MATSCIENCE REPORT 35

LECTURES ON

A MODIFIED MODEL OF EUCLIDEAN QUANTUM FIELD THEORY

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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20 (INDIA)

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A MODIFIED MODEL OF EUCLIDEAN QUANTUM FIELD THEORY

Lecture 1:

1. Introduction

I assume that you all know more or less what relativistic quantum field theory is or rather what we would like it to be, since no such theory with a non-trivial S-matrix is known to exist. However, if we assume that the theory under consideration, which may be characterized by the existence of certain symmetry groups besides the Lorentz group, also satisfies a certain set of postulates[1)] then it possesses an S-matrix which should, in general, be a non-trivial one. This holds for the following set of postulates.

1. Relativistic invariance
2. Existence of local field operators
3. Mass spectrum conditions.

We can then attempt to deduce as many consequences for the S-matrix as possible from these postulates. This is one important aspect of this so called 'Axiomatic Field Theory', its main result being the analytic properties of scattering amplitudes expressed, e.g., as dispersion relations[2)].

In this approach we are studying the most general objects of a rather broadly defined class not knowing if this class is actually non-empty. This is what Heisenberg calls the approach from the exterior, where one is deliberately silent about any dynamics at the beginning and expects to discover it, perhaps, after an entirely abstract analysis. He strongly advocates the opposite way, viz., to go from the interior to the exterior, start with the microscopic dynamics and deduce the macroscopic one, including symmetries and the corresponding conserved quantities from it. His non-linear theory is based on a specific (unrenormalizable) Lagrangian.[3]

During the past year, as a consequence of the developments around SU_3[4] the Lagrangian approach, though not necessarily that of Heisenberg, has redrawn the attention of many theoreticians. Namely, even if Chew's belief that the symmetries and dynamics do flow from the S-matrix standpoint[5], properly interpreted is correct the way from the present status of this approach to what one needs in connection with SU_3 in certainly distressingly long. One is willing to start on a less ambitious level, namely, a Lagrangian theory where SU_3 and its violations are already built in or where only SU_3 is built in and its violation is "spontaneous". Even with a definite Lagrangian, the theory so defined can be studied only by using approximations of generally doubtful validity. (There exist also attempts to derive symmetries "self-consistently" without explicit use of a Lagrangian. Characteristically, however, all of these work with off-mass-shell quantities).

At this point, it might be useful to recall the most of our confidence in the concepts evolved in connection with quantum field theory, even if they are abstracted later into an "S-matrix theory", comes from the success of quantum electrodynamics (QED), since in the realm of strong interactions, a verification of the predictions of quantum field theory has been possible, at best, only in a qualitative manner. (Certain precise predictions like TCP invariance and the spin-statistics connection, are in themselves only qualitative ones). QED is a Lagrangian theory soluble by renormalized perturbation theory which, one feels, might yield an asymptotic series. That is, the existence of QED as a mathematically well-defined theory is not known. However, the concept of renormalizability developed in perturbation theory has been of great interest. Some say a Lagrangian theory should be renormalizable to be taken seriously; others, on the contrary, say the more unrenormalizable a theory is, the better. But neither opinion has good arguments in its favour or against it. Clearly, outside of purely axiomatic field theory, the study of any Lagrangian theory that is not exactly soluble, (and only theories with a trivial S-matrix are soluble) without use of perturbation theory, is most desirable.

I made an attempt towards such a study; but the results are very meager as yet. Nevertheless, I shall present them since some of the results seem to merit thinking about. These investigations may also be described as a mathematical study of problems formulated in terms of an infinite system of coupled integral equations of a certain type suggested by Lagrangian quantum field theory.

We may represent the relation between the conventional or Minkowski Quantum Field Theory (MQFT) and the present modified version of Euclidean Quantum Field Theory (EQFT) as follows ($t \to -it$ means, ~~so~~ explained later, analytic continuation in terms of Green's functions

MQFT

$\downarrow$ $t \to -it$

EQFT

$\downarrow$

Modified EQFT $\leftrightarrow$ Results

defined and existence shown

by Friedricks-Shapiro integral

There is an analogy to the transition from the Schrodinger equation in non-relativistic quantum mechanics (NRQM) to the heat equation. (More details are given in lecture VIII.)

NRQM $\equiv$ Schrodinger equation $\longleftrightarrow$ Feynman path integrals

$\Big\downarrow t \to -it$

"Euclidean" NRQM $\equiv$ heat equation $\longleftrightarrow$ Wiener integral

The modifications that lead from EQFT to ...

(a) Regularization (of the Lagrangian)

(b) Introduction of a finite space-time volume

The infinite set of coupled integral equations for the Green's function (this is the form of which MQFT with a given Lagrangian can be rewritten) [6] has so far been, of little help because of our inability to formulate properly the boundary conditions on such systems to make them mathematically meaningful. There does not also seem be any justifiable way of truncating these equations which leads to reliable answers. Further each single equation is not well-defined. In momentum space they involve at best only conditionally convergent integrals while in co-ordinate space they involve products of distributions. Dyson [7] has shown how this last difficulty can be overcome in perturbation theory by a rotation of the path of integration (in momentum space). More generally one can perform an analytic continuation and define the original functions by the boundary values of their continuations. Continuing to imaginary time and energy we obtain the Euclidean Green's functions studied by Schwinger [2] and Nakano. [3] These functions can be defined even without recourse to a Lagrangian and may be used to define axiomatic EQFT, the characteristic symmetry group of which is the four-dimensional and orthogonal group (instead of the Lorentz group). An operator formulation for the Lagrangian case will be given in Lecture IX. Transition from MQFT to EQFT does not, however, remove the ultra violet difficulties of MQFT but only makes them appear in terms of divergent, instead of meaningless integrals. In our approach we regularize the theory by the method of Pais and Uhlenbec[1] The indefinite metric introduced in MQFT does not carry over to EQFT where the metric remains positive definite. Also, certain properties of EQFT we derive seem to be independent of the regularization. The restriction to a finite space-time volume is more drastic, but still, a few of properties of the EQFT Green's function seem to be insensitive to even this and certain inequalities which arise in our model must be valid also for the unmodified model and hence for the corresponding MQFT, if its Green's functions exist at all.

The infinite system of integral equations mentioned before is equivalent to a functional differential equation for the generating functional of the Green's functions, the Schwinger functional. In MQFT this is formally solved by the Feynman history integral[10] but this solution seems to be difficult to analyze mathematically. The corresponding Wiener history integral, which solves EQFT is, however, well-defined for the modified theory.

The following topics will be covered in these lecture;

I Axiomatic formulation

II General considerations

III A model. The Friedrichs-Shapiro (FS) integral

IV The (FS) integral (contd.)

V The (FS) integral (contd.) Modifications of the model

VI The regularised Yukawa operator. Solutions of the modified model.

VII General properties of the solution: Positive definiteness and uniqueness

VIII General properties (contd). The semigroup property. Comparison with the Feynman path integral and the Feynman-Kac formula. Analyticity in the coupling constant. Question of measure.

IX Special properties: Entire analyticity in Dirichlet data: Bounds based on convexity. Variational bounds. Operator formulation. Conclusion.

2. FORMULATION OF EQFT

Some symbols used in this section have the following meaning

V^+ = futur light cone, $\forall$ = for all, $\exists$ = there exists
$\bigcup$ = Union, $\bigcap$ = interaction, $\in$ = contained in

To formulate EQFT, we adopt the axiomatic approach to relativistic quantum field theory developed by Wightman[1] and for simplicity confine ourselves to the theory of a single scalar Hermitian field $A(x)$. The Wightman functions which are the vacuum expectation values.

$$\langle A(x_0)\, A(x_1) \cdots A(x_n) \rangle ;$$

$$x_i = (x_i^0 , x_i^1 , x_i^2 , x_i^3) \qquad (\text{I}.1)$$

of products of field operators are considered as functions of $\xi_i = x_{i-1} - x_i$, $(i = 1, \ldots\ldots n)$ boundary values of analytic functions $W_n((\zeta))$ where (ζ) is the n-tuple of points $(\xi_1, \ldots \xi_n)$, (each of which is a complex variable) with analyticity in the tube

$$\mathcal{R}_n = \{ (\zeta) : \mathfrak{I}_m \zeta_i \in V^+ , \forall_i \} \qquad (\text{I}.2)$$

i.e. $\mathfrak{I}_m \zeta_i^0 > 0$, $(\mathfrak{I}_m \zeta_i)^2 < 0$ with $g_{\mu\nu} = \delta_{\mu\nu}(-1)^{\delta_{\mu 0}}$

This is due to the stability of the vacuum, the spectral condition and the assumed temperedness of these functions as distributions.

Due to relativistic invariance, $W_n((\zeta))$ is analytic in the extended tube

$$R_n' = \{ \zeta : \exists \Lambda + (c) \text{ such that } (\zeta) = (\Lambda + (c)\zeta') \qquad (\text{I}.3)$$
$$(\zeta') \in \mathcal{R}_n \}$$

where $\Lambda_+(c)$ is a proper homogeneous complex Lorentz transformation i.e.

$$\Lambda_+^T(c)\, g\, \Lambda_+(c) = g\,, \qquad \det \Lambda_+(c) = 1 \tag{I.4}$$

Because of local commutivity, $W_n((\zeta))$ is analytic and single-valued in the permuted extended tube

$$\mathcal{R}_n'' = \bigcap_{\text{all } P} P\,\mathcal{R}_n' \tag{I.5}$$

with

$$P\,\mathcal{R}_n' = \left\{ (\zeta) : (\zeta) = (P\zeta')\,, (\zeta') \in \mathcal{R}_n' \right\} \tag{I.6}$$

where $P \in S_{n+1}$ is a permutation

$$P : (0\,1 \cdots n) \rightarrow \big(P(0)\,P(1) \ldots\ldots P(n)\big) \tag{I.7}$$

If $\zeta_i = Z_{i-1} - Z_i$ then $P\zeta_i = Z_{P(i-1)} - Z_{P(i)}$ (I.8)

In $\mathcal{R}_n''$ we have

$$W_n((P\zeta)) = W_n((\zeta)) \tag{I.9}$$

since we have only one field.

We now define the Schwinger points as

$$(\zeta_s) : \operatorname{Re} \zeta_i^0 = 0\,, \quad \operatorname{Im} \zeta_i^{1,2,3} = 0\,, \quad \forall_i \tag{I.10}$$

They lie in the interior of $\mathcal{R}_n''$ if

$$\zeta_{s(i+1)} + \cdots\cdots + \zeta_{sk} \neq 0 \text{ for all } 1 \leq i+1 \leq k \leq n \tag{I.11}$$

We may write

$$\zeta_{si} = x_{i-1} - x_i \,, \quad x = (x^1, x^2, x^3, x^4) \qquad (1.12)$$

and introduce the Schwinger functions

$$W_n\big((\zeta_0)\big) \equiv S(x_0\, x_1 \,\ldots\, x_n) \qquad (I.13)$$

These Euclidean Green's functions are symmetries in their $n+1$ four-vector arguments; they are invariant under the proper in-homogeneous orthogonal group in four-dimensions (the Euclidean group). Namely, if g is the Lorentz metric,

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \qquad (I.14)$$

introduce the Eucledian metric by multiplying all coordinate vectors by

$$h = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & i \end{pmatrix} \; ; \qquad h = h^T \; ; \quad h^2 = g \qquad (I.15)$$

If Λ is a complex Lorentz transformation

$$\Lambda^* g \Lambda = \Lambda^T h\, h\, \Lambda = g = h\, h \qquad (I.16)$$

or

$$\underbrace{h^{-1} \Lambda^T h}\; \underbrace{h \Lambda h^{-1}} = 1 \qquad (I.17)$$

Thus $\quad h \Lambda h^{-1} \in O_4(c)$, $\qquad$ (I.18)

an (in general complex) orthogonal transformation, and using the theorem of Hall Wightman [1] the Schwinger functions are invariant under these transformations.

This provides a proof that the non-coincident S-points lie in $\mathcal{R}_n''$, since O_4 allows the points (ξ) to have for a suitable ordering, of arguments the 4-components decreasing. Transformation back to Minkowski space by the application of makes the 0-components imaginary and to increase, that is, we end up in the tube .

An especially simple way to obtain the Schwinger functions is from the time-ordered functions.

$$F(x_0, x_1, \ldots x_n) = \langle T(A(x_0) \cdots A(x_n)) \rangle \tag{I.19}$$

and make the substitution

$$x_i^0 \rightarrow x_i^0 e^{-i\alpha}, \qquad \alpha = 0, \ldots \pi/2 \tag{I.20}$$

or from the functions with anti-time ordering

$\langle \overline{T}(A(x_0) \ldots A(x_n)) \rangle$ by the substitution

$$x_i^0 \rightarrow x_i^0 \rightarrow e^{i\alpha}, \quad \alpha = 0, \ldots \pi/2 \tag{I.21}$$

Comparing these two expressions and using the uniqueness of the analytic Wightman function gives the symmetry the relations

$$S(x_0, \ldots x_n) = S(x_0^t \ldots x_n^t)^*$$

$$= S(x_0^s \ldots x_n^s)^* = S(-x_0, \ldots -x_n) \tag{I.22}$$

where x^t is the time-reflected and x^s the space-reflected position coordinates. Thus if the theory is invariant under time reversal or space-reflection the Schwinger functions are real and so are their fourier transforms (FT).

