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SOME TOPICS
IN
THE STRONG AND WEAK INTERACTIONS
OF
ELEMENTARY PARTICLES

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PREFACE

This thesis comprises the work done by the author during the years 1959-1962 on the strong and weak interactions of elementary particles under the guidance of Professor Alladi Ramakrishnan, formerly Professor of Physics at the University of Madras and now the Director of the Institute of Mathematical Sciences, Madras.

It consists of three parts, the first dealing with strong interactions, the second with weak interactions and the third with the concept of causality in deterministic, stochastic and quantum mechanical processes. Ten papers relating to the subject-matter of this thesis have been published by the author and five more are in the course of publication. The available reprints are enclosed in the form of a booklet. Collaboration in these papers either with my guide Professor Alladi Ramakrishnan or with my colleagues was necessixtated by the nature and range of problems dealt with in this thesis and due acknowledgment of this collaboration has been made in each chapter.

I am deeply indebted to Professor Alladi Ramakrishnan for his guidance throughout the course of this work. I am very grateful to the University of Madras and the Institute of Mathematical Sciences, Madras, for providing me with excellent facilities for research work and to the University of Madras and the Atomic Energy Commission, Government of India, for the award of Research Fellowships during the period of study.

The Institute of Mathematical Sciences, Madras
August, 1962.

A.P.Balachandran

(A.P.Balachandran)

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CHAPTER I

INTRODUCTION

1. Objectives

This thesis is devoted to the study of some topics in the strong and weak interactions of elementary particles with special emphasis on the application of techniques recently developed in two new fields of quantum field theory, namely,

1) Dispersion relations, and

2) Symmetry principles.

These techniques were necessitated by the difficult situation created by the gross inadequacy of the conventional perturbation theoretical methods in the study of strong interactions and by the discovery of new particles whose mutual interactions are not clearly understood.

It is a curious fact that while in the early stages of the evolution of non-perturbative methods, the principle of causality was assumed to play a fundamental role, recently the point of view has been stressed that the analytic properties of a scattering amplitude are governed by the "principle of maximum smoothness".[1] It is not however clear whether this attitude really amounts to an elimination from our thinking of all conventional concepts of elementarity of

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1) G.F.Chew and S.C.Frautschi, Phys. Rev. Letters, 8 , 41 (1962).

particles and the evolutionary nature of a scattering process. Therefore it seems worthwhile to examine the meaning of causality ab initio drawing analogues from stochastic processes which are inherently evolutionary in character. For this, we are constrained to appeal to the perturbative nature of a scattering process.

To appreciate the need for a departure from perturbative concepts and realize the depth and extent of the new methods, it seems appropriate to outline briefly the concept of a quantum mechanical collision in a perturbative theory emphasizing features which are actually discussed in the thesis.

2. The Perturbative Approach

The study of any scattering process in quantum theory is ultimately concerned with determining the state of the system in the infinite future given its state in the infinite past. The most natural method that suggests itself to study this evolution is to postulate, in analogy with classical mechanics, the possible changes in the system in an infinitesimal time interval Δt and obtain the change in a finite or infinite interval of time by a process of integration. In both classical and quantum mechanics, the specification of the state of the system involves a knowledge of its dynamical attributes like the number of particles of different types

in the state, their energies, momenta or angular momenta and intrinsic attributes like spins and isotopic spins. In a classical picture, by a stationary state, we imply that these observables which characterize it remain constant in time. On the other hand, in quantum theory a state is said to be stationary if its time dependence is of the exponential type e^{-iEt} where t denotes time and E is a real parameter to be identified with the energy of the system. Thus the time variation of a stationary state consists only in a change of its phase, consequently the expectation values of the dynamical variables in this state are independent of time. Thus if in analogy with classical mechanics, we postulate that the time evolution of the system is determined by a Hamiltonian, we require that the state in the infinite past which consists of non-interacting particles be an eigenstate of the free field Hamiltonian H_0 so that its time dependence is of the type e^{-iEt} where E is the total energy of the system of particles. We can then postulate that the changes in the state of the system are brought about by changes in it in infinitesimal intervals of time Δt by the operation of an interaction Hamiltonian H' where H' represents the interactions between the components particles. The total Hamiltonian is thus the sum of two parts H_0 and H' . It follows that the state

of the system at time t prior to which H' has operated n times, namely at times $t_1, t_2, \cdots, t_n$ the system having evolved under H_0 in between these intervals, that is, between t_1 and t_2, t_2 and t_3, $\cdots$, t_n and t, is given by

$$e^{-iH_0(t-t_n)} H'(t_n)\Delta t_n\, e^{-iH_0(t_n-t_{n-1})} H'(t_{n-1})\Delta t_{n-1} \cdots e^{-iH_0(t_2-t_1)} H'(t_1)\Delta t_1\, e^{-iE(t_1-t_0)} |\rangle$$

where $|\rangle$ denotes the state of the system of non-interacting particles at time t_0. Clearly to obtain the state at $t \to \infty$ given its state at $t_0 = -\infty$, we have to sum over all n, integrate over t_1, t_2, $\cdots$, t_n over the permissible range consistent with the time ordering $t_n > t_{n-1} > \cdots > t_1 > t_0$ and take the limits $t \to \infty$ and $t_0 \to -\infty$ in the above expression (the limits being defined as usual by the adiabatic hypothesis). This gives us immediately the state of the system at $t = \infty$ as the perturbation expansion of the scattering matrix operating on the state at $t_0 = -\infty$. Such a way of looking at a process is inherently perturbative in character. The perturbative method has been universally employed in electrodynamics and the weak interactions of leptons. We shall have occasion to refer to this method in detail in a later chapter when we discuss the concept of causality.

III. Non-perturbative Approaches

Perturbation theory signally fails in the domain of strongly interacting particles. Here it is no longer adequate for instance to try to build up a "dressed" proton by attaching meson by meson to a "bare" proton as we do to build up a dressed electron in quantum electrodynamics. The reason for this is that the concept of scattering which we have sketched involves a series development of the scattering matrix in powers of the coupling constant occurring in H' which has therefore to be small in magnitude. For strong interactions, however, the effect of H' is no longer a small perturbation and the series therefore does not converge. Thus an adequate solution of even the simplest of scattering problems namely, the elastic scattering of two strongly interacting particles, will require not only the complete solution of the equations in terms of the interactions between the particles involved in the scattering, but also in terms of the interactions between all the other strongly interacting particles as well since quantum field theory allows for the transmutation of the initial system into any other system of strongly interacting particles provided only that it is consistent with selection rules. Intuitively one anticipates that the existence of new channels which are accessible to the initial state will alter the scattering properties of the initial state. A quantitative formulation of this statement is the well-known unitarity condition

which has thus to be adequately taken into account in the solution of the scattering amplitude. It is clear from this brief description that any treatment of strongly interacting systems which has to have any chance of success at all is bound to be very complicated. Consequent upon the complete failure of the Hamiltonian formalism in this domain, two other approaches of a non-perturbative character have been tried. One is the use of symmetry principles which can give selection rules and relations between observable amplitudes which can be experimentally checked. The scope of this method is however severely limited since it can make no detailed quantitative predictions regarding the characteristics of any process. The more ambitious approach has been to study in detail the analytic properties of scattering amplitudes implied by such very general assumptions underlying local field theory like microscopic causality (local commutativity), unitarity and other well-established principles of invariance. This has now grown into an important branch of elementary particle physics known as dispersion relations. Hopefully, one expects that a complete specification of the analytic properties of the scattering amplitude as a function of both energy and momentum transfer variables coupled with the unitarity condition can be used to develop a dynamical theory of elementary particles since the two assumptions which the theory of dispersion relations makes regarding the form of the Hamiltonian, namely, its locality and Lorentz

invariance, are in fact sufficient to specify the Hamiltonian in the more usual field theories to within a small number of coupling constants provided one demands in addition that the Hamiltonian theory be renormalizable. The conjective regarding the analyticity of the two-particle scattering amplitudes with respect to both energy and momentum transfer variables is due to Mandelstam [2] and coupled with a unitarity approximation (which is to be contrasted with the perturbation theoretic approximation which is of a very different nature) seems to provide a plausible low energy approximation for scattering processes.

4. Outline of the Problems Discussed

The thesis is divided into three parts. Part I is devoted to the study of strong interactions (mostly by non-perturbative methods) while in Part II , some topics in the theory of weak interactions are discussed using symmetry principles as well as non-perturbative methods. Finally, Part III contains a semi-expository article on the role of causality in quantum theory.

In Chapter II, which is the first chapter in Part I , we derive partial wave dispersion relations for the scattering processes associated with the lambda-nucleon system. Most of the complications in using dispersion relations for such processes

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2) S. Mandelstam, Phys. Rev., 112 , 1344 (1958).

are of kinematical originx, like those arising from the spins of the scattering particles. In this chapter, we discuss in detail the choice of the linear momentum and angular momentum amplitudes for these processes, derive the analytic properties of the partial wave amplitudes and finally write down the dispersion relations for these amplitudes.

Chapter III is devoted to developing effective range formulae for the recently observed Λ-π and Σ-π resonances using dispersion theory. These formulae are the analogues of the Chew-Low effective range formula for π-N scattering. We also develop approximate expressions for the scattering amplitudes of the process $\gamma + \Lambda \rightarrow \Lambda + \pi$ whose knowledge is of importance in the study of processes like $\gamma + N \rightarrow \Lambda + K + \pi$.

In Chapter IV , we carry out an analysis of the low energy K- nucleon scattering data after approximating the scattering amplitude by the contributions to it arising from the Λ- and Σ- poles and the Λ-π , Σ-π and π-π resonances. The analysis predicts a definite behaviour of the S -wave $\overset{K\text{-}N}{\Lambda}$ scattering amplitude in the isotopic spin zero state and a curve showing the behaviour is presented.

In Chapter V , we discuss the recently observed K-π resonance within the framework of the Mandelstam representation assuming that it is in a state of angular momentum unity. An approximation scheme is used which enables one to compute the K-π scattering phase shifts in a simple fashion. The

results for these phase shifts in the resonant channel are presented in the form of curves for various values of the parameters of the theory.

Chapter VI is concerned with the derivation of an integral equation for the production amplitude in any multi-channel reaction in terms of the scattering amplitudes of the initial and final systems. The equations presented in this chapter are non-perturbative.

In Chapter VII, we make some comments on a theory of strong interactions which Sakurai has recently proposed. It is shown in particular that in such a theory only two of the fields among the baryons and spinless mesons can be elementary and the rest have to be composite systems if the theory is not to contradict experiments.

In Chapter VIII , we derive some rigorous consequences of the sign of the relative parity of the K -meson with respect to the Σ- or Λ- hyperon in some reactions which involves these particles using the density matrix formalism. These results suggest some methods which can help decide the K -hyperon relative parities.

In Chapter IX , we discuss some exact consequences of the symmetries of the unrenormalized Lagrangian on the mass renormalization term for fermions and bosons. A perturbation theoretic calculation of the mass difference of the charged and neutral cascade particles arising from the mass difference of the charged and neutral K -particles is presented. This mass difference calculation involves no cut-off since the resultant integrals are finite.

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3) J.J.Sakurai, Ann. Phys., 11 , 1 (1960).

Part II of the thesis is devoted to the study of weak interactions. In Chapter X which is the first chapter of this Part , we explicitly construct a model in which the renormalization of the axial vector coupling constant in β-decay can be calculated non-perturbatively even though the corresponding current is not conserved. An equivalence theorem is proved which shows how the fermion mass term can be parametrized by an unobservable field so as to ensure its γ_5 -invariance. A model is suggested to explain the low rates of most of the leptonic decay modes of the strange particles.

In Chapter XI , we develop an isotopic spin scheme for leptons by introducing a new transformation called the τ_3 -transformation. The scheme explains the absence of strong interactions for leptons. On the basis of this scheme, a set of phenomenological rules is suggested to explain the low leptonic decay rates of most of the strange particles. The idea involved in the τ_3 -transformation is then extended to the hypercharge quantum number to develop a four-dimensional isotopic spin formalism for the Gell-Mann-Nishijima scheme in which leptons are also included in a natural way. Within the framework of this formalism, a theory of weak interactions is suggested and the structure of the weak interaction currents implied by the theory is discussed. An isotopic spin classification is also provided for vector mesons with charged components only postulated in connection with the theory of weak interactions. The scheme forbids the strong interactions of these mesons.

Chapter XII deals with the application of dispersion theory to the non-leptonic decays of some of the strange particles. A pole approximation is presented for the decay of the Ξ particle into a Λ and a π. It is shown how one can attempt to explain the Σ -decay asymmetries as arising from a parity clash between diagrams involving Λ and those involving Ξ, the crucial point that is exploited here being that the asymmetry parameters of Λ and of Ξ are opposite in sign. Finally it is pointed out that dispersion theory yields a finite answer for the mass differences of elementary particles in some cases. This is illustrated by calculating the $K_1^0 - K_2^0$ mass difference in the pole approximation.

Chapter XIII which is the concluding chapter of this Part deals with some compound models for elementary particles. In the first section of this chapter, we analyze the Ξ - decay asymmetries assuming that Ξ is the bound state of a Λ and a $\bar{K}$. In the second section, we suggest a few methods which can give some indication of whether or not a particle is compound.

Part III consists of Chapter XIV which deals with the concept of causality in deterministic, stochastic and quantum mechanical processes. It is found worthwhile to discuss ab initio the meaning of causal connection between events which

in turn necessitates the clarification of an event in the quantum mechanical description. A point of view is stressed that the integrand of the S -matrix represents the amplitude for a pattern of events though these events are not realisable or observable in a scattering process. We find it useful to divide such events into two classes : (1) Causative and (2) Resultant , and the procedure of determining amplitudes consists in connecting a causative event to its corresponding resultant or vice-versa. This we feel is the essence of the Feynman $\mathcal{R}$ formalism which therefore brings out the role of causality in a scattering process in a perspicuous manner. In the course of the discussion, a combinational problem relating to Feynman diagrams is posed and solved.

Finally in the Appendix is included a discussion on a possible Ξ-π resonance.

P A R T I

THE STRONG INTERACTIONS OF ELEMENTARY PARTICLES

CHAPTER II.

PARTIAL WAVE DISPERSION RELATIONS FOR THE LAMBDA-NUCLEON SYSTEM*

1. Introduction

This chapter is devoted to the derivation of partial wave dispersion relations for lambda-nucleon scattering and the associated processes of lambda-antinucleon scattering and lambda-antilambda annihilation into a nucleon-antinucleon pair. Analogous considerations for nucleon-nucleon scattering have already been presented by Amati, Leader and Vitale[1] and by Goldberger, Grisaru, MacDowell and Wong[2]. Compared to the nucleon-nucleon problem, the unequal masses of the particles involved and the absence of Pauli principle causes many additional complications in the case of the Λ-N system. On the other hand, the fact that there is no bound state of a lambda and a nucleon implies that there is no analogue of the deuteron pole in this case.

Briefly, the contents of this chapter are as follows: In Section II, we summarize the various kinematical relations which will be frequently used in the rest of the chapter. In Section III, we discuss the resolution of the Feynman amplitudes in each of the three channels into a convenient set of invariant functions. The kinematical singularities

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* Alladi Ramakrishnan and A.P.Balachandran, Nuovo Cimento (in press)

1) D. Amati, E. Leader and B.Witale, Nuovo Cimento, 17, 68 (1960); 18 , 409 , 458 (1960).

2) M.L.Goldberger, M.T.Grisaru, S.W.MacDowell and D.Y.Wong, Phys. Rev., 120 , 2250 (1960).

of these functions are then investigated in fourth order perturbation theory and a set of functions $\bar{F}_i$ which are free of such singularities is defined. In doing this, we demonstrate the connection between two different sets of amplitudes F_i and $\bar{F}_i$ defined through suitably chosen basic matrices. It is shown that while both these sets are free of kinematical singularities in the $N-N$ problem, it is only the $\bar{F}_i$'s that are suitable for writing dispersion relations for the $\Lambda-N$ system. The crossing relations between the amplitudes are established. The scattering amplitudes are then resolved into partial waves using the formalism recently developed by Jacob and Wick[3] and their relation to the previously chosen invariant amplitudes is derived.

In Section IV, we discuss the analytic properties of the various amplitudes. The occurrence of anomalous thresholds in our problem and the modification in the Mandelstam representation[4] implied by it are briefly discussed. The analytic properties of the partial wave amplitudes are then investigated and their discontinuities across the branch cuts are expressed in terms of the appropriate absorptive parts. The region of convergence of the Legendre polynomial expansion of the absorptive parts is then identified. Finally the partial wave dispersion relations implied by these considerations are written down.

This analysis brings us to a stage where numerical calculations on the scattering processes associated with the $\Lambda-N$ system can be done with suitable techniques.

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3) M. Jacob and G.C.Wick, Ann. Phys., 7 , 404 (1959).
4) S.Mandelstam, Phys. Rev., 112 , 1344 (1958); also 115 , 1752 (1959).

II. Kinematics

In the Mandelstam representation, it is assumed that the scattering amplitudes for the following three processes are boundary values of the same analytic function :

$$\Lambda + N \rightarrow \Lambda + N \qquad (I),$$

$$\Lambda + \bar{N} \rightarrow \Lambda + \bar{N} \qquad (II),$$

$$\Lambda + \bar{\Lambda} \rightarrow N + \bar{N} \qquad (III) \qquad (2\cdot1)$$

Here the bars designate antiparticles. For process I, let us denote by P_1 and P_1' the four-vector momenta of the incident and outgoing lambda and by P_2 and P_2', the four-vector momenta of the incident and outgoing nucleon. Define the variables

$$s = -(P_1 + P_2)^2 \qquad (2\cdot2),$$

$$\bar{t} = -(P_1 - P_2')^2 \qquad (2\cdot3),$$

$$t = -(P_1 - P_1')^2 \qquad (2\cdot4).$$

Throughout this thesis, the scalar product A.B. of two four-vectors A and B is defined as $\vec{A}\cdot\vec{B} + A_4 B_4$ where $A_4 = iA_0$ so that the metric tensor $g_{\mu\nu}$ is equal to $\delta_{\mu\nu}$. s denotes the square of the centre-of-mass energy while $\bar{t}$ and t denote the squares of the four-momentum transfers for this process. Let k denote the corresponding centre-of-mass momentum and θ the angle of scattering. We have then the following kinematical relations[5] :

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5) See for example W.R.Frazer and J.R.Fulco, Phys.Rev. 119 , 1420 (1960).

$$t = -2k^2(1-\cos\theta) \qquad (2.5),$$

$$k^2 = \frac{[s-(m_\Lambda+m_N)^2][s-(m_\Lambda-m_N)^2]}{4s} \qquad (2.6),$$

$$\bar{t} = \frac{1-\cos\theta}{2}\,\frac{(m_\Lambda^2-m_N^2)^2}{s} - \frac{1+\cos\theta}{2}(s-2m_\Lambda^2-2m_N^2) \qquad (2.7),$$

$$s+\bar{t}+t = 2m_\Lambda^2+2m_N^2 = \Sigma s_{ij} \qquad (2.8)$$

$\bar{t}$ is the square of energy in the centre-of-mass system of reaction II while t is the corresponding variable in the centre-of-mass system of reaction III. If l denotes the centre-of-mass momentum and φ the scattering angle for process II, we have the following analogues of equations (2.5 - 2.8) :

$$t = -2l^2(1-\cos\varphi) \qquad (2.9),$$

$$l^2 = \frac{[\bar{t}-(m_\Lambda+m_N)^2][\bar{t}-(m_\Lambda-m_N)^2]}{4\bar{t}} \qquad (2.10),$$

$$s = \frac{1-\cos\varphi}{2}\,\frac{(m_\Lambda^2-m_N^2)^2}{\bar{t}} - \frac{1+\cos\varphi}{2}(\bar{t}-2m_\Lambda^2-2m_N^2) \qquad (2.11)$$

For process III, let p and q denote the centre-of-mass moments of Λ and N respectively and η the scattering angle. Then we have

$$t = 4(p^2+m_\Lambda^2) = 4(q^2+m_N^2) \qquad (2.12),$$

$$s = -p^2-q^2-2pq\cos\eta = -\tfrac{1}{2}t - \tfrac{1}{2}(t-4m_\Lambda^2)^{\frac{1}{2}}(t-4m_N^2)^{\frac{1}{2}}\cos\eta + m_\Lambda^2+m_N^2 \qquad (2.13),$$

$$\bar{t} = -p^2 - q^2 + 2pq\cos\eta = -\tfrac{1}{2}t + \tfrac{1}{2}(t-4m_\Lambda^2)^{\frac{1}{2}}(t-4m_N^2)^{\frac{1}{2}}\cos\eta + m_\Lambda^2 + m_N^2 \qquad (2\cdot 14)$$

III. The choice of amplitudes

We now turn to the question of the choice of the basic amplitudes with which we can characterize each of the three reactions. Isotopic spin causes no complications since the reaction proceeds through only one isotopic spin channel in each of the three processes. It is then easy to show that for reactions I and II, six amplitudes are necessary to specify the scattering while for reaction III, only five independent amplitudes exist.

Consider for instance reaction I . For a given total angular momentum J , the system can either be in a singlet $(S=0)$ or a triplet $(S=1)$ state (Here S denotes total spin). For the singlet state, the orbital angular momentum ℓ is equal to J while for the triplet state, $\ell = J \pm 1$ or J . Transitions between states of even and odd ℓ are forbidden by parity conservation while time-reversal invariance implies that the amplitude for transition from $\ell = \ell'$, $S = S'$ to $\ell = \ell''$, $S = S''$ is equal to the amplitude for the inverse transition, $\ell = \ell''$, $S = S''$ to $\ell = \ell'$, $S = S'$. We are thus left with six amplitudes corresponding to the transitions

$$\ell = J,\ S=0 \to \ell = J,\ S=0\ ;\ \ell = J+1,\ S=1 \to \ell = J+1,\ S=1\ ;$$

$$\ell = J,\ S=1 \to \ell = J,\ S=1\ ;\ \ell = J-1,\ S=1 \to \ell = J-1,\ S=1\ ;$$

$$\ell = J,\ S=0 \to \ell = J,\ S=1\ ;$$

$$\ell = J+1,\ S=1 \to \ell = J-1,\ S=1 \qquad (3\cdot 1)$$

Exactly the same considerations apply also to reaction II.

For reaction III, we notice that initial and final states involve a fermion and its own anti-fermion. Such states are eigenstates of charge conjugation with eigen-value $(-1)^{\ell+S}$. Clearly then, transitions between states with the same ℓ with S -values differing by an odd number are forbidden if charge conjugation is a good operation. Therefore the singlet-triplet amplitude of the previous paragraph becomes zero in this case. We are finally left with five amplitudes with which to characterize the reaction.

Our next step is to resolve the covariant matrix (which, when its indices are saturated by the initial and final spinors, gives the Feynman amplitude for the process) as a sum of a complete set of suitably chosen basic matrices. One such set of basic matrices is provided by

$$P_1 = 1^{\Lambda}\, 1^{N}\ ,\qquad P_2 = i\gamma^{\Lambda}\, P_2'\, 1^{N}\ ,$$

$$P_3 = 1^{\Lambda}\, i\gamma^{N} P_1'\ ,\qquad P_4 = i\gamma^{\Lambda}\, P_2'\, i\gamma^{N} P_1'\ ,$$

$$P_5 = \gamma^{\Lambda}\, \gamma^{N}\ ,\qquad P_6 = \gamma_5^{\Lambda}\, \gamma_5^{N} \qquad (3\cdot 2)$$

These are the analogues of the perturbative invariants of Amati, Leader and Vitale[1]. Here the matrices with the index

Λ are to be taken between the initial and final lambda spinors and those with the index N are to be taken between nucleon spinors. The Feynman amplitude for process I can therefore be written as

$$\begin{aligned}
\mathcal{M}^{I}(s,\bar{t},t) &= \bar{F}_1^{I}(s,\bar{t},t)\,\bar{u}_\Lambda(p_1')\,u_\Lambda(p_1)\,\bar{u}_N(p_2')\,u_N(p_2) \\
&+ \bar{F}_2^{I}(s,\bar{t},t)\,\bar{u}_\Lambda(p_1')\,\not{p}_2'\,u_\Lambda(p_1)\,\bar{u}_N(p_2')\,u_N(p_2) \\
&+ \bar{F}_3^{I}(s,\bar{t},t)\,\bar{u}_\Lambda(p_1')\,u_\Lambda(p_1)\,\bar{u}_N(p_2')\,\not{p}_1'\,u_N(p_2) \\
&+ \bar{F}_4^{I}(s,\bar{t},t)\,\bar{u}_\Lambda(p_1')\,\not{p}_2'\,u_\Lambda(p_1)\,\bar{u}_N(p_2')\,\not{p}_1'\,u_N(p_2) \\
&+ \bar{F}_5^{I}(s,\bar{t},t)\,\bar{u}_\Lambda(p_1')\,\gamma_\mu\,u_\Lambda(p_1)\,\bar{u}_N(p_2')\,\gamma_\mu\,u_N(p_2) \\
&+ \bar{F}_6^{I}(s,\bar{t},t)\,\bar{u}_\Lambda(p_1')\,\gamma_5\,u_\Lambda(p_1)\,\bar{u}_N(p_2')\,\gamma_5\,u_N(p_2) \qquad (3\cdot3).
\end{aligned}$$

Our notation is such that in any function $\varphi(x,y,z)$ the first variable denotes the square of the centre-of-mass energy of reaction I, the second that of II and the third that of III. The spinors have the conventional normalization $\bar{u}u=1$.

The choice (3·3) is motivated by perturbation theory considerations since fourth order perturbation theory shows that $\bar{F}_i^{I}$'s are free of kinematical singularities[1]. We shall illustrate this for the particular fourth order graph shown below.

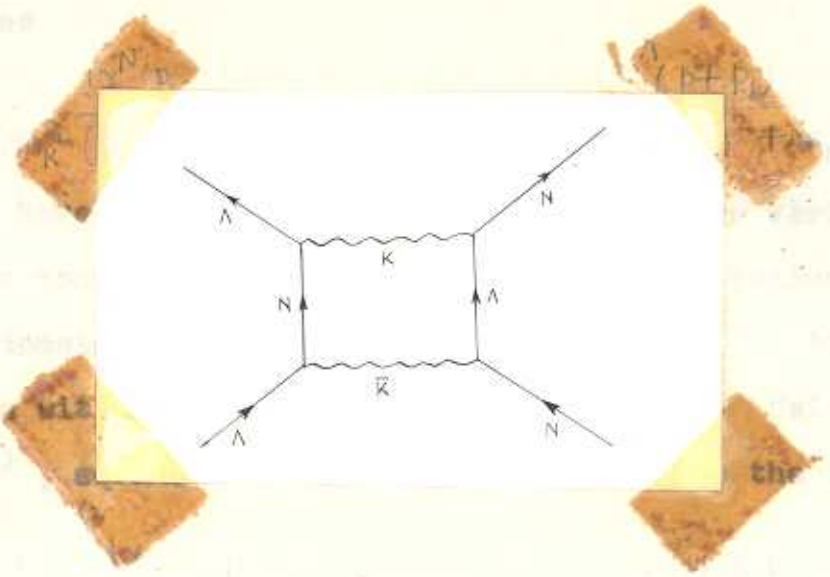

Fig. 1.

We shall see later that this diagram is of importance in another contexxt also in our problem, namely, in fixing the threshold of integration of the δ -variable (cf. Section IV). Ignoring a constant factor, the contribution

of Fig. 1 to the Feynman amplitude for the process can be written as

$$I = \int d^4p \, \frac{1}{p^2+m_K^2} \, \frac{(\gamma^N(P_2-p)-m_\Lambda)}{(P_2-p)^2+m_\Lambda^2} \, \frac{1}{(p-P_2-P_2')^2+m_K^2} \, \frac{(\gamma^\Lambda(p+P_1)-m_N}{(p+P_1)^2+m_N^2} \quad (3\cdot4)$$

where we have omitted possible γ_5's at the vertices which are of no consequence in the following considerations. Further, we have for a moment denoted the external momenta by small p_i's to avoid confusion with the P_i's of equation (3·2). Using standard methods[6], equation (3·4) can be reduced to the form

$$I = 6 \int_0^1 d\alpha_1 \int_0^{\alpha_1} d\alpha_2 \int_0^{\alpha_2} d\alpha_3 \int d^4p \, \frac{1}{(p^2+a^2)^4} \Big[\{m_\Lambda + m_N(1+\alpha_2+\alpha_3-2\alpha_1)\}\{m_\Lambda\alpha_1+m_N\}P_1$$

$$- \{m_\Lambda + m_N(1+\alpha_2+\alpha_3-2\alpha_1)\}\{\alpha_2+\alpha_3-2\alpha_1\}P_2$$

$$+(\alpha_1-1)(m_\Lambda\alpha_1+m_N)P_3-(\alpha_1-1)(\alpha_2+\alpha_3-2\alpha_1)P_4$$

$$+\tfrac{1}{4}p^2P_5\Big]$$

where

$$a^2 = m_K^2(\alpha_1+\alpha_2+\alpha_3)+(m_N^2-m_\Lambda^2)(1+\alpha_3-\alpha_1-\alpha_2)+2(P_2P_2'-m_N^2)(\alpha_1-\alpha_2)-K^2$$

with

$$K = P_1(1-\alpha_1)+P_2(\alpha_3-\alpha_1)+P_2'(\alpha_2-\alpha_1) \quad (3\cdot5)$$

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6) These are discussed in detail in J.M.Jauch and F.Rohrlich, The Theory of Photons and Electrons (Addison-Wesley, 1955), Appendix A 5. See also Chapter IX of this thesis.

Equation (3·5) reveals that the part of $\bar{F}_c^I$ coming from Fig. 1 is free of kinematical singularities. We have checked that this is indeed true for any arbitrary fourth order graph. However, in our considerations, it is found more convenient to deal with the following set of invariants instead of (3·5), since many of the matrix elements involved have been evaluated in ref. [2] :

$$J_1 = 1^\Lambda \cdot 1^N \quad , \qquad J_2 = \gamma^\Lambda \cdot \gamma^N ,$$

$$J_3 = \tfrac{1}{2}\sigma_{\mu\nu}^\Lambda \sigma_{\mu\nu}^N \quad , \qquad J_4 = i\gamma_5^\Lambda \gamma_\mu^\Lambda \, i\gamma_5^N \gamma_\mu^N ,$$

$$J_5 = \gamma_5^\Lambda \gamma_5^N \quad , \qquad J_6 = i\gamma^\Lambda \cdot P_2' \, 1^N \qquad (3·6)$$

This gives

$$\begin{aligned} \mathcal{F}^I(s,\bar{t},t) = {} & F_1^I(s,\bar{t},t)\, \bar{u}_\Lambda(P_1')\, u_\Lambda(P_1)\, \bar{u}_N(P_2')\, u_N(P_2) \\ & + F_2^I(s,\bar{t},t)\, \bar{u}_\Lambda(P_1')\, \gamma_\mu\, u_\Lambda(P_1)\, \bar{u}_N(P_2')\, \gamma_\mu\, u_N(P_2) \\ & + F_3^I(s,\bar{t},t)\, \tfrac{1}{2}\, \bar{u}_\Lambda(P_1')\, \sigma_{\mu\nu}\, u_\Lambda(P_1)\, \bar{u}_N(P_2')\, \sigma_{\mu\nu}\, u_N(P_2) \\ & + F_4^I(s,\bar{t},t)\, \bar{u}_\Lambda(P_1')\, i\gamma_5\gamma_\mu\, u_\Lambda(P_1)\, \bar{u}_N(P_2')\, i\gamma_5\gamma_\mu\, u_N(P_2) \\ & + F_5^I(s,\bar{t},t)\, \bar{u}_\Lambda(P_1')\, \gamma_5\, u_\Lambda(P_1)\, \bar{u}_N(P_2')\, \gamma_5\, u_N(P_2) \\ & + F_6^I(s,\bar{t},t)\, \bar{u}_\Lambda(P_1')\, iP_{2\mu}'\, \gamma_\mu\, u_\Lambda(P_1)\, \bar{u}_N(P_2')\, u_N(P_2) \end{aligned} \qquad (3·7)$$

We have now to verify that the $F_i^{I,\Delta}$ of equation (3·7) are free of kinematical singularities. For this, following ref. [1] , let us define the following set of four mutually orthogonal four-vectors :

$$M = P_1 + P_2 = P_1' + P_2' \quad , \qquad K = P_2 - P_1' = P_2' - P_1 \quad ,$$

$$\Delta = P_1' - P_1 = P_2 - P_2' \quad , \qquad L_\lambda = \epsilon_{\lambda\mu\nu\rho} M_\mu K_\nu \Delta_\rho \qquad (3·8)$$

with

$$M^2 = -s, \quad K^2 = -\bar{t}, \quad \Delta^2 = -t, \quad L^2 = -s\bar{t}t \qquad (3\cdot9)$$

Here $\epsilon_{\lambda\mu\nu\rho}$ is the Levi-Civita symbol. We can now write each of the γ_μ's in equation (3·6) as

$$\gamma_\mu = -\frac{1}{s}(\gamma\cdot M)M_\mu - \frac{1}{\bar{t}}(\gamma\cdot K)K_\mu - \frac{1}{t}(\gamma\cdot\Delta)\Delta_\mu - \frac{1}{s\bar{t}t}(\gamma\cdot L)L_\mu \qquad (3\cdot10)$$

Noticing that[7]

$$(\gamma\cdot L) = \frac{1}{4}\left\{(\gamma\cdot\Delta), [(\gamma\cdot K),(\gamma\cdot M)]_-\right\}_+\gamma_5$$

$$= (\gamma\cdot K)(\gamma\cdot M)(\gamma\cdot\Delta)\gamma_5 \qquad (3\cdot11)$$

where the minus denotes the commutator and plus the anti-commutator, we can express J_2, J_3 and J_4 in terms of a set of basic matrices analogue to those used by Goldberger, Nambu and Oehme[8]. Defining

$$G_1 = P_1 \quad ; \quad G_2 = P_2 \, ,$$

$$G_3 = P_3 \, , \quad G_4 = P_4 \, ,$$

$$G_5 = \gamma_5^\Lambda (\gamma\cdot N)\gamma_5^N (\gamma\cdot P) \, , \quad G_6 = P_6 \qquad (3\cdot12)$$

where

$$P = \frac{1}{2}(P_1 + P_1') \, ,$$

$$N = \frac{1}{2}(P_2 + P_2') \qquad (3\cdot13) \, ,$$

we find

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7) M.L.Goldberger, Y. Nambu and R. Oehme, Ann.Phys., 2 , 226 (1957).

$$J_2 = \frac{\bar{t}-s}{s\bar{t}} m_\Lambda m_N G_1 - \frac{\bar{t}+s}{s\bar{t}} m_N G_2 - \frac{\bar{t}+s}{s\bar{t}} m_\Lambda G_3$$

$$+ \frac{\bar{t}-s}{s\bar{t}} G_4 + \frac{t}{s\bar{t}} G_5 - \frac{4 m_\Lambda m_N (m_\Lambda^2 - m_N^2)^3}{s\bar{t}t} G_6,$$

$$J_3 = \frac{1}{s\bar{t}t} \left\{ (\bar{t}-s) \left[(m_\Lambda^2 + m_N^2)^2 - s\bar{t} \right] - (m_\Lambda^2 - m_N^2)^2 t \right\} G_1$$

$$- \frac{2m_\Lambda}{s\bar{t}t} \left[(m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2s\bar{t} \right] G_2$$

$$- \frac{2m_N}{s\bar{t}t} \left[(m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2s\bar{t} \right] G_3$$

$$+ \frac{4m_\Lambda m_N}{s\bar{t}t} (\bar{t}-s) G_4 + \frac{4}{s\bar{t}} m_\Lambda m_N G_5$$

$$+ \frac{1}{st} \left[s^2 - (m_\Lambda^2 - m_N^2)^2 \right] G_6,$$

$$J_4 = - \frac{m_\Lambda m_N}{s\bar{t}t} (s-\bar{t})^2 G_1 + \frac{m_N}{s\bar{t}t} (s-\bar{t})(t-4m_\Lambda^2) G_2$$

$$+ \frac{m_\Lambda}{s\bar{t}t} (s-\bar{t})(t-4m_N^2) G_3 + \frac{1}{s\bar{t}t} \left[4(m_\Lambda^2 - m_N^2)^2 - (s+\bar{t})^2 \right] G_4$$

$$+ \frac{s-\bar{t}}{s\bar{t}} G_5 + \frac{4 m_\Lambda m_N}{t} G_6 \qquad (3\cdot14a)$$

A check on the correctness of these equations is provided by the fact that if we set $m_\Lambda = m_N$, the coefficients of $(G_2 - G_3)$ in these equations vanish as required by charge symmetry[7]. Further in this limit, these reduce to the equations appropriate to the nucleon-nucleon problem

and given in ref. [1] Equation (3.14a) together with the definitions (3.2), (3.6) and (3.12) now gives G_5 in terms of the J_i's:

$$G_5 = \frac{\alpha_1}{\beta} J_1 + \frac{\alpha_2}{\beta} J_2 + \frac{\alpha_3}{\beta} J_3 + \frac{\alpha_5}{\beta} J_5 + \frac{\alpha_6}{\beta} J_6 \qquad (3.14b)$$

where

$$\alpha_1 = \frac{a_1}{b_1} - \frac{a_2}{b_2},$$

$$\alpha_2 = \frac{a_3}{b_2} - \frac{a_4}{b_1},$$

$$\alpha_3 = \frac{a_3}{b_1} - \frac{a_5}{b_2},$$

$$\alpha_5 = \frac{a_6}{b_1} - \frac{a_7}{b_2},$$

$$\alpha_6 = \frac{a_8}{b_1} - \frac{a_9}{b_2},$$

$$\beta = \frac{c_1}{b_2} - \frac{c_2}{b_1} \qquad (3.14c)$$

with

$$a_1 = \frac{1}{(s\bar{t}t)^2}\Big[m_\Lambda (s-\bar{t})(t-4m_N^2)\Big\{(\bar{t}-s)\Big[(m_\Lambda^2+m_N^2)^2 - s\bar{t}\Big] - (m_\Lambda^2-m_N^2)^2 t\Big\}$$
$$- 2m_\Lambda m_N^2 (s-\bar{t})^2\Big\{(m_\Lambda^2+m_N^2)(s+\bar{t}) - 2s\bar{t}\Big\}\Big],$$

$$a_2 = \frac{1}{(s\bar{t})^2 t}\Big[m_\Lambda (\bar{t}+s)\Big\{(\bar{t}-s)\Big[(m_\Lambda^2+m_N^2)^2 - s\bar{t}\Big] - (m_\Lambda^2-m_N^2)t\Big\}$$
$$- 2m_\Lambda m_N^2 (\bar{t}-s)\Big\{(m_\Lambda^2+m_N^2)(s+\bar{t}) - 2s\bar{t}\Big\}\Big],$$

$$a_3 = -\frac{2m_N}{s\bar{t}t}\Big[(m_\Lambda^2+m_N^2)(s+\bar{t}) - 2s\bar{t}\Big],$$

$$a_4 = \frac{m_\Lambda}{s\bar{t}t}(s-\bar{t})(t-4m_N^2),$$

$$a_5 = -\frac{\bar{t}+s}{s\bar{t}}\, m_\Lambda ,$$

$$a_6 = \frac{1}{(st)^2\bar{t}}\Big[m_\Lambda \Big\{ s^2 - (m_\Lambda^2 - m_N^2)^2 \Big\} (s-\bar{t})(t-4m_N^2)$$

$$+ 8 m_\Lambda m_N^2 s \Big\{ (m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2s\bar{t} \Big\} \Big] ,$$

$$a_7 = \frac{1}{(s\bar{t}t)^2}\Big[8 m_\Lambda m_N^2 (m_\Lambda^2 - m_N^2)^2 \Big\{ (m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2s\bar{t} \Big\}$$

$$+ m_\Lambda \bar{t} t (s+\bar{t}) \Big\{ s^2 - (m_\Lambda^2 - m_N^2)^2 \Big\} \Big] ,$$

$$a_8 = \frac{1}{(s\bar{t})^2 t}\, 2(m_N^2 - m_\Lambda^2)(s-\bar{t}) \Big[(m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2s\bar{t} \Big] ,$$

$$a_9 = \frac{1}{(s\bar{t})^2 t}\, 2(m_N^2 - m_\Lambda^2)(s+\bar{t}) \Big[(m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2s\bar{t} \Big] ,$$

$$b_1 = \frac{1}{(s\bar{t}t)^2}\Big[4 m_\Lambda^2 m_N (s-\bar{t})^2 (4m_N^2 - t) +$$

$$+ 2 m_N \Big\{ (m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2s\bar{t} \Big\} \Big\{ 4(m_\Lambda^2 - m_N^2)^2 - (s+\bar{t})^2 \Big\} \Big] ,$$

$$b_2 = \frac{1}{(s\bar{t})^2 t}\Big[2 m_N (m_\Lambda^2 - m_N^2)(\bar{t}-s)^2 + 4 m_N s\bar{t}(\bar{t}-s) \Big] ,$$

$$c_1 = \frac{1}{(s\bar{t})^2}\Big[2 m_N (m_\Lambda^2 - m_N^2)(s+\bar{t}) + 4 m_N s\bar{t} \Big] ,$$

$$c_2 = \frac{s-\bar{t}}{(s\bar{t})^2 t}\left[8 m_\Lambda^2 m_N (m_\Lambda^2 - m_N^2) - 4 m_N s\bar{t} + 2(s+\bar{t}) m_N (m_N^2 - m_\Lambda^2)\right] \quad (3\cdot14\,d)$$

We also have

$$J_3 = \sum_{i=1}^{6} \lambda_i P_i$$

where

$$\lambda_1 = \frac{1}{\left[s\bar{t} - 4 m_\Lambda m_N \frac{\alpha_3}{\beta}\right]}\left[\frac{(\bar{t}-s)\left[(m_\Lambda^2+m_N^2)^2 - s\bar{t}\right] - (m_\Lambda^2 - m_N^2)^2 t}{t} + 4 m_\Lambda m_N \frac{\alpha_3}{\beta}\right],$$

$$\lambda_2 = \frac{2 m_\Lambda}{\left[s\bar{t} - 4 m_\Lambda m_N \frac{\alpha_3}{\beta}\right]}\left[\frac{2s\bar{t} - (m_\Lambda^2+m_N^2)(s+\bar{t})}{t} + \frac{\alpha_6}{\beta} 2 m_N\right],$$

$$\lambda_3 = -\frac{2 m_N}{t\left[s\bar{t} - 4 m_\Lambda m_N \frac{\alpha_3}{\beta}\right]}\left[(m_\Lambda^2 + m_N^2)(s+\bar{t}) - 2 s\bar{t}\right],$$

$$\lambda_4 = \frac{4 m_\Lambda m_N (\bar{t}-s)}{t\left[s\bar{t} - 4 m_\Lambda m_N \frac{\alpha_3}{\beta}\right]},$$

$$\lambda_5 = \frac{4 m_\Lambda m_N}{\left[s\bar{t} - 4 m_\Lambda m_N \frac{\alpha_3}{\beta}\right]}\frac{\alpha_2}{\beta},$$

$$\lambda_6 = \frac{1}{\left[s\bar{t} - 4 m_\Lambda m_N \frac{\alpha_3}{\beta}\right]}\left[4 m_\Lambda m_N \frac{\alpha_5}{\beta} + \frac{\bar{t}\left[s^2 - (m_\Lambda^2 - m_N^2)^2\right]}{t}\right] \quad (3\cdot14\,e)$$

Using equations (3·14) , we can now express the $\bar{F}_i^I$'s in terms of the F_i^I's :

$$\bar{F}_1^I = F_1^I + \lambda_1 F_3^I + \left[-\frac{m_\Lambda m_N}{s\bar{t}t}(s-\bar{t})^2 + \frac{s-\bar{t}}{s\bar{t}}\left(\frac{\alpha_1}{\beta} + \frac{\alpha_3}{\beta}\lambda_1\right)\right]F_4^I,$$

$$\bar{F}_2^I = \lambda_2 F_3^I + \left[\frac{m_N}{s\bar{t}t}(s-\bar{t})(t-4m_\Lambda^2) + \frac{s-\bar{t}}{s\bar{t}}\left(\frac{\alpha_6}{\beta} + \lambda_2\frac{\alpha_3}{\beta}\right)\right]F_4^I + F_6^I,$$

$$\bar{F}_3^I = \lambda_3 F_3^I + \left[\frac{m_\Lambda}{s\bar{t}t}(s-\bar{t})(t-4m_N^2) + \frac{s-\bar{t}}{s\bar{t}} \frac{\alpha_3}{\beta} \lambda_3 \right] F_4^I,$$

$$\bar{F}_4^I = \lambda_4 F_3^I + \left[\frac{4(m_\Lambda^2-m_N^2)^2-(s+\bar{t})^2}{s\bar{t}t} + \frac{s-\bar{t}}{s\bar{t}} \frac{\alpha_3}{\beta} \lambda_4 \right] F_4^I,$$

$$\bar{F}_5^I = F_2^I + \frac{4}{s\bar{t}} m_\Lambda m_N \left(\frac{\alpha_2}{\beta} + \frac{\alpha_3}{\beta} \lambda_5 \right) F_3^I + \frac{s-\bar{t}}{s\bar{t}} \left(\frac{\alpha_2}{\beta} + \frac{\alpha_3}{\beta} \lambda_5 \right) F_4^I,$$

$$\bar{F}_6^I = F_5^I + \left[\frac{1}{st} \left\{ s^2 - (m_\Lambda^2-m_N^2)^2 \right\} + \frac{4}{s\bar{t}} m_\Lambda m_N \left\{ \frac{\alpha_5}{\beta} + \frac{\alpha_6}{\beta} \lambda_6 \right\} \right] F_3^I$$

$$+ \left[\frac{4 m_\Lambda m_N}{t} + \frac{s-\bar{t}}{s\bar{t}} \left(\frac{\alpha_3}{\beta} \lambda_6 + \frac{\alpha_5}{\beta} \right) \right] F_4^I \qquad (3\cdot15a)$$

These equations may be compactly written in the form

$$\bar{F}_i^I(s,\bar{t},t) = R_{iJ}(s,\bar{t},t) F_i^I(s,\bar{t},t) \qquad (3\cdot15b)$$

where $R_{iJ}(s,\bar{t},t)$ is the appropriate matrix indicated by equation (3·15a). Notice also that using the above equations, it may be shown that the F_i's have kinematical singularities at the zeros of the polynomial

$$\gamma(s,\bar{t},t) = (m_N^2-m_\Lambda^2)\, s\bar{t}(s+\bar{t}) + s(m_\Lambda^2-m_N^2)\, s\bar{t} + (m_\Lambda^2-m_N^2)^2 (s+\bar{t}) - 2(s\bar{t})^2 \qquad (3\cdot16)$$

Such singularities are absent for the corresponding set of amplitudes in the $N-N$ case.[1), 2)]

Turning now to reaction II, let us denote the moments of the incident Λ and $\bar{N}$ by P_1 and P_2 and those of the outgoing Λ and $\bar{N}$ by P_1' and P_2' respectively. The Feynman amplitude for reaction II can then be written as

$$\mathcal{J}^{II}(\bar{t},s,t) = F_1^{II}(\bar{t},s,t)\,\bar{u}_\Lambda(P_1')\,u_\Lambda(P_1)\,\bar{u}_N(P_2')\,u_N(P_2)$$
$$+ F_2^{II}(\bar{t},s,t)\,\bar{u}_\Lambda(P_1')\,\gamma_\mu\,u_\Lambda(P_1)\,\bar{u}_N(P_2')\,\gamma_\mu\,u_N(P_2)$$
$$+ F_3^{II}(\bar{t},s,t)\,\tfrac{1}{2}\,\bar{u}_\Lambda(P_1')\,\sigma_{\mu\nu}\,u_\Lambda(P_1)\,\bar{u}_N(P_2')\,\sigma_{\mu\nu}\,u_N(P_2)$$
$$+ F_4^{II}(\bar{t},s,t)\,\bar{u}_\Lambda(P_1')\,i\gamma_5\gamma_\mu\,u_\Lambda(P_1)\,\bar{u}_N(P_2')\,i\gamma_5\gamma_\mu\,u_N(P_2)$$
$$+ F_5^{II}(\bar{t},s,t)\,\bar{u}_\Lambda(P_1')\,\gamma_5\,u_\Lambda(P_1)\,\bar{u}_N(P_2')\,\gamma_5\,u_N(P_2)$$
$$+ F_6^{II}(\bar{t},s,t)\,i(P_{2\mu}'-P_{1\mu}')\,\bar{u}_\Lambda(P_1')\,\gamma_\mu\,u_\Lambda(P_1)\,\bar{u}_N(P_2')\,u_N(P_2) \qquad (3.17)$$

For reaction III, let P_1 and P_2 denote the incident $\bar{\Lambda}$ and Λ momenta and P_1' and P_2' those of the outgoing N and $\bar{N}$ respectively. Then if $\mathcal{J}^{III}$ denotes the amplitude for transition in the state with isotopic spin zero, we write

$$-\frac{1}{\sqrt{2}}\,\mathcal{J}^{III}(t,\bar{t},s) = F_1^{III}(t,\bar{t},s)\,\bar{v}_\Lambda(P_1)\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,v_N(P_2')$$
$$+ F_2^{III}(t,\bar{t},s)\,\bar{v}_\Lambda(P_1)\,\gamma_\mu\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,\gamma_\mu\,v_N(P_2')$$
$$+ \tfrac{1}{2}\,F_3^{III}(t,\bar{t},s)\,\bar{v}_\Lambda(P_1)\,\sigma_{\mu\nu}\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,\sigma_{\mu\nu}\,v_N(P_2')$$
$$+ F_4^{III}(t,\bar{t},s)\,\bar{v}_\Lambda(P_1)\,i\gamma_5\gamma_\mu\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,i\gamma_5\gamma_\mu\,v_N(P_2')$$
$$+ F_5^{III}(t,\bar{t},s)\,\bar{v}_\Lambda(P_1)\,\gamma_5\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,\gamma_5\,v_N(P_2')$$
$$+ F_6^{III}(t,\bar{t},s)\,i(P_{2\mu}'-P_{1\mu}')\,\bar{v}_\Lambda(P_1)\,\gamma_\mu\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,v_N(P_2') \qquad (3.18)$$

Here v denotes the negative energy spinor with the normalization $\bar{v}\,v=-1$. We can as before show that the amplitudes

$$\bar{F}_i^{II}(\bar{t},s,t) = R_{ij}(\bar{t},s,t)\,\Delta^{II\,-1}_{jk}\,F^{II}_k(\bar{t},s,t), \quad (3\cdot19a),$$

$$\bar{F}_i^{III}(t,\bar{t},s) = R_{ij}(t,\bar{t},s)\,\Delta^{III\,-1}_{jk}\,F^{III}_k(t,\bar{t},s) \quad (3\cdot19b)$$

are free of kinematical singularities. Here Δ^{II} and Δ^{III} are the crossing matrices for the F_i's which will be derived below. The definitions (3·19a,b) immediately give us the results

$$\bar{F}_i^{II}(\bar{t},s,t) = \bar{\Delta}^{II}_{ij}\,\bar{F}_j^{I}(\bar{t},s,t),$$

$$\bar{F}_i^{III}(t,\bar{t},s) = \bar{\Delta}^{III}_{ij}\,\bar{F}_j^{I}(t,\bar{t},s) \quad (3\cdot20a)$$

where

$$\bar{\Delta}^{II}_{ij} = \bar{\Delta}^{III}_{ij} = \delta_{ij} \quad (3\cdot20b)$$

These equations show in an alternative fashion that $\bar{F}_i^{II}$ and $\bar{F}_i^{III}$ are free of spurious singularities at least in fourth order perturbation theory since we have already shown that $\bar{F}_i^{I}$'s are free of such singularities. Later on when we write down dispersion relations, we shall always maintain a distinction between $\bar{F}_i^{I}$,and $\bar{F}_i^{II}$ and $\bar{F}_i^{III}$. This is done in order that the equations which we derive may be valid also for any other choice of amplitudes.

