

One may also define the Poincare group by its action on a function of x. By definition $f(x)$ is a scalar function if:

$$f'(x') = f(x) \tag{3.22}$$

If we let $x' = \Lambda x + a$ according to (3.1) and further consider the infinitesimal Lorentz transformation given by (3.12), we have

$$f'(x') = f'(x+\delta x) = f'(x) + (\partial^\mu f(x))\, \delta x_\mu$$

$$= f'(x) + (\partial^\mu f(x))\, (M_{\alpha\beta})^\nu_\mu\, x_\nu\, \delta\varepsilon^{\alpha\beta} = f(x)$$

where $\partial^\mu = \partial/\partial x_\mu$. Using (3.16) one obtains:

$$\delta f(x) = f'(x) - f(x) = -(x_\beta \partial_\alpha - x_\alpha \partial_\beta) f(x)\, \delta\varepsilon^{\alpha\beta}$$

Hence the generators are represented by

$$L_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha \tag{3.23}$$

These too satisfy the commutation relation (3.18).

Applying (3.22) to infinitesimal translations:

$$f'(x') = f'(x+\delta x) = f'(x) + (\partial^\mu f(x))\, \delta x_\mu$$

$$= f'(x) + \partial^\mu f(x)\, \delta a_\mu = f(x)$$

$$\delta f(x) = f'(x') - f(x) = -\partial^\mu f(x)\, \delta a_\mu$$

Hence $$P_\mu = \partial^\mu \tag{3.24}$$

(The sign is irrelevant)

From (3.23) and (3.24) it follows that

$$[P_\mu, P_\nu] = 0$$

$$[L_{\alpha\beta}, P_\mu] = g_{\mu\beta} P_\alpha - g_{\mu\alpha} P_\beta$$

As pointed out earlier, since the commutation relations do not depend on a particular representation, these relations are true in general.

We have seen earlier that P_μ transforms like a four vector under the group L. Hence, (3.25) gives the transformation law of an operator which transforms like a four-vector under infinitesimal transformations. Similarly (3.18) is the transformation law of an antisymmetric tensor.

We next consider the transformation of a tensor function of x . Instead of (3.1), (3.2) and (3.22) we now have

$$\varphi'^{\mu\nu\ldots}_{\rho\sigma\ldots}(x') = \Lambda^{\rho'}_{\rho} \Lambda^{\sigma'}_{\sigma} \cdots (\Lambda^{-1})^{\mu}_{\mu'} (\Lambda^{-1})^{\nu}_{\nu'} \cdots \varphi^{\mu'\nu'\ldots}_{\rho'\sigma'\ldots}(x) \tag{3.26}$$

Note that the translations are represented in a trivial way. (only the argument x is changed)

For irreducible tensors, one may introduce a collective index A, $A = 1, \ldots, N$, where N stands for the number of independent components.

$$\varphi'_A(x') = S_A^B(x)\,\varphi_B(x) \tag{3.27}$$

For infinitesimal Lorentz transformation this reads:

$$S_A^B = \delta_A^B + (\Sigma_{\alpha\beta})_A^B\,\delta\varepsilon^{\alpha\beta} \tag{3.28}$$

We now have

$$\varphi'_A(x') = \varphi'_A(x+\delta x) = \varphi'_A(x) + \partial^\mu \varphi_A(x)\,\delta x_\mu$$

$$\delta\varphi_A(x) = \varphi'_A(x) - \varphi_A(x)$$

$$= \left[(x_\alpha\partial_\beta - x_\beta\partial_\alpha)\varphi_A(x) + (\Sigma_{\alpha\beta})_A^B\,\varphi_B(x)\right]\delta\varepsilon^{\alpha\beta} \tag{3.29}$$

$$\left.\begin{array}{ll} L_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha & \text{is called the orbital part} \\ (\Sigma_{\alpha\beta})_A^B & \text{is called the spin part} \end{array}\right\} \tag{3.30}$$

In the rest system, the Fourier transform of $(x_\alpha\partial_\beta - x_\beta\partial_\alpha) \to 0$

But in the rest system $\Sigma_{\alpha\beta} \not\to 0$, and this part is zero only if spin is zero.

Lecture 4.

The homomorphism of SL(2,C) and the Lorentz group.

A relativistic formulation of SU(6) theory has to contain orbital angular momentum and spin mixed in a Lorentz invariant manner. For this to be possible the ordinary spin group SU(2) has to be extended to SL(2,C) which is the "universal covering group" of the Lorentz group. In this lecture we will first show that SL(2,C) is homomorphic to the Lorentz group.

SL(2,C) is the group of 2x2 complex matrices (Λ) with determinant one. Hence, it is a six parameter group. Consider a hermitian 2x2 matrix X (which is not an element of SL(2,C)). The hermiticity will be preserved by the transformations of SL(2,C) if its transformation law is:

$$X' = \Lambda X \Lambda^{+} \tag{4.1}$$

Furthermore, since $\det \Lambda = 1$,

$$\det X' = \det X \tag{4.2}$$

The matrix X can always be written in terms of 4 real numbers (x_μ) as:

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} \tag{4.3}$$

which is the most general hermitian 2x2 matrix.

$$\det X = x_0^2 - x_3^2 - x_1^2 - x_2^2 =$$
$$= g^{\mu\nu} x_\mu x_\nu = \text{invariant} \qquad (4.4)$$

One sees that the group SL(2,C) has the same number of parameters as the Lorentz group and leaves invariant the same quadratic form. One may also write:

$$X = x_\mu \sigma^\mu \qquad (4.5)$$

with

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The transformation of X can be written as either:

$$X' = x_\mu A \sigma^\mu A^+ \qquad (4.6)$$

$$\text{or} \quad X' = x'_\mu \sigma^\mu \qquad (4.7)$$

The second transformation (4.6) will leave det X invariant , only if

$$x'_\mu = L_\mu{}^\nu x_\nu$$

since only the Lorentz group leaves $g^{\mu\nu} x_\mu x_\nu$ invariant, Therefore, we have established a two-to-one correspondence between the group SL(2,C) and the Lorentz group.

$$\text{i.e.} \quad \pm A \longrightarrow L \qquad (4.8)$$

Thus, i.e. SL(2,C) has a discrete centre of 2 elements.

Notice that it follows from (4.6) and (4.7) that

$$x_{\mu} A \sigma^{\mu} A^{+} = x_{\mu} L^{\mu}_{\ \nu} \sigma^{\nu}$$

This relation should hold for any x . Hence

$$A \, (L^{-1})^{\nu}_{\ \mu} \, \sigma^{\mu} A^{+} = \sigma^{\nu} \tag{4.9}$$

This shows that σ^{μ} is a numerically invariant object under the combined A and L transformation. It is sometimes called an isometry and provides a link between two and 4 dimensional representations of SL(2.C).

The unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as well as the matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are elements of SL(2,C) . These two elements, I and -I, form an invariant sub-group C_2 of SL(2,C). (4.7) shows that

$$\frac{SL(2,C)}{C_2} = L \tag{4.10}$$

If we consider transformations of A in the neighbourhood of the identity, then -A is outside it. Hence in this neighbourhood there is a one-to-one correspondence between elements of SL(2,C) and L. This means that the Lie algebras of the two groups are the same.

From now on, we make the replacement

$$\begin{aligned} M_{\alpha\beta} &\longrightarrow i\, M_{\alpha\beta} \\ M_i &\longrightarrow i\, M_i \\ N_i &\longrightarrow i\, N_i \end{aligned} \qquad (4.11)$$

Then the commutation relations (3.18) and (3.21) acquire an additional factor i. It is easy to verify that the following set of 2x2 matrices have the correct commutation relations:

$$\begin{aligned} M_i &= \tfrac{1}{2}\,\sigma_i \\ N_i &= \tfrac{1}{2}\, i\sigma_i \end{aligned} \qquad (4.12)$$

$$\begin{aligned} [M_i, M_j] &= [\tfrac{1}{2}\sigma_i, \tfrac{1}{2}\sigma_j] = i\varepsilon_{ijk}(\tfrac{1}{2}\sigma_k) = i\varepsilon_{ijk} M_k \\ [M_i, N_j] &= [\tfrac{1}{2}\sigma_i, \tfrac{i}{2}\sigma_j] = i\varepsilon_{ijk}(\tfrac{i}{2}\sigma_k) = i\varepsilon_{ijk} N_k \\ [N_i, N_j] &= [\tfrac{i}{2}\sigma_i, \tfrac{i}{2}\sigma_j] = -i\varepsilon_{ijk}(\tfrac{1}{2}\sigma_k) = -i\varepsilon_{ijk} M_k \end{aligned} \qquad (4.13)$$

A general finite transformation Λ is then written as:

$$\begin{aligned} A &= e^{i(M_i\varepsilon_i + N_i\varepsilon_i')} \\ &= e^{i(\frac{1}{2}\sigma_i\varepsilon_i + \frac{i}{2}\sigma_i\varepsilon_i')} \end{aligned} \qquad (4.14)$$

The exponential in (4.14) is defined by the power series

$$\sum_n \frac{1}{n!} \left(\frac{i}{2} \sigma_i \varepsilon_i + \frac{i^2}{2} \sigma_i \varepsilon_i' \right)^n$$

The matrices A which satisfy

$$A^+ A = I \tag{4.15}$$

form the unitary three parameter subgroup SU_2. Its generators are: $M_i = \frac{1}{2} \sigma_i$. This is a compact sub-group. SU_2 contains also the invariant subgroup C_2 with the elements I and $-I$. In the same way as we have shown the homomorphism between $SL(2,C)$ and L, one shows that:

$$SU_2 / C_2 = SO_3 \tag{4.16}$$

Remark 1: Again one has a Schreier extension because SO_3 is not a subgroup of SU_2 . We show this for the one-parameter subgroup of SU_2, in which case the representation is:

$$e^{\frac{1}{2} i \varepsilon} \tag{4.17}$$

The corresponding representation for the rotation around any axis is:

$$e^{i \varepsilon} \tag{4.18}$$

So, a rotation of 2π corresponds to -1 in first case and $+1$ in the second case. Hence for (4.17), ε varies from 0 to 4π , in order to satisfy the group property. One can divide (4.17) into two sets:

$$e^{\frac{1}{2} i\varepsilon} \quad ; \quad e^{i\left(\frac{1}{2}\varepsilon + 2\pi\right)} = -e^{\frac{1}{2} i\varepsilon}$$

$$\text{for} \quad 0 \leq \varepsilon \leq 2\pi \tag{4.19}$$

One can make a correspondence between the first set and (4.18). However, this does not satisfy the group property. Since, for e.g., for $\varepsilon = 3\pi/2$,

$$\left(e^{i\frac{1}{2}\cdot\frac{3\pi}{2}}\right)^4 = e^{i\,3\pi} = -1 \,,$$ which is

in the second set. The same difficulty appears even if the group is divided in a different way. One can repeat the argument for the other parameters of the subgroup SU_2.

<u>Remark 2</u>: Lorentz group is a simple group and thus has no invariant subgroup, so that L cannot be written as the product of two groups. But the group O_4 is a semi-simple group and hence one can write:

$$O_4 = O_3 \otimes O_3 .$$

The only difference between O_4 and L is that, for O_4:

$$\det X = x_0^2 + x_1^2 + x_2^2 + x_3^2 = \text{invariant},$$

instead of (4.4).

Remark 3: SL(2,C) is the "Universal Covering Group" of the Lorentz group and is homomorphic to L. All representations of L will be representations of SL(2,C). But the converse is not true. For example: the two dimensional representation of SL(2,C) is only atwo valued representation of L. Similarly the spin $\frac{1}{2}$ representations of SU_2 are double valued representations of SO_3.

The Unitary representations of the Poincare algebra

To each transformation $x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu$ will correspond a unitary transformation $\mathcal{U}(\Lambda, a)$ such that

$$\mathcal{U}(\Lambda', a')\, \mathcal{U}(\Lambda, a) = \mathcal{U}(\Lambda'\Lambda, \Lambda' a + a') \qquad (4.20)$$

$$\mathcal{U}^+ \mathcal{U} = 1$$

Instead of the infinitesimal operators $M_{\alpha\beta}$ and P_μ, one often introduces

$$M_{\alpha\beta} = i\, \underline{M}_{\alpha\beta} \quad \text{and} \quad P_\mu = i\, \underline{P}_\mu$$

If $\mathcal{U}$ is unitary, $M_{\alpha\beta}$ and P_μ are hermitian. The commutation relation (3.25) will now acquire an additional factor i.

We now give an elementary proof that the noncompact Lorentz group has no finite dimensional, unitary representations (except the trivial representations). This proof was communicated to us by Dr.N.Cabibbo.

From the commutation relation (3.18) it follows:

$$[M_{12}, M_{03}] = 0 \; ; \quad [M_{23}, M_{01}] = 0$$

$$[M_{12} + iM_{03}, M_{23} - iM_{01}] = 0 \tag{4.21}$$

$$[M_{12} + iM_{03}, M_{23} + iM_{01}] = -2(M_{02} + iM_{31})$$

If two finite dimensional, hermitian matrices commute, they can be diagonalized simultaneously. The same is true for normal operators, i.e.

$$N = A + iB \;, \quad A^+ = A, \quad B^+ = B, \quad [A,B] = 0.$$

Hence $M_{12} + i\,M_{03}$ and $M_{23} - i\,M_{01}$ can be diagonalized simultaneously. Since M_{23} and M_{01} are hermitian , and $[M_{23}, M_{01}] = 0$, it follows from $(M_{23} - iM_{01})$ diagonal that M_{23} and M_{01} are separately diagonal. Similarly, M_{12} and M_{03} have to be diagonal. This leads therefore to

$$[M_{12} + iM_{03}, M_{23} + iM_{01}] = 0.$$

which contradicts the last equation in (4.21). It ofcourse follows that the Poincare group also can have no non-trivial, finite dimensional, unitary representation.

Lecture 5

Representations of the Algebra.

Let us now examine the representations of the Lie algebra of the Poincare' group according to Shirokov's method. The first step is to look for Casimir operators, i.e. the operators which commute with all the generators. We have seen that the Lorentz group has two invariant tensors, $g_{\mu\nu}$ and $\varepsilon_{\mu\nu\rho\sigma}$, so that all operators which commute with $M_{\alpha\beta}$ will be functions of

$$g^{\mu\nu} P_\mu P_\nu = P^\nu P_\nu \tag{5.1}$$

$$\left.\begin{aligned} \frac{1}{4|m|^2} g^{\mu\nu} \varepsilon_{\mu\alpha\beta\rho} M^{\alpha\beta} P^\rho \varepsilon_{\nu\gamma\delta\sigma} M^{\gamma\delta} P^\sigma &= \Gamma_\nu \Gamma^\nu \\ \text{with} \quad \Gamma_\mu = \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta\rho} M^{\alpha\beta} P^\rho & \end{aligned}\right\} \tag{5.2}$$

and
$$g^{\alpha\beta} g^{\gamma\delta} M_{\alpha\gamma} M_{\beta\delta} = M_{\alpha\gamma} M^{\alpha\gamma} \tag{5.3}$$

$$\varepsilon_{\alpha\beta\gamma\delta} M^{\alpha\beta} M^{\gamma\delta} \tag{5.4}$$

However, the operators (5.3) and (5.4) do not commute with P_μ. Therefore, the operators (5.1) and (5.2) are the two Casimir operators of the Poincare' group. The eigen values of these operators will characterize an irreducible representation of the group.

The next step is to find a complete set of commuting operators. Besides the two Casimir operators (5.1) and (5.2), one may take the three momentum operators P_i (since $P_\mu P^\mu$ is an invariant, only three among the four momenta are independent and we take these three to be P_i). Obviously none of the $M_{\alpha\beta}$ commutes with all P_i. But,

$$
\begin{aligned}
[T_\mu, P^\nu] &= \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta\rho} [M_{\alpha\beta} P^\rho, P^\lambda] \\
&= \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta\rho} \{ [M_{\alpha\beta}, P^\lambda] P^\rho + M_{\alpha\beta} \underbrace{[P^\rho, P^\lambda]}_{= 0} \} \\
&= \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta\rho} [M_{\alpha\beta}, P^\lambda] P^\rho \\
&= \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta\rho} (g^{\alpha\lambda} P^\beta - g^{\beta\lambda} P^\alpha) P^\rho \\
&= 0
\end{aligned}
$$

Hence all T^μ commute with all P^ν. Further, the conditions:

(i) $\quad T^\mu P_\mu = 0$

Implies that only three T_μ are independent

and (ii) $\quad [T_\mu, T_\nu] = \frac{i}{m} \varepsilon_{\sigma\mu\nu\lambda} T_\sigma P_\lambda$

that only one Γ_μ can be taken diagonal. This we take to be Γ_3. So that, one may choose for the complete set of commuting operators:

$$P_\mu P^\mu , \Gamma_\mu \Gamma^\mu, \quad P_i \ (i = 1,2,3) \quad \text{and} \quad \Gamma_3 \tag{5.5}$$

The operators P_μ and P^2 take all real values between $-\infty$ and $+\infty$. For $P^2 = m^2 \geqslant 0$, their physical meaning is obvious: they represent the four-momentum and the mass of a free particle. An eigen state will hereafter be charzcterized by $| m,s,p_i,s_3 >$ where m, s, p_i and s_3 stand for the eigenvalues of the complete set of commuting operators (5.5).