For the two-point function we have

$$W(\xi) = \frac{K(m(-\xi^2)^{1/2})}{(-\xi^2)^{1/2}} \frac{m}{4\pi^2}$$

$$= \frac{1}{(2\pi)^4} \int \frac{e^{-ipx}\, d^3p}{p^2+m^2}$$

(I.23)

and hence the FT is perfectly analytic. (K is the Kelvin function) (Contrast the situation for $W(\xi)$ in MQFT,

$$W(\xi) = \frac{M_1^{(2)}[m(-\xi^2 - i\varepsilon\xi_0)^{1/2}]}{(-\xi^2 - i\varepsilon\xi_0)^{1/2}} \frac{m}{4\pi^2}$$

(I.24)

which has the FT, $\delta(p^2-m^2)\,\theta(p_0)$. Generally Ruelle[18] has shown that the Fourier transform

$$\tilde{F}(p_1 \ldots p_n) = \int dx_1 \ldots dx_n\, e^{i\sum x_i p_i} F(0, x_1 \ldots x_n)$$

(I.25)

is the boundary value of an analytic function which is invariant under the proper homogeneous Lorentz group. The Schwinger points in momentum space

$$p_s : \operatorname{Re} p_i^0 = 0, \quad \vartheta_m p_i^{1,2,3} = 0, \quad p_i^0 = -i p_i^4 \, , \forall_i \tag{I.26}$$

lie inside the analyticity domain except for points where a non-empty partial sum of the vectors $p s_i$ vanishes. We write

$$\tilde{F}((p_0)) = \zeta(p_0, p_1 \cdots p_n) \tag{I.27}$$

Then

$$(2\pi)^4 \zeta(p_0, \cdots p_n) \delta(p_1 + \cdots p_n) = \int dx_0 \cdots dx_n \, e^{-i\sum x_i p_i} \, S(x_0, \cdots x_n) \tag{I.28}$$

where

$$x_i p_i = x_i^1 p_i^1 + x_i^2 p_i^2 + x_i^3 p_i^3 + x_i^4 p_i^4 .$$

We can also define the truncated Wightman functions W^T and truncated Green's functions F^T as those which contain only the connected parts; then the Fourier transform $\tilde{F}^T$ have no singularities at Schwinger points. Hence the Fourier transform

$\tilde{S}^T(p_0, \cdots p_n) = \tilde{F}^T((p_0))$ are real analytical functions which are invariant under the proper orthogonal group and satisfy because of (1.22)

$$\tilde{S}^T(p_0, \ldots p_n) = \tilde{S}^T(p_0^T \ldots p_n^T)^* = \tilde{S}^T(p_0^s, \ldots p_n^s)^* = \tilde{S}^T(-p_0, \ldots \ldots -p_n). \tag{I.29}$$

They can be analytically continued into the tube,

$$(\vartheta_m p) \in D^m \equiv \bigcap_{\text{all } I} D_I^m \tag{I.30}$$

where

$$D_I^m = \left\{ (\vartheta_m p) \sum_{k=1}^{4} \left(\sum_{i \in I} \vartheta_m p_i^k \right)^2 < m^2 \right\} \tag{I.31}$$

I is a proper subset of $\{0, \ldots \ldots n\}$ and $m > 0$ is the lowest mass in the spectrum except for the vacuum. Hence if $\min_{i \neq j} (x_i - x_j)^2 > \epsilon$ we have

$$\lim_{D \to \infty} S^T(x_0, \ldots x_n) e^{\sum_{i=0}^{n} \alpha_i \cdot n_i} = 0, \quad q_i \in D^m \tag{I.32}$$

where

$$q_i x_i = q_i^1 x_i^1 + \ldots + q_i^4 x_i^4 \,; \; D = \left(\max_{i,j} (x_i - x_j)^2 \right)^{1/2} \tag{I.33}$$

(I.32) shows that S^T must decrease exponentially for increasing distance between its arguments is immediately seen for the two point function (I.23).

Lecture II:

1. GENERAL CONSIDERATIONS

Having established in the last lecture the existence of Euclidean Green's functions in every theory that satisfies Wightman's postulates, we will proceed more heuristically which seems justified as no physically notrivial example of a Wightman theory is known. This will enable us to derive an interesting inequality for the generating functional of Euclidean Green's functions. The following considerations (apart from the analytic continuation, they are thoroughly familiar, e.g. [1]) apply to any theory merely invariant under time translation.

With a real function $j(\vec{x},t)$, $\vec{x}=(x^1,x^2,x^3)$, we define the two-parameter family of operators $U(t,t')$, $t \geq t'$ by

$$\left(\frac{\partial}{\partial t}\right) U(t,t') = i\int d\vec{x}\, A(\vec{x},t)\, j(\vec{x},t)\, U(t,t') \tag{II.1a}$$

$$U(t',t') = 1 \tag{II.1b}$$

or

$$U(t,t') = 1 + i\int_{t'}^{t} d\tau \int d\vec{x}\, A(\vec{x},\tau)\, j(\vec{x},\tau)\, U(\tau,t') \tag{II.2}$$

From (II.2) follows, if we assume uniqueness of its solution,

$$\frac{\partial}{\partial t'} U(t,t') = -i\int d\vec{x}\, U(t,t')\, A(\vec{x},t')\, j(\vec{x},t') \tag{II.3}$$

from (II.1) and (II.3) we have, for $t \geq t'' \geq t'$

$$U(t,t'')\, U(t'',t') = U(t,t') \tag{II.4}$$

From (II.1)

$$U(t,t')^{\dagger}\, U(t,t') = 1$$

and from (II.3)

$$U(t,t')\, U(t,t')^{+} = 1$$

(II.1) r (II.3) and the consequence of (II.2) and (II.4)

$$\left[\frac{\delta}{\delta j(x)}\right] U(t,t') = i\,\theta(t-x^0)\,\theta(x^0-t')\,U(t,x^0)\,A(x)\,U(x^0,t') \tag{II.5}$$

substantiate the symbolic formula

$$U(t,t') = T\exp\left[i\int_{t'}^{t} d\tau \int d\vec{x}\, A(\cdot\vec{x},\tau)\, j(\vec{x},\tau)\right] \tag{II.6}$$

• The formal limit

$$F\{j\} \equiv \lim_{\substack{t\to+\infty \\ t'\to-\infty}} \langle U(t,t')\rangle \tag{II.7a}$$

is the generating functional of the Green's functions $F(x_1 \cdots x_n)$ considered in Lecture 1.

$$F\{j\} = 1 + \sum_{n=1}^{\infty} (n!)^{-1}\, i^n \int\cdots\int dx_1 \cdots dx_n\, j(x_1)\cdots j(x_n)\, F(x_1\cdots x_n) \tag{II.7b}$$

Similarly,

$$F^T\{j\} = \ln F\{j\} \tag{II.8}$$

is the generating functional of the truncated Green's functions. (The limit $t \to +\infty$, $t' \to -\infty$ of $U(t, t')$ will actually exist as weak, and since $U(t, t')$ is unitary, also as strong operator limit provided $j(x, \tau)$ vanishes sufficiently strongly for $|\tau| \to \infty$

The functional derivative of $F\{j\}$ is given by

$$\frac{\delta}{\delta j(x)} F\{j\} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(F\{j'\} - F\{j\}\right)$$

where $j'(x') = j(x') + \varepsilon\,\delta(x - x')$

Alternatively the functional derivative as a linear form may be defined by

$$\int \frac{\delta}{\delta j(x')} F\{j\} f(x)\, dx = \lim_{\epsilon \to 0} \frac{1}{\varepsilon}\left(F\{j + \varepsilon f\} - F\{j\}\right).$$

if it exists and has the requisite properties (linearity in f so that the Hahn-extension theorem is applicable. The simplest case to visualise is obtained by pu[illegible] j [illegible] f [illegible] expanding the right hand side in powers of ε -this gives for $j = 0$ and $\varepsilon = 1$ the Maclaurin (or Volterra) series:

$$F\{j\} = F\{0\} + \int j(y) F(y)\, dy + \frac{1}{2!}\iint j(y_1)\, j(y_2)\, F(y_1, y_2)\, dy_1\, dy_2 + \cdots$$

where

$$F(y_1, y_2) = \frac{\delta^n}{\delta j(y_1)\, \delta j(y_2)} F\{j\} \qquad j=0$$

and so on. The functional derivative prescription then gives

$$\frac{\delta^n F\{j\}}{\delta j(x_1) \ldots \delta j(x_n)} = F(x_1, \ldots x_n) + \int j(y) F(x_1 \cdots x_n, y)\, dy$$
$$+ \frac{1}{2!} \int j(y_1) j(y_2) F(x_1, x_2, \cdots x_n, y_1, y_2)\, dy_1 dy_2$$

One can also verify that

$$F\{j\} = F\{0\} + \int_0^1 d\varepsilon\, j(x) \frac{\delta}{\delta j(x)} F\{\varepsilon j\}\, dx$$

which is actually the Taylor's formula with ε as variable for $\varepsilon = 1$

We define the unitary operators

$$V(t, t') = e^{-iHt}\, U(t, t')\, e^{iHt'} \qquad \text{(II.9)}$$

where H is the Hamiltonian, which satisfies with ground state $|\rangle_0$; $H|\rangle_0 = 0.$ (II.10)

(Though we are not using a Lagrangian theory the Hamiltonian enters since $H = P_0$ itself.)

From

$$A(\vec{x}, t) = e^{iHt} A(\vec{x}, 0)\, e^{-iHt}$$

and (4) we find

$$V(t, t') = 1 - i \int_{t'}^{t} H\{j_\tau\}\, V(\tau, t') \qquad \text{(II.11)}$$

where $\dot{j}_\tau(\vec{x}) = \dot{j}(\vec{x},\tau)$ and

$$H\{\dot{j}\tau\} = H - \int d\vec{x}\, A(\vec{x},0)\,\dot{j}(\vec{x},\tau) \tag{II.12}$$

is the energy operator perturbed by a ("time-independent") source term. The V's have the same semigroup properties as the U's. It may be noted that also

$$\lim_{\substack{t\to+\infty \\ t\to-\infty}} \langle V(t,t')\rangle = \langle e^{-iHt}\, U(t,t')\, e^{iHt'}\rangle$$
$$= F\{\dot{j}\}.$$

We now replace (II.11) by

$$V_z(t,t') = 1 - iz\int_{t'}^{t} d\tau\, H\{\dot{j}_\tau\} V_z\{\tau,t'\} \tag{II.13}$$

with the properties of $V_Z(t\,t')$

$$V_z(t,t') = V_z(t,t'')\,V_z(t'',t') \tag{II.14}$$
$$t \geq t'' \geq t',$$

$$\left[\delta/\delta\dot{j}(x)\right] V_z(t,t') = iz\,\theta(t-x^0)\,\theta(x^0-t')\,V_z(t,x^0)\,A(\vec{x},0)\,V_z(x^0,t') \tag{II.15}$$

$$\left(\frac{\partial}{\partial z}\right) V_z(t,t') = -i\int_{t'}^{t} d\tau\, V_z(t,\tau)\, H\{\dot{j}_\tau\}\, V_z(t,t')$$

and the representation analogous to II.6

$$V_z(t,t') = T\exp\cdot\left[-iz\int_{t'}^{t} d\tau\, H\{\dot{j}_\tau\}\right] \tag{II.16}$$

From (13) we have

$$\left(\frac{\partial}{\partial t}\right) V_z(t, t') = -i z H\{j_\tau\} V_z(t, t')$$

and thus

$$\frac{\partial}{\partial t} \langle \phi | [V_z(t,t')^+ V_z(t,t')] | \phi \rangle = \tag{II.17}$$
$$= 2(\mathrm{Im}\, z) \langle \phi | V_z(t,t')^+ H\{j_\tau\} V_z(t,t') | \phi \rangle$$

Let

$$E_0\{j_\tau\} = g.l.b. \{\langle \phi | H\{j_\tau\} | \phi \rangle \langle \phi | \phi \rangle^{-1}\} \tag{II.18}$$

Then from (17) follows provided $\mathrm{Im}\, Z \leq 0$

$$\left(\frac{\partial}{\partial t}\right) \langle \phi | V_z(t,t')^+ V_z(t,t') | \phi \rangle$$
$$\leq 2\,\mathrm{Im}\, Z \langle \phi | V_z(t,t')^+ H\{j_\tau\} V_z(t,t') | \phi \rangle$$
$$\leq 2(\mathrm{Im}\, z) \langle \phi | V_z^+ V_z | \phi \rangle E_0\{j_\tau\}$$

By integration we obtain

$$\langle \phi | V_z(t,t')^+ V_z(t,t') | \phi \rangle \leq \langle \phi | \phi \rangle \cdot \tag{II.19}$$
$$\cdot \exp\left[2\,\mathrm{Im}\, Z \int_{t'}^{t} E_0\{j_\tau\} d\tau\right]$$

and therefore the Cauchy inequality gives for any matrix element of V_z kind is

$$|\langle \phi | V_z(t,t') | \phi' \rangle| = \|\phi\| \, \|\phi'\| \exp\left[(\mathrm{Im}\, z) \int_{t'}^{t} E_0\{j_\tau\} d\tau\right] \tag{II.20}$$

As can be seen by comparing (I.19-20), (II.9), (II.11) and (II.13)

$$\lim_{\substack{t\to+\infty \\ t'\to-\infty}} \langle V_{-i}(t,t')\rangle = S\{j\} \qquad \text{(II.21)}$$

is the generating functional of the Euclidean Green's functions of Lecture 1.