We shall now proceed to derive the crossing relations between F_i^{I}, F_i^{II} and F_i^{III}. First consider $\mathcal{F}^{I}$ and $\mathcal{F}^{II}$. Let

$$\mathcal{F}^{I}(s,\bar{t},t) = \bar{u}_\Lambda^{\alpha}(P_1')\, T_{\alpha\beta}^{I}(s,\bar{t},t)\, u_\Lambda^{\beta}(P_1) \qquad (3\cdot21a)$$

Application of the asymptotic condition then shows us that[2)]

$$(2\pi)^4 \delta(P_1'+P_2'-P_1-P_2)\, T_{\alpha\beta}^{I}(s,\bar{t},t)$$
$$= i\left[\frac{P_{10}'P_{10}}{m_\Lambda^2}\right]^{1/2}\int d^4x\, d^4y\, e^{-iP_1'x}\left[\gamma_\mu\frac{\overrightarrow{\partial}}{\partial x_\mu}+m_\Lambda\right]_{\alpha\alpha'}\times$$
$$\times\langle P_2'(N)|\,T\left(\psi_\Lambda^{\alpha'}(x)\,\bar{\psi}_\Lambda^{\beta'}(y)\right)|\,P_2(N)\rangle\left[-\tilde{\gamma}_\mu\frac{\overleftarrow{\partial}}{\partial y_\mu}+m_\Lambda\right]_{\beta\beta'}e^{iP_1y} \qquad (3\cdot21b)$$

Here the symbol T denotes the time-ordered product while the symbol N in $|P_2(N)\rangle$ and $|P_2'(N)\rangle$ denotes that it is a nucleon state. Further the field variables of Λ and N will hereafter be denoted by ψ_Λ and ψ_N while the corresponding masses will be denoted by m_Λ and m_N. $\tilde{\gamma}_\mu$ is the transposed γ_μ while the zeroth components of the four-vector momenta as usual correspond to energy.

For $\mathcal{F}^{II}$ also, we have the similar expressions

$$\mathcal{F}^{II}(\bar{t},s,t) = \bar{u}_\Lambda^{\alpha}(P_1')\, T_{\alpha\beta}^{II}(\bar{t},s,t)\, u_\Lambda^{\beta}(P_1) \qquad (3\cdot22)$$

with

$$(2\pi)^4 \delta(P_1' + P_2' - P_1 - P_2)\, T^{II}_{\alpha\beta}(\bar{t}, s, t)$$

$$= \iota \left[\frac{P_{10}\, P_{10}'}{m_\Lambda^2} \right]^{\frac{1}{2}} \int d^4x\, d^4y\, e^{-\iota P_1' x} \left[\gamma_\mu \frac{\vec{\partial}}{\partial x_\mu} + m_\Lambda \right]_{\alpha\alpha'}$$

$$\langle P_2'(\bar{N}) | T\left(\psi_\Lambda^{\alpha'}(x)\, \bar{\psi}_\Lambda^{\beta'}(y) \right) | P_2(\bar{N}) \rangle \left[-\tilde{\gamma}_\mu \frac{\overleftarrow{\partial}}{\partial y_\mu} + m_\Lambda \right]_{\beta\beta'} e^{\iota P_1 y} \qquad (3.23)$$

Now $|P_2(\bar{N})\rangle = \mathcal{C} |P_2(N)\rangle$, $|P_2'(\bar{N})\rangle = \mathcal{C} |P_2'(N)\rangle$ where $\mathcal{C}$ is the unitary operator which effects the charge conjugation of state vectors. It induces the following transformation on the field operators :

$$\mathcal{C}^{-1} \psi_\Lambda(x)\, \mathcal{C} = e\, \bar{\psi}_\Lambda(x),$$

$$\mathcal{C}^{-1} \bar{\psi}_\Lambda(x)\, \mathcal{C} = e^{-1} \psi_\Lambda(x),$$

$$\mathcal{C}^{-1} \psi_N(x)\, \mathcal{C} = \iota \tau_2\, e\, \bar{\psi}_N(x),$$

$$\mathcal{C}^{-1} \bar{\psi}_N(x)\, \mathcal{C} = \iota \tau_2\, e^{-1} \psi_N(x) \qquad (3.24)$$

where e is the unitary and antisymmetric matrix with the property

$$e^{-1} \gamma_\mu\, e = -\tilde{\gamma}_\mu \qquad (3.25)$$

while τ_2 is the usual two-by-two Pauli matrix. Using (3.24) and (3.25), we find from (3.21) and (3.23) that

$$T^{II}(\bar{t}, s, t) = e\, \tilde{T}^{I}(\bar{t}, s, t)\, e^{-1} \qquad (3.26)$$

where the transposition is in the spinor space of the Λ's. (3.26) when worked out gives the following relation between

F_i^{II} and F_i^{I} : or equivalently between $\bar{F}_i^{II}$ and $\bar{F}_i^{I}$.

$$F_i^{II}(\bar{t}, s, t) = \Delta^{II}_{ij} F_j^{III}(\bar{t}, s, t) \qquad (3.27a)$$

with

$$\Delta^{II} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad (3.27b)$$

The derivation of the crossing relations between the F_i^{III} and F_i^{I} proceeds along similar lines. We againwrite

$$\psi'^{III}(t, \bar{t}, s) = \overline{u_N}^{\alpha}(P_1') \, T'^{III}_{\alpha\beta}(t, \bar{t}, s) \, u_\Lambda^{\beta}(P_2) \qquad (3.28)$$

with

$$(2\pi)^4 \delta(P_1' + P_2' - P_1 - P_2) \, T'^{III}_{\alpha\beta}(t, \bar{t}, s)$$

$$= i \left[\frac{P_{10}' P_{20}}{m_N m_\Lambda}\right]^{\frac{1}{2}} \int d^4x \, d^4y \, e^{-iP_1' x} \left[\gamma_\mu \frac{\vec{\partial}}{\partial x_\mu} + m_N\right]_{\alpha\alpha'} \times$$

$$\times \langle P_2'(\bar{N}) | T'\left(\psi_N^{\alpha'}(x) \, \bar{\psi}_\Lambda^{\beta'}(y)\right) | P_1(\bar{\Lambda})\rangle \left[-\tilde{\gamma}_\mu \frac{\overleftarrow{\partial}}{\partial y_\mu} + m_\Lambda\right]_{\beta\beta'} e^{iP_2 y} \qquad (3.29)$$

The dash for ψ'^{III} and T'^{III} denotes that the nucleon-antinucleon state fed in is of the firm $p\bar{p}$ or $n\bar{n}$ and not of the form $\frac{-p\bar{p} - n\bar{n}}{\sqrt{2}}$. We have also suppressed the isotopic spin index of the nucleon in T'^{III} for simplicity.

(3.29) can be reduced to the relation

$$(2\pi)^4 \delta(P_1' + P_2' - P_1 - P_2)\, T'^{\,III}_{\alpha\beta}(t,\bar{t},s)$$

$$= -i\left[\frac{P_{10}' P_{20}}{m_\Lambda m_N}\right]^{\frac{1}{2}} \int d^4x\, d^4y\, e^{iP_2 y}\, C^{-1}_{\beta\beta'} \left[\gamma_\mu \frac{\overrightarrow{\partial}}{\partial y_\mu} + m_\Lambda\right]_{\beta'\beta''} i\tau_2^{IJ} \times$$

$$\times \langle P_2'(N) | T\left(\psi_\Lambda^{\beta''}(y)\, \bar{\psi}_N^{\alpha''}(x)\right) | P_1(\Lambda)\rangle \left[-\tilde{\gamma}_\mu \frac{\overleftarrow{\partial}}{\partial x_\mu} + m_N\right]_{\alpha'\alpha''} \times \qquad (3.30a)$$

$$\times\, e^{-iP_1' x}\, C_{\alpha'\alpha}$$

i.e.

$$T'^{\,III}_{\alpha\beta}(t,\bar{t},s) = -i\tau_2^{IJ}\, C^{-1}_{\beta\beta'}\, T'^{\,I}_{\beta'\alpha'}(t,\bar{t},s)\, C_{\alpha'\alpha} \qquad (3.30b)$$

where i denotes the nucleon and j the antinucleon isotopic spin index and $T'^{\,I}_{\alpha\beta}$ is defined through the relation

$$\mathcal{T}^I(t,\bar{t},s) = \bar{u}_\Lambda^{\alpha}(-P_2)\, T'^{\,I}_{\alpha\beta}(t,\bar{t},s)\, u_N^{\beta}(-P_1') \qquad (3.31)$$

The matrix element of (3.30b) between states with isotopic spin $I = 0$ (with the convention that $\frac{-p\bar{p} - n\bar{n}}{\sqrt{2}}$ denotes the $I = 0$ state) gives

$$-\frac{1}{\sqrt{2}}\, T^{III}_{\alpha\beta}(t,\bar{t},s) = C^{-1}_{\beta\beta'}\, T'^{\,I}_{\beta'\alpha'}(t,\bar{t},s)\, C_{\alpha'\alpha} \qquad (3.32)$$

where $T^{III}_{\alpha\beta}(t,\bar{t},s)$ denotes the $I=0$ amplitude. Using the relations $\bar{u}_\beta C_{\beta\alpha} = -v_\alpha$ and $C^{-1}_{\alpha\beta} u_\beta = \bar{v}_\beta$ and carrying out the indicated operations in (3.32), we obtain

$$-\frac{1}{\sqrt{2}}\,\mathcal{H}^{III}(t,\bar{t},\delta) = -F_1^I(t,\bar{t},\delta)\,\bar{v}_\Lambda(P_1)\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,v_N(P_2')$$
$$-F_2^I(t,\bar{t},\delta)\,\bar{v}_\Lambda(P_1)\,\gamma_\mu\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,\gamma_\mu\,v_N(P_2')$$
$$-F_3^I(t,\bar{t},\delta)\,\tfrac{1}{2}\,\bar{v}_\Lambda(P_1)\,\sigma_{\mu\nu}\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,\sigma_{\mu\nu}\,v_N(P_2')$$
$$-F_4^I(t,\bar{t},\delta)\,\bar{v}_\Lambda(P_1)\,i\gamma_5\gamma_\mu\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,i\gamma_5\gamma_\mu\,v_N(P_2')$$
$$-F_5^I(t,\bar{t},\delta)\,\bar{v}_\Lambda(P_1)\,\gamma_5\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,\gamma_5\,v_N(P_2')$$
$$-F_6^I(t,\bar{t},\delta)\,iP_{2\mu}'\,\bar{v}_\Lambda(P_1)\,\gamma_\mu\,u_\Lambda(P_2)\,\bar{u}_N(P_1')\,v_N(P_2') \qquad (3\cdot33)$$

Therefore

$$F_i^{III}(t,\bar{t},\delta) = \Delta_{iJ}^{III}\,F_J^I(t,\bar{t},\delta) \qquad (3\cdot34\text{a})$$

where

$$\Delta^{III} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \qquad (3\cdot34\text{b})$$

Later we shall have occasion to actually evaluate the matrix elements occurring in (3·7), (3·17) and (3·33). For convenience of this evaluation, we shall bring the first five terms of (3·33) into a form analogous to those in (3·7) and (3·17). Let S, V, T, A, P denote the scalar, vector, tensor, axial vector and pseudoscalar Fermi matrix elements in the ordering $\bar{\psi}_A\,\mathcal{J}_i\,\psi_b\,\bar{\psi}_c\,\mathcal{J}_i\,\psi_D$ (where i is not summed). Further let $\tilde{S}$, $\tilde{V}$, $\tilde{T}$, $\tilde{A}$, $\tilde{P}$ denote the corresponding invariants in the ordering $\bar{\psi}_A\,\mathcal{J}_i\,\psi_D\,\bar{\psi}_c\,\mathcal{J}_i\,\psi_B$. We then have

the familar result [8]

$$\begin{bmatrix} \tilde{S} \\ \tilde{V} \\ \tilde{T} \\ \tilde{A} \\ \tilde{P} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} S \\ V \\ T \\ A \\ P \end{bmatrix} \qquad (3.36)$$

If we make this transformation on the first five terms of (3.33), we find that it brings the matrix elements in question into the form $\bar{v}_\Lambda J_i v_N \bar{u}_N J_i u_\Lambda$. Using the relation between the u's and v's which we quoted a moment before, we find

$$-\frac{1}{\sqrt{2}} \mathcal{H}^{III}(t,\bar{t},s) = F_1'^{III}(t,\bar{t},s)\, \bar{u}_N(P_2')\, u_\Lambda(P_1)\, \bar{u}_N(P_1')\, u_\Lambda(P_2)$$
$$+ F_2'^{III}(t,\bar{t},s)\, \bar{u}_N(P_2')\, \gamma_\mu u_\Lambda(P_1)\, \bar{u}_N(P_1')\, \gamma_\mu u_\Lambda(P_2)$$
$$+ F_3'^{III}(t,\bar{t},s)\, \tfrac{1}{2}\, \bar{u}_N(P_2')\, \sigma_{\mu\nu} u_\Lambda(P_1)\, \bar{u}_N(P_1')\, \sigma_{\mu\nu} u_\Lambda(P_2)$$
$$+ F_4'^{III}(t,\bar{t},s)\, \bar{u}_N(P_2')\, i\gamma_5 \gamma_\mu u_\Lambda(P_1)\, \bar{u}_N(P_1')\, i\gamma_5 \gamma_\mu u_\Lambda(P_2)$$
$$+ F_5'^{III}(t,\bar{t},s)\, \bar{u}_N(P_2')\, \gamma_5 u_\Lambda(P_1)\, \bar{u}_N(P_1')\, \gamma_5 u_\Lambda(P_2)$$
$$+ F_6'^{III}(t,\bar{t},s)\, iP_{2\mu}'\, \bar{v}_\Lambda(P_1)\, \gamma_\mu u_\Lambda(P_2)\, \bar{u}_N(P_1')\, v_N(P_2') \qquad (3.37)$$

where

$$F_i'^{III}(t,\bar{t},s) = \bar{\Delta}'^{III}_{iJ}\, F_J^{I}(t,\bar{t},s) \qquad (3.38a)$$

with

8) M. Fierz, Zeit fur Phys., 104, 553 (1937).

with

$$\Delta^{III} = \begin{bmatrix} \frac{1}{4} & 1 & \frac{3}{2} & 1 & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -1 & \frac{3}{2} & -1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (3.38\,b)$$

The Feynman amplitude $\mathcal{J}^I$ is related to the differential cross-section for reaction I through the relation

$$\frac{d\sigma^I}{d\Omega} = \frac{m_\Lambda^2 m_N^2}{(2\pi)^2 (E_\Lambda + E_N)^2} \left| \mathcal{J}^I \right|^2 \quad (3.39)$$

where E_Λ and E_N denote the energy of Λ and N respectively.

The above considerations refer to a linear momentum representation of the initial and final states and we have now to go over to an angular momentum representation to make a partial wave analysis of the scattering process. For this, it is convenient to use the helicity amplitudes of Jacob and Wick[3] rather than the conventional amplitudes involving the specification of the spin directions along the y -axis. These amplitudes are related to the differential cross-section through

$$\frac{d\sigma^I}{d\Omega} = \left| \langle \lambda_1' \lambda_2' \left| \varphi^I \right| \lambda_1 \lambda_2 \rangle \right|^2 \quad (3.40)$$

It follows that

$$f^I = 2\pi \frac{E_\Lambda + E_N}{m_\Lambda m_N} \langle \lambda_1' \lambda_2' | \varphi^I | \lambda_1 \lambda_2 \rangle \quad (3.41)$$

Here λ_1 and λ_1' denote the helicities of the incoming and outgoing Λ respectively while λ_2 and λ_2' denote those of N. The expansion of $\langle \lambda_1' \lambda_2' | \varphi^I | \lambda_1 \lambda_2 \rangle$ in terms of amplitudes for transition between states with definite total angular momentum J and definite helicities reads

$$\langle \lambda_1' \lambda_2' | \varphi^I | \lambda_1 \lambda_2 \rangle = \frac{1}{k} \sum_J (2J+1) \langle \lambda_1' \lambda_2' | T_J^I(W) | \lambda_1 \lambda_2 \rangle \times \times d^J_{\lambda\lambda'}(\theta) \quad (3.42)$$

where $W = E_\Lambda + E_N$ and $d^J_{\lambda\lambda'}(\theta)$ is the reduced rotation matrix with $\lambda = \lambda_1 - \lambda_2$ and $\lambda' = \lambda_1' - \lambda_2'$.

The definition (3.42) of $\langle \lambda_1' \lambda_2' | T_J^I(W) | \lambda_1 \lambda_2 \rangle$ is in conformity with that of ref. [2]) $d^J_{\lambda\lambda'}(\theta)$ has the following

useful symmetry properties[9]

$$d^J_{\lambda\lambda'}(\theta) = d^J_{-\lambda-\lambda'}(\theta) = (-1)^{\lambda-\lambda'} d^J_{\lambda'\lambda}(\theta) \qquad (3.43)$$

Now time-reversal invariance implies that the matrix $\langle \lambda_1' \lambda_2' | T^I_J(w) | \lambda_1 \lambda_2 \rangle$ is symmetric in the space of the λ's while parity conservation implies that

$$\langle \lambda_1' \lambda_2' | T^I_J(w) | \lambda_1 \lambda_2 \rangle = \langle -\lambda_1' - \lambda_2' | T^I_J(w) | -\lambda_1 - \lambda_2 \rangle \qquad (3.44)$$

Consistent with these symmetry properties we can then choose the following six linearly independent φ^I's :

$$\varphi^I_1 = \langle +\tfrac{1}{2} +\tfrac{1}{2} | \varphi^I | +\tfrac{1}{2} +\tfrac{1}{2} \rangle = \frac{1}{k} \sum_J (2J+1) \langle +\tfrac{1}{2} +\tfrac{1}{2} | T^I_J(w) | +\tfrac{1}{2} +\tfrac{1}{2} \rangle d^J_{00}(\theta),$$

$$\varphi^I_2 = \langle +\tfrac{1}{2} +\tfrac{1}{2} | \varphi^I | -\tfrac{1}{2} -\tfrac{1}{2} \rangle = \frac{1}{k} \sum_J (2J+1) \langle +\tfrac{1}{2} +\tfrac{1}{2} | T^I_J(w) | -\tfrac{1}{2} -\tfrac{1}{2} \rangle d^J_{00}(\theta),$$

$$\varphi^I_3 = \langle +\tfrac{1}{2} -\tfrac{1}{2} | \varphi^I | +\tfrac{1}{2} -\tfrac{1}{2} \rangle = \frac{1}{k} \sum_J (2J+1) \langle +\tfrac{1}{2} -\tfrac{1}{2} | T^I_J(w) | +\tfrac{1}{2} -\tfrac{1}{2} \rangle d^J_{11}(\theta),$$

$$\varphi^I_4 = \langle +\tfrac{1}{2} -\tfrac{1}{2} | \varphi^I | -\tfrac{1}{2} +\tfrac{1}{2} \rangle = \frac{1}{k} \sum_J (2J+1) \langle +\tfrac{1}{2} -\tfrac{1}{2} | T^I_J(w) | -\tfrac{1}{2} +\tfrac{1}{2} \rangle d^J_{-11}(\theta),$$

$$\varphi^I_5 = \langle +\tfrac{1}{2} +\tfrac{1}{2} | \varphi^I | +\tfrac{1}{2} -\tfrac{1}{2} \rangle = \frac{1}{k} \sum_J (2J+1) \langle +\tfrac{1}{2} +\tfrac{1}{2} | T^I_J(w) | +\tfrac{1}{2} -\tfrac{1}{2} \rangle d^J_{10}(\theta),$$

$$\varphi^I_6 = \langle +\tfrac{1}{2} +\tfrac{1}{2} | \varphi^I | -\tfrac{1}{2} +\tfrac{1}{2} \rangle = \frac{1}{k} \sum_J (2J+1) \langle +\tfrac{1}{2} +\tfrac{1}{2} | T^I_J(w) | -\tfrac{1}{2} +\tfrac{1}{2} \rangle d^J_{-10}(\theta)$$

---- (3.45)

The first five amplitudes in (3.45) correspond to the choice of ref.[2] for the nucleon-nucleon problem. φ^I_5 and φ^I_6 however become degenerate for nucleon-nucleon scattering.

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9) See for example, M.E.Rose, Elementary Theory of Angular Momentum, John Wiley (1957), page 54.

To establish the relation between the Φ_i^I and F_i^I we proceed as in ref.2 . The γ -matrices used are given in terms of two sets of 2 x 2 Pauli matrices ρ_i and σ_i through the relations

$$\gamma_i = \rho_2 \times \sigma_i \ (i=1,2,3),\ \gamma_4 = \rho_3 \times 1,\ \gamma_5 = -\rho_1 \times 1 \qquad (3.46)$$

(This is actually the Dirac representation of the γ's). The Dirac spinors in this representation read[2)]

$$u_{\lambda_1} = \frac{1}{N_1}\begin{bmatrix} E_\Lambda + m_\Lambda \\ 2k\lambda_1 \end{bmatrix} \chi_{\lambda_1},\quad u_{\lambda_2} = \frac{1}{N_2}\begin{bmatrix} E_N + m_N \\ 2k\lambda_2 \end{bmatrix} \chi_{-\lambda_2},$$

$$u_{\lambda_1'} = \frac{1}{N_1}\begin{bmatrix} E_\Lambda + m_\Lambda \\ 2k\lambda_1' \end{bmatrix} e^{-i\sigma_y\theta/2} \chi_{\lambda_1'},\ u_{\lambda_2'} = \frac{1}{N_2}\begin{bmatrix} E_N + m_N \\ 2k\lambda_2' \end{bmatrix} e^{-i\sigma_y\theta/2} \chi_{-\lambda_2'} \qquad (3.47)$$

where $N_1 = [2m_\Lambda(E_\Lambda + m_\Lambda)]^{\frac{1}{2}}$ and $N_2 = [2m_N(E_N + m_N)]^{\frac{1}{2}}$ and χ_λ is an eigenstate of $\frac{1}{2}\sigma_y$ with eigenvalue λ . The matrix elements we require are listed below[2)]:

$$\bar{u}_{\lambda'} u_\lambda = \frac{1}{2m}\left[(E+m) - 4\lambda\lambda'(E-m)\right]\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\theta/2} \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \gamma_k u_\lambda = -i\frac{k}{m}(\lambda+\lambda')\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\theta/2}\sigma_k \chi_{\pm\lambda}\right] \ (k=1,2,3),$$

$$\bar{u}_{\lambda'} \gamma_4 u_\lambda = \frac{1}{2m}\left[(E+m) + 4\lambda\lambda'(E-m)\right]\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\theta/2} \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \sigma_{ij} u_\lambda = \frac{1}{2m}\left[(E+m) - 4\lambda\lambda'(E-m)\right]\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\theta/2} \sigma_k \chi_{\pm\lambda}\right]$$

$$[i,j,k = 1,2,3 \text{ and cyclic permutations thereof}],$$

$$\bar{u}_{\lambda'} \sigma_{i4} u_\lambda = \frac{k}{m}(\lambda-\lambda')\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\theta/2}\sigma_i \chi_{\pm\lambda}\right] \ (i=1,2,3),$$

$$\bar{u}_{\lambda'} \gamma_5 \gamma_k u_\lambda = -\frac{i}{2m}\left[(E+m) + 4\lambda\lambda'(E-m)\right]\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\theta/2}\sigma_k \chi_{\pm\lambda}\right] \ (k=1,2,3),$$

$$\bar{u}_{\lambda'} \gamma_5 \gamma_4 u_\lambda = \frac{k}{m}(\lambda'+\lambda)\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\theta/2} \chi_{\pm\lambda}\right] \qquad (3.48)$$

$E = E_\Lambda$ or E_N depending on the spinor involved and similarly $m = m_\Lambda$ or m_N. The positive sign in $\chi_{\pm\lambda}$ or $\chi_{\mp\lambda'}$ refers to particle 1 and the negative sign to particle 2. The matrix elements involving χ's read[2]

$$\chi^\dagger_{\lambda'} e^{i\sigma_y\theta/2}\chi_\lambda = |\lambda'+\lambda|\cos\frac{\theta}{2} + (\lambda'-\lambda)\sin\frac{\theta}{2},$$

$$\chi^\dagger_{\lambda'} e^{i\sigma_y\theta/2}\vec{\sigma}\chi_\lambda = (\lambda'+\lambda)\left[\epsilon_z\cos\frac{\theta}{2} + i(\lambda+\lambda')\epsilon_y\sin\frac{\theta}{2} + \epsilon_x\sin\frac{\theta}{2}\right]$$

$$-|\lambda'-\lambda|\left[\epsilon_z\sin\frac{\theta}{2} + i(\lambda'-\lambda)\epsilon_y\cos\frac{\theta}{2} + \epsilon_x\cos\frac{\theta}{2}\right] \qquad (3.49)$$

where the ϵ's are the unit vectors along the three axes. Using the above results, we find the following relations between the φ_i^I and F_i^I :

$$4\pi W\varphi_1^I = m_\Lambda m_N F_1^I + (3k^2 + E_\Lambda E_N)F_2^I - 3m_\Lambda m_N F_3^I - 3E_\Lambda E_N F_4^I$$
$$- m_N(E_\Lambda E_N + k^2)F_6^I + \{m_\Lambda m_N F_1^I + (E_\Lambda E_N - k^2)F_2^I + m_\Lambda m_N F_3^I$$
$$+ E_\Lambda E_N F_4^I - m_N(E_\Lambda E_N + k^2)F_6^I\}\cos\theta,$$

$$4\pi W\varphi_2^I = -E_\Lambda E_N F_1^I - m_\Lambda m_N F_2^I + 3(E_\Lambda E_N + k^2)F_3^I + 3m_\Lambda m_N F_4^I$$
$$- k^2 F_5^I + m_\Lambda E_N^2 F_6^I + \{E_\Lambda E_N F_1^I + m_\Lambda m_N F_2^I + (E_\Lambda E_N + k^2)F_3^I$$
$$+ m_\Lambda m_N F_4^I + k^2 F_5^I - m_\Lambda E_N^2 F_6^I\}\cos\theta,$$

$$4\pi W\varphi_3^I = m_\Lambda m_N F_1^I + (k^2 + E_\Lambda E_N)F_2^I + m_\Lambda m_N F_3^I + E_\Lambda E_N F_4^I$$
$$- (m_\Lambda k^2 + m_N E_N E_\Lambda)F_6^I + \{m_\Lambda m_N F_1^I + (k^2 + E_\Lambda E_N)F_2^I + m_\Lambda m_N F_3^I$$
$$+ E_\Lambda E_N F_4^I - (m_\Lambda k^2 + m_N E_N E_\Lambda)F_6^I\}\cos\theta,$$

$$4\pi W \varphi_4^I = E_\Lambda E_N F_1^I + m_\Lambda m_N F_2^I + (E_\Lambda E_N - k^2) F_3^I$$

$$+ m_\Lambda m_N F_4^I - k^2 F_5^I - m_\Lambda E_N^2 F_6^I + \{ - E_\Lambda E_N F_1^I$$

$$- m_\Lambda m_N F_2^I - (E_\Lambda E_N - k^2) F_3^I - m_\Lambda m_N F_4^I + k^2 F_5^I$$

$$+ m_\Lambda E_N^2 F_6^I \} \cos\theta ,$$

$$4\pi W \varphi_5^I = \{ - m_\Lambda E_N F_1^I - m_N E_\Lambda F_2^I - 3 m_\Lambda E_N F_3^I$$

$$- 3 m_N E_\Lambda F_4^I + (k^2 E_N + E_N^2 E_\Lambda) F_6^I \} \sin\theta ,$$

$$4\pi W \varphi_6^I = \{ m_N E_\Lambda F_1^I + m_\Lambda E_N F_2^I + 3 m_N E_\Lambda F_3^I$$

$$+ 3 m_\Lambda E_N F_4^I - m_\Lambda m_N E_N F_6^I \} \sin\theta \qquad (3.50)$$

To obtain the customary amplitudes involving singlet and triple states we notice that these states can be written down as linear combinations of helicity states. We have[3)]

Singlet state : $\frac{1}{\sqrt{2}} \left\{ |J + \tfrac{1}{2} + \tfrac{1}{2}\rangle - |J - \tfrac{1}{2} - \tfrac{1}{2}\rangle \right\}$

Triplet state : $J=\ell : \frac{1}{\sqrt{2}}\left\{ |J+\frac{1}{2}-\frac{1}{2}\rangle - |J-\frac{1}{2}+\frac{1}{2}\rangle \right\}$

$$J=\ell+1 : \frac{1}{\sqrt{2}}\left\{ |J+\frac{1}{2}+\frac{1}{2}\rangle + |J-\frac{1}{2}-\frac{1}{2}\rangle \right\}$$

$$J=\ell-1 : \frac{1}{\sqrt{2}}\left\{ |J+\frac{1}{2}-\frac{1}{2}\rangle + |J-\frac{1}{2}+\frac{1}{2}\rangle \right\} \qquad (3.51)$$

We thus obtain the following six amplitudes corresponding to transitions for a given ℓ and total ℓS : S

a) Singlet - singlet :

$$f_0^J = \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J+\tfrac{1}{2}+\tfrac{1}{2}\rangle - \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J-\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

b) Triplet - triplet :

$$J=\ell : \quad f_1^J = \langle J+\tfrac{1}{2}-\tfrac{1}{2}|\, T_J^I \,|J+\tfrac{1}{2}-\tfrac{1}{2}\rangle - \langle J+\tfrac{1}{2}-\tfrac{1}{2}|\, T_J^I \,|J-\tfrac{1}{2}+\tfrac{1}{2}\rangle$$

$$J=\ell+1 : f_2^J = \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J+\tfrac{1}{2}+\tfrac{1}{2}\rangle + \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J-\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$J=\ell-1 : f_3^J = \langle J+\tfrac{1}{2}-\tfrac{1}{2}|\, T_J^I \,|J+\tfrac{1}{2}-\tfrac{1}{2}\rangle + \langle J+\tfrac{1}{2}-\tfrac{1}{2}|\, T_J^I \,|J-\tfrac{1}{2}+\tfrac{1}{2}\rangle$$

$$J=\ell+1 \rightleftarrows J=\ell-1 : f_{23}^J = \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J+\tfrac{1}{2}-\tfrac{1}{2}\rangle + \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J-\tfrac{1}{2}+\tfrac{1}{2}\rangle$$

c) Singlet - triplet :

$$J=\ell : f_{01}^J = \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J+\tfrac{1}{2}-\tfrac{1}{2}\rangle - \langle J+\tfrac{1}{2}+\tfrac{1}{2}|\, T_J^I \,|J-\tfrac{1}{2}+\tfrac{1}{2}\rangle$$

... (3.52)

Here we have used the symmetry properties of the scattering amplitudes implied by time-reversal invariance and parity conservation (cf. equation (3·44)). Using the orthogonality of the $d^J_{\lambda\lambda'}$'s :

$$\int_0^\pi d^J_{\lambda\lambda'}(\theta)\, d^{J'}_{\lambda\lambda'}(\theta) \sin\theta\, d\theta = \delta_{JJ'} \frac{2}{2J+1} \qquad (3\cdot 53)$$

we can express the f_i^J's in terms of the φ_i^I's :

$$f_0^J(s) = \tfrac{1}{2} k \int_0^\pi [\varphi_1^I(s,\theta) - \varphi_2^I(s,\theta)]\, d^J_{00}(\theta) \sin\theta\, d\theta,$$

$$f_1^J(s) = \tfrac{1}{2} k \int_0^\pi [\varphi_3^I(s,\theta)\, d^J_{11}(\theta) - \varphi_4^I(s,\theta) d^J_{-11}(\theta)] \sin\theta\, d\theta,$$

$$f_2^J(s) = \tfrac{1}{2} k \int_0^\pi [\varphi_1^I(s,\theta) + \varphi_2^I(s,\theta)]\, d^J_{00}(\theta) \sin\theta\, d\theta,$$

$$f_3^J(s) = \tfrac{1}{2} k \int_0^\pi [\varphi_3^I(s,\theta)\, d^J_{11}(\theta) + \varphi_4^I(s,\theta)\, d^J_{-11}(\theta)] \sin\theta\, d\theta,$$

$$f_{01}^J(s) = \tfrac{1}{2} k \int_0^\pi [\varphi_5^I(s,\theta) + \varphi_6^I(s,\theta)]\, d^J_{10}(\theta) \sin\theta\, d\theta,$$

$$f_{23}^J(s) = \tfrac{1}{2} k \int_0^\pi [\varphi_5^I(s,\theta) - \varphi_6^I(s,\theta)]\, d^J_{10}(\theta) \sin\theta\, d\theta \qquad (3\cdot 54)$$

where in (3·54) , we have used the symmetries of the $d^J_{\lambda\lambda'}$ given by (3·43) .

The above analysis shows us now to make the transition from the Feynman amplitude for process I to the corresponding amplitudes characterized by a given total J . Exactly similar results also hold for the amplitudes in process II as can be seen from the similarity of the forms of equations (3·7) and (3·17) . For process III, of course, the analysis has to be re-done. Let ω be the energy of any one of the particles

in the centre-of-mass system of this reaction. [Notice that in this case, all the particles carry equal energy in the centre-of-mass system.] Let $\vec{p}, \lambda_1$ and $-\vec{p}, \lambda_2$ denote the momentum and helicity of Λ and $\bar{\Lambda}$ respectively. Correspondingly let $\vec{q}$, λ_1' and $-\vec{q}$, λ_2' denote the momentum and helicity of the outgoing N and $\bar{N}$. The differential cross-section is then given by

$$\frac{d\sigma^{III}}{d\Omega} = \frac{m_\Lambda^2 m_N^2}{(2\pi)^2 4\omega^2} \left(\frac{q}{p}\right) |y^{III}|^2$$

$$= |\langle \lambda_1' \lambda_2' | \varphi^{III} | \lambda_1 \lambda_2 \rangle|^2 \qquad (3.55)$$

where $\langle \lambda_1' \lambda_2' | \varphi^{III} | \lambda_1 \lambda_2 \rangle$ denotes the helicity amplitude for the process [3] . (3.55) gives

$$y^{III} = \frac{4\pi\omega}{m_\Lambda m_N} \left(\frac{p}{q}\right)^{\frac{1}{2}} \langle \lambda_1' \lambda_2' | \varphi^{III} | \lambda_1 \lambda_2 \rangle \qquad (3.56)$$

$\langle \lambda_1' \lambda_2' | \varphi^{III} | \lambda_1 \lambda_2 \rangle$ can be expanded in the form

$$\langle \lambda_1' \lambda_2' | \varphi^{III} | \lambda_1 \lambda_2 \rangle = \frac{1}{p} \sum_J (2J+1) \langle \lambda_1' \lambda_2' | T_J^{III}(E) | \lambda_1 \lambda_2 \rangle d^J_{\lambda\lambda'}(\eta) \qquad (3.57)$$

where $E = 2\omega$, $\lambda = \lambda_1 - \lambda_2$ and $\lambda' = \lambda_1' - \lambda_2'$. The basic set of φ_i^{III} can now be chosen as in (3.45) . To establish the connection between the F_i^{III} and φ_i^{III} s, we proceed as in the case of reaction I. Our spinors are

$$u_{\lambda'} = \frac{1}{\left[2m_N(E_N+m_N)\right]^{\frac{1}{2}}} \begin{bmatrix} \omega+m_N \\ 2p\lambda' \end{bmatrix} e^{-i\sigma_y\eta/2}\, \chi_{\pm\lambda'},$$

$$u_{\lambda} = \frac{1}{\left[2m_\Lambda(E_\Lambda+m_\Lambda)\right]^{\frac{1}{2}}} \begin{bmatrix} \omega+m_\Lambda \\ 2p\lambda \end{bmatrix} \chi_{\pm\lambda} \qquad (3.58)$$

(with $E_\Lambda = E_N = \omega$)

where the plus sign in $\chi_{\pm\lambda}$ or $\chi_{\pm\lambda'}$ again refers to particle 1 and the minus sign to particle 2 . We then have

$$\bar{u}_{\lambda'} u_\lambda = \frac{1}{2\sqrt{m_N m_\Lambda}} \left[(\omega+m_N)^{\frac{1}{2}}(\omega+m_\Lambda)^{\frac{1}{2}} - 4\lambda\lambda'(\omega-m_N)^{\frac{1}{2}}(\omega-m_\Lambda)^{\frac{1}{2}}\right] \times$$
$$\times \left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\sigma_R\, \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \gamma_R u_\lambda = -\frac{1}{\sqrt{m_N m_\Lambda}} \left[\lambda'(\omega+m_\Lambda)^{\frac{1}{2}}(\omega-m_N)^{\frac{1}{2}} + \lambda(\omega+m_N)^{\frac{1}{2}}(\omega-m_\Lambda)^{\frac{1}{2}}\right] \times$$
$$\times \left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\sigma_R\, \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \gamma_4 u_\lambda = \frac{1}{2\sqrt{m_N m_\Lambda}} \left[(\omega+m_N)^{\frac{1}{2}}(\omega+m_\Lambda)^{\frac{1}{2}} + 4\lambda\lambda'(\omega-m_N)^{\frac{1}{2}}(\omega-m_\Lambda)^{\frac{1}{2}}\right] \times$$
$$\times \left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\, \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \sigma_{Rg} u_\lambda = \frac{1}{2\sqrt{m_N m_\Lambda}} \left[(\omega+m_N)^{\frac{1}{2}}(\omega+m_\Lambda)^{\frac{1}{2}} - 4\lambda\lambda'(\omega-m_N)^{\frac{1}{2}}(\omega-m_\Lambda)^{\frac{1}{2}}\right] \times$$
$$\times \left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\sigma_R\, \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \sigma_{i4} u_\lambda = \frac{1}{\sqrt{m_N m_\Lambda}} \left[\lambda(\omega-m_\Lambda)^{\frac{1}{2}}(\omega+m_N)^{\frac{1}{2}} - \lambda'(\omega-m_N)^{\frac{1}{2}}(\omega+m_\Lambda)^{\frac{1}{2}}\right] \times$$
$$\times \left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\sigma_i\, \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \gamma_5 \gamma_R u_\lambda = -\frac{1}{2\sqrt{m_N m_\Lambda}} \left[(\omega+m_N)^{\frac{1}{2}}(\omega+m_\Lambda)^{\frac{1}{2}} + 4\lambda\lambda'(\omega-m_N)^{\frac{1}{2}}(\omega-m_\Lambda)^{\frac{1}{2}}\right] \times$$
$$\times \left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\sigma_R\, \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'} \gamma_5 \gamma_4 u_\lambda = \frac{1}{\sqrt{m_N m_\Lambda}} \left[\lambda'(\omega-m_N)^{\frac{1}{2}}(\omega+m_\Lambda)^{\frac{1}{2}} + \lambda(\omega-m_\Lambda)^{\frac{1}{2}}(\omega+m_N)^{\frac{1}{2}}\right] \times$$
$$\times \left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\, \chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda'}\gamma_5 u_\lambda = \frac{1}{\sqrt{m_N m_\Lambda}}\left[\lambda'(\omega-m_N)^{\frac12}(\omega+m_\Lambda)^{\frac12} - \lambda(\omega-m_\Lambda)^{\frac12}(\omega+m_N)^{\frac12}\right]\times$$
$$\times\left[\chi^\dagger_{\pm\lambda'} e^{i\sigma_y\eta/2}\chi_{\pm\lambda}\right],$$

$$\bar{u}_{\lambda_1'} v_{\lambda_2'} = \frac{q}{m_N}(\lambda_1'+\lambda_2')\left[|\lambda_1'-\lambda_2'|\sin\eta - 2\lambda_1'|\lambda_1'+\lambda_2'|\cos\eta\right],$$

$$\bar{v}_{\lambda_2}\gamma_R u_{\lambda_1} = -\frac{1}{2m_\Lambda}\left[(\omega+m_\Lambda) - 4\lambda_1\lambda_2(\omega-m_\Lambda)\right]\left[|\lambda_1-\lambda_2|2\lambda_2\epsilon_x\right.$$
$$\left. + |\lambda_1-\lambda_2|\epsilon_y + 2|\lambda_1+\lambda_2|\epsilon_z\right],$$

$$\bar{v}_{\lambda_2}\gamma_4 u_{\lambda_1} = 0 \qquad (3.59)$$

In evaluating the matrix elements involving v's , use has been made of the fact that the matrix $\mathcal{C}$ of equation (3.25) is given by $\mathcal{C} = i\rho_1\times\sigma_y$ in the Dirac representation. The connection between the $\mathcal{P}_i^{III}$ and $F_i'^{III}$ is now simple to establish. We find

$$-\frac{1}{\sqrt2}\frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac12}\mathcal{P}_1^{III} = F_1'^{III}\frac{1}{4m_\Lambda m_N}\left[\omega^2+m_\Lambda m_N-(\omega^2-m_\Lambda^2)^{\frac12}(\omega^2-m_N^2)^{\frac12}\right]\times$$
$$\times(1+\cos\eta)$$
$$+F_2'^{III}\left\{\frac{1}{2m_\Lambda m_N}\left[2\omega^2-m_\Lambda m_N+2(\omega^2-m_\Lambda^2)^{\frac12}(\omega^2-m_N^2)^{\frac12}\right]+\frac12\cos\eta\right\}$$
$$-F_3'^{III}\frac{1}{4m_\Lambda m_N}\left\{2\omega^2-2(\omega^2-m_\Lambda^2)^{\frac12}(\omega^2-m_N^2)^{\frac12}\right\}(3-\cos\eta)$$
$$+F_4'^{III}\left[-\frac{1}{2m_\Lambda m_N}\left\{2\omega^2+m_\Lambda m_N+2(\omega^2-m_N^2)^{\frac12}(\omega^2-m_\Lambda^2)^{\frac12}\right\}+\frac12\cos\eta\right]$$
$$+F_5'^{III}\frac{1}{4m_\Lambda m_N}\left[\omega^2-m_\Lambda m_N-(\omega^2-m_\Lambda^2)^{\frac12}(\omega^2-m_N^2)^{\frac12}\right](1+\cos\eta)$$
$$+F_6'^{III}\frac{q^2}{m_N}\cos^2\eta,$$

$$-\frac{1}{\sqrt{2}}\ \frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_2^{\overline{III}}$$

$$= -F_1'^{\overline{III}}\ \frac{1}{4m_\Lambda m_N}\left[\omega^2 + m_\Lambda m_N + (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](1-\cos\eta)$$

$$+F_2'^{\overline{III}}\left\{-\frac{1}{2m_\Lambda m_N}\left[2\omega^2 - m_\Lambda m_N - 2(\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right]+\frac{1}{2}\cos\eta\right\}$$

$$+F_3'^{\overline{III}}\left\{+\frac{1}{2m_\Lambda m_N}\left[\omega^2 + (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](3+\cos\eta)\right\}$$

$$+F_4'^{\overline{III}}\left\{\frac{1}{2m_\Lambda m_N}\left[2\omega^2 + m_\Lambda m_N - 2(\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right]+\frac{1}{2}\cos\eta\right\}$$

$$-F_5'^{\overline{III}}\ \frac{1}{4m_\Lambda m_N}\left[\omega^2 - m_\Lambda m_N + (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](1-\cos\eta)$$

$$+F_6'^{\overline{III}}\ \frac{q^2}{m_N}\cos^2\eta,$$

$$-\frac{1}{\sqrt{2}}\ \frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_3^{\overline{III}}$$

$$= F_1'^{\overline{III}}\ \frac{1}{4m_\Lambda m_N}\left[\omega^2 + m_\Lambda m_N - (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](1+\cos\eta)$$

$$+F_2'^{\overline{III}}\ \frac{1}{2m_\Lambda m_N}\left[\omega^2 + (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](1+\cos\eta)$$

$$+F_3'^{\overline{III}}\ \frac{1}{2}(1+\cos\eta)\ +$$

$$+F_4'^{\overline{III}}\ \frac{1}{2m_\Lambda m_N}\left[\omega^2 + (\omega^2-m_N^2)^{\frac{1}{2}}(\omega^2-m_\Lambda^2)^{\frac{1}{2}}\right](1+\cos\eta)$$

$$+F_5'^{\overline{III}}\ \frac{1}{+m_\Lambda m_N}\left[(\omega^2-m_N^2)^{\frac{1}{2}}(\omega^2-m_\Lambda^2)^{\frac{1}{2}} + m_\Lambda m_N - \omega^2\right](1+\cos\eta),$$

$$-\frac{1}{\sqrt{2}}\,\frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_4^{\text{III}}$$

$$= F_1'^{\text{III}}\,\frac{1}{4m_\Lambda m_N}\left[\omega^2 + m_\Lambda m_N + (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](1-\cos\eta)$$

$$+ F_2'^{\text{III}}\,\frac{1}{2m_\Lambda m_N}\left[\omega^2 - (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](1-\cos\eta)$$

$$+ F_3'^{\text{III}}\,\frac{1}{2}(1-\cos\eta)$$

$$+ F_4'^{\text{III}}\,\frac{1}{2m_\Lambda m_N}\left[\omega^2 - (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right](1-\cos\eta)$$

$$+ F_5'^{\text{III}}\,\frac{1}{4m_\Lambda m_N}\left[m_\Lambda m_N - \omega^2 - (\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}\right] \times (1-\cos\eta),$$

$$-\frac{1}{\sqrt{2}}\,\frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_5^{\text{III}}$$

$$= -F_1'^{\text{III}}\,\frac{m_\Lambda + m_N}{4m_\Lambda m_N}\,\omega\sin\eta$$

$$+ F_2'^{\text{III}}\,\frac{m_\Lambda - 2m_N}{2m_\Lambda m_N}\,\omega\sin\eta$$

$$- F_3'^{\text{III}}\,\frac{3}{2}\,\frac{\omega}{m_N}\sin\eta$$

$$- F_4'^{\text{III}}\,\frac{m_\Lambda + 2m_N}{2m_\Lambda m_N}\,\omega\sin\eta$$

$$+ F_5'^{\text{III}}\,\frac{m_N - m_\Lambda}{4m_\Lambda m_N}\,\omega\sin\eta$$

$$- F_6'^{\text{III}}\,\frac{1}{2m_\Lambda m_N}\,q^2\,\omega\sin 2\eta,$$

$$-\frac{1}{\sqrt{2}}\frac{4\pi w}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_6^{III}$$

$$= F_1'^{III}\frac{m_\Lambda + m_N}{4m_\Lambda m_N} w\sin\eta$$

$$+ F_2'^{III}\frac{2m_N - m_\Lambda}{2m_\Lambda m_N} w\sin\eta$$

$$+ F_3'^{III}\frac{3}{2}\frac{w}{m_N}\sin\eta$$

$$+ F_4'^{III}\frac{m_\Lambda + 2m_N}{2m_\Lambda m_N} w\sin\eta$$

$$- F_5'^{III}\frac{m_N - m_\Lambda}{4m_\Lambda m_N} w\sin\eta$$

$$+ F_6'^{III}\frac{1}{2m_\Lambda m_N} q^2 w\sin 2\eta = +\frac{1}{\sqrt{2}}\frac{4\pi w}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_5^{III} \qquad (3.60)$$

Using (3.54b) and (3.58b) we can obtain the expressions for q_i^{III} in terms of F_i^{III}. We have

$$-\frac{1}{\sqrt{2}}\cdot\frac{4\pi w}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_1^{III} = F_1^{III}\frac{1}{m_\Lambda m_N}(\omega^2 - m_\Lambda^2)^{\frac{1}{2}}(\omega^2 - m_N^2)^{\frac{1}{2}}$$

$$- F_2^{III}\cos\eta - F_3^{III}\frac{1}{m_\Lambda m_N}\left[\omega^2 - (\omega^2 - m_\Lambda^2)^{\frac{1}{2}}(\omega^2 - m_N^2)^{\frac{1}{2}}\right]\cos\eta$$

$$- F_4^{III} - F_5^{III}\frac{4}{m_\Lambda m_N}\omega^2 + F_6^{III}\frac{q^2}{m_N}\cos\eta,$$

$$-\frac{1}{\sqrt{2}}\frac{4\pi w}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}} q_2^{III} = F_1^{III}\frac{1}{m_\Lambda m_N}(\omega^2 - m_\Lambda^2)^{\frac{1}{2}}(\omega^2 - m_N^2)^{\frac{1}{2}}$$

$$- F_2^{III}\cos\eta - F_3^{III}\frac{1}{m_\Lambda m_N}\left[\omega^2 + (\omega^2 - m_\Lambda^2)^{\frac{1}{2}}(\omega^2 - m_N^2)^{\frac{1}{2}}\right]\cos\eta$$

$$+ F_4^{III} + F_5^{III}\frac{\omega^2}{m_\Lambda m_N} + F_6^{III}\frac{q^2}{m_N}\cos^2\eta$$

$$-\frac{1}{\sqrt{2}}\frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}}\Phi_3^{III} = -F_2^{III}\frac{\omega^2}{m_\Lambda m_N}(1+\cos\eta) - F_3^{III}(1+\cos\eta)$$

$$+F_4^{III}\frac{1}{m_\Lambda m_N}(\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}(1+\cos\eta),$$

$$-\frac{1}{\sqrt{2}}\frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}}\Phi_4^{III} = -F_2^{III}\frac{\omega^2}{m_\Lambda m_N}(1-\cos\eta)$$

$$-F_3^{III}(1-\cos\eta) - F_4^{III}\frac{1}{m_\Lambda m_N}(\omega^2-m_\Lambda^2)^{\frac{1}{2}}(\omega^2-m_N^2)^{\frac{1}{2}}(1-\cos\eta),$$

$$-\frac{1}{\sqrt{2}}\frac{4\pi\omega}{m_\Lambda m_N}\left(\frac{p}{q}\right)^{\frac{1}{2}}\Phi_5^{III} = \frac{3}{2}F_2^{III}\frac{\omega\sin\eta}{m_\Lambda} + \frac{3}{2}F_3^{III}\frac{\omega\sin\eta}{m_N}$$

$$-F_4^{III}\frac{\omega\sin\eta}{2m_\Lambda} - F_5^{III}\frac{\omega\sin\eta}{2m_N} - F_6^{III}\frac{1}{2m_\Lambda m_N}q^2\omega\sin 2\eta,$$

$$\Phi_6^{III} = -\Phi_5^{III} \qquad (3.61)$$

From the analogues of equations (3.54) for the partial wave amplitudes for this process (which we shall denote by g_i^J), we find that the result $\Phi_6^{III} = -\Phi_5^{III}$ implies that $g_{01}^J = 0$ or the singlet-triplet amplitude vanishes. This as we saw before is a direct consequence of charge conjugation invariance and merely implies that the form for F_i^{III} which has been assumed is charge conjugation invariant.