There will be four classes of representations, according to the value of m^2. We shall simply enumerate them, before coming back in more detail to the first among the following four cases:

I. $m^2 > 0$: P_μ is time-like.
II. $m^2 = 0$, $P_\mu \neq 0$: P_μ is on light-cone.
III. $m^2 < 0$. : P_μ is space-like
IV. $m^2 = 0$, $P_\mu = 0$: P_μ is null-vector.

<u>Case I. $m^2 > 0$</u>.

The four vector P_μ is time-like. Therefore, there is a further invariant s_H, which is the sign of the energy:

$$s_H = P_0 / | P_0 | \tag{5.6}$$

In order to find the eigenvalues of T^2 and T^3, we choose the particular vector

$$p^R_\mu = (m,\ 0,\ 0,\ 0\ ) \tag{5.7}$$

where P_μ^R is the eigen value of operator P_μ and the super-fix R stands for 'Rest' system. Since P_0 is the only component which has a non-zero eigen value, in this rest system of the particle, we have:

$$T_\mu |m, s, p_i, s_3\rangle = T_\mu^R\ |m, s, p_i, s_3\rangle$$

$$= \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta 0} M^{\alpha\beta} p^0 |m, s, p_i, s_3\rangle$$

$$= \frac{1}{2} \varepsilon_{\mu\alpha\beta 0} M^{\alpha\beta} |m, s, p_i, s_3\rangle$$

or $$T_\mu = \frac{1}{2} \varepsilon_{\mu\alpha\beta 0} M^{\alpha\beta}$$

Therefore, $$T_0^R = 0$$

and $$\left.\begin{aligned} T_0^R &= 0 \\ T_k^R &= \frac{1}{2} \varepsilon_{ijk} M^{ij} \end{aligned}\right\} \tag{5.8}$$

$$[T_i^R, T_j^R] = i\,\varepsilon_{ijk} T_k^R \tag{5.9}$$

Eq.(5.9) shows that T_i^R has the commutation relations of an angular momentum. Since, it is non-zero for $\vec{P} = 0$, it represents an intrinsic angular momentum of the particle, i.e. the <u>spin</u>. Further (5.9) defines the algebra of the group SO_3 and SU_2. These groups leave the <u>particular momentum</u>, P_μ^R, invariant. SO_3 is the <u>little group</u> of the Poincare' group, with respect to the momentum P_μ^R. SU_2 is the universal covering group of SO_3. The invariance of P_μ^R is obvious from the fact that (5.8) defines the rotation sub group of L and that P_μ^R has no space components. From (5.9) and the theory of the rotation group it follows that the eigenvalues of $(T_i^R)^2$ and T_3^R are:

$$\left.\begin{aligned} (T_i^R)^2 &= s(s+1), \quad s = 0, \tfrac{1}{2}, 1, \ldots. \\ T_3^R &= s_3 \, , \quad -s \lesssim s_3 \lesssim s \end{aligned}\right\} \qquad (5.10)$$

We have evaluated $T_\mu T^\mu = (T^\mu)^2$ in the particular system where $P_\mu = P_\mu^R$. Since it is a relativistic invariant we have in general

$$T_\mu T^\mu = s(s+1) \qquad (5.11)$$

Therefore the irreducible unitary representations are labelled by

$$\left.\begin{aligned} \infty > m^2 > 0 \\ s = 0, 1/2, 1, \ldots \\ s_H = \pm 1 \end{aligned}\right\} \qquad (5.12)$$

Within the irreducible representation, the states are labelled by continuous values of P_μ and discrete values of s_3. Obviously, the group generated by (5.8) leaves only P_μ^R invariant. To know the little group which leaves any P_μ invariant, we start with a four vector orthogonal to P_μ :

$$n_{(i)}^{\mu} P_{\mu} = 0$$

and define $$\Gamma_i^{(P)} = \Gamma_{\mu} n_{(i)}^{\mu} \tag{5.13}$$

The explicit form of $n_{(i)}^{\mu}$ is:

$$n_{(i)}^{\mu} = \delta_i^{\mu} - \frac{P_i}{m(m+P_0)} (m \delta_0^{\mu} + P^{\mu}) \tag{5.14}$$

The $\Gamma_i^{(P)}$ satisfy the commutation relation:

$$[\Gamma_i^{(P)}, \Gamma_j^{(P)}] = i \varepsilon_{ijk} \Gamma_k^{(P)} \tag{5.15}$$

and
$$\begin{aligned}
(\Gamma_{(i)}^{P})_{\lambda}^{\sigma} P_{\sigma} &= (\Gamma_{\mu} n_{(i)}^{\mu})_{\lambda}^{\sigma} P_{\sigma} \\
&= \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta\rho} (M^{\alpha\beta})_{\lambda}^{\sigma} P^{\rho} n_{(i)}^{\mu} P_{\sigma} \\
&= \frac{1}{2|m|} \varepsilon_{\mu\alpha\beta\rho} (g^{\alpha\sigma} \delta_{\lambda}^{\beta} - g^{\beta\sigma} \delta_{\lambda}^{\alpha}) \cdot \\
&\qquad \cdot P^{\rho} P_{\sigma} n_{(i)}^{\mu} \\
&= \frac{1}{2|m|} (\varepsilon_{\mu\alpha\lambda\rho} P^{\rho} P^{\alpha} - \varepsilon_{\mu\lambda\beta\rho} P^{\rho} P^{\beta}) n_{(i)}^{\mu} \\
&= 0
\end{aligned}$$

(because $P^{\rho} P^{\alpha}$ is symmetric in ρ and α while $\varepsilon_{\mu\alpha\lambda\rho}$ is completely anti-symmetric in all indices).

A little group can be found for any momentum provided $P^2 = m^2$. All the little groups which belong to the same mass are related through similaritytransformations.

For the sake of completeness, we will briefly discuss the other representations of the Poincare' group, although we shall not need them.

<u>Case II:</u> $m^2 = 0$, $P_\mu \neq 0$.

This is also a physical case which applies to neutrinos and photons. The four-vector P_μ is now on the light cone. One has to delete the factor $\frac{1}{m}$ in (5.2). Let us consider a particular vector $P_\mu^{(0)} = (1, 1, 0, 0)$. Then (5.2) gives:

$$T_\mu^{(0)} = \frac{1}{2}\varepsilon_{\mu\alpha\beta 0} M^{\alpha\beta} - \frac{1}{2}\varepsilon_{\mu\alpha\beta 1} M^{\alpha\beta}$$

$$T_0^{(0)} = -\frac{1}{2}\left(\varepsilon_{0231} M^{23} + \varepsilon_{0321} M^{32}\right) = -M^{23}$$

(since $\varepsilon_{0123} = 1$, cyclic)

$$T_1^{(0)} = \frac{1}{2}\left(\varepsilon_{1230} M^{23} + \varepsilon_{1320} M^{32}\right) = -M^{23}$$

$$T_2^{(0)} = \frac{1}{2}\left(\varepsilon_{2130} M^{13} + \varepsilon_{2310} M^{31}\right) - \frac{1}{2}\left(\varepsilon_{2031} M^{03} + \varepsilon_{2301} M^{30}\right)$$

$$= M^{13} + M^{03}$$

$$T_3^{(0)} = \frac{1}{2}\left(\varepsilon_{3120} M^{12} + \varepsilon_{3210} M^{21}\right) - \frac{1}{2}\left(\varepsilon_{3021} M^{02} + \varepsilon_{3201} M^{20}\right)$$

$$= -M^{12} - M^{02}$$

The group generated by $\Gamma_i^{(0)}$ (i = 1, 2, 3) leaves $P_\mu^{(0)}$ invariant. Its commutation relations are:

$$[\Gamma_2^{(0)}, \Gamma_3^{(0)}] = 0$$

$$[\Gamma_2^{(0)}, \Gamma_1^{(0)}] = i\,\Gamma_3^{(0)} \qquad (5.16)$$

$$[\Gamma_3^{(0)}, \Gamma_1^{(0)}] = -i\,\Gamma_2^{(0)}$$

Comparing the general formulas:

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, M_{\alpha\beta}] = g_{\mu\alpha} P_\beta - g_{\mu\beta} P_\alpha$$

with

$$[\Gamma_2^{(0)}, M_{32}] = -i\, g_{22} \Gamma_3$$

$$[\Gamma_3^{(0)}, M_{32}] = +i\, g_{33} \Gamma_2$$

one sees that $\Gamma_2^{(0)}$ and $\Gamma_3^{(0)}$ correspond to translations in a plane and $\Gamma_1^{(0)}$ (or M^{23}) to a rotation around the axis orthogonal to the plane. Hence, the algebra defined by (5.16) is ismorphic to the algebra of the group of <u>Eucledian transformations of the plane</u>. One can again study the representations of this group by using the technique of the little group, since $\Gamma_2^{(0)}$ and $\Gamma_3^{(0)}$ form an invariant subgroup. The only representation with physical interest are those for which the eigen values of $\Gamma_2^{(0)}$ and $\Gamma_3^{(0)}$ are zero:

$$\text{i.e.}\quad T_2^{(0)} \, | m, s, p_i, s_3 \rangle = 0 = T_3^{(0)} \, | m, s, p_i, s_3 \rangle \tag{5.17}$$

In this case the little group, i.e. the subgroup of the Eucledian group which leaves the two component vector $(T_2^{(0)} = 0,\ T_3^{(0)} = 0)$ invariant, is just generated by $T_1^{(0)}$. Since this is a rotation around axis 1, one has

$$T_1^{(0)} = \varepsilon = 0,\ \pm 1/2,\ \pm 1,\ \ldots\ldots$$

Each value of ε generates an irreducible representation, the sign of $T_1^{(0)}$ being also an invariant. Because of (5.17):

$$T_\mu^{(0)} = \varepsilon\, P_\mu^{(0)} \tag{5.18}$$

where ε is the projection of the spin on the 'momentum, i.e. the helicity.

Thus a particle with spin S and mass zero has only one helicity state, which is very different from the situation of particles with mass non-zero. Quantum field theory shows that the particle with opposite helicity is the anti-particle: this is the case of the neutrino. For the photon, particle and anti-particle are identical. Thus the photon has only two helicity states. In the classical theory this corresponds to the two possible polarizations viz, left and right circular polarizations.

There exist other unitary representations for $P^2 = 0$, but they do not seem to be of physical interest.

Lecture 6:

We have seen how the representations of the Poincare group fall into four different classes and we briefly discussed the cases which are of physical interest for free particles. For the sake of completeness, let us now consider the other two cases first.

Case III: $m^2 < 0$

The vector P_μ is now space-like. Let us consider the particular vector

$$P_\mu = (0, 0, 0, 1)$$

We have for (5.2), now:

$$\Gamma_\mu = \frac{1}{2} \varepsilon_{\mu\alpha\beta 3} M^{\alpha\beta}$$

or explicitly:

$$\Gamma_0 = \frac{1}{2} (\varepsilon_{0123} M^{12} + \varepsilon_{0213} M^{21}) = M^{12}$$

$$\Gamma_1 = \frac{1}{2} (\varepsilon_{1023} M^{02} + \varepsilon_{1203} M^{20}) = -M^{02}$$

$$\Gamma_2 = \frac{1}{2} (\varepsilon_{2013} M^{01} + \varepsilon_{2103} M^{10}) = M^{01}$$

$$\Gamma_3 = 0$$

Thus, Γ_0 represents a rotation around axis 3, while Γ_1 and Γ_2 represent Lorentz transformations along the axis 1 and 2 respectively. This group is called the "pseudo-rotation group" in three dimensions and is denoted by SO(2,1) (Note: In this notation, the Lorentz group will be denoted by SO(3,1)). In other words, this is the "Lorentz group in three dimensions" since the quadratic form which is left invariant, now, is:

$x_0^2 - x_1^2 - x_2^2$. This case is not of physical interest since no particles have negative mass. Therefore, we shall not discuss the representation of this group further.

Case IV. $m^2 = 0, \quad P_\mu = 0$

The little group here is just the homogeneous Lorentz group. Again no physical interpretation is known, till now.

The Unitary Representations of the Poincare group with $m^2 > 0$.

We follow here the global treatment of Joos[1] which is nearer to the original work of Wigner, i.e. we consider the representations of the group and not only those of the algebra. Since

$$[P_\mu, P_\nu] = 0$$

and the operator $P_\mu P^\mu$ commutes with the whole group, the operators P_μ are diagonal in some basis

$$\begin{aligned} P_\mu \,| p, \xi ; M, \varepsilon \rangle &= p_\mu \,| p, \xi ; M, \varepsilon \rangle \\ P^2 \,| p, \xi ; M, \varepsilon \rangle &= m^2 \,| p, \xi ; M, \varepsilon \rangle \end{aligned} \tag{6.1}$$

1) Hans Joos, Fortschritte der Physik 10, 65 (1962).

where $P = (P_\mu) = (\varepsilon\sqrt{M^2+\vec{p}^{\,2}}\,,\ \vec{p}\,)$ in which $\varepsilon = p_0/|p_0|$ and ξ is degeneracy parameter which is to be fixed later on. Consider now a Lorentz transformation L(p) which brings the four vector P_μ to the form P_μ^R, without rotating the plane (P, P^R):

$$L_\mu{}^\nu(p)\ p_\nu = p_\mu^R \quad ; \qquad p_\mu^R = (m, 0, 0, 0) \tag{6.2}$$

To this transformation corresponds a unitary transformation of the state vectors in Hilbert space:

$$\mathcal{U}\big(L(p)\big)\,|p, \xi; M, \varepsilon\rangle = |p^R, \xi; M, \varepsilon\rangle \tag{6.3}$$

One verifies indeed that if $P_\mu\,|p\rangle = p_\mu\,|p\rangle$, then $P_\mu\,|p^R\rangle = p^R\,|p^R\rangle$, because the relation

$$\mathcal{U}^{-1}(\Lambda)\,P_\mu\,\mathcal{U}(\Lambda) = L_\mu{}^\nu P_\nu \tag{6.4}$$

implies

$$\begin{aligned} P_\mu\,\mathcal{U}\big(L(p)\big)\,|p, \xi; M, \varepsilon\rangle &= \mathcal{U}\big(L(p)\big)L_\mu{}^\nu P_\nu\,|p, \xi; M, \varepsilon\rangle \\ &= \mathcal{U}\big(L(p)\big)L_\mu{}^\nu p_\nu\,|p, \xi; M, \varepsilon\rangle \\ &= p_\mu^R\,\mathcal{U}\big(L(p)\big)\,|p, \xi; M, \varepsilon\rangle \end{aligned} \tag{6.5}$$

Equation (6.5) expresses the fact that P_μ is at the same time a four-vector and an operator in Hilbert space.

The vectors $| p^R, \xi \rangle$ are transformed amongst themselves by three-dimensional rotations:

$$\mathcal{U}(R)\, | p^R, \xi \rangle = | p^R, \xi' \rangle \, \mathcal{D}_{\xi'\xi}(R) \tag{6.6}$$

where $\mathcal{D}_{\xi'\xi}(R)$ is the rotation matrix. This is indeed the case because:

$$P_i \, \mathcal{U}(R)\, | p^R, \xi \rangle = \mathcal{U}(R)\, R_i{}^j P_j \, | p^R, \xi \rangle = 0$$

The subgroup of L which leaves the given momentum P^R invariant is the Little group. Any homogeneous Lorentz transformation can be written as:

$$\Lambda = L^{-1}(\Lambda p)\, R(\Lambda, p)\, L(p)$$

with $\quad R(\Lambda, p) = L(\Lambda p)\, \Lambda\, L^{-1}(p) \qquad (6.7)$

It follows from $L_\mu{}^\nu p_\nu = p_\mu{}^R$ that $R(\Lambda, p)\, p^R = p^R$ so that $R(\Lambda, p)$ is indeed a rotation.