$$S\{j\} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int\int\int\cdots\int dx_1 \cdots dx_n\, j(x_1)\, j(x_2) \cdots j(x_n) \cdot S(x_1 \cdots x_n)$$

and (II.20) gives

$$|S\{j\}| \leq e^{-\int E_0\{j_\tau\} d\tau} \qquad \text{(II.22)}$$

The V'_Λ of the above description can be shown to exist for certain models; for instance when the Hamiltonian is given by

$$H = \frac{1}{2}\int d\vec{x}\left[A(\vec{x})^2 + (\vec{\nabla}A(\vec{x}))^2 + m^2 A(\vec{x})^2 + V(\vec{x})\, A(\vec{x})^2\right] + \text{Const} \qquad \text{(II.23)}$$

We can solve explicitly in terms of the Green's function.

$$-\Delta G(\vec{x}, \vec{x}') + V(\vec{x})\, G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$$

From (II.16) we have

$$\left|\langle T|\exp\left[-\int H\{j_\tau\}d\tau\right]\rangle\right| \leq \exp\left[-\int E_0(j_\tau)\, d\tau\right] \qquad \text{(II.24)}$$

which is intuitively evident

This inequality is the same as the one used by Feynman[3] for the
it an
polaron problem he derived from the upper Bound for the polaron ground state energy. Now we give a list of inequalities[4] which will be made use of later

$$|\overline{fg}| \le |\overline{f^2}|^{1/2}\,|\overline{g^2}|^{1/2} \qquad \text{(Cauchy's inequality)}$$

or more generally

$$|\overline{fg}| \le |\overline{f^\alpha}|^{1/\alpha}\,\left|\overline{g^{\frac{\alpha}{\alpha-1}}}\right|^{\frac{\alpha-1}{\alpha}} \qquad 1 \le \alpha \le \infty$$

(Hölder's inequality)

$$\overline{|f+g|^\alpha}^{\,\frac{1}{\alpha}} \le |\overline{f^\alpha}|^{1/\alpha} + |\overline{g^\alpha}|^{1/\alpha}$$

$$\ge \left|\, |\overline{f^\alpha}|^{1/\alpha} - |\overline{g^\alpha}|^{1/\alpha} \,\right| ,$$

$(\alpha > 1)$

(Minkowski's inequality)

(II.25)

Lecture III

1. A Model

Last time we wrote the Volterra expansion for the F and S functionals

$$F\{j\} = 1 + \sum \frac{(i)^n}{n!} \iint dx_1 \cdots dx_n \, F(x_1, \ldots x_n) \cdot j(x_1) \ldots (jx_n)$$

$$= \langle T e^{i\int A(x)\, j(x)\, dx} \rangle \qquad \text{(III.1)}$$

$$S\{j\} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \iint dx_1 \ldots dx_n \cdot S(x_1; \cdots x_n) \cdot j(x_1) \ldots j(x_n)$$

$$= \langle T e^{-i\int_{-\infty}^{+\infty} d\tau \, H\{j_\tau\}} \rangle \qquad \text{(III.2)}$$

the latter obeying the inequality

$$|\langle T e^{-\int_{-\infty}^{+\infty} d\tau \, H\{j_\tau\}} \rangle| \leq e^{-\int d\tau \, E_0\{j_\tau\}} \qquad \text{(III.3)}$$

Instead of the vacuum expectation values we can also consider arbitrary matrix elements of these, e.g.

$$\mathrm{Tr}\; e^{-\int_0^B d\tau \, H\{j_\tau\}}$$

(real-time interval only in $0-\beta$))

which can be related by analytic continuation to the quantity

$$\mathrm{Tr}\; e^{i\int_0^B d\tau \, H\{j_\tau\}}$$

The considerations made so far have made no multilations on the theory. We now consider a model (one which was also considered by Caianiello) in another connection). The Lagrangian density is given by

$$L = \frac{1}{2} (\partial_0 A)^2 - \frac{1}{2} \vec{\nabla} A \cdot \vec{\nabla} A - \frac{m^2}{2} A^2 - \frac{g}{4} A^4 \qquad \text{(III.4)}$$

(A term cubic in the field A is physically unacceptable as leading term, see 1)
The variation of this leads to the equation of motion for the field

$$(\partial_0^2 - \nabla^2 + m^2)\, A(x) + g A^3(x) = 0. \qquad \text{(III.5)}$$

and leads together with the canonical commutation relations to ... to the equation for the Schwinger functional

$$(\partial_0^2 - \nabla^2 + m^2)(-i) \frac{\delta F\{j\}}{\delta j\{x\}} + g(-i)^3 \frac{\delta^3 F\{j\}}{\delta j(x)^3}$$
$$= j(x) F\{j\} \qquad \text{(III.6)}$$

We have the normalization $F\{0\} = 1$

$\lim\limits_{\substack{t \to +\infty \\ t' \to -\infty}} \langle V(t,t') \rangle$ satisfies the same equation.

Similarly for $F_z\{j\} = \lim\limits_{\substack{t \to +\infty \\ t' \to -\infty}} \langle V_z(t,t') \rangle$

we have the equation

$$(z^{-2}\partial_0^2 - \nabla^2 + m^2)(iz)^{-1} \frac{\delta F_z\{j\}}{\delta j(x)}$$
$$+ (iz)^{-3} \left[\frac{\delta^3 F_z\{j\}}{\delta j(x)^3} \right] = j(x) F_z\{j\}. \qquad \text{(III.7)}$$

with

$$F_z\{0\} = 1$$

For $z = -i$, we have

$$(-\partial_0^2 - \nabla^2 + m^2)\left[\frac{\delta}{\delta j(x)}\right] S\{j\} + g\,\frac{\delta^3 S\{j\}}{\delta j(x)^3}$$

$$= j(x)\, S\{j\}; \quad S(0) = 1$$

(III.8)

The equation for the Schwinger functional, is difficult to solve because we do not know what the boundary conditions in function space are which we must use. But the corresponding equation in EQFT for $S\{j\}$ can be solved in terms of the Wiener history integral and we shall be mainly concerned with the solution of this equation. It cannot be solved as it is (except in perturbation theory) but by modifying it we can obtain a solution. This is the main content of the rest of these lectures.

If we write

$$(-\partial_0^2 - \nabla^2) = -\Delta \qquad \text{(III.9)}$$

and the Yukawa operator in four-dimensions

$$K_x = -\Delta + m^2 \qquad \text{(III.10)}$$

we have on using the expansion for $S\{j\}$ (and when the external source is made zero) the infinite set of equations

$$K_x\, S(x) + g\, S(xxx) = 0 \qquad \text{(odd)}$$

(III.11)

$$K_x \, S(xy) + g \, S(xxxy) = \delta(x-y)$$

(even)

$$K_x \, S(xy_1y_2) + g \, S(xxxy_1y_2)$$
$$= \delta(x-y_1) \, S(y_2) + \delta(x-y_2) \, S(y_1)$$

(odd)

$$S(x) = S(xy_1y_2) = S(xy_1y_2y_3y_4) = \cdots = 0 \qquad \text{(III.12)}$$

is a consistent solution of the equations for odd-functions, and holds if the invariance of the Lagrangian and the equation of motion for the substitution $A(x) \longrightarrow -A(x)$ is maintained.

The equations can be integrated using the free field Green's function $G_0(x-y)$ of the Yukawa operator. In the Euclidean space, the only boundary condition is that G_0 vanish at ∞. G_0 is given by

$$G_0(x-y) = \frac{1}{(2\pi)^4} \int \frac{e^{-ipx} \, d^4p}{p^2+m^2} \qquad \text{(III.13)}$$

We have

$$S(x,y) = G_0(x-y) - g \int G_0(x-x') \, S(x'x'x'y) \, dx'$$

$$S(xy_1y_2y_3) = G_0(x-y_1) \, S(y_2y_3) + G_0(x-y_2) \, S(y_1y_3)$$
$$+ G_0(x-y_3) \, S(y_1y_2) - g \int G_0(x-x') \, S(x'x'x'y_1y_2y_3) \times dx' \qquad \text{(III.14)}$$

and in general

$$S(x\, y_1 \cdots y_n) = \int G_0(x-y)\left[-g\, S(y\,y\,y\,y_1 \cdots y_n)\right] dy$$

$$+ \sum_{\nu=1}^{n} \int G_0(x-y_\nu)\, S(y_1 \cdots y_{\nu-1}\, y_{\nu+1} \cdots y_n) \qquad \text{(III.15)}$$

The solution has the form

$$S\{j\} = e^{\frac{1}{2}\iint j(x)\, G_0(x-y)\, j(y)\, dx\, dy} \times S_{red}\{j\} \qquad \text{(III.16)}$$

where $S_{red}\{j\} = 1$ if $g = 0$

For perturbation theoretical purposes a more effective substitution is

$$S\{j\} = e^{S^T\{j\}}$$

in terms of the generating functional of the truncated (involving only connected parts) Green's functions, which then obeys the equation

$$\frac{\delta}{\delta j(x)} S^T\{j\} = \int G(x-y)\Big[-g \frac{\delta^3}{\delta j(y)^3} S^T\{j\}$$

$$-3g\, \frac{\delta S^T\{j\}}{\delta j(y)}\, \frac{\delta S^T\{j\}}{\delta j(y)^2} - g\left(\frac{\delta S^T}{\delta j(y)}\right)^3 + j(y)\Big]\, dy \qquad \text{(III.17)}$$

However the nonlinearity of this equation makes it for us no better than equation (III.8) with which we started. Returning to the latter we may try a solution of the type

$$S\{j\} = \frac{N\{j\}}{N\{0\}} \qquad \text{(III.18)}$$

where

$$N\{j\} = \int e^{-\frac{1}{2}(\phi|K\phi) - \frac{g}{4}(\phi^4) + (j\phi)} D_W(\phi) \qquad \text{(III.19)}$$

The bracket

$$(\phi K \phi) = \int [(\bar{\nabla}\phi(x))^2 + m^2 \phi(x)] \, dx$$

$$(\phi^4) = \int \phi^4(x)\, dx \, ; \, (\phi \, j) = \int j(x) \phi(x) \, dx \qquad \text{(III.20)}$$

$D_W(\phi) \; exp\left[-\frac{1}{2}(\phi K \phi)\right]$ is the measure differential (quasi-interval) of a suitably generalized Wicher integral (which in our case will be the Fridrichs-Shapiro integral).

We have

$$K_x \frac{\delta}{\delta j(x)} N\{j\} \equiv K_x N_x\{j\}$$

$$= \int K_x \phi(x) \, exp\left[-\frac{1}{2}(\phi K \phi) - \frac{g}{4}(\phi^4)\right] e \times D_W(\phi)$$

$$= \int \left\{ -\frac{\delta}{\delta \phi(x)} exp\left[-\frac{1}{2}(\phi K \phi) - \frac{g}{4}(\phi^4)\right] - g\,\phi(x)^3 \, exp\left[\;\text{''}\;\right]\right\} e^{(j\phi)} D_W$$

$$= j(x) \int exp\left[-\frac{1}{2}(\phi K \phi) - \frac{g}{4}(\phi^4) + (j\phi)\right] D_W$$

$$- g \frac{\delta^3}{\delta j(x)^3} \int exp\left[-\frac{1}{2}(\phi K \phi) - \frac{g}{4}(\phi^4) + (j\phi)\right] D_W(\phi)$$

$$= -g \frac{\delta^3}{\delta j(x)^3} N\{j\} + j(x) N\{j\} \qquad \text{(III.}$$

so that $N\{j\}$ is indeed a solution of the equation (III.8) In the above we have assumed that (a) the integral $N\{j\}$ is well-defined, (b) that functional differentiation with respect to j can be carried out under the integral sign, and (c)

$$\int \frac{\delta}{\delta\phi(x)} \exp\left[-\frac{1}{2}(\phi K \phi) - \frac{g}{4}(\phi^4) + j\phi\right] D_W(\phi) = 0$$

We shall see, however, that for $N\{j\}$ can only be given the value zero so that we have to modify the integrals and hence the model first before studying the unmodified model. To get oriented we may first look at the perturbation solutions of the model.

First we have the zero-space, zero-time case in which the whole space-time shrinks to a point. This corresponds to Caianiello's "numerical" model which is solved in terms of the special functions of mathematical physics.

Next we have the zero-space, (onlytime) model in which the the 3-space shrinks to a point. This corresponds to the case of the anharnomic oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m^2 q^2}{2} + \frac{g}{4} q^4 \tag{III.22}$$

Everything is finite in this problem, but still it cannot be solved analytically but only numerically. (E.g. the ground state energy has the form $E_0(g) = g^{1/3} f(g^{-2/3})$

Where $f(z)$ is entire analytic. This shows that $E_0(g)$ is no

Next we have the case of 1 space + time. (the only case in which an explicit solution can then be obtained is for the Thirring model) The operator $A^3(x)$ is not well-defined and so we have to use Wick's normal product.

$$
\begin{aligned}
:A^3(x): &= A^3(x) - 3\Delta_F(0)\, A(x) \\
&= \lim_{\xi_1,\xi_2,\xi_3\to 0} \left[A(x+\xi_1)\, A(x+\xi_2)\, A(x+\xi_3) \right. \qquad \text{(III.23)} \\
&\quad - \Delta_F(\xi_1\xi_2)\cdot A(x+\xi_3) - \Delta_F(\xi_1\xi_3)\, A(x+\xi_2) \\
&\quad \left. - \Delta_F(\xi_2\xi_3)\, A(x+\xi_1) \right)
\end{aligned}
$$

The approach to zero of the ξ's is through space-like directions. A perturbation-theoretical solution can be obtained in this case.