IV. Partial Wave Dispersion Relations

Before we write down dispersion relations for the appropriate amplitudes, it is necessary to examine how the presence of anomalous thresholds modifies the integral representations of the scattering amplitudes. That there are anomalous thresholds in this problem is already clear from the fact that Λ does not satisfy the superstability condition since $m_\Lambda^2 > m_N^2 + m_K^2$ [10] In order to locate the threshold of the dispersion integrals in this case, we use the results of Karplus, Sommerfield and Wichmann[10]. Thus we consider a general fourth order graph with the variables labelled as in the figure below.

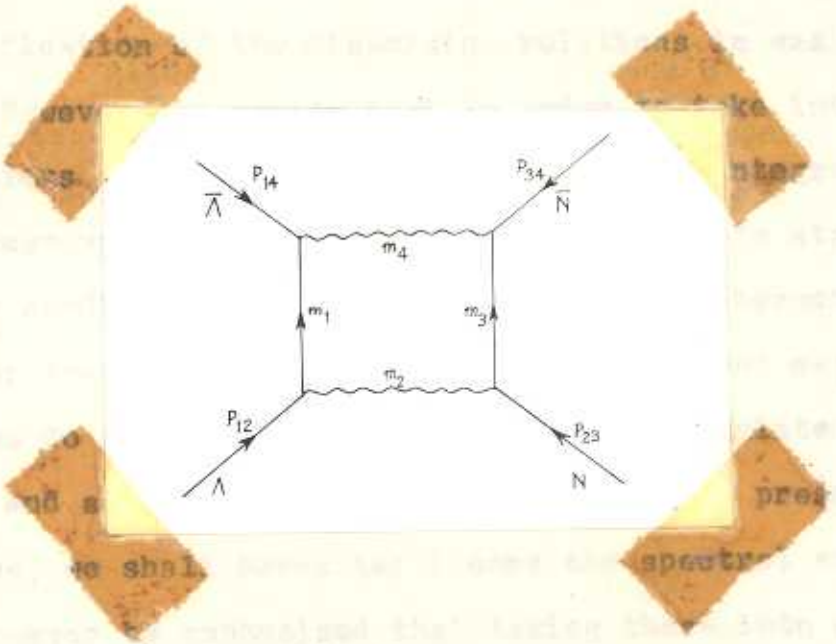

Fig.2.

If we define

$$\gamma_{k\ell} = \frac{m_k^2 + m_\ell^2 + t_{k\ell}^2}{2 m_k m_\ell} \qquad (4.1)$$

the results of ref.[10] show that the thresholds of integration

10) R. Karplus, C.M.Sommerfield and E.M.Wichmann, Phys. Rev., 111, 1187 (1958).

in the dispersion relations are given by

$$\frac{m_1^2+m_3^2-s}{2m_1m_3} = \min.\left\{ y_{12}y_{23} - [(1-y_{12}^2)(1-y_{23}^2)]^{\frac{1}{2}},\ y_{14}y_{34} - [(1-y_{14}^2)(1-y_{34}^2)]^{\frac{1}{2}} \right\},$$

$$\frac{m_2^2+m_4^2-t}{2m_2m_4} = \max.\left\{ y_{12}y_{14} + [(1-y_{12}^2)(1-y_{14}^2)]^{\frac{1}{2}},\ y_{23}y_{34} + [(1-y_{23}^2)(1-y_{34}^2)]^{\frac{1}{2}} \right\} \quad (4\cdot2)$$

An analysis of the fourth order graphs using the above formulae shows that the anomalous threshold makes its appearance only for the graph of Fig. 1 and that the threshold of the s -integration is changed due to its presence from $s = (m_\Lambda + m_N)^2$ to

$$s = (m_\Lambda + m_N)^2 - 0\cdot2008\, m_\Lambda m_N \quad (4\cdot3)$$

This modification of the dispersion relations is easily taken into account. However, we notice that in order to take into account the anomalous contributions to the dispersion integrals consistently, it is necessary to include the $K-\bar{K}$ intermediate state in the unitarity condition in the t -channel. This intermediate state is heavier than seven pion masses and the present calculational techniques do not seem adequate to treat it consistently. Consequently, and also since we wish our formulae to present a neat appearance, we shall hereafter ignore the spectral anomalies. It may however be emphasized that taking these into account in writing down partial wave dispersion relations presents no serious difficulty[11].

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11) See in this connection the work of R. Blankenbecler and Y. Nambu, Nuovo Cimento, 18, 595 (1961) on anamalous thresholds. See also S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

We have shown in Section III that the amplitudes $\bar{F}_i$ are free of kinematical singularities in the fourth order perturbation theory. It will now be assumed that this is generally true and that $\bar{F}_i^{I}$, $\bar{F}_i^{II}$ and $\bar{F}_i^{III}$ have a Mandelstam representation. We have for

$$\bar{F}_i^{I}(s,\bar{t},t) = \frac{1}{\pi^2}\int_{(m_\Lambda+m_N)^2} ds' \int_{(m_K+m_\pi)^2} d\bar{t}' \frac{\rho_{12}^{i}(s',\bar{t}')}{(s'-s)(\bar{t}'-\bar{t})}$$

$$+\frac{1}{\pi^2}\int_{(m_\Lambda+m_N)^2} ds' \int_{4m_\pi^2} dt' \frac{\rho_{13}^{i}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{(m_K+m_\pi)^2} d\bar{t}' \int_{4m_\pi^2} dt' \frac{\rho_{23}^{i}(\bar{t}',t')}{(\bar{t}'-\bar{t})(t'-t)}$$

$$\pm \Delta_{i1}^{II-1} \frac{g_{\Lambda NK}^2}{\bar{t}-m_K^2} \qquad (4\cdot4)$$

The last term represents the K -meson pole contribution and the $\pm$ signs correspond to K mesons of odd and even parity respectively (the odd case being, of course, most probably the true one)[12] The presence of subtractions in (4·4) has been ignored since it does not affect the considerations which follow. From (4·4), one can derive the following one dimensional dispersion relations:

a) Fixeds:

$$\bar{F}_i^{I}(s,\bar{t},t) = \frac{1}{\pi}\Delta_{iJ}^{II-1}\int_{(m_K+m_\pi)^2} d\bar{t}' \frac{\rho_J(\bar{t}',\Sigma-s-\bar{t}')}{\bar{t}'-\bar{t}}$$

$$+\frac{1}{\pi}\Delta_{iJ}^{III-1}\int_{4m_\pi^2} dt' \frac{\sigma_J(t',\Sigma-s-t')}{t'-t} \pm \Delta_{i1}^{II-1}\frac{g_{\Lambda NK}^2}{\bar{t}-m_K^2} \qquad (4\cdot5)$$

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12) Block et. al., Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester, page 419.

where

$$\bar{\Delta}^{II\,-1}_{ij}\; B_j(\bar{t}', \Sigma - s - \bar{t}') = \frac{1}{\pi}\int_{(m_\Lambda+m_N)^2} ds'\, \frac{\rho^{i}_{12}(s', \bar{t}')}{s'-s} + \frac{1}{\pi}\int_{4m_\pi^2} dt'\, \frac{\rho^{i}_{23}(\bar{t}', t')}{t'-t},$$

$$\bar{\Delta}^{III\,-1}_{ij}\; C_j(t', \Sigma - s - t') = \frac{1}{\pi}\int_{(m_\Lambda+m_N)^2} ds'\, \frac{\rho^{i}_{13}(s', t')}{s'-s} + \frac{1}{\pi}\int_{(m_K+m_N)^2} d\bar{t}'\, \frac{\rho^{i}_{23}(\bar{t}', t')}{\bar{t}'+t'+s-\Sigma} \qquad (4\cdot6)$$

b) Fixed $\bar{t}$:

$$\bar{F}^{I}_{i}(s, \bar{t}, t) = \frac{1}{\pi}\int_{(m_\Lambda+m_N)^2} ds'\, \frac{A_i(s', \Sigma - \bar{t} - s')}{s'-s} + \frac{1}{\pi}\,\bar{\Delta}^{III\,-1}_{ij}\int_{4m_\pi^2} dt'\, \frac{C_j(t', \bar{t})}{t'-t} \qquad (4\cdot7)$$

where

$$A_i(s', \Sigma - \bar{t} - s') = \frac{1}{\pi}\int_{4m_\pi^2} dt'\, \frac{\rho^{i}_{13}(s', t')}{t'+s'+\bar{t}-\Sigma} + \frac{1}{\pi}\int_{(m_K+m_K)^2} d\bar{t}'\, \frac{\rho^{i}_{12}(s', \bar{t}')}{\bar{t}'-\bar{t}} \qquad (4\cdot8)$$

From (4·7) , we have the following dispersion relation for $\bar{F}^{II}_{i}$.

$$\bar{F}^{II}_{i}(s, \bar{t}, t) = \frac{1}{\pi}\,\bar{\Delta}^{II}_{ij}\int_{(m_\Lambda+m_N)^2} ds'\, \frac{A_j(s', \Sigma - \bar{t} - s')}{s'-s} + \frac{1}{\pi}\,\bar{\Delta}^{II}_{ik}\,\bar{\Delta}^{III\,-1}_{kj}\int_{4m_\pi^2} dt'\; \times$$

$$\times\; \frac{C_j(t', t)}{t'-t} \pm \frac{g^2_{\Lambda NK}}{\bar{t}-m_K^2} \qquad (4\cdot9)$$

c) Fixed t :

$$\bar{F}^{I}_{i}(s, t, \bar{t}) = \frac{1}{\pi}\int_{(m_\Lambda+m_N)^2} ds'\, \frac{A_i(s', t)}{s'-s} + \frac{1}{\pi}\,\bar{\Delta}^{III\,-1}_{ij}\int_{(m_K+m_K)^2} d\bar{t}'\, \frac{B_j(\bar{t}', t)}{\bar{t}'-\bar{t}}$$

$$\pm\; \bar{\Delta}^{III\,-1}_{i1}\; \frac{g^2_{\Lambda NK}}{\bar{t}-m_K^2} \qquad (4\cdot10)$$

which gives for $\bar{F}^{III}_{i}$:

$$\bar{F}_i^{III}(s,\bar{t},t) = \frac{1}{\pi}\bar{\Delta}^{III}_{ij}\int_{(m_\Lambda+m_N)^2} ds' \frac{A_j(s',t)}{s'-s} + \frac{1}{\pi}\bar{\Delta}^{III}_{ik}\bar{\Delta}^{II-1}_{kj} \times$$

$$\times \int_{(m_K+m_\pi)^2} d\bar{t}' \frac{B_j(\bar{t}',t)}{\bar{t}'-\bar{t}} \pm \bar{\Delta}^{III}_{ik}\bar{\Delta}^{II-1}_{k1}\frac{g^2_{\Lambda NK}}{\bar{t}-m_K^2} \quad (4\cdot11)$$

Here A_i, B_i and C_i are the absorptive parts of $\bar{F}_i^{I}$, $\bar{F}_i^{II}$ and $\bar{F}_i^{III}$ for reactions I, II and III respectively.

a) Reaction I :

A serious difficulty which we encounter in setting up the partial wave dispersion relations for the three processes with which we are concerned is that we are unable to define any amplitude simply related to the partial wave amplitudes introduced in the previous section which have no kinematical singularities and the correct asymptotic behaviour. Thus we are obliged to work with the amplitudes

$$\bar{F}^{I\,J}(s) = \int_{-1}^{1} d\cos\theta\, d^J(\theta)\,\bar{F}^{I}(s,\bar{t},t),$$

$$\bar{F}^{II\,J}(\bar{t}) = \int_{-1}^{1} d\cos\varphi\, d^J(\varphi)\,\bar{F}^{II}(\bar{t},s,t),$$

$$\bar{F}^{III\,J}(t) = \int_{-1}^{1} d\cos\eta\, d^J(\eta)\,\bar{F}^{III}(t,\bar{t},s) \quad (4\cdot12)$$

where the indices i,λ,λ' have been suppressed.

The chief disadvantage in dealing with these amplitudes is that it is the partial wave amplitudes of the last section for which the unitarity condition takes a simple form.

The singularities of $\bar{F}_i^{IJ}(s)$ in the s -plane implied by the Mandelstam representation are easy to investigate and has been discussed at length for example by Frazer and Fulco[5]. The cut in $\bar{F}_i^{I}$ associated with the s -variable clearly gives a cut in the s -plane for $\bar{F}^{IJ}$ in the region

$$s \geq (m_\Lambda + m_N)^2 \qquad (4\cdot13)$$

To obtain the singularities of the partial waves in the s -plane due to the $\bar{t}$- and t- cuts of $\bar{F}_i^{I}$, we follow the method of Frazer and Fulco[5]. Thus the integral relevant for the investigation of the $\bar{t}$- cut reads

$$\int_{-1}^{1} dw\, s\, \theta\, d^J(\theta)\, \frac{1}{\pi}\, \bar{\Delta}_{iJ}^{\pi-1} \int_{(m_K+m_\pi)^2}^{\infty} d\bar{t}'\, \frac{B_J(\bar{t}', \Sigma - s - \bar{t}')}{\bar{t}' - \bar{t}}$$

$$= \frac{1}{2R^2} \int_{\frac{(m_\Lambda^2 - m_N^2)^2}{s}}^{2(m_\Lambda^2+m_N^2)-s} d\bar{t}\, d^J(\theta)\, \frac{1}{\pi}\, \bar{\Delta}_{iJ}^{\pi-1} \int_{(m_K+m_\pi)^2}^{\infty} d\bar{t}'\, \frac{B_J(\bar{t}', \Sigma - s - \bar{t}')}{\bar{t}' - \bar{t}} \qquad (4\cdot14)$$

where we have performed a change of variables using equation (2·7). We now ask ourselves the question : what are the values of s for which the limits of the $\bar{t}$ -integration are such that the denominator of the integrand vanishes? These will then be the values of s for which there is a cut for $\bar{F}_i^{IJ}(s)$ in the s -plane. The procedure can be repeated for the t -cut of $\bar{F}_i^{I}(s)$ also. We list below the singularities of $\bar{F}_i^{IJ}(s)$ obtained inthis fashion.

The K -mesonpole in the $\bar{t}$- variable in $\bar{F}_i^{I}$ maps into the region

$$\frac{(m_\Lambda^2 - m_N^2)^2}{m_K^2} \leq s \leq 2m_\Lambda^2 + 2m_N^2 - m_K^2 \qquad (4\cdot15)$$

and

$$s \leq 0 \qquad (4\cdot16)$$

The cut in the $\bar{t}$- variable gives a cut in the s -plane in the region

$$s \leq 2m_\Lambda^2 + 2m_N^2 - (m_K + m_\pi)^2 = \Sigma - (m_K + m_\pi)^2 \quad (4\cdot17)$$

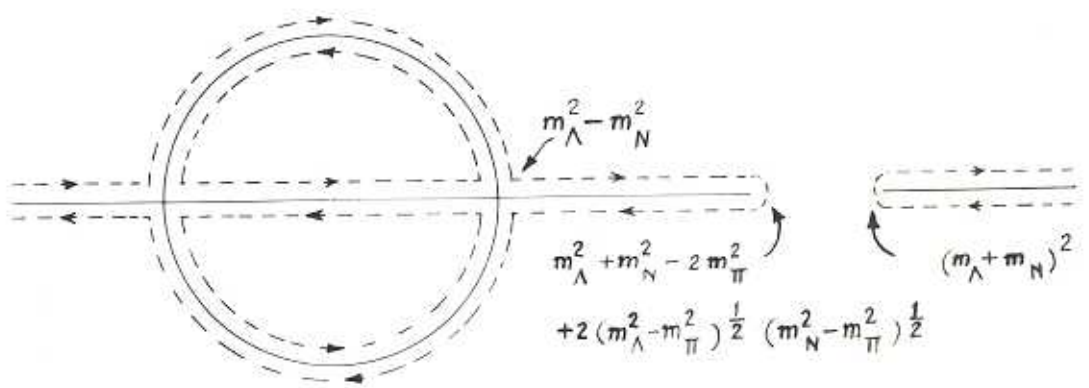
Finally the cut in the t -variable maps into the regions

$$s \leq 0, \quad (4\cdot18a)$$

$$m_\Lambda^2 - m_N^2 \leq s \leq m_\Lambda^2 + m_N^2 + 2(m_\Lambda^2 - m_\pi^2)^{\frac{1}{2}}(m_N^2 - m_\pi^2)^{\frac{1}{2}} - 2m_\pi^2 \quad (4\cdot18b)$$

and a circle of radius $r = m_\Lambda^2 - m_N^2$ centred at the origin in the s -plane.

The nature of the singularities of $\bar{F}_i^{IJ}(s)$ together with the contour of integration to be used is indicated in the figure below:



Contour of integration for $\bar{F}_i^{IJ}(s)$ in the s-plane

Fig. 3

The contribution of the pole term to $\bar{F}_l^{IJ}$ can be calculated explicitly and will not be discussed further.

Along the real axis, the discontinuity across the cut (4·17) due to the vanishing of the denominators involving the $\bar{t}$- variable is given by

$$\text{Im } I_1(s) = -\frac{1}{\pi}\int_{-1}^{1} d\cos\theta \left[\int_{(m_\Lambda+m_N)^2}^{\infty} ds' \frac{\rho_{12}(s',\bar{t})}{s'-s} + \mathcal{P}\int_{4m_\pi^2}^{\infty} dt' \frac{\rho_{23}(\bar{t},t')}{t'-t} \right]$$

$$= -\int_{-1}^{1} d\cos\theta \; d^J(\theta)\, \bar{\Delta}^{I} \, \text{Re } B(\bar{t},t) \qquad (4\cdot19)$$

since $\bar{\Delta}^{I-1} = \bar{\Delta}^{I}$. actually the regions of the cuts (4·17) and (4·18) overlap and $\text{Im } \bar{F}^{IJ}$ would have had contributions from the vanishings of denominators involving both $\bar{t}$ and t . We shall however treat the contributions due to the latter separately. Hence the occurrence of $\text{Im } I_1(s)$ instead of $\text{Im } \bar{F}^{IJ}(s)$ on the left hand side of (4·19).

The contribution of the cut (4·17) to $\bar{F}^{IJ}$ is therefore :

$$I_1(s) = -\frac{1}{\pi}\int_{-\infty}^{\Sigma-(m_K+m_\pi)^2} \frac{ds'}{s'-s} \int_{-1}^{1} d\cos\theta \; \bar{\Delta}^{I} \, d^J(\theta) \, \text{Re } B(\bar{t},t') \qquad (4\cdot20)$$

Using equation (2·7) , the integration over $\cos\theta$ can be converted into an integral over $\bar{t}$:

$$I_1(s) = \frac{1}{\pi}\int_{-\infty}^{\Sigma-(m_K+m_\pi)^2} \frac{ds'}{s'-s} \frac{1}{2k'^2} \int_{\frac{(m_\Lambda^2-m_N^2)^2}{s'}}^{\Sigma-s'} d\bar{t}' \, \bar{\Delta}^{I} \, d^J(\theta) \, \text{Re } B(\bar{t},t') \qquad (4\cdot21)$$

We shall rewrite this integral in the form

$$I_1(s) = \frac{1}{\pi}\int_0^{(m_\Lambda - m_N)^2} \frac{ds'}{s'-s}\,\frac{1}{2k'^2}\int_{(m_\Lambda^2-m_N^2)^2/s'}^{\Sigma - s'} d\bar{t}'\, d^J(\theta')\, \mathrm{Re}\,B(\bar{t}',t') \times$$

$$* \frac{1}{\pi}\left[\int_{-\infty}^{0} + \int_{(m_\Lambda - m_N)^2}^{\Sigma-(m_K+m_\pi)^2}\right]\frac{ds'}{s'-s}\,\frac{1}{2k'^2}\int_{(m_\Lambda^2-m_N^2)^2/s'}^{\Sigma - s'} d\bar{t}' \times$$

$$\times \bar{\Delta}^{III}\, d^J(\theta')\, \mathrm{Re}\, B(\bar{t}',t') \qquad (4\cdot 22)$$

An examination of the first integral in (4·22) reveals that it involves only the physical region of reaction II. This follows since the $\bar{t}'$ integration ranges only over the region $\bar{t}' \geq (m_\Lambda + m_N)^2$ while $s' \leq (m_\Lambda - m_N)^2$. Therefore in this region, $B(\bar{t}', t')$ can be expanded in Legendre polynomials with coefficients involving $\mathrm{Im}\,\bar{F}^{III\,J}$ where $\bar{F}^{III\,J}$ are the analogues of $\bar{F}^{I\,J}$ for process II . A serious complication however arises in this integral in that the lower limit for $\bar{t}'$ becomes infinite for $s' = 0$ permitting contributions to from states of arbitrarily high energy. This point has been emphasized by Frazer and Fulco[3] in the π-N problem and does not arise for the scattering of particles with equal mass.

The last two integrals in (4·22) involve unphysical regions of reaction II and we need an analytic continuation of the absorptive part into this region. This can be accomplished

by the Legendre expansion which converges within an ellipse with foci at $\cos\theta = \pm 1$, the size of the ellipse being determined by the nearest singularity of the absorptive part. The boundary of the region where the spectral functions fail to vanish and B becomes complex therefore also gives the boundary for the region of convergence of the Legendre expansion. This region has already been computed by Mandelstam[13] and we can read off the results we require from his paper. The diagram which causes this singularity arises from the sequence $\Lambda + \bar{N} \to \bar{K} + \pi \to \Lambda + \bar{N}$ and is shown in the figure below.

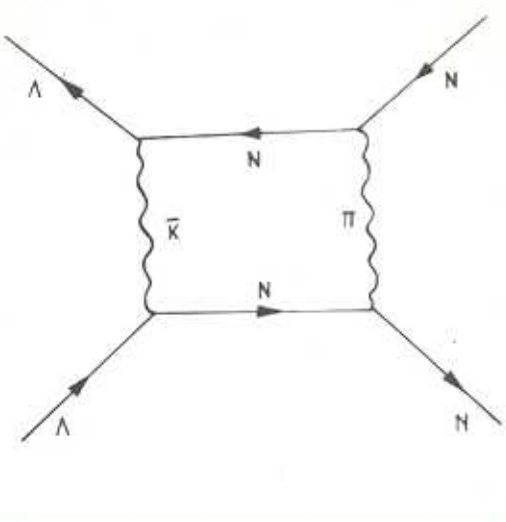


Fig.4

13) S. Mandelstam, Phys. Rev., 115 , 1741 (1959).

The Legendre expansion is then seen to converge in the region

$$\cos\varphi < 2\gamma'^2 - 1 \qquad (4\cdot23a)$$

where

$$\gamma' = \frac{\left[l^2 + q_1'^2 + m_N^2 - \left\{ (m_\Lambda^2 + l^2)^{\frac{1}{2}} - (m_K^2 + q_1'^2)^{\frac{1}{2}} \right\}^2 \right]}{2q'l} \qquad (4\cdot23b)$$

and

$$q_1'^2 = \frac{\left[\bar{t} - (m_K + m_\pi)^2 \right] \left[\bar{t} - (m_K - m_\pi)^2 \right]}{4\bar{t}} \qquad (4\cdot23c)$$

The expression for $\cos\varphi$ and l^2 in terms of s and $\bar{t}$ is given in (2·10 , 2·11). The resulting formulae are evidently complicated.

The contribution to the discontinuity across the real axis from the vanishing of denominators involving t is

$$\mathrm{Im}_{\bar{t}}^{c}(s) = -\int_{-1}^{1+\frac{2m_\pi^2}{k^2}} d\cos\theta \, d^J(\theta) \, \bar{\Delta}^{III} \, \mathrm{Re}\, \mathcal{C}(t, \bar{t}) \qquad (4\cdot24)$$

since $\bar{\Delta}^{III-1} = \bar{\Delta}^{III}$. The upper limit in the $\cos\theta$ integration extends only upto $1 + \frac{2m_\pi^2}{k^2}$ since beyond this point, the denominator involving t does not vanish[2]. Notice also that $k^2 \leq 0$ in the region under consideration. The contribution to $\bar{F}^{IJ}$ from (4·24) is

$$I_2(s) = -\frac{1}{\pi}\left[\int_{-\infty}^{0} + \int_{m_\Lambda^2 - m_N^2}^{\left[m_\Lambda^2 + m_N^2 + 2(m_\Lambda^2 - m_\pi^2)^{1/2} (m_N^2 - m_\pi^2)^{1/2} \right]} \right] \frac{ds'}{s' - s} \times$$

$$\times \int_{-1}^{1+\frac{2m_\pi^2}{k'^2}} d\cos\theta \, d^J(\theta) \, \bar{\Delta}^{III} \, \mathrm{Re}\, \mathcal{C}(t', \bar{t}') \qquad (4\cdot25)$$

Using (2·5), we write this in the form

$$I_2(s) = -\frac{1}{\pi}\left[\int_{-\infty}^{0} + \int_{m_\Lambda^2-m_N^2}^{m_\Lambda^2+m_N^2+2(m_\Lambda^2-m_\pi^2)^{1/2}(m_N^2-m_\pi^2)^{1/2}}\right]\times$$

$$\times \frac{ds'}{s'-s}\frac{1}{2k^2}\int_{-4k'^2}^{+m_\pi^2} dt'\, d^J(\theta')\,\overline{\Delta}^{III}\,\mathrm{Re}\,\mathcal{C}(t',\bar{t}') = \mathcal{I}_1 + \mathcal{I}_2 \quad \text{say} \qquad (4\cdot26)$$

By a change in the order of integrations, $\mathcal{I}_1$ can be written as

$$\mathcal{I}_1 = -\frac{1}{\pi}\left[\int_{+m_\pi^2}^{+m_\Lambda^2} dt' \int_{0}^{-\infty} ds' + \int_{+m_\Lambda^2}^{\infty} dt' \int_{0}^{\phi_1(t')} ds' + \int_{+m_\Lambda^2}^{\infty} dt' \int_{\phi_2(t')}^{-\infty} ds'\right]\times$$

$$\times\left[\frac{1}{s'-s}\frac{1}{2k'^2}\, d^J(\theta')\,\overline{\Delta}^{III}\,\mathrm{Re}\,\mathcal{C}(t',\bar{t}')\right] \qquad (4\cdot27a)$$

where

$$\phi_1(t') = \frac{2m_\Lambda^2 + 2m_N^2 - t' + \left[\left\{t' - 2(m_\Lambda^2+m_N^2)\right\}^2 - 4(m_\Lambda^2-m_N^2)^2\right]^{\frac{1}{2}}}{2} \qquad (4\cdot27b)$$

and

$$\phi_2(t') = \frac{2m_\Lambda^2 + 2m_N^2 - t' - \left[\left\{t' - 2(m_\Lambda^2+m_N^2)\right\}^2 - 4(m_\Lambda^2-m_N^2)^2\right]^{1/2}}{2} \qquad (4\cdot27c)$$

An examination of the last integral in (4·27a) reveals that it is inthe physical region of reaction III $\left[i.e.\ t' \geq +m_\Lambda^2,\ s' \leq -(m_\Lambda^2-m_N^2)\right]$ and consequently we can expand the absorptive part $\mathcal{C}$ in terms of Legendre polynomials.

The region of convergence of the polynomial expansion in the first two integrals in (4·27a) and in $\mathcal{I}_2$ can be

obtained by considering the following diagram :

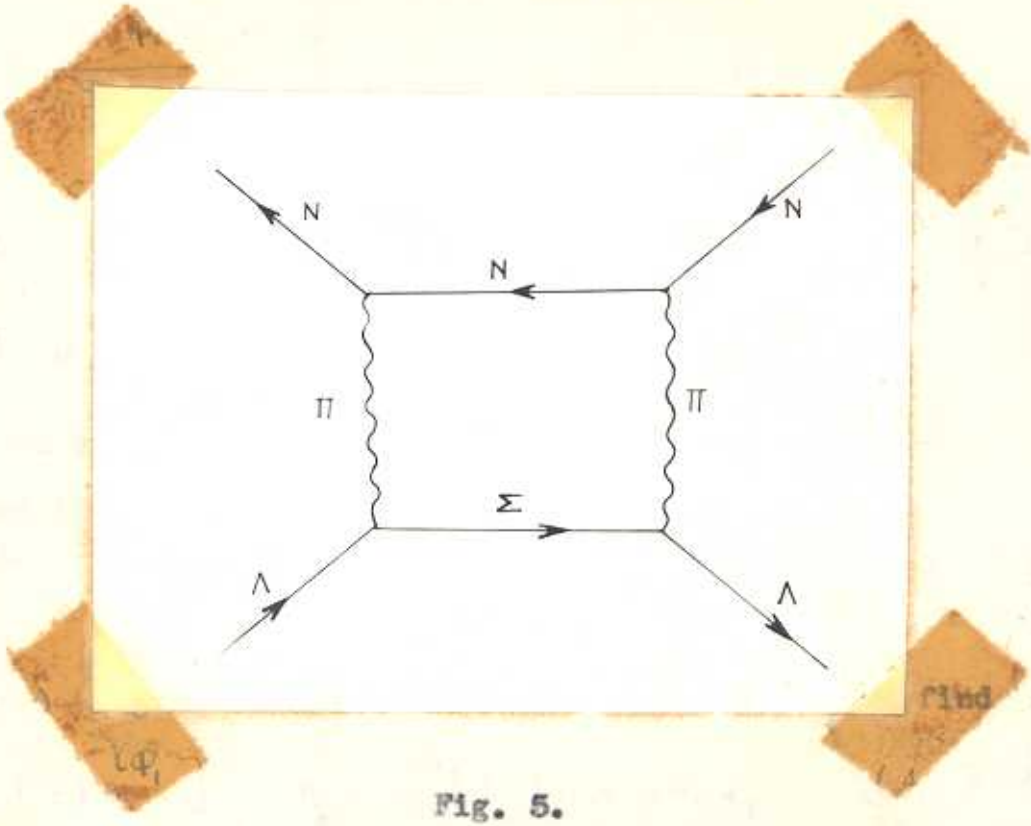


Fig. 5.

Before proceeding to writing down the region of convergence of the polynomial expansion, we notice that a serious complication arises if we wish to take into account the two pion contribution to the absorptive part $\mathcal{C}(t,\bar{t})$. The simplest diagram which can give this contribution is Fig. 5 and even this necessarily involves the Σ-N intermediate state in the channel $\Lambda + N \to \Lambda + N$. The elastic unitarity condition on the Λ-N rescattering cut will not therefore suffice if we wish to investigate 2π intermediate states in $\mathcal{C}(t,\bar{t})$. This can constitute a serious obstacle to any calculation which attempts to take into account the cut in t .

We write down below the region where we can analytically continue $\mathcal{C}(t,\bar{t})$ via the Legendre expansion :

$$\frac{2(m_\Lambda^2+m_N^2)-t-2s}{(t-4m_\Lambda^2)^{1/2}(t-4m_N^2)^{1/2}} < \frac{(\frac{t}{2}+m_\Sigma^2-m_\Lambda^2-m_\pi^2)(\frac{t}{2}-m_\pi^2)}{(\frac{t}{4}-m_\pi^2)(t-4m_\Lambda^2)^{\frac{1}{2}}(t-4m_N^2)^{\frac{1}{2}}}$$

$$+ \frac{1}{(\frac{t}{4}-m_\pi^2)(t-4m_\Lambda^2)^{1/2}(t-4m_N^2)^{1/2}} \times$$

$$\left[m_\Sigma^2\left\{m_\Sigma^2-2m_\Lambda^2-2m_\pi^2\right\}+m_\Lambda^2(m_\Lambda^2-2m_\pi^2)+m_\pi^4+m_\Sigma^2 t\right]^{\frac{1}{2}} \times$$

$$\times\left[+m_N^2(\frac{t}{4}-m_\pi^2)+m_\pi^4\right]^{\frac{1}{2}} \qquad (4\cdot 28)$$

We can now proceed to compute the discontinuity across the circle. This discontinuity is given by

$$\mathrm{Im}\,\bar{F}^{IJ}(re^{i\varphi_1}) = \frac{\bar{F}^{IJ}[(r+\epsilon)e^{i\varphi_1}]-\bar{F}^{IJ}[(r-\epsilon)e^{i\varphi_1}]}{2i} \qquad (4\cdot 29)$$

When $s = re^{i\varphi_1}$ is replaced by $(r\pm\epsilon)e^{i\varphi_1}$, we find

$$k^2[(r\pm\epsilon)e^{i\varphi_1}] = k^2(re^{i\varphi_1}) \pm i\epsilon\,\mathrm{sm}\,\varphi_1 \qquad (4\cdot 30a)$$

where

$$k^2(re^{i\varphi_1}) = \frac{1}{4}[2r\cos\varphi_1-\Sigma] \qquad (4\cdot 30b)$$

These formulae are given in ref. [5]. It follows that

$$\mathrm{Im}\,\bar{F}^{IJ}(re^{i\varphi_1}) = -\epsilon(\mathrm{sm}\,\varphi_1)\int_{-1}^{1+\frac{2m_\pi^2}{k^2}} d\cos\theta\, d^J(\theta)\,\bar{\Delta}^{III}\,\mathrm{Re}\,\mathcal{C}(t,\bar{t}) \qquad (4\cdot 31a)$$

The contribution to $\bar{F}^{IJ}(s)$ from (4·31a) therefore reads

$$I_s(s) = \frac{r}{\pi}\int_0^{2\pi} \frac{i d\varphi_1\,\epsilon(\mathrm{sm}\,\varphi_1)}{[r-se^{-i\varphi_1}]}\,\frac{1}{2k'^2}\int_{-4k'^2}^{+\infty} dt'\,d^J(\theta')\,\bar{\Delta}^{III}\,\mathrm{Re}\,\mathcal{C}(t',\bar{t}') \qquad (4\cdot 31b)$$

The relevant region of convergence of the polynomial expansion of $\mathrm{Re}\,\mathcal{C}$ is given by our previous formula (4·28).

Finally we write down the partial wave dispersion relation for $\bar{F}^{IJ}(s)$:

$$\bar{F}^{IJ}(s) = \bar{F}_0^{IJ}(s) + I_1(s) + I_2(s) + I_3(s) + \frac{1}{\pi}\int_{(m_\Lambda+m_N)^2}^{\infty} ds' \frac{\mathrm{Im}\,\bar{F}^{IJ}(s')}{s'-s-i\epsilon} \qquad (4\cdot 32)$$

where $\bar{F}_0^{IJ}(s)$ is the K -meson pole contribution. (4·32) has been written down formally leaving out questions of subtractions and so on. In any actual calculation, of course, advantage should be taken of the fact that a partial wave amplitude of order J corresponding to $l=J$ or $l=J-1$ to $l=J+1$ bahaves like k^{2J} in the vicinity of $k=0$ while the amplitude for $l=J-1$ and $l=J+1$ behave like $k^{2(J-1)}$ and $k^{2(J+1)}$ respectively to construct new amplitudes which have better behaviour at infinity. In making subtractions, one can as usual take advantage of the boundedness of the amplitudes implied by the unitarity condition.

b) Reaction II. The analogues of $\bar{F}_l^{IJ}(s)$ for this reaction has been previously denoted by $\bar{F}_l^{IIJ}(\bar{t})$. The singularities of $\bar{F}_l^{IIJ}(\bar{t})$ in the $\bar{t}$ - plane are easily enumerated by a procedure analogous to that we used for $\bar{F}_l^{IJ}(s)$. We have, first, the pole term and the cut in $\bar{t}$ in $\bar{F}_l^{II}$ giving for $\bar{F}_l^{IIJ}(\bar{t})$ a pole at

$$\bar{t} = m_K^2 \qquad (4\cdot 33)$$

and a cut in the region

$$\bar{t} \geq (m_K + m_\pi)^2 \qquad (4\cdot 34)$$

The cut in the Λ -variable in $\bar{F}_i^{III}$ maps into the region

$$\bar{t} \leq (m_\Lambda - m_N)^2 \qquad (4\cdot 35)$$

Finally the cut in t in $\bar{F}_i^{III}$ gives cuts in $\bar{t}$ in the regions

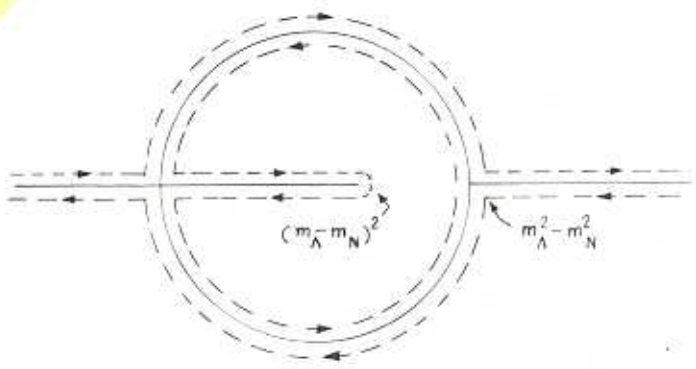
$$(m_\Lambda^2 - m_N^2) \leq \bar{t} \leq m_\Lambda^2 + m_N^2 - 2m_\pi^2 + 2(m_\Lambda^2 - m_\pi^2)^{\frac{1}{2}}(m_N^2 - m_\pi^2)^{\frac{1}{2}}, \qquad (4\cdot 36a)$$

$$\bar{t} \leq 0 \qquad (4\cdot 36b)$$

and a circle of radius

$$r = m_\Lambda^2 - m_N^2 \qquad (4\cdot 36c)$$

centred at the origin. The figure below illustrates these singularities.



Contour of integration for $\bar{F}_i^{III}(\bar{t})$ in the $\bar{t}$-plane

Fig.6.

The contribution to $\mathrm{Im}\,\bar{F}^{\mathrm{II\,J}}$ arising from the cut in s reads

$$\mathrm{Im}\,K_1(\bar{t}) = -\int_{-1}^{1} d\cos\varphi\; d^{J}(\varphi)\,\bar{\Delta}^{\mathrm{II}}\,\mathrm{Re}\,A(s,t) \qquad (4\cdot37)$$

so that

$$K_1(\bar{t}) = -\frac{1}{\pi}\int_{-\infty}^{(m_\Lambda-m_N)^2}\frac{d\bar{t}'}{\bar{t}'-\bar{t}}\int_{-1}^{1} d\cos\varphi\; d^{J}(\varphi)\,\bar{\Delta}^{\mathrm{II}}\,\mathrm{Re}\,A(s',t')$$

$$= \frac{1}{\pi}\int_{-\infty}^{(m_\Lambda-m_N)^2}\frac{d\bar{t}'}{\bar{t}'-\bar{t}}\;\frac{1}{2\lambda'^2}\int_{\frac{(m_\Lambda^2-m_N^2)^2}{\bar{t}'}}^{\Sigma-\bar{t}'} ds'\; d^{J}(\varphi')\,\bar{\Delta}^{\mathrm{II}}\,\mathrm{Re}\,A(s',t') \qquad (4\cdot38)$$

where a change of variables has been effected using (2·11).

The integral can be rewritten as

$$K_1(\bar{t}) = \frac{1}{\pi}\left[\int_{-\infty}^{0} + \int_{0}^{(m_\Lambda-m_N)^2}\right]\frac{d\bar{t}'}{\bar{t}'-\bar{t}}\int_{\frac{(m_\Lambda^2-m_N^2)^2}{\bar{t}'}}^{\Sigma-\bar{t}'} ds' \;\times$$

$$\times\; d^{J}(\varphi')\,\bar{\Delta}^{\mathrm{II}}\,\mathrm{Re}\,A(s',t') \qquad (4\cdot39)$$

where the first integral is in the physical region of reaction I while in the second integral, the Legendre expansion converges for

$$\frac{\left[\Sigma - s' - \bar{t}' - 16m_\pi^2\right]\left[s'-(m_\Lambda+m_N)^2\right]\left[s'-(m_\Lambda-m_N)^2\right]}{4s'} \leq 16m_\pi^4 \qquad (4\cdot40)$$

the condition (4·40) emerging from the diagram shown below.

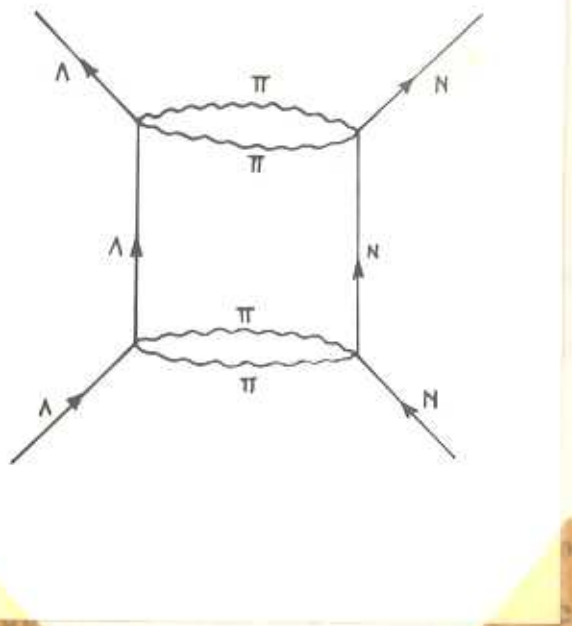


Fig.7.

The first integral in (4·39) as before involves contributions from large values of s' .

Along the real axis, the cut in t in $\bar{F}_i^{II}$ makes a contribution to $\bar{F}^{IIJ}$ given by

$$\mathrm{Im}\, K_2(\bar{F}) = -\int_{-1}^{1+\frac{2m_K^2}{t^2}} d\cos\varphi\, d^J(\varphi)\, \bar{\Delta}^{II}\bar{\Delta}^{III}\, \mathrm{Re}\, \mathcal{C}(t,\bar{F}) \qquad (4\cdot41)$$

We can therefore write,

$$K_2(\bar{t}) \equiv \frac{1}{\pi}\left\{-\int_{-\infty}^{\bar{t}_1} - \int_{\bar{t}_1}^{\bar{t}_2}\right\}\frac{d\bar{t}'}{\bar{t}'-\bar{t}} \times$$

$$\times \int_{-1}^{1+\frac{2m_\pi^2}{l'^2}} d\cos\varphi\, d^J(\varphi)\, \bar{\Delta}^{II}\bar{\Delta}^{III}\, \mathrm{Re}\, \mathcal{C}(t',\bar{t}') \quad (4\cdot42)$$

$$= \mathcal{J}_1 + \mathcal{J}_2 \text{ say}$$

where

$$\bar{t}_1 = m_\Lambda^2 - m_N^2,$$

$$\bar{t}_2 = m_\Lambda^2 + m_N^2 - 2m_\pi^2 + 2\left(m_\Lambda^2 - m_\pi^2\right)^{\frac{1}{2}}\left(m_N^2 - m_\pi^2\right)^{\frac{1}{2}} \quad (4\cdot43)$$

Here equation (2·9) has been used to make a change of variables. Equation (4·42) is very similar to (4·26) and we can split off the physical region in $\mathcal{J}_1$ as in (4·27a) . The region of convergence of the polynomial expansion can be obtained from (4.28) by replacing δ in terms of t and $\bar{t}$ using (2·8).

Finally for the discontinuity across the circle $\bar{t} = r e^{i\varphi_2}$ we have

$$\mathrm{Im}\, \bar{F}^{II\,J}(\bar{t}) = -\epsilon(\sin\varphi_2)\int_{-1}^{1+\frac{2m_\pi^2}{l^2}} d\cos\varphi\, d^J(\varphi)\, \bar{\Delta}^{II}\bar{\Delta}^{III}\, \mathrm{Re}\, \mathcal{C}(t,\bar{t}) \quad (4\cdot44)$$

so that the contribution of (4·44) to $\bar{F}^{II\,J}$ is

$$K_3(\bar{t}) = \frac{r}{\pi}\int_0^{2\pi}\frac{i\,d\varphi_2}{r-\bar{t}e^{-i\varphi_2}}\,\frac{\epsilon(\sin\varphi_2)}{2l'^2}\int_{-4l'^2}^{+m_\pi^2} dt'\, d^J(\varphi')\,\bar{\Delta}^{II}\bar{\Delta}^{III} \times$$

$$\times\, \mathrm{Re}\, \mathcal{C}(t',\bar{t}') \quad (4\cdot45)$$

Thus we can write

$$\bar{F}^{II\,J}(\bar{t}) = \bar{F}_0^{II\,J}(\bar{t}) + K_1(\bar{t}) + K_2(\bar{t}) + K_3(\bar{t}) + \frac{1}{\pi}\int_{(m_K+m_\pi)^2}^{\infty} d\bar{t}'\,\frac{\mathrm{Im}\,\bar{F}^{II\,J}(\bar{t}')}{\bar{t}'-\bar{t}-i\epsilon} \quad (4\cdot46)$$

Here $\bar{F}_0^{II\,J}(\bar{t})$ is the contribution from the pole term. The second integral in (4.~~39~~ 46) involves imaginary values of l in the region $(m_K+m_\pi)^2 \le \bar{t} < (m_\Lambda+m_N)^2$

and the validity of the usual methods of calculation of $B(\bar{t}, t)$ in this region has been discussed by Grisaru[14]. (4.46) has again been written down only formally.

c) Reaction III.

The amplitudes $\bar{F}^{III\,J}$ for this process have been defined previously. We shall now enumerate its singularities by using a method similar to that employed in the other two cases.

The cut in the t -plane for $\bar{F}^{III\,J}$ arising from the cut in t in $\bar{F}^{III}$ lies in the region

$$t \geq 4m_{\pi}^2 \qquad (4.47)$$

In the region $4m_{\pi}^2 \leq t < 4m_{\Lambda}^2$, p is complex while in the region $4m_{\pi}^2 \leq t < 4m_{N}^2$, q is complex. As remarked earlier the calculation of absorptive parts in the region by the usual formulae needs some justification.

14) M.T. Grisaru, Phys. Rev., 111, 1719 (1958).

The cut in s in $\bar{F}^{III}$ becomes a cut for $\bar{F}^{III\,J}$ in the t -plane in the region

$$t \leq 0 \qquad (4.48)$$

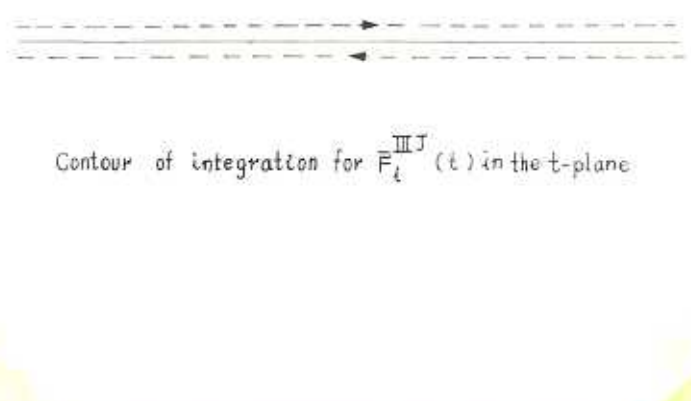
The pole term maps into the region

$$t \leq \frac{4m_\Lambda^2 m_N^2 - (m_\Lambda^2 + m_N^2 - m_K^2)^2}{m_K^2} \qquad (4.49)$$

Finally the branch cut in $\bar{F}^{III}$ starting at $\bar{t} = (m_K + m_\pi)^2$ gives a branch cut for $\bar{F}^{III\,J}$ in the region

$$t \leq \frac{4m_\Lambda^2 m_N^2 - [m_\Lambda^2 + m_N^2 - (m_K + m_\pi)^2]^2}{(m_K + m_\pi)^2} \qquad (4.50)$$

The singularities of $\bar{F}^{III\,J}$ and the contour of integration to be used in the t- plane are shown in the figure below.



Contour of integration for $\bar{F}_t^{III\,J}(t)$ in the t-plane

Fig.8.

The discontinuity across the cut $t \leq 0$ due to the vanishing of the denominators involving s is clearly

$$Im\, L_1(t) = -\int_{-1}^{1} d\cos\eta\, d^J(\eta)\, \bar{\Delta}^{III}\, Re\, A(s,t) \qquad (4.51)$$

Therefore

$$L_1(t) = -\frac{1}{\pi}\int_{-\infty}^{0} \frac{dt'}{t'-t} \int_{-1}^{1} d\cos\eta\, d^J(\eta)\, \bar{\Delta}^{III}\, Re\, A(s't')$$

$$= \frac{1}{\pi}\int_{-\infty}^{0} \frac{dt'}{t'-t}\, \frac{1}{2p'q'} \int_{\delta_1(t')}^{\delta_2(t')} ds'\, d^J(\eta')\, \bar{\Delta}^{III}\, Re\, A(s',t') \qquad (4.52a)$$

where equation (2.13) has been used and

$$\delta_1(t') = -\frac{1}{2}t' + \frac{1}{2}(t'-4m_\Lambda^2)^{1/2}(t'-4m_N^2)^{1/2} + m_\Lambda^2 + m_N^2 \qquad (4.52b)$$

$$\delta_2(t') = -\frac{1}{2}t' - \frac{1}{2}(t'-4m_\Lambda^2)^{1/2}(t'-4m_N^2)^{1/2} + m_\Lambda^2 + m_N^2 \qquad (4.52c)$$

By changing the order of integration, $L_1(t)$ can be rewritten as

$$L_1(t) = \frac{1}{\pi}\left[\int_{(m_\Lambda+m_N)^2}^{(m_\Lambda-m_N)^2} ds' \int_{-\infty}^{0} dt' + \int_{(m_\Lambda-m_N)^2}^{0} ds' \int_{-\infty}^{t_1(s')} dt' + \int_{(m_\Lambda+m_N)^2}^{\infty} ds' \int_{-\infty}^{t_1(s')} \right] \times$$

$$\times \frac{1}{t'-t}\cdot\frac{1}{2p'q'}\, d^J(\eta')\, \bar{\Delta}^{III}\, Re\, A(s',t') \qquad (4.53)$$

where

$$t_1(s') = \frac{4m_\Lambda^2 m_N^2 - (s'-m_\Lambda^2-m_N^2)^2}{s'} \qquad (4.54)$$

Here L_3 is in the physical region of reaction I. L_1 and L_2 involve unphysical regions of reaction I so that we have again to determine the region where the polynomial expansion converges. This region has already been computed in equation (4.40) which may be rewritten as

$$\frac{[t-16m_\pi^2][s-(m_\Lambda+m_N)^2][s-(m_\Lambda-m_N)^2]}{4s} < 16m_\pi^2 \qquad (4.55)$$

In this way, we find that the polynomial expansion is convergent for δ_2 , while δ_3 should be rewritten as

$$\delta_1 = \frac{1}{\pi}\left\{\int_{(m_\Lambda+m_N)^2}^{s_1} ds' \int_{-\infty}^{t(s')} dt' + \int_{s_1}^{s_2} ds' \int_{-\infty}^{0} dt' + \int_{s_2}^{(m_\Lambda-m_N)^2} ds' \int_{-\infty}^{t(s')} dt'\right\} \times$$

$$\times \frac{1}{t'-t}\,\frac{1}{2p'q'}\, d^J(\eta')\, \mathrm{Re}\, A(s',t') + \frac{1}{\pi}\left\{\int_{(m_\Lambda+m_N)^2}^{s_1} ds' \int_{t(s')}^{\infty} dt' + \right.$$

$$\left. + \int_{s_2}^{(m_\Lambda-m_N)^2} ds' \int_{t(s')}^{0} dt'\right\} \frac{1}{t'-t}\,\frac{1}{2p'q'}\, d^J(\eta')\, \mathrm{Re}\, A(s',t') \qquad (4\cdot56)$$

where

$$s_1 = m_\Lambda^2 + m_N^2 - 2m_\pi^2 + 2(m_\Lambda^2 - m_\pi^2)^{\frac{1}{2}}(m_N^2 - m_\pi^2)^{\frac{1}{2}},$$

$$s_2 = m_\Lambda^2 + m_N^2 - 2m_\pi^2 - 2(m_\Lambda^2 - m_\pi^2)^{\frac{1}{2}}(m_N^2 - m_\pi^2)^{\frac{1}{2}} \qquad (4\cdot57)$$

For the integrals in the second bracket in (4·57), the polynomial expansion converges while it does not converge for those in the first bracket.