The unitary representation of Λ will be:

$$\begin{aligned}
\mathcal{U}(\Lambda)\, | p, \xi ; M, \varepsilon \rangle &= \mathcal{U}(L^{-1}(\Lambda p))\, \mathcal{U}(R(\Lambda, p))\, \mathcal{U}(L(p))\, | p, \xi ; M, \varepsilon \rangle \\
&= \mathcal{U}(L^{-1}(\Lambda p))\, \mathcal{U}(R(\Lambda, p))\, | p^R, \xi ; M, \varepsilon \rangle \\
&= \mathcal{U}(L^{-1}(\Lambda p))\, | p^R, \xi' ; M, \varepsilon \rangle\, \mathcal{D}_{\xi'\xi}(R(\Lambda, p)) \\
&= | \Lambda p, \xi' ; M, \varepsilon \rangle\, \mathcal{D}_{\xi'\xi}(R(\Lambda, p))
\end{aligned} \tag{6.8}$$

Therefore, an irreducible unitary representation of the Lorentz group is determined by an irreducible representation of the little group SO_3, belonging to the momentum p_μ^R. We have already seen that, the sign of the energy, ε is an invariant for $m^2 > 0$. Hence, one can summarize the situation as follows: In the Hilbert space of an irreducible unitary representation, one has a basis with the following properties:

a) $$P_\mu \,|\, p, s_3; m, s, \varepsilon \rangle = p_\mu \,|\, p, s_3; m, s, \varepsilon \rangle$$
$$p^2 = m^2 \quad ; \quad \varepsilon = p_0 / |p_0| = \pm 1$$

b) $$|\, p, s_3; m, s, \varepsilon \rangle = \mathcal{U}(L^{-1}(p)) \,|\, p^R, s_3; m, s, \varepsilon \rangle$$
$$p_\mu^R = L_\mu{}^\nu \, p_\nu$$

c) $$\vec{M}^2 \,|\, p^R, s_3; m, s, \varepsilon \rangle = s(s+1) \,|\, p^R, s_3; m, s, \varepsilon \rangle$$
$$s = 0, \tfrac{1}{2}, 1, \ldots$$

d) $$M_3 \,|\, p^R, s_3; m, s, \varepsilon \rangle = s_3 \,|\, p^R, s_3; m, s, \varepsilon \rangle$$
$$(M_1 \pm i M_2) \,|\, p^R, s_3; m, s, \varepsilon \rangle = \left[(s \mp s_3)(s \pm s_3 + 1) \right]^{1/2} |\, p^R, s_3 \pm 1; m, s, \varepsilon \rangle$$
$$M_i = T_i^{(R)}$$

The three operators $M_i = T_i^{(R)}$ are the generators of the little group in the "rest system", i.e. they leave the particular momentum
$$p_\mu^R = (\varepsilon m, 0, 0, 0)$$
invariant.

$$T_i^{(R)} \,|P^R, s_3; m, s, \varepsilon\rangle = |P^R, s_3'; m, s, \varepsilon\rangle (T_i^R)_{s_3' s_3}$$

$$= T_i \,|P^R, s_3; m, s, \varepsilon\rangle \quad (6.9)$$

From this it follows

$$\mathcal{U}^{-1}(L(p))\, T_i\, \mathcal{U}(L(p))\, \mathcal{U}(L^{-1}(p))\, |P^R, s_3; m, s, \varepsilon\rangle$$

$$= \mathcal{U}(L^{-1}(p))\, |P^R, s_3'; m, s, \varepsilon\rangle\, (T_i^R)_{s_3' s_3} \quad (6.10)$$

Now one has, for any four vector which is an operator in Hilbert space:

$$\mathcal{U}^{-1}(\Lambda)\; T_\mu \;\mathcal{U}(\Lambda) = \Lambda_\mu{}^\nu\, T_\nu \quad (6.11)$$

Hence

$$\mathcal{U}^{-1}(L(p))\, T_i\, \mathcal{U}(L(p)) = L_i{}^\nu\, T_\nu = T_i^{(p)} \quad (6.12)$$

where $L_\mu{}^\nu P_\nu = P_\mu{}^R$ and the explicit form for $L_\mu{}^\nu$ is given by:

$$L_\mu^\nu = \delta_\mu^\nu - \frac{1}{m(m+P_0)}\Big[m^2 \delta_\mu^0 \delta_0^\nu + m P_\mu \delta_0^\nu + P_\mu P^\nu$$

$$- (m + 2P_0)\, \delta_\mu^0\, P^\nu \Big] \quad (6.13)$$

It may be verified that

$$T_i^{(P)} = T_\mu\, n_{(i)}{}^\mu$$

where as already defined in an earlier lecture

$$n_{(i)}^{\ \mu} = \delta_i^{\ \mu} - \frac{p_i}{m(m+p_0)}(m\,\delta_0^{\ \mu} + p^\mu)$$

$\Pi_i^{(p)}$ are the generators of the little group which leave P_μ invariant, as eas shown in the preceding paragraph. From (6.10) and (6.12) it follows that:

e) $$\Pi_i^{(p)}\,|p, s_3; m, s, \varepsilon\rangle = |p, s_3; m, s, \varepsilon\rangle \left(\Pi_i^R\right)_{s_3' s_3} \tag{6.14}$$

so that the matrix elements of $\Pi_i^{(p)}$ in the basis spanned by $|p, s_3\rangle$ are the same as those of Π_i^R in the basis $|p^R, s_3\rangle$.

f) The basis vectors are normalised according to:

$$\langle p, s_3; m, s, \varepsilon\,|\,p', s_3'; m, s, \varepsilon\rangle = 2|p_0|\,\delta(\vec{p} - \vec{p}\,')\,\delta_{s_3 s_3'} \tag{6.15}$$

(All the states are fixed up-to a phase).

For finite translations:

$$\mathcal{U}(a)\,|p, s_3; m, s, \varepsilon\rangle = e^{i P_\mu a^\mu}\,|p, s_3; m, s, \varepsilon\rangle \tag{6.16}$$

and for finite Lorentz transformations:

$$\mathcal{U}(\Lambda)\,|p, s_3; m, s, \varepsilon\rangle = |\Lambda p, s_3'; m, s, \varepsilon\rangle\, D^{s}_{s_3' s_3}(R(\Lambda, p)) \tag{6.17}$$

where $D(R)$ is a representation of the rotation group in three dimensions $SO(3)$.

From (6.16) and (6.17) one gets the generators of the Poincare group.

$$P_\mu \,|\, p, s_3; m, s, \varepsilon\rangle = p_\mu \,|\, p, s_3; m, s, \varepsilon\rangle$$

$$\vec{M} \,|\, p, s_3; m, s, \varepsilon\rangle = \left[i(\vec{p} \times \vec{\nabla}_p) + \vec{\Gamma}^{(p)} \right] |p, s_3; m, s, \varepsilon\rangle \qquad (6.18)$$

$$\vec{N} \,|\, p, s_3; m, s, \varepsilon\rangle = \left[i p_0 \vec{\nabla}_p - \frac{\varepsilon}{m + |p_0|} \vec{p} \times \vec{\Gamma}^{(p)} \right] |p, s_3; m, s, \varepsilon\rangle$$

This gives all the representations.

Lecture 7:

Clebsch-Gordon coefficients of Poincare' group.

Out of the irreducible representations of a group one obtains further representations by forming their direct sums and direct products. In quantum mechanical systems, the relativistic particles are described by the products of unitary representations of the inhomogeneous Lorentz group. Therefore the examination of the direct products of representations of the Poincare' group is of practical importance.

To every inhomogeneous Lorentz transformation (Λ, a) is associated a unitary transformation $\mathcal{U}(\Lambda, a)$ in a Hilbert space $\mathcal{H}$ given the basis vectors $\{\Phi\}$. A basis in the product space of $\mathcal{H}' \otimes \mathcal{H}''$ can be built from the product of the canonical basis states: $|p', s_3'; m', s'\rangle$ and $|p'', s_3''; m'', s''\rangle$ of the representations $\mathcal{U}'(\Lambda, a)$ and $\mathcal{U}''(\Lambda, a)$. A general vector $|\Phi\rangle$ in the product space $\mathcal{H}' \otimes \mathcal{H}''$ has the form:

$$|\Phi\rangle = \sum_{s_3', s_3''} \iint \frac{d^3p'}{2|p_0'|} \frac{d^3p''}{2|p_0''|} \; |p', s_3'; m', s'\rangle \otimes |p'', s_3''; m'', s''\rangle \times$$

$$\times \langle p', s_3'; p'', s_3'' \mid \Phi\rangle \qquad (7.1)$$

where the coefficients $\langle p', s_3'; p'', s_3'' \mid \Phi\rangle$ are defined for:

$$p'^2 = M'^2, \quad p_0' > 0 \quad ; \quad p''^2 = M''^2, \quad p_0'' > 0$$

$$-s' \leq s_3' \leq s' \quad ; \quad -s'' \leq s_3'' \leq s''$$

(In (7.1) we have introduced a measure which is Lorentz invariant). In the direct product space, $\mathcal{H}' \otimes \mathcal{H}''$, a unitary representation $U(\Lambda, a)$ is defined by

$$U(\Lambda,a)\,|\Phi\rangle = \sum_{s_3', s_3''} \iint \frac{d^3p'}{2|p_0'|} \frac{d^3p''}{2|p_0''|} \left(\mathcal{U}'(\Lambda,a)\,|p', s_3'; m', s'\rangle\right) \times$$

$$\times \left(\mathcal{U}''(\Lambda,a)\,|p'', s_3''; m'', s''\rangle\right) \langle p', s_3'; p'', s_3'' \mid \Phi\rangle$$

(7.2)

with the direct product of representations:

$$\mathcal{U}'(\Lambda,a) \otimes \mathcal{U}''(\Lambda,a) = U(\Lambda,a) \qquad (7.3)$$

Such a product does not belong to an Irreducible representation of Poincare group. So, a canonical basis for $U(\Lambda, a)$ in $\mathcal{H}' \otimes \mathcal{H}''$ is introduced by:

$$|p, s_3 ; m, s, \eta\rangle = \sum_{s_3', s_3''} \iint \frac{d^3p'}{2|p_0'|} \frac{d^3p''}{2|p_0''|} \, |p', s_3' ; m', s'\rangle \otimes$$

$$\otimes |p'', s_3''; m'', s''\rangle \begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \\ p' & s_3' & ; & p'' & s_3'' & \Big| & p & s_3 & \eta \end{pmatrix} \tag{7.4}$$

where the coefficients $\begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \\ p' & s_3' & ; & p'' & s_3'' & \Big| & p & s_3 & \eta \end{pmatrix}$

are called the Clebsch-Gordon coefficients of the inhomogeneous Lorentz group. The parameter η denotes the degeneracy. The infinitesimal generators are additive so that:

$$\begin{aligned} P_\mu &= P_\mu' \otimes 1'' + 1' \otimes P_\mu'' = P_\mu' + P_\mu'' \\ M_{\mu\nu} &= M_{\mu\nu}' \otimes 1'' + 1' \otimes M_{\mu\nu}'' = M_{\mu\nu}' + M_{\mu\nu}'' \end{aligned} \tag{7.5}$$

Due to this, the Clebsch-Gordon coefficients for the Poincare' group obey the following relations:

$$\begin{aligned} (P_\mu' + P_\mu'' - P_\mu) \begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \\ p' & s_3' & ; & p'' & s_3'' & \Big| & p & s_3 & \eta \end{pmatrix} &= 0 \\ \left((p' + p'')^2 - M^2\right) \begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \\ p' & s_3' & ; & p'' & s_3'' & \Big| & p & s_3 & \eta \end{pmatrix} &= 0 \end{aligned} \tag{7.6}$$

$$\begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \eta \\ p' & s_3' & ; & p'' & s_3'' & \Big| & p & s_3 & \end{pmatrix} = \sum_{\bar{s}_3', \bar{s}_3''} \begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \eta \\ L(p)p' & \bar{s}_3' & ; & L(p)p'' & s_3'' & \Big| & P_R & s_3 & \end{pmatrix} \times$$

$$\times \; D^{s'}_{s_3' \bar{s}_3'}\left(R(p',p)\right) D^{s''}_{s_3'' \bar{s}_3''}\left(R(p'',p)\right) \qquad (7.7)$$

This relation connects the C.G. coefficients for P to those for P_R. The relation (7.7) is obvious if we recollect the following relationships due to a general Lorentz transformation:

$$\mathcal{U}(\Lambda)\, |P, s_3; m, s\rangle = \sum_{s_3'} |\Lambda P, s_3'; m, s\rangle\, D_{s_3 s_3'}\left(R(P)\right)$$

$$\mathcal{U}(L(p))\, |p', s_3'; m', s'\rangle = \sum_{\bar{s}_3'} |L(p)p, s_3'; m', s'\rangle\, D_{s_3', \bar{s}_3'}\left(R(p',p)\right)$$

where $\quad R(p, p') = L(p')\, L^{-1}(p)\, \left(L(p)p'\right)$

which satisfies $\quad R\,(P, p')\, P_R = P_R$

The total operator will obey the relations:

$$\sum_{\bar{s}_3', \bar{s}_3''} \left(\vec{p}\,' \times \frac{1}{i} \vec{\nabla}_{p'} + \vec{p}\,'' \times \frac{1}{i} \vec{\nabla}_{p''} + \vec{S}\,' + \vec{S}\,''\right)^2_{s_3', s_3''; \bar{s}_3', \bar{s}_3''} \times \begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \eta \\ p' & \bar{s}_3' & ; & p'' & \bar{s}_3'' & \Big| & P_R & s_3 & \end{pmatrix} =$$

$$= s(s+1) \begin{pmatrix} m' & s' & ; & m'' & s'' & \Big| & m & s & \eta \\ p' & s_3' & ; & p'' & s_3'' & \Big| & P_R & s_3 & \end{pmatrix} \qquad (7.8)$$

$$\sum_{\bar{s}_3', \bar{s}_3''} \left(\vec{p}\,' \times \frac{1}{i} \vec{\nabla}_{p'} + \vec{p}\,'' \times \frac{1}{i} \vec{\nabla}_{p''} + \vec{s}\,' + \vec{s}\,'' \right)_q \Big|_{s_3', s_3''; \bar{s}_3', \bar{s}_3''} \times$$

$$\times \left(\begin{matrix} m' & s' ; & m'' & s'' \\ p' & \bar{s}_3' ; & p'' & \bar{s}_3'' \end{matrix} \middle| \begin{matrix} m & s & \\ p_R & s_3 & \eta \end{matrix} \right) =$$

$$= - \sqrt{s(s+1)}\,(1, q; s, s_3 | s, s_3 + q) \left(\begin{matrix} m' & s' ; & m'' & s'' \\ p' & s_3' ; & p'' & s_3'' \end{matrix} \middle| \begin{matrix} m & s & \eta \\ p_R & s_3 + q & \end{matrix} \right) \qquad (7.9)$$

Due to the existence of a relative orbital momentum ℓ, the product contains the same representation more than once, hence the degeneracy noted by η. One first combines the two spins s_1 and s_2 and the resultant vector with ℓ. This shows that spin is not conserved. The Clebsch-Gordon coefficients have the orthogonality property

$$\sum_{s_3', s_3''} \iint \frac{d^3p'}{2|p_0'|} \frac{d^3p''}{2|p_0''|} \left(\begin{matrix} m' & s' ; & m'' & s'' \\ p' & s_3' ; & p'' & s_3'' \end{matrix} \middle| \begin{matrix} \bar{m} & \bar{s} & \\ \bar{p} & \bar{s}_3 & \bar{\eta} \end{matrix} \right)^* \left(\begin{matrix} m' & s' ; & m'' & s'' \\ p' & s_3' ; & p'' & s_3'' \end{matrix} \middle| \begin{matrix} m & s & \\ p & s_3 & \eta \end{matrix} \right)$$

$$= 2|p_0|\, \delta(m - \bar{m})\, \delta_{\eta\bar{\eta}}\, \delta_{s_3 \bar{s}_3}\, \Theta(m - m' - m'') \qquad (7.10)$$

The inner product for the basis vectors becomes fixed because of (7.1) in the following way

$$\langle \Phi_1 | \Phi_2 \rangle = \sum_{s_3', s_3''} \iint \frac{d^3p'}{2|p_0'|} \frac{d^3p''}{2|p_0''|} \langle p', s_3'; p'', s_3'' | \Phi_1 \rangle^* \times$$

$$\times \langle p', s_3' ; p'', s_3'' | \Phi_2 \rangle \qquad (7.11)$$

Lecture 8

We have seen that the group SL(2,C) is homomorphic to L. The generators of SL(2,C) in the fundamental two-dimensional representation were:

$$M_i = \frac{1}{2}\sigma_i = M_i^+$$

$$N_i = \frac{1}{2}\sigma_i = -N_i^+$$

$$\delta\varphi = i M_i \delta\varepsilon^i \varphi \tag{8.1}$$

and $$\delta\varphi = i N_i \delta(\varepsilon')^i \varphi$$

Instead of (8.1) one can also use the 4x4 Dirac matrices and the use of Dirac matrices turns out to be more advantageous. These matrices are defined by:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2 g_{\mu\nu} \tag{8.2}$$

(where $g_{\mu\nu}$ is (+,-,-,-) as before). For unitary representations of the γ's we adopt the convention:

$$\gamma_\mu^+ = \gamma_\mu^{-1} \; ; \; \gamma_0^+ = \gamma_0 \text{ and } \gamma_i^+ = -\gamma_i \tag{8.3}$$

In terms of Dirac matrices, the generators of SL(2,C) can be defined as:

$$M_{ij} = \frac{i}{4}(\gamma_i\gamma_j - \gamma_j\gamma_i) = M_{ij}^+$$

$$N_i = M_{0i} = \frac{i}{4}(\gamma_0\gamma_i - \gamma_i\gamma_0) = -N_i^+ \tag{8.4}$$

Let us choose the particular representation:

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad ; \quad \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \; ; \gamma_5 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \gamma_0\gamma_1\gamma_2\gamma_3 \tag{8.5}$$

Then,

$$M_{ij} = \left(\begin{array}{c|c} \frac{1}{2}\varepsilon_{ijk}\sigma_k & \\ \hline & \frac{1}{2}\varepsilon_{ijk}\sigma_k \end{array}\right) ; \; N_i = \left(\begin{array}{c|c} -\frac{i}{2}\sigma_i & 0 \\ \hline 0 & \frac{i}{2}\sigma_i \end{array}\right) \tag{8.6}$$

So, one gets a four-dimensional reducible representation of SL(2,C). However, it becomes irreducible if one includes the parity operation, as will be seen in a minute.