For two space-dimensions, a perturbation solution exists with redefined subtraction terms in the definition of $:A^3(x):$, namely, the mass renormalization is infinite. For three space dimensions the coupling constant and amplitude renormalizations also become infinite. Still upon redefinition of the interaction term, a perturbation theoretical solution exists [4]. In more dimensions the interaction term in (III.5) would have to be replaced by a series of infinite order in the field operator.

$- g'/6\, A^6 - g''/8\, A^8 + \ldots$) and the meaning of even the perturbation theoretical solution becomes unclear.

All the above statements in MQFT are also true for EQFT.

Before proceeding further we introduce certain notations and notions regarding functional integration.

h = Separable real Hilbert space,

x = an element of h, $x \in h$

E = Linear finite dimensional subspace of

$P_E x \in E$, P_E the orthogonal projector on E

R = collection of all E

In terms of the orthogonal basis $e_\nu, \nu = 1 \cdots n$.

where n is the dimension of E

$$P_E x = \sum_{\nu=1}^{n} e_\nu (e_\nu^E, x) \; ; \; (e_\nu^E, e_\mu^E) = \delta_{\mu\nu}$$

(III.24)

We shall confine ourselves to real Hilbert spaces without loss of generality so that the basis functions are real.

Introduce the measure

$$d\mu_E = \left(\frac{a}{\pi}\right)^{\frac{n}{2}} e^{-a(x_1^2 + \cdots + x_n^2)} dx_1 \cdots dx_n \qquad \text{(III.25)}$$

which is a Gaussian. It is rotationally invariant and hence independent of the basis. Any functional $f(x)$ is a cylinder functional iff $\exists\, E$ such that

$$f(P_E x) = f(x), \quad \forall x$$

Then $f(x)$ is said to have a base in E.

For any $\alpha \geq 1$, C_α^E is the set of all cylinder functionals which have a base in E

$$C_\alpha^E = \{ f(x) ; \; f(x) = f(P_E x), \; \forall x$$

$$\int | f(P_E x)|^\alpha d\mu_E < \infty \}$$

Let $C_\alpha = \bigcup_{E \in \mathcal{R}} C_\alpha^E$ we may write

$$I_E(|f|^\alpha) = \int |f(P_E x)|^\alpha d\mu_E = I_H(|f|^\alpha)$$

and $I_H(f) = I_E(f)$

(Reference to the precise subspace has been suppressed and replaced by $\mathcal{H}$ since once the orthonormal basis is chosen appropriate to E an increase in the dimension of the space does not change the integration result).

Consider the sequence of projections (P_n) corresponding to the sequence $g_1, \ldots\ldots g_n$ of orthonormal bases in Hilbert space.

$$x \in h; \qquad x = \sum_{i=1}^{\infty} x_i g_i \; ; \qquad P_N x = \sum_{i=1}^{N} x_i g_i \; ;$$

For $\alpha \geq 1$

$$\therefore \; C_\alpha^{(P)} = \bigcup_n C_\alpha^{(P_n)}$$

$$_\alpha\|f\|_{(P_n)} = \left[I_{\mathcal{H}} \left(|f|^\alpha \right) \right]^{1/\alpha}$$

Adjoin to this space all Cauchy sequences in this space as "ideal" functionals. This completed space is called $\mathcal{L}_\alpha^{(P)}$ There is in general no finite dimensional subspace such that the relation

$$f(P_E x) = f(x)$$

holds if $f \in \mathcal{L}_\alpha^{(P)}$.

Now let $f(x)$ be an arbitrary (generally not cylinder-) functional. $f(x)$ is defined to be α-invariant iff

(a) for all basic systems, (P_n); $\{ f(P_n x) \} \in \mathcal{L}_\alpha^{(P)}$

(b) for any two basic systems (P_n), (Φ_m)

$$\lim_{n,m \to \infty} I_{\mathcal{H}} (| f(P_n) - f(\Phi_m K) |) = 0$$

For α-invariant functionals

$$I_{\mathcal{H}} (|f|^{\alpha}) = \lim_{n \to \infty} I_{\mathcal{H}} (| f(P_n) |^{\alpha})$$

$$I_{\mathcal{H}} (f) = \lim_{n \to \infty} I_{\mathcal{H}} (f(P_n))$$

Two Theorems: (1) Integrable Cylinder functionals are invariant.

(2) Polynomials (of finite order) are invariant if the coefficients are suitably restricted.

LECTURE IV

In the last lecture we defined α -invariance of a functional. It is a simple matter to prove that α-invariant functions are contained in $\bigcup_{\text{all } P_n} \mathcal{L}_\alpha^{(P)}$. The inverse is not known to be true. In fact, Friedrichs and Shapiro have introduced the concept of semi-invariant functions as:

(1) $f(x) \in \mathcal{L}_\alpha^{(P)}$ all P

(2) $\lim\limits_{n\to\infty} I_n(|f(P_n)|^\alpha)$ and $\lim\limits_{n\to\infty} I_n(f(P_n))$

both exist and are independent of the basic system P_n.

Invariance may be a stronger requirement than semi-invariance, for our purposes semi-invariance would suffice.

We have for

$$f = \int x(s)\, x(s')\, d\mu(s)\, d\mu(s')\, K(s,s') \tag{IV.1}$$

with

$$(x,y) = \int x(s)\, y(s)\, d\mu(s) \tag{IV.2}$$

$$I_n(P) = \int d\mu(s)\, K(s,s') \tag{IV.3}$$

Here we can ask whether one is actually integrating over Hilbert space or not. One may, e.g. ask if this integral is countably additive which is defined as follows:

If $f = \sum_{i=1}^{\infty} f_i$ converges for all $X \in \mathcal{G}$, $I_{\mathcal{N}}(f)$ exists and $I_{\mathcal{N}}(f_i), \forall_i$ exists, $\sum_i I_{\mathcal{N}}(f_i)$ converges and $\sum_i I_{\mathcal{N}}(f_i) = I_{\mathcal{N}}(f)$ then the integral is said to be countably additive. That this is not true in general for our integral is shown by the example:

$$1 = \Theta(1 - X^2) + \Theta(2 - X^2)\,\Theta(X^2 - 1) + \ldots \quad \text{(IV.4)}$$

For every $X^2 < \infty$, the series trivially converges. But each term integrated separately gives zero, while for the series as a whole, it gives 1. Hence the integral is not countably additiv over Hilbert space. FS have defined as "Corona Integral"(at th edge of the Hilbert space) which is countably additive and applicable to invariant functionals. In this case the above paradox is removed since the Hilbert space (in which only (IV.4) is true) has measure zero now.

Now we shall use the elementary inequalities (Holder and Minkowski in particular) for $I_{\mathcal{N}}$. Since $I_{\mathcal{N}}$ is defined as the limit of Cauchy sequences of finite-dimensional integrals. for which those inequalities hold, they hold also for $I_{\mathcal{N}}$. We shall also use functional differentiation under the integral sign and partial integration and prove their validity in our cases, using again the approximating sequences for which it is easily established.

We now give without proof a number of lemmas which will be required for the sequel.

Lemma 1

If a functional is α-invariant, it is also β-invariant for $1 \leq \beta < \alpha$

Lemma 2

If $f(x) \in \mathcal{L}_\alpha$ and if for any γ, with $1 \leq \gamma < \infty$ $\gamma+\epsilon \| f(P_E) \|_E$ has a uniform bound for all E and some $\epsilon > 0$, then $f(x) \in \mathcal{L}_\gamma$

Lemma 3

If $f_i \in \mathcal{L}_\alpha$, $i = 1, \ldots, n$ and if $1 \leq \beta < \alpha$ and if for a set of non-negative numbers c_i with $\sum_{i=1}^{n} c_i = 1$ the norms $\frac{\alpha\beta}{(\alpha-\beta)\,c_i} \| f_i(P_E) \|_E$ have for all E uniform upper bounds, then

$$\prod_{i=1}^{n} f_i \in \mathcal{L}_\beta$$

Lemma 4

If $f \in \mathcal{L}_\alpha$, if $1 \leq \beta < \alpha$ and if $\frac{\alpha\beta}{\alpha-\beta} \| \exp f(P_E) \|_E$ has for all a uniform upper bound then

$$e^f \in \mathcal{L}_\beta$$

(Proofs of these lemmas, omitted here, require only the inequalities mentioned at the end of lecture II and use of the inequality

$$|e^f - e^{f'}| \leq \sqrt{q}\, |f - f'| \operatorname{Max}(e^{Re f}, e^{Re f'})$$

LECTURE V

In the previous lectures we went into the mathematical tools necessary for studying our functional differential equation and defined the Friedrichs-Shapiro integral,

$$I_{\mathcal{H}}(f) = \lim_{n\to\infty} \int d\mu_{\mathcal{H}} f(P_n(x))$$

$$= \lim_{n\to\infty} \left(\frac{a}{2\pi}\right)^{n/2} \int e^{-\frac{a}{2}(x_1^2+\cdots+x_n^2)} \qquad \text{(V.1)}$$

where $f(P_n x) = f(x_1 \ldots x_n)$

$f(x)$ is invariant if

(a) $\int d\mu_{\mathcal{H}} \, | f(P_n x)|^{\alpha}$ converges as a Lebesgue integral for all basic systems,

(b) $\lim_{\substack{m\to\infty \\ n\to\infty}} I_{\mathcal{H}}(|f(P_n) - f(Q_m)|^{\alpha}) = 0$

(V.2)

For completeness we shall introduce the concept of uniform convergence in Hilbert space of an FS integral and give some theorems regarding it.

Let $\mathcal{U}$ be a set of parameters in a parameter space

Let $f(x,u) \in \mathcal{L}_{\alpha} \left[\equiv \underset{\alpha \geq 1}{C_{\alpha}^{(P_n)}} \right]$ be much that for all

we have

(c) $_{\alpha}\| f(P_n, u) \|_{(P_n)} < M(U)$, for all basic systems, (P_n)

and

(d) for any two basic systems $(P_n), (Q_m)$

$$\alpha \| f(P_n', u) - f(Q_m', u) \|_{\mathcal{H}} < \varepsilon \; ; \; \mathcal{H} \equiv (P_n \otimes Q_m)$$

if

$$n > n(\alpha, \varepsilon, P, U) \; ; \quad m > m(\alpha, \varepsilon, Q, U) \qquad \text{(V.3)}$$

Then $I_{\mathcal{H}}(f(\cdot, u))$ is uniformly α -convergent with respect to U in U .

Theorems:

(1) . A uniformly α-convergent FS integral $I_{\mathcal{H}}(f(\cdot,u))$ may be integrated under the integral sign if $\int_U dm(U) < \infty$ and the resulting integral is α -invariant

(V.4)

(2) An FS integral can be differentiated under the integral sign if the resulting FS integral is uniformly convergent.

We shall apply the first theorem to FS integration over Hilbert space by using it for approximating finite-dimensional integrals. The second theorem will be used for functional differentiation.

We also give here an important formula, viz. if $P\{\alpha\}$ is a polynomial, then

$$\int_{\mathcal{H}} e^{\left[\int f(s)\,x(s)\,dm(s) - \frac{1}{2}\iint x(s)\,x(s')\,A(s,s')\,dm(s)\,dm(s')\right]} \cdot P\{x\}\,d\mu_{\mathcal{H}}$$

$$= P\left\{\frac{\delta}{\delta(f\,dm)}\right\} \times \exp\left[\frac{1}{2}\iint f(s)\,f(s')\left([1+A]^{-1}\right) \times f(s,s')\,dm(s)\,dm(s') - \frac{1}{2}\int dm(s)\left(\log[1+A]\right)\right]$$

(V.5)

whenever the integrand is invariant. We can rewrite (V.5) in the form

$$\int \exp\left[(f\varphi) - \frac{1}{2}(\varphi A \varphi)\right] P\{\varphi\} \exp\left[-\frac{1}{2}(\varphi\varphi)\right] \mathcal{D}_{fs}(\varphi)$$

$$= P\left\{\frac{\delta}{\delta J}\right\} \exp\left[\frac{1}{2}\left(f[1+A]^{-1}f\right) - \frac{1}{2}\,\mathrm{Tr}\log(1+A)\right]$$

(V.6)

which introduces the notation we shall be using.

Modifications of the Model

We shall modify our model by changing the functional differential equation for $N\{f\}$ and then show that the removal of the modifications leads to $N\{f\} = 0$. The modifications we shall make are (a) regularization and (b) use of finite-space-time volume.

(a) The regularization of the Lagrangian,

$$L = \frac{1}{2}\partial^{\mu}A\,\partial_{\mu}A - \frac{1}{2}m^2A^2 - \frac{g}{4}A^4 \qquad \text{(V.7)}$$

can be made in many ways. One of them would be to introduce form factors. Another is to introduce higher derivatives in the Lagrangian. This method was used by Pais and Uhlenbeck and we shall adopt it here. The method consists in replacing the Klein-Gordon operators in the field equation by a product of such operators

$$K_x \rightarrow \prod_{i=1}^{N}(\partial_0^2 - \nabla^2 + m_i^2)$$

$$= \sum_{n=0}^{N}(\partial_0^2 - \nabla^2)^n S_{N-n}$$

(V.8)

and then using canonical quantization. Under the above replacement (in which all the m_i's are positive), for EQFT we have in (III.19)

$$(\varphi K \varphi) \rightarrow \int \varphi(x)\prod_{i=1}^{N}(\overleftarrow{\nabla}\overrightarrow{\nabla} + m_i^2)\varphi(x)\,dx \qquad \text{(V.9)}$$

The replacement (V.8) together with the corresponding commutation relations lead, for $g = 0$ and $N \geq 2$ either to an indefinite energy operator, or necessitates the introduction of an indefinite metric whereby the positivity of the eigenvalues of the energy operator can be maintained.