For the contribution to $\overline{F}^{III\,J}$ from the cut in $\bar{t}$ in $\overline{F}^{III}$, we find

$$\mathrm{Im}\, L_2(t) = -\int_{-1}^{1} d\cos\eta\; d^J(\eta)\; \overline{\Delta}^{III}\,\overline{\Delta}^{III}\, \mathrm{Re}\, B(\bar{t},t) \qquad (4\cdot58)$$

We may therefore write

$$L_2(t) = -\frac{1}{\pi} \int_{-\infty}^{t_0[(m_K+m_\pi)^2]} \frac{dt'}{t'-t} \int_{-1}^{1} d\cos\eta \, d^J(\eta) \bar{\Delta}^{III} \bar{\Delta}^{II} \, Re\, B(\bar{t}', t')$$

$$= -\frac{1}{\pi} \int_{-\infty}^{t_0[(m_K+m_\pi)^2]} \frac{dt'}{t'-t} \int_{f_2(t')}^{f_1(t')} \frac{d\bar{t}'}{2p'q'} \, d^J(\eta') \bar{\Delta}^{III} \bar{\Delta}^{II} \, Re\, B(\bar{t}', t') \quad (4.59)$$

where $f_1(t)$ and $f_2(t)$ are defined in equations (4.52 b, c) and

$$t_0(x) = \frac{4 m_\Lambda^2 m_N^2 - [m_\Lambda^2 + m_N^2 - x]^2}{x} \quad (4.60)$$

$L_2(t)$ may equivalently be written as

$$L_2(t) = -\frac{1}{\pi} \left\{ \int_0^{(m_\Lambda - m_N)^2} d\bar{t}' \int_{-\infty}^{t_0(\bar{t}')} dt' + \int_{(m_\Lambda - m_N)^2}^{(m_\Lambda + m_N)^2} d\bar{t}' \int_{-\infty}^{0} dt' \right.$$

$$\left. + \int_0^{t_0[(m_K+m_\pi)^2]} dt' \int_{f_2(t')}^{f_1(t')} d\bar{t}' + \int_{(m_\Lambda+m_N)^2}^{\infty} d\bar{t}' \int_{-\infty}^{t_0(\bar{t}')} dt' \right\} \frac{1}{t'-t} \frac{1}{2p'q'} \times$$

$$\times \, d^J(\eta') \bar{\Delta}^{III} \bar{\Delta}^{II} \, Re\, B(\bar{t}', t') \quad (4.61)$$

The last integral in (4.61) is in the physical region of reaction II . For the other three integrals the region where the Legendre expansion is valid can be computed from (4.23 a, b, c) written as a function of $\bar{t}'$ and t' .

Finally the dispersion relations for $\bar{F}^{III\,J}(t)$ may be formally written as

$$\bar{F}^{III\,J}(t) = \bar{F}_0^{III\,J}(t) + L_1(t) + L_2(t) + \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' \frac{Im\, \bar{F}^{III\,J}(t')}{t'-t-i\epsilon} \quad (4.62)$$

where $\bar{F}_0^{III\,J}(t)$ is the contribution from the K -meson pole.

5 Conclusion

We have thus given the complete partial wave analysis of the Λ-N problem and brought it to a stage where numerical calculations can be performed. This task will become worthwhile especially when there is now the prospect of obtaining data on Λ-N scattering.

CHAPTER III

ON THE Y^* RESONANCES*

1. Introduction

Recently evidence has come to light showing the existence of a Λ-π resonance (called the Y_1^*) with a mass of about 1385 Mev[1] and a Σ-π resonance (called the Y_0^*) with a mass of about 1405.[2] In this chapter, we shall investigate these resonances using dispersion theory and develop effective range formulae for the scattering amplitudes in the resonance channels in the static approximation which are the analogues of the Chew-Low effective range formula for π-N scattering.[3] The chapter is divided into three sections. In section II, we study the Y_1^* and Y_0^* resonances assuming that they are in the S -state. The derivation of the effective range formulae in this case follows closely the method suggested by Frazer and Fulco for π-N scattering.[4] In the next section, we develop approximate formulae for the scattering amplitudes of the process $Y + \Lambda \rightarrow \Lambda + \pi$ assuming a dominant S -wave

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* Alladi Ramakrishnan, T.K.Radha, R. Thunga and A.P.Balachandran, Nuovo Cimento (in press).

1) Alston et. al., Phys. Rev. Letters, 5, 520 (1960);
Berge et. al., Phys. Rev. Letters, 6 , 557 (1961);
Block et. al., Nuovo Cimento, 20, 724 (1961);
Dahl et. al., Phys. Rev. Letters, 6, 142 (1961);
M.H.Alston and M.Ferro-Luzzi, Rev.Mod. Phys., 33 , 416 (1961)

2) Alston et. al., Phys. Rev. Letters, 6 , 698 (1961)

3) G.F.Chew and F.E. Low, Phys. Rev., 101, 1570 (1956).

4) W.R.Frazer and J.R.Fulco, Phys. Rev., 119, 1420 (1960).

Λ-π interaction. Though this process cannot be observed directly, it is of importance in the study of reactions like $\gamma + N \rightarrow \Lambda + K + \pi$ and the electroproduction of pions on Λ -hyperons. Finally in Section IV, we study the possibility of resonances in the Λ-π system in the $J = 3/2$ and $J = 5/2$ states within the scope of our model where J denotes the total angular momentum.

II. Effective Range Formulae for S -wave Y-π Scattering

In this section, we shall assume that the Y-π resonances (where $Y = \Lambda$ or Σ) are S -wave resonances. The initial motivation for this study was that at one stage, Block et. al.[1] presented some tentative evidence showing that Y_1^* has $J = 1/2$ and odd parity. We shall illustrate the derivation of the effective range formula for Λ-π scattering for odd Σ-Λ parity. The derivation in the other cases is similar. Let k_1 and k_1 denote the incident and k_2 and k_2 the outgoing Λ and π four-vector momenta respectively. Following ref.[5], we can write the T -matrix as

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5) G.F.Chew, M.L.Goldberger, F.E.Low and Y. Nambu, Phys. Rev., 106 , 1337 (1957).

$$T^1 = -A^1 + i\frac{k_1+k_2}{2}B^1 \qquad (1)$$

where the superscripts denote the isotopic spin of the channel, a notation which we shall use throughout the section. These superscripts are not really necessary for $\Lambda\pi$ scattering since only one isotopic spin channel is available in this case. These are however introduced in order to distinguish these quantities from those of $\Sigma\pi$ scattering in the $I=0$ state which will occur later). Further let W denote the total energy and E the energy of the Λ in the centre-of-mass system. Define also

$$A'_\ell = \int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, A,$$

$$B'_\ell = \int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, B \qquad (2)$$

where θ is the scattering angle. Then following Frazer and Fulco[4], we form the amplitude

$$f^1_\ell(W) = \frac{W}{E+m_\Lambda}\,\frac{e^{i\delta^1_{\ell+}}\sin\delta^1_{\ell+}}{k^{2\ell+1}} = \frac{1}{16\pi}\left[\frac{A'_\ell}{k^{2\ell}} + (W-m_\Lambda)\frac{B'_\ell}{k^{2\ell}} + (E-m_\Lambda)^2\left\{-\frac{A'_{\ell+1}}{k^{2\ell+2}} + (W+m_\Lambda)\frac{B'_{\ell+1}}{k^{2\ell+2}}\right\}\right] \qquad (3)$$

where δ^1_{l+} is the phase shift in a state of total angular momentum $J = l + \frac{1}{2}$ and isotopic spin 1.

For S - waves, (3) becomes

$$h^1_0(W) = \frac{1}{16\pi}\left\{ A^1_0 + (W - m_\Lambda) B^1_0 + \right.$$

$$\left. + (E - m_\Lambda)^2 \left[- \frac{A^1_1}{k^2} + (W + m_\Lambda) \frac{B^1_1}{k^2} \right] \right\} \qquad (4)$$

where $h^1_0(W)$ is related to the * S -wave phase shift δ^1_0 through

$$h^1_0(W) = \frac{W}{E + m_\Lambda} \frac{e^{i\delta^1_0} \sin \delta^1_0}{k} \qquad (5)$$

In developing the effective range formula, we shall completely neglect the cuts associated with the energy variables of the crossed processes. Similarly the Σ -pole associated with the crossed Λ-π scattering will also be neglected. This is consistent with the spirit of the effective range approximation which retains only the nearest singularities. In the Chew-Low formula, the "crossed" nucleon pole gives rise to branch cuts in the W- plane for the partial wave amplitudes of which the part lying in the region

$$m_N - \frac{1}{m_N} \leq W \leq (m_N^2 + 2)^{\frac{1}{2}} \qquad (6)$$

is retained† and becomes the static pole of the theory[4].

The analogous region in our case is

$$\frac{m_\Lambda^2 - 1}{m_\Sigma} \leq W \leq (2m_\Lambda^2 + 2 - m_\Sigma^2)^{\frac{1}{2}} \qquad (7)$$

Estimates show that the contribution of this cut is indeed small compared to the contribution of the "direct" Σ -pole and

† Note further that we work in units in which the pion mass is unity and denote the N, Λ and Σ masses by m_N, m_Λ and m_Σ respectively.

hence is neglected. With these approximations, the theory requires only single variable dispersion relations. For odd Σ-Λ parity, the part of $h_0^1(w)$ arising from the Σ -pole occurring in the w -variable reads

$$J_0^1(w) = \frac{g^2_{\Sigma\Lambda\pi}}{8\pi} \frac{1}{m_\Sigma - w} \qquad (8)$$

where $g_{\Sigma\Lambda\pi}$ is the renormalized Σ-Λ-π coupling constant. Thus we may write for $h_0^1(w)$, with the pion mass set equal to unity,

$$h_0^1(w) = J_0^1(w) + \frac{1}{\pi} \int_{(m_\Lambda+1)}^{\infty} dw' \frac{Im\, h_0^1(w')}{w' - w} \qquad (9)$$

where we have as yet made no subtractions. The N/D solution[6] of equation (9) with two subtractions for D reads

$$\frac{g^2_{\Sigma\Lambda\pi}}{8\pi} \frac{1}{m_\Sigma - w} (E + m_\Lambda) \frac{k}{w} \cot \delta_0^1 = 1 - \frac{w - m_\Sigma}{\omega_\Lambda} + \frac{(w - m_\Sigma)^2}{\pi} P \int_{(m_\Lambda+1)}^{\infty} dw' \frac{Im\, D(w')}{(w' - m_\Sigma)^2 (w' - w)} \qquad (10)$$

where ω_Λ is a subtraction constant. We have normalized D such that $D(m_\Sigma) = 1$. This implies a definition of the renormalized Σ-Λ-π coupling constant by an analogue of the Lepore-Watson convention for the π-N coupling constant[7]. In a static approximation, (10) becomes

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6) G.F.Chew and S.Mandelstam, Phys. Rev., 119 , 467 (1960).
7) K.M.Watson and J.V.Lepore, Phys. Rev., 76 ,1157 (1949).

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\frac{k}{m_\Sigma - W}\cot\delta_0^1 = 1 - \frac{W-m_\Sigma}{\omega_\Lambda} \qquad (11)$$

where the Σ-Λ relative parity $P_{\Sigma\Lambda} = -1$. Similarly one derives the corresponding formula for even Σ-Λ parity:

$$-\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\frac{k}{m_\Sigma + W}\cot\delta_0^1 = 1 - \frac{W-m_\Sigma}{\omega_\Lambda} \quad (P_{\Sigma\Lambda} = +1) \qquad (12)$$

ω_Λ is fixed by the requirement that $\delta_0^1 = \frac{\pi}{2}$ at resonance. This gives

$$\omega_\Lambda = 1{\cdot}357 \qquad (13)$$

The formula for Σ-π scattering in the resonance channel can now be written down. We have

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\frac{k}{m_\Lambda - W}\cot\delta_0^0 = 1 - \frac{W-m_\Lambda}{\omega_\Sigma} \quad (P_{\Sigma\Lambda} = -1) \qquad (14)$$

and

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\frac{k}{m_\Lambda + W}\cot\delta_0^0 = 1 - \frac{W-m_\Lambda}{\omega_\Sigma} \quad (P_{\Sigma\Lambda} = +1) \qquad (15)$$

with

$$\omega_\Lambda = 2{\cdot}076 \qquad (16)$$

where the variables now are those of Σ-π scattering.

We give below curves showing the variation of the S-wave cross-section with the centre-of-mass momentum (in natural units) for various values of $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$ for both Λ-π and Σ-π scattering. We also give the plots of $\frac{k}{m_\Sigma \mp W}\cot\delta_0^1$ and $\frac{k}{m_\Lambda \mp W}\cot\delta_0^0$ with $(W - m_\Sigma)$ and $(W - m_\Lambda)$ respectively as given by the equations (11), (12), (14) and (15).

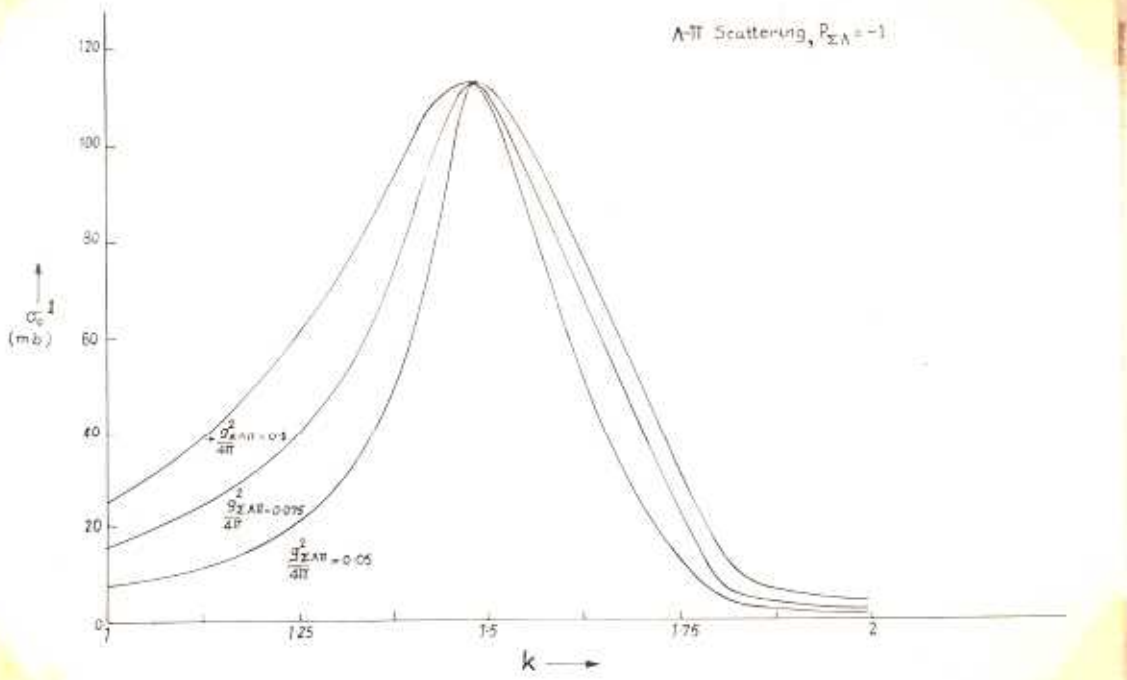
Λ-π Scattering, $P_{\Sigma\Lambda} = -1$
120
100
80
60
40
20
0
σ_0 t (mb)
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.5$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.075$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.05$
1
1.25
1.5
1.75
2
k ⟶

Fig. 1.

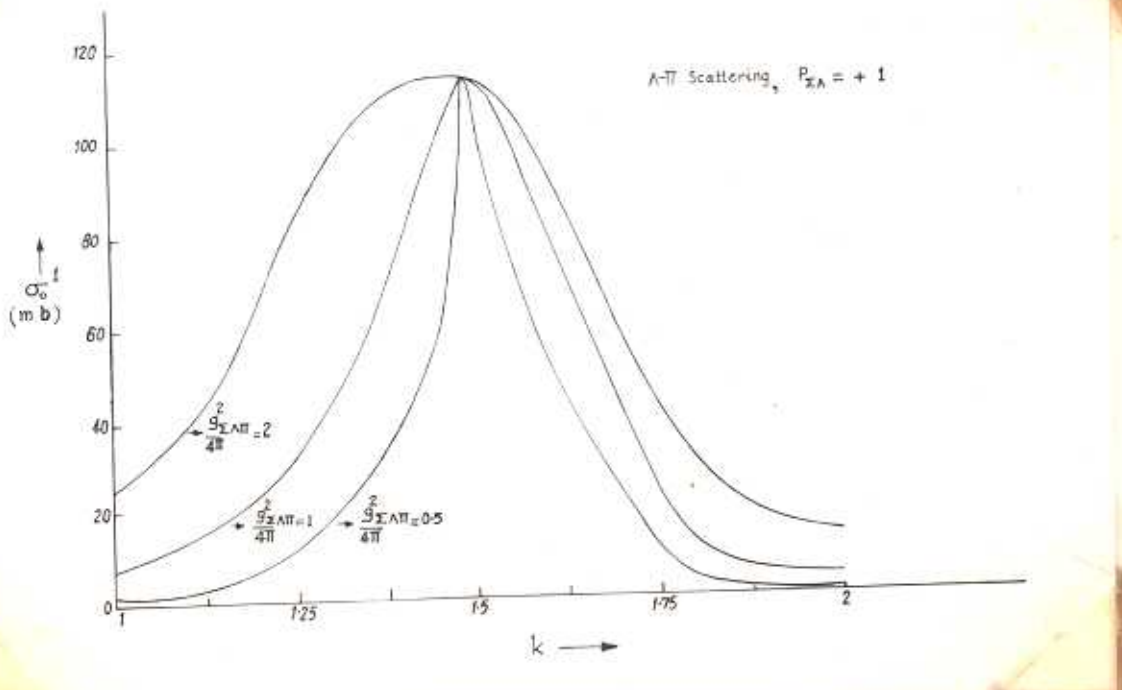
Λ-π Scattering, $P_{\Sigma\Lambda} = +1$
120
100
80
60
40
20
0
σ_0 t (mb)
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 2$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.5$
1
1.25
1.5
1.75
2
k ⟶

Fig. 2.

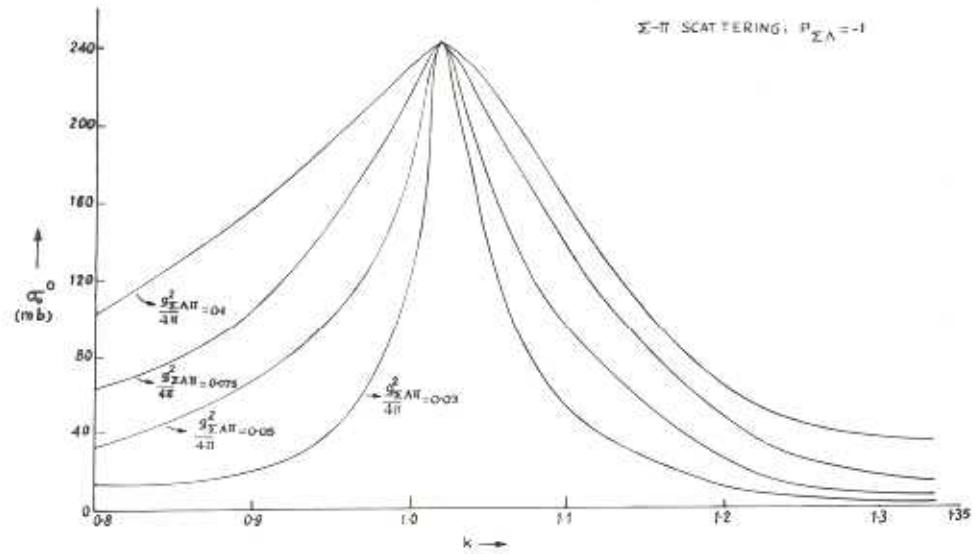
Σ-π SCATTERING; $P_{\Sigma\Lambda}=-1$
240
200
160
120
80
40
0
σ_0 (mb)
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.4$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.075$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.05$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.03$
0·8
0·9
1·0
1·1
1·2
1·3
1·35
k

Fig. 3.

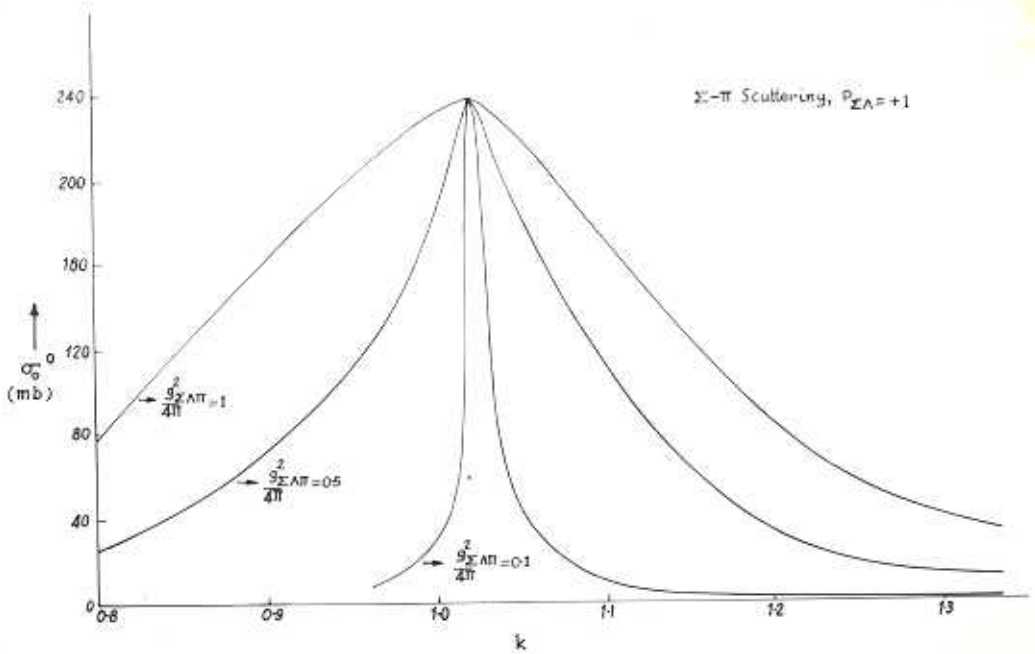
Σ-π Scattering, $P_{\Sigma\Lambda}=+1$
240
200
160
120
80
40
0
σ_0 (mb)
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.5$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.1$
0·8
0·9
1·0
1·1
1·2
1·3
k

Fig.4.

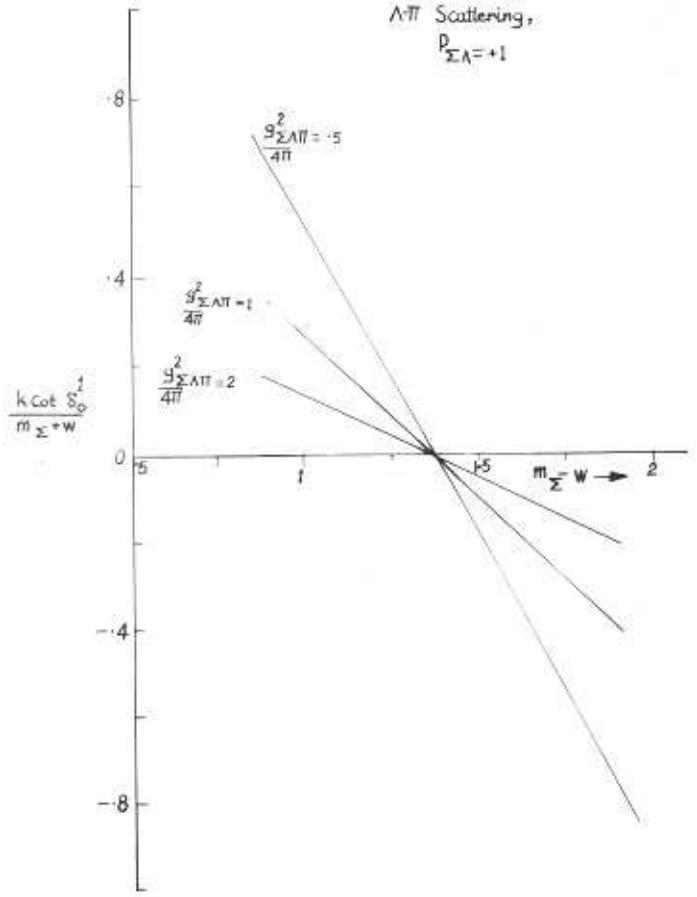
Λ-π Scattering,
$p_{\Sigma\Lambda} = +1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = .5$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 2$
$\frac{k \cot \delta^1_0}{m_{\Sigma^+} w}$
.8
.4
0
-.4
-.8
.5
1
1.5
2
$m_{\Sigma^-} w \rightarrow$

Fig. 6.

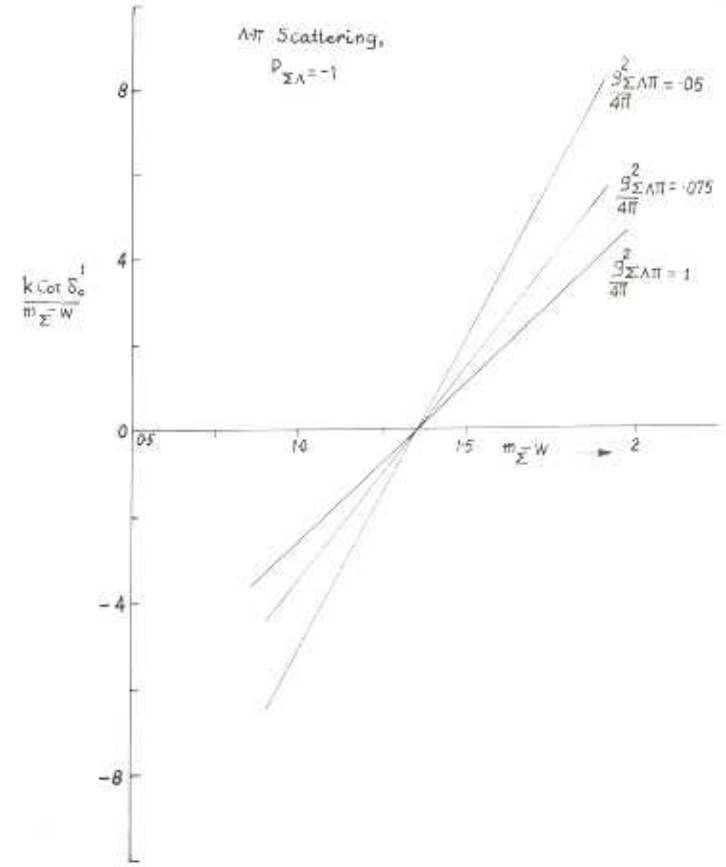
Λ-π Scattering,
$p_{\Sigma\Lambda} = -1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = .05$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = .075$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = .1$
$\frac{k \cot \delta^1_0}{m_{\Sigma^-} w}$
8
4
0
-4
-8
.05
1.0
1.5
2
$m_{\Sigma^-} w \rightarrow$

Fig. 5.

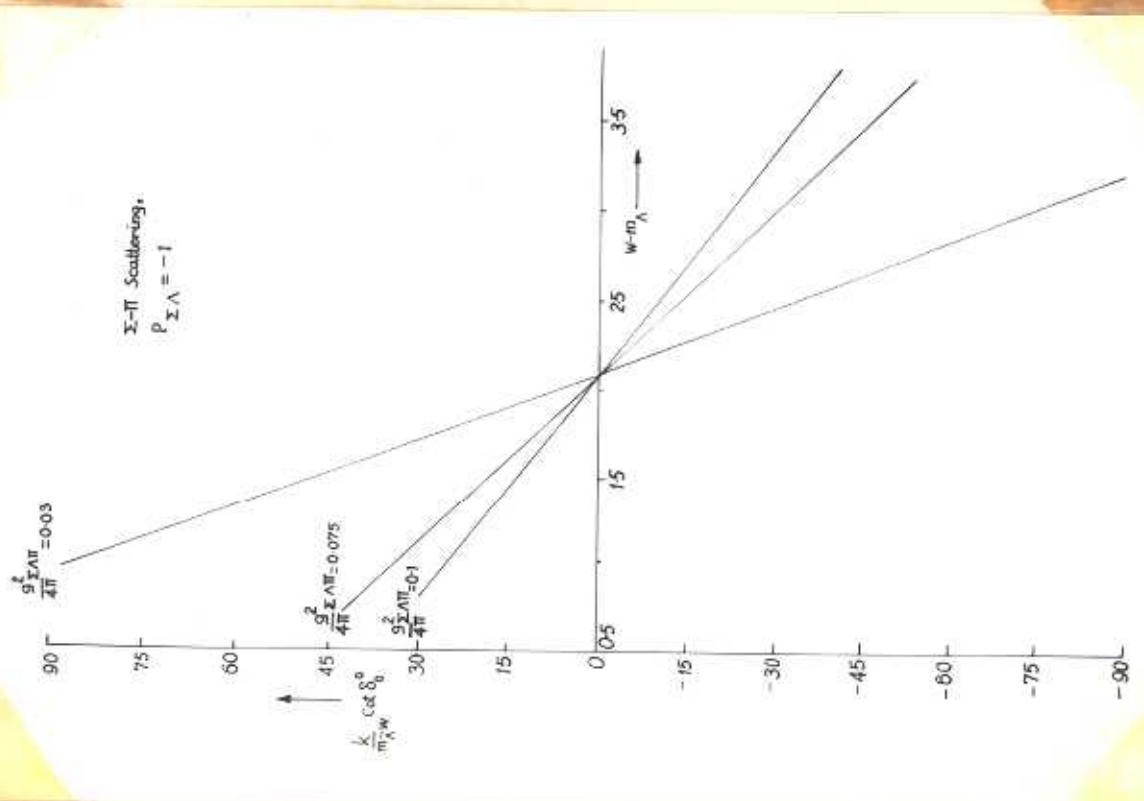
Σ-π Scattering.
$P_{\Sigma\Lambda} = -1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$=0·03
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$=0·075
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$=0·1
$\frac{k}{m_\pi w}$ Cot δ_0^0
$w-m_\Lambda$
90 75 60 45 30 15 0 -15 -30 -45 -60 -75 -90
0·5 1·5 2·5 3·5

Fig. 7.

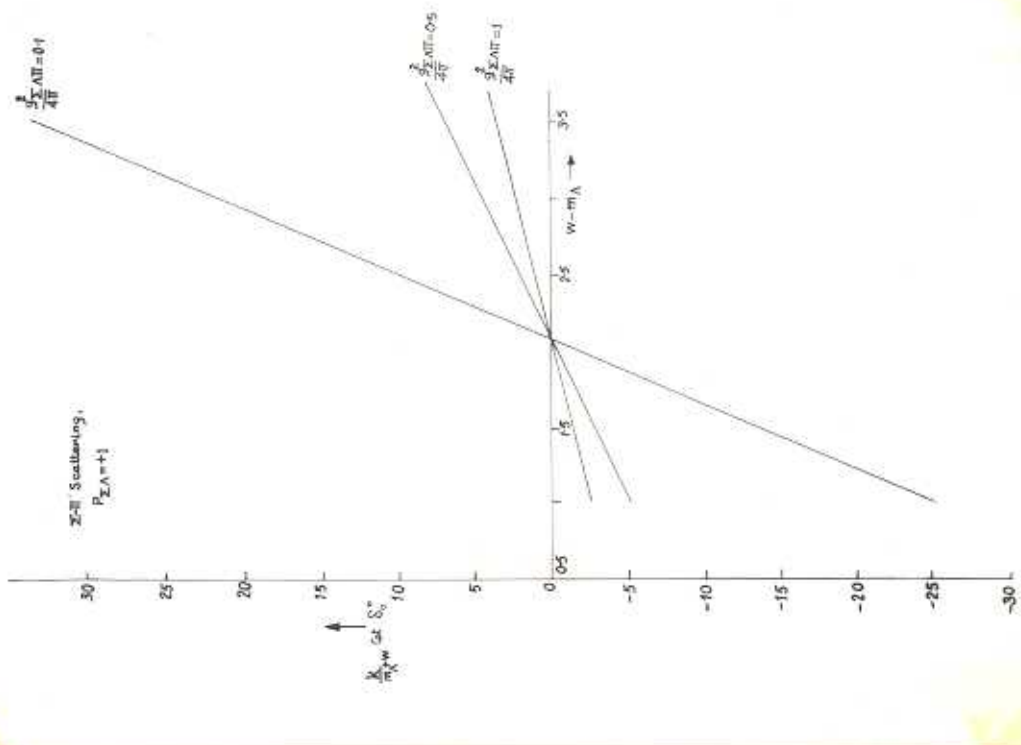
Σ-π Scattering,
$P_{\Sigma\Lambda} = +1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$=0·1
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$=0·5
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$=1
$\frac{k}{m_\pi w}$ Cot δ_0^0
$w-m_\Lambda$
30 25 20 15 10 5 0 -5 -10 -15 -20 -25 -30
0·5 1·5 2·5 3·5

Fig. 8.

From the experimental results on the widths of the Y_1^* and Y_0^* resonances, it is possible to deduce the value of $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$. Thus taking the half-width of the resonance to be 25 Mev[8], we find $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.075$ for $P_{\Sigma\Lambda} = -1$ and $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 1$ for the $P_{\Sigma\Lambda} = +1$. The former gives a Y_0^* resonance with a half-width of about 5 Mev and the latter with a half-width of about 10.3 Mev. Clear-cut data on the half-width of Y_0^* xcan thus prove or disprove the theory, especially since the theory predicts that the Y_0^* resonance should be much narrower than the Y_1^* resonance. Present evidence suggests a half-width of about 10 Mev for Y_0^* [2] which thus fits in with the theory provided the Σ-Λ relative parity is even.

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8) R.H.Dalitz and D. H. Miller, Phys. Rev. Letters, 6, 562 (1961)

III. PHOTO PRODUCTION OF PIONS ON Λ -HYPERONS*

In the previous section we have set up two-parameter effective range formulae for Λ-π and Σ-π scattering amplitudes for both even and odd Σ-Λ relative parities assuming that Y_1^* and Y_0^* are S -wave resonances. These parameters were then evaluated by fitting the position and width of the Y_1^* and Y_0^* . We shall now study the process $\gamma + \Lambda \rightarrow \Lambda + \pi$ assuming again a dominant S -wave Λ-π interaction. Since the unitarity condition then relates this process to S -wave Λ-π scattering amplitude, we can solve for the photoproduction amplitudes in terms of phase shifts of the corresponding S -wave, Λ-π amplitude. The main features of this process which are different from those of the photoproduction on nucleons (which has been studied previously)[9] are :

(i) While the amplitudes generated by magnetic moment are non-vanishing, those generated by the electric charge e are zero since the Λ is a neutral particle.

(ii) Since the pole is at m_Σ , the amplitudes are all proportional to the Σ-Λ transition magnetic moment.

(iii) Since the mass difference $m_\Sigma - m_\Lambda$ does not vanish, terms which are zero in the nucleon case where the pole occurs at the nucleon mass itself give contributions in our process.

(iv) The relative Σ-Λ parity is not yet fixed and hence we give the amplitudes for both odd and even Σ-Λ relative parity.

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* Alladi Ramakrishnan, T.K.Radha, R.Thunga and A.P.Balachandran, Nuovo Cimento (in press)

9) G.F.Chew, M.L.Goldberger, F.E.Low and Y.Nambu, Phys. Rev., 106 , 1345 (1957).

Let P_1 and K denote the four-vector momenta of the incoming Λ and γ, P_2 and Q those of the out-going Λ and π and ϵ the polarization vector of the photon. Then with Chew, Goldberger, Low and Nambu[9] we write the transition matrix for the process in the form

$$T = M_A A + M_B B + M_C C + M_D D \qquad (17)$$

where

$$M_A = \iota\gamma_5 \, \gamma\cdot\epsilon \; \gamma\cdot K ,$$

$$M_B = 2\iota\gamma_5 \left[P\cdot\epsilon \; Q\cdot K - P\cdot K \; Q\cdot\epsilon \right] ,$$

$$M_C = \gamma_5 \left[\gamma\cdot\epsilon \; Q\cdot K - \gamma\cdot K \; Q\cdot\epsilon \right] ,$$

$$M_D = 2\gamma_5 \left[\gamma\cdot\epsilon \; P\cdot K - \gamma\cdot K \; P\cdot\epsilon - \iota m_\Lambda \, \gamma\cdot\epsilon \; \gamma\cdot K \right] \qquad (18)$$

Here $P = \frac{1}{2}(P_1 + P_2)$ and m_Λ denotes the mass of the Λ. As in ref.[9], we may also introduce a matrix related to the differential cross-section in the centre-of-mass system through

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left| \langle 2 | \mathcal{F} | 1 \rangle \right|^2 \qquad (19)$$

where q and k denote the magnitude of the centre-of-mass three-vector momenta of the incident γ and outgoing π respectively and the matrix element of $\mathcal{F}$ is to be taken between initial and final Pauli spinors. $\mathcal{F}$ may be written as

$$\mathcal{F} = i\vec{\sigma}\cdot\vec{\epsilon}\; \mathcal{J}_1 + \frac{\vec{\sigma}\cdot\vec{q}\;\vec{\sigma}\cdot(\vec{k}\times\vec{\epsilon})}{qk}\mathcal{J}_2 + \frac{i\vec{\sigma}\cdot\vec{k}\;\vec{q}\cdot\vec{\epsilon}}{qk}\mathcal{J}_3 + \frac{i\vec{\sigma}\cdot\vec{q}\;\vec{q}\cdot\vec{\epsilon}}{q^2}\mathcal{J}_4 \qquad (20)$$

where the σ 's are Pauli matrices and $\vec{q}$ and $\vec{k}$ are the meson and photon three-momenta. The relation between the $\mathcal{J}_\ell$'s and A , B , C and D is easily derived[9]. The $\mathcal{J}_\ell$'s are related to transitions involving multipole radiations through

$$\mathcal{J}_1 = \sum_{\ell=0}^{\infty}\left\{\left[\ell M_{\ell+} + E_{\ell+}\right]P'_{\ell+1}(\cos\theta) + \left[(\ell+1)M_{\ell-} + E_{\ell-}\right]P'_{\ell-1}(\cos\theta)\right\},$$

$$\mathcal{J}_2 = \sum_{\ell=1}^{\infty}\left\{(\ell+1)M_{\ell+} + \ell M_{\ell-}\right\}P'_\ell(\cos\theta),$$

$$\mathcal{J}_3 = \sum_{\ell=1}^{\infty}\left\{\left[E_{\ell+} - M_{\ell+}\right]P''_{\ell+1}(\cos\theta) + \left[E_{\ell-} + M_{\ell-}\right]P''_{\ell-1}(\cos\theta)\right\},$$

$$\mathcal{J}_4 = \sum_{\ell=2}^{\infty}\left[M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-}\right]P''_\ell(\cos\theta) \qquad (21)$$

Here θ is the scattering angle in the centre-of-mass system and $M_{\ell\pm}$ and $E_{\ell\pm}$ refer to transitions initiated by magnetic and electric radiation respectively leading to final states of orbital angular momentum ℓ and total angular momentum $J = \ell \pm 1/2$.

With these preliminaries which are to be found in ref.[9] and which we have reproduced for completeness, let us proceed to the calculation of the matrix elements for our process. We shall illustrate the prodedure first for odd Σ-Λ parity. In a static approximation, for odd Σ-Λ relative parity, we find that we can conveniently ignore the contributions arising from the energy variables of the crossed channels, since in this case,

these only modify the scattering amplitude by corrections of the order of $1/m_\Lambda$ compared to the terms which are retained. Thus if μ denotes the Σ-Λ transition magnetic moment, the pole terms for A , B , C and D for odd Σ-Λ parity read

$$A^B = -\mu g_{\Sigma\Lambda\pi} \frac{m_\Lambda + m_\Sigma}{m_\Sigma^2 - \Delta} ,$$

$$B^B = 0 ,$$

$$C^B = -\frac{\mu g_{\Sigma\Lambda\pi}}{m_\Sigma^2 - \Delta} ,$$

$$D^B = -\frac{\mu g_{\Sigma\Lambda\pi}}{m_\Sigma^2 - \Delta} = C^B \qquad (22)$$

Here $\Delta = -(P_1 + K)^2$ and $g_{\Sigma\Lambda\pi}$ is the renormalized Σ-Λ-π coupling constant. A straight-forward calculation then reveals the contributions of (22) to the individual multipole amplitudes to be

$$E_{0+}^B = \frac{\mu g_{\Sigma\Lambda\pi}}{2\pi} \frac{W - m_\Lambda}{W - m_\Sigma} \frac{m_\Lambda}{W} ,$$

$$M_{1+}^B = \frac{1}{32\pi} \frac{q}{W} \mu g_{\Sigma\Lambda\pi} \frac{W - m_\Lambda}{W + m_\Sigma} ,$$

$$M_{1-}^B = \frac{1}{16\pi} \frac{q}{W} \mu g_{\Sigma\Lambda\pi} \frac{W - m_\Lambda}{W + m_\Sigma} ,$$

$$E_{1+}^B = 0 \qquad (23)$$

whereas in the rest of the calculation, we retain only those multipole amplitudes leading to final states with $l = 0$ or 1. In equation (23), we have denoted the total centre-of-mass energy by W. We notice that in a static approximation the leading Born term is E_{0+}^{B}, the others being of the order of $1/m_\Lambda$ compared to it. Therefore, keeping only E_{0+}, we may write

$$E_{0+}(\omega) = E_{0+}^{B}(\omega) + \frac{\omega}{\pi} \int_1^\infty d\omega' \frac{\mathrm{Im}\, E_{0+}(\omega')}{\omega'(\omega' - \omega)} \qquad (24)$$

where $\omega = W - m_\Lambda$ and we have set the pion mass equal to unity. $\mathrm{Im}\, E_{0+}(\omega)$ can as usual be calculated using the unitarity condition in which only the Λ-π intermediate state is retained. We find, in a static approximation,

$$\mathrm{Im}\, E_{0+}(\omega) = e^{-i\delta_0} \sin\delta_0 \; E_{0+}(\omega) \qquad (25)$$

where we have explicitly assumed that the Λ-π interaction proceeds dominantly through S-waves and denoted the S-wave Λ-π scattering phase shift by δ_0. Equation (24) now reduces to the familiar mapping problem discussed by Omnes[10] whose solution reads

$$E_{0+}(\omega) = \left[\frac{f g_{\Sigma\Lambda\pi}}{2\pi} \frac{\omega}{\omega - \delta m} \cos\delta(\omega) + \frac{\omega}{\pi} \exp[P(\omega)] \times \right.$$
$$\left. \times \mathcal{P} \int_1^\infty d\omega' \frac{f g_{\Sigma\Lambda\pi}}{2\pi} \frac{\sin\delta(\omega') \exp[P(\omega')]}{(\omega' - \delta m)(\omega' - \omega)} \right] e^{i\delta(\omega)} \qquad (26)$$

10) R. Omnes, Nuovo Cimento, 8 , 316 (1953).

where

$$P(\omega) = \frac{\omega}{\pi} \mathcal{P} \int_1^\infty d\omega' \frac{\delta(\omega')}{\omega'(\omega' - \omega)} \qquad (27)$$

Here $\delta m = m_\Sigma - m_\Lambda$ and E_{0+}^B has also been replaced by its static form.

For even Σ-Λ relative parity, it is found that the contributions from the crossed channel are of the same order of magnitude as those from the direct channel. Bearing in mind that under crossing $\Lambda \to A$; $C \to -C$ and $D \to D$, the Born terms for the multipoles in the static approximation turn out to be

$$E_{0+}^B(\omega) = \frac{\mu g_{\Sigma\Lambda\pi}}{2\pi} \omega ,$$

$$E_{1+}^B(\omega) = 0 ,$$

$$M_{1+}^B(\omega) = -\frac{\mu g_{\Sigma\Lambda\pi}}{6\pi} \frac{kq}{\omega + \delta m} ,$$

$$M_{1-}^B(\omega) = -\frac{\mu g_{\Sigma\Lambda\pi}}{2\pi} \frac{1}{\omega + \delta m} \qquad (28)$$

Since we are retaining only the S -wave Λ-π scattering amplitude in the unitarity condition, it is easy to see that the different multipole amplitudes do not get coupled by it , so that for each of them we have a relation of the form (25) . Finally the solutions of the dispersion integrals for these amplitudes read

$$E_{0+}(\omega) = 0 \; \Big[\frac{\mu g_{\Sigma\Lambda\pi}}{2\pi} \omega \cos\delta(\omega) + \frac{\omega}{\pi} \exp[P(\omega)] \times$$

$$\times \mathcal{P} \int_1^\infty \frac{\mu g_{\Sigma\Lambda\pi}}{2\pi} \frac{\sin\delta(\omega') \exp[P(\omega')]}{(\omega' - \omega)} d\omega' \Big] e^{i\delta(\omega)} ,$$

$$E_{1+}(\omega) = 0 ,$$

$$M_{1+}(\omega) = -\Bigg[\frac{f g_{\Sigma\Lambda\pi}}{6\pi} \frac{kq}{\omega+\delta m} \cos\delta(\omega) + \frac{\omega}{\pi} \exp[\rho(\omega)] \times$$

$$\times \mathcal{P} \int_1^\infty d\omega' \frac{f g_{\Sigma\Lambda\pi}}{6\pi} \frac{k' q' \sin\delta(\omega') \exp[\rho(\omega')]}{\omega'(\omega'+\delta m)(\omega'-\omega)} \Bigg] e^{i\delta(\omega)},$$

$$M_{1-}(\omega) = -\Bigg[\frac{f g_{\Sigma\Lambda\pi}}{2\pi} \frac{1}{\omega+\delta m} \cos\delta(\omega)$$

$$+ \frac{\omega}{\pi} \exp[\rho(\omega)] \mathcal{P} \int_1^\infty d\omega' \frac{f g_{\Sigma\Lambda\pi}}{2\pi} \frac{\sin\delta(\omega') \exp\rho(\omega')}{\omega'(\omega'+\delta m)(\omega'-\omega)} \Bigg] e^{i\delta(\omega)}$$

(29)

where $\rho(\omega)$ is again given by (27).

We have calculated above the multipole amplitudes for the process $\gamma + \Lambda \to \Lambda + \pi$ for low values of l in terms of the Λ-π scattering phase shifts on the assumption that the Λ-π scattering is dominated by S-wave interaction. Since we have already presented effective range formulae for S-wave Λ-π scattering, we can easily calculate the actual values of the appropriate cross-sections from these formulae. The influence of the process $\gamma + \Lambda \to \Lambda + \pi$ on processes like $\gamma + N \to \Lambda + K + \pi$ is the subject of present investigation.

IV. Effective Range Formulae for Higher Orbital Angular Momentum States*

In Section II, we have investigated the possibility that the recently observed Λ-Π resonance is an S-wave resonance using the static model and the effective range approximation. Since at present there seems to be some experimental indication that the Y_1^* has $J \geq 3/2$ [11], we shall here develop analogous effective range formulae for Λ-Π scattering for the $J = 3/2$ and $J = 5/2$ channels in the orbital angular momentum states $l = 2$ and 3. The possibility of $J = 3/2$, $l = 1$ for the Y_1^* has been the subject of extensive investigations by other workers in the context of the global symmetry model. For orbital angular momentm states other than $l = 0$ we know that the direct pole does not contribute and hence we have to take the residue of the crossed pole only. In the static limit, for $P_{\Sigma\Lambda} = +1$, we have, for the pole term,

$$h_{1+}^{0}(\omega) = \frac{1}{16\pi k^2}\left[A_1 + (\omega - m_\Lambda)B_1 + \frac{(E - m_\Lambda)^2}{k^2}\left\{-A_2 + (\omega - m_\Lambda)B_2\right\}\right]$$

$$= \frac{g_{\Sigma\Lambda\Pi}^2}{16\pi}\frac{2}{(8.5 + 16\omega)^3}\left[3.2 + 11.8\omega + 10.7\omega^2\right] \qquad (30)$$

where

$$A_1 = -g_{\Sigma\Lambda\Pi}^2 (m_\Lambda - m_\Sigma)\frac{2}{3}\frac{a}{b^2},$$

$$B_1 = g_{\Sigma\Lambda\Pi}^2 \frac{2}{3}\frac{a}{b^2},$$

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* Alladi Ramakrishnan, T.K.Radha, R.Thunga and A.P.Balachandran, Nuovo Cimento (in press).