If a transformation of SL(2,C) is:

$$\delta\psi = i\, M_i \,\delta\varepsilon^i \,\psi$$
$$\text{and} \quad \delta\psi' = i\, N_i \,\delta(\varepsilon')^i \,\psi \tag{8.7}$$

where ψ is a four-component spinor, then it is easy to see that

$$\bar{\psi}\psi = \text{invariant.} \qquad (\bar{\psi} = \psi^+ \gamma_0)$$
$$\bar{\psi}\gamma_\mu\psi = \text{four-vector} \tag{8.8}$$

space reflection operator is one which anticommutes with γ_i but commutes with γ_0, so that, one obvious choice of the parity operator is γ_0, which is non-diagonal. (This is why Dirac matrices are more advantageous, since they lead to a simple form for the parity operator.)

Remark:

The relation between SL(2,C) and L was shown to arise from the form-invariance of the four matrices σ_μ . Here, the analogous relation is:

$$S^{-1} L_\mu{}^\nu \gamma_\nu S = \gamma_\mu \tag{8.9}$$

where S is an element of SL(2,C). Equation (8.9) leads also to Eq.(8.8).

A four-component Dirac spinor describes a particle with positive energy and spin ½. Hence, two of the components are related to two others, the link being the Dirac equation:

$$(i \gamma^\mu \partial_\mu + m) \psi(x) = 0 \tag{8.10}$$

In momentum space

$$\psi(x) = u(p) e^{ip\cdot x}$$

so that equation (8.10) takes the form:

$$(\gamma^\mu p_\mu - m) u(p) = 0 \tag{8.11}$$

(In fact, the whole purpose of the Dirac equation is to get rid of the two redundant components in the four component spinor by bringing in a relationship between the two upper and two lower components of the Dirac four spinor.)

Let us no take the representation in which

$$\gamma_0' = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{8.12}$$

Then in the rest system, $P_\mu = P_\mu^R = (m,0,0,0)$. Eq.(8.11) gives no restriction for the first two components of u, so that u_1 and u_2 are arbitrary, while it implies $u_3 = u_4 = 0$ So, in the rest system, one has just a two component spinor describing a spin $\frac{1}{2}$ particle.

The invariance of $\overline{\psi}\psi$ can also be expressed in the following way·

$$\psi' = S\psi$$

$$\overline{\psi}'\psi' = \overline{\psi}\psi \Rightarrow \underline{S^+\gamma_0 S = \gamma_0} \tag{8.13}$$

The fact that $\overline{\psi}\gamma_\mu\psi$ is a four-vector also implies:

$$\overline{\psi}'\gamma_\mu p^\mu \psi' = \overline{\psi}\gamma_\mu p^{\mu'}\psi$$

$$S^+\gamma_0\gamma_\mu p^\mu S = \gamma_0\gamma_\mu p^{\mu'}$$

$$\text{or } S^{-1}\gamma_\mu p^\mu S = \gamma_\mu p^{\mu'} \quad ; \quad p^{\mu'} = (L^{-1})^\mu{}_\nu p^\nu \tag{8.14}$$

Actually Eq. (8.14) follows not only from Eq.(8.13) but also from the additional restriction (8.9). If one leaves this restriction, one gets the larger group $\mathcal{U}(2,2)$. This notation means that $\mathcal{U}(2,2)$ leaves a pseudo-hermitian form with two plus signs and two minus signs invariant, as is apparent from Eqns.(8.12) and (8.13). The group $\mathcal{U}(2,2)$ is non compact and leaves the pseudo-hermitian form invariant:-

$$\psi^{+}\gamma_{o}\psi = u_1^{+}u_1 + u_2^{+}u_2 - u_3^{+}u_3 - u_4^{+}u_4 \quad (8.15)$$

We have to look now for that group of transformations which leave Eq.(8.14) invariant to find the generators of $\mathcal{U}$ (2,2). From Eq.(8.13) one can find the sixteen generators of $\mathcal{U}$ (2,2) as follows:

$$\delta\psi = i\,\Gamma_A\,\delta\varepsilon^A\,\psi$$

$$(1 - i\,\Gamma_A^{+}\delta\varepsilon^A)\,\gamma_0\,(1 + i\,\Gamma_A\,\delta\varepsilon^A) = \gamma_0,$$

which should hold for any ε . Therefore,

$$\gamma_0\,\Gamma_A = \Gamma_A^{+}\,\gamma_0 \quad (8.16)$$

$$\therefore \Gamma_A : \quad \mathbb{1}\,,\ \gamma_\mu\,,\ \gamma_5\,,\ \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\,,\ i\gamma_\mu\gamma_5 \quad (8.17)$$

The group SU(2,2) is the one which does not have $\Gamma_A = \mathbb{1}$ as a generator. $\mathcal{U}$(2,2) or SU(2,2) has an irreducible fifteen-dimensional representation. One may use it to define 15 momenta through a formula analogous to (8.14):

$$S^{-1}\,\Gamma_A\,P^A S = \Gamma_A\,P^{A'}\,, \quad A = 1, \ldots, 15 \quad (8.18)$$

Of course, this procedure is not unique. ($\Gamma_0 = \mathbb{1}$, has to be omitted). The motivation for Eq.(8.18) is the following:

We want the set of 15 momenta

(a) to form an irreducible, real representation of $U(2,2)$

(b) to contan the four physical momenta P_μ. Eq.(8.14) shows that this is the case.

These 15 momenta may be thought of as translations in a 15-dimensional real space so that one is dealing with the semi-direct product:

$$SU(2,2) \boxed{\sigma} T_{15} \tag{8.19}$$

Using the 15 momenta, one can write down a "Dirac" equation, which is invariant under $SU(2,2)$, as:

$$(T_A P^A - m)\psi = 0 \tag{8.20}$$

<u>Remark</u>:

Within the set of 16 generators in (8.17) the set of 8 generators:

$$\mathbb{1}, \; \gamma_0, \; i\gamma_i\gamma_j \;\text{ and }\; i\gamma_i\gamma_5 \tag{8.21}$$

are hermitian and these form a compact subgroup. An equivalent set of 8 generators are:

$$(1+\gamma_0)\gamma_i\gamma_j \quad , \quad (1+\gamma_0) \tag{8.22a}$$

and

$$(1-\gamma_0)\gamma_i\gamma_j \quad , \quad (1-\gamma_0) \tag{8.22b}$$

The set (8.22a) and (8.22b) are separate sets of generators for a $\mathcal{U}(2)$ group. Therefore,

$$\mathcal{U}(2,2) \supset \mathcal{U}(2) \otimes \mathcal{U}(2) \tag{8.23}$$

We are now in a position to consider the relativistic generalization of the internal symmetry group SU93). This can be done in two ways:

(I) SL(2,C) $\longrightarrow$ SL(6,C)

(II) U(2,2) $\longrightarrow$ U(6,6).

The first way is that chosen by Ruhl et al, and Fulton and Wess. The second way has beem by Salam et al, Beg and Pais, Sakita and Wali and Cornwall et al. There is no logical reason in favour of the second approach. On the contrary, it leads to contradictions with unitarity and crossing symmetry.

Lecture 9.

In this lecture we will discuss the SL(6,C) and U(6,6) groups. It is based on the papers of Bell and Ruegg[1)] and Ruhl[2)].

I. Homogeneous SL(6,C) group:

We first try the algebra

$$\frac{i}{2}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right)\otimes\lambda_a \quad , (a=0,1,\ldots,8) \qquad (9.1a)$$

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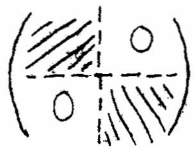
1) J.S.Bell and H.Ruegg, CERN, TH 559, 30 April 1965, to appear in Nuovo Cimento.

2) W.Ruhl, Nuovo Cimento, 37, 301-332 (1965).

where λ_a (a = 1, ..., 8) are the eight 3x3 matrices generating SU(3) together with the 3x3 unit matrix λ_0 . The set of generators (9.1a) are not closed under commutation, so one has to add the generators

$$\mathbb{1} \otimes \lambda_a \quad \text{and} \quad \gamma_5 \otimes \lambda_a \tag{9.1b}$$

Thus one gets a set of 72 matrices which form a 12x12 <u>reducible</u> representation of GL(6,c). They reduce according to:



(9.1c)

where each block is 6x6. One gets SL(6,C) group by assuming that each block be traceless. This means taking out of the set (9.1b) the two matrices:

$$\mathbb{1} \otimes \lambda_0 \quad \text{and} \quad \gamma_5 \otimes \lambda_0 \tag{9.1d}$$

The set (9.1a) with (9.1b) is, however, irreducible if one adds the parity operation:

$$P : \qquad \gamma_0 \otimes \lambda_0 = \Gamma_0 \tag{9.2}$$

which is non-diagonal. Each block in (9.1c) is of the form:

$$\left(\frac{1+i\gamma_5}{2}\right)\Sigma \qquad \text{and} \qquad \left(\frac{1-i\gamma_5}{2}\right)\Sigma \tag{9.3}$$

where Σ is the set of 12x12 matrices given by (9.1a) and (9.1b). Parity thus interchanges the two blocks. The situation is completely analogous to the case of SL(2,C), where one had two component and four component spinors.

II. Inhomogeneous SL(6,C) group:

If ψ is a 12 component spinor then its finite transformations are:

$$\psi' = S\psi \tag{9.4}$$

where S is an element of the group SL(6,C). Using (9.1) one gets:

$$\left.\begin{aligned} S^{+}\Gamma_0 S &= \Gamma_0 \\ S^{-1}(\gamma_5 \otimes \lambda_0) S &= \gamma_5 \otimes \lambda_0 \\ \det S &= 1 \end{aligned}\right\} \tag{9.5}$$

For this group the invariants are:

$$\bar{\psi}\psi = \psi^{+}\Gamma_0\psi = \text{invariant} \tag{9.6}$$

$$\text{and} \quad \bar{\psi}(\gamma_5 \otimes \lambda_0)\psi = \text{invariant} \tag{9.7}$$

(9.7) is however pseudo-scalar under reflections of space.

Exactly as in the case of SL(2,C), one can introduce momenta by considering a hermitian 6x6 matrix X. Thus, one gets 36 momenta. However, in this case, parity transforms this set into a new, inequivalent set of 36 momenta. The proof for this has been given by Ruhl[1]. So, if one wants the momenta to have the correct behaviour under space reflection, one has to introduce 72 momenta:

$$\Gamma_0(\gamma_\mu \otimes \lambda_a)P^{\mu a} \quad \text{and} \quad \Gamma_0(i\gamma_5)(\gamma_\mu \otimes \lambda_a)\hat{P}^{\mu a} \tag{9.8}$$

1) W.Ruhl, Nuovo Cimento, 37, 301-332 (1965)

The factor Γ_0 is necessary for the hermiticity ($\gamma_i^+ = -\gamma_i$ and $\gamma_o^+ = \gamma_o$). Using these 72 momenta, one may again write a "Dirac" equation which is invariant under SL(2,C). So, we finally arrive at the group:

$$SL(6,C) \boxed{\sigma} T_{72} \tag{9.9}$$

It should be noticed that, while SL(2,C) was homomorphic to the Lorentz group SO(3,1), not such homomorphism exists here. There is no real, quadratic form left invariant by SL(6,C).

A second difference comes from the fact that $\det S = 1$, has different consequences for SL(2,C) and SL(6,C):

In the case of 2x2 matrices,

$$\det S \, \varepsilon_{ab} = S_a{}^{a'} \, S_b{}^{b'} \, \varepsilon_{a'b'}$$

and if $\det S = 1$, ε_{ab} is an invariant and

$$\varepsilon_{ab} = -\varepsilon_{ba}$$

This implies that a quadratic antisymmetric form is left invariant.

In the case of 6x6 matrices,

$$\det S. \, \varepsilon_{abcdef} = S_a{}^{a'} \, S_b{}^{b'} \, S_c{}^{c'} \, S_d{}^{d'} \, S_e{}^{e'} \, S_f{}^{f'} \, \varepsilon_{a'b'c'd'e'f'}$$

Therefore, here $\det S = 1$ implies the invariance of a form of degree six. This is the mathematical reason why the sets:

$$\left(\frac{1+i\gamma_5}{2}\right)\Gamma_o(\gamma_\mu \otimes \lambda_a)\, P^{\mu a} \quad \text{and} \quad \left(\frac{1-i\gamma_5}{2}\right)\Gamma_o(i\gamma_5)(\gamma_\mu \otimes \lambda_a)\, \hat{P}^{\mu a} \tag{9.10}$$

are inequivalent. In the case of SL(2,C) the corresponding sets are related by an equivalence transformation.

It may also be seen that, while the sets (9.10) have definite transformation properties under parity, the transformations properties of (9.8) are:

$$S^{+}\Gamma_0(\gamma_\mu \otimes \lambda_a)P^{\mu a}S = \Gamma_0(\gamma_\mu \otimes \lambda_a)P^{\mu a'}$$
$$\text{and} \quad S^{+}\Gamma_0(i\gamma_5)(\gamma_\mu \otimes \lambda_a)\hat{P}^{\mu a}S = \Gamma_0(i\gamma_5)(\gamma_\mu \otimes \lambda_a)\hat{P}^{\mu a'} \qquad (9.11)$$

which change the two sets of momenta into each other. ((9.11) is analogous to (8.14)). Hence, the requirement of parity and SL(6,C) covariance entails the existence of 72 momenta.

II. Inhomogeneous U(6,6) group:

Until now, we have incorporated SU(3) by starting from SL(2,C). This is the most natural thing to do since the group with which we end up is SL(6,C) $\boxed{\sigma}$ T_{72} which is the smallest group which contains:

(1) the Poincare group: L $\boxed{\sigma}$ T_4 $\sim$ SL(2,C) $\boxed{\sigma}$ T_4

(2) the group $\mathfrak{U}(3)$.

(3) parity.

A number of authors, however, start with $\mathfrak{U}(2,2)$ instead of SL(2,C). They then arrive at the algebra of the group $\mathfrak{U}(6,6)$:

$$\Gamma_A \otimes \lambda_a \quad \text{where} \quad A = 1, \ldots, 16$$
$$a = 0, \ldots\ldots, 8 \qquad (9.12)$$

where Γ_A are the sixteen 4x4 matrices given by (8.17) and λ_a the nine 3x3 matrices of $\mathcal{U}(3)$. These are the 144 generators of the $\mathcal{U}(6,6)$ group in the irreducible 12 dimensional representation.

All the unitary representations of the inhomogeneous $\mathcal{U}(6,6)$ group have been found by Ruhl[1] by generalizing the procedure given in these lectures for the Poincare group. Ruhl has also found all the unitary representations for the inhomogeneous SL(6,C) group[2].

With $\mathrm{tr}\ \Gamma_A \otimes \lambda_a = 0$, which is equivalent to eliminating the generator $\mathbb{1} \otimes \lambda_0$, one has 143 generators which form the algebra of $S\mathcal{U}(6,6)$.

The finite transformations:

$$\psi' = S\psi$$

leave again the form:

$$\overline{\Psi}\psi = \psi^{+}(\gamma_0 \otimes \lambda_0)\psi = \text{invariant}$$

without any additional restriction. Exactly as in the case of U(2,2), one may use the adjoint real representation of U(6,6), to define 143 momenta. The formula analogous to (8.18) is now:

$$S^{-1}(\Gamma_A \otimes \lambda_a) S P^{Aa} = (\Gamma_A \otimes \lambda_a)(P^{Aa})'$$

$$\mathrm{tr}(\Gamma_A \otimes \lambda_a) = 0 \tag{9.13}$$

$$(P^{Aa})' = O^{Aa}_{Bb}(S)\, P^{Bb} \tag{9.14}$$

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1) W.Ruhl, Nuovo Cimento, 38, 675 (1965).
2) W.Ruhl, Nuovo Cimento, 39, 307 (1965).

where $O(s)$ is the 143 dimensional representation of $U(6,6)$.

One may now write a formal "Dirac" equation, invariant under $U(6,6)$, as :

$$[(\Gamma_A \otimes \lambda_a) P^{Aa} - m]\psi = 0 \qquad (9.15)$$

So far we have been doing no physics. Let us hope that some physical content can be extracted from the schemes presented till now.