We now show that for $g > 0$, and assuming the theory to possess a unique vacuum, the metric must be indefinite if $N \geq 2$. We have for the expectation value of the product of two A's the Lehmann-like spectral representation

$$\langle A(x) A(y) \rangle = (2\pi)^{-3} \int dk \, e^{-ik(x-y)} \cdot \rho(k^2)\, \theta(k^0)\, \theta(k^2)$$

$$= \int_0^{\infty} dk^2 \; \rho(k^2)\, i \Delta_k^{(f)}(x-y) \tag{V.10}$$

The canonical commutation relations give for

$$\int_0^{\infty} dk \; \rho(k^2)\,(k^2)^n \equiv \sigma_n \,, \quad n = 0, \cdots, 2N-1 \tag{V.11}$$

the values

$$\sigma_n = 0, \quad n = 0, \cdots\cdots, N-2$$

$$\sigma_{N-1} = (-1)^{N-1}$$

$$\sigma_{N+\ell} = (-1)^{N-1} \det \alpha_\rho \,, \quad \rho = 0, \cdots N-1 \tag{V.12}$$

where (α_ℓ) is the $(\ell+1)(\ell+1)$ matrix

$$(\alpha_\ell)_{ij} = s_{i+1-j}\,, \quad s_0 = 1,\ s_{-k} = 0,\quad k = 1,2.$$

Hence $\rho(k^2)$ must change sign at least $N-1$ times for $N \geqslant 2$ as it does for $g = 0$ when $\rho(k^2)$ is a sum of N δ-functions of alternating signs. (For the unregularized case $\sigma_0 = 1, \sigma_1 = m^2$, m = bare mass

$$\rho(k^2) = \delta(k^2 - m_\rho^2) + \theta(k^2 - (3m_\rho)^2)\,\rho(k^2)$$

with m_ρ the true particle mass)

The solutions of (V.11) are obtained by additing to the known $\rho(k^2)$ for $g = 0$ an arbitrary function of k^2 the first $2N-1$ moments of which vanish in the interval $(0,\infty)$.

(b) <u>Finite space-time volume</u> :

We can achieve this in two ways. (1) We may restrict the integration in all the terms $(\varphi K \varphi)$, (φh) etc. in the definition of $N\{J\}$ to one over a bounded domain Ω with $\int_\Omega dx = V_\Omega < \infty$, or more generally (2) we may have different restrictions on the terms in the integrand, making e.g.,

$$g\,(\varphi^4) \to \int g(x)\,\varphi(x)^4\,dx \qquad \text{(V.13)}$$

where $g(x) = 1$ for $|x| < R$, $g(x) = 0$ for $|x| > R' \gg R$ and $g(x)$ infinitely differentiable with $R \to \infty$. The region of integration for the factor containing $\frac{g}{4}(\varphi^4)$ is $\Omega < \infty$. But in such a case it can be shown that there is no Schrodinger equation. Hence we use the first alternative and make $\Omega < \infty$ for the whole integral.

We must know the properties of the solutions of the equations for the elliptic Yukawa operator appearing in the integrand of the modified (regularized) $N\{J\}$. We have

$$\dot{K}_{mod} = \prod_{i=1}^{N} (-\Delta + m_i^2) \equiv \sum_{i=0}^{N} S_i (-\Delta)^{N-i}$$

The generalization of Green's theorem for our case reads

$$\int \varphi(x) \overleftarrow{\vartheta} \cdot \overrightarrow{\vartheta} \psi(x) dx + \int \varphi(x) \overrightarrow{K} \psi(x) dx = \int \varphi(x) \overleftrightarrow{K} \psi(x) dx \qquad \text{(V.14)}$$

where the $\overleftarrow{\vartheta}$, $\overrightarrow{\vartheta}$ are certain normal derivatives, each N in number.

LECTURE VI

Last time we began the discussion of the regularized Yukawa operator

$$K = \prod_{i=1}^{N} (-\Delta + m_i^2) \equiv \sum_{i=0}^{N} S_{N-1} (-\Delta)^i ,$$

$$\overleftrightarrow{K} = \sum_{i=0}^{N/2} S_{N-2n} (\overleftarrow{\Delta})^i (\overrightarrow{\Delta})^i + \sum_{u=0}^{N-2/2} S_{N-1-m} (\overleftarrow{\Delta}) \overleftarrow{\partial} . \overrightarrow{\partial} (\overrightarrow{\Delta})^u$$

$$\overleftarrow{d}_K = (\overleftarrow{\Delta})^{[K/2]} (\overleftarrow{\partial}_n)^{K-2[\frac{K}{2}]} \qquad \text{(Dirichlet derivative) (VI.1)}$$

$$\overrightarrow{D}_K = \sum_{u=0}^{N-K-1} (-1)^u S_{N-1-K-u} \quad) (\overrightarrow{\Delta})^{u+[\frac{K+1}{2}]} ,$$

$$(\varphi K \psi) \equiv (\varphi \overleftrightarrow{K} \psi) = \int dx\, \varphi \overrightarrow{K} \psi \; + \; + \oint dO(s)\, \varphi\, \overleftarrow{\vartheta} . \overrightarrow{\vartheta}\, \psi ; \qquad \text{(Neumann derivative)}$$

$$\overleftarrow{\vartheta} \cdot \overrightarrow{\vartheta} = \sum_{K=0}^{N-1} d_K \cdot \overrightarrow{D}_K \quad ;$$

We had to regularize to make $N\{f\}$ finite. Recall that already for $d = 2$ the gA^4 theory requires us to use the Wick product,

$$: A^4 : = A^4 - 6A^2 \langle A^2 \rangle_0 + 3 \langle A^2 \rangle_0^2 \qquad \text{(VI.2)}$$

With $\langle\,\rangle_0$ (the interaction free-value) and $\langle A^2 \rangle_0$ infinite. So to be able to operate with finite quantities it is quite natural to have to regularize since then $\langle A^2 \rangle_0$ is finite. I proved last time that the regularized MQFT must possess an indefinite metric, but the metric in EQFT, which we shall introduce at the very end where we discuss the operator approach to EQFT, will be seen to be positive definite nevertheless.

The Green's functions for the problem are introduced as

$$(j\, G_D\, j) = \underset{\vec{\partial}\varphi = 0 \text{ on } d\Omega}{\underset{\max}{\text{l.u.b.}}} \frac{(j\varphi)^2}{(\varphi K \varphi)} \qquad \text{(VI.3)}$$

Then

$$\vec{K}\, G_D(x, y) = \delta(x-y) \qquad \text{(VI.4)}$$

$$\Big|_{x \to s} \vec{\partial}\, G_D(x, y) = 0 \qquad \text{(VI.5)}$$

Further

$$\Big|_{x \to s} \vec{\partial}\, G_D(x, y)\, \overleftarrow{\partial} \Big|_{y \to s'} = -\delta(s, s') \qquad \text{(VI.6)}$$

after $x \to s$ a Dirac delta function on the boundary of

Using the following formal substitution in (III.19), with Dirichlet data a, which is an n-component function

$$\varphi = G_D^{1/2}\, \psi + f \qquad \text{(VI.7)}$$

where

$$f = G j - G \overleftarrow{\partial} \cdot a$$

We have

$$N\{a, j\} = e^{-\frac{1}{2} a \cdot \theta_j \cdot a - j G \overleftarrow{\partial} \cdot a + \frac{1}{2}(j G j)}$$

$$\int e^{-\frac{1}{2}(\psi\psi) - \frac{1}{4} g\left((H\psi + f)^4\right)} \mathcal{D}_{FS}(\psi)$$

with

$$\theta_j^{(ss')} = -\left| \vec{\partial}\, G\, \overleftarrow{\partial} \right|_{\substack{x \to s \\ y \to s'}} \qquad \text{(VI.8)}$$

Here $H \equiv G_0^{1/2}$ and e.g.

$$H\,\psi(x) = \int H(x,y)\,\psi(y)\,dy \qquad \text{(VI.9)}$$

It is to be mentioned that

$\overline{}_{x\to s}\left| \overrightarrow{\partial}\, G(x,y)\, \overleftarrow{\partial} \right|_{y\to s'}$ is a positive definite $N \times N$ matrix integral operator inspite of the negative sign. $e^{-\frac{1}{2}(\psi\psi)}\, \mathcal{D}_{FS}(\psi)$ is the measure $d\mu_{\mathcal{H}}$ of V.I with $\alpha = 1$.

The integrand is given by

$$e^{-\frac{1}{4}g\,(H\psi+f)^4}$$

This is the exponential of a polynomial and hence the integration goes through by Lemma 4 of Lecture IV but we need apriori bounds for the integral. We shall derive bounds not for the integral VI.8 but for the integral where the fourth power is made into a Wick product.

$$\left((H\psi+f)^4\right) \to \left((H\psi+f)^4\right) - 6\,G_0(0)\left((H\psi+f)^2\right)$$

where

$$G_0(x-y) = \frac{1}{(2\pi)^4}\int \frac{e^{-ik(x-y)}\,dk}{\prod_{i=1}^{N}(k^2+m_i^2)}$$

given already in (III.13) is the Euclidean free field contraction function. Thus, we shall henceforth discuss the integral

$$N\{a, j\}\, e^{\frac{1}{2} a \cdot \mathcal{G} \cdot a + j G \overleftarrow{\partial} \cdot a - \frac{1}{2}(j G j)}$$

(VI.10)

$$= \int \exp\left\{-\frac{1}{2}(\psi\psi) - \frac{g}{4}\left((H\psi + f)^4\right) + \frac{3g}{2} G_0(0)\left((H\psi+f)^2\right)\right\} \cdot \mathcal{D}_{FS}(\psi).$$

complete a square in the exponent and dropt it.

$$-\frac{g}{4}\left((H\psi + f)^4\right) + \frac{3g}{2} G_0(0)\left((H\psi + f)^2\right) \leq \frac{9g}{4} G_0(0)^2 \int_\Omega dx$$

and thus the integral in (VI.10) has the apriori upper bound

$$e^{\frac{9g}{4} G_0(0)^2 \int_\Omega dx}$$

which is finite due to regularization $\left(G_0(0) < \infty\right)$ and the choice of a finite space-"time"-volume $\left(\int_\Omega dx < \infty\right)$. We also need to know that the integral is strictly positive, since otherwise ratios such as $N\{a, j\}/N\{a, 0\}$ would still not exist. For this we use the inequality of the arithmetic and geometic mean:

$$\int e^{-\frac{1}{2}(\psi\psi) - \frac{g}{4}\left((H\psi+f)^4\right) + 3\frac{g}{2} G_0(0)\left((H\psi+f)^2\right)} \mathcal{D}_{FS}(\psi)$$

$$\geq e^{\int e^{-\frac{1}{2}(\psi\psi)}\left[-\frac{g}{4}\left((H\psi+f)^4\right) + \frac{3g}{2} G_0(0)\left((H\psi+f)^2\right)\right] \mathcal{D}_{FS}(\psi)}$$

Note that the lower bound goes to $+\infty$ if $\int_C dx \to \infty$ or if we try to remove regularisation: $G_\varphi(0) \to \infty$

which can be calculated with (VI.6). The result is finite if e.g. f is bounded, and we introduce restrictions on f such that f is bounded and even continuous, which permits to satisfy also the other conditions for (VI.10) to exist as a FS integral:

$$(jGj) < \infty , \qquad a \cdot g \cdot a < \infty ,$$

besides, of course, $g \geq 0$, $\int_\Omega dx < \infty$, and finally

$$\underset{\varphi, x}{\text{l.u.b}} \frac{\varphi(x^2)}{(\varphi K \varphi)} < \infty ,$$

the last being a condition on the smoothness of the boundary of Ω.

We can now show that $N\{a, j\}$ defined by

$$N\{a,j\} = \int \exp\left[-\tfrac{1}{2}(\varphi K \varphi) - \tfrac{g}{4}(\varphi^4) + j(\varphi) + \right.$$
$$\left. + \tfrac{3g}{2}(\varphi^2)\, G_0(0)\right] \mathcal{D}_W(\varphi) \qquad \text{(VI.11)}$$

satisfies the equation

$$\left\| K\left[\tfrac{\delta}{\delta j}\right] N + g\left[\delta^3/\delta j^3\right] N - 3g\, G_0(0)\left[\delta/\delta j\right] N - jN \right\|_{-N} = 0 \qquad \text{(VI.12)}$$

where

$$\| j \|_{-N} = (jGj) \quad \text{and}$$

$$\left| \vec{\vartheta}\left[\tfrac{\delta}{\delta j}\right] N - aN \right|_{N-\frac{1}{2}} = 0$$

(Dirichlet boundary condition)

(VI.13)

Equivalently the integral equation satisfied by N is

$$\left\| \frac{\delta}{\delta j} N + g\, G\left[\frac{\delta^3}{\delta j^3} \right] N - 3g\, G_0(0)\, G \frac{\delta}{\delta j} N - - G j N + G \overleftarrow{\vartheta} \cdot N \right\|_N = 0$$

(VI.14)

The Neumann derivatives satisfy the relation

$$\overrightarrow{\vartheta} \frac{\delta}{\delta j} N + \frac{\delta}{\delta a} N \Big|_{-N+\frac{1}{2}} = 0$$

(VI.15)

This last equation is important in connection with this Schrodinger equation $N\{a, j\}$ satisfies.