11) Introductory talk by M.G.K.Menon at the Cosmic Ray Symposium, Madras (1961).

$$A_2 = g^2_{\Sigma\Lambda\pi}\,(m_\Lambda - m_\Sigma)\,\frac{4}{15}\,\frac{a^2}{b^3}\,,$$

$$B_2 = -\,g^2_{\Sigma\Lambda\pi}\,\frac{4}{15}\,\frac{a^2}{b^3} \qquad (31)$$

with

$$a = m_\Sigma^2 - m_\Lambda^2 - 1 + 2E_\Lambda\omega$$

$$= 8\cdot5 + 16\,\omega \qquad (32),$$

$$b = 2k^2 = 2(\omega^2 - 1) \qquad (33)$$

and

$$\omega = W - m_\Lambda \qquad (34)$$

and the notation is as in Section II

Here we have set the pion mass equal to unity. It follows that $h_{1+}(\omega)$ satisfies the approximate dispersion relation

$$h_{1+}(\omega) = h^0_{1+}(\omega) + \frac{1}{\pi}\int_1^\infty d\omega'\,\frac{q_{\omega'}\,h_{1+}(\omega')}{\omega' - \omega} \qquad (35)$$

$h^0_{1+}(\omega)$ being given by (30). This may be solved by the N/D method[6] and the solution in the one meson approximation reads

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{(3\cdot2 + 11\cdot8\,\omega + 10\cdot7\omega^2)}{(8\cdot5 + 16\omega)^3}\,\frac{8}{\omega+8}\,k^3\cot\delta_{1+}(\omega)$$

$$= 1 - \frac{16\omega + 8\cdot5}{\pi}\,\mathcal{P}\int_1^\infty d\omega'\,\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{(3\cdot2 + 11\cdot8\,\omega' + 10\cdot7\omega'^2)\,8\,k'^3 v^2(k'^2)}{(8\cdot5 + 16\omega')^4\,(\omega'+8)\,(\omega' - \omega)} \qquad (36)$$

where we have chosen the subtraction point such that renormalised Σ-Λ-π coupling constant gets defined by an analogue of the Lepore-Watson convention for the π-N coupling constant[7] and $v^2(k^2)$ is the usual cut-off function. As in the Chew-Low theory, we may now approximately write

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\ \frac{(3\cdot2+11\cdot8\omega+10\cdot7\omega^2)}{(8\cdot5+16\omega)^3}\ \frac{8}{\omega+8}\ k^3\cot\delta_{1+}=1-\frac{16\omega+8\cdot5}{\omega_T} \qquad (37)$$

where the constant ω_T is positive. The right hand side of equation (37) now shows the possibility of a resonance in the system corresponding to the vanishing of $\cot\delta_{1+}$ for an appropriate value of ω_T. A similar calculation yields for even Σ-Λ parity for $J = 3/2$ and $l = 2$,

$$\frac{g^2_{\Sigma\Lambda\pi}}{16\pi}\ \frac{(0\cdot072+0\cdot26\omega+0\cdot084\omega^2)}{(8\cdot5+16\omega)^3}\ \frac{16}{\omega+8}\ (\omega^2-1)^{5/2}\cot\delta_{2-}=1-\frac{16\omega+8\cdot5}{\omega_T} \qquad (38)$$

which can also give rise to a resonance for a suitable ω_T.

The results for $J = 5/2$ and even Σ-Λ parity are

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\ \frac{[-1\cdot28-4\cdot7\omega-4\cdot2\omega^2]}{(8\cdot5+16\omega)^4}\ \frac{16}{\omega+8}\ (\omega^2-1)^{5/2}\cot\delta_{2+}=1+\frac{16\omega+8\cdot5}{\omega_T} \qquad (39a),$$

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\ \frac{[-0\cdot003-0\cdot02\omega-0\cdot01\omega^2]}{(8\cdot5+16\omega)^4}\ \frac{1}{\omega+8}\ (\omega^2-1)^{7/2}\cot\delta_{3-}=1+\frac{16\omega+8\cdot5}{\omega_T} \qquad (39b)$$

where we have used

$$A_3 = -g^2_{\Sigma\Lambda\pi}\,(m_\Lambda-m_\Sigma)\,\frac{4}{35}\,\frac{b^3}{a^4},$$

$$B_3 = g^2_{\Sigma\Lambda\pi}\,\frac{4}{35}\,\frac{b^3}{a^4} \qquad (40)$$

Here equation (39a) corresponds to $l = 2$ and (39b) to $l = 3$.

For odd Σ-Λ parity, the corresponding formulae are

$$-\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\ \frac{[95+163\omega-11\omega^2]}{(8\cdot5+16\omega)^3}\ \frac{8}{\omega+8}\ (\omega^2-1)^{3/2}\cot\delta_{1+}=1+\frac{16\omega+8\cdot5}{\omega_T}$$

$$(J=3/2,\ l=1),$$

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\ \frac{[4+0\cdot1\omega+0\cdot02\omega^2]}{(8\cdot5+16\omega)^3}\ \frac{16}{\omega+8}\ (\omega^2-1)^{5/2}\cot\delta_{2-}=1-\frac{16\omega+8\cdot5}{\omega_T}$$

$$(J=3/2,\ l=2),$$

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{[38+72\omega-4\cdot25\omega^2]}{(8\cdot5+16\omega)^4}\cdot\frac{16}{\omega+8}(\omega^2-1)^{5/2}\cot\delta_{2+}=1-\frac{16\omega+8\cdot5}{\omega_T}$$

$(J=5/2,\ l=2)$;

$$-\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{(2+0\cdot16\omega+0\cdot01\omega^2)}{(8\cdot5+16\omega)^4}\,\frac{16}{\omega+8}(\omega^2-1)^{7/2}\cot\delta_{3-}$$

$$=1+\frac{16\omega+8\cdot5}{\omega_T}\quad\left(J=\tfrac{5}{2},\ l=3\right)\quad(41)$$

In all these, ω_T denotes an appropriate positive constant. We have also written down the P -wave, $J=$ 3/2 effective range formula for odd Σ-Λ parity in equation (41).

Equations (39) and (41) now indicate that within the scope of our approximations, a $J=$ 5/2 resonance is forbidden for even Σ-Λ parity. Similarly we find that if the Σ-Λ parity is odd, there can be no P -wave resonance with $J=$ 3/2 or an F- wave resonance with $J=$ 5/2. However, a P- or a D- wave resonance with $J=$ 3/2 is possible for even Σ-Λ parity while a D -wave resonance with $J=$ 3/2 or 5/2 is possible for odd Σ-Λ parity.

We give below curves showing the variation of the scattering cross-section in the $J=$ 3/2, $l=$ 2 channel with the centre-of-mass ~~energy~~ momentum (in natural units) for various values for the Σ-Λ-π coupling constant and for both even and odd parities.

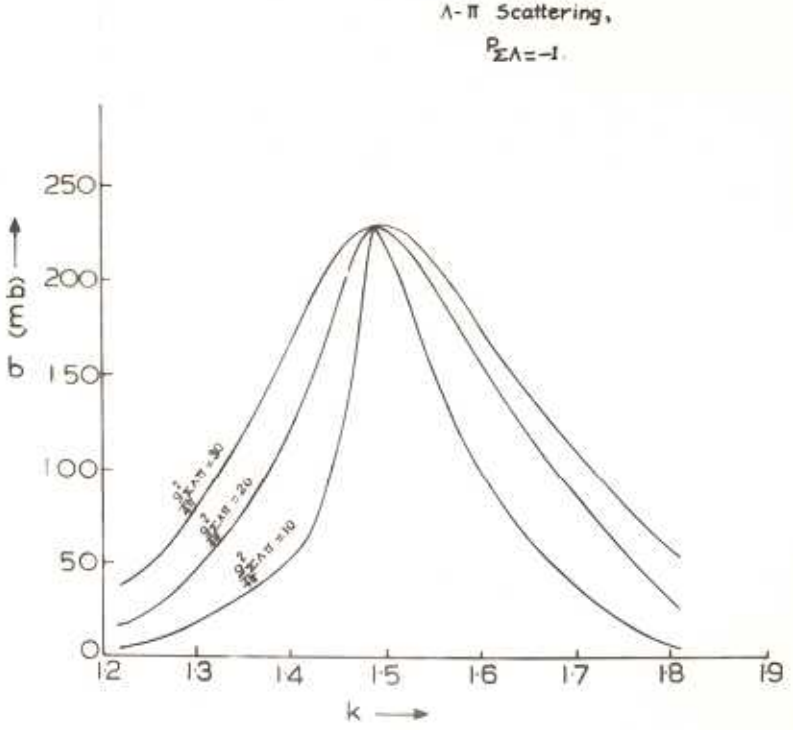
Λ-π Scattering,
$P_{\Sigma\Lambda}=-1$
250
200
150
100
50
0
σ (mb)
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
k
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=30$
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=20$
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=10$

Fig.9.

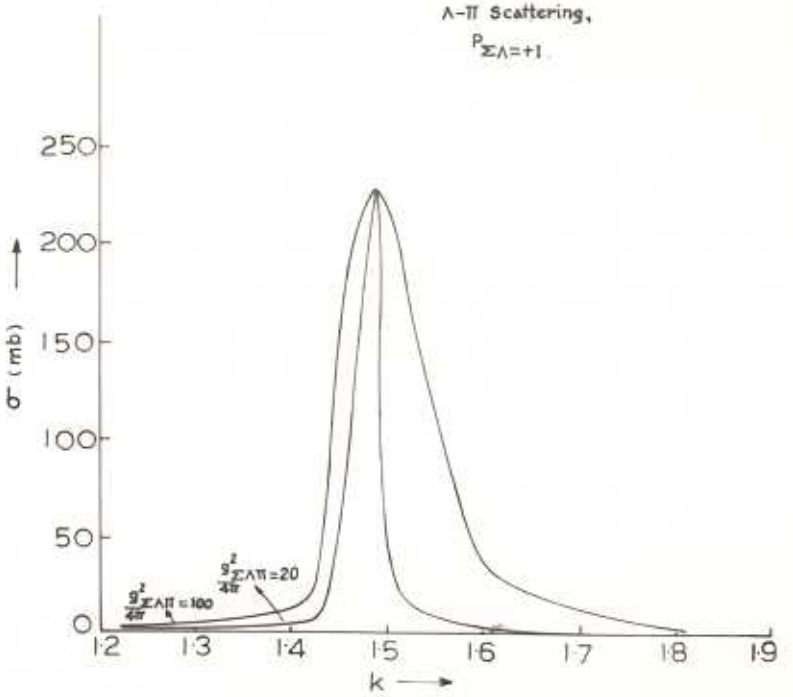
Λ-π Scattering,
$P_{\Sigma\Lambda}=+1$
250
200
150
100
50
0
σ (mb)
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
k
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=100$
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=20$

Fig. 10

The curves show that we can reproduce the observed half-width of 25 Mev for the Λ-π resonance with $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} \approx 20$ for odd Σ-Λ patrity. However for even Σ-Λ parity $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$ has to be of the order of 100 if we are to obtain the 25 Mev half-width for the Y_1^*. The very large value of the coupling constant required to obtain a D-wave $J = 3/2$, Y_1^* resonance with the observed width for even $\Sigma\Lambda$ parity indicates that such a resonance is precluded within the scope of our model. It is worthwhile noticing that the one meson approximation which we have made implies that wehave completely neglected the presence of the $\bar{K}$-N channel and its influence on the Λ-π scattering. If it should subsequently turn out that the Y_1^* resonance has the parameters $J = 3/2$ and $\ell = 2$ and that the Σ-Λ relative

parity is even, it may well be an indication that this resonance can be understood only if we appropriately take into account the $\bar{K}$-N intermediate sta tes.

CHAPTER IV.

LOW ENERGY $K^{\pm}$ NUCLEON SCATTE RING*

1. Introduction

Recently a variety of resonances have been observed in systems involving strongly interacting particles. Thus there is a Λ-π resonance (called the Y_1^*) with isotopic spin 1 and a mass of about 1385 Mev[1], a Σ-π resonance (called the Y_0^*) with zero isotopic spin and a mass of about 1400 Mev[2], and a π-π resonance (called the ρ -meson) with spin 1 and isotopic spin 1[3]. We devote this chapter to an analysis of low-energy S -wave $K^{\pm}N$ scattering data, approximating the scattering amplitude by the contributions from these resonances and the Λ- and Σ- pole terms. Using the analytical form of the amplitude provided by the approximation, we attempt to deduce the unknown parameters by fitting the results of Rodberg and Thaler[4] on the low energy S -wave $K^{\pm}N$ scattering phase shifts. Such an analysis seems to be of particular interest in view of the success of the resonance and pole approximations in explaining the isovector part of the nucleon electromagnetic form factor[5] and the life-time of the charged pion[6].

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* Alladi Ramakrishnan, A.P.Balachandran and K.Raman, Nuovo Cimento (~~in press~~). 24, 369 (1962).

1) Alston et. al., Phys. Rev. Letters, 5, 520 (1960);
Berge et. al., Phys. Rev. Letters, 6 , 557 (1961);
Block et. al., Nuovo Cimento, 20 , 724 (1961);
Dehl et. al., Phys. Rev. Letters, 6 , 142 (1961);
M.H.Alston and M. Ferro-Luzzi, Rev.Mod.Phys., 33 , 416 (1961).

2) Alston et. al., Phys. Rev. Letters, 6 , 698 (1961).

3) Erwin et. al., Phys. Rev. Letters, 6 , 628 (1961).

4) L.S.Rodberg and R.M.Thaler, Phys. Rev. Letters, 4,372 (1960).

5) W.R.Frazer and J.R.Fulco, Phys. Rev., 117, 1609 (1960).

6) J.Bernstein, S.Fubini, M.Gell-Mann and W.Thirring, Nuovo Cimento, 17 , 757 (1960). See also J.Bernstein, M.Gell-Mann and L.Michel, Nuovo Cimento, 16, 560 (1960); M.Gell-Mann and M. Levy, Nuovo Cimento, 16, 705 (1960); Y.Nambu, Phys. Rev. Letters, 4 , 380 (1960);.

Evidence has appeared recently for a 3π bound state[7] and two a 3π resonant states[8]. We shall tentatively neglect the contributions from these states, essentially to restrict the number of unknowns.

2. The calculation

Let p_1 and q_1 denote the incident four-vector momenta and p_2 and q_2 the outgoing four-vector momenta of the nucleon and K- meson respectively. Define the variables

$$s = -(p_1 + q_1)^2,$$
$$\bar{s} = -(p_1 - q_2)^2,$$
$$t = -(p_1 - p_2)^2,$$
$$Q = \frac{q_1 + q_2}{2} \qquad (1)$$

s is the square of the total centre-of-mass energy in the process $K + N \to K + N$ (which we shall call process I) while $\bar{s}$ and t are the momentum transfers for this process. In the centre-of-mass system of the reaction $\bar{K} + N \to \bar{K} + N$ (the reaction II), $\bar{s}$ is the square of total energy while in the centre-of-mass system of the reaction $K + \bar{K} \to N + \bar{N}$ (the reaction III), t is the corresponding variable. Following Chew, Goldberger, Low and Nambu[9], we write the T -matrix for reaction I as

$$T^{I} = -A + i\gamma Q\, B \qquad (2)$$

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7) A. Abashian, N.E.Booth and K.M.Crowe, Phys. Rev. Letters, 5 , 258 (1960); Rev. Mod. Phys., 33 , 393 (1961).
8) Maglic et. al., Phys. Rev. Letters, 7 , 178 (1961).
9) G.F.Chew, M.L.Goldberger, F.E.Low and Y. Nambu, Phys. Rev., 106 , 1337 (1957).

A and B can be decomposed in terms of the amplitudes for definite isotopic spin using triplet and singlet projection operators :

$$A = A^0 \frac{1 - \vec{\tau}_N \cdot \vec{\tau}_K}{4} + A^1 \frac{3 + \vec{\tau}_N \cdot \vec{\tau}_K}{4} \qquad (3)$$

where A^I denotes the amplitude for isotopic spin and $\vec{\tau}_N$ and $\vec{\tau}_K$ are Pauli matrices operating between the nucleon and K -meson isotopic spin wave functions. A similar decomposition holds for B . An analogous decomposition also gives us the $I=0$ and $I=1$ amplitudes $\bar{A}^0$, $\bar{B}^0$ and $\bar{A}^1$, $\bar{B}^1$ of process II . $\bar{A}^0$ and $\bar{A}^1$ are related to A^0 and A^1 through the equation $\bar{A}^i(s,\bar{s},t) = \Gamma^{II}_{ij} \bar{A}^j(s,\bar{s},t)$ ($i,j = 0,1$) where the crossing matrix Γ^{II} is given by

$$\Gamma^{II} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad (4)$$

Our notation is such that in a function $F(x,y,z)$ the first variable denotes the energy of reaction I , the second that of II and the third that of III . The crossing matrix for the B is $+\Gamma^{II}$ while the crossing matrix from reaction III to reaction I reads

$$\Gamma^{III} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad (5)$$

Equation(4) can be derived as follows.[10] Let $T^{\beta i\, \mathrm{II}}_{\beta\alpha}$ denote the T -matrix element for the process $N_\alpha + \bar{K}_i \to K N_\beta + \bar{K}_j$ where the indices i , j are the isotopic spin indices of $\bar{K}$ while α and β are to be understood as a collective symbol denoting both spinor and isotopic spin indices of the nucleon. We can then write

$$(2\pi)^4 \delta(p_2 + q_2 - p_1 - q_1)\, T^{\beta i\, \mathrm{II}}_{\beta\alpha}(\vec{x}, \vec{y}, t)$$
$$= i \left[\frac{p_{10}\, p_{20}}{m_N^2}\right]^{\frac{1}{2}} \int d^4x\, d^4y\, e^{-ip_2 x} \left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m_N\right)_{\beta\beta'}$$
$$\langle \bar{K}_j(q_2) | T\left(\psi^N_{\beta'}(x)\, \bar{\psi}^N_{\alpha'}(y)\right) | \bar{K}_i(q_1)\rangle\, e^{ip_1 y} \times$$
$$\times \left[-\tilde{\gamma}_\mu \frac{\overleftarrow{\partial}}{\partial y_\mu} + m_N\right]_{\alpha\alpha'} \qquad (6)$$

where the q's in $|\bar{K}(q)\rangle$ denote the momentum of the in that state. Assuming charge conjugation invariance, we can write $|\bar{K}_i(q_1)\rangle = \mathcal{C}\,|K_i(q_1)\rangle$, $\langle \bar{K}_j(q_2)| = \langle K_j(q_2)|\,\mathcal{C}^{-1}$ where $\mathcal{C}$ is the unitary operator which effects the charge conjugation of the state vectors. Since

$$\mathcal{C}^{-1}\, \psi^N(x)\, \mathcal{C} = i\tau_2\, C\, \bar{\psi}^N(x), \qquad (7),$$

$$\mathcal{C}^{-1}\, \bar{\psi}^N(x)\, \mathcal{C} = i\tau_2\, C^{-1}\, \psi^N(x) \qquad (8).$$

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10) See for example, M.L.Goldberger, M.T.Gnsaru, S.W.MacDowell and D.Y.Wong, Phys. Rev., 120 , 2250 (1960).

where C is the usual unitary antisymmetric matrix with the property

$$C^{-1}\gamma_\mu C = -\tilde{\gamma}_\mu \qquad (9),$$

we find

$$T^{II\ ji}_{\beta\alpha}(\bar{\delta},\delta,t) = -C^{-1}_{\alpha\alpha'}\,\tau_2^{\alpha\alpha'}\,T^{I\ ji}_{\alpha'\beta'}(\bar{\delta},\delta,t)\,\tau_2^{\beta'\beta}\,C_{\beta'\beta}$$

$$= \tau_2^{\beta\beta'}\Big[\Big\{A^0(\bar{\delta},\delta,t)\mp i\gamma_Q\bar{B}^0(\bar{\delta},\delta,t)\Big\}\frac{\delta_{\beta'\alpha'}\delta_{ji}-\vec{\tilde{\tau}}_N^{\,\beta'\alpha'}\vec{\tau}_K^{\,ji}}{4}$$

$$+\Big\{A^1(\bar{\delta},\delta,t)\mp i\gamma_Q\bar{B}^1(\bar{\delta},\delta,t)\Big\}\frac{3\delta_{\beta'\alpha'}\delta_{ji}+\vec{\tilde{\tau}}_N^{\,\beta'\alpha'}\vec{\tau}_K^{\,ji}}{4}\Big]\tau_2^{\alpha'\alpha}$$

$$=\Big\{-\bar{A}^0(\bar{\delta},\delta,t)+i\gamma_Q\bar{B}^0(\bar{\delta},\delta,t)\Big\}\frac{\delta_{\beta\alpha}\delta_{ji}-\vec{\tau}_N^{\,\beta\alpha}\vec{\tau}_K^{\,ji}}{4}$$

$$+\Big\{-\bar{A}^1(\bar{\delta},\delta,t)+i\gamma_Q\bar{B}^1(\bar{\delta},\delta,t)\Big\}\frac{3\delta_{\beta\alpha}\delta_{ji}+\vec{\tau}_N^{\,\beta\alpha}\vec{\tau}_K^{\,ji}}{4} \qquad (10).$$

(4) now follows. (5) is also derived along similar lines

We shall now evaluate the pole and resonance contributions to A and B. There is a strong experimental indication for an odd $K-\Lambda$ relative parity[11]. Assuming this to be true and denoting by $g_{KN\Lambda}$ the $K-N-\Lambda$ coupling constant, we see that this pole contributes to A and B expressions of the form

$$A_\Lambda^0 = \frac{1}{2} g_{KN\Lambda}^2 \frac{m_\Lambda - m_N}{m_\Lambda^2 - \bar{\delta}},$$

$$B_\Lambda^0 = \frac{1}{2} g_{KN\Lambda}^2 \frac{1}{m_\Lambda^2 - \bar{\delta}},$$

$$A_\Lambda^1 = -\frac{1}{2} g_{KN\Lambda}^2 \frac{m_\Lambda - m_N}{m_\Lambda^2 - \bar{\delta}},$$

$$B_\Lambda^1 = -\frac{1}{2} g_{KN\Lambda}^2 \frac{1}{m_\Lambda^2 - \bar{\delta}} \qquad (11)$$

Here the subscript Λ on A and B denotes that the contribution is from the Λ- pole while the superscript denotes the isotopic spin state. The masses of the particles have been denoted by m with the particle symbol as the subscript. We shall follow a similar notation in what follows also.

There is some tentative indication for an odd $\Sigma-\Lambda$ parity[12, 13] with $\frac{g_{KN\Sigma}^2}{4\pi} \approx 0.6$[13] where $g_{KN\Sigma}$ is the $K-N-\Sigma$ coupling constant. If we accept this, we can write for the Σ- pole contribution,

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11) Block et. al., Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester, page 419.

12) J.J.Sakurai and Y. Nambu, Phys. Rev. Letters 6, 377 (1961).

13) J.J.Sakurai, Nuovo Cimento, 20, 1212 (1961)

$$A^0_\Sigma = \frac{3}{2} g^2_{KN\Sigma} \frac{m_N + m_\Sigma}{m^2_\Sigma - \bar{s}},$$

$$B^0_\Sigma = -\frac{3}{2} g^2_{KN\Sigma} \frac{1}{m^2_\Sigma - \bar{s}},$$

$$A^1_\Sigma = \frac{1}{2} g^2_{KN\Sigma} \frac{m_N + m_\Sigma}{m^2_\Sigma - \bar{s}},$$

$$B^1_\Sigma = -\frac{1}{2} g^2_{KN\Sigma} \frac{1}{m^2_\Sigma - \bar{s}} \qquad (12)$$

For the Λ-$\bar{\pi}$ resonance, we assume a contribution to process II of the form

$$\bar{A}^1_{Y_1^*} = \frac{\lambda_1}{m^2_{Y_1^*} - \bar{s}} \qquad (13)$$

while for the Σ-$\bar{\pi}$ resonance, the contribution is assumed to be of the form

$$\bar{A}^0_{Y_0^*} = \frac{\lambda_2}{m^2_{Y_0^*} - \bar{s}} \qquad (14)$$

Finally, for the π-π resonance contribution to process III, we assume the form

$$\mathcal{C}^1_\rho = \frac{\lambda_3}{m^2_\rho - t} \qquad (15)$$

where $\mathcal{C}$ is the scattering amplitude for process III. Here we have systematically neglected the widths of the resonances. The total resonance and pole contributions A_r and B_r to A and B can now be written down. We have

$$A_\pi^0 = \frac{1}{2} g^2_{KN\Lambda} \frac{m_\Lambda - m_N}{m_\Lambda^2 - \bar{s}} + \frac{3}{2} g^2_{KN\Sigma} \frac{m_N + m_\Sigma}{m_\Sigma^2 - \bar{s}} - \frac{3}{2} \frac{\lambda_1}{m^2_{Y_1^*} - \bar{s}}$$

$$+ \frac{1}{2} \frac{\lambda_2}{m^2_{Y_0^*} - \bar{s}} + \frac{3}{2} \frac{\lambda_3}{m_\rho^2 - t} + \qquad (16),$$

$$B_\pi^0 = \frac{1}{2} g^2_{KN\Lambda} \frac{1}{m_\Lambda^2 - \bar{s}} - \frac{3}{2} g^2_{KN\Sigma} \frac{1}{m_\Sigma^2 - \bar{s}} + \qquad (17),$$

$$A_\pi^1 = -\frac{1}{2} g^2_{KN\Lambda} \frac{m_\Lambda - m_N}{m_\Lambda^2 - \bar{s}} + \frac{1}{2} g^2_{KN\Sigma} \frac{m_N + m_\Sigma}{m_\Sigma^2 - \bar{s}} \qquad (18),$$

$$- \frac{1}{2} \frac{\lambda_1}{m^2_{Y_1^*} - \bar{s}} - \frac{1}{2} \frac{\lambda_2}{m^2_{Y_0^*} - \bar{s}} - \frac{1}{2} \frac{\lambda_3}{m_\rho^2 - t}$$

$$B_\pi^1 = -\frac{1}{2} g^2_{KN\Lambda} \frac{1}{m_\Lambda^2 - \bar{s}} - \frac{1}{2} g^2_{KN\Sigma} \frac{1}{m_\Sigma^2 - \bar{s}} \qquad (19).$$

Defining

$$A_\ell^I(s) = \int_{-1}^{1} d\cos\theta \; P_\ell(\cos\theta) \, A(s, \cos\theta) \qquad (20)$$

and similarly for B_ℓ^I, we can form the amplitude suggested by Frazer and Fulco[14]:

$$h_\ell^I(k^2) = \frac{W}{E + m_N} \frac{1}{k^{2\ell}} \frac{e^{i\delta^I_{\ell+}} \sin\delta^I_{\ell+}}{k}$$

$$= \frac{1}{16\pi} \left[\frac{A_\ell^I}{k^{2\ell}} + (W - m_N) \frac{B_\ell^I}{k^{2\ell}} + (E - m_N)^2 \left\{ - \frac{A^I_{\ell+1}}{k^{2\ell+2}} + (W + m_N) \frac{B^I_{\ell+1}}{k^{2\ell+2}} \right\} \right] \qquad (21)$$

where W is the total and E the nucleon energy in the centre-of-mass system of reaction I and k is the corresponding momentum. $\delta^I_{\ell+}$ is the K-N scattering phase shift in a state of total angular momentum $J = \ell + \frac{1}{2}$. (cf. ref. 9)) and isotopic

14) W.R.Frazer and J.R.Fulco, Phys. Rev., 119, 1420 (1960).

spin I . The contribution $J_0^1(k^2)$ from the poles and resonances to $h_0^1(k^2)$ is therefore given by

$$J_0^1(k^2) = \frac{1}{16\pi}\left[F_0^1 + (W-m_N)G_0^1 + (E-m_N)^2\left\{-\frac{F_1^1}{k^2} + \frac{W-m_N}{k^2}G_1^1\right\}\right] \quad (22)$$

where

$$F_0^1 = -\beta_1\frac{m_\Lambda - m_N}{2k^2}\log x_\Lambda + \beta_2\frac{m_N+m_\Sigma}{2k^2}\log x_\Sigma + \frac{\alpha_1}{2k^2}\log x_{y_1^*}$$

$$+\frac{\alpha_2}{2k^2}\log x_{y_0^*} + \frac{\alpha_3}{2k^2}\log\left(\frac{4k^2}{m_\pi^2}+1\right),$$

$$G_0^1 = -\frac{\beta_1}{2k^2}\log x_\Lambda - \frac{\beta_2}{2k^2}\log x_\Sigma,$$

$$F_1^1 = f_1\left(k^2, m_\Lambda, -\beta_1[m_\Lambda - m_N]\right) + f_1\left(k^2, m_\Sigma, \beta_2[m_N+m_\Sigma]\right)$$

$$+ f_1\left(k^2, m_{y_1^*}, \alpha_3\right) + f_1\left(k^2, m_{y_0^*}, \alpha_4\right) + f_2\left(k^2, m_\pi, \alpha_5\right)$$

and

$$G_1^1 = f_1\left(k^2, m_\Lambda, -\beta_1\right) + f_1\left(k^2, m_\Sigma, -\beta_2\right) \quad (23)$$

Here

$$x_y = \frac{\delta + m_y^2 - 2(m_N^2+m_K^2)}{m_y^2 - \frac{(m_N^2-m_K^2)^2}{\delta}},$$

$$f_1(k^2, m, \alpha) = \frac{\alpha}{2k^2}\left[1 + \frac{2m_N^2+2m_K^2-m^2-\delta}{2k^2}\right]\log x_m$$

$$-\frac{\alpha}{4k^4}\left[2m_N^2+2m_K^2-\delta - \frac{(m_N^2-m_K^2)^2}{\delta}\right]$$

and

$$f_2(k^2, m, \alpha) = \frac{\alpha}{2k^2}\left(2 + \frac{m^2+2k^2}{2k^2}\log\frac{m^2}{4k^2+m^2}\right) \quad (24)$$

The α's and β's in these equations are related to the coupling constants through

$$\alpha_i = -\frac{1}{2}\lambda_i \qquad (i = 1, 2, 3),$$

$$\beta_1 = \frac{1}{2} g^2_{KN\Lambda},$$

$$\beta_2 = \frac{1}{2} g^2_{KN\Sigma} \qquad (25)$$

Approximating the singularities associated with the $\bar{\nu}$ and t variables by those arising from the poles and resonances, we can write down the dispersion relation for $h_0^1(k^2)$:

$$h_0^1(k^2) = J_0^1(k^2) + \frac{1}{\pi}\int_0^\infty dk'^2 \frac{\mathrm{Im}\, h_0^1(k'^2)}{k'^2 - k^2 - i\epsilon} \qquad (26)$$

where we have as yet made no subtractions. The N/D solution[15] of (26) with the elastic unitarity condition on the right-hand cut and one subtraction for D reads

$$J_0^1(k^2)\,\frac{k}{w}(E+m_N)\cot\delta_0^1(k^2)$$
$$= 1 - \frac{(k^2 - k_0^2)}{\pi} P\int_0^\infty dk'^2 \frac{J_0^1(k'^2)\,k'(E'+m_N)}{w'(k'^2 - k_0^2)(k'^2 - k^2)} \qquad (27)$$

where k_0^2 is the subtraction point and we have normalized $D(k^2)$ to unity there. As remarked in the introduction, in this calculation we will neglect the rescattering correctness completely and attempt to fit the data from the contributions coming from the resonances and poles alone. In such an approximation, (27) becomes

$$J_0^1(k^2)\,\frac{k}{w}(E+m_N)\cot\delta_0^1(k^2) = 1 \qquad (28)$$

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15) G.F.Chew and S.Mandelstam, Phys. Rev., 119 , 467 (1960)

Analogous to (23), we have for the S -wave, $I=0$ scattering phase shift the formula

$$J_0^0(k^2)\,\frac{k}{W}\,(E+m_N)\cot\delta_0^0(k^2) = 1 \qquad (29)$$

where

$$J_0^0(k^2) = \frac{1}{16\pi}\left[F_0^0 + (W-m_N)G_0^0 + (E-m_N)^2\left\{-\frac{F_1^0}{k^2} + \frac{(W+m_N)}{k^2}G_1^0\right\}\right] \qquad (30)$$

Here

$$F_0^0 = \frac{\beta_1(m_\Lambda - m_N)}{2k^2}\log x_\Lambda + \frac{3\beta_2(m_N+m_\Sigma)}{2k^2}\log x_\Sigma$$

$$+\frac{3\alpha_3}{2k^2}\log x_{Y_1^*} - \frac{\alpha_4}{2k^2}\log x_{Y_0^*} + \frac{3\alpha_5}{2k^2}\log\left(\frac{m_\rho^2}{4k^2+m_\rho^2}\right),$$

$$G_0^0 = \frac{\beta_1}{2k^2}\log x_\Lambda - \frac{3\beta_2}{2k^2}\log x_\Sigma,$$

$$F_1^0 = f_1\left[k^2, m_\Lambda, \beta_1(m_\Lambda - m_N)\right] + f_1\left[k^2, m_\Sigma, 3\beta_2(m_N+m_\Sigma)\right]$$

$$+ f_1\left[k^2, m_{Y_1^*}, 3\alpha_3\right] + f_1\left[k^2, m_{Y_0^*}, -\alpha_4\right] + f_2\left[k^2, m_\rho, -3\alpha_5\right]$$

and

$$G_1^0 = f_1\left(k^2, m_\Lambda, \beta_1\right) + f_1\left(k^2, m_\Sigma, -3\beta_2\right) \qquad (31)$$

The approximation used here makes the cross-sections go to zero at zero momentum for which reason it is probably not good at very low energies. We therefore attempt to fit the data above 50 Mev . The $I=1$ scattering data obtained from K^+ scattering on hydrogen has been fitted by Rodberg and Thaler [4] by an S -wave effective range formula (the amplitudes for $l>0$ being negligible in this isotopic spin sxtate) :

$$k\cot\delta_0^1 = -\frac{1}{a_0^1} + \frac{1}{2}r_0^1 k^2 \qquad (32)$$

with the scattering length a_0^I : 0.34 fermis and the effective range r_0^I : 0.50 fermis, a_0^I and r_0^I being accurate to about 10 % . The results were used by us between 50 and 210 Mev laboratory energy together with the rough estimates of the $I=0$, S -wave phase shifts at three values of the K -meson energy given by Rodberg and Thaler[4] to evaluate our unknown parameters.

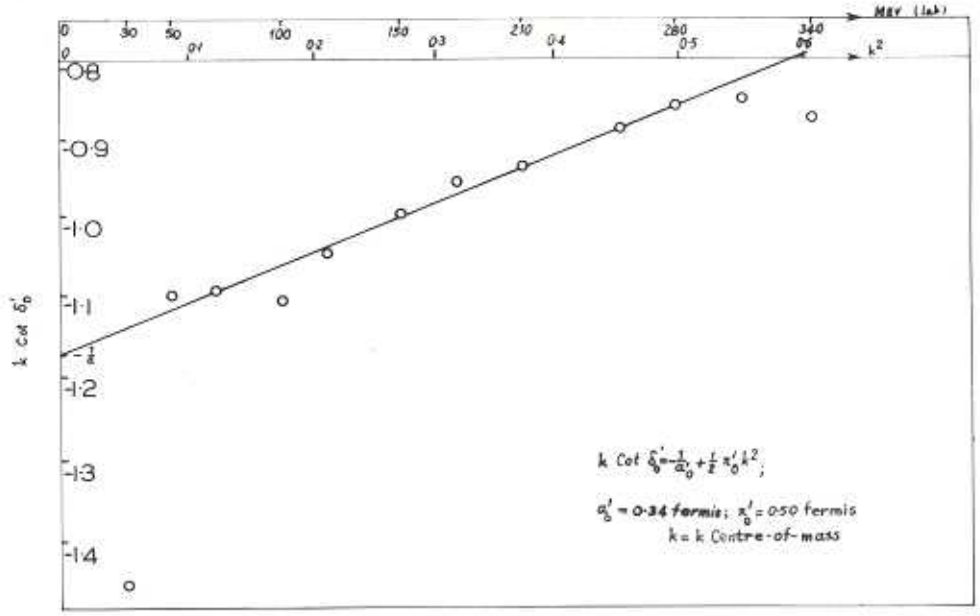

Fig.1.

Of the five constants to be determined, we accept Sakurai's estimate[13] for $g_{KN\Sigma}$, namely $\frac{g^2_{KN\Sigma}}{4\pi} = 0.6$. Of the remaining four, the $I=1$ phase shifts alone determine two of the constants and the sum of the other two; the $I=0$ phase shifts were used to separate the last two constants.

Our results are:

$$\frac{g^2_{KN\Lambda}}{4\pi} = 9.55,$$

$$\lambda_1 = -2.28,$$

$$\lambda_2 = 1000,$$

$$\lambda_3 = -210$$

where we have assumed

$$\frac{g^2_{KN\Sigma}}{4\pi} = 0.6 \qquad (33)$$

We emphasise that these are only order of magnitude estimates as the data used are not accurate enough. The S -wave $\overset{K-N}{\Lambda}$ cross-section predicted by these estimates is shown in Fig. 2 while Fig. 3 shows the S -wave contribution to the $I=0$ cross-section.

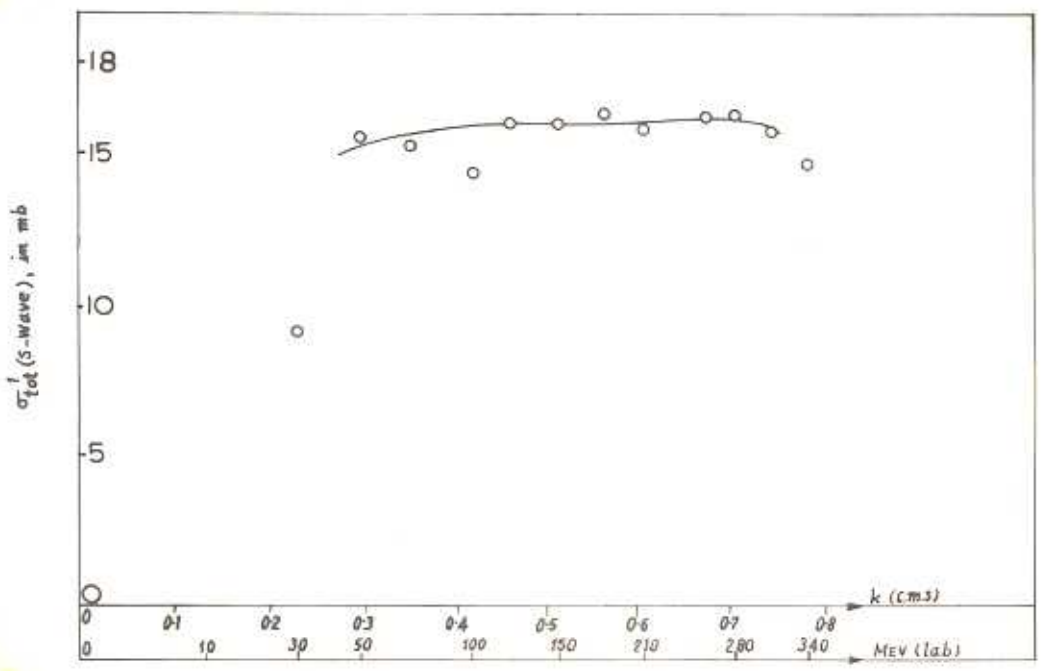

Fig.2.

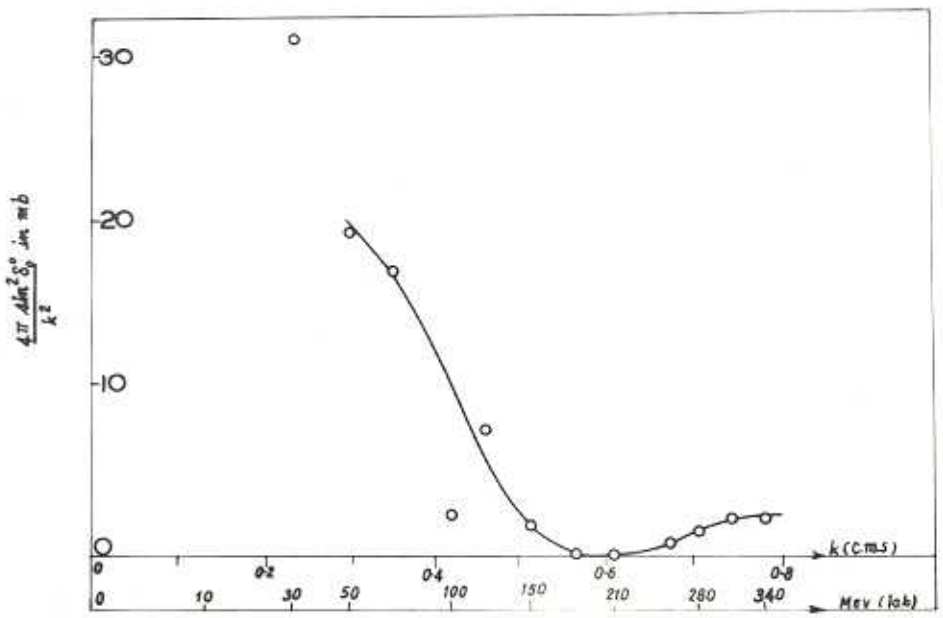

Fig. 3.

As far as we are aware, no detailed analysis of the data giving $\sim$ the $I=0$ cross-section is as yet available. Therefore it will be desirable to carry out comparisons of these formulae directly with the total cross-sections for $K^+ - n$ elastic and charge exchange scattering to find out how good such approximations are. Such a task is at present being attempted with the p-wave contributions where we explicitly taken into account.

CHAPTER V.

The K- meson-Pion scattering*

1. Introduction

Recently a pion-pion resonance in the isotopic spin $I = 1$ and angular momentum $J = 1$ state, long suspected from the electromagnetic structure data[1] has been experimentally confirmed[2]. A $K-\pi$ resonance with $I = \frac{1}{2}$ has also been observed at Berkeley by Alston et. al.[3]. While no conclusive evidence is as yet available as to the spin of this resonance, from the copiousness of its production and its narrow width, C.H.Chan[4] has made out a case for assigning this particle $J = 1$. In this paper we shall assume that this assignment is correct and attempt to develop an approximation for the $K-\pi$ scattering amplitude in the resonance channel using the Mandelstam representation[5]. Our procedure is to replace the contributions arising from the singularities in the crossed $(\text{energy})^2$ variables by the $K-\pi$ and $\pi-\pi$ resonance terms at the appropriate energy and with zero widths for the resonances. With these assumptions, it is easy to write down an integral

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* A.P.Balachandran, Nuovo Cimento (in press)

1) W.R.Frazer and J.R.Fulco, Phys. Rev., 117, 1609 (1960).
2) Erwin et. al., Phys. Rev.Letters, 6, 628 (1961).
3) Alston et. al., Phys. Rev.Letters, 6, 600 (1961).
4) C.H.Chan, Phys. Rev. Letters, 6, 383 (1961).
5) Previous calculations on this process using the Mandelstam representation are those of B.W.Lee, Phys. Rev., 120, 325 (1960) and B.W.Lee and K.S.Cho, Nuovo Cimento, 20, 553 (1961).

equation for the $K-\pi$ scattering amplitude in the resonance channel. The equation is solved by the N/D method[6] with two subtractions for D one of which is then fixed by normalization and the other by the requirement that there be a resonance in the $K-\pi$ system at the appropriate energy. This gives a two-parameter approximation for the scattering amplitude and the variation of the phase shift for various values of these parameters is presented. The main defect of the approximation employed is that it does not preserve crossing symmetry. The calculation may very well be inadequate because of this. However it may be of some interest to see the sort of results such an approximation leads to. Further, as will be indicated later, it can be used as the first step in a self-consistency calculation.

The kinematical results we require in this calculation have been already tabulated in our discussion of the Λ-N scattering. We shall use the same notation here also with $K_1 + \pi_1 \rightarrow K_2 + \pi_2$ denoting process I, $K_1 + \pi_2 \rightarrow K_2 + \pi_1$ denoting process II and $K_1 + \bar{K}_2 \rightarrow \pi_1 + \pi_2$ denoting process III. This notation will be briefly recalled once again at the appropriate places. We will be writing the dispersion relations for the $I=\frac{1}{2}$, p wave amplitude in the k^2 complex plane where the physical branch cut extends from $k^2 = 0$ to ∞. Here k is the centre-of-mass momentum for process I. It is not essential to know the singularities of the partial waves due to singularities of the total amplitude

6) G.F.Chew and S.Mandelstam, Phys.Rev., 119, 467 (1960)

in the $\bar{t}$ and t variables in the approximation we employ since we will be explicitly evaluating these contributions using a zero width resonance approximation.

2. The s-wave, $I = 1/2$ scattering amplitude

The S-matrix for the process is as usual written as

$$S_{fi} = \delta_{fi}\, \delta(P_f - P_i) - i (2\pi)^4 \delta(P_f - P_i) \frac{1}{4\omega_K \omega_\pi} T_{fi} \qquad (1)$$

where P_f is the final and P_i the initial four-momentum vector. We are working the centre-of-mass system and ω_K denotes the energy of the K-meson (incident or outgoing) and ω_π that of the π-meson in this system. Let α and β denote the isotopic spin indices of the pion. Then we may write T as a matrix in the isotopic spin space of the kaon in the following way[7].

$$T_{\beta\alpha} = \delta_{\beta\alpha} A^+ + \frac{1}{2} [\tau_\beta, \tau_\alpha] A^- \qquad (2)$$

where the τ's are the usual Pauli spin matrices. The relation between the eigen amplitudes $A^{\frac{1}{2}}$ and $A^{\frac{3}{2}}$ for scattering in the isotopic spin states 1/2 and 3/2 in channel I and the $A^{\pm}$ of equation (2) are given by[7]

$$A^{\frac{1}{2}} = A^+ + 2A^-,$$
$$A^{3/2} = A^+ - A^- \qquad (3).$$

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7) G.F.Chew, M.L.Goldberger, F.E.Low and Y. Nambu, Phys. Rev., 106, 1337 (1957).

The analogous relations for Channel II are

$$\bar{A}^{\frac{1}{2}} = A^{+} - 2A^{-},$$

$$\bar{A}^{3/2} = A^{+} + A^{-} \qquad (4).$$

For Channel III , if B^0 and B^1 denote the isotopic spin amplitudes for $I=0$ and $I=1$, we have[8]

$$B^0 = \sqrt{6}\, A^{+},$$

$$B^1 = 2A^{-} \qquad (5).$$

For reaction I , the ℓ th partial wave amplitude in the isotopic spin state I can be defined in terms of the A^I of equation (3) as follows[6] :

$$A_\ell^I(s) = \frac{1}{2}\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, A^I(s,\bar{t},t)$$

$$= \frac{\sqrt{s}}{k}\, e^{i\delta_\ell^I} \sin\delta_\ell^I \qquad (6).$$

The variables have the same significance as in Λ-N scattering. Thus s is the square of the centre-of-mass energy, θ the scattering angle and k the centre-of-mass momentum of reaction I. $\bar{t}$ and t are the square of the energies in the centre-of-mass systems of reactions II and III . A relation analogous to (6) Holds for reaction II between $\bar{A}_\ell^I(\bar{t})$ and $\bar{A}^I(s,\bar{t},t)$. Note that as usual $s+\bar{t}+t = 2m^2+2$, m and 1 denoting the K -meson and pion masses (where as in what follows, the pion mass is taken to be unity).

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8) W.R. Frazer and J.R. Fulco, Phys. Rev., 117 , 1603 (1960).

's remarked earlier, we approximate the contributions from the singularities in the $\bar{t}$ and t variables by the ones due to the $K-\pi$ and $\pi-\pi$ resonances of zero widths. This would imply for instance that for the crossed $K-\pi$ process in the p -wave, $I = 1/2$ state, we can write

$$\mathrm{Im}\, \bar{A}_1^{\frac{1}{2}}(\bar{t}) = \frac{\sqrt{\bar{t}}}{\ell}\, \gamma \ell^3\, \pi\, \delta(\bar{t} - s_r) \qquad (7)$$

where ℓ denotes the centre-of-mass momentum of process II and $\sqrt{s_r}$ denotes the mass of the $K-\pi$ resonance which is taken to be 885 Mev.[3] The contribution to process I from (7) is clearly

$$\alpha_1^{\frac{1}{2}} = -\frac{1}{3}\,\frac{1}{\pi}\,\frac{1}{2}\int_{-1}^{1} d\cos\theta\, \cos\theta \int_{(m+1)^2}^{\infty} d\bar{t}'\, 3\cos\varphi'\, \frac{\sqrt{\bar{t}'}}{\ell'}$$

$$\times\ \frac{\gamma \ell'^3\, \pi\, \delta(\bar{t}' - s_r)}{\bar{t}' - \bar{t}} \qquad (8)$$

where φ is the scattering angle in channel II and the factor - 1/3 comes fromthe crossing matrix. The latter is easily derived by an inspection of equations (3) and (4). We have:

$$\begin{bmatrix} A^{\frac{1}{2}} \\ A^{\frac{3}{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \bar{A}^{\frac{1}{2}} \\ \bar{A}^{3/2} \end{bmatrix} \qquad (9).$$

The integration over $\cos\theta$ in (8) can be written as on integral over $\bar{t}$ using the relation

$$\bar{t} = \frac{1-\cos\theta}{2}\,\frac{(m^2-1)^2}{s} - \frac{1+\cos\theta}{2}\,(s - 2m^2 - 2) \qquad (10).$$

so that we have

$$\alpha_1^{\frac{1}{2}} = \frac{1}{12k^2}\,\frac{1}{\pi}\int_{L_1(s)}^{L_2(s)} d\bar{t}\left(1+\frac{\Sigma - s - \bar{t}}{2k^2}\right)\int_{(m+1)^2}^{\infty} d\bar{t}'\; 3\cos\varphi' \times$$

$$\times\ \frac{\sqrt{\bar{t}'}}{\ell'}\ \frac{\gamma \ell'^3\,\pi\,\delta(\bar{t}'-t_r)}{\bar{t}'-\bar{t}} \qquad (11)$$

where $\Sigma = 2m^2+2$ and

$$L_1(s) = \frac{(m^2-1)^2}{s},$$

$$L_2(s) = 2m^2+2-s \qquad (12)$$

The integrations in (11) are easy to perform and finally for $\alpha_1^{\frac{1}{2}}$, we have the expression

$$\alpha_1^{\frac{1}{2}} = -\frac{3\gamma\cos\varphi_r\sqrt{s_r}\,\ell_r^2}{12k^2}\left[\log\frac{2m^2+2-s-s_r}{\frac{(m^2-1)^2}{s}-s_r}\times\right.$$

$$\left.\times\left\{1+\frac{\Sigma-s}{2k^2}-s_r\right\}-2m^2-2+s+\frac{(m^2-1)^2}{s}\right] \qquad (13)$$

where ℓ_r is the centre-of-mass momentum in channel II at the resonance energy and $\cos\varphi_r$ is defined through

$$s = \frac{1-\cos\varphi_r}{2}\,\frac{(m^2-1)^2}{s_r} - \frac{1+\cos\varphi_r}{2}\,(s_r - 2m^2 - 2) \qquad (14)$$

Similarly we may assume for the $K+\bar{K}\to\pi+\pi$ amplitude a resonant form and write for its contribution to $A_1^{\frac{1}{2}}$,

$$\beta_1^{\frac{1}{2}} = \frac{1}{4k^2}\int_{-4k^2}^{0} dt'\left(1+\frac{t'}{2k^2}\right)\frac{\lambda}{t'-t_r}\,3\cos\eta \qquad (15)$$

Here we have used the relation

$$t = -2k^2(1-\cos\theta) \qquad (16)$$

to convert the $\cos\theta$ integration into a t integration. η is the scattering angle in the centre-of-mass system of reaction III and t_π is the square of the mass of the $\pi-\pi$ resonance which we take to be 22 as suggested by Bowcock, Cottingham and Lurie[9]. As an examination of equations (3) and (5) show, the crossing matrix in this case is given by

$$\begin{bmatrix} A^{\frac{1}{2}} \\ A^{\frac{3}{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & 1 \\ \frac{1}{\sqrt{6}} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} B^0 \\ B^1 \end{bmatrix} \qquad (17)$$

so that the coefficient in (15) arising from this is unity.