We have to start with negative statements. To our knowlege, nobody has explicitly shown that one can build consistent quantum field theory which is invariant under $SL(6,C)$ or $SU(6,6)$. For example, the free Lagrangian of a Dirac particle contains the term.

$$\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi$$

where μ runs from 1 to 4 and not 1 to 72 or 1 to 143.

One may however postulate that the scattering <u>operator</u> be invariant under $SL(6,C)$ or $SU(6,6)$. How the Lagrangian should look, in order to satisfy this, is as yet an unsolved problem. One may, of course, forget the Lagrangian altogether.

However, one knows that free particles have only four momenta. Therefore, one has to impose the <u>supplementary condition</u>, that for physical <u>states</u>, only the usual operators P_{μ} have non-zero eigen values. Even then, one may run into trouble. Coleman has argued that one gets a continuous mass spectrum. Alles and Amati have shown, in a particular case (qq and $q\bar{q}$ elastic scattering), that invariance under inhomogeneous $U(6,6)$ group is incompatible with unitarity together with crossing symmetry.

Lecture 10

It is well known that one can judge the success of a theory after making a comparison with the results predicted by the theory with the existing experimental data. If any schemes yields some predictions which agree with the experimental situation, it is definitely worthwhile studying it.

Let us suppose that the scattering operator is invariant under a group G, and the states are restricted by supplementary conditions.

a) Considering first a free particle at rest, it is to be noted that whatever be the group with which we deal and whatever may be the momenta to start with, the free particle at rest has only one non-vanishing momentum whole value is equal to the mass m. The sub-group of G which leaves this momentum invariant is just the little group of G. The little group, as we have seen, enables us to find all the irreducible unitary representations of the inhomogeneous group G. Thus it will serve to classify the particles.

b) Consider particles moving all in the same direction, say along the Z-axis. This will be thecase for a two body disintegration process like $N^* \rightarrow N + \pi$; or forward and backward scattering; or for vertex functions, which describe the contribution of strong interactions to electromagnetic and weak processes. In this case, an even smaller subgroup, which we call the collinear group, will leave P_0 and P_1 invariant. The S-matrix will therefore be invariant under this smaller group.

c) An even smaller subgroup will leave P_0, P_1 and P_2 invariant. This subgroup we call as the coplanar subgroup. For coplanar processes, therefore, the S-matrix will be invariant under a still smaller subgroup.

d) For general processes - i.e. processes in which particles are moving in all three directions - only U(3) symmetry will be left.

We first give the results of our discussion which is to follow:

Group G:	U(2,2)	U(6,6)	SL(6,C)
Little group of G:	$U(2) \otimes U(2)$	$U(6) \otimes U(6)$	SU(6)
Collinear group of G:	U(2)	U(6)	$SU(3) \otimes SU(3)$
Coplanar group of G:	$U(1) \otimes U(1)$	$U(3) \otimes U(3)$	SU(3)

i.e. We indicate the decrease in symmetry due to increase of generality of the process, for example starting with U(2,2) symmetry, by:

$$U(2,2) \supset U(2) \otimes U(2) \supset U(2) \supset U(1) \otimes U(1).$$

This telescoping of the groups is called the "Hierachy of groups" and was first discussed by Lipking and Meshkov and Dashen and Gell-Mann.

Let us start with the little group of U(2,2). We have already seen that the momenta transform according to:

$$S^{-1} \Gamma_A \; P^A \; S = \Gamma_A \; P^{A'} \qquad (10.1)$$

The subgroup which leaves P_A^R invariant, where the only non-vanishing component of P_A^R is $P_0^R = m$, obeys:

$$S^{-1} \gamma_0 S = \gamma_0 \tag{10.2}$$

Using the representation:

$$\gamma_0 = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right)$$

where, as before, the blocks denote 2x2 matrices, we have:

$$S = \left(\begin{array}{c|c} S_a & 0 \\ \hline 0 & S_b \end{array}\right) \tag{10.3}$$

where S_a and S_b are arbitrary 2x2 matrices,

As an application, consider a set of wave functions $\phi_\alpha{}^\beta$ transforming as the adjoint representation of U(2,2), i.e. as the product of spinors $\psi_\alpha \bar{\psi}^\beta$. In the rest system let us write the matrix ϕ in the form:

$$\phi = \left(\begin{array}{c|c} \phi_a & \phi_c \\ \hline \phi_d & \phi_b \end{array}\right) \tag{10.4}$$

Then under the little group, $\phi_a, \cdots, \phi_d$ transform separated into one another and so are not connected by the symmetry. To obtain the minimum multiplicity dictated by the group we put all but one of the $\phi_a, \phi_b, \phi_c, \phi_d$ equal to zero. This can be done by requiring

$$(\gamma_0 P_0 \pm m)\, \phi = 0$$

$$\phi\, (\gamma_0 P_0 \pm m) = 0 \tag{10.5}$$

with some choice of signs. (For each choice of sign we select one of the 4 pieces in (10.4)). In the general frame of reference this transforms into the pair of Wigner-Bargmann equations:

$$(\Gamma_A \; P^A \pm m) \; \phi = 0$$
$$\phi \; (\Gamma_A \; P^A \pm m) = 0 \qquad (10.6)$$

Under the parity transformation:

$$\psi' = \gamma_0 \psi$$

we have $$\phi \Rightarrow \pm \phi$$

according to whether the right and left-hand Dirac equations have the same or opposite signs for M.

Thus to obtain odd parity we require for example

$$(\Gamma_A \; P^A - m) \; \phi = \phi \; (\Gamma_A \; P^A + m) = 0$$

In the rest system ϕ takes the form:

$$\left(\begin{array}{c|c} 0 & \phi_c \\ \hline 0 & 0 \end{array}\right)$$

Under the little group right and left-hand indices of the 2x2 matrix ϕ_c are transformed by independent unitary matrices S_1 and S_2^{-1}. (i.e. $\phi_c' = S_1 \phi_c S_2^{-1}$) Therefore the trace is not invariant and the dimensionality of the representation is 4 = 1 + 3. (36 in U(6,6)). Note that meson wave functions which in the rest system take the alternative form

$$\left(\begin{array}{c|c} \phi_a & 0 \\ \hline 0 & 0 \end{array}\right)$$

are obtained by changing the sign of the mass in one of the Wigner-Bargmann equations. They are equally acceptable representations of the symmetry and give a degree of degeneracy of only 3 rather than 4 because $\Phi'_a = S_1 \Phi_a S_1^{-1}$ and the trace is invariant under $U(2) \otimes U(2)$. [35 rather than 36 in $U(6,6)$]. However this representation does not readily arise from a simple quark-antiquark bound state model. This can be seen in various ways, for example from the remarks below that in the static limit the quark and antiquark transform separately and so give four-fold degeneracy. On the other hand, the baryons regarded as bound states of three quarks (no antiquarks), are no more degenerate in $U(2) \otimes U(2)$ than in $SU(2)$, or in $U(6) \otimes U(6)$ than in $SU(6)$, or in $U(6,6)$ than in $SL(6,C)$.

Remarks on charge conjugation:

To have the substitution rule, which together with some analyticity gives crossing symmetry, we require that a "particle of negative energy" in the final state can be interpreted as an antiparticle in the initial state. Thus an initial antiquark is represented in the matrix element by a factor $U^*(-p)$, where $U(-p)$ is a solution of the Dirac equation with momentum $-p$. To treat particles and antiparticles on a more equal footing we introduce the notation

$$\underline{U^*(-p) = C\ U^c(p)}$$

where C is the usual charge conjugation matrix, which reduces to unity in the Majorana representation, or to

$$C = \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

in the Pauli representation. Then, for physical momenta $U^c(p)$ is a solution of the Dirac equation with momentum $+p$, and for particles or antiparticles in the initial state, and in the same state of spin, the appropriate spinors U and U^c are numerically equal. For ordinary Lorentz transformations U and U^c transform identically, but not so for the more general transformations of $U(2,2)$. The transformation

$$U \rightarrow SU$$

induces

$$U^c \rightarrow S^c\, U^c$$

with

$$S^c = C^{-1}\, S^*\, C$$

and S^* denotes the complex (not Hermitian) conjugate of S. For this reason the two Dirac equations for general momentum are conjugate rather than identical:

$$(m + \Gamma_A\, P^A)\, U = 0$$

$$(m + \Gamma^c_A\, P^A)\, U^c = 0$$

where

$$C^{-1}(\Gamma^*_A\, P^{A*})\, C = \Gamma_A^{\,c}\, P^A$$

for each value of $A = 1, \ldots, 15$; thus the Γ_A^C and Γ_A differ in sign for some A. The fact that U and U^c belong to inequivalent representations of the group is no objection to the theory, but must be carefully noted. The difficulties of Riazuddin et al[1)] arises from supposing that U^c must transform as either U or $\overline{U}$. The inequivalence of U and U^c takes an interesting form for a system of quarks and antiquarks in slow relative motion, so that their various little groups can be approximately identified. The U and U^c transform under the little group by matrices

$$\left(\begin{array}{c|c} S_a & \\ \hline & S_b \end{array}\right) \qquad \text{and} \qquad \left(\begin{array}{c|c} \sigma_z \; S_b^* \sigma_z & \\ \hline & \sigma_z \; S_a^* \sigma_z \end{array}\right)$$

or, if we ignore the small components in each case, by S_a and $\sigma_z \; S_b^* \; \sigma_z$. Thus U and U^c are transformed by quite independent 2×2 unitary matrices, illustrating that the static limit of $\widetilde{U}(4)$ is $U(2) \otimes U(2)$. In the same way the static limit of $U(6,6)$ is $U(6) \otimes U(6)$ rather than just $U(6)$.

In the case of the mesons we define the charge conjugate wave function by

$$\phi^c(p)\,\gamma_0 = C^{-1}\,\{\phi(-p)\,\gamma_0\}^*\,C$$

and note that $\phi^c(p)$ and $\phi(p)$ satisfy positive energy Wigner-Bargmann equations.

The transformation

$$\phi \rightarrow S\phi S^{-1}$$

induces

$$\phi^c \rightarrow S^c \phi^c S^{c-1}$$

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1) Riazuddin, L.K.Pandit and S.Okubo, Rochester preprint (April 1965).

Note further that the quantity $C \phi^c \gamma_0 C$ transforms identically with the transpose of $\phi \gamma_0$, so that particle and antiparticle multiplets are in this case equivalent and can be consistently identified.

The extension of these considerations to U(6,6) is mainly a matter of notation. We now require $S^{\dagger} \gamma_0 \otimes \mathbb{1} S = \gamma_0 \otimes \mathbb{1}$. Where $\mathbb{1}$ denotes the unit matrix with respect to SU(3) indices. The 16 matrices Γ_A are replaced by 144 matrices Γ_μ which are direct products of Γ_A with SU(3) matrices $\mathbb{1}, \lambda_1, \ldots\ldots, \lambda_8$. With the 143 Γ_μ's other than the unit matrix are associated 143 momenta P_μ, and physical particles have again only $P_1 \ldots, P_4$ non-zero. The little group for time-like momenta is $U(6) \otimes (U(6)$. The reducibility of the tensor representations under the little group is expressed by the requirement that acting on each tensor index one of the two Dirac operators $(m \pm \Gamma_\mu P^\mu)$ gives zero — the Bargmann-Wigner equations. The new charge conjugation matrix C is the direct product of the old with the SU(3) unit matrix.

We note in passing that if the additional requirement

$$S^{-1} \gamma_5 \otimes \mathbb{1} S = \gamma_5 \otimes \mathbb{1}$$

is imposed on the transformation matrices, and determinant is required to be unity, one obtains SL(6,C). Working in the representation:

$$\Gamma_4 = \gamma_0 \otimes \mathbb{1} = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right)$$

$$\gamma_5 \otimes \mathbb{1} = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right)$$

and for the little group

$$S = \left(\begin{array}{c|c} S_a & 0 \\ \hline 0 & S_b \end{array}\right)$$

where the blocks denote 6x6 matrices, the extra requirements give

$$S_a = S_b$$

in addition to $S_a^+ \; S_a = S_b^+ \; S_b = 1$

Thus the little group is then just SU(6).

<u>Remark</u>:

The adjoint representation of U(6,6) is 143, which splits under U(6) $\otimes$ U(6) into:

$$143 = 35 + 35 + 1 + 36 + 36$$

The 36 accommodates the nonet of pseudoscalar (0^-) mesons and the nonet of vector (1^-) mesons. But in SU(6), the mesons are accommodated in the 35 dimensional irreducible representation. The missing meson in SU(6) is χ_0 ($\sim$ 980 MeV). Therefore in U(6) $\otimes$ U(6), we have $\chi_0 - \eta$ mixing analogous to the $\omega - \varphi$ mixing.

We mention here about the "irregular" couplings[1]. The quantity $P_\mu \Gamma'^\mu$ transforms as the adjoint representation of U(6,6), and can be so used in the construction of invariant matrix elements. Its use in this way is just as legitimate as its use in the Wigner-Burgmann equations. Ruhl has shown that with the introduction of such couplings the notorious unitarity difficulty of the theory is somewhat shifted. For Beg and Pais[2] the most obvious difficulty was the appearance in the summation over final states of projection operators involving the quantity $P_\mu \Gamma^\mu$. But these are formally invariant when the ir--regular couplings are admitted.

To be explicit we give here the generators of these groups

U(2,2) : $\mathbb{1}\ ,\ \gamma_5\ ,\ \frac{1}{2}\gamma_\mu\gamma_\nu\ ,\ \gamma_\mu\ ,\ i\gamma_\mu\gamma_5\ :\ \Gamma_A$

U(6,6): $\Gamma_A \otimes \lambda_a$

GL(6,C): $\mathbb{1}\otimes\lambda_a\ ,\ \gamma_5\otimes\lambda_a\ ,\ \frac{1}{2}\gamma_\mu\gamma_\nu\otimes\lambda_a$

U(6): $\mathbb{1}\otimes\lambda_a\ ,\ \frac{1}{2}\gamma_i\gamma_j\otimes\lambda_a$

For collinear groups we want γ_0 and γ_1 to be invariant. In the case of the U(2,2) group, the little group is U(2) $\otimes$ U(2) which has the set of 8 generators $\mathbb{1}$, γ_0, $\gamma_i\gamma_j$ and $i\gamma_5\gamma_i$ which commute with γ_0. The collinear group is U(2), which has the set of 4 generators: $\mathbb{1}$, $\gamma_2\gamma_3$, $i\gamma_5\gamma_2$ and $i\gamma_5\gamma_3$ which commute with γ_1. Similarly for the group U(6,6), the collinear group is $U_W(6)$.

1) W.Ruhl, CERN preprint 65/475/5-Th.536 (March 1965).
2) M.A.B.Beg and A.Pais, Rockefeller preprint (February 1965),

For Coplanar group, besides γ_0 and γ_1 , γ_2 has to be invariant. The only generators of U(2) $\otimes$ U(2) which commute with γ_2 are: $\mathbb{1}$ and $\gamma_5 \gamma_3$. These form the Coplanar group U(1) $\otimes$ U(1).

The little group tells about specification of particles while collinear and coplanar groups give restrictions on S-matrix. The collinear group gives restrictions on two particle decays, forward and backward scattering and vertex functions. The coplanar group gives restriction on two-to-two particle scattering processes.

Remark: The coplanar group of U(6,6) is U(3) $\otimes$ U(3) which gives trouble with unitarity (Alles and Amati). The lowest representation is the 12-quark representation. We consider the elastic quark-quark scattering and quark-antiquark scattering and postulate that the quarks are the lightest particles in the world so that the quark and antiquark do not annihilate each other to form any other particle. If now the same analytic function satisfies both these processes, (i.e. we assume crossing symmetry) then the unitarity conditions in the two processes are different. Alles and Amati have shown that they are incompatible. Therefore, unitarity, crossing and U(6,6) symmetry are not compatible. This is of course only a mathematical model but shows that something may be wrong. But, in SL(6,C) this difficulty has not been shown to be present.

Lecture 11

The group SU(6).