LECTURE VII

Today, we shall study some of the general properties of the solution of our functional differential equation, (the word "general" here signifying that the properties do not depend on the precise form of the interaction term)

$$S\{j\} = \frac{N\{j\}}{N\{0\}} \tag{VII.1}$$

$$N\{j\} = \int e^{-\frac{1}{2}(\varphi K \varphi) - \frac{g}{4}(\varphi^4) + (j\varphi)} \mathcal{D}_W(\varphi) \tag{VII.2}$$

$-\frac{g}{4}(\varphi^4)$ in the exponent of the integrand is replaced for reasons mentioned earlier by the Wick product which is equal to

$$-\frac{g}{4}(\varphi^4) + \frac{3g}{2} G_0(0)(\varphi^2)$$

As mentioned in the last lecture, we make the formal substitution

$$\varphi = G^{1/2}\psi + f = H\psi + f \tag{VII.3}$$

where

$$f = Gj - G\overleftarrow{\partial}\cdot a$$

and obtain

$$N\{a,j\} = \exp\{-\tfrac{1}{2}\, a\cdot\mathcal{G}\cdot a - jG\overleftarrow{\vartheta}\cdot a + \tfrac{1}{2}(jGj)\}$$

$$\cdot \int \exp\Big[-\tfrac{1}{2}(\psi\psi) - \tfrac{g}{4}\big((H\psi+f)^4\big) +$$

$$+ \tfrac{3g}{2} G_0(0)\big((H\psi+f)^2\big)\Big] \, \mathcal{D}_{FS}(\psi)$$

(VII.4)

In (VII.3), the a represent a set of n-component data given by $\overrightarrow{\vartheta}\varphi = a$ on the boundary $\partial\Omega$, if ψ is sufficiently well behaved. The behaviour of φ is dictated by the ψ that contribute in the FS integral. Hence, the corresponding property of the functional are to be proved on the basis of the integral.

(VII.4) does satisfy the functional differential equation ( ) in an appropriately generalized form:

$$\Big\| K\frac{\delta}{\delta j} N\{j\} + g\,\frac{\delta^3}{\delta j^3} N\{j\} - 3g\, G_0(0)\,\frac{\delta}{\delta j} N\{j\} - j\,N\{j\}\Big\|_{-N} = 0. \quad \text{(VII.5)}$$

with the boundary condition

$$\Big\| \overrightarrow{\vartheta}\frac{\delta}{\delta j} N\{j\}\Big|_{x\to s} - a\,N\{j\}\Big\|_{-N+\frac{1}{2}} = 0 \quad \text{(VII.6)}$$

or equivalently, it satisfies the integral equation

$$\left\| \frac{\delta}{\delta j} N\{j\} + g\, G \frac{\delta^3}{\delta j^3} - 3 g\, G_0(0) \frac{\delta}{\delta j} N - G j N + G \overleftarrow{\partial} \cdot a \; N \right\|_N = 0 \tag{VII.7}$$

The conditions for the existence of the integral (VII.4) are: for $g \geq 0$

$$\int dx < \infty \; ; \quad a \cdot \mathcal{G} \cdot a < \infty \; ; \quad (j G j) < \infty \; ;$$

$$\text{l.u.b}\; G_N (x, x) < \infty \tag{VII.8}$$

Here $(j G_N j) = \underset{\varphi}{\text{l.u.b}} \dfrac{(\varphi j)}{\| \varphi \|_N^2}$ is the quadratic form that defines the Neumann Green's function

$$\Big[K G_N = \delta \; ; \quad \vec{\partial}\, G_N = 0 \; ;$$

$$\Big| \vec{\partial}\, G(x, y) \overleftarrow{\partial} \Big|_{\substack{x \to s \\ y \to s'}} = 1\, \delta (s, s') \Big]$$

The bound on $G_N (x, y)$ implies that the cone condi- tion (see e.g. Courant-Hilbert) is satisfied on the boundary. It also implies that $G (x, y) < \infty$

For the norms appearing in (VII.5, 6 or 7) we have, e.g.,

$$\| \rho \|_N^2 = \| K \rho \|_{-N}^2 + | \vec{\partial} \rho |^2_{N - \frac{1}{2}} \tag{VII.9}$$

The surface norms can also be defined as follows (which are similar to Schecter ( ))

$$|a|^2_{N-\frac{1}{2}} \equiv a \cdot \mathcal{G} \cdot a = \underset{\vec{\partial}\varphi = a}{g.l.b} \; \|\varphi\|^2_N \qquad \text{(VII.10)}$$

$$|b|^2_{-N+\frac{1}{2}} = b \cdot \mathcal{G}^{-1} b = \underset{\varphi}{l.u.b} \; \frac{|b \cdot \vec{\partial}\varphi|}{\|\varphi\|^2_N}$$

Also

$$\left| \vec{\partial} \frac{\delta}{\delta j} N\{j\} + \frac{\delta}{\delta a} N\{j\} \right|_{-N+\frac{1}{2}} = 0 \qquad \text{(VII.11)}$$

which essentially expresses the fact that the remaining integral in $N\{a, j\}$ depends only on f instead of j and a separately.

We wish to remark here that if j and a are sufficiently smooth functions, we can replace $\|f\|_{-N} = 0$ by $f(x) = 0$, $\|Gf\|_N = 0$ by $(Gf)(x) = 0$ and similarly $|a|_{N-\frac{1}{2}} = 0$ by $a(s) = 0$ and $|b|_{-N+\frac{1}{2}} = 0$ by $b(s) = 0$ in the formulae given earlier with f, a and b appropriately substituted. In the further analysis of the properties of the model, however, we have also to integrate over the Dirichlet data and here a which are not smooth functions play the main role. Hence the necessity for the sharper statements made above.

For some later purposes, it is convenient to write $N\{a,j\}$ in the following way.

$$N\{a,j\} = \exp\left[-\tfrac{1}{2}\, a\cdot\mathcal{G}\cdot a - j\, G\overleftarrow{\partial}\cdot a\right]\cdot$$
$$\cdot\int \exp\Big[-\tfrac{1}{2}(\psi\psi) - \tfrac{1}{4}g\left((H\psi - G\overleftarrow{\partial}\cdot a)^4\right)$$
$$+\frac{3g}{2}G_0(0)\left((H\psi - G\overleftarrow{\partial}\cdot a)^2\right) + (jH\psi)\Big]\cdot$$
$$\cdot\ \mathcal{D}_{FS}(\psi)$$

(VII.12)

The ψ appearing in (VII.12) is

$$\psi = \psi_{new} = \psi_{old} + Hj$$

Hj is an element of the Hilbert space since we have $(jHHj) = (jGj) < \infty$ by assumption. In general, the argument can be shifted in an FS integral if the shift function is square integrable. There is no permissible shift that would, e.g., eliminate the $G\overleftarrow{\partial}\cdot a$ term from the integrands in the interaction. (This is why the entire analyticity of $N\{a,j\}$ in j is a general property of the solution whereas the entire analyticity in a is only a special property. The latter property holds since our perturbation term is itself entire analytic. This would not be valid in the modifications of the perturbation of the kind proposed by Fradkin and Efimov, e.g.

$$gA^4 \longrightarrow \frac{gA^4}{(1+cA^2)^{3/2}} \sim |A| \quad \text{asymptotically})$$

The form of the new integral makes the proof of the entire analyticity of $N\{\mathcal{g}, \dot{j}\}$ in $\dot{j}$ simple. A functional of $\dot{j}$ is entire analytic in $\dot{j}$ if it possesses a Volterra series expansion that converges whenever a positive definite quadratic form of $\dot{j}$ is finite. In our case $(\bar{j} G j) < \infty$.

We shall give here the convergence proof as the simplest example of the application of simple inequalities, which will be repeatedly used in later proofs (which will be omitted in these lectures). The sequence of inequalities goes as follows.

For the apriori upper bound, we have

$$\int \exp\cdot\left[-\frac{1}{2}(\psi\psi) - \frac{1}{4}g((h)^4) + \frac{3g}{2}G_0(0)((h)^2)\right]\cdot$$

$$\cdot\,|(\dot{j}H\psi)|^n\cdot\mathcal{D}_{FS}(\psi) \leq e^{\frac{9g}{4}G_0(0)^2\int dx}\,\cdot$$

$$\cdot\int e^{-\frac{1}{2}(\psi\psi)}\,|(\dot{j}H\psi)|^n\,\mathcal{D}_{FS}(\psi) \quad \text{(VII.13)}$$

and

$$\int e^{-\frac{1}{2}(\psi\psi)}\,|(\dot{j}H\psi)|^n\,\mathcal{D}_{FS}(\psi)$$

$$\leq\left[\int e^{-\frac{1}{2}(\psi\psi)}\left((\mathrm{Re}\,\dot{j}H\psi)^2 + (\mathrm{Im}\,\dot{j}H\psi)^2\right)^n\mathcal{D}_{FS}(\psi)\right]^{1/2}$$

$$\leq\left\{\left[\int e^{-\frac{1}{2}(\psi\psi)}\,(\mathrm{Re}\,\dot{j}H\psi)^{2n}\,\mathcal{D}_{FS}(\psi)\right]^{1/n} + \right.$$

$$\left. + \left[\int \exp\left[-\frac{1}{2}(\psi\psi)\right](\mathrm{Im}\,\dot{j}H\psi)^{2n}\,\mathcal{D}_{FS}(\psi)\right]^{1/n}\right\}^{n/2}$$

$$(\text{where } h = H\psi - G\overleftarrow{\partial}\cdot a)$$

On using the Minkowski inequality

$$\left(\overline{(A^2+B^2)^n}^{\frac{1}{n}} \leq \overline{A^{2n}}^{\frac{1}{n}} + \overline{B^{2n}}^{\frac{1}{n}} \right)$$

$$= \left\{ (\mathrm{Re}\, j G\, \mathrm{Re}\, j) + (\mathrm{Im}\, j\, G\, \mathrm{Im}\, j) \right\}^{n/2} \cdot \left(\frac{2n!}{n!\, 2^n} \right)^{1/2} \qquad \text{(VII.14)}$$

upon evaluation with (V.5).

Summing these bounds over n from 0 to ∞, we get a behaviour $\sim \exp(\bar{j}\, G\, j)$ which proves the convergence. The existence of the integrals themselves and of the integral for the sum can be shown simiarly using the the inequalities given earlier.

It is to be noted that the absolute convergence condition for the Volterra series involves $(\bar{j}\, G\, j)$ while the absolute value of the original sum has a bound $\sim e^{\frac{1}{2}(\mathrm{Re}\, j\, G\, \mathrm{Re}\, j)}$ indepndent of $\mathrm{Im}\, j$. In general

$$|N\{a, j\}| \leq N\{a, \mathrm{Re}\, j\} \qquad \text{(VII.15)}$$

We now discuss the positive definiteness and uniqueness of our solution. A function $f(x)$ is said to be positive definite if for i, $(i = 1, \ldots n)$ and for all sets c_i and real x_i out of a suitably restricted class we have

$$\sum_{i,j=1}^{n} \bar{c}_i c_j F(x_i - x_j) \geq 0. \qquad \text{(VII.16)}$$

In the case of our function $N\{\Omega, ij\}$ we have $(j_i\, G\, j_i) < \infty, \forall i$. The above definition means that $N\{\Omega, ij\}$ is a positive definite $n \times n$ matrix and generalizes the concept of positive definiteness of functions of, e.g., one varaible [3]. We can even make a stronger statement than (VII.16), viz.,

$$\sum_{ij=1}^{n} \bar{c}_i c_j \, N\{\Omega, \bar{j}_i + j_j\} \geq 0 \qquad \text{(VII.17)}$$

Now a theorem of Bochner states that for such an $f(x)$ the Fourier transform exists.

$$f(x) = \int e^{iwx} \tilde{f}(w)\, dw \quad \text{with } \tilde{f}(w) \geq 0 \qquad \text{(VII.18)}$$

or

$$= \int e^{iwx} \, d\rho(w)$$

with $\rho(w)$ non-decreasing measure.

The Bochner theorem can be generalized to integrals over Hilbert space 4). Then the functional (for which also certain continuity conditions should be fulfilled) is the functional Fourier transform of a non-negative measure.

Thus $N(\Omega, j)$ for imaginary j is the functional Fourier transform of a non-negative measure. (To be on Hilbert space, we may take

$$N'\{\Omega, j\} = N\{\Omega, j\} \exp[\tfrac{1}{2}\Omega\, \mathcal{G}\, \Omega + j\, G\, \overleftarrow{\mathcal{D}} \cdot \Omega]$$

and HTj instead of j). Substituting the Fourier representation of $N'\{\Omega, j\}$, in (VII.7) and carrying out the differentiations under the integral sign and performing a partial integration gives a functional differential equation for the measure, the solution of which is $N\{\Omega, j\}$ (with j imaginary) unique

LECTURE VIII

In this lecture we shall cover the following topics; the semigroup property of the solution, canonical commutation relations, analyticity in g, divergence of the g-expansion, the question of measure, entire analyticity in the Dirichlet data bounds based on convenity and on variational methods.