Now, if we regard the $\pi-\pi$ resonance as an elementary particle (which we shall denote by the symbol ρ) with unit spin and isotopic spin, we can compute the Born approximation amplitude due to the $\pi-\pi$ resonant intermediate state. Let $g_{\rho\pi}$ and $g_{\rho K}$ denote the renormalized $\pi-\pi-\rho$ and $K-K-\rho$ coupling constants. Then the Born amplitude reads

$$2 g_{\rho\pi} g_{\rho K} \frac{s-\bar{t}}{t-m_\rho^2} = 2 g_{\rho\pi} g_{\rho K} \frac{4pq\cos\eta}{t-m_\rho^2} \qquad (18)$$

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9) J. Bowcock, W.N.Cottingham and D.Lurie, Phys. Rev. Letters, 5 , 386 (1960); Nuovo Cimento, 16, 918 (1960).

Here m_ρ is the mass of the ρ -meson and p and q are the centre-of-mass momenta of the K -meson and π -meson in process III which are related to t through

$$p = \tfrac{1}{2}(t - 4m^2)^{\frac{1}{2}},$$
$$q = \tfrac{1}{2}(t - 4)^{\frac{1}{2}} \qquad (19)$$

Comparison of (15) and (18) shows that in (15), we have set $pq \approx p_r q_r$ where p_r and q_r correspond to the point of resonance. Thus from (18), we may write

$$\cos\eta \approx \frac{s - t}{4 p_r q_r} = \frac{s - t}{(t_r - 4m^2)^{1/2}(t_r - 4)^{\frac{1}{2}}} \qquad (20)$$

It follows that

$$\beta_1^{\frac{1}{2}} = \frac{3\lambda}{4k^2(t_r - 4m^2)^{\frac{1}{2}}(t_r - 4)^{\frac{1}{2}}} \left\{ (2s - \Sigma)\log\frac{t_r}{4k^2 + t_r} + (2s - \Sigma)\left(2 + \frac{t_r}{2k^2}\log\frac{t_r}{4k^2 + t_r}\right) + \left(4k^2 + t_r \log\frac{t_r}{4k^2 + t_r}\right) + \frac{1}{2k^2}\left(-8k^4 + 4t_r k^2 + t_r^2 \log\frac{t_r}{4k^2 + t_r}\right)\right\} \qquad (21)$$

If we denote the quantity $\alpha_1^{\frac{1}{2}} + \beta_1^{\frac{1}{2}}$ by $p_1^{\frac{1}{2}}$, the p -wave, $I = \frac{1}{2}$ scattering amplitude for the K-π system has the representation

$$A_1^{\frac{1}{2}}(k^2) = p_1^{\frac{1}{2}}(k^2) + \frac{1}{\pi}\int_0^\infty dk'^2 \frac{\mathrm{Im}\, A_1^{\frac{1}{2}}(k'^2)}{k'^2 - k^2 - i\varepsilon} \qquad (22)$$

where we have as yet made no subtractions. The N/D solution of (22)[6] with two subtractions for D is

$$p_1^{\frac{1}{2}}(k^2)\,\frac{k}{\sqrt{s}}\,\cot\delta_1^{\frac{1}{2}}(k^2) = 1 + \Gamma(k^2 - k_0^2) - \frac{(k^2-k_0^2)^2}{\pi} P\int_0^\infty dk'^2\,\frac{p_1^{\frac{1}{2}}(k'^2)\,k'}{\sqrt{s'}\,(k'^2-k_0^2)^2\,(k'^2-k^2)} \qquad (23)$$

where we have used the unitarity condition with neglect of all inelastic channels, Γ is the subtraction constant and $D(k^2)$ is normalized to unity at $k^2 = k_0^2$. We shall conveniently choose $k_0^2 = -1$. The integral in (23) has been computed numerically by hand. We can now impose the condition that its left-hand side vanish at $k^2 = k_r^2 = 4.08$ which is the value at which the $K-\pi$ system develops a resonance. This gives

$$\Gamma = -0\cdot20 - 0\cdot20\,\Gamma_1 - 0\cdot03\,\Gamma_2 \qquad (24)$$

where

$$\Gamma_1 = \frac{\gamma\sqrt{s_r}\,k_r^2}{8},$$

$$\Gamma_2 = \frac{3}{4}\,\frac{\lambda}{(t_r-4m^2)^{1/2}\,(t_r-4)^{\frac{1}{2}}} \qquad (25)$$

γ and λ are related to the usual coupling constants through the relations

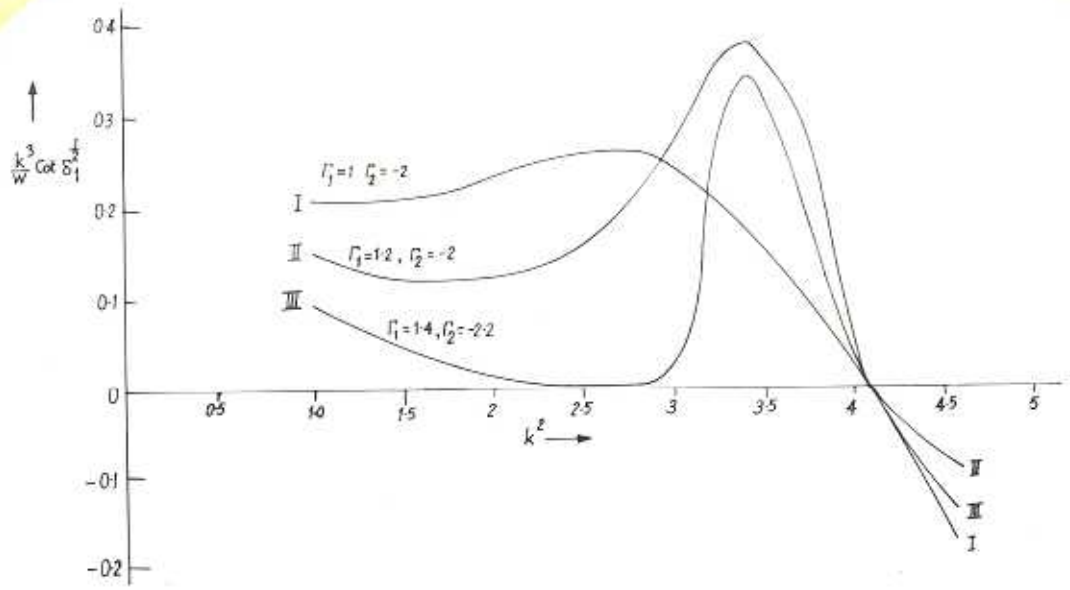
$$\gamma = \frac{4\,g_{K'}^2}{\sqrt{s_r}},$$

$$\lambda = \frac{8}{3}\,g_{\rho\pi}\,g_{\rho K}\,p_r\,q_r = \frac{2}{3}\,g_{\rho\pi}\,g_{\rho K}\,(t_r-4m^2)^{\frac{1}{2}}\,(t_r-4)^{\frac{1}{2}} \qquad (26)$$

Here $g_{K'}$ is the renormalized $K-K'-\pi$ coupling constant with

the $K-\pi$ resonance (denoted by K') being regarded as elementary. The relation for λ is obtained by comparing (15) and (18) while a similar comparisonof equation (7) with the corresponding Born amplitudes yields the relation for γ . Equation (26) indicates that γ is positive while λ depends on the relative sign of $g_{\rho\pi}$ and $g_{\rho K}$. It follows that Γ_1 is also positive.

We give below some curves showing the variation of $\frac{k^3}{w}\cot\delta_1^{1/2}$ with k^2 (in natural units) / where $W=\sqrt{s}$ is the total centre-of-mass energy of reaction I . The values of Γ_1 and Γ_2 are chosen to correspond to values of $\frac{g^2_{\rho\pi}}{4\pi}$ and $\frac{g^2_{\rho K}}{4\pi}$ of the order of magnitude of 1/3 and of $\frac{g^2_{K'}}{4\pi}$ of the order of magnitude of 1/25. The curves are <u>very sensitive</u> to the values of Γ_1 and Γ_2 essentially because $\phi_1^{\frac{1}{2}}(k^2)$ is sensitive to small changes in their values. <u>Therefore to obtain reliable results, it seems essential to evaluate the integrals to a higher degree of accuracy than what can be achieved by hand computation</u> (which is the method which has been adopted in this paper for lack of better facilities).



Conceivably, we can use such a calculation as the first step in a self-consistency procedure where the phase shifts we have obtained with the assumption of a predominant zero width resonance in the channel II is used as the starting point of a second calculation where channel II is now assumed to have these phase shifts. It is of course crucial in such a procedure that the zero width approximation for the π-π resonance in channel III be a good one.

We have not discussed here the K-π scattering amplitudes in the other channels. These amplitudes are presumably small compared to the amplitude in the resonant channel. A detailed examination of this point is at present in progress by Mr. K. Raman of our group.

CHAPTER VI.

A NOTE ON SCATTERING AND PRODUCTION AMPLITUDES*

We derive below a few exact results in scattering theory involving the production amplitude for an arbitrary process with n final channels and the elastic scattering amplitudes of the initial and final states in such a process. This is an extension of some results of Sucher and Day[1] who discussed the problem when only one production channel is open. We consider a process in which a given initial state $|i\rangle$ can go over into any one of the final states $|f_{(\beta)}\rangle$ $(\beta = 1, 2, \cdots, n)$ The scattering amplitude from $|i\rangle$ to $|f_{(\beta)}\rangle$ can as usual be defined by

$$M_{f_{(\beta)} i} = \langle f_{(\beta)} | T | i \rangle \qquad (1)$$

where T is the T -matrix satisfying the integral equation

$$T = H_I + H_I G T \qquad (2)$$

Here H_I is the interaction Hamiltonian and

$$G = (E - H_0 + i\varepsilon)^{-1} \qquad (3)$$

H_0 being the free field Hamiltonian. Associated with $M_{f_{(\beta)} i}$

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* A.P.Balachandran and N.R.Ranganathan, Nuclear Physics, 18, 81, (1960).

1) J. Sucher and T.B.Daym, Nuovo Cimento, 13 , 1111 (1959).

we can define a "pure" production amplitude $P_{f_{\sigma(\beta)} i}$ which would be the value of $M_{f_{\sigma(\beta)} i}$ if there were no initial or final state interactions. $P_{f_{\sigma(\beta)} i}$ can be defined through the relation

$$P_{f_{\sigma(\beta)} i} = \langle f_{\sigma(\beta)} | T_p | i \rangle \qquad (4)$$

where T_p is the solution of the equation

$$T_p = H_I + H_I G_p T_p \qquad (5)$$

where

$$G_p = G(1 - \Lambda_{fi}) \qquad (6)$$

Here Λ_{fi} is the projection operator with unit eigen value for any one of the initial or final states and zero eigen value for any other state.

We can also define a scattering amplitude $\mathcal{S}_i$ in the initial state which ~ involves none of the final particles in its intermediate states. We have

$$\mathcal{S}_i = \langle i | T_S | i \rangle \qquad (7)$$

where

$$T_S = H_I + H_I G_S T_S \qquad (8)$$

with

$$G_S = G\left[1 - \sum_{\beta=1}^{n} \Lambda_{f_{\sigma(\beta)}}\right] \qquad (9)$$

and

$$\Lambda_{f_{\sigma(\beta)}} | f_{\sigma(\beta)} \rangle = | f_{\sigma(\beta)} \rangle, \quad \Lambda_{f_{\sigma(\beta)}} | \rangle = 0 \qquad (10)$$

In the last formula, $|\rangle$ denotes any state other than $| f_{\sigma(\beta)} \rangle$.

Elementary manipulations then give us the following formulae :

$$M_{\beta_{(\beta)} i} = P_{\beta_{(\beta)} i} + P_{\beta_{(\beta)} i} \cdot \partial_i + \sum_{k=1}^{n} M_{\beta_{(\beta)} \beta_{(k)}} \cdot P_{\beta_{(k)} i}$$

$$+ \sum_{k=1}^{n} M_{\beta_{(\beta)} \beta_{(k)}} \cdot P_{\beta_{(k)} i} \cdot \partial_i \qquad (11),$$

$$M_{ii} = \partial_i + \sum_{k=1}^{n} P_{i \beta_{(k)}} \cdot M_{\beta_{(k)} i} + \sum_{k=1}^{n} \partial_i \cdot P_{i \beta_{(k)}} \cdot M_{\beta_{(k)} i} \qquad (12)$$

The "dot" multiplication in (11) and (12) implies for example that

$$M_{\beta_{(\beta)} \beta_{(k)}} \cdot P_{\beta_{(k)} i} = \sum \frac{M_{\beta_{(\beta)} \beta_{(k)}} (q_\beta, q_k) P_{\beta_{(k)} i} (q_k, q_i)}{E - E_k + i\epsilon} \qquad (13)$$

where the sum is over the complete range of all the quantum numbers of the k th final state and q_τ is used as a collective symbol to denote the complete set of commuting observables specifying the τ th state. If we now eliminate ∂_i from equations (11) and (12) , we have

$$M_{\beta_{(\beta)} i} = P_{\beta_{(\beta)} i} + P_{\beta_{(\beta)} i} \cdot M_{ii}$$

$$+ \sum_{k=1}^{n} \left[M_{\beta_{(\beta)} \beta_{(k)}} \cdot P_{\beta_{(k)} i} + M_{\beta_{(\beta)} \beta_{(k)}} \cdot P_{\beta_{(k)} i} \cdot M_{ii} \right.$$

$$\left. - M_{\beta_{(\beta)} i} \cdot P_{i \beta_{(k)}} \cdot M_{\beta_{(k)} i} \right] \qquad (14)$$

A result of some interest for a process like γ-γ scattering

is given by (12) . Let us define

$$|\gamma\gamma\rangle = |i\rangle, \quad |e^-e^+\rangle = |f\rangle \qquad (15)$$

(12) gives (if we neglect the interaction of the γ -ray with the charged fields other than the electron),

$$M_{ii} = \partial_i + P_{if} \cdot M_{fi} + \partial_i \cdot P_{if} \cdot M_{fi} \qquad (16)$$

where M_{ii} is the complete γ-γ scattering amplitude. Remembering that ∂_i denotes that part of the γ-γ scattering amplitude which does not involve electron-positron pairs in the intermediate states, we have $\partial_i = 0$ since even the lowest order diagram for γ-γ scattering (the square diagram) involves such a pair in the intermediate state. Thus

$$M_{ii} = P_{if} \cdot M_{fi} \qquad (17)$$

To order e^2 , $P_{if} = M_{if}$ so that to order e^4, (17) reduces to the familiar result

$$M_{ii} = M_{if} \cdot M_{fi} \qquad (18)$$

Equations like (14) can be of some use in deducing unknown scattering amplitudes provided a method of continuing these amplitudes off the energy-shell is available. An example will be to try to deduce something regarding the π-π scattering amplitude knowing the annihilation amplitude $N + \bar{N} \to \pi + \pi$ $(= M_{fi})$ and the scattering amplitude $N + \bar{N} \to N + \bar{N}$ $(= M_{ii})$ (say from experimental data) and assuming some approximate expression for P_{fi} which is defined so as to involve no π-π intermediate state

(Here $|\alpha\rangle = |N\bar{N}\rangle$ and $|\beta\rangle = |\pi\pi\rangle$). (14) will be then be an integral equation for the $\pi\text{-}\pi$ scattering amplitude $M_{\beta\beta}$ which can therefore be solved for. Of crucial importance in such a procedure is clearly to know before hand the matrix elements $M_{\beta i}$ and M_{ii} off the energy shell also. As far as we are aware, there is no method of obtaining matrix elements off the energy shell from the experimentally known matrix elements which are on the energy shell. If such techniques can be developed, one anticipates that equations like (14) will be of considerable interest.

CHAPTER VII.

SOME REMARKS ON SAKURAI'S THEORY OF STRONG INTERACTIONS*

Recently a great deal of interest has been evinced in gauge theories of elementary particles following the fundamental work of Yang and Mills[1] on the localization of isotopic spin rotations. We mention in particular the work of Gell-Mann[2], Neeman[3], Sakurai[4] and Salam and Ward[5]. The idea involved in all these speculations is a well-known one in electrodynamics which states that the existence of an electromagnetic field coupled to the current of the charged field is a necessary consequence of demanding the invariance of the Lagrangian under gauge transformations of the first kind:

$$\psi(x) \rightarrow e^{ie\Lambda(x)} \psi(x) \qquad (1)$$

where e denotes the charge of the field ψ and Λ is the gauge variable which is a function of x. With constant Λ, the invariance of the Lagrangian under (1) will imply that charge is conserved. However when Λ is taken to be a function of x, the gauge acquires a deep and beautiful significance. It implies that the phase of the field can be altered arbitrarily from one space-time point to another, that is, the local phase of the field is not a quantity of significance. The notion of

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* A.P.Balachandran and N.G.Deshpande, Nuovo Cimento, 20, 1022, (1961)

1) C.N.Yang and R.L.Mills, Phys. Rev., 96 , 191 (1954), see also ref. 6)
2) M.Gell-Mann, Phys. Rev. (to be published)
3) Y.Neeman, Nuclear Physics 26, 222(1961).
4) J.J.Sakurai, Ann. Phys., 11 , 1 (1960)
5) A. Salam and J.C.Ward, Nuovo Cimento, 19 , 165 (1961)

locality thus acquires a very much richer meaning if we take our gauge to be local and insist that our Lagrangian be invariant under it. Let us now examine the nature of the consequences that such an assumption would imply. We will consider for simplicity the case where ψ is a spin 1/2 field. Analogous considerations apply to the other types of fields also. The free Lagrangian

$$\mathcal{L}_f = -\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi \qquad (2)$$

of ψ is not invariant under (1) so that we are constrained to replace $\mathcal{L}_f$ by

$$\mathcal{L} = -\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi + i e \bar{\psi} \gamma_\mu \psi A_\mu \qquad (3)$$

where A_μ is a new vector field which may be identified with the electromagnetic field and which undergoes the gauge transformation of the second kind

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x) \qquad (4)$$

when ψ transforms as in (1) . (4) further implies that A_μ is massless. In this sense then, the existence of a massless electromagnetic field is a direct consequence of the conservation of charge. It may be noted that what we have done in generating the electromagnetic field is to replace the ordinary derivative ∂_μ occurring (2) by the covariant derivative $(\partial_\mu - i e A_\mu)$ [6] corresponding to the space-time dependant transformation (1) , a procedure well-known in general relativity.

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6) R. Utiyama , Phys. Rev., 101 , 1597 (1956).

Yang and Mills[1] whose paper we mentioned earlier attempted a localization of isotopic spin rotations in a way analogous to the localization of the phase transformation associated with charge conservation while Sakurai[4] with whose work we are primarily concerned applied these ideas to the three conservation laws characteristic of strong interactions, namely, those associated with baryon number, hypercharge and isotopic spin. His interaction Lagrangian which results from these considerations reads

$$\mathcal{L}_{int} = f_B \left\{ i\bar{N}\gamma_\mu N + i\bar{\Lambda}\gamma_\mu \Lambda + i\bar{\Sigma}\gamma_\mu \Sigma + i\bar{\Xi}\gamma_\mu \Xi \right\} B_\mu^{(B)}$$

$$+ f_Y \left\{ i\bar{N}\gamma_\mu N - i\bar{\Xi}\gamma_\mu \Xi + i\left(\partial_\mu K^+ K - K^+ \partial_\mu K\right) \right.$$

$$\left. - f_Y K^+ K B_\mu^{(Y)} \right\} B_\mu^{(Y)}$$

$$+ f_I \left\{ i\bar{N}\frac{\vec{\tau}}{2}\gamma_\mu N - \vec{\Sigma}\times\gamma_\mu\vec{\Sigma} + i\bar{\Xi}\frac{\vec{\tau}}{2}\gamma_\mu \Xi \right.$$

$$+ \vec{\pi}\times\partial_\mu\vec{\pi} - f_I \vec{\pi}\cdot\vec{\pi}\,\vec{B}_\mu^{(I)} + i\left(\partial_\mu K^+ \frac{\vec{\tau}}{2} K - K^+ \frac{\vec{\tau}}{2}\partial_\mu K\right)$$

$$\left. + - f_I K^+ K \vec{B}_\mu^{(I)} + \vec{f}_{\mu\nu}^{(I)} \times \vec{B}_\nu^{(I)} \right\} \vec{B}_\mu^{(I)}$$

where

$$\vec{f}_{\mu\nu}^{(I)} = \partial_\nu \vec{B}_\mu^{(I)} - \partial_\mu \vec{B}_\nu^{(I)} - f_I \vec{B}_\mu \times \vec{B}_\nu \qquad (6)$$

The fields are here denoted by the corresponding particle symbols. $B_\mu^{(B)}$ denotes the $I = 0$ vector field associated with baryon conservation and $B_\mu^{(Y)}$ the $I = 0$ vector field

associated with hypercharge conservation while $\vec{B}_\mu^{(I)}$ which is the Yang-Mills meson[1], is an isotopic spin triplet and is associated with isotopic spin conservation.

We wish to show that in such a theory, it is essential that only two of the fields involved (apart from those of the vector mesons) can be elementary. Here we regard isotopic spin multiplets as constituting a single field. In the Lagrangian (5), all the fields make their appearance since they are all regarded as elementary. (5) shows that any baryon or spinless meson field always occurs bilinearly so that if we denote any one of these fields by φ, the Lagrangian is invariant under the transformation

$$\varphi \rightarrow -\varphi \qquad (7)$$

Thus any reaction which involves a final state which is different from the initial state is forbidden. For example, consider the reaction $A + B \rightarrow C + D$. We have

$$\langle C D | A B \rangle = - \langle C D | A B \rangle = 0 \qquad (8)$$

Here we assume of course that this is not a reaction of some such form, as for example $p + \bar{p} \rightarrow \pi^+ + \pi^-$ or $A + B \rightarrow A + B + B_\mu$. Thus if all the fields are elementary, $K^- + p$ cannot go into $\Sigma^\mp + \pi^\pm$ or the vector fields $B_\mu^{(B)}$ or $B_\mu^{(Y)}$ decay into an odd number of pions. These results have an obvious analogy to the well-known fact that in the presence of electromagnetic interactions alone, the decay $\mu^\pm \rightarrow e^\pm + \gamma$ is forbidden since there too the

the electromagnetic interaction is generated by local gauge transformations and consequently have the form i.e. $ie\left[\bar{e}\gamma_\mu e + \bar{\mu}\gamma_\mu\mu\right]A_\mu$. Further, just as electromagnetic interactions do not define the μ-e relative parity, the Lagrangian (5) also does not define any of the relevant relative parities between the strongly interacting fields. Thus the reaction $\pi^- + D \rightarrow n + n$ [7] whose study yielded the pion parity experimentally, is forbidden in such a theory. Also this Lagrangian cannot give an effective interaction of the form $ig\bar{N}\gamma_5\vec{\tau}N\cdot\vec{\pi}$ or any other interaction which is linear in the individual baryon or spinless meson fields, as such interactions do not possess the symmetries of the primitive Lagrangian and give rise to reactions forbidden by it.

The above discussion assumes that all the fields are elementary. One verifies that the difficulties pointed out in such a case are removed if we assume that the baryon and spinless meson fields are built out of two basic fields (which is the minimum number necessary to give us all the quantum numbers). One may for instance assume the Sakata model[8] or the Goldhaber-Gyorgyi model[9] for these fields. We emphasize that what we state here is at variance with Sakurai's point of view[4] that it does not matter whether "elementary" particles are really elementary or not in his

7) W.K.H.Panofsky, R.L.Aamodt and J.Hadley, Phys. Rev., 81 , 565 (1951).
8) S. Sakata, Progr. Theoret. Phys.(Kyoto), 16, 686 (1956).
9) M.Goldhaber, Phys. Rev., 92 , 1279 (1953); 101, 433 (1956); G. Gyorgyi, Zh. eksper. teor. fiz., 32,152(1957).

theory. Even with this compound model, however, the theory does not explain why there is no bound state of an $\bar{n}$ and a Σ^+ in a simple way. This could of course be due to the detailed dynamics of the interaction of which one can as usual say very little with confidence. The most unsatisfactory feature of these theories (which we mention for completeness) is a well-known one, namely, that the vector mesons associated with these gauge transformations (except possibly the Yang-Mills meson) turn out to be massless while experimentally there is very strong evidence against the existence of such massless vector fields. It would seem that here once again, we have an indication of a breakdown of the basic concepts underlying local field theories. It is true that the definition of a local field does not necessarily require that its local phase should be a matter of irrelevance. However the latter principle, which gives such a beautiful and deep significance to the notion of a local field, works very well indeed for electromagnetism and gravitation[6]. Therefore its failure for the other classes of interactions should be regarded as an extremely unhappy feature of the present field theories.

CHAPTER VIII.

A Note on the K -Hyperon Relative Parities *

We present below two possible methods of determining the $K-\Sigma$ relative parity $P_{K\Sigma}$ and the $K-\Lambda$ relative parity $P_{K\Lambda}$. In the first method, we study the reactions $\pi^- + p \to \Sigma^0 + K^0$, $\Sigma^0 \to \Lambda^0 + \gamma$ and show how the polarization of the Λ^0 can be used to decide the sign of $P_{K\Sigma}$. This is an adaptation of the method of Capps[1] who studies the reaction $\pi^- + p \to \Lambda^0 + K^0$. In the second method, which can equally well be used to obtain the $K-\Lambda$ relative parity, we study the reactions $K^- + p \to \Sigma^0 + \gamma$, $\Sigma^0 \to \Lambda^0 + \gamma$ which are useful to determine $P_{K\Sigma}$ and the reaction $K^- + p \to \Lambda^0 + \gamma$ which is of interest in determining $P_{K\Lambda}$.

1) Capps[1] has shown that if in the reaction $\pi^- + p \to \Sigma^0(\Lambda^0) + K^0$, the target protons are polarized perpendicular to the incident pion beam, then the $\Sigma^0(\Lambda^0)$'s emitted in the directions $\theta = 0$ or π (the angle θ being measured from the direction of the incident pion beam) are polarized tranversely, the polarization being given by

$$P_\Sigma(\mp, \vec{m}) = \mp P_t(\vec{m}) \qquad (1)$$

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* A.P.Balachandran and N.R.Ranganathan, Nuovo Cimento, 16 , 1142 (1960).

1) R.H.Capps, Phys. Rev., 115 , 736 (1959)

where $P_i(\vec{m})$ denotes the polarization of the target in the direction $\vec{m}$ (i.e. in the direction transverse to its momentum.) The minus and plus signs in $P_\Sigma(\mp, \vec{m})$ correspond to the cases when $p_{K\Sigma} = -1$ and $+1$ respectively. For completeness, we shall sketch the derivation of these results of Capps[1]. Thus the T- matrix has the following general form in the centre-of-mass system:

$$T = i C \vec{\sigma} \cdot \vec{\nu}_i + i D \vec{\sigma} \cdot \vec{\nu}_f \qquad (2)$$

if $p_{K\Sigma} = +1$ and

$$T = A\,1 + i B \vec{\sigma} \cdot \vec{\nu}_i \times \vec{\nu}_f \qquad (3)$$

if $p_{K\Sigma} = -1$. Here $\vec{\nu}_i$ and $\vec{\nu}_f$ are unit vectors in the directions of the initial and final meson momenta and $\vec{\sigma}$ and 1 are the Pauli spin operator and the unit operator operating between the nucleon and hyperon spin states. A, B, C and D are complex functions of $\vec{\nu}_i \cdot \vec{\nu}_f = \cos\theta$ and the centre-of-mass energy ω.

Let us denote by ρ_N and ρ_Σ the density matrices of the nucleon and Σ. If we normalize ρ_N such that $Sp.[\rho_N] = 1$ where $Sp[x]$ denotes the spur of the spin matrix x, the initial polarization in a direction $\vec{k}$ is given by

$$P_N(\vec{k}) = Sp[\vec{\sigma} \cdot \vec{k}\, \rho_N] \qquad (4)$$

ρ_Σ is related to ρ_N through the relation $\rho_\Sigma = T \rho_N T^\dagger$. The final polarization in a direction $\vec{n}$ is given by

$$P_{\Sigma}(\vec{n}, \omega, \theta) = \frac{Sp\left[\vec{\sigma}.\vec{n}\ \rho_{\Sigma}(\omega,\theta)\right]}{Sp\left[\rho_{\Sigma}(\omega,\theta)\right]} \qquad (5)$$

Notice that ρ_{Σ} and P_{Σ} are in general functions of the energy ω and the scattering angle θ. We have also the following expression for ρ_N in terms of P_N from (4) :

$$\rho_N = \frac{1}{2}\left[1 + P_N\ \vec{\sigma}.\vec{k}\right] \qquad (6)$$

With these formulae, the final polarization P_{Σ} can be immediately expressed in terms of the initial polarization P_N. The procedure is straight-forward and the details are given in Capps' paper[1]. One obtains finally the result (1) when the protons are transversely polarized and $\theta = 0$ or π.

Now consider the decay of Σ^0 in its own rest system into a Λ^0 and a γ. Let $\vec{q}$ denote the unit vector in the direction of the Λ^0 momentum and $\vec{\epsilon}$ the polarization of the γ -ray. We have then the relation $\vec{\epsilon}.\vec{q} = 0$. The T -matrix for even $\Sigma - \Lambda$ parity is

$$T = A'\ \vec{\tau}\ [\vec{\epsilon} \times \vec{q}\ ] \qquad (7)$$

while for odd $\Sigma - \Lambda$ parity, it reads

$$T = B'\ \vec{\tau}.\vec{\epsilon} \qquad (8)$$

It is early seen that T must be homogeneous in $\vec{\epsilon}$. Here A' and B' are functions analogous to the A , B , C , D of equations (2) and (3) while the Pauli matrix $\vec{\tau}$ operates between the Σ and Λ spin states. Thus the polarization of Λ in a direction $\vec{n}$ is given by[2]

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2) G. Feldman and T. Fulton, Nuclear Physics, 8 ,106 (1958)

$$P_\Lambda(\vec{m}) = -\vec{m}\cdot\vec{P}_\Sigma + 2\,\vec{m}\cdot(\vec{\epsilon}\times\vec{q})\;\vec{P}_\Sigma\cdot(\vec{\epsilon}\times\vec{q}) \qquad (9)$$

for even Σ-Λ parity and

$$P_\Lambda(\vec{m}) = -\vec{m}\cdot\vec{P}_\Sigma + 2(\vec{m}\cdot\vec{\epsilon})\;(\vec{P}_\Sigma\cdot\vec{\epsilon}) \qquad (10)$$

for odd Σ-Λ parity.

The method of derivation is similar to that used in obtaining equation (1) . Here the notation is that $\vec{m}\cdot\vec{P}_\Sigma = P_\Sigma(\vec{m})$ equation (1) where we drop the $\mp$ in P_Σ temporarily .

We have thus the cascade of ~~decays~~ reactions $\pi^- + p \to \Sigma^0 + K^0$, $\Sigma^0 \to \Lambda^0 + \gamma$ and we wish to study P_Λ in terms of P_N and $P_{K\Sigma}$. Now consider the Λ^0 's emitted in the direction $\vec{m}$ so that $\vec{q} = \vec{m}$. If we further observe the polarization of Λ^0 's in the direction $\vec{m}$, we have also $\vec{n} = \vec{m}$. Since $\vec{\epsilon}\cdot\vec{q} = 0$, equation(9) shows us that $P_\Lambda = -P_\Sigma$ independent of the relative Σ-Λ parity. Thus finally under these circumstances and with the nucleons polarized transversely to the incident pion beam ,

$$P_\Lambda(\vec{m}) = +P_N(\vec{m}) \quad \text{for } P_{K\Sigma} = +1,$$

$$= -P_N(\vec{m}) \quad \text{for } P_{K\Sigma} = -1 \qquad (11)$$

where $\vec{m}$, it may be emphasized once again, denotes the direction transverse to the direction of the incident π^- beam in the reaction $\pi^- + p \to \Sigma^0 + K^0$. Thus one notices that the sign of the polarization is the same as the sign of $P_{K\Sigma}$. One possible method of detecting the Λ^0 -polarization would consist in observing the asymmetry in the Λ^0 decaying into a nucleon and a pion as suggested by Capps[1]. For this, it is necessary

that one knows beforehand the correlation between the Λ^0- spin and the direction of the decay pion. Unfortunately this point seems to be a matter of some experimental controversy[3].

If we do not select Λ^0's in any particular direction, (11) becomes

$$P_\Lambda(\vec{m}) = \tfrac{1}{3} P_N(\vec{m}) \quad \text{for } p_{K\Sigma} = +1,$$
$$= -\tfrac{1}{3} P_N(\vec{m}) \quad \text{for } p_{K\Sigma} = -1 \qquad (12)$$

so that as before the sign of P_Λ will give us the sign of $p_{K\Sigma}$. However in this case the polarization is very much reduced.

2) Let us now consider the reaction $K^- + p \to Y^0 + \gamma$ where $Y^0 = \Sigma^0$ or Λ^0. Clearly the reaction will be rather rare since it involves an electromagnetic interaction. A qualitative estimate would be that its cross-section would be about $1/137$ times the competing strong interaction cross-sections. Therefore what we are discussing here may not be completely impossible experimentally.

Let $\vec{\nu}_i$ and $\vec{\nu}_f$ denote the incident K^- and outgoing γ -ray directions in the centre-of-mass system. Further let $\vec{\tau}$ denote the spin operator operating between the nucleon and Y^0 states and $\vec{\epsilon}$ denote the polarization of the γ-ray

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3) E. Boldt, H.C.Bridge, D.O.Caldwell and Y. Pal, Phys. Rev. Letters, 1, 256 (1958) and R.Birge and W.Fowler, Phys. Rev. Letters, 5, 254 (1960) obtain contradictory results. See however Leitner et al., Phys Rev Letters 7, 264 (1961); Beall et al., Phys Rev Letters 7, 285 (1961).

with $\vec{\epsilon} \cdot \vec{\nu}_f = 0$. The transition matrix is homogeneous in $\vec{\epsilon}$. Thus for even K-Σ parity, it has the general form

$$T = i E \, \vec{\tau} \cdot (\vec{\nu}_i \times \vec{\epsilon}) + i F \, \vec{\tau} \cdot (\vec{\nu}_f \times \vec{\epsilon}) \qquad (13)$$

Let $P_N(\vec{m})$ denote the polarization of the nucleon in the direction $\vec{m}$. Thus the density matrix for γ is calculated to be

$$\begin{aligned}
\rho_\gamma &= T \, \tfrac{1}{2} \left[1 + P_N(\vec{m}) \, \vec{\tau} \cdot \vec{m} \right] T^\dagger \\
&= |E|^2 \sin^2\theta + |F|^2 + \vec{\nu}_\perp \cdot (\vec{\nu}_f \times \vec{\epsilon}) (E F^* + E^* F) \sin\theta \\
&\quad - i (E F^* - E^* F) (\vec{\tau} \cdot \vec{\epsilon}) (\vec{\nu}_\perp \cdot \vec{\nu}_f) \sin\theta \\
&\quad + P_N(\vec{m}) \Big[|E|^2 \left\{ 2 (\vec{\tau} \cdot \vec{\nu}_\perp)(\vec{m} \cdot \vec{\nu}_\perp) - \vec{\tau} \cdot \vec{m} \right\} \sin^2\theta \\
&\quad + |F|^2 \left\{ 2 \, \vec{\tau} \cdot (\vec{\nu}_f \times \vec{\epsilon}) \, \vec{m} \cdot (\vec{\nu}_f \times \vec{\epsilon}) - \vec{\tau} \cdot \vec{m} \right\} \\
&\quad + E F^* \left\{ (\vec{\tau} \cdot \vec{\nu}_\perp) \, \vec{m} \cdot (\vec{\nu}_f \times \vec{\epsilon}) + i (\vec{\nu}_\perp \cdot \vec{\nu}_f)(\vec{m} \cdot \vec{\epsilon}) \right. \\
&\quad - \vec{\tau} \cdot (\vec{\nu}_\perp \times \vec{\nu}_f)(\vec{m} \cdot \vec{\epsilon}) - i (\vec{\nu}_\perp \cdot \vec{\epsilon})(\vec{m} \cdot \vec{\nu}_f) \\
&\quad \left. + \vec{\tau} \cdot (\vec{\nu}_\perp \times \vec{\epsilon})(\vec{m} \cdot \vec{\nu}_f) \right\} \\
&\quad + E^* F \left\{ \vec{\tau} \cdot (\vec{\nu}_f \times \vec{\epsilon})(\vec{m} \cdot \vec{\nu}_\perp) + i (\vec{\nu}_f \times \vec{\epsilon}) \cdot (\vec{m} \times \vec{\nu}_\perp) \right. \\
&\quad \left. - \vec{\tau} \cdot \vec{\epsilon} \; \vec{\nu}_f \cdot (\vec{m} \times \vec{\nu}_\perp) + (\vec{\tau} \cdot \vec{\nu}_f) \, \vec{\epsilon} \cdot (\vec{m} \times \vec{\nu}_\perp) \right\} \Big] \qquad (14)
\end{aligned}$$

Here θ is the scattering angle and $\vec{\nu}_\perp$ is the unit vector normal to the scattering plane, i.e. $\vec{\nu}_\perp \sin\theta = \vec{\nu}_i \times \vec{\nu}_f$.
This gives us via equation (5) the following expression for $P_y(\vec{m})$:

$$P_y(\vec{m}) = \frac{N_y(\vec{m})}{D_y(\vec{m})}$$

where

$$N_y(\vec{m}) = 2\,\mathrm{Im}(EF^*)\,(\vec{m}\cdot\vec{\epsilon})\;\vec{\nu}_f\cdot(\vec{\nu}_i\times\vec{\epsilon})$$
$$+P_N(\vec{m})\Big[|E|^2\Big\{2\big[\vec{m}\cdot(\vec{\nu}_i\times\vec{\epsilon})\big]^2-(\vec{\nu}_i\times\vec{\epsilon})^2\Big\}$$
$$+|F|^2\Big\{2\big[\vec{m}\cdot(\vec{\nu}_f\times\vec{\epsilon})\big]^2-1\Big\}+EF^*\Big\{\vec{m}\cdot(\vec{\nu}_i\times\vec{\epsilon})\,\vec{m}\cdot(\vec{\nu}_f\times\vec{\epsilon})$$
$$+\vec{m}\cdot[\vec{\nu}_f\times(\vec{\nu}_i\times\vec{\epsilon})]\,\vec{m}\cdot\vec{\epsilon}-\vec{m}\cdot[\vec{\epsilon}\times(\vec{\nu}_i\times\vec{\epsilon})]\,\vec{m}\cdot\vec{\nu}_f\Big\}$$
$$+E^*F\Big\{\vec{m}\cdot(\vec{\nu}_f\times\vec{\epsilon})\,\vec{m}\cdot(\vec{\nu}_i\times\vec{\epsilon})-(\vec{m}\cdot\vec{\epsilon})\,\vec{\nu}_f\cdot[\vec{m}\times(\vec{\nu}_i\times\vec{\epsilon})]$$
$$+\vec{m}\cdot\vec{\nu}_f\;\vec{\epsilon}\cdot[\vec{m}\times(\vec{\nu}_i\times\vec{\epsilon})]\Big\},$$

$$D_y(\vec{m}) = |E|^2\sin^2\theta+|F|^2+2\,\mathrm{Re}(EF^*)\,(\vec{\nu}_i\times\vec{\epsilon})\cdot(\vec{\nu}_f\times\vec{\epsilon})$$
$$+P_N(\vec{m})\Big[2\,\mathrm{Im}(E^*F)\big[\vec{\nu}_f\cdot(\vec{\nu}_i\times\vec{\epsilon})\big](\vec{m}\cdot\vec{\epsilon})\Big] \qquad (15)$$

For the odd parity case the corresponding expressions are

$$T = iG\,\vec{\tau}\cdot\vec{\epsilon}, \qquad (16),$$

$$P_y = \tfrac{1}{2}|G|^2\Big[1+P_N(\vec{m})\Big\{2(\vec{\tau}\cdot\vec{\epsilon})(\vec{m}\cdot\vec{\epsilon})-\vec{\tau}\cdot\vec{m}\Big\}\Big] \qquad (17),$$

and

$$P_y(\vec{m}) = P_N(\vec{m})\left[2(\vec{m}\cdot\vec{\epsilon})^2 - 1\right] \qquad (18).$$

Thus even if the initial nucleon is not ~~paralysed~~ polarized, for the even parity case, y^0 is in general polarized with

$$P_y(\vec{m}) = \frac{2\,\mathrm{Im}(EF^*)(\vec{m}\cdot\vec{\epsilon})\,\vec{\nu}_f\cdot(\vec{\nu}_i\times\vec{\epsilon})}{|E|^2\sin^2\theta + |F|^2 + 2\,\mathrm{Re}(EF^*)(\vec{\nu}_i\times\vec{\epsilon})\cdot(\vec{\nu}_f\times\vec{\epsilon})} \qquad (19)$$

while for odd P_{Ky}, $P_y = 0$ if $P_N = 0$. For the decay $\Sigma^0 \to \Lambda^0 + \gamma$, one sees from equations (9) and (10) that the Λ^0 polarization vanishes if $P_{\Sigma^0} = 0$ independent of the Σ-Λ parity. Thus there will not be any asymmetry in the decay of Λ^0 in such a case. Consequently if one can observe the sequence of reactions $K^- + p \to \Sigma^0 + \gamma$, $\Sigma^0 \to \Lambda^0 + \gamma$, $\Lambda^0 \to N + \pi$ and no asymmetry is observed in the weak decay of the Λ^0, one has a clear-cut evidence for odd $P_{K\Sigma}$ while if there is an asymmetry, one can conclude that $P_{K\Sigma} = +1$. Similarly if the reactions are $K^- + p \to \Lambda^0 + \gamma$, $\Lambda^0 \to N + \pi$, the absence and presence of asymmetries in the Λ^0-decay will unambiguously indicate odd and even $P_{K\Lambda}$ respectively. One of course has already very good evidence for odd $P_{K\Lambda}$ from hyperfragment experiments[4]. Experimental information on the sign of $P_{K\Sigma}$ is however practically nil. The method

4) Block et. al., Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester, page 419.

suggested here , while unambiguous , unfortunately involves a reaction with a small cross-section, as remarked earlier.

CHAPTER IX

Some Remarks on the Masses of Elementary Particles*

1. Introduction

Among the most urgent problems in elementary particle physics today is the understanding of the mass spectrum of the elementary particles or at least of the mass differences among charge multiplets. It would seem that conventional local field theories are totally inadequate to tackle this problem since one always encounters divergences in any attempt at the calculation of dynamical effects on the masses of the particles. It appears very improbable that the theory does in fact contain finite results due to capricious cancellation of the various divergent integrals. Therefore we have here a very strong indication of the failure of the space-time concepts underlying the present local field theories. There are at present a variety of attempts to obtain finite results from an inherently infinite theory by putting in suitable structure factors or cut-offs on the self energy integrals to render them finite. All these go beyond the framework of local field theory and are inconsistent for instance with relativistic causality. Among the earliest of such attempts was that Feynman and Speisman[1] who calculated

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* A.P.Balachandran, Proceedings of the Cosmic Ray Symposium at Ahmedabad (March 1960), page 243 (The paper was read by Mr. V. Devanathan)

1) R.P.Feynman and G. Speisman, Phys. Rev., 94 , 500 (1954)

the $\pi^+ - \pi^0$ mass difference from the self-energy of π^+ arising from the second order electromagnetic self-energy graph. To obtain a finite result, they replaced the photon propagator $\frac{1}{k^2}$ by $\frac{1}{k^2}\left(\frac{\Lambda^2}{\Lambda^2+k^2}\right)^2$ and found that they could get the observed mass difference of about 4.6 Mev with the cut-off Λ of about one nucleon mass. Here the charged member of the multiplet is heavier than the ~~virtual~~ neutral π^+ member which is consistent with the fact that the electromagnetic self-energy by itself is positive. In the case of the nucleons, the situation is more complicated since proton and neutron carry anomalous magnetic moments. Feynman and Speisman[1] calculated the proton-neutron mass difference by assuming that they obey Dirac equations with additional Pauli terms to take into account the anomalous magnetic moments. By using seperate cut-offs for the anomalous magnetic moment coupling term and the photon propagator, they were able to get a neutron heavier than the proton. Following Feynman and Speisman, quite a few papers have appeared on the mass differences of the other particles as well. In particular, Heitler and co-workers have attempted to develop a consistent field theory out of this cut-off model[2]. They find that in all cases, the qualitative trend in the mass differences ~~are~~ is in the correct direction with a cut-off of the order of the nucleon mass.

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2) E. Arnous, and W. Heitler, Nuovo Cimento, 11, 443 (1959); E. Arnous, W. Heitler and Y. Takahashi, Nuovo Cimento, 16, 671 (1960); E. Arnous, W. Heitler and L. O'Raifeartaugh, Nuovo Cimento, 16, 785 (1960).

They then notice that Chew[3] has proposed an extended model to explain the low-energy pion-nucleon scattering with a cut-off of the same order of magnitude. In view of these facts, a convergent, non-local theory has been suggested by Arnous and Heitler[4] with a universal cut-off of the order of magnitude of the nucleon mass. The rest of the series of papers is an attempt to develop this idea consistently. Its correctness or otherwise will perhaps be decided by future developments. In this connection, we would also like to mention the very ambitious non-linear spinor theory of Heisenberg and co-workers[5] which should in principle be able to predict the entire mass spectrum. Its internal consistency however seems very controversial.[6]

2. On the Masses of Elementary Particles.

Let us now discuss whether it is possible to generate the entire mass of an elementary particle as a self-energy effect. Without concerning ourselves with the means whereby the theory is going to yield finite results, it is possible to single out the types of interaction which can possibly give rise to mass terms. Consider for instance a fermion

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3) G.F.Chew, Phys.Rev., 94 , 1748, 1755 (1954)
4) E. Arnous and W. Heitler, loc. cit.
5) H.P.Dürr , W.Heisenberg, H.Mitter, S.Schbeder and K. Yamazaki, Z.f. Naturforsch, 14 a, 441 (1959).
6) W. Pauli, Proceedings of the High Energy Physics Conference at CERN (1958).

of bare mass zero. Its free Lagrangian can be written as

$$\mathcal{L} = -\bar{\psi}\,\gamma_\mu \partial_\mu\,\psi \qquad (1)$$

This is invariant under the γ_5- transformation[7] of the fermion field :

$$\psi \to \pm \gamma_5 \psi \qquad (2)$$

Therefore if the ψ -field has interactions which are also γ_5- invariant, the unrenormalized Lagrangian will be γ_5 -invariant and therefore the MS renormalization procedure cannot generate mass terms since these are not γ_5-invariant[8]†. Thus if the interaction of the electron or muon with the electromagnetic field is $ie\bar{\psi}\gamma_\mu\psi A_\mu$, it is invariant under

$$\psi \to \pm \gamma_5 \psi, \quad A_\mu \to A_\mu \qquad (3)$$

On the other hand, if the interaction is purely through magnetic moment and reads $\lambda\,\bar{\psi}\,\sigma_{\mu\nu}\,\psi\,F_{\mu\nu}$ where, of course, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the interaction is again γ_5 -invariant with

$$\psi \to \pm \gamma_5 \psi, \quad A_\mu \to -A_\mu \qquad (4)$$

Thus a pure current or magnetic moment type of electromagnetic interaction can never generate the electron or muon masses if their bare masses are zero. If both these types of terms are present, however, the Lagrangian is no longer γ_5 - invariant and there is a possibility of an electromagnetic origin of the masses of these particles. Similar arguments can be applied

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7) A. Salam, Nuovo Cimento, 5 , 299 (1957).

8) R.P.Feynman, Proceedings of the 1958 Annual International Conference on High Energy Physics at CERN.

† At least in perturbation theory.

to the interactions of other fields. Thus for direct Yukawa couplings like $ig\bar{N}\gamma_5\vec{\tau}N\cdot\vec{\pi}$, we have invariance under

$$N \to \pm\gamma_5 N\,, \quad \vec{\pi} \to -\vec{\pi} \qquad (5a)$$

where the γ_5 -transformation is made on all the fields simultaneously. The latter is necessary since we can have couplings like $\lambda g\bar{N}\gamma_5\Lambda K$ and has the following physical significance. The γ_5 -transformation takes over any fermion field to a field of opposite intrinsic parity[9] . However the relative parities of the different fermion fields is a meaningful concept in strong interactions. To leave the relative parities unaltered, we have therefore to change the intrinsic parities of <u>all</u> the fermion fields to ones of opposite sign. This implies that the γ_5-transformation has to be applied on all the fields simultaneously. Now for derivative couplings, the transformation(5) is to be replaced by

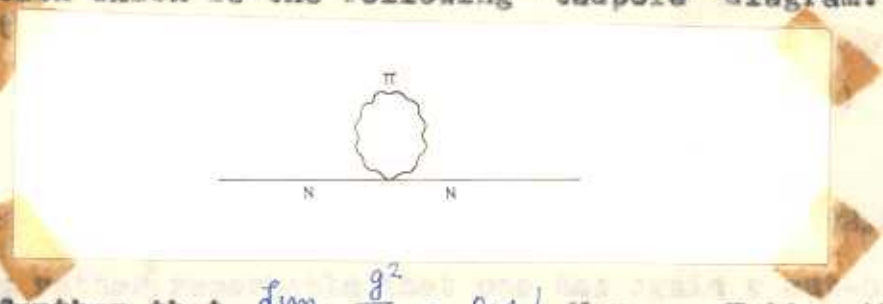
$$N \to \pm\gamma_5 N\,, \quad \vec{\pi} \to \vec{\pi} \qquad (5b)$$

so that if both these types of terms are present, γ_5 -invariance is again lost and we have the possibility of obtaining a non-zero mass from a zero bare mass.