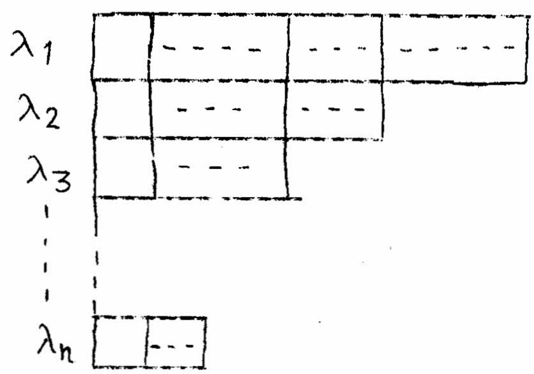
We shall now consider the model based on the inhomogeneous SL(6,C) group. Its chain in the Hierarchy of groups is:

$$SL(6,C) \supset SU(6) \supset SU(3) \otimes SU(3) \supset SU(3)$$

The little group of SL(6,C) is SU(6). Therefore, we shall just give a summary on SU(6). and its irreducible representations

Notions on Young tableau:

The group U(n) is the group of nxn unitary matrices and SU(n) is the subgroup of unitary <u>unimodular</u> matrices (i.e. determinant is one). The irreducible representations of U(n) are characterized by Young tableau, denoted by $(\lambda_1, \lambda_2, \cdots, \lambda_n)$



where λ_i are pos~~t~~itive integers which give the number of boxes in a row, with

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots\cdots \lambda_n \geqslant 0 .$$

A Young tableau of N boxes corresponds to a tensor, where each index runs from 1 to n. Such a tensor may be written as:

$$t_{ABC\cdots\cdots}$$

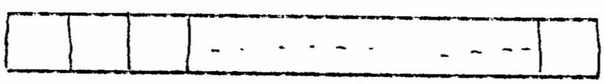
This tensor is defined by its transformation property

$$t'_{ABC\ldots\ldots} = U_A{}^a U_B{}^b U_c{}^c \ldots\ldots t_{abc\ldots} \qquad (11.1)$$

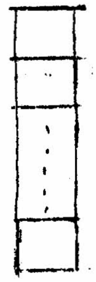
(A tensor with N suffixes is of order N.). Then, a Young tableau ($\lambda_1, \lambda_2, \cdots, \lambda_n$) will corresponds to a mixed symmetric tensor:

$$t_{\underbrace{A_1A_2\ldots A_n}_{\lambda_1} \cdot \underbrace{B_1B_2\ldots B_n}_{\lambda_2} \underbrace{\ldots\ldots}_{\lambda_3} \underbrace{\ldots\ldots}_{----} \underbrace{\ldots\ldots}_{\lambda_n}}$$

To make this tensor an irreducible tensor, there are two equivalent prescriptions: One may first symmetrize the λ_i indices among themselves (i.e. first symmetrize the rows), and then antisymmetrize the indices in different lines. Or, one may first antisymmetrize the indices in different lines (i.e. antisymmetrize the columns of the Young tableau) and then symmetrize the λ_i 's among themselves. The production of irreducible tensors in this way is a complicated process for any but the simplest cases. We prefer the former prescription. It follows, by definition of the prescription, that Young tableau with only one row of boxes:

 $\Rightarrow t_{\lambda_1}$

will correspond to a completely symmetric tensor. A Young tableau with only column :



$\Longrightarrow \quad t_{111......1}$

will corresponds to a completely antisymmetric tensors.

Contravariant tensors are introduced by the definition:

$$t'^{A} = (U^{-1})^{A}{}_{B} \; t^{B} \tag{11.2}$$

And of course,

$$t_{A} \, t^{A} = \text{invariant.}$$

Since the group is unitary

$$t'^{A} = (U^{-1})^{A}_{B} \; t^{B} = (U^{+})^{A}{}_{B} \; t^{B} \pm (U^{*})_{B}{}^{A} \; t^{B} \tag{11.3}$$

Comparing (11.3) with (11.1) one sees that the contravariant representation is equivalent to the conjugate (or complex conjugate) representation, for unitary groups. In general, there is no way of raising or lowering an index at a time, except in the case of SU(2).

The determinant of a nxn matrix is given by:

$$\det U \; \varepsilon_{A'B'C'......} = U_{A'}{}^{A} \, U_{B'}{}^{B} \, U_{C'}{}^{C} ...... \varepsilon_{ABC.....} \tag{11.4}$$

For the subgroup SU(n), which has det U = 1, the Levi-Civita tensor (which is a completely antisymmetric tensor) is an invariant. Hence a Young tableau with n vertical boxes is equivalent to the trivial (unit) representation. Due to this fact, a column of n boxes in a ("proper") Young tableau can always be deleted. Therefore, an irreducible representation of SU(n) will have $\lambda_n = 0$.

Thus, in the case of SU(n), one can introduce instead of the set λ_i, the set P_i with

$$\left.\begin{aligned} P_i &= \lambda_i - \lambda_{i+1} \; ; \quad i = 1, \ldots . \; n-1 \\ P_n &= \lambda_n = 0 \end{aligned}\right\} \quad (11.5)$$

The P_i are the coordinates of the highest weight vector of the irreducible representation (weights are eigenvalues of the operators of the Cartan subalgebra, i.e. the set of generators of SU(n) which are mutually diagonal).

The dimension of an irreducible representation of SU(n) is:

$$d = \frac{1}{2!\, 3! \ldots\ldots (n-1)!} (\lambda_1 - \lambda_2 + 1)(\lambda_1 - \lambda_3 + 2) \ldots\ldots \cdot (\lambda_1 - \lambda_n + n-1) \times$$

$$\times (\lambda_2 - \lambda_3 + 1)(\lambda_2 - \lambda_4 + 2) \ldots . (\lambda_2 - \lambda_n + n-2)$$

$$\ldots\ldots (\lambda_i - \lambda_j + j - i) \ldots\ldots (\lambda_{n-1} - \lambda_n + 1)$$

$$(11.6a)$$

Or, in terms of the P_i's defined by (11.5):

$$d = \frac{1}{2!\,3!\ldots(n-1)!}\,(p_1+1)\,(p_1+p_2+2)\ldots\ldots(p_1+p_2+\ldots+p_{n-1}+n-1)\times$$
$$\times(p_2+1)\,(p_2+p_3+2)\ldots\ldots(p_2+p_3+\ldots+p_{n-1}+n-2)\times$$
$$\ldots\ldots\ldots(p_i+p_{i+1}+p_{i+2}+\ldots+p_{j-1}+j-i)\times\ldots$$
$$\ldots\ldots\ldots\ldots(p_{n-1}+1)\,. \qquad (11.6b)$$

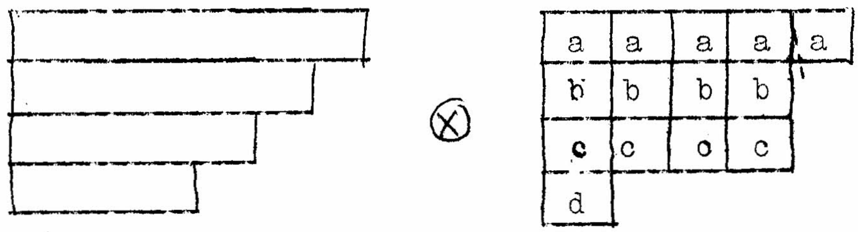
The dimension of an irreducible representation of SU(6) is given by the corresponding formulae:

$$d = \frac{1}{2!3!4!5!}\,(\lambda_1-\lambda_2+1)\,(\lambda_1-\lambda_3+2)\,(\lambda_1-\lambda_4+3)\,(\lambda_1-\lambda_5+4)\cdot$$
$$\cdot(\lambda_1-\lambda_6+5)\times$$
$$\times(\lambda_2-\lambda_3+1)\,(\lambda_2-\lambda_4+2)\,(\lambda_2-\lambda_5+3)\,(\lambda_2-\lambda_6+4)\times$$
$$\times(\lambda_3-\lambda_4+1)\,(\lambda_3-\lambda_5+2)\,(\lambda_3-\lambda_6+3)\times$$
$$\times(\lambda_4-\lambda_5+1)\,(\lambda_4-\lambda_6+2)\times$$
$$\times(\lambda_5-\lambda_6+1) \qquad (11.7a)$$

Or,

$$d = \frac{1}{2!3!4!5!}\,(p_1+1)\,(p_1+p_2+2)\,(p_1+p_2+p_3+3)\,(p_1+p_2+p_3+p_4+4)\cdot$$
$$\cdot(p_1+p_2+p_3+p_4+p_5+5)\times$$
$$\times(p_2+1)\,(p_2+p_3+2)\,(p_2+p_3+p_4+3)\,(p_2+p_3+p_4+p_5+4)$$
$$\times(p_3+1)\,(p_3+p_4+2)\,(p_3+p_4+p_5+3)$$
$$\times(p_4+1)\,(p_4+p_5+2)$$
$$\times(p_5+1).$$

The product of two irreducible representations can be reduced in the following way. Consider the Young diagrams of the two irreducible representations:



The following is the procedure, to be adopted in order to find all the irreducible representations into which the product can be decomposed:

1) Give names to the N_2 boxes of the Young diagram of the second irreudicble representation. (giving the same name to boxes in a row).

2) Add the boxes of the second irreducible representation to the N_1 first ones, in all possible ways, starting with those labelled a, then b, etc., in such a way that:

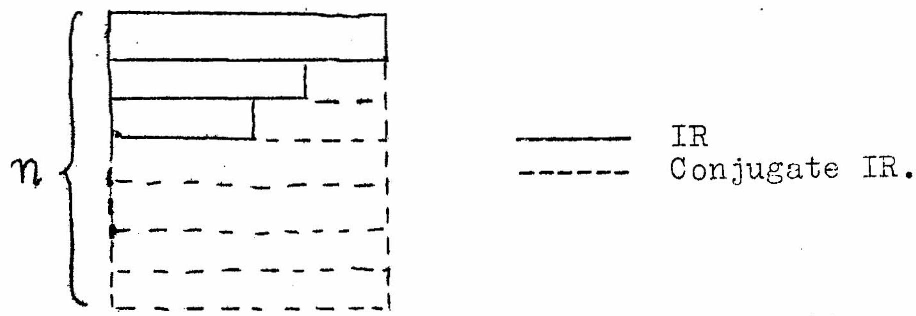
3) At each step the resulting Young diagram should be an allowed one (an allowed Young diagram is one with non-increasing numbers of squares and symbols in successive rows). i.e. $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_n \geqslant 0$.

4) Two boxes with the same label should <u>never</u> be in the same column (vertical line).

5) When reading the resulting diagram from right to left and in consecutive rows from top to bottom, the number of times 'a' occurs should be $\geqslant$ the number of times 'b' occurs $\geqslant$ the number of times 'c' occurs and so on.

6) The sum of the dimensionalities of all possible Young diagrams (which have $N_1 + N_2$ boxes) in the decomposition will be equal to the product of the dimensionalities of the two Young diagrams which is being decomposed.

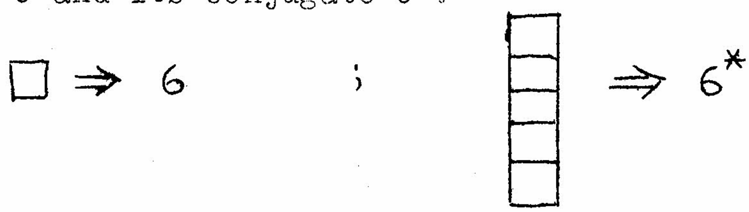
As can be seen from equations (11.1) and (11.3), the product of an irreducible representation with its conjugate representation contains the unit (trivial) representation. Conversely, only the product of an irreducible representation with its conjugate irreducible representation contains 1. Using rules 1) to 6), one sees that an irreducible representation and its conjugate have the following Young diagrams.

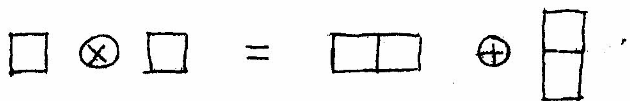


(Ofcourse, the dotted diagram has to be rotated in the anti-clockwise direction through 180° to make it an allowed Young diagram). In formulae, the conjugate irreducible representation has the Young tableau ($\lambda'_1, \lambda'_2, \ldots\ldots, \lambda'_{n-1}$):

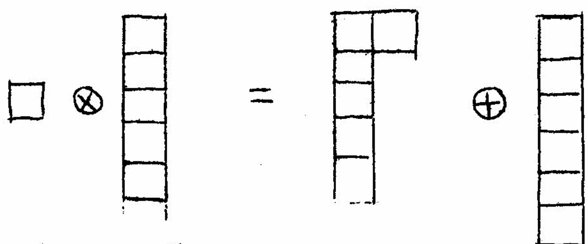
$$\left.\begin{array}{lll} & \lambda'_i = \lambda_1 - \lambda_{n-i+1} ; \\ \text{or} & P'_i = P_{n-i} \end{array}\right\} \qquad (11.8)$$

The dimension formula (11.6) is symmetric under transformation (11.8). So, conjugate I.R. has the same dimenion as the I.R. We can now study the lowest irreducible representations of SU(6). Starting with the fundamental (defining) irreducible representation of dimension 6 and its conjugate 6*:

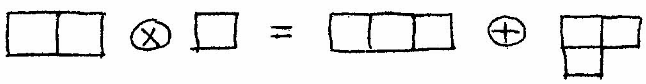




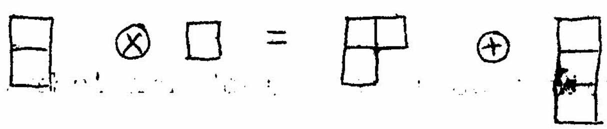
$$6 \otimes 6 = 21 \oplus 15$$



$$6 \otimes 6^* = 35 \oplus 1$$



$$21 \otimes 6 = 56 \oplus 70$$

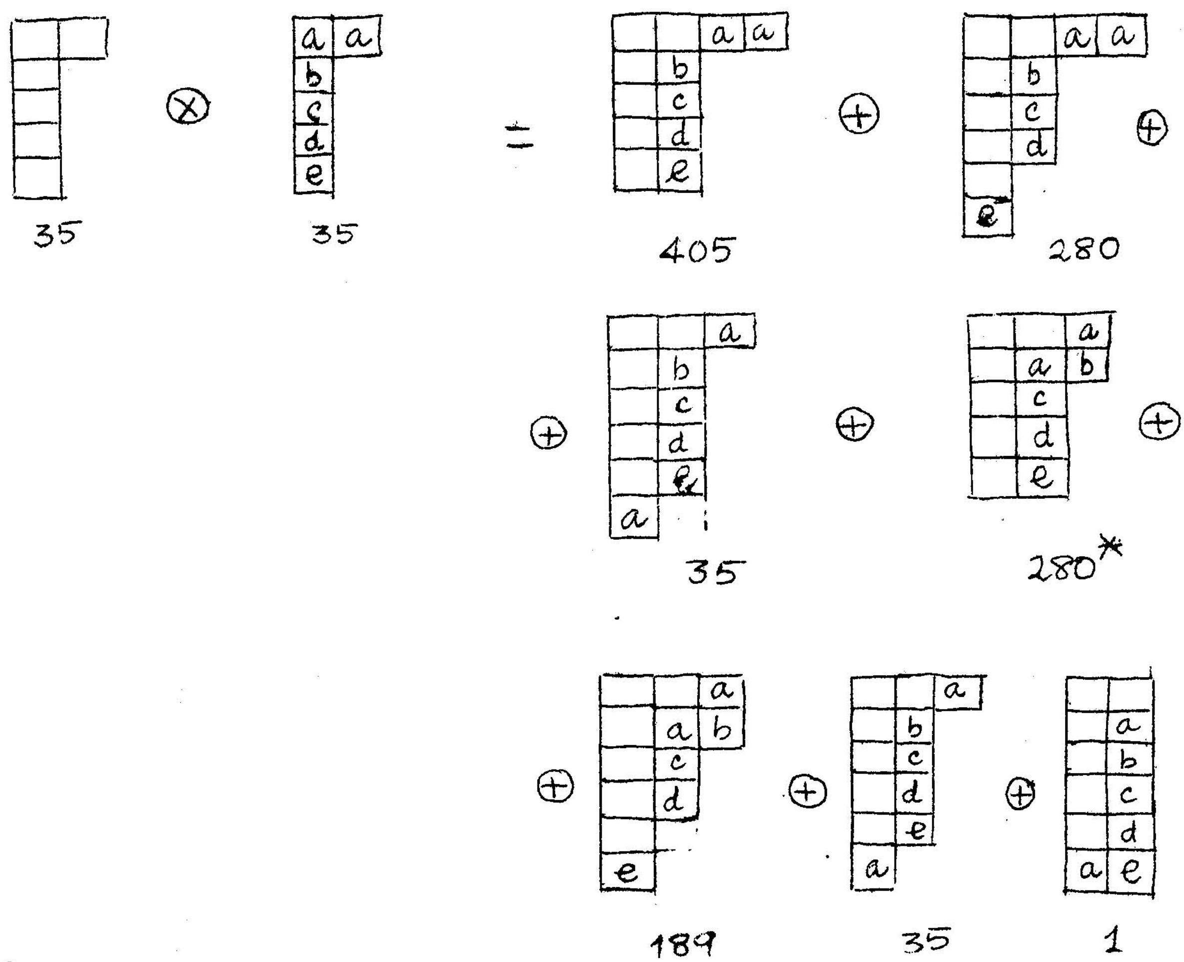


$$15 \otimes 6 = 70 \oplus 20$$

(11.9)

(Notice that irreducible representations 35 and 20 are self-conjugate representations).