Last time we write down the solution for the Dirichlet data.

$$N\{\partial, j\} = \exp\left[-\frac{1}{2}\,\partial\cdot g\cdot\partial - \int G\,\overleftarrow{\vartheta}\cdot\partial + \frac{1}{2}(jGj)\right]\cdot$$

$$\cdot\int \exp\left[-\frac{1}{2}(\psi\psi) - \frac{g}{4}\left((H\psi+f)^4\right) + \frac{3}{2}\,g\,G_0(0)\left((H\psi+f)^2\right)\right]\cdot$$

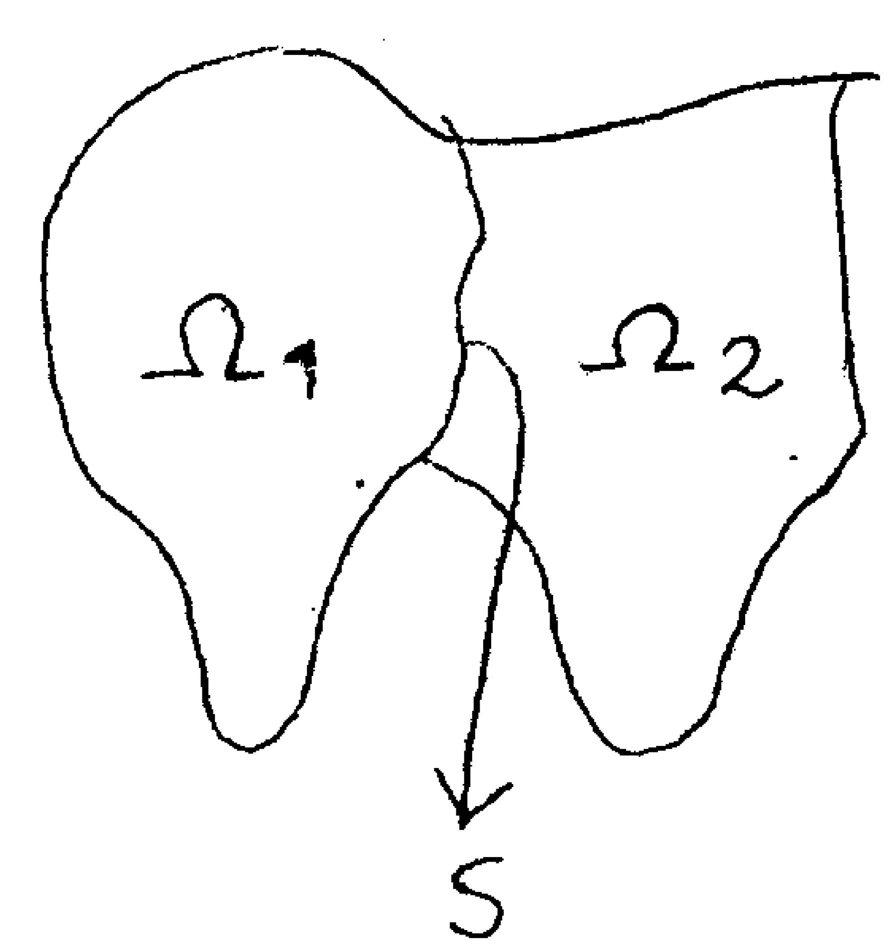
$$\cdot\,\mathcal{D}_{FS}(\psi) \tag{VIII.1}$$

We can prove that

$$N\{\partial_1, \partial_2, j_1, j_2\} = C_{12}\int N_1\{\partial_1, \partial_3, j_1\}\cdot$$

$$\cdot\,N_2\{\partial_3, \partial_2, j_2\}\;\mathcal{D}_{FS}(\mathcal{F}, \partial_s) \tag{VIII.2}$$

where C_{12} is a finite normalization factor. The combination of the $\partial\cdot g\cdot\partial$ term gives the Gaussian factor of the FS integral. For $g=0$ we can carry out the resulting Gaussian integral exactly.



The proof requires the use of a number of relations that connect the Dirichlet Green's functions in $\Omega_1 + \Omega_2$ to these in $\Omega_1 + \Omega_2$; e.g.

$$G_{1+2} = G_1 + G_1 \overleftarrow{\vartheta}_{(s)} \cdot \overrightarrow{\vartheta} G_1 + 2 \overleftarrow{\vartheta}_{(s} \cdot \overrightarrow{\vartheta} G_1$$

$$= G_1 + G_1 \overleftarrow{\vartheta}_{(s)} \cdot \left(\overrightarrow{\vartheta} G_1 + 2 \overleftarrow{\vartheta} \right)^{-1}_{s(\dot{s})} \cdot \overrightarrow{\vartheta} G_1 \tag{VIII.3}$$

where

$$\overrightarrow{\vartheta} G_{1+2} \overleftarrow{\vartheta} = \left(g_1 + \xi . g_2 \cdot \xi \right)^{-1}_{s} \tag{VIII.4}$$

Here $(\quad)^{-1}_{s}$ means that the inverse of the $N \times N$ matrix integral operator in brackets is taken on the common boundary surface S and likewise $(\dot{s})$ means that the convolution is calculated on that surface only. E.g. $A_{(\dot{s})} (A)^{-1}_{s} = 1$ the $N \times N$ unit operator $\begin{pmatrix} \delta(s,s') & & 0 \\ & \delta(s^2, s') & \\ 0 & & \delta(s,s') \end{pmatrix}$ on S (ξ is a matrix that simply taken into consideration that an outward normal on S with respect to Ω_1 is an inward normal with respect to Ω_2 .)

The formula means, in words, that one has to integrate the product of the functionals over two adjoining regions with the Dirichlet data on the common boundary identical over these common data in the Friedrichs-Shapiro is manner. By this the Gaussian convergence factor is supplied in these functionals themselves, namely, in the "free-field" factor in front of the integral in $N\{\partial, j\}$. The above semigroup property is the generalization to a rather more complicated setting of the property of the fundamental solution

$$K(x,x';-t,t') = \frac{1}{\sqrt{2\pi(t-t')}^{3}}\, e^{-\frac{(x-x')^2}{2(t-t')}} \qquad \text{(VIII.5)}$$

of the heat equation (in 3 space dimensions, $x = x_1, x_2, x_3$)

$$\int K(x,x';t,t')\,K(x',x'';t',t'')\,dx' = \\ = K(x,x'';t,t'') \qquad \text{(VIII.6)}$$

which is immediately verified. It is not so trivial to verify that the formula also holds the heat equation has an additional potential term

$$\frac{\partial U(x,t)}{\partial t} = \left[\frac{1}{2}\Delta - V(x)\right]U(x,t) \qquad \text{(VIII.7)}$$

where $K(x,x';t,t') = U(x,t)$ with the initial condition $U(x,t') = \delta(x-x')$. You heard lectures on semigroups, so I do not need to explain that term. I only want to contrast it with the group property of solution of the Schrodinger equation

$$-i\,\frac{\partial U_s(x,t)}{\partial t} = \left[\frac{1}{2}\Delta - V(x)\right]U_s(x,t) \qquad \text{(VIII.8)}$$

Here it is not necessary that the time interval in the fundamental solution, which is

$$K_s(x,t;x',t') = \sqrt{\frac{i}{2\pi(t-t')}} \cdot e^{i\frac{(x-x')^2}{2(t-t')}} \tag{VIII.9}$$

in the case of $V(x) = 0$, be positive in other words, the semigroup property becomes a group property in the Schrodinger case. Note that the heat equation arises from the Schrodinger equation by the replacement $t \to -it$ which is precisely the same replacement, to be interpreted as an analytic continuation that leads from MQFT to EQFT. In fact, the well-known Feynman path integral,

$$K_s(x,x';t,t') = \int_{\substack{x(t')=x' \\ x(t)=x}} \mathcal{D}_{Paths}\, e^{\frac{i}{2}\int_{t'}^{t} \dot{x}(\tau)^2 d\tau - i\int_{t'}^{t} V(x(\tau))d\tau} \tag{VIII.10}$$

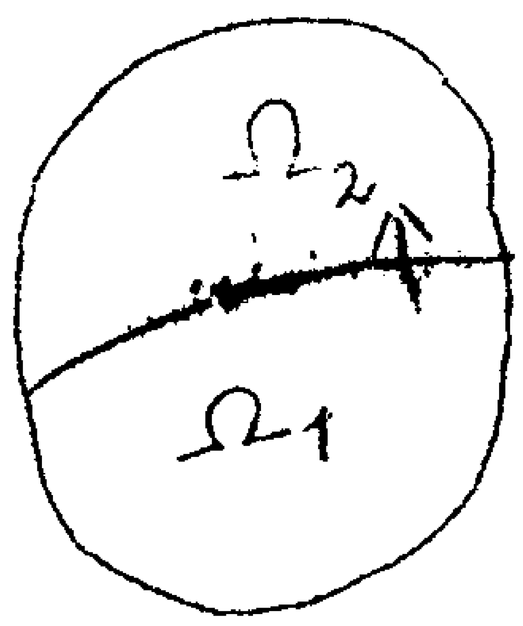
goes, under the same substitution $\tau \to -i\tau$, $t \to -it$, $t' \to -it'$, into the "Wiener-Kac" formula for the solution of the heat equation problem.

$$K(x,x';t,t') = \int \mathcal{D}_W(x)\, e^{-\frac{1}{2}\int_{t'}^{t} \dot{x}(\tau)^2 d\tau - \int_{t'}^{t} V(x(\tau))d\tau} \tag{VIII.11}$$

which is just the one-dimensional case $(d = 1)$ of our functional integral in the φ -form, the only difference being that now we have a three component function instead of one. The important thing is that there is only one independent variable, time, so our field theoretical case is solved by a Wiener-integral in multi-dimensional time, which we call the Wiener history integration analogy to the Feynman history integral. It is interesting to observe that in the exponent of the Feynman integral the Lagrangian appears whereas it is the Hamiltonian which appears in the exponent of the Wiener integral.

The proof of the semigroup property in our case proceeds by verifying that the expression on the right hand side of the equation satisfies all equations that $N_{1+2}\{a, j\}$ satisfies. If these equations have a unique solution then this integral must be proportional to $N_{1+2}\{a, j\}$. In this connection the positive-definiteness becomes important. It is well-known that the matrix $A_{ik} B_{ik}$ is positive definite if A_{ik} and B_{ik} are positive definite, and from this follows that our integral is a positive definite functional in the sense explained earlier. Since the positive definite solution is unique, our statement is proved. If the enlargement of $\Omega_1 \rightarrow \Omega_1 + \Omega_2$ is infinitesimal then from the semigroup property, a Schrodinger equation or more precisely a heat equation in multi-dimensional time is obtained. I hope that it may lead to a proof that $\dfrac{N_\Omega\{a, j\}}{N_\Omega\{a, 0\}}$ has a limit as $\Omega \rightarrow \infty$, and that the limit would be the desired $S\{j\}$ in infinite space-"time"-but stil Euclidean and regularized. It seems that none of the other modifications can be successfully tackled before the first one regarding finite space-"time", is removed.

There is also an Euclidean analogue to the canonical commutation relations of (regularized) MQFT. Its precise form is



$$\Big|_{\substack{x\to s\\ \text{from } \Omega_1}} \vec{\partial}_x \frac{\delta}{\delta j(x)} \frac{\delta}{\delta j(y)} N\{a,j\} \overleftarrow{\partial}_y \Big|_{\substack{y\to s'\\ \text{from } \Omega_2}} -$$

$$- \Big|_{\substack{x\to s\\ \text{from } \Omega_2}} \vec{\partial}_x \frac{\delta}{\delta j(x)} \frac{\delta}{\delta j(y)} N\{a,j\} \overleftarrow{\partial}_y \Big|_{\substack{y\to s'\\ \text{from } \Omega_1}} = 1\, \delta(s,s') \cdot N\{a,j\} \qquad \text{(VIII.12)}$$

(with normal direction from Ω_1 outwards). Here, in our c-number formulation, the question of position of an operator in MQFT is replaced by performing limiting processes in specific and alternative manners. Formal transition from imaginary to real times, as is possible for a flat surface, gives the MQFT canonical commutation relations in a form where the position of operators is determined by time ordering, e.g.

$$\partial_{x^0} \langle T A(x_1) \cdots A(x_n) A(x) A(y) \rangle \left(\Big|_{y\to x+0} - \Big|_{y\to x-0} \right) =$$

$$= i\, \delta(\vec{x} - \vec{y}) \langle T A(x_1) \cdots A(x_n) \rangle \qquad \text{(VIII.13)}$$

for the unregularized case.

Next, we consider the analytic properties of $N\{a,j\}$ as a function of g. We can prove by straightforward techniques that $N\{a,j\}$ is analytic for $Re\ g > 0$, satisfies

$$\left| N_g \{a,j\} \right| \leq N_{Re\,g} \{a,j\} \qquad \text{(VIII.14)}$$

and possesses still derivatives of all orders along the imaginary g-axis. From this follows that the formal g-expansion at $g = 0$ is asymptotic, in fact for $\left| \arg g \right| \leq \frac{\pi}{2}$, $g \to 0$. It follows then that also the expansion of $\frac{N\{a,j\}}{N\{a,0\}}$ or $\frac{N\{a,j\}}{N\{0,0\}}$ (which are meromorphic in $Re\ g > 0$) is asymptotic, since the formal ratio of two asymptotic series is asymptotic (and $N\{a,0\}$ can be shown not to vanish in a neighbourhood of $g = 0$ this includes a portion of the imaginary g-axis.

The proof of the for these series encounter the difficulty in the case we are considering that $G(x,x')$ is not always positive in the regularized c se $(N \geq 2)$. However, if we set up the modified model differently, namely by switching g off in infinite space-"time" instead of making space-"time" free finite, then the divergence proof works, since now the perturbation theoretical expansion involves graphs with $G_0(x-y)$-lines instead of $G(x,y)$-lines, and a simple consideration shows that Caianiello's "numerical model", with suitable choice of parameters gives a g-expansion that minimises the one in question. The g-expansion of the "numerical model" diverges, however.