Consider now a pion-nucleon interaction of the form $\frac{g^2}{2m}\bar{N}N\pi^2$ where g is the usual pion-nucleon coupling constant, m the nucleon mass and N and π are the nucleon and pion fields respectively. This term has been considered previously in an attempt to bring the usual pseudoscalar theory into agreement

9) E.C.G.Sudarshan and R.E.Marshak, Proc. of the Padua-Venice Conference (Septembera 1957); Phys. Rev. <u>109</u> , 1860 (1958).

with the S -wave data in low-energy pion-nucleon scattering[10] Thus in the pseudoscalar theory, the (second order) S -wave scattering length for scattering without charge exchange is $-\frac{g^2}{m}$ while experimentally this is a very small quantity. The term $\frac{g^2}{2m}\bar{N}N\pi^2$ exactly cancels out the second order effect of the first order coupling. It seems that this cancellation of terms like $\frac{g^2}{m}$, $\frac{g^4}{m}$, etc. occurs to all orders. The point with which we are concerned about this term however is that it cannot be made invariant[11] and can therefore give rise to nucleon mass. Let us therefore try to calculate the whole of the nucleon mass term assuming that it arises from the lowest order graph due to this interaction which is the following "tadpole" diagram:



Assume further that $\lim_{m\to 0}\frac{g^2}{2m} = 0.1$ Mev . Notice that there are three such diagrams corresponding to π^+, π^0 and π^- emissions. We have then the following expression for the nucleon mass :

$$m = 3\,\frac{ig^2}{2m}\,\frac{1}{(2\pi)^4}\int d^4k\,\frac{1}{k^2}\left(\frac{\Lambda^2}{\Lambda^2+k^2}\right)^2 \qquad (6)$$

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10) A.Klein, Phys. Rev., 99 , 938 (1955); S.D.Drell, M.H. Friedman and F.Zachariasen, Phys. Rev.,104, 236 (1956)

11) The following refers to some unpublished work of Alladi Ramakrishnan, A.P.Balachandran and N.R.Ranganathan

Here we assume that the bare pion mass too is zero.

$\left(\frac{\Lambda^2}{\Lambda^2+k^2}\right)^2$ is a convergence factor introduced to make the integrals finite. We evaluate (6) by the usual Feynman techniques.[12] We write

$$\frac{1}{k^2(k^2+\Lambda^2)^2} = 2\int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{1}{[k^2+\Lambda^2(1-x_2)]^3} \qquad (7)$$

so that

$$m = 6\frac{ig^2}{2m}\frac{\Lambda^4}{(2\pi)^4}+\int_0^1 dx_1\int_0^{x_1} dx_2 \int d^4k \frac{1}{[k^2+\Lambda^2(1-x_2)]^3} \qquad (8)$$

But

$$\int \frac{d^4k}{[k^2+\Lambda^2(1-x_2)]^3} = \frac{i\pi^2}{2\Lambda^2(1-x_2)} \qquad (9)$$

(8) thus becomes

$$m = \frac{3\pi^2}{(2\pi)^4}+\frac{g^2}{2m}\Lambda^2 \qquad (10)$$

For $m \approx$ 938 Mev ., we find from (10) that $\Lambda \approx$ 702.5 Mev. It seems rather remarkable that one has again a cut-off of the order of the nucleon mass.

The general dependence of the fermion mass renormalization term δm on the bare mass m for an interaction which is γ_5 -invariant can be deduced by simple arguments. Let us for instance consider a theory in which a fermion of mass m interacts with a boson of mass μ . The Lagrangian is then invariant under the mass reversal transformation[13] :

$$\psi \to \pm\gamma_5\psi, \quad m \to -m \qquad (11)$$

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12) These are discussed at great length in J.H.M Jauch and F. Rohrlich, 'Theory of Photons and Electrons', (Addison-Wesley, 1955)

13) J.J.Sakurai, Nuovo Cimento, 7 , 649 (1958).

along with an appropriate transformation for the boson field. (11) should also ~~x~~ leave the renormalization term δm invariant so that δm should be an odd function of m . Further the Lagrangian is invariant under the replacement $\mu \to -\mu$ so that δm can only be a function of μ^2 . Since the theory is γ_5 invariant if $m = 0$, we must also have $\lim_{m \to 0} \delta m = 0$. Thus δm should be of the form

$$\delta m = a m + b \frac{m^3}{\mu^2} + \cdots \qquad (12)$$

where a , b, ... are dimensionless . We can now assume that δm gains no new singularity as $\mu \to 0$ which is the case for instance in electrodynamics. It follows that δm should be independent of μ and be proportional to m , a result well-known in electrodynamics. Also, the constant a , which is possibly divergent must be such that $\lim_{m \to 0} a m = 0$ i.e. a should diverge less strongly than linearly. It could for instance go to infinity like a logarithm does. ~~Analogous conclusions can be drawn regarding the degree of divergence of , , also.~~ ~~All~~ These results are a consequence of the fact that $\lim_{m \to 0} \delta m = 0$. ~~Now that~~ if we use a cut-off mass Λ , we would have in general

$$\delta m = m f\left(\frac{m^2}{\Lambda^2}, \frac{\mu^2}{\Lambda^2} \right) \qquad (13)$$

where f has no singularity as μ tends to zero and $\lim_{\substack{\Lambda \to \infty \\ m \to 0}} m f\left(\frac{m^2}{\Lambda^2}, \frac{\mu^2}{\Lambda^2}\right) = 0$. f may for instance be composed of terms which tend to a logarithmic infinity or to a constant as $\Lambda \to \infty$.

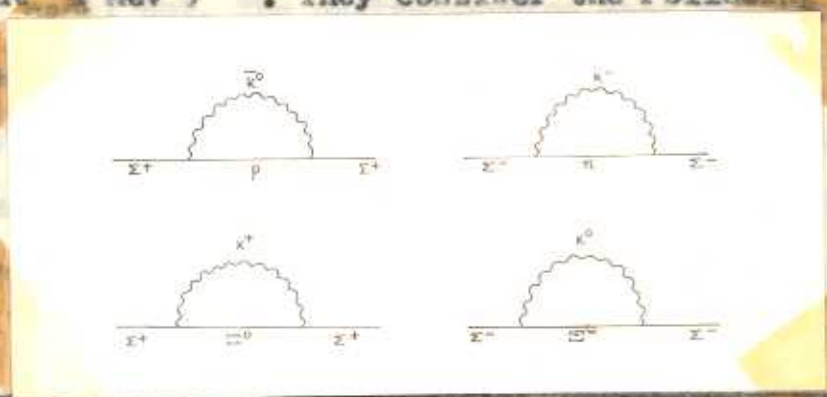
We can now consider the case of boson mass terms. The transformation invariance under which ensures zero boson mass[14] is

$$\varphi \to \varphi + \lambda \qquad (14)$$

where φ is the boson field and λ a constant . (14) is the analogue of the gauge transformation of the second kind on the electromagnetic field which guarantees the masslessness of the photon. Derivative Yukawa couplings are invariant under (14) and therefore cannot generate boson masses if the bare boson mass is zero. However electromagnetic interactions of charged bosons, direct Yukawa couplings etc. are not left invariant under (14) and hence can give rise to non-zero renormalization terms even if we start with massless bosons.

3. On the $\Xi^- - \Xi^0$ Mass Difference.

In this section, we shall discuss a suggestion of Bransden and Moorhouse[15] that a large part of for example the $\Sigma^- - \Sigma^+$ mass difference (which is of about 7 Mev.) may be due to the large $\bar{K}^0 - K^-$ and $K^0 - K^+$ mass difference (which is of about 4 Mev)[16]. They consider the following diagrams:



14) Such transformations were first considered by K.Nishijima, Nuovo Cimento, 11 , 910 (1959).
15) B.H.Bransden and R.G.Moorhouse, Phys. Rev. Letters, 2, 431 (1959).
16) W.H.Barkas and A.H.Rosenfeld, Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester, page 878.

Writing

$$m_{K^0} = m_{\bar{K}^0} = m_K + \frac{\delta m_K}{2}\,, \quad m_{K^+} = m_{K^-} = m_K - \frac{\delta m_K}{2} \qquad (15a)$$

where m_K is the mean mass of the K-mesons, and

$$\frac{1}{k^2 + m_{K^0}^2} = \frac{1}{k^2 + m_K^2}\left[1 - \frac{m_K \delta m_K}{k^2 + m_K^2}\right],$$

$$\frac{1}{k^2 + m_{K^-}^2} = \frac{1}{k^2 + m_K^2}\left[1 + \frac{m_K \delta m_K}{k^2 + m_K^2}\right] \qquad (16)\ (15b)$$

etc., we find, for the mass-difference of Σ^- and Σ^+ arising from the $\bar{K}^0 - K^-$ and $K^0 - K^+$ mass differences, the finite results:

$$m(\Sigma^-) - m(\Sigma^+) = -\frac{i g_{N\Sigma K}^2}{4\pi^3} m_K \delta m_K \int \gamma' \frac{i\gamma(p-k) - m_N}{(p-k)^2 + m_N^2} \gamma' \frac{d^4k}{(k^2+m_K^2)^2}$$

$$+ \frac{i g_{\Xi\Sigma K}^2}{4\pi^3} m_K \delta m_K \int \gamma' \frac{i\gamma(p-k) - m_\Xi}{(p-k)^2 + m_\Xi^2} \gamma' \frac{d^4k}{(k^2+m_K^2)^2} \qquad (17)\ (15c)$$

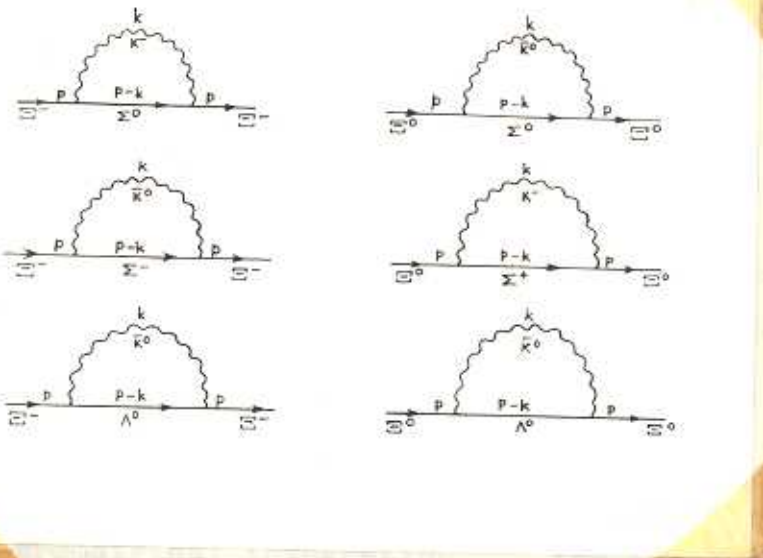
Here $\gamma' = 1$ or $i\gamma_5$ depending upon the relative parities of the particles involved. The zeroeth order terms in δm_K clearly cancel out. The neutron-proton mass difference which is rather small is also neglected. They find that the most favourable case when $m(\Sigma^-) - m(\Sigma^+)$ turns out to be largest is when the K-nucleon relative parity is odd and K-cascade relative parity is even. In such a case,

$$m(\Sigma^-) - m(\Sigma^+) = 0{\cdot}18\, g_{N\Sigma K}^2 + 2{\cdot}14\, g_{\Xi\Sigma K}^2 \qquad (18)\ (15d)$$

which gives the required 7 Mev for $\frac{g_{N\Sigma K}^2}{4\pi} \approx 4$, $\frac{g_{\Xi\Sigma K}^2}{4\pi} \approx 3$.

Let us now apply the above method to the $\Xi^- - \Xi^0$ mass

differences.[17] The relevant diagrams are shown below:



The diagrams involving Σ's give contributions which tend to cancel out so that practically all the contribution comes from the last two diagrams. The latter can be written as

$$m(\Xi^-) - m(\Xi^0) = -\frac{i g^2_{\Xi\Lambda K}}{2\pi^3} m_K \delta m_K \int \gamma' \frac{-i\gamma k \mp m_\Xi - m_\Lambda}{(p-k)^2 + m_\Lambda^2} \gamma' \frac{d^4k}{(k^2+m_K^2)^2} \qquad (15c)$$

where we have set $i\gamma p = \mp m_\Xi$ for $\gamma' = \begin{matrix} 1 \\ i\gamma_5 \end{matrix}$. The evaluation of the integral in (15c) will be illustrated for even K-Ξ parity, (i.e. $\gamma' = 1$), the odd parity case being quite similar. Using the formula[12]

$$\frac{1}{a_1 a_2^2} = 2 \int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{1}{[a_1 x_2 + a_2(1-x_2)]^3} \qquad (16)$$

17) A.P.Balachandran, Proceedings of the Cosmic Ray Symposium at Ahmedabad (March, 1960), page 249.

we have,

$$m(\Xi^-) - m(\Xi^0) = -\,i\,\frac{g^2_{\Xi\Lambda K}}{\pi^3}\, m_K\, \delta m_K \times$$

$$\times \int d^4k \int_0^1 dx_1 \int_0^{x_1} dx_2 \; \frac{-\gamma k - m_\Xi - m_\Lambda}{\left[(k - p x_2)^2 + p^2 x_2 (1-x_2) + m_K^2 (1-x_2) + m_\Lambda^2 x_2\right]^3} \tag{17}$$

Shifting the origin by replacing k by $k + p x_2$[12], we find

$$m(\Xi^-) - m(\Xi^0) = -\,i\,\frac{g^2_{\Xi\Lambda K}}{\pi^3}\, m_K\, \delta m_K \times$$

$$\times \int d^4k \int_0^1 dx_1 \int_0^{x_1} dx_2 \; \frac{m_\Xi (x_2 - 1) - m_\Lambda}{\left[k^2 + p^2 x_2 (1-x_2) + m_K^2 (1-x_2) + m_\Lambda^2 x_2\right]^3} \tag{18}$$

Since by a symmetrical integration[12], the integrals involving γk becomes zero. Now[12]

$$\int \frac{d^4k}{(k^2 + a^2)^3} = \frac{i\pi^2}{2a^2} \tag{19}$$

so that

$$m(\Xi^-) - m(\Xi^0) = -\,\frac{g^2_{\Xi\Lambda K}}{2\pi}\, m_K\, \delta m_K \times$$

$$\times \int_0^1 dx \; \frac{m_\Xi x^2 + m_\Lambda x}{m_\Xi^2 x^2 + (m_K^2 - m_\Xi^2 - m_\Lambda^2) x + m_\Lambda^2} \tag{20}$$

where we have set $p^2 = -m_\Xi^2$ and performed one of the x-integrations. For the pseudoscalar case, we find, similarly,

$$m(\Xi^-) - m(\Xi^0) = -\,\frac{g'^2_{\Xi\Lambda K}}{2\pi}\, m_K \delta m_K \times$$

$$\times \int_0^1 \frac{-m_\Xi x^2 + m_\Lambda x}{m_\Xi^2 x^2 - (m_\Xi^2 + m_\Lambda^2) x + m_K^2 + m_\Lambda^2} \tag{21}$$

where $g'_{\Xi\Lambda K}$ denotes the pseudoscalar coupling constant.

Now,

$$\int_0^1 \frac{x^2\,dx}{ax^2+bx+c} = \frac{1}{a}\left\{1 - \frac{b}{2a}\log\frac{a+b+c}{a} + \frac{b^2-2ac}{a}\frac{1}{\sqrt{4ac-b^2}}\left(\tan^{-1}\frac{2a+b}{\sqrt{4ac-b^2}} - \tan^{-1}\frac{b}{\sqrt{4ac-b^2}}\right)\right\},$$

$$\int_0^1 \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\log\frac{a+b+c}{c} - \frac{b}{a}\frac{1}{\sqrt{4ac-b^2}}\left[\tan^{-1}\frac{2a+b}{\sqrt{4ac-b^2}} - \tan^{-1}\frac{b}{\sqrt{4ac-b^2}}\right] \quad (22)$$

where the integrals are evaluated noting that $4ac > b^2$ in our case. Using these formulae and taking m_Ξ = 1315 Mev., m_Λ = 1115 Mev and m_K = 496 Mev., we finally get

$$m(\Xi^-) - m(\Xi^0) = 1{\cdot}424\, g^2_{\Xi\Lambda K} \text{ Mev. for } \gamma' = 1,$$
$$= 0{\cdot}236\, g'^2_{\Xi\Lambda K} \text{ Mev. for } \gamma' = \gamma_5 \quad (23)$$

In either case, Ξ^- comes out to be the heavier particle. There seems to be some experimental evidence[16] that Ξ^- is heavier than Ξ^0 by about 7 Mev though the quoted error for the mass of the Ξ^0 is so large that no definite conclusions can be drawn. In any case, (23) indicates that the mechanism considered here can make a contribution to the mass difference which is not negligible especially if the Ξ-Λ-K coupling is scalar.

P A R T II

THE WEAK INTERACTIONS OF ELEMENTARY PARTICLES

CHAPTER X

ON THE VALUE OF THE AXIAL VECTOR RENORMALIZATION IN β-DECAY*

1. Introduction

The decay of the μ-meson is the only process which has been studied in weak interactions which does not involve any strongly interacting particle and the resultant complications arising from strong interactions. All evidence for this process supports the Lagrangian[1]

$$\mathcal{L}_\mu = 2^{-\frac{1}{2}} G_\mu \left[\bar{\nu} \gamma_\alpha (1+\gamma_5) e\right]\left[\bar{\nu}' \gamma_\alpha (1+\gamma_5) \mu\right]^\dagger + h.c. \qquad (1)$$

where ν and ν' can ~~in general~~ be different[2].

For processes arising from the weak interactions involving the neutron (like the nuclear β-decay, K-capture and inverse β-decay, μ-capture by nuclei and the decay of the charged pion) experiments suggest that the interaction is essentially of the form[1]

$$\mathcal{L}_{int} = 2^{-\frac{1}{2}} G' \left[\bar{p} \gamma_\alpha (1+\gamma_5) n\right]\left[\bar{\nu} \gamma_\alpha (1+\gamma_5) e + \bar{\nu}' \gamma_\alpha (1+\gamma_5) \mu\right]^\dagger + h.c. \qquad (2)$$

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* A.P.Balachandran, Nuovo Cimento. (in press)

1) This Lagrangian was first suggested by R.P.Feynman and M.Gell-Mann, Phys. Rev., 109, 193 (1958); E.C.G.Sudarshan and R.E.Marshak, Phys. Rev., 109, 1860 (1958).

2) B. Pontecorvo, Soviet Phys. JETP, 6 429 (1958); K. Nishijima, Phys. Rev., 108, 907 (1957); G. Feinberg, Phys. Rev., 10, 1482 (1958).

(except for possible extra terms)
when the renormalization effects due to strong interactions are neglected. With the inclusion of strong interactions, (2) would become

$$\mathcal{L}_{int} = 2^{-\frac{1}{2}} \left[G_V \bar{p} \gamma_\alpha n - G_A \bar{p} \gamma_\alpha \gamma_5 n \right] \left[\bar{\nu} \gamma_\alpha (1+\gamma_5) e + \bar{\nu}' \gamma_\alpha (1+\gamma_5) \mu \right]^\dagger + h.c. \quad (3)$$

where G_V and $-G_A$ are the renormalized vector and axial vector coupling constants.

There is strong experimental evidence for the equality of the vector coupling constant G_V in nuclear β -decay and the coupling constant G_μ in μ -decay. The best support so far for the "universality" of this coupling comes from the decay of O^{14} $(I=1,\ J=0^+)$ to N^{*14} $(I=1,\ J=0^+)$ 3) which involves only the Fermi coupling constant since it is a $J=0^+ \rightarrow J=0^+$ transition. The nuclear matrix element can be exactly evaluated if we neglect charge-dependent corrections since O^{14} and N^{*14} are members of the same isotopic spin multiplet. The measured ft value is 3071 $\pm$ 16 which gives for the Fermi coupling constant G_V a value of $1.415 \pm 0.004 \times 10^{-49}$ erg cm^3 which in turn predicts the life-time of the μ -meson to be $\tau_\mu = 2.251$ 0.012 secs. if we assume the V-A theory 1) and set $G_\mu = G_V$. The value of the μ -meson mass used here is

3) R.P.Feynman, Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester, page 501.

206.76 m_e . The measured value of τ_μ is 2.210 ± 0.003 secs.[3] which is very close to the theoretically predicted value. Since strongly interacting particles are involved in nuclear β -decay, it would seem rather surprising that even if the bare coupling constants for μ -decay and β -decay are equal, strong interactions do not strongly distort this equality. It has been suggested[4] that this lack of renormalization of G_V arises from the fact that the vector part of the current in β -decay is the charged component of the isotopic vector current which is exactly conserved in the presence of strong interactions alone. This would imply that the renormalization of G_V is nil as will be shown below. The discrepancy between the μ -decay coupling constant G_μ and the β -decay vector coupling constant G_V should therefore arise from charge-dependent electromagnetic corrections in both these decays. Berman[5] has calculated the electromagnetic corrections to μ -decay and this unfortunately actually enhances the 2 % discrepancy in the calculated and observed value of the μ life-time into a 4 % discrepancy. Coulomb corrections for the decay of O^{14} based on the shell-model are not enough to account for this discrepancy. However there are other sources of error, in particular the mass difference between charged

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4) S.S.Gershtein and J.B.Zeldovich, Soviet Physics JETP 2, 576 (1957); R.P.Feynman and M.Gell-Mann, loc.cit.

5) S.M.Berman, Phys. Rev. 112 , 267 (1958).

and neutral pions which can give an appreciable correction to the charge independence of the short range nuclear forces[6]. Further there are corrections arising from the change in the nuclear radius in going from O^{14} to N^{*14}. In view of this it seems plausible to assume that $G_V = G_\mu$ in the absence of electromagnetic effects.

With regard to the axial vector part of the current in β-decay, the following remarks can be made. Experimentally $-\frac{G_A}{G_V} = 1.25 \pm 0.06$[7] so that if the bare axial vector and vector coupling constants are equal, the renormalization of $-G_A$ is not large. Further the axial vector current cannot be a conserved current since if it is conserved, the rate of decay of the charged pion will vanish[8]. It is therefore rather remarkable that the axial vector renormalization is so small and suggests that perhaps in some approximation, it should be exactly calculable. We develop below a model in which this is indeed possible provided only that the strong part of the interaction Lagrangian is invariant under the Touschek transformation[9]

$$\psi(x) \rightarrow e^{i\alpha\gamma_5}\psi(x) \qquad (4)$$

on any one of the strongly interacting fields. The result emerges that if G is the unrenormalized axial vector coupling constant, $-\frac{G_A}{G} = -1$. This would imply that the unrenormalized

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6) L.L.Foldy, Proceedings of the 1960 Annual International Conference on High Energy Physics, Page 506.

7) M.T.Burgy et. al., Phys. Rev. 110, 1214 (1958); see also C.S.Wu, Rev. Mod. Phys., 31, 783 (1959).

8) J.C.Taylor, Phys. Rev., 110, 1216 (1958); M.L.Goldberger and S.B.Treiman, Phys. Rev., 110, 1478 (1958).

9) B.F.Touschek, Nuovo Cimento, 5, 1281 (1957).

β -decay interaction is $V+A$. However the model involves the introduction of new and unobserved fields and it is quite difficult to say whether it has any correspondence with reality or not.

2. The Generalized Ward Identities

We wish to derive for our currents the analogues of the Ward identity in electrodynamics[10] which implies that the charge renormalization constants arising from the vertex and electron self-energy graphs cancel each other exactly. This shows that if the bare electric charges of all the elementary particles are equal, the renormalized charges are also equal since the only charge renormalization factor now comes from the photon self-energy which is however common to all the particles. This explains why, for instance, the charge of the electron and proton are exactly equal. Physically this means that since the electromagnetic current is conserved, when the proton goes into a neutron and a π^+ for example, the total charge is again equal to the charge of the proton so that the mean value of charge is unaltered by these transitions.

The method we employ in deriving the generalized ward identity is that of Bernstein, Gell-Mann and Michel[11] . We shall sketch their derivation briefly for the conserved vector

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10) J.C.Ward, Phys. Rev., 78 , 182 (1950)
11) J. Bernstein, M.Gell-Mann and L.Michel, Nuovo Cimento, 16 , 560 (1960).
12) M. Gell-Mann and M. Levy, Nuovo Cimento, 16 , 705 (1960).

current and then pass on to the derivation of the analogous result in our case. It may be shown[12] that if we make an infinitesimal gauge transformation

$$N(x) \rightarrow \left[1 + i \vec{\tau} \cdot \vec{u}(x)\right] N(x) \qquad (5)$$

on the unrenormalized nucleon field $N(x)$ with the gauge variable u a function of x, the Lagrangian undergoes the change

$$\mathcal{L} \rightarrow \mathcal{L} - i \vec{V}_\alpha \cdot \partial_\alpha \vec{u} \qquad (6)$$

where $V_\alpha = \bar{N} \gamma_\alpha \vec{\tau} N$. Notice that (5) is an infinitesimal, space-time dependent isotopic spin rotation. Also for convenience, we neglect the existence of other strongly interacting particles. To take the latter into account, (5) must be supplemented by the corresponding transformations on these fields also. These modifications however will not alter any of our conclusions.

Under the transformation (5), the unrenormalized nucleon propagator

$$S_F(x-y) = i \left\langle T\left(N(x)\,\bar{N}(y)\right)\right\rangle \qquad (7)$$

undergoes the change

$$\Delta S_F(x-y) = i \vec{\tau} \cdot \vec{u}(x) S_F(x-y) - i S_F(x-y) \vec{\tau} \cdot \vec{u}(y) \qquad (8)$$

$\Delta S_F(x-y)$ can also be evaluated by computing the change in the right hand side of (7) by adding a first order perturbation to the Lagrangian as given by (6). This gives us, in the

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12) M. Gell-Mann and M. Levy, Nuovo Cimento, 16, 705 (1960)

momentum space,

$$i\vec{\tau}\,S_F(p) - i\,S_F(p')\vec{\tau} = -S_F(p')\,k_\alpha\,\vec{\Gamma}_\alpha(p',p)\,S_F(p) \qquad (9)$$

where $k_\alpha = p'_\alpha - p_\alpha$ and $\vec{\Gamma}_\alpha(p',p)$ is the unrenormalized vertex function associated with the current $\vec{V}_\alpha$. Let Z_2 be the charge renormalization factor arising from nucleon self-energy graphs and $\frac{1}{Z_1}$ the charge renormalization factor arising from the vertex graphs. On passing over to renormalized quantities, (9) gives

$$S_{Fc}^{-1}(p')\vec{\tau} - \vec{\tau}\,S_{Fc}^{-1}(p) = i k_\alpha\,\vec{\Gamma}_\alpha(p',p) \qquad (10)$$

where $S_{Fc} = \frac{1}{Z_2} S_F$ is the renormalized nucleon propagator and $\vec{\Gamma}_\alpha(p',p) = Z_2\,\vec{\Gamma}_\alpha(p',p) = \frac{Z_2}{Z_1}\,\vec{\Gamma}_{\alpha c}(p',p)$ is the effective vertex function, $\vec{\Gamma}_{\alpha c}$ being the renormalized vertex function which between free spinors acts like $\vec{\tau}\gamma_\alpha$. Clearly $\frac{Z_2}{Z_1}$ is the charge renormalization we wish to compute. If we denote the renormalized and unrenormalized coupling constants by G_V and G, we have

$$\frac{Z_2}{Z_1} = \frac{G_V}{G} \qquad (11)$$

Taking $k_\alpha = (p'-p)_\alpha$ to be an infinitesimal quantity, (10) gives

$$\vec{\tau}\,\frac{\partial}{\partial p_\alpha} S_{Fc}^{-1}(p) = i\,\vec{\Gamma}_\alpha(p,p) \qquad (12)$$

which is the Ward identity. Near the mass shell, $S_{Fc}^{-1}(p) = (i\gamma p + m) + O(i\gamma p + m)^2$ so that (12) taken between free spinors gives

$$\bar{u}_\beta \vec{\tau} \gamma_\alpha u_\alpha = \frac{Z_2}{Z_1} \bar{u}_\beta \vec{\tau} \gamma_\alpha u_\alpha$$

$$i.e. \quad \frac{Z_2}{Z_1} = 1 \qquad (13)$$

which is the xx

This shows us that if the current is ~~x~~ conserved, the charge is not renormalized. If the current were not conserved, the gauge transformation (5) would have added terms proportional to $\vec{u}$ in (6) so that the result would no longer be true. In fact by an application of the Euler-Lagrange principle to the gauge variable $\vec{u}$ [12], we find

$$\partial_\alpha \frac{\delta \mathcal{L}}{\delta(\partial_\alpha \vec{u})} = \frac{\delta \mathcal{L}}{\delta \vec{u}}$$

or

$$\partial_\alpha \vec{V}_\alpha = 0$$

where

$$\vec{V}_\alpha = \frac{\delta \mathcal{L}}{\delta(\partial_\alpha \vec{u})} \qquad (14)$$

$\vec{V}_\alpha$ clearly would have non-vanishing divergence if $\frac{\delta \mathcal{L}}{\delta \vec{u}} \neq 0$. Thus the Ward identity is a consequence of the fact that the current in question is conserved.

We will now pass on to the case of the pseudovector current. In the course of the proof, we will make use of an equaivalence theorem due to Salam[13] which may be briefly summarized as follows[14] : Consider the Lagrangian

$$\mathcal{L} = -\bar{\psi}(\gamma_\mu \partial_\mu + m)\psi - ig\bar{\psi}\gamma_5 \gamma_\mu \psi \partial_\mu B$$

$$-\tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\mathcal{L}^2 B^2 \qquad (15)$$

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13) A. Salam, Nuclear Physics, 18 , 681 (1960).

14) The following version of the proof is due to S. Kamefuchi, Nuclear Physics, 18, 691 (1960).

where ψ is a spinor field, B a pseudo scalar field and m and κ denote the masses of the spinor and pseudo scalar fields respectively. The current

$$j_\mu = -i\bar{\psi}\gamma_5\gamma_\mu\psi \qquad (16)$$

is not conserved since

$$\partial_\mu j_\mu = 2im\bar{\psi}\gamma_5\psi \qquad (17)$$

Passing now to the interaction representation and denoting the variables in this representation with a subscript I, we have, for the interaction Hamiltonian,

$$H'(x) = m\bar{\psi}_I\psi_I + ig\bar{\psi}_I\gamma_5\gamma_\mu\psi_I\partial_\mu B_I - \tfrac{1}{2}g^2\left[\bar{\psi}_I\gamma_5\gamma_\mu\psi_I n_\mu\right]^2 \qquad (18)$$

where n_μ is the unit vector normal to the space-like surface σ through x and where we have included the fermion mass term in H'. The current $j_{I\mu}$ in this interaction representation is clearly conserved. It has been shown by Glauber[15] and Umezawa[16] that if there are no terms in H' which violate the invariance under the gauge transformation induced by the current $j_{I\mu}$ which is given by

$$U(\sigma,\lambda)\,\psi_I(x)\,U^{-1}(\sigma,\lambda) = e^{ig\gamma_5\lambda}\psi_I(x) \qquad (19)$$

where

$$U(\sigma,\lambda) = \exp\left[-ig\int_\sigma d\sigma_\mu'\, j_{I\mu}(x')\,\lambda\right] \qquad (20)$$

15) R.J.Glauber, Prog. Theor. Phys., 9, 295 (1953)

16) H. Umezawa, Prog. Theor. Phys., 7, 551 (1952); Quantum Field Theory, North-Holland Publishing Co., Amsterdam (1956), page 204.

then under the unitary transformation

$$\Psi_I(\sigma) \rightarrow U(\sigma; B_I(x))\, \Psi_I(\sigma) \qquad (21)$$

on the interaction representation state-vector, the transformed H' becomes zero. Notice that when (19) leaves H_0 and H' invariant, the current j_μ in the Heisenberg representation is exactly conserved. The physical content of the theorem is simply that the coupling of a spinless field to a conserved current is equivalent to a null coupling for scattering processes.

[The interaction representation state vectors in the infinite past and the infinite future are unaltered since by the adiabatic hypothesis, $U(\sigma; B_I(x))$ becomes equal to unity at these times. Therefore the S-matrices calculated with the two Hamiltonians should be identical.]

In our case, H' contains a term $m\bar{\psi}_I \psi_I$ which is not invariant under (19) so that the transformed Hamiltonian instead of being zero, becomes

$$H''(x) = m\,\bar{\psi}_I(x)\, e^{2ig\gamma_5 B_I(x)}\, \psi_I(x) \qquad (22a)$$

The new Lagrangian can thus be written as

$$\mathcal{L}' = -\bar{\psi}\left(\gamma_\mu \partial_\mu + m e^{2ig\gamma_5 B}\right)\psi \qquad (22b)$$

With these preliminaries, we proceed to the construction of our model. In the first instance, we shall prove our result for an I-O vertex for simplicity. Subsequently the proof for

the $I = 1$ vertex, which is the case of physical interest, will be outlined. Consider the Lagrangian

$$\mathcal{L} = -\bar{N}(\gamma_\mu \partial_\mu + m)N + ig\,\bar{N}\gamma_\mu\gamma_5 N\,\partial_\mu B + ig'\,\bar{N}\gamma_\mu\gamma_5 N A_\mu$$
$$-\tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{4}[\partial_\mu A_\nu - \partial_\nu A_\mu][\partial_\mu A_\nu - \partial_\nu A_\mu] + \mathcal{L}_{int} \qquad (23)$$

The fields B and A_μ are massless and the coupling constants g and g' may be taken to be as small as is necessary to avoid any contradiction with experiments. $\mathcal{L}_{int}$ denotes any strong interaction Lagrangian which is invariant under the Touschek transformation (equation (4) or (19)) which we shall also call the γ_5 -gauge transformation. One such $\mathcal{L}_{int}$ is provided by the Sakurai Lagrangian[17] in the absence of strong isotopic spin couplings . Since $\mathcal{L}_{int}$ is γ_5 -invariant, we still have, after the unitary transformation (21) ,

$$\mathcal{L}' = -\bar{N}(\gamma_\mu \partial_\mu + m e^{2ig\gamma_5 B})N + ig'\,\bar{N}\gamma_\mu\gamma_5 N A_\mu$$
$$-\tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \mathcal{L}_{int} \qquad (24)$$

To calculate $\frac{Z_2}{Z_1}$ for the $I=0$ vertex, we consider the infinitesimal gauge transformations

$$N(x) \to [1 + i\gamma_5 v(x)]N(x),$$
$$B(x) \to B(x) - \frac{v(x)}{g},$$
$$A_\mu(x) \to A_\mu(x) + \frac{1}{g'}\partial_\mu v(x) \qquad (25)$$

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17) J.J.Sakurai, Ann. Phys., 11 , 1 (1960).

under which the unrenormalized nucleon propagator

$$S_F(x-y) = i \langle T\left(N(x)\,\bar{N}(y)\right)\rangle \qquad (26)$$

undergoes the change

$$\Delta S_F(x-y) = i\gamma_5 S_F(x-y)\,v(x) + i S_F(x-y)\,\gamma_5\, v(y) \qquad (27)$$

while

$$\mathcal{L}' \to \mathcal{L}' + \frac{1}{g}\,(\partial_\mu B)(\partial_\mu v) \qquad (28)$$

It may be noted how the use of the unitary equivalence has enabled us to get rid of the nucleon mass term which would otherwise have appeared in $\mathcal{L}$ under the transformation (25)[18] As before, we can once again calculate $\Delta S_F(x-y)$ by adding a first order perturbationto $\mathcal{L}'$ as given by (28). This gives

$$\frac{\Delta S_F(x-y)}{\Delta v(z)} = \frac{1}{g}\,\partial_\mu^2(z)\,\langle T\left(N(x)\,\bar{N}(y)\,B(z)\right)\rangle \qquad (29)$$

where $\partial_\mu^2(z)$ acts on the z-variable. To evaluate this, we use the S-matrix in the representation(23) and retain terms only to order g^2 (which is permissible since g may be taken to be a very small quantity). By an application of Wick's theorem[19] we find that the contraction symbol between the B-field in (29) and the one arising from the S-matrix gives ~~in~~ the vertex function for the current $\bar{N}\gamma_\mu\gamma_5 N$. Finally, in momentum representation, we have,

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18) Such a parametrization of the fermion mass term to ensure its γ_5-gauge invariance is due to K.Nishijima, Nuovo Cimento, 11, 910 (1959).

19) G.C.Wick, Phys. Rev., 80, 268 (1950).

$$i\gamma_5 S_F(p) + i S_F(p')\gamma_5 = k_\mu S_F(p') \Gamma_{\mu 5}(p', p) S_F(p) \quad (30)$$

which leads to the result

$$\frac{Z_2}{Z_1} = -1 \quad (31)$$

by arguments analogous to the ones employed in getting (13) . Note that we have here treated the coupling due to g' exactly and the one due to g only to order g^2 .

The above result is true for an $I=0$ vertex. To prove it for an $I=1$ vertex, we consider the Lagrangian

$$\mathcal{L} = -\bar{N}(\gamma_\mu \partial_\mu + m) N + ig \bar{N} \gamma_\mu \gamma_5 \vec{\tau} N \partial_\mu \vec{B}$$
$$+ ig' \bar{N} \gamma_\mu \gamma_5 \vec{\tau} N \vec{A}_\mu - \frac{1}{2}(\partial_\mu \vec{B})^2 - \frac{1}{4}[\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu][\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu]$$
$$+ \mathcal{L}_{int} \quad (32)$$

where $\vec{B}$ and $\vec{A}_\mu$ are now isotopic spin triplets. The unitary transformation (21) is now replaced by

$$\Psi_I(\sigma) \rightarrow \exp\left[i \int_\sigma d\sigma_\mu' \, \vec{j}_{I\mu}(x') B_I(x')\right] \Psi_I(\sigma) \quad (33)$$

where

$$\vec{j}_{I\mu} = ig \bar{N}_I \gamma_5 \gamma_\mu \vec{\tau} N_I \quad (34)$$

We shall hereafter neglect terms of the order of g^2 , gg' etc. and neglect the couplings of $\vec{B}$ and $\vec{A}_\mu$ in calculating the divergence of $\vec{j}_\mu$. It follows that $\partial_\mu \vec{j}_\mu = 0$ if $m = 0$. Under the transformation (33), $\mathcal{L}$ gets transformed into

$$\mathcal{L}' = -\bar{N}(\gamma_\mu \partial_\mu + m e^{2ig\gamma_5 \vec{\tau}\cdot\vec{B}}) N + ig' \bar{N} \gamma_\mu \gamma_5 \vec{\tau} N \vec{A}_\mu$$
$$- \frac{1}{2}(\partial_\mu \vec{B})^2 - \frac{1}{4}[\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu][\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu] + \mathcal{L}_{int} \quad (35)$$

(25) is now replaced by

$$N(x) \to [1 + i\gamma_5 \vec{\tau}\cdot\vec{v}(x)]\, N(x),$$
$$\vec{B}(x) \to \vec{B}(x) - \frac{\vec{v}(x)}{g},$$
$$\vec{A}_\mu(x) \to \vec{A}_\mu(x) + \frac{1}{gi}\partial_\mu \vec{v}(x) \qquad (36)$$

The rest of the derivation is similar and we again obtain $\frac{Z_2}{Z_1} = -\frac{G_A}{G_V} = -1$. This rather curious result, if it bears x any relation to reality, would imply that the unrenormalized current in β decay is $V+A$. It may be remarked that in Sakurai's theory, the $\mathcal{L}_{int}$ we have considered involves the neglect of the weakest of the family of couplings viz., the isotopic couplings so that in such a theory of strong interaction, this model is not inconsistent with the fact that $\left|\frac{G_A}{G_V}\right|$ departs only by a small amount from unity.

It is interesting in this connection to note that a parametrization of the fermion was term similar to that in equation (24) to ensure its γ_5 -gauge invariance can be made even if the ψ -field has no interaction with the B -field. We may in fact choose the auxiliary field suchthat all its quantum numbers are identical with the vacuum. Consider the Lagrangian

$$\mathcal{L} = -\bar{\psi}(\gamma_\mu\partial_\mu + m)\psi - \frac{1}{2}(\partial_\mu B)^2 \qquad (37)$$

where we have set the mass of the B -field equal to zero. The unitary transformation (21) now gives

$$\mathcal{L}' = -\bar{\psi}(\gamma_\mu\partial_\mu + m e^{2ig\gamma_5 B})\psi + g\bar{\psi}\gamma_5\gamma_\mu\psi\partial_\mu B - \frac{1}{2}(\partial_\mu B)^2 \qquad (38)$$

which proves the assertion made earlier since (38) is invariant under the transformations $\psi \to e^{i\lambda\gamma_5}\psi$, $B \to B - \frac{\lambda}{g}$ with λ a constant. It is in fact the analogue of the second

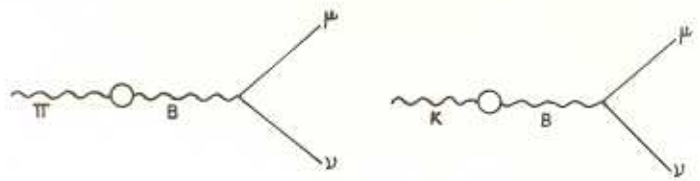
term in (38) which cancelled out the last two terms in (18) to give us (22). The B -field of course is completely unobservable since we have assumed that it has no interaction with any other field.

3. Conclusion

We conclude the discussion with a brief speculation on the role scalar mesons may play in weak interactions. Experimentally the strangeness-changing leptonic decays of hyperons seem very x rare [20] while the rates for the leptonic decays of the K -mesons are not negligible. This can find a simple explanation if strangeness-changing leptonic decays are mediated by scalar mesons since then the hyperonic decays would get suppressed due to the smallness of the momentum transfers involved. (Decays other than the $|\Delta S| = 1$ leptonic decays may be assumed to proceed for instance through a contact interaction between the currents). Assuming that the coupling constant $\frac{g}{m_B}$ of this boson of mass m_B to the corresponding currents is equal to $\sqrt{g_\beta}$ where g_β is the β-decay coupling constant, we may write

$$\frac{\tau_{\pi\to\mu+\nu}}{\tau_{K\to\mu+\nu}} = f^2 \frac{m_\pi}{m_K} \left[\frac{m_K^2}{m_B^2 - m_K^2}\right]^2 \frac{m_\pi^2 + m_\mu^2}{m_K^2 + m_\mu^2} \frac{\rho(K)}{\rho(\pi)} \qquad (39)$$

for the corresponding decay life-times. Here f denotes the relative strength of the strong K- and π- vertices. ρ is the density of final states. (39) is computed from the following diagrams.



(39) gives for $m_B \approx$ 571.5 Mev, $f^2 \approx$ 1/10 which is reasonable in view of the usual estimates regarding the relative strength of the $K-$ and $\pi-$ strong couplings. For the $\Lambda-$ and $\Sigma-$ leptonic decay modes, the largest momentum transfer at the $\Lambda-N$ -vertex is $\approx$ 175 Mev, at the $\Sigma-N$ vertex, $\approx$ 250 Mev, both occurring when the nucleon is produced at rest. In the latter case, the factor by which the rate is depressed relative to the universal rate is $\leq \left[\frac{250^2}{571.5^2 - 250^2}\right]^2 \approx 0.06$ if it is computed for a coupling of the form $\frac{g}{m_B}\ \bar{\Sigma}^- \gamma_\mu (1+\gamma_5) n \partial_\mu B^- + h.c.$. All this seems consistent with the present scanty experimental data. Finally, we notice that the introduction of such a meson would imply that the $|\Delta S| = 1$ currents to which it is coupled can have no $\frac{\Delta Q}{\Delta S} = -1$ components since otherwise there would be a contradiction with the observed small $K_1^0 - K_2^0$ mass difference[21].

21) L.B.Okun' and B.M.Pontecorvo, Soviety Phys. JETP 34, 763 (1958); Muller et. al., Phys. Rev. Letters, 4, 418 (1960).

CHAPTER XI.

ON AN ISOTOPIC SPIN SCHEME FOR LEPTONS AND A FOUR-DIMENSIONAL ISOTOPIC SPIN FORMALISM

I. This chapter is devoted to a discussion of an isotopic spin scheme for leptons and a four-dimensional isotopic spin formalism and is divided into two sections. In Section II , the isotopic spin scheme for leptons is developed and used in deriving some phenomenological rules for the leptonic decays of strange particles. In Section III, an extension of the ideas of Section II is considered which gives rise to a four-dimensional isotopic spin scheme. The structure of weak interactions is then discussed within the framework of this scheme.

II. On an Isotopic Spin Scheme for Leptons*

1. Introduction

From time to time, there have been attempts to develop an isotopic spin scheme for leptons. At first sight, there seems to be some difficulty in classifying these particles in such a

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* Alladi Ramakrishnan, A.P.Balachandran and N.R.Ranganathan, Proc. Ind. Acad. Sc., 53 , 1 (1961); Alladi Ramakrishnan, A.P.Balachandran, N.R.Ranganathan and N.G.Deshpande, Nuclear Physics, 26 , 52 (1961).

scheme since, for instance, the muon and electron have no neutral counterparts. Most of the schemes which have hitherto been proposed have assumed the existence of the neutral counterpart of the muon (the μ^0) and joined the electron and neutrino into a doublet[1] or in some analogous way tried to build up multiplets out of particles which are or a widely different nature.[2] We propose below an isotopic spin scheme which seems to be a natural extension of the idea involved in the two-component theory of the neutrino[3] to this internal symmetry space. A consequence of the scheme is that interactions involving μ or e cannot conserve isotopic spin so that if an interaction must conserve it if it is to be strong, these particles can have no strong interactions. Similarly if strong interactions should also conserve parity, the neutrino can have no strong interactions.[3] This therefore seems to account rather naturally for the absence of strong interactions for leptons.

2. The Isotopic Spin Scheme

It may be first remarked that the absence of the neutral counterparts of μ and e has an analogue in the space-time structure of the neutrino field where it is found that only the left-helicity state of the neutrino is realized in

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1) R.E.Marshak and E.C.G.Sudarshan, preprint, University of Rochester (1957).
2) S.A.Bludman, Nuovo Cimento, 1 , 433 (1958).
3) A. Salam, Nuovo Cimento, 5 , 299 (1957);
L. Landau, Nuclear Physics, 3 , 127 (1957);
T.D.Lee and C.N.Yang, Phys. Rev., 105 , 1671 (1957).

nature. This has been interpreted to be a consequence of the γ_5- invariance of this field:[3]

$$\nu = \gamma_5 \nu \qquad (1)$$

Since the formalism of isotopic spin has many formal similarities to that of ordinary spin, it seems plausible to assume in analogy with the neutrino field that μ^- and e^- are components of two distinct isodoublets μ and e whose neutral counterparts do not exist. This can be achieved if μ and e are invariant under the $-\tau_3$- transformation:

$$\mu = -\tau_3 \mu, \qquad e = -\tau_3 e \qquad (2)$$

where, as usual,

$$\tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (3)$$

(2) implies that μ and e can be written as

$$\mu = \begin{bmatrix} 0 \\ \mu^- \end{bmatrix}, \quad e = \begin{bmatrix} 0 \\ e^- \end{bmatrix} \qquad (4)$$

The exclusion of the $I_3 = +\frac{1}{2}$ components of the μ and e doublets implies that the allowed wave functions do not completely spam the isotopic spin rotation space. Consequently no interaction can be formed which is a scalar in the isotopic spin space involving these fields, i.e. interactions involving μ or e cannot conserve isotopic spin. The situation is similar to the case of the neutrino where equation (1) implies that all the interactions of the neutrino are parity non-conserving.

The neutrino and the photon are included in the scheme as isosinglets. If one assumes a relation between the charge Q, the y-component of the isotopic spin I_y and the lepton number l of these particles analogous to the corresponding relation which is known to exist for strongly interacting particles, we have

$$Q = I_y + \frac{l+S}{2} \qquad (5)$$

where S plays the role of strangeness for leptons. Assigning $l=+1$ for μ^-, e^- and ν, we have $S=-2$ for μ^- and e^- and -1 for ν. For photon, of course, $Q = I_y = l = S = 0$

3. Strong and Electromagnetic Interactions

A consequence of the scheme which has been proposed above is that we can forbid the strong interactions of leptons with the following additional assumptions :

a) Strong interactions must conserve isotopic spin. This implies that the muon and electron can have no strong interactions.

b) They must conserve parity. The neutrino can then have no strong interactions since it always occurs in the combination $\frac{1}{2}(1+\gamma_5)\nu$.

The electromagnetic interactions of strongly interacting particles are known to transform as a scalar plus the third component of a vector when the particles are isospinors, an example being the nucleon whose electromagnetic interaction reads i.e. $ie\bar{N}\gamma_\mu \frac{1+\tau_3}{2} N A_\mu$. This is exactly the transformation

property for the electromagnetic interactions of the muon and electron in this scheme which read $ie\bar{\mu}\gamma_\mu\mu A_\mu$ and $ie\bar{e}\gamma_\mu e A_\mu$ or equivalently, $ie\bar{\mu}\gamma_\mu\frac{1+\tau_3}{2}\mu A_\mu$ and $ie\bar{e}\gamma_\mu\frac{1+\tau_3}{2}e A_\mu$. The classification therefore is consistent as far these interactions are concerned. The electromagnetic interactions of the neutrino can then be forbidden by assuming that these interactions must transform as a scalar plus the third component of a vector or as the third component of a vector (The latter situation occurs among strongly interacting particles for isovectors like the Σ- hyperon).

It is interesting to note that only leptons seem to possess invariance under transformations which eliminate one of their components (the γ_5- invariance of the neutrino or the $-\tau_3$- invariance of the muon and electron) while for baryons, such invariances seem to be a property of some of their interactions and not of the fields themselves.

4. Weak Interactions

Present evidence in weak interactions seems to rule out the existence of neutral lepton currents[4]. We shall accept this as a given fact and assume that the only lepton currents (which we shall denote by J_α^l) relevant for weak interactions are $\bar{\mu}\nu$ and $\bar{e}\nu$.

4) T.D.Lee and C.N.Yang, Phys. Rev., 119, 1410 (1960)

(Here we have omitted the γ -matrices for convenience of notation). These transform as one of the components of an isospinor. The $|\Delta S|=1$ strangeness-changing currents involving strongly interacting particles (which we shall call J_α^{SNC}) necessarily carry an isotopic spin 1/2 or 3/2 since these are built up of an isospinor and an isotensor (scalar or vector). Thus the interaction $J_\alpha^{SNC\dagger}\, I_\alpha^{\ell}$ which induces the $|\Delta S|=1$ strangeness-changing leptonic decays of strongly interacting particles will give the rule $|\Delta \vec{I}| = 0$, $\blacksquare$ 1 or 2 .