The decomposition of $35 \otimes 35$ is given below, since it is necessary for later application:



Hence,

$$35 \otimes 35 = 1 \oplus 35 \oplus 35 \oplus 189 \oplus 280 \oplus 280^* \oplus 405 \tag{11.10}$$

In the same way one gets

$$56 \otimes 56^* = 1 \oplus 35 \oplus 405 \oplus 2695 \tag{11.11}$$

We now have all the important representations for physical applications. The whole purpose of the game is to include SU(3) into SL(6,C) which has SU(6) as its little group. In particular, the physicists are interested in the subgroup:

$$SU(6) \supset SU(3) \otimes SU(2) \qquad (11.12)$$

Where SU(2) is the little group of the Poincare group. It is found that all the known particles are in the faithful representations of $SU(3)/C_3$ (C_n is the cyclic group of n elements) This group $SU(3)/C_3$ is interesting in physics because it gives all the representations of the octet model, viz., 1,8,10,10*,27, ... It does not contain 3, which is the quark representation -- the quarks have not been seen so far. In the case of (11.12) we have:

$$\frac{SU(6)}{C_3} \supset \frac{SU(3)}{C_3} \otimes SU(2) \quad \text{: No. of boxes - 3mod. 6}$$

$$\frac{SU(6)}{C_2} \supset SU(3) \otimes \frac{SU(2)}{C_2} \sim SO_3 \quad \text{: No. of boxes - 2 mod.6.}$$

$$\frac{SU(6)}{C_6} \supset \frac{SU(3)}{C_3} \otimes \frac{SU(2)}{C_2} \quad \text{: No. of boxes - 0 mod.6.}$$

The quark representations (6) belongs to SU(6) only. 35 and 1 are faithful representations of $SU(6)/C_6$ and these accommodate all the known mesons with integer spin (1 accommodates χ_0) 20, 56 and 70 are faithful representations of $SU(6)/C_3$. In this case 56 is found to be the suitable candidate for accommodating the low lying particles (fermions) with half integer spins. Some resonances could be accommodated in 70 but it is not full. Only 1, 35 and 56 dimensional representations are full. There is no rational explanation for 56 being full while the lower dimensional representation 20 is empty.

Lecture 12

In the last lecture we stated that for the physical applications we are interested in the subgroup $SU(3) \otimes SU(2)$, where $SU(2)$ is the little group of the Poincare group which describes spin. Furthermore, $SU(3)/C_3$ describes particles of the Octet model of Gell-Mann and Ne'eman, and $SU(2)/C_2$ describes particles of integer spin. Therefore, one has to consider the following subgroups, and their faithful representations which are contained in the Young diagrams of $SU(6)$ with the following limitations:

	Subgroup.	No. of boxes in Young diagram	Example of I.R. of SU(6).
1)	$SU(3) \otimes SU(2)$	any	$6, 6^*$
2)	$\frac{SU(3)}{C_3} \otimes SU(2)$	3 (mod. 6)	20, 56, 70
3)	$SU(3) \otimes \frac{SU(2)}{C_2}$	2 (mod. 6)	21, 15
4)	$\frac{SU(3)}{C_3} \otimes \frac{SU(2)}{C_2}$	2 (mod. 6)	1, 35

Therefore, Fermions of the octet model will be of category 2) and octet Bosons in category 4).

Remembering the way the fundamental representation of $SU(6)$ was built: the 6x6 hermitian matrices which are the generators were obtained as the direct product of 3x3 matrices (generators of $SU(3)$)and 2x2 matrices (generators of $SU(2)$). Hence, the irreducible representation 6 of $SU(6)$ contains an $SU(3)$ triplet of spin 1/2 particles, the "quarks" and we note:

$$6 = (3, 2) \quad \text{and} \quad 6^* = (3^*, 2) \qquad (12.1)$$

(Note: Irreducible representations of SU(2) are self-conjugate)

In general, the notation will be:

$$d\,(\,SU(6)\,) = \sum_{\oplus} \Big(d\,(SU(3))\,,\; d\,(SU(2))\Big) \qquad (12.2)$$

where d denotes the dimension of the irreducible representation. Such a formula gives the reduction of an irreducible representation of SU(6) into irreducible representations of SU(3) $\otimes$ SU(2). We have seen in the last lecture that

$$6 \otimes 6^* = 35 \oplus 1$$

This is the reduction into irreducible representations of SU(6). The same reduction into irreducible representations of SU(3) $\otimes$ SU(2) is written as:

$$\begin{aligned} 6 \otimes 6^* &= (3, 2) \otimes (3^*, 2) \\ &= (8, 3) \oplus (1, 3) \oplus (8, 1) \oplus (1, 1). \end{aligned}$$

Hence, the irreducible representation 35 of SU(6) when reduced with respect to SU(3) $\otimes$ SU(2) gives:

$$35 = (8, 3) \oplus (1, 3) \oplus (8, 1) \qquad (12.3)$$

Since SU(6) commutes with parity, all the particles accommodated in 35 will have the same parity. From Eq.(12.3) we conclude that 35 contains:

$$\begin{aligned} &\text{an octet of } 1^- : (\rho\,, K^*, \omega_o) \\ &\text{a singlet of } 1^- : \varphi_o \qquad (12.4) \\ &\text{and an octet of } 0^- : (\pi\,, K, \eta\,) \end{aligned}$$

where by ω_0 we mean a particular combination of ω and φ . and by φ_0 another combination of ω and φ. Hence, 35 accommodates the low lying mesons. It is very satisfactory, in view of the mixing, that these particles belong to the same irreducible representation of SU(6).

One also has:

$$6 \otimes 6 = 15 \oplus 21$$

$$= (3,2) \otimes (3,2) = (3^*,1) \oplus (3^*,3) \oplus (6,1) \oplus (6,3)$$

We have seen in the last lecture that the irreducible representation 15 is antisymmetric $\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$, while 21 is symmetric $(\square\square)$ As far as SU(3) is concerned, irreducible representation 3*, obtained in the decomposition of the product $3 \otimes 3$, is antisymmetric and 6 is symmetric. From the point of view of SU(2), 1 is antisymmetric and 3 is symmetric. All this follows from the general rules and multiplication of Young diagrams for SU(n), as given above. Hence:

$$21 = (3^*, 1) \oplus (6,3)$$

$$15 = (3^*, 3) \oplus (6,1) \qquad (12.5)$$

No physical particles corresponding to these irreducible representations have been found yet, since all particles we know belong to the irreducible representations of $SU(3)/C_3$. Let us now consider:

$$21 \otimes 6 = 56 \oplus 70$$

$$= [(3^*,1) \oplus (6,3)] \otimes (3,2) = (8,2) \oplus (10,4) \oplus (1,2) \oplus (12.6) \oplus$$
$$\oplus (8,2) \oplus (8,4) \oplus (10,2)$$

Likewise

$$15 \otimes 6 = 20 \oplus 70$$

$$= [(3^*,3) \oplus (6,1)] \otimes (3,2) = (8,4) \oplus (8,2) \oplus (1,4) \oplus (1,2) \oplus \oplus (10,2) \oplus (8,2) \qquad (12.7)$$

Since 70 occurs in both decompositions, one can easily find that:

$$70 = (1,2) \oplus (8,2) \oplus (8,4) \oplus (10,2) \qquad (12.8)$$

From Eqs. (12.6), (12.7) and (12.8) it immediately follows that:

$$56 = (8,2) \oplus (10,4)$$

$$20 = (8,2) \oplus (1,4) \qquad (12.9)$$

Although it seems more natural to assign the low lying baryons into the irreducible representation 20, it just happens that the irreducible representation 56 is favoured empirically: while there is no known SU(3) singlet with spin 3/2 to fit in 20, there is the famous SU(3) decuplet with spin 3/2 to fit into 56, together with the "stable" octet of baryons of spin 1/2. This was the choice of Gursey and Radicati. At present, all that one can say about the irreducible representation 70, is that it may contain higher resonances.

<u>Mass formulae:</u>

One of the greatest successes of SU(3) is the mass formula which facilitates the classification of particles. For SU(6), there are three positive points which support the above classification:

a) ω and φ are in the same irreducible representation of SU(6), thus justifying $\omega - \varphi$ mixing.

b) Empirically, one has

$$m_{K^*}^2 - m_K^2 = m_\rho^2 - m_\pi^2$$

which is well satisfied. It is gretifying that K^*, K, ρ and π are in the same irreducible representation of SU(6).

c) If one writes the Gell-Mann-Okubo formula for the spin 1/2, baryon octet:

$$m = m_0^{(8)} + a^{(8)} Y + b^{(8)} \left[I(I+1) - \frac{Y^2}{4} \right] \qquad (12.10)$$

and for the spin 3/2 decuplet:

$$m = m_0^{(10)} + (a^{(10)} + \frac{3}{2} b^{(10)}) + 2 b^{(10)} \qquad (12.11)$$

where $I = \frac{Y}{2} + 1$ has been used in the mass formula for the decuplet. One finds, with $a^{(8)} = a^{(10)}$, $b^{(8)} = b^{(10)}$ the correct spacing between the members of the decuplet.

In SU(3) one gets the Gell-Mann-Okubo mass formula by a simple assumption, viz., octet dominance. But, in the case of SU(6), there is no such simple principle which would yield a), b) and c). (See Beg and Singh). Therefore, there is no natural and simple way of deriving a mass formula for SU(6).

<u>Tensor form for the basis of irreducible representations:</u>

It is useful, for applications, to write the irreducible representations 35 and 56 in tensor notation. In this notation, one denotes a quark representation of SU(6) by:

$$q_A = q_{a\alpha} \quad \text{where} \quad \begin{array}{l} A = 1, \ldots, 6. \\ a = 1, 2, 3 \\ \alpha = 1, 2. \end{array} \qquad (12.12)$$

Since 35 is contained in $6 \otimes 6^*$:

$$6 \otimes 6^* = 35 \oplus 1$$

$$q_A \, q'^B = (t_A{}^B - \tfrac{1}{6}\,\delta_A{}^B\, t_c{}^c) + \tfrac{1}{6}\,\delta_A{}^B\, t_c{}^c$$

Hence, the basis for the irreducible representation 35 is a traceless, mixed tensor of rank two:

$$35 : \; t_A{}^B - \frac{1}{6}\,\delta_A{}^B\, t_c{}^c = M_A{}^B \qquad (12.13)$$

Further, since

$$35 = (8,1) \oplus (8,3) \oplus (1,3),$$

$$M_A{}^B = \frac{1}{2}\, P_a{}^b\, \delta_\alpha{}^\beta + V_a{}^b (\chi_8)_\alpha^\beta + \frac{1}{3}\, \delta_a{}^b\, (\chi_1)_\alpha^\beta \qquad (12.14)$$

where $\delta_a{}^b$, $V_a{}^b$ and $P_a{}^b$ are tensors in SU(3)-spin space. χ_8 and χ_1 are spin one wave functions of the octet and singlet. P and V stand for pseudoscalar and vector respectively.

$$V_a{}^a = P_a{}^a = 0.$$

The tensor $M_A{}^B$ is normalized by:

$$M_A{}^B\, M_B{}^A = 3.$$

We have already seen that 56 splits under the group $SU(3) \otimes SU(2)$ as:

$$56 = (10,4) \oplus (8,2)$$

We saw also that the basis for 56 ($\boxed{}\boxed{}\boxed{}$) is a third rank tensor: B_{ABC} completely symmetric in all three indices. 10 is completely symmetric in the $SU(3)$ indices and 4 is completely symmetric in the $SU(2)$ indices. Therefore, this part is also trivial. An $SU(3)$ octet is given by a traceless mixed symmetric tensor B^a_b, and spin 1/2 by a one-component spinor $(\chi)_\gamma$. Hence a spin 1/2 octet is written as: $B^a_b\ (\chi_8)_\alpha$. In order to get the indices A, B, C $\sim$ $a\alpha$, $b\beta$, $c\gamma$, one has to multiply this by invariant tensors, which are ε_{bcd} and $\varepsilon_{\beta\gamma}$. Then, one must symmetrize all the three pairs of indices, $a\alpha$, $b\beta$ and $c\gamma$, in order to get the completely symmetric tensor B_{ABC}. Hence, B_{ABC} can be decomposed uniquely as:

$$\begin{aligned} B_{ABC} &= B_{a\alpha,\, b\beta,\, c\gamma} \\ &= D_{abc}\,(\chi_{10})_{\alpha\beta\gamma} + \frac{1}{3\sqrt{2}}\Big[N_a^{\ d}\, \varepsilon_{dbc}\,(\chi_8)_\alpha\, \varepsilon_{\beta\gamma} + \\ &\qquad + N^d_b\, \varepsilon_{dca}\,(\chi_8)_\beta + - N^d_c\, \varepsilon_{dab}\,(\chi_8)_\gamma\, \varepsilon_{\alpha\beta}\Big] \end{aligned} \tag{12.15}$$

where D_{abc} and $(\chi_{10})_{\alpha\beta\gamma}$ are completely symmetric and $N_a^{\ d}$ is traceless.

These tensors are used to set up a mass formula for $SU(6)$. They are also useful for $(SU(6))_W$.

The collinear group $U(3) \otimes U(3)$:

$U(3) \otimes U(3)$ is the collinear group of the inhomogeneous $SL(6,C)$ group. For collinear processes, one still has some symmetry left. We have shown that this follows from the invariance of the S-operator under the inhomogeneous group $SL(6,C) \boxed{\sigma} T_{72}$ and the supplementary condition for physical states. In this form the Lorentz invariance of the model is obvious. A necessary condition, for getting $U(3) \otimes U(3)$ from inhomogeneous $SL(6,C)$, and also for satisfying the unitarity condition, is to include the so-called "irregular couplings" (see Ruhl, loc. cit.) which involve the 72 momenta. $U(3) \otimes U(3)$ symmetry restricts scattering amplitudes and vertex functions. We shall discuss examples, following Ruegg and Volkov[1].

The generators of $U(6)$ which has $U(3) \otimes U(2)$ as a subgroup are: $\lambda_i \otimes \sigma_\mu$. For particles travelling in the same direction, we have the collinear subgroup $U(3) \otimes U(3)$, which commutes with Lorentz transformations along a particular direction, say, the z-axis, which can be chosen as the axis of quartization. The generators of $U(3) \otimes U(3)$ are:

$$\lambda_i \otimes \frac{1 \pm \sigma_z}{2} \qquad (12.16)$$

The + sign stands for that $U(3)$ group which acts on the spin direction parallel to the z-axis and the - sign stands for the

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1) H.Ruegg and D.V.Volkov, CERN Th.616, October 1965

other U(3) group which acts on antiparallel spin. In tensor notation, the quarks of these two representations will be denoted by q_{a1} and q_{a2}. Then,

$$
\begin{aligned}
q_{a1} &= i\,(\lambda_i)_a^b\; q_{b1}\,\delta\varepsilon_{i1} \\
q_{a2} &= i\,(\lambda_i)_a^b\; q_{b2}\,\delta\varepsilon_{i2}
\end{aligned}
\qquad (12.17)
$$

We will, hereafter, denote:

$$
\begin{aligned}
q_{a1} &\equiv q_a \\
q_{a2} &\equiv q_{\bar{a}}
\end{aligned}
\qquad (12.18)
$$

where q_a has helicity +1 and is the basis for the representation (3,1) while $q_{\bar{a}}$ has helicity -1 and is the basis for the representation (1,3). Also,

$$q^{a1} = q^a \in (3^*,1) \text{ and } q^{a2} = q^{\bar{a}} \quad (1,3^*)$$

Note that parity relates q_a and $q_{\bar{a}}$.

The meson representation 35 decomposed under $U(3) \otimes U(3)$ as:

$$35 = (8,1) \oplus (1,8) \oplus (3,3^*) \oplus (3^*,3) \oplus (1,1) \qquad (12.19)$$

One sees why this is so, if one writes down this decomposition explicitly using the tensor notation for the meson representation 35:

$$M_B^A = M_{b\beta}^{a\alpha} .$$

$$M_B^A \Big|_{\alpha=1,\beta=1} = M_b^a + \frac{1}{\sqrt{6}}\,\delta_b^a\, M$$

(Note: B is a covariant index and its spin index takes +1, so it denotes +ve helicity. But A is a contravariant index and its spin index takes -1, so it denotes -ve helicity).

$$M^{A}_{B}\Big|_{\alpha=2,\ \beta=2} = M^{\bar{a}}_{\bar{b}} + \frac{1}{\sqrt{6}}\ \delta^{\bar{a}}_{\bar{b}}\ M \tag{12.20}$$

$$M^{A}_{B}\Big|_{\alpha=1,\ \beta=2} = M^{a}_{\bar{b}}$$

$$M^{A}_{B}\Big|_{\alpha=2,\ \beta=1} = M^{\bar{a}}_{b}$$

where the following identification can be made:

$$M^{a}_{b} = (8,\ 1) \quad ; \quad M = (1,\ 1)$$

$$M^{\bar{a}}_{\bar{b}} = (1,\ 8) \quad ; \quad M^{\bar{a}}_{b}\ (3,\ 3^{*})$$

$$M^{a}_{\bar{b}} = (3^{*},\ 3)$$

One can be even more explicit and write:

$$M^{a}_{b} = \frac{1}{\sqrt{2}}\ P^{a}_{b} + V^{a}_{b}\ (\chi_{8})^{1}_{1}$$

$$M^{\bar{a}}_{\bar{b}} = \frac{1}{\sqrt{2}}\ P^{a}_{b} + V^{a}_{b}\ (\chi_{8})^{2}_{2}$$

$$M^{a}_{\bar{b}} = V^{a}_{b}\ (\chi_{8})^{2}_{1} + \frac{1}{\sqrt{3}}\ \delta^{a}_{b}\ (\chi_{1})^{2}_{1} \tag{12.21}$$

$$M^{\bar{a}}_{b} = V^{a}_{b}\ (\chi_{8})^{1}_{2} + \frac{1}{\sqrt{3}}\ \delta^{a}_{b}\ (\chi_{1})^{1}_{2}$$

$$M = \sqrt{2}\ (\chi_{1})^{1}_{1} = -\sqrt{2}\ (\chi_{1})^{2}_{2}$$

where $(\chi_{8})^{1}_{1} = -(\chi'_{8})^{2}_{2}$ because of tracelessness. From this it is clear that M stands for the spin zero singlet of SU(3); $M^{a}_{\bar{b}}$ and $M^{\bar{a}}_{b}$ are the spin 1 octet and singlet of SU(3); M^{a}_{b} and $M^{\bar{a}}_{\bar{b}}$ are the spin zero and spin one octets of SU(3). These accommodate the mesons.