Regarding the next point I shall consider, viz. the "question of measure" I mentioned already that every positive definite functional on Hilbert space, as defined earlier, is (under certain continuity assumptions that are fulfilled in our case) the Fourier transform of a non-negative measure is via finite-dimensional projections, very similar to the constructions one uses in the definition of the FS-integral. More explicitly, one defines

$$h(q_1, \cdots q_n) = \frac{\int e^{-\frac{1}{2}(\psi\psi)} e^{-\cdots} \prod \delta\left(q_i - (f_i \psi)\right) \mathcal{D}_{FS}(\psi)}{\int e^{-\frac{1}{2}(\psi\psi)} e^{-\cdots} \mathcal{D}_{FS}(\psi)} \tag{VIII.15}$$

where f_i $(i = 1, \cdots n)$ is a set of orthonormal functions, and $\cdots$ stands for the terms $-\frac{1}{4} g\left((H\psi + f)^4\right)$ etc. Then for any cylinder functional $f(x)$ such that

$$f(P_n x) = f(x), \qquad P_n x = x_1 f_1 + \cdots + x_n f_n \tag{VIII.16}$$

We can define the modified Hilbert space-integral

$$\int d\hat{\mu}_{\mathcal{H}} f(x) = \int dx_1 \ldots dx_n\, f(P_n x)\, h(x_1 \ldots x_n) \qquad \text{(VIII.17)}$$

and define the integral of more general functionals by limiting processes. If $g = 0$, $h(x_1 \ldots x_n) = e^{-x_1^2/2 - \cdots - x_n^2/2}$, if $g > 0$ I can prove the existence of and number of properties, among them the upper and lower bounds for this function, but I cannot prove that $h(x_1 \ldots x_n)$ has a limit as $\Omega \to \infty$ in fact then the upper and lower bounds go to the infinitely and zero, respectively. If this mathematically quite well-problem could be solved, it would mean a big step forward in the direction of understanding quantum field theory.

Lecture IX

We now come to 'special' properties of the solution of the model, these are properties that depend (at least in their precise form) on the choice we made for the interaction term, namely a simple fourth power of the field operator. I mentioned already that entire analyticity in λ is tied up with the same property for the interaction term. I shall nog go further into the proof of this property, which requires techniques which one are similar to the ones I used earlier, and also we have to require that for complete $\bar{\lambda} \cdot g \cdot a < \infty$ instead of $\lambda \cdot g \cdot \lambda < \infty$

The next topic is 'bounds based on convexity'. For lack of time, I will only present ideas and results. I defined earlier what a converse function as follows: $f(x)$ is converse if $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y)$. Here x may be a one- or several or even infinitely dimensional variables. A lemma

I mentioned in connection with the FS-integral gives e.g.

$$\ln \int e^{-\frac{1}{2}(\varphi K \varphi) - g(\varphi^4) + \frac{3g}{2}(\varphi^2) + (j\varphi) G_0(0)} \mathcal{D}_W(\varphi) \qquad (\text{IX}.1)$$

is a convex function of g and a convex functional of j and convex function of U if a term $e^{UF\{\varphi\}}$ is inserted into the integral with $F\{\varphi\}$ real. For convex functions of one variable we have

$$\underset{y<x}{\text{l.u.b}} \frac{f(y)-f(x)}{y-x} \leq \frac{d}{dx} f(x) \leq \text{g.l.b} \frac{f(y)-f(x)}{y-x} \qquad (\text{IX}.2)$$

with both l.u.b. and g. l.b actually reached for $y \to x$.

Using this in conjunction with the techniques used earlier, completing a square in the exponent of the functional integral and dropping it, we are able to using the arithmetic-mean geometic-mean inequality, to derive a number of bounds. They are of limited interest in themselves (although these techniques are of interest the quantum statistical mechanics, and some of them have been used earlier by Feynman with success), but indicative as the lend support to the conjecture that the limit $\Omega \to \infty$ does really exist. Namely:

For $N\{a, j\}$ upper and lower bounds are found that grow exponentially as $\Omega \to \infty$.

However, for $\ln \frac{N\{a,j\}}{N\{a,0\}}$ we find an upper bound that grows only like $\Omega^{1/4}$ for $\Omega \to \infty$ and a lower bound that possesses a finite limit for $\Omega \to \infty$.

The lower bound is actually obtained using a variational method due to Feynman (1.c): Write in the numerator integral $\psi \to \psi + \psi_0$, ψ_0 a function to be chosen later, and use the arithmetic-mean-geometric-mean inequality. In brief, setting e.g. $a = 0$:

$$\frac{\int \exp\left[-\frac{1}{2}(\varphi K \varphi) - \frac{g}{4}(\varphi^4) + \frac{3g}{2}G_0(0)(\varphi^2) + (j\varphi)\right]\cdot \mathcal{D}_W(\varphi)}{\int \exp\left[-\frac{1}{2}(\varphi K \varphi) - \frac{g}{4}(\varphi^4) + \frac{3g}{2}G_0(0)(\varphi^2)\right]\cdot \mathcal{D}_W(\varphi)}$$

$$= \exp\left[-\frac{1}{2}(\varphi_0 K \varphi_0) - \frac{g}{4}(\varphi_0^4) + \frac{3g}{2}G_0(0)(\varphi_0^2) + (j\varphi_0)\right]$$

$$\text{Mean}\left[\exp\left\{-g(\varphi_0\varphi^3) - \frac{3g}{2}(\varphi_0^2\varphi^2) - g(\varphi_0^3\varphi) + 3G_0(0)(\varphi\varphi_0) + (j\varphi)\right\}\right] \qquad (\text{IX}.3)$$

$$\geq \exp\left[-\frac{1}{2}(\varphi_0 K \varphi_0) - \frac{g}{4}(\varphi_0^4) + \frac{3g}{2}G_0(0)(\varphi_0^2) + (j\varphi_0)\right]\cdot$$
$$\cdot \exp\left\{\text{Mean}\left[-g(\varphi_0\varphi^3) - \frac{3g}{2}(\varphi_0^2\varphi^2) - g(\varphi_0^3\varphi) + 3g\,G_0(0)(\varphi\varphi_0) + (j\varphi)\right]\right\} =$$

$$= \exp\left[-\frac{1}{2}(\varphi_0 K \varphi) - \frac{g}{4}(\varphi_0^4) + \frac{3g}{2}G_0(0)(\varphi_0^2) + (j\varphi_0) - \frac{3g}{2}\int \varphi_0(x)^2 S(x,x)\, dx\right]. \qquad (\text{IX}.4)$$

Using an independently derived upper bound for $S(X, X)$; we can look for the φ_0 that maximises this expression. Then, apart from the "quantum mechanical Correction" that $S(X,X) \neq G_0(0)$ the maximising φ_0 is precisely the solution of the "classical" problem

$$K\varphi_0 + g\varphi_0^3 = j$$

that stands to MQFT in the same relation as a classical nonlinear oscillator stands to a non-linear quantu m oscillator.

I now give a few inequalities on which an interesting observation can be made.

We write

$$\lim_{\Omega \to \infty} S(X,X) = S(0,0)$$

We write

$$\lim_{\Omega \to \infty} S(X\,X) = S(0\,0)\,,$$

$$\lim_{\Omega \to \infty} S(X\,X\,X\,X) = S(0000)\,,$$

, assuming that these limits exist and then are of course X -independent. Then we have the following bounds

$$G_0(0) \leq S(00) \leq \left[\frac{3}{2} + \sqrt{\frac{3}{2}}\right] G_0(0)$$

$$3\left[S(00) - G_0(0)\right]^2 \leq -S(0000) + 3\,S(00)^2 \qquad \text{(IX.5)}$$

$$\leq 2\,S(00)^2$$

$$\frac{\partial}{\partial g}\left\{3\,S(00)^2 - S(0000) - 3\left[S(00) - G_0(0)\right]^2\right\} \geq 0$$

The quantities here have a direct meaning even in (regularized) MQFT, since time difference zero is the same whether the time is real or imaginary. The left and middle term of the second equations and thus the quantity in the third equation, should even be finite in 2-dimensional unregularized, MQFT if one believes perturbation theory. Incidentally

$$\int_0^g S_{g'}(0000)\,dg' \qquad \text{(IX.6)}$$

is the quantity Caianiello has been studying and is (apart from regularization) also the quantity

$$\lim_{V,T\to\infty} \cdot \frac{1}{iVT} \ln \langle 0 | e^{-i\int H_{int}\, dx} | 0 \rangle$$

of MQFT perturbation theory, $\langle 0|$ the bare vacuum, where V and T are space volume and time interval, respectively. (Also, the logorithmand is the infinite phase factor from vacuum graphs). Note, however, that our proofs at no state use perturbation expansions but are rigorous (apart from the assumptions about $\Omega \to \infty$ expressly stated).

We can obtain bounds on the measure $h(q_1, \ldots q_n)$. The unsolved problem here is to prove the existence of a limit of h as $V_\Omega \to \infty$ which would yield a measure (perhaps not a Gaussian) which can be physical interest.

Finally, I give an operator formulation of EQFT. We introduce, in 4-dimensional space the Hermitian operator field $P(x)$ and $Q(X)$ which satisfy

$$\begin{aligned} [Q(x), Q(y)] &= 0 \\ [Q(x), P(y)] &= i\delta(x-y) \\ [P(x), P(y)] &= 0. \end{aligned} \qquad \text{(IX.7)}$$

Then

$$P(x) - \frac{i}{2} K\, Q(x) - \frac{1}{2} g\, Q(x)^3 + \frac{3ig}{2} G_0(0)\, Q(x) | \rangle$$

$$\equiv C(x) | \rangle \qquad \text{(IX.8)}$$

defines a state $| \rangle$ (suppress for brevity the dependence on Dirichlet data   ). Then

$$S\{j\} = \frac{\langle | e^{\int j(x) Q(x) dx} | \rangle}{\langle | \rangle} \qquad \text{(IX.9)}$$

as shown by diagonalization of Q, the "Schrodinger' representation"

$$\langle \varphi | Q(x) = \varphi(x) \langle \varphi | ,$$

$$\langle \varphi | P(x) = -i \frac{\delta}{\delta \varphi(x)} \langle \varphi | \qquad \text{(IX.10)}$$

gives, it inserted e.g. as

$$| \rangle = \int F\{\varphi\} | \varphi \rangle\, \mathcal{D}_W(\varphi) \qquad \text{(IX.11)}$$

with $F\{\varphi\}$ to be determined directly by the functional integral ratio for $S\{j\}$ whereby in the space the Q, P operat in, the metrices is puritive definite. I do not know, however, how useful this operator formulation will be to obtain results on the model, particularly on $\Omega \to \infty$ beyond these obtained from direct study of functional integrals.

In conclusion it seems that the problem of existence of nontrivial quantum field theories is not entirely inaccessible if sufficiently abstract mathematical methods of <u>representation</u> (here in terms of functional integrals) are used. E.g. The Friedrichs-Shapiro integral permits us to deal with the present model that is entirely nontrivial and not solvable exactly. I believe that the search for more such representations should be a challenge to mathematical physicists.

R E F E R E N C E S

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ERRATA

page 2	3th line	-	Insert 'alone' after 'interpreted'.
page 2	14th line	-	Omit 'the theory' after Lagrangian
,, 2	19th line	-	Replace 'recall the' by 'recall that'
,, 3	23rd line	-	Omit 'so' before 'as'
,, 4	6th line	-	Add 'modified EQFT are' after 'FQFT to'
	10th line	-	Replace 'form of' by 'form in'
,, 5	2nd and 3rd	-	Omit 'and'
,, 7	2nd line	-	Replace ' V ' by ' $\forall$ ' and ' $\mathcal{F}$ ' by ' $\exists$ '
,, 10	4 "	-	Add ' k^{-1} ' at the end of the line
	16th line	-	Omit 'the' after 'symmetry'
,, 13	15th line	-	Add 'as' after 'arguments'
,, 19	16th line	-	Omit 'kind is'
,, 21	2nd line	-	Insert semicolon after 'polaron problem'
,, 24	6th line	-	Insert ' $F\{\gamma\}$ ' after 'functional'
,, 27	7th line	-	Replace 'wicher' by 'Wiener'
	8th line	-	Insert 2) (Ref.) at the end of the line
,, 28	6th line	-	Insert ' $g > 0$ ' after 'that for'
	13th line	-	Insert '3)' (Ref.) after, 'model'
,, 31	1st line	-	Insert ' 3)" after 'subspace'
,, 36	15th line	-	Replace 'much' by 'such' and add ' $u \in U$ ' at the end of the line
,, 39		-	Insert '1)' (Ref.) after 'Uhlenbeck'

page 43 - 6th line - Insert "$\left(\vec{\partial}_n\right)^{K-1-2\left[\frac{K+1}{2}\right]}$"

after "$S_{N-1-K-n}$"

,, 51 -13th line - Insert 1a) (Ref.) after 'Courant-Hilbert'

,, 52 - 2nd line - Insert '$\not{b}$' and '1b)' (Ref.) after 'Schechter'

,, 53 -13th line - Insert '2a) (Ref.) after 'Fradkin' and '2b)' (Ref.) after 'Efimov'.

,, 63 -14th line - Omit 'of the' after 'proof'

22nd line - Insert '1)' (Ref.), after 'model'

,, 66 -15th line - Insert '1)' (Ref.) after 'success'

-16th line - Replace 'the lend' by 'they lend'

,, 71 - 4th line - Insert 'I' before 'suppress'

- 5th line - Insert '∂' after 'data'

-15th line - Replace 'metrices is puritive' by 'metric is positive'