Let us now try to deduce some phenomenological rules for strangeness-changing leptonic decays using our isospin classification of leptons. Experimentally, such decays of the hyperons and the three-body decays of the K -meson ($K \to \pi + e + \nu$ and $K \to \pi + \mu + \nu$) seem rare while the $K \to \mu + \nu$ decay is not negligible [5]. All these decays can be forbidden if we impose on them the selection rule $|\Delta \vec{I}| =$ 1/2 or 3/2 since we have seen that these necessarily have $|\Delta \vec{I}| =$ an integer. Such a rule will however be hard to reconcile with the comparitively large decay rate of the $K \to \mu + \nu$ mode. Alternatively we may assume that all the isotopic spin amplitudes are allowed in the final state. These amplitudes may further be assumed to be such that

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5) D.A.Glaser, Intern. Conf. on High Energy Physics, Kiev (1960);
L.B.Okun', Annual Review of Nuclear Science, 9,82 (1959).

$$\alpha_0 \approx \alpha_{\frac{1}{2}} \approx \alpha_1 \approx \alpha_{3/2} \qquad (5)$$

upto a sign

where α_0 , $\alpha_{1/2}$, α_1 , $\alpha_{3/2}$ denote the amplitudes for the I= 0, 1 and 3/2 final states that. Assume further that these amplitudes always interfere destructively. One then finds that the decays

$$\Lambda^0 \to p + \ell^- + \bar{\nu},$$
$$\Sigma^+ \to n + \ell^+ + \nu \qquad (6)$$

are forbidden (where ℓ denotes μ or e) while the decays

$$K^+ \to \pi^0 + \ell^+ + \nu,$$
$$K^- \to \pi^0 + \ell^- + \bar{\nu},$$
$$K_1^0 \to \pi^+ + \ell^- + \bar{\nu},$$
$$K_1^0 \to \pi^- + \ell^+ + \nu,$$
$$K_2^0 \to \pi^+ + \ell^- + \bar{\nu},$$
$$K_2^0 \to \pi^- + \ell^+ + \nu,$$
$$\Xi^- \to \Sigma^0 + \ell^- + \bar{\nu},$$
$$\Xi^0 \to \Sigma^+ + \ell^- + \bar{\nu},$$
$$\Xi^0 \to \Sigma^- + \ell^+ + \nu \qquad (7)$$

are reduced to about 1/18 of the universal rate if the α_i are taken to be of the same order of magnitude as the expected

universal decay amplitudes. The derivation of these results is straight forward. For instance for the $K^{\pm}$ decay, the decay amplitude can be analyzed in terms of two amplitudes corresponding to the $I = 1/2$ and $I = 3/2$ states of the final system. It can therefore be written as $\sqrt{\frac{2}{3}}\alpha_{3/2} - \sqrt{\frac{1}{3}}\alpha_{1/2}$ which coupled with (5) gives us the factor 1/18 quoted above. The assumption made does not however suppress the following decays:

$$K^+ \to l^+ + \nu,$$
$$K^- \to l^- + \bar{\nu},$$
$$\Sigma^- \to n + l^- + \bar{\nu},$$
$$\Xi^- \to \Lambda^0 + l^- + \bar{\nu} \qquad (8)$$

since the final system in each of these cases is in a definite isotopic spin state. This is consistent with the rate for $K^+ \to l^+ + \nu$ and $K^- \to l^- + \bar{\nu}$ which are known to be appreciable.

Alternatively, instead of (5) , we may assume

$$\alpha_0 \approx \alpha_{\frac{1}{2}}\ , \quad \alpha_1 \approx \frac{7}{12}\alpha_0\ , \quad \alpha_{\frac{3}{2}} \approx \frac{7}{12}\alpha_{\frac{1}{2}} \qquad (9)$$

with destructive interference between the α_i's . This will allow for the decays listed in equation (6) which were forbidden above, although with a rate 1/12th of that expected with a universal Fermi coupling. The rates for the modes

$$K^+ \to \pi^0 + \ell^+ + \nu,$$
$$K^- \to \pi^0 + \ell^- + \bar{\nu},$$
$$\Xi^- \to \Sigma^0 + \ell^- + \bar{\nu} \qquad (10)$$

are now about 1/100 times the universal rates. Experimentally the three-body decay modes of $K^{\pm}$ seem to be suppressed by a factor of 170 relative to the universal rate. Finally the decays of the K_1^0, K_2^0 and Ξ^0 listed in equation (7) are now suppressed by a factor of 1/4 . Experimentally, the leptonic decays of hyperons seem to be an order of magnitude smaller than the universal rate[5].

In this discussion, we have not assumed the $\frac{\Delta Q}{\Delta S} = 1$ rule for strangeness-changing currents[6]. Also if $|\Delta S| = 2$ leptonic decays are allowed, $\Xi^- \to n + \ell^- + \bar{\nu}$ should be fast while $\Xi^0 \to p + \ell^- + \bar{\nu}$ should be about 1/12 of the universal rate. In the {current}x{current} picture of weak interactions[6], we should have $\frac{\Delta Q}{\Delta S} = 1$ rule rigorously for the strangeness-changing currents and no $|\Delta S| = 2$ currents since the non-leptonic decays do not seem to exist. The most convincing evidence for this comes from the $K_1^0 - K_2^0$ mass difference. It has been observed by Okun' and Pontecorvo[7 &] that if there are such currents, there will be a transition from K^0 to $\bar{K}^0$ in the first order in weak interactions which would imply a $K_1^0 - K_2^0$ mass difference of the order of 10 eV . If there are no $|\Delta S| = 2$ transitions to the first order in the weak coupling, the $K_1^0 - K_2^0$ mass difference is proportional

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6) R.P.Feynman and M. Gell-Mann, Phys. Rev., 109 , 193 (1958)
7) L. B. Okun' and B.M.Pontecorvo, Soviety Physics, JETP, 34 , 763 (1958).

to the square mof the weak coupling constant and of the order of 10^{-5} Mev . Evidence that the mass difference is of this order has been reported recently[8] thereby ruling out $|\Delta S| = 2$ non-leptonic transitions through a first-order weak coupling.

III. A FOUR DIMENSIONAL ISOTOPIC SPIN FORMALISM AND WEAK INTERACTIONS*

1. The four-dimensional Isotopic Spin Formalism

An extension of the ideas suggested in our discussion on the isotopic spin scheme for leptons will now be described which enables us to develop a four-dimensional isotopic spin formalism for elementary particles. This is a modification of the Salam-Polkinghorne classification of elementary particles[9] but is distinct from it in that it implies effectively no more assumption than those contained in the Gell-Mann - Nishijima scheme and also includes leptons in a natural way.

We start by considering the usual relation between the charge Q of an elementary particle and the 3 -component of its isotopic spin:

$$Q = I_3 + U_3 \qquad (12)$$

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8) Muller et. al., Phys. Rev. Letters 4 , 413 (1969).

* This work was done in collaboration with Alladi Ramakrishnan and N.G.Deshpande.

9) A. Salam and J.C.Polkinghorne, Nuovo Cimento, 2 , 685 (1955); J.C.Polkinghorne, Nuovo Cimento, 6 , 864 (1957).

where $U_3 = \frac{N+S}{2}$ for strongly interacting particles and $= \frac{l+S}{2}$ for the lepton family. Here N is the baryon number, l the lepton number, S in the case of heavy mesons and baryons is the usual strangeness while S for the leptons has been defined previously. For the photon, $U_3 = 0$. Now on the right-hand side of the relation (12), the first term is of the 3-component of an angular momentum. It seems therefore most suggestive to think of U_3 too as the 3-component of an angular momentum since then I_3 and U_3 which appear on a symmetrical footing in (12) would have symmetrical significance as quantum numbers. These considerations lead one to introduce a three-dimensional hypercharge space associated with the quantum number U_3. (The definition of hypercharge here differs by a factor of 1/2 from the usual definitions). This was done by Salam and Polkinghorne for strongly interacting particles[9]. In this space, Σ, Λ, ν, π and A_μ form scalars. The nucleon has $U_3 = +\frac{1}{2}$ and Ξ, $U_3 = -\frac{1}{2}$. We shall include these particles also in this scheme by extending the idea we used previously for leptons. Namely, we shall assume that N and Ξ are the components of hypercharge doublets N' and Ξ' where N' and Ξ' are such that

$$N' = \sigma_3 N' , \qquad \Xi' = -\sigma_3 \Xi' \qquad (13)$$

where

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (14)$$

now operates in the hypercharge space. Thus N' and Ξ' are of the form

$$N' = \begin{pmatrix} N \\ 0 \end{pmatrix}, \qquad \Xi' = \begin{pmatrix} 0 \\ \Xi \end{pmatrix} \qquad (15)$$

the μ and e will be hypercharge doublets like Ξ while K and $\bar{K}$ may be considered together to form a doublet as shown below:

$$K' = \begin{pmatrix} K \\ \bar{K} \end{pmatrix} \qquad (16)$$

These quantum number assignments for baryons and mesons are identical with those in the Salam-Polkinghorne scheme except that they consider N and Ξ as components of the same doublet which implies assumptions not involved in the Gell-Mann - Nishijima classification. With Salam and Polkinghorne, we may now take the direct product of the isospin and hypercharge rotation groups to obtain a four-dimensional real rotation group in which Ξ, N, μ, e and K form four-vectors, Σ and Π form self-dual antisymmetric tensors and Λ, ν and A_μ form scalars. This then completes the classification in the four-dimensional space in question.

It has been suggested by d'Espagnat, Prentki and Salam[10] that the space generated by the infinitesimal operators formed by adding the infinitesimal operators in the I- and U-spaces may have some significance in weak interactions. They have called this space the M-space. Consider for

10) B. d'Espagnata, J.Prentki and A. Salam, Nuclear Physics, 5, 447 (1958).

instance, the $|\Delta S| = 1$ non-leptonic decays of strange particles and assume that weak interactions form scalars in the M -space. The conserved quantum number associated with this space is $\vec{I} + \vec{U}$. But then since $\vec{U}$ changes ~~in much~~ by half a unit in such an interaction, $\vec{I}$ too will change by half a unit, i.e. the Lagrangian will transform like an isospinor. This therefore gives us the well-known $|\Delta \vec{I}| = \frac{1}{2}$ rule in non-leptonic decays[11]. Thus the M -space explains the $|\Delta \vec{I}| = 1/2$ rule as an invariance principle. It may therefore be of some interest to investigate the structure of other types of weak interactions too on the assumption that they are also scalars in M -space. It is however to be noted that it is not possible to form such scalars in a model such as ours since some of the fields have missing components. Nevertheless we may still talk of M -space scalars in the sense that we may introduce some fictitious fields in place of the missing components, form scalars in M -space and finally take into account equations (13) . As we shall see below (this will once again give us the $|\Delta \vec{I}| = 1/2$ rule for non-leptonic decays. The situation is not satisfactory. The primary motivation for such a procedure is that we shall formulate a conjecture due to Bludman[12] made in connection with

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11) M. Gell-Mann and A. Pais, Proc. Conf. Nuclear and Meson Physics, Glasgow, 1954, page 342 (London, 1955)

12) S. A. Bludman, Phys. Rev., 115 , 468 (1959).

non-leptonic decays in an equivalent way in the M -space and then proceed to see the sort of consequences this implies in all the other weak interactions as well. We shall see that this gives the correct structure of the weak interactions for β -decay and μ -decay and forbids $\mu \to 3e$. We shall in fact start not with Bludman's conjecture, but with the known form of the weak interaction for β -decay and see that this naturally leads to Bludman's result. The latter considers his Lagrangian to be phenomenological since no account is taken of the influence of strong interactions on the weak interaction symmetries. For this reason, our Lagrangians too should be regarded as phenomenological. We may however note the following. We are primarily interested in the parity structure of the currents. Thus we wish to know how if the primary interaction for $\Sigma \to N + \pi$ is such that $\Sigma^{\pm} \to n + \pi^{\pm}$ is parity conserving and $\Sigma^{+} \to p + \pi^{0}$ is a $V-A$ interaction, strong interactions do not make the neutron decay mode of Σ also parity non-conserving through processes like $\Sigma^{+} \to p + \pi^{0} \to n + \pi^{+}$ or $\Sigma^{-} \to \Lambda^{0} + \pi^{-} \to p + \pi^{-} + \pi^{-} \to n + \pi^{-}$. One sees immediately that such effects can be minimized to a large extent if strong interactions are largely independent of isotopic spin and strangeness i.e. if processes in which particle character changes have comparitively low probability. There is some qualitative evidence from strong interaction processes that this may indeed be so to some extent. Thus ΛK and ΣK production in πN and N-N collisions, Ξ-K production in $K^{-}p$ collisions and $K^{-}p$ charge exchange

scattering seem to be suppressed while K^+ and K^- scattering cross-sections are large. If these indications are eventually seen to imply the isotopic spin and strangeness independance of strong interactions to a good approximation, the interactions we discuss below may be regarded as primitive.

2. Weak Interactions

The group of rotations in M -space we are considering is one which leaves the fourth axis invariant. Thus for this sub-group of the four-dimensional group, the four-vector becomes a scalar and the self-dual antisymmetric tensor becomes a three-vector. We shall denote by $\vec{N}$ and N_5 the three-vector and scalar parts of the nucleon four-vector in M- space with a similar notation for the other four-vectors. Explicit expressions for these multiplets in terms of the component fields in a suitable representation is given in d'Espagnat, Prentki and Salam's paper[10]. For the nucleon, we have

$$\vec{N} = \begin{bmatrix} \frac{-p+\alpha^-}{\sqrt{2}} \\ \frac{p+\alpha^-}{i\sqrt{2}} \\ \frac{-n-\alpha^0}{\sqrt{2}} \end{bmatrix} \qquad (17),$$

$$N_5 = \frac{-n+\alpha^0}{i\sqrt{2}} \qquad (18)$$

where α^- and α^0 are the dummy fields of which we spoke of before. For Ξ , $\vec{\Xi}$ and Ξ_5 read

$$\vec{\Xi} = \begin{bmatrix} \frac{-\beta^+ + \Xi^-}{\sqrt{2}} \\ \frac{\beta^+ + \Xi^-}{i\sqrt{2}} \\ \frac{-\beta^0 - \Xi^0}{\sqrt{2}} \end{bmatrix} \qquad (19)$$

$$\Xi_5 = \frac{-\beta^0 + \Xi^0}{i\sqrt{2}} \qquad (20)$$

where the β 's are the dummy fields. Similar expressions can be written down for μ and ϱ . For $\vec{K}$ and K_5 the expressions are obtained from (17) and (18) by the replacements $p \to K^+$, $n \to K^0$, $\alpha^0 \to \bar{K}^0$ and $\alpha^- \to K^-$. For Σ , the expression is

$$\vec{\Sigma} = \begin{bmatrix} \frac{-\Sigma^+ - \Sigma^-}{\sqrt{2}} \\ \frac{\Sigma^+ - \Sigma^-}{i\sqrt{2}} \\ \Sigma^0 \end{bmatrix} \qquad (21)$$

An analogous expression obtains for Π .

In considering weak interactions, we shall often find that more than one M -space scalar can be formed out of the given fields. In such cases, we shall assume that these are to be added together with equal coefficients, but with a sign which we shall leave unspecified for the moment.

Consider now the β -decay Lagrangian. It reads

$$\mathcal{L}_\beta^N = g \left\{ (\vec{N} \times \vec{N}) \pm \bar{N}_0 \vec{N} \right\} (\bar{\nu} \vec{e}) + h.c. \quad (22)$$

where the space-time structure of the currents is left unspecified. Now it is well-known that β-decay and μ-decay interactions are V-A [13), 14)]. This is immediately incorporated into our scheme if we assume that the vector components in M -space of the four vectors of the four-space always occur with the projection operator $\frac{1}{2}(1+\gamma_5)$ and the corresponding scalar components occur with $\frac{1}{2}(1-\gamma_5)$, the current involved being always of the vector type. Because of the well-known property of γ_μ that it cannot connect fields of opposing chirality[14)], we find that $\bar{N}_0 \gamma_\mu \vec{N} = 0$. $\mathcal{L}_\beta^N$ thus reads

$$\mathcal{L}_\beta^N = -\frac{ig}{4\sqrt{2}} (\bar{n} O_\alpha p)(\bar{\nu} O_\alpha e^-) + h.c. \quad (23)$$

where $O_\alpha = \gamma_\alpha (1+\gamma_5)$. This is the usual β -decay Lagrangian. Notice that, here, as in what follows, μ can equivalently be replaced by e and vice-versa. For μ -decay, the Lagrangian reads

$$\mathcal{L}_\beta^\mu = g \left[\vec{\mu} \gamma_\alpha \frac{1+\gamma_5}{2} \nu \right] \left[\bar{\nu} \gamma_\alpha \frac{1+\gamma_5}{2} \vec{e} \right] + h.c.$$

$$= \frac{g}{4} (\bar{\mu}^- O_\alpha \nu)(\bar{\nu} O_\alpha e^-) + h.c. \quad (24)$$

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13) R.P.Feynman and M. Gell-Mann, loc. cit.
14) E.C.G.Sudarshan and R.E.Marshak, Phys. Rev. 109 , 1860 (1957).

which is again consistent with experimental facts.

Consider now the non-leptonic decays of Λ. We have, with our assumptions,

$$\mathcal{L}_\Lambda = g(\bar{\Lambda} O_\alpha \vec{N})\,\partial_\alpha \vec{\pi} + h.c.$$

$$= g\left[\bar{\Lambda} O_\alpha p\,\partial_\alpha \pi^- - \sqrt{2}\,\frac{1}{\sqrt{2}}\,\bar{\Lambda} O_\alpha n\,\partial_\alpha \pi^0\right] + h.c. \quad (25)$$

(In these considerations, we are assuming that any particular decay occurs through an interaction made up of the constituent fields. This may be regarded as phenomenological. See however the discussion on previous pages). $\mathcal{L}_\Lambda$ clearly transforms as an isospinor so that we have the usual $|\Delta \vec{I}| = \frac{1}{2}$ rule. (25) also predicts that the nucleon from Λ -decay must have left-helicity. The experimental situation on this point is still unsettled.[15] The decay mode $\Xi \to \Lambda + \pi$ would also have a current structure identical to that of the mode $\Lambda \to N + \pi$. Consider now the decay of Σ into a nucleon and a π. We have,

$$\mathcal{L}_\Sigma = g\left\{(\bar{\vec{\Sigma}} O_\alpha \times \vec{N}) \pm (\bar{\vec{\Sigma}} O_\alpha N_5)\right\}\partial_\alpha \vec{\pi} + h.c.$$

$$= \frac{ig}{\sqrt{2}}\left\{\bar{\Sigma}^+ (O_\alpha \pm O'_\alpha)\, n\,\partial_\alpha \pi^+ - \bar{\Sigma}^- (O_\alpha \mp O'_\alpha)\, n\,\partial_\alpha \pi^-\right.$$

$$\left. + \bar{\Sigma}^0 O'_\alpha n\,\partial_\alpha \pi^0\right\} + ig\left\{\bar{\Sigma}^+ O_\alpha p\,\partial_\alpha \pi^0 - \bar{\Sigma}^0 O_\alpha p\,\partial_\alpha \pi^-\right\} + h.c. \quad (26)$$

where $O'_\alpha = \gamma_\alpha(1-\gamma_5)$. We have again the $|\Delta \vec{I}| = 1/2$ rule. (26) gives $V-A$ interaction for the π^0-decay mode of Σ^+

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15) The results of Boldt et. al., Phys. Rev. Letters, 1, 256 (1958) indicate a left-helicity for the proton while R. Birge and W. Fowler, Phys. Rev. Letters, 5, 254 (1960) obtain precisely the opposite result. See however Leitner et.al., Phys. Rev Letters 1, 264 (1961); Beall et. al., Phys Rev. Letters 1, 285 (1961).

(as for Λ -decay) and pure $V(A)$ for the π^+ decay mode of Σ^+ and pure $A(V)$ for the π^- -decay mode of Σ^- depending on the relative sign chosen in adding the two scalars. Experimental results indicate that the decay $\Sigma^+ \to p+\pi^0$ indeed has a large up-down asymmetry while the decays $\Sigma^\pm \to n+\pi^\pm$ are very nearly parity-conserving.[16]

The Lagrangians (25), (26) and the one for Ξ^- decay coincide with those of Bludman.[12] A closer examination in fact shows that the two formulations are completely equivalent for non-leptonic decays.

We will now show that the process $\mu \to 3e$ can be forbidden in this scheme provided we assume that the two scalars which can be formed are to be added with a plus sign. We have

$$\mathcal{L}_\mu^{\mathcal{W}} = g\left\{ (\vec{\mu} O_\alpha \vec{e})(\vec{e} O_\alpha \vec{e}) + (\vec{\mu} \times O_\alpha \vec{e})(\vec{e} \times O_\alpha \vec{e}) \right\} + h.c.$$

$$= 0 \qquad (27)$$

since with this choice of sign, the two terms in (27) cancel each other. However for the $K \to \mu + e$ or $K \to e + e$,

$$\mathcal{L}_K^{\mathcal{W}} = g\left\{ a_\alpha \vec{K} (\vec{\mu} \times O_\alpha \vec{e}) \mp a_\alpha K_\delta (\vec{\mu} O_\alpha \vec{e}) \right\} + h.c.$$

$$\neq 0 \qquad (28)$$

There is a good experimental evidence that this decay is in fact forbidden[17] so that in (28) we have to arbitrarily set $g = 0$. This is very unsatisfactory.

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16) The $\Sigma^\pm$ experiments are those of Cool et. al., Phys. Rev., 114 , 912 (1959) while the Σ^- -experiments are those of Franzini et. al., Bull. Am. Phys. Soc. 5, 224 (1969) See also Beall et al., loc cit.

17) Bardob et. al., Ann. Phys., 5 , 156 (1958).

Let us now discuss the leptonic decays of π and the strange particles. For the decay modes $\pi \to l + \nu$ and $K \to l + \nu$, one has

$$\mathcal{L}_{\pi l_2} = \frac{g}{2}\left[\bar{\vec{l}}\, O_\alpha \nu\right] \partial_\alpha \vec{\pi} + h.c. = -g(\bar{l}^- O_\alpha \nu)\partial_\alpha \pi^- + h.c.,$$

$$\mathcal{L}_{K l_2} = g\left(\bar{\vec{l}}\, O_\alpha \nu\right) \partial_\alpha \vec{K} + h.c.$$

$$= g(\bar{\vec{l}}^- O_\alpha \nu)\partial_\alpha K^- + h.c. \qquad (29)$$

For K_{l_3},

$$\mathcal{L}_{K l_3} = \frac{g}{2}\left[(\vec{K} \times \partial_\alpha \vec{\pi}) + K_3 \partial_\alpha \vec{\pi}\right](\bar{\vec{l}}\, O_\alpha \nu) + h.c.$$

$$= -\tfrac{1}{2} g\left[K^- \partial_\alpha \pi^0 - \sqrt{2}\, \bar{K}^0 \partial_\alpha \pi^-\right](\bar{l}^- O_\alpha \nu) + h.c. \qquad (30)$$

Here we have chosen the appropriate sign to give the $\frac{\Delta Q}{\Delta S} = 1$ [13] and $|\Delta \vec{I}| = \tfrac{1}{2}$ [18] rules (Notice that the plus sign has been chosen in (27) as well as (30). The same sign in (26) will then predict V for $\Sigma^+ \to n + \pi^+$ and A for $\Sigma^- \to n + \pi^-$). The consequences of these rules are discussed in detail in ref.[18] We shall see presently that it is in fact impossible to maintain these rules for baryon decays within the present scheme so that confirmation of this rule for K_{l_3} may not mean much. However this is again linked with the question of strong interaction effects in weak decays. If these are serious, one would then expect a large admixture of $|\Delta \vec{I}| = 3/2$

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18) S. Okubo, R.E.Marshak, E.C.G.Sudarshan, W.B.Teusch and S. Weinberg, Phys. Rev., 112, 665 (1958).

currents in K_{l_3} in spite of the fact that $\ell_{K_{l_3}}$ written down in (30) has $|\Delta \vec{I}| = \frac{1}{2}$ (which implies $\frac{\Delta Q}{\Delta S} = +1$) . The present experimental situation[19] is not sufficiently precise to decide whether or not the rule is correct.

Let us now discuss the baryonic currents for the corresponding leptonic decays. Since $\partial_\alpha \vec{\pi}$ and $(\bar{\nu} \partial_\alpha \vec{l})$ are similar to each other, one deduces that the baryonic currents in leptonic and non-leptonic decays are identical. There is one slight difference however since l^- and l^+ belong to different multiplets in contra distinction to π^- and π^+ so that we have to write down two Lagrangians for instance for $\Sigma^\pm$ leptonic decays, one for $\Sigma^+ \to n + l^+ + \nu$ and the other for $\Sigma^- \to n + l^- + \bar{\nu}$. A glance at (25) shows that the Λ-N current in the leptonic decay of Λ will be V-A . (26) similarly shows that $\Sigma^- n$ and $\Sigma^+ n$ currents will be V or A . (As remarked a moment before, there are two Lagrangians, one each for Σ^+ and Σ^- decays instead of the one in (26) so that there is no necessary correlation between $\Sigma^+ n$ and $\Sigma^- n$ currents here unlike in (26)) . (26) also shows that the $|\Delta \vec{I}| = \frac{1}{2}$ and $\frac{\Delta Q}{\Delta S} = 1$ rules for strangeness non-conserving currents are no longer valid. Thus for instance Σ^+ can decay into $n + e^+ + \nu$. Since Ξ is in many ways similar to N in our scheme, we deduce also that the $\Xi^- \Sigma^0$ current (like the $\Sigma^0 p$ current, cf.

19) Crawford et. al., Phys. Rev. Letters 2, 361 (1959).

equation (26)) is $V-A$ while $\Xi^0-\Sigma^+$ and $\Xi^0-\Sigma^-$ currents (like the $\Sigma^+ n$ and $\Sigma^- n$ currents) may be V or A . The $\Xi-\Lambda$ current is of course again $V-A$. Notice that in the parity-conserving decays involved here, we can arrange things so that only γ_4 appears in the baryonic currents. For equal values of the coupling constants, this would suppress these decays by a factor of 1/4 compared to the $V-A$ case.[20] This seems consistent with the observed fact that the hyperon decays seem surprisingly small compared to what the universal $V-A$ theory predicts.[21), 22] In particular we see that this factor can be got for the β -decays of Σ . However it cannot be got for the β -decay of Λ and hence offers no explanation for its low rate.[22]

The presence of $\frac{\Delta Q}{\Delta S} = -1$ components implies that the scheme suggested is inconsistent with the $\{$ current $\}$ x $\{$ current $\}$ picture of weak interactions widely discussed in the literature. This is because the current certainly contains $\frac{\Delta Q}{\Delta S} = 1$ components (as shown for instance by the $K \to \mu + \nu$ decay) so that the interference term between the $\frac{\Delta Q}{\Delta S} = 1$ and $\frac{\Delta Q}{\Delta S} = -1$ currents will give rise to $|\Delta S| = 2$ transitions in the first order in weak interaction coupling since the

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20) A. Pais, Nuovo Cimento, 18 , 1003 (1960)
21) For the status of hyperon β- decays, see L.B.Okun', Ann. Rev. Nucl. Sci., 9 , 82 (1959).
22) For an alternative mechanism which can depress these decays, see A.P.Balachandran, Nuovo Cimento, (in press); also chapter X , page .

total interaction has of course $\Delta Q = 0$. Therefore there can be a $K^0 \to \overline{K^0}$ transition in the first order in the weak coupling which, as discussed in Section II , will contradict the observed $K_1^0 - K_2^0$ mass difference. Therefore a scheme like the one considered here cannot be generalized into a $\{$ current $\}$ x $\{$ current $\}$ picture. Thus it specifically precludes the case where weak interactions are mediated by a quartet of vector mesons with the properties discussed by Lee and Yang.[23]

We now proceed to determine the structure of the Ξ-Ξ , Σ-Σ and Σ-Λ currents. As in the nucleon case, one sees that the Ξ-Ξ current is V-A . Now equation (26) implies that $\vec{\Sigma}$ cannot be associated with $\frac{1}{2}(1+\gamma_5)$ or $\frac{1}{2}(1-\gamma_5)$ since one or the other of the brackets in (26) will then vanish. It should therefore be associated with 1 or $i\gamma_5$. Therefore the Σ-Σ current should be a pure vector. By the conserved vector current hypothesis[13] , the coupling constant which occurs here can be related to the vector coupling constant in the β -decay of the neutron. The decay which is observable is of course $\Sigma^- \to \Sigma^0 + e^- + \bar{\nu}$ since Σ^0 is heavier than Σ^+ while Σ^- is heavier than Σ^0 . The low momentum transfer involved in this decay also implies that the weak form factor can be taken at its zero momentum transfer limit where its value is unity. Thus the decay characteristics can be exactly predicted and could in

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23) T.D.Lee and C.N.Yang, Phys. Rev. 119 , 1410 (1960).

principle constitute a very good test of our scheme and of the conserved vector current theory. However, unfortunately, because of the very small Q -value involved, Σ^- is very long-lived with respect to this mode and the competing dominant non-leptonic decay mode will completely swamp it. We can also determine the Σ-Λ current by such arguments. Thus (25) shows that Λ occurs with 1, γ_5 or $\frac{1}{2}(1+\gamma_5)$. Λ however is a particle very similar to ν in our classification (since both are scalars). By the two-component theory[24], $\frac{1}{2}\nu$ always occurs with $\frac{1}{2}(1+\gamma_5)$. It is therefore plausible to assume that Λ occurs with $\frac{1}{2}(1+\gamma_5)$. Thus the Σ-Λ current should be V-A . This can be tested experimentally, but there are again unpredictable strong interaction effects to be taken into account.

We conclude by noting that the isotopic spin formalism developed in the first part of this chapter provides a natural classification for the vector bosons with charged components only conjectured by d' Espagnat[25] in connection with a theory of weak interactions. It would also automatically forbid the strong interactions of such bosons which combined with their massiveness, could explain why they have not hitherto been observed.

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24) A. Salam, Nuovo Cimento, 5 , 299 (1957);
L. Landau, Nuclear Physics, 3 , 127 (1957);
T. D. Lee and C.N.Yang, Phys. Rev., 105 , 1671 (1957).

25) B. d' Espagnat, Nuovo Cimento, 18 , 287 (1960).

CHAPTER XII

A Note on the Non-leptonic Decay Modes of the Strange Particles*

1. Introduction

Recently a great deal of interest has been shown in the application of the "pole" approximation of dispersion theory to problems in weak interactions following the work of Bernstein et. al.[1] On the Goldberger-Treiman formula for the rate of decay of the charged pion[2] in terms of the strong pion-nucleon coupling constant and the weak axial vector constant. The derivation of the formula (which is in excellent agreement with experiment) given by Goldberger and Treiman themselves were based on a number of dubious assumptions which made its success very surprising, but it was shown subsequently that the formula results if one assumes the dominance of the one-pion pole in the matrix element $\langle p\bar{n} | \partial_\alpha P_\alpha | 0 \rangle$ where P_α is the axial vector current and $|p\bar{n}\rangle$ and $|0\rangle$ denote the proton - anti-neutron and vacuum states. Shortly after, the pole approximation was applied by Feldman, Matthews and Salam to Σ- and Λ- non-leptonic

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* A.P.Balachandran and K. Venkatesan, Prog.Theor.Physics 26, 792(1961) ~~(in press)~~

1) J. Bernstein, S.Fubini, M.Gell-Mann and W.Thirring, Nuovo Cimento, 17 , 757 (1960). See also J.Bernstein, M.Gell-Mann and L.Michel, Nuovo Cimento, 16 , 560 (1960); M.Gell-Mann and M.Levy, Nuovo Cimento, 16 , 705, (1960); Y. Nambu, Phys. Rev.Letters, 4 ,380 (1960).

2) M.L.Goldberger and S.B.Treiman, Phys. Rev., 110 , 1478 (1958); See also M.L.Goldberger, Rev. Mod. Phys., 31, 797 (1959).

decays.[3] The non-leptonic decay modes of the Σ -hyperon possess a number of remarkable features. Thus the modes $\Sigma^+ \to n + \pi^+$ and $\Sigma^- \to n + \pi^-$ show no up-down asymmetry while the mode $\Sigma^+ \to p + \pi^0$ seems to have an asymmetry parameter very close to unity.[4] The relative life-times of Σ^+ and Σ^- indicate equality among all three rates to within 10 %.[5] These results are consistent with the $|\Delta \vec{I}| = \frac{1}{2}$ rule and shows that the Gell-Mann-Rosenfeld triangle[6] is an almost perfect $45^o - 45^o - 90^o$ one with one arm along the S -axis and the other along the p- axis. (One of course still does not know which of the modes $\Sigma^+ \to n + \pi^+$ and $\Sigma^- \to n + \pi^-$ has an S -wave amplitude and which a P -wave amplitude). Unless this is a most unusual accident arising from capricious renormalization effects, it is very suggestive of a deeper symmetry underlying strong and weak interactions. There are at present several schemes[7] attempting to explain this. The work of Feldman, Matthews and Salam[3] is also concerned with these matters and they show that one can arrive at the Σ -decay asymmetries with the assumption of global symmetry for pion couplings.[8] By choosing essentially one more

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3) G. Feldman, P.T.Matthews and A. Salam, Phys. Rev., 121, 302 (1961)

4) The Σ^+ -experiments are due to Cool et. al., Phys. Rev. 114,912 (1959) while the Σ^- -experiments are those of Franzini et. al., Bull. Am. Phys. Soc., 5 , 224 (1960).

5) Alvarez et. al., Proceedings of the International Conference on High Energy Physics, Kiev (1959).

6) M.Gell-Mann and A.H.Rosenfeld, Annual Review of Nuclear Science, 7 , 454 (1957).

7) F. Gursey, Nuovo Cimento, 16, 230 (1960); preprint, Institute for advanced Study (1960); A. Pais, Nuovo Cimento, 18 , 1003 (1960); UCRL - 9460 (1960).

8) M. Gell-Mann, Phys. Rev., 106 , 1296 (1957).

parameter, they are able to correlate the Λ- and Σ- decay times and asymmetries. (See also A. Pais, ref.[7]) . While this is most suggestive, very tentative and preliminary evidence seems to indicate an odd Σ-Λ parity.[9] If this is confirmed, it will invalidate their discussion and those of ref.[7] and a plausible explanation of these data would seem very much more difficult.

Because of the success of the pole approximation in explaining the life-times of the $\pi^{\pm}$ decays, it seems worth while to estimate the vertex parameters in other decays as well in such an approximation. Recently data has been presented by Fowler et. al.[10] on the magnitude of the asymmetry parameter as well as the life-time of the decay $\Xi^- \to \Lambda^0 + \pi^-$. We present below a calculation for this decay mode using the pole approximation following Feldman, Matthews and Salam[3]. Further another experimental result of Fowler et. al.[10], namely that the decay asymmetries in $\Xi^- \to \Lambda^0 + \pi^-$ and $\Lambda^0 \to p + \pi^-$ or $n + \pi^0$ have opposing signs, is used to suggest a mechanism which can be responsible for the Σ -decay asymmetries. Finally we note that dispersion theory can give finite answers for the mass differences of elementary particles in certain cases. We illustrate this for the $K_1^0 - K_2^0$ mass difference by assuming that it comes mainly from the one-pion pole diagram.

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9) Y. Nambu and J.J.Sakaurai, Phys. Rev. Letters, J.J.Sakurai, Nuovo Cimento, 20, 1212, (1961); R. Adair, Rev.Mod.Phys., 33, 406 (1961).

10) Fowler et. al., Phys. Rev. Letters, 6 , 134 (1961)

2. The Decay Modes

We shall first discuss the case of Ξ^- decaying into a Λ^0 and a π^-. The pole diagrams are shown in Fig. The diagram (b)

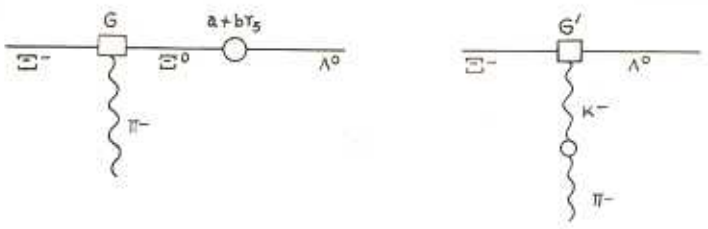

Fig1 Here and in figures 2 and 3, □ denotes the strong and ○ the weak vertex.

with the K -meson pole makes a comparatively small contribution for comparable values of the coupling constants and is neglected in order to reduce the number of unknowns. G denotes the renormalized $\Xi - \Xi - \pi$ coupling constant and $a + b\gamma_5$ denotes the $\Xi - \Lambda$ (weak) vertex. The asymmetry parameter α_Ξ and the life-time τ are calculated by usual methods. We have,

$$\alpha_\Xi = - \frac{2 P_\Lambda \, Re\,(BA^*)}{B^2 (E_\Lambda + m_\Lambda) + A^2 (E_\Lambda - m_\Lambda)} \qquad (1),$$

$$\frac{1}{\tau} = \frac{G^2 E_\pi P_\Lambda}{2\pi m_\Xi m_\pi} \left[\frac{1}{2 m_\Xi E_\pi - m_\pi^2} \right]^2 \left[B^2 (E_\Lambda + m_\Lambda) + A^2 (E_\Lambda - m_\Lambda) \right] \qquad (2).$$

Here P_Λ is the momentum of the outgoing Λ, E_Λ and E_π

are the energies of the Λ and π and

$$A = -(m_\Lambda + m_\Xi)\, a \,,$$

$$B = -(m_\Xi - m_\Lambda)\, b \qquad (3).$$

The masses of the particles are denoted by the symbol m with the particle symbol as the subscript. From the kinematics, one deduces that $E_\Lambda = 124.5$ MeV., $E_\Lambda = 1121$ MeV. and $E_\pi = 196$ MeV. The value of Fowler et. al.[10] for α_Ξ is $\mp$ 0.69 (The sign ambiguity of α_Ξ has not been resolved). Using this value, from equation (1) we deduce a quadratic equation for $\frac{a}{b}$ with the solutions $\frac{a}{b} = \mp$ 0.58 or $\mp$ 4.42. Using equation (2) and the value $\tau = 1.28 \times 10^{-10}$ secs. given by Fowler et. al.,[10] we arrive at the following values for the dimensionless parameter $\frac{G^2}{4\pi}\left(\frac{b}{m_\pi}\right)^2$:

$$\frac{G^2}{4\pi}\left(\frac{b}{m_\pi}\right)^2 = 0.24 \times 10^{-11} \text{ for } \left|\frac{a}{b}\right| = 0.58$$

and

$$\frac{G^2}{4\pi}\left(\frac{b}{m_\pi}\right)^2 = 0.31 \times 10^{-12} \text{ for } \left|\frac{a}{b}\right| = 4.42 \qquad (4).$$

As equation (2) shows, τ is independent of the sign of $\frac{a}{b}$. Experimental results also indicate that the asymmetry parameters α_Ξ and α_Λ in Ξ and Λ decays, have opposing signs.[10] However the sign of α_Λ is uncertain, different groups of workers obtaining contradictory results.[11] The sign of α_Ξ is therefore uncertain. Note that the

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11) Boldt et. al., Phys. Rev. Letters, 1 , 256 (1958); R.W.Birge and W.B.Fowler, Phys. Rev. Letters, 5 , 254 (1960). See however Leitner et al., Phys Rev. Letters 7, 264 (1961); Beall et al., Phys Rev Letters 7, 285 (1961)

values of these parameters for the mode $\Xi^0 \to \Lambda^0 + \pi^0$ follow from above if we use the $|\Delta \vec{I}| = \frac{1}{2}$ rule.

It is possible that the observed pattern of the Σ^- decay asymmetries arises due to a parity clash between diagrams involving Λ and those involving Ξ. Fig. (2) indicates the relevant diagrams.

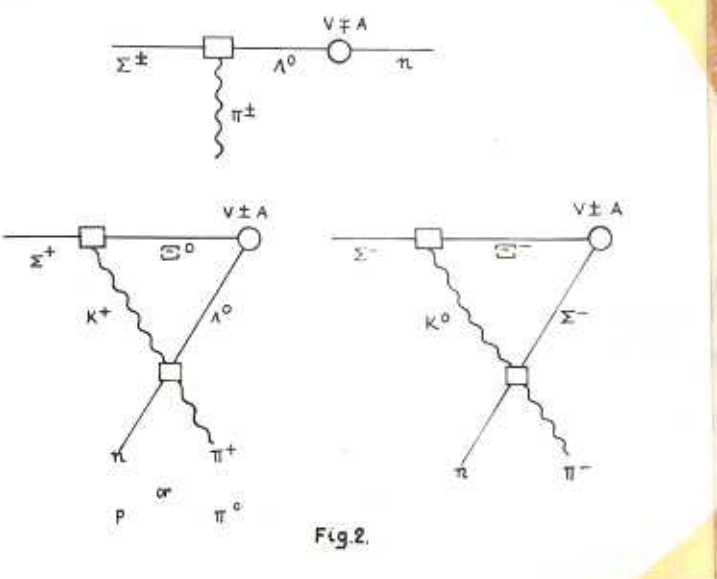

Fig.2.

For the Λ -intermediate state, the pole diagram has been drawn and is assumed to be dominant. For Ξ - intermediate states, no pole diagram exists and those next in order of simplicity are taken to be the important ones. It is clear that for suitable values of the parameters involved these three diagrams can yield parity-conserving $\Sigma^\pm \to n + \pi^\pm$

decays and parity-violating $\Sigma^+ \to p + \pi^0$ decay, the proton from the Σ^+ - decay and the Λ from Ξ - decay having helicities of the same sign. In Fig. 2 , we could have included the diagrams with the nucleon and K -meson poles in which case there will be an even an larger number of unknowns than in Fig. 2 to fit the few data and the required results can certainly be obtained. The validity of such approximations is very uncertain and would imply for instance that relations like $R(\Sigma^+ \to p + \pi^0) \approx 2\, R(\Lambda^0 \to p + \pi^-)$ for the corresponding decay rates [12] are in the nature of dynamical accidents.

We conclude by calculating the $K_1^0 - K_2^0$ mass difference in the pole approximation. This mass difference is related to twice the transition matrix element from K^0 to $\bar{K}^0$. This is shown as follows. In the $K_1^0 - K_2^0$ representation, the mass matrix is diagonal and is shown below:

$$\begin{array}{c} \\ K_1^0 \\ K_2^0 \end{array} \begin{array}{c} \begin{array}{cc} K_1^0 & K_2^0 \end{array} \\ \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \end{array} = \frac{m_1 + m_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{m_1 - m_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (5)$$

Here m_1 and m_2 are the masses of K_1^0 and K_2^0 . In the $K^0 - \bar{K}^0$ representation, this matrix becomes

$$\begin{array}{c} \\ K^0 \\ \bar{K}^0 \end{array} \begin{array}{c} \begin{array}{cc} K^0 & \bar{K}^0 \end{array} \\ \begin{bmatrix} \frac{m_1+m_2}{2} & 0 \\ 0 & \frac{m_1+m_2}{2} \end{bmatrix} \end{array} + \begin{array}{c} \\ K^0 \\ \bar{K}^0 \end{array} \begin{array}{c} \begin{array}{cc} K^0 & \bar{K}^0 \end{array} \\ \begin{bmatrix} 0 & \frac{m_1-m_2}{2} \\ \frac{m_1-m_2}{2} & 0 \end{bmatrix} \end{array} \qquad (6)$$

12) S.B.Treiman, Nuovo Cimento, 15, 916 (1960);
A. Pais, loc. cit.

which proves the assertion made earlier. The one-pion pole diagram giving rise to this mass difference is shown in Fig.3.

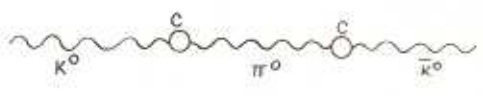

FIG.3

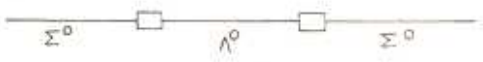

FIG.4. Here □ denotes a vertex which arises from a Combination of strong and electromagnetic interactions.

If c denotes the K-π vertex, the mass difference δm is given by $\frac{c^2}{m_K}\left(\frac{1}{m_K^2 - m_\pi^2}\right)$. For δm of the order of 10^{-5} eV.[13], the dimensionless quantity $\left(\frac{c}{m_\pi^2}\right)^2$ has the value 0.3×10^{-11}. A precise value of δm would allow us to determine c more accurately. Similar calculations with finite results can be performed for the self-mass of Σ^0 arising from the combined action of electromagnetic and strong interactions. The pole diagram involves Λ^0 and is shown in Fig.4. However since we have no idea as to how much of the Σ^0 mass is due to electromagnetism, there seems to be no way of estimating the strength of the $\Sigma^0\Lambda^0$ vertex.

13) Muller et. al., Phys. Rev. Letters, 4 , 418 (1960).

CHAPTER XIII.

ON THE COMPOUND MODELS FOR ELEMENTARY PARTICLES

1. Introduction

It has always seemed attractive to attempt at a reduction in the number of basic fields in elementary particle theory. The great number of heavy particles which are similar to each other in many respects appears very unwieldy so that one is tempted to assume that only a few of them are elementary and the rest compound. Among the earliest attempts in this direction was that of Fermi and Yang[1] who set up a model for the pion-nucleon system with the pion as the bound state of a nucleon and anti-nucleon. With the advent of strange particles, Sakata[2] proposed an extension of the Fermi-Yang model with the lambda hyperon and the nucleon as the two basic fields. The Sakata model has enjoyed considerable popularity especially in theories of weak interactions, since it explains in a natural way the conserved vector current hypothesis for strangeness-conserving decays like nuclear β -decays[3] and the $|\Delta \vec{I}| = \frac{1}{2}$ rule of the non-leptonic decays. It is also very suitable for the construction of

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1) E. Fermi and C.N.Yang, Phys. Rev., 76 , 1739 (1949)

2) S. Sakata, Prog. Theor. Phys. (Kyoto), 16 , 686 (1956)

3) S.S.Gershtein and J.B.Zeldovich, Sov. Phys. JETP, 2 , 576 (1957); R.P.Feynman and M. Gell-Mann, Phys. Rev., 109 , 193 (1958).

theories where vector and axial vector currents appear in a symmetrical way[4] and predicts the proposed $\frac{\Delta Q}{\Delta S} = 1$ and $|\Delta \vec{I}| = \frac{1}{2}$ rules for the strangeness non-conserving currents[5), 6)] in the leptonic decays of strange particles. Evidence for these latter two rules is however very scanty and any observation of events like the Σ^+ decaying into $n + e^+ + \nu$, K^0 decaying into a π^+ plus leptons or $\bar{K}^0$ into a π^- plus leptons will invalidate these rules and make the Sakata model very unplausible.

Another, but much less popular model is that of Goldhaber and Gyorgyi[7)]. Here the fundamental fields are the nucleon and K -meson. Compared to the Sakata model this is rather less attractive since it involves two very different types of objects like a baryon and a boson as its basic constituents.

There are a variety of other papers on compound models among which the work of Gursey[8)] and Thirring[9)] are of particular interest. These works are purely group-theoretical in character and investigate the number of basic fields necessary to give all the conserved quantities that we know of. Some of the elementary fields themselves can be unobservable. The basic unanswered problem in these models is to show that the number of asymptotic fields exceeds the number of local fields.

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4) M. Gell-Mann, Proceedings of the 1960 Annual International Conference at Rochester, page 740.
5) R.P.Feynman and M.Gell-Mann, loc. cit.
6) S.Okubo, R.E.Marshak, E.C.G.Sudarshan, W.B.Teutsch and S. Weinberg, Phys. Rev., 112 , 665 (1958).
7) S.Goldhaber, Phys. Rev., 101 , 433 (1956); G.Gyorgyi, Zh. eksper. teor. fiz., 32 , 152 (1957)
8) F. Gursey, Nuclear Physics, 8 , 675 (1958) .

and these smaller number of local fields already forms a complete set[9]. On these matters however, very little work has as yet been done.

With the discovery of many new resonances in the strong interactions of mesons and hyperons[10], it has become an urgent task to decide which of these new particles are new dynamical degrees of freedom (which in a Lagrangian approach will show themselves as new fields in the Lagrangian whose masses and quantum numbers are such that they are unstable) and which are the resonances ~~xixix~~ arising from the nature of the forces between the resonating particles. In a dispersion theoretic approach, this distinction shows itself not in the writing down of the equations (as in the Lagrangian theory), but in their solution, through the Castillejo, Dalitz, Dyson ambiguities[11]. Thus the zeroes of the scattering amplitude which these ambiguities imply seem to be associated with the existence of new degrees of freedom in the Lagrangian, i.e. new elementary particles which are however unstable[11]. The urgency of finding out which is an elementary particle and which a compound one is already evident in connection with the observed P -wave pion-pion resonance in the $I=1$ state[12]. Chew, Mandelstam and Noyes[13] have proposed that it arises out of a "bootstrap"

9) W. Thirring, Nuclear Physics, 10 , 97 (1959); 14, 565,(1959/60)
10) Alston et. al., Phys. Rev. Letters 5 , 520 (1960); Abashmian et. al., ibid 5 , 258 (1960); Erwin et. al., ibid, 6,528(1961); Ferro-Luzzi et. al., Bull.Am.Phys. Soc., 5 , 509 (1960). See also Rev. Mod. Phys., 33 (1961).
11) Castillejo, Dalitz and Dyson, Phys. Rev., 101, 453 (1956); N.G.Van Kampen, Physica, 23 , 157 (1957). See also S. Mandelstam, Phys. Rev., 112 , 1344 (1958).
12) Erwin et. al., loc. cit.
13) G.F.Chew, S.Mandelstam and H.P.Noyes, Phys. Rev., 119 , 478 (1960); see also F. Zachariasan, Phys. Rev. Letters, 7 , 112 (1961).

mechanism with the attractive force between the resonating pions being produced by the exchange of a resonating pion pair. On the other hand, such a particle appears as an elementary particle in the vector meson theories of Yang and Mills and Sakurai[14] with a universal coupling to the isotopic spin vector current. The problem involved is a fundamental one for which as yet no solution has been suggested.

With this introduction, we present in Section II, a compound model for Ξ^- decays and in Section III, some possible experimental tests to detect whether a particle is compound or not.

II. On a model for Ξ^- decays*

Recently in an attempt to explain the observed pattern of the Σ -decay asymmetries[15], Barshay and Schwartz[16] considered a model for the Σ -particle where it is treated as the bound state of a Λ and a π, the mechanism which induces Σ to decay being the decay of Λ. In view of the apparent success of this model, it becomes of interest to investigate an analogous model for the decay of the Ξ^- hyperon with the Ξ treated as the bound state of a Λ and a $\bar{K}$.[17] We shall discuss below the consequences of such an assumption on the decay characteristics

14) C.N.Yang and R.L.Mills, Phys. Rev., 96, 191 (1954); J.J.Sakurai, Ann. Phys., 11, 1 (1960).
*A.P.Balachandran and N.R.Ranganathan, Nuovo Cimento, 18, 1269 (1960).
15) Cool et. al., Phys. Rev., 114, 912 (1959); Franzini et.al., Bull. Am. Phys. Soc., 5, 224 (1960); Beall et al., Phys Rev Letters 1, 285 (1961).
16) S. Barshay and M. Schwartz, Phys. Rev. Letters, 4, 618 (1960).
17) N. Dallaporta, Nuovo Cimento, 7, 200 (1959); P.Budini, N. Dallaporta and L. Fonda, Nuovo Cimento, 9, 316 (1958).

of Ξ leaving aside the question of the strong interaction properties of Ξ implied by this model.

We first consider the case where the intrinsic parity of the $(\Lambda\bar{K})$ system relative to the Ξ- hyperon is even. In such a case, the bound state will be an S -state. We may further assume that the K-Λ relative parity is odd. Then the decay of the Ξ- particle will proceed by the Λ^0 within the Ξ decaying into a pion and a nucleon and the nucleon subsequently absorbing the $\bar{K}$ in a P -state relative to it. Let us denote by A and $\pm A$ the amplitudes for Λ^0 decaying into an S- and a P -wave pion-nucleon system respectively. The S- and P -wave amplitudes are chosen to be equal in magnitude