The baryons, as we have seen earlier, belong to the representation 56, which is formed out of three quarks. So, the spin directions can be all parallel or all antiparallel (spin 3/2). Also, two spins can be parallel and antiparallel which would be a mixture of spin 3/2 and spin 1/2. The 56 representation decomposes under $U(3) \otimes U(3)$ as:

$$56 = (10,\ 1) \oplus (1,\ 10) \oplus (6,\ 3) \oplus (3,\ 6) \qquad (12.22)$$

The 56 dimensional baryon representation in the tensor notation is denoted by $B_{ABC} = B_{a\alpha\ ,b\beta\ ,c\gamma}$ (12.23)

which is a symmetric third rank tensor. To be explicit:

$$\begin{aligned}
B_{abc} &= D_{abc}\ (\chi_{10})_{111} \\
B_{\bar{a}\bar{b}\bar{c}} &= D_{abc}\ (\chi_{10})_{222} \qquad (12.24) \\
B_{ab\bar{c}} &= \sqrt{3}\ D_{abc}\ (\chi_{10})_{112} + \frac{1}{\sqrt{6}} (N_a^{\ d}\, \varepsilon_{dbc} + N_b^d\, \varepsilon_{dbc}) \times \\
&\qquad \times (\chi_8)_1 \\
B_{a\bar{b}\bar{c}} &= \sqrt{3}\ D_{abc}\ (\chi_{10})_{122} + \frac{1}{\sqrt{6}} (N_c^d\, \varepsilon_{dab} + N_b^d\, \varepsilon_{dac}) \times \\
&\qquad \times (\chi_8)_2
\end{aligned}$$

These relations are obtained from Eq.(12.15).

Lecture 12 (bis).

We have seen that the irreducible representations 35 and 56 of SU(6) decompose with respect to the collinear group U(3) $\otimes$ U(3) as:

$$35 = (8,1) \oplus (1,8) \oplus (3,3^*) \oplus (3^*,3) \oplus (1,1)$$
$$56 = (10,1) \oplus (1,10) \oplus (6,3) \oplus (3,6). \qquad (12.25)$$

The requirement that in the rest system particles are classified according to SU(6) restricts the freedom in the subgroup U(3) $\otimes$ U(3). The classification of particles is the same in this moving system also, since Lorentz transformations along the z-axis commute with the generators of U(3) $\otimes$ U(3).

We will now consider two applications of this collinear group:

(1) Baryon-Pseudo-scalar meson scattering in the forward and backward direction, and

(2) electro-magnetic form factors.

Baryon-Pseudo-scalar meson scattering in the forward and backward direction:

The pseudo-scalar mesons belong to the (8,1) representation, which in the tensor notation is denoted by M^a_b. (A lower index stands for spin up and an upper index for spin down). This tensor has zero spin projection, so that it contains the pseudo-scalar and vector particles:

$$M^a_b = \frac{1}{\sqrt{2}} P^a_b + V^a_b \, (\chi_8)^1_1 \quad , \quad M^a_a = 0 \qquad (12.26)$$

Parity transforms $q_a \rightarrow q_{\bar{a}}$ i.e. it changes the sign of the helicity. Hence, the contribution with opposite baryon helicity- viz. $M^{\bar{a}}_{\bar{b}}$ - may be obtained by a parity transformation.

The representation 56 of U_6 contains a decuplet of spin 3/2 particles and an octet of spin 1/2 baryons. Under $U(3) \otimes U(3)$, we have seen that 56 splits into $(10,1) \oplus (1,10) \oplus \oplus (6,3) \oplus (3,6)$:

$$B_{abc} = D_{abc} \, (\chi_{10})_{111} \qquad (12.27)$$

$$B_{ab\bar{c}} = \sqrt{3} \, D_{abc} \, (\chi_{10})_{112} + $$

$$+ \frac{1}{\sqrt{6}} (N_a^{\ d} \, \varepsilon_{dbc} + N_b^{\ d} \, \varepsilon_{dac}) \, (\chi_8)_1$$

As before, $B_{\bar{a}\bar{b}\bar{c}}$ and $B_{a\bar{b}\bar{c}}$ may be obtained by a parity transformation of the above relations.

The best available target is the proton p and the best available beam of incoming particles is the beam of π or K mesons. Therefore, the most important thing to study is the Baryon-Pseudo-scalar meson scattering. The octet of pseudo-scalar mesons are in M^a_b. One thus gets for baryon-pseudo-scalar meson scattering:

$$A_1 \, \bar{B}^{ab\bar{c}} \, B_{ab\bar{c}} \, \bar{M}^d_{\ e} \, M^e_{\ d} + A_2 \, \bar{B}^{ab\bar{c}} \, B_{ab\bar{c}} \, \bar{M}^{\bar{d}}_{\bar{e}} \, M^{\bar{e}}_{\bar{d}}$$

$$+ A_3 \, \bar{B}^{ab\bar{c}} \, B_{de\bar{c}} \, \bar{M}^d_{\ a} \, M^e_{\ b}$$

$$+ A_4 \, \bar{B}^{ab\bar{c}} \, B_{db\bar{c}} \, \bar{M}^d_{e} \, M^e_{\ a} + A_5 \, \bar{B}^{ab\bar{c}} \, B_{db\bar{c}} \, M^d_e \, \bar{M}^e_a$$

$$+ A_6 \, \bar{B}^{ab\bar{c}} \, B_{ab\bar{d}} \, \bar{M}^{\bar{d}}_{\bar{e}} \, M^{\bar{e}}_{\bar{c}} + A_7 \, \bar{B}^{ab\bar{c}} \, B_{ab\bar{d}} \, M^{\bar{d}}_{\bar{c}} \, \bar{M}^{\bar{e}}_{\bar{c}} \qquad (12.28)$$

$$+ A_8 \, \bar{B}^{ab\bar{c}} \, B_{ad\bar{e}} \, \bar{M}^d_b \, M^{\bar{e}}_{\bar{c}} + A_9 \, \bar{B}^{ab\bar{c}} \, B_{ad\bar{c}} \, M^d_b \, \bar{M}^e_{\bar{c}}$$

The different terms here are not necessarily independent.

When (12.26) and (12,27) are applied to (12.28), one gets for Baryon-pseudo-scalar meson scattering expressions of the form $\mathrm{tr}\ \overline{B}\, B\, \overline{P}\, P$, $\mathrm{tr}\ \overline{B}\, \overline{P}\ \mathrm{tr}\ BP$ and various permutations of these, each term being multiplied by an invariant matrix element. The following term is absent due to $U(3) \otimes U(3)$.

$$\mathrm{tr}\ \overline{B}\ \overline{P} \cdot \mathrm{tr}\ B\, P - \mathrm{tr}\ \overline{B}\, P \cdot \mathrm{tr}\ B\overline{P} \tag{12.29}$$

because the first three terms of Eq. (12.28) are symmetric in the interchange of P and $\overline{P}$. The next four terms contain the product $P\overline{P}$ or $\overline{P}P$. Direct computation shows that the last two terms do not contribute.

Using standard SU(3) calculations[1], the absence of the term (12.29) implies for the amplitudes of elastic scattering in the forward direction:

$$(K^+ P) - (K^- P) = (\pi^+ P) - (\pi^- P) + (K^\circ P) - (\overline{K}^\circ P) \tag{12.30}$$

Where we mean by (PB) the elastic scattering $P + B \rightarrow P + B$.

Using the <u>optical theorem</u>, which relates the imaginary part of the forward scattering amplitude to the total cross-section, Eq.(12.30) is also true for total cross-sections.

This relation was also found by Ruhl, using the inhomogeneous SL(6,C) group. It is very well satisfied experimentally[2] from 6 Gev/c to 18 Gev/c, kinetic energy of the incident meson.

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1) C.A.Levinson, H.J.Lipkin and S.Meshkov, Phys. Letters. 1,<u>44</u>, (1962)
P.G.O.Freund, H.Ruegg, D.Speiser and A.Morales, Nuovo Cimento, <u>25</u>, 307 (1962).

2) W.Galbraith et al., BNL 8744 (February)

The relation (12.30) is a combination of the two Johnson-Treiman relations:

$$\frac{1}{2}\left[(K^+ P) - (K^- P)\right] = (\pi^+ P) - (\pi^- P)$$

$$\frac{1}{2}\left[(K^+ P) - (K^- P)\right] = (K^\circ P) - (\bar{K}^\circ P)$$

which are relations among total cross sections.

The absence of Eq.(12.29) implies for the differential cross-sections in the forward and backward direction:

$$\frac{1}{4} \sigma (K^- P \rightarrow K^\circ \Xi^-) = (K^- P \rightarrow K^\circ \Xi^\circ) = \\ = \sigma(\pi^- P \rightarrow K^+ \Sigma^-) \qquad (12.31)$$

Using SU(3) invariance:

$$\sigma(K^- P \longrightarrow K^\circ \Xi^\circ) = \sigma(K^- P \rightarrow \pi^+ \Sigma^-) \qquad (12.32)$$

Eq.(12.31) was derived in Blankenbecler et al[1] under much more restrictive hypothesis.

The relation $1/4\ \sigma(K^- P \rightarrow K^+ \Xi^-) = (K^- P \rightarrow \pi^+ \Sigma^-)$ does not seem to be very well satisfied experimentally[2]. The same is true for Eq.(12.32), which requires only SU(3). However, the best test would be provided by the relation:

$$(K^- P \longrightarrow K^+ \Xi^-) = 4 \quad (K^- P \rightarrow K^\circ \Xi^\circ).$$

where the kinematical factors are the same for both processes, so that this cannot be attributed to mass-breaking. A rough estimate of the experimental results[2), 3)] at 1.5 Gev/c and at 3.0 Gev/c gives:

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1) R.Blankenbecler, M.C.Goldberger, K.Johnson and S.B.Treiman, Phys. Rev. Letters. <u>14</u>, 5-8 (1965)
2) J.D.Jackson, University of Illinois, Preprint.
3) M.L.Stevenson, UCRL - 11493, June 1964

$$4\ \frac{\frac{d\sigma}{d\Omega}(K^-P \rightarrow K^\circ\ \Xi^\circ)}{\frac{d\sigma}{d\Omega}(K^-P \rightarrow K^+\ \Xi^-)} =$$

1.5 Gev/c		3.0 Gev/c	
0°	180°	0°	180°
>4	1.5	16	1.4

where the value of the experimental ratio for θ = 180° is consistent with the predicted value 1, within experimental errors, while there is a large discrepancy for θ = 0°. If these values are confirmed, it is surprising that the Johnson-Treiman relation holds so well. The main difficulty in the case of the forward scattering, for the experimentalists is in detecting Ξ° which is not seen in the bubble chamber since it is a neutral particle. Ξ° has to be observed only from its decay:

$$\Xi^\circ \rightarrow \Lambda + K^\circ$$

But unfortunately Λ is again neutral particle which has to be observed only from its decay:

$$\Lambda \rightarrow P + \pi^-$$

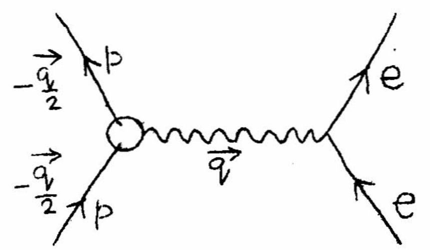
Furthermore, in the decay mode of K° what one observes is the decay of K_1° (now called K_s° where 's' stands for 'short lived') Therefore, we do not know whether we see K° or $\bar{K}^\circ$. The experimentalists look for K_s° and for each event they calculate the missing mass. Hence, it is difficult to distinguish Λ from N^*, since the mass of N^* is nearly the same as that of Λ. So, large uncertainities in Λ may be responsible for the unfavourable results for forward scattering. So, further results are being awaited where the Λ will be directly detected.

In the case of inhomogeneous U(6,6), we have the group $(U(6))_W$ for the collinear processes, which means that we have more symmetry for collinear processes than what we had for SL(6,C). Due to this one gets more predictions. Jackson finds that for $(U(6))_W$ • there are many processes which are in violent disagreement with experimental results. Already $(U(6))_W$ violates unitarity, so it is not surprising to note that it may not be good for scattering also.

Vertex functions:

The results in this case are found to be good both for $(U(6))_W$ and $U(3) \otimes U(3)$. May be, ultimately, these groups will survive for classification of particles and vertex functions.

We will now derive the value for the ratio of magnetic form-factors of the proton and the neutron. We consider electron-proton scattering. In the Breit-system, the two baryons which interact with the photon have opposite momenta. Hence, one can use the collinear groups.



The matrix element:

$$\langle p \,|\, j_\mu \,|\, p \rangle = F_1(q^2)\, \bar{U}\, \gamma_\mu U + iF_2(q^2)\, \bar{U}\, \sigma_{\mu\nu} q^\nu U \tag{12.33}$$

where $F_1(q^2)$ and $F_2(q^2)$ are the electric and magnetic moment form factors of the proton. Following Sachs[1], we define:

$$G_E(q^2) = F_1(q^2) - \left(\frac{q^2}{2M} \right) F_2(q^2)$$
$$G_M(q^2) = \frac{F_1(q^2)}{2m} + F_2(q^2) \qquad (12.34)$$

where m is the nucleon mass.

In order to get predictions for the electromagnetic form factors, one has not only to assume $U(3) \otimes U(3)$ symmetry for the strong interactions, but make a specific assumption on the transformation properties of the current. These will be:

1) The current transforms according to irreducible representations of $U(3) \otimes U(3)$ contained in I.R. 35 of $SU(6)$

2) As far as $SU(3)$ is concerned, only the octet contributes to the current

3) If s is the ordinary spin and s_z its projection, the charge form factor transforms as $s = 0$, and the magnetic moment form factor, as $s = 1$, $s_z = 1$.

Hence, one gets for the magnetic moment form factor the unique value

$$\mu(q^2) = G_M(q^2) \; \overline{B}^{a\bar{b}\bar{c}} \left(\frac{\vec{q}}{2} \right) B_{a\bar{b}d} \left(\frac{-\vec{q}}{2} \right) \; Q^d_{\bar{c}} \qquad (12.35)$$

where $Q^d_{\bar{c}}$ is the $SU(3)$ charge matrix.

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1) R.G.Sachs, Phys. Rev. 126, 2256 (1962)

[Remark: In SU(3) we had two form factors. We looked for the number of octets which occur in the decomposition of $8 \otimes 8$. Since two 8's occured in the decomposition, we had two different couplings (F and D) giving rise to two form factors. But $U(3) \otimes U(3)$ gives only one form factor.] The contribution of (12.35) to the nucleus is given by Eq.(12.27).

$$\begin{aligned}\mu(q^2) &= G_M(q^2) \; (5 \text{ tr } \bar{N}QN + \text{tr } \bar{N}NQ) \\ &= G_M(q^2) \Big[3 \text{ tr } \bar{N}(QN + NQ) \\ &\qquad + 2 \text{ tr } \bar{N}(QN - NQ) \Big] \end{aligned} \tag{12.36}$$

where $(QN + NQ)$ is the symmetric combination of two octets in SU(3), called D-type coupling and (QN-NQ) is the F-type coupling. Therefore,

$$\frac{D}{F} = \frac{3}{2}$$

This gives the ratio of the magnetic form factors for the proton and netron as:

$$\frac{\mu_p(q^2)}{\mu_n(q^2)} = -\frac{3}{2} \tag{12.37}$$

This result gives correctly the ratio of magnetic moments ($\mu\,(q^2=0)$ and this also agrees with the empirical fact that the ratio of the form factors is independent of q^2.

Certainly the simplest way of deriving this result is to combine in one group spin and unitary spin, as is the case for both $U(3) \otimes U(3)$ and $(U(6))_W$.

Proceedings of Matscience Symposia

Edited By

ALLADI RAMAKRISHNAN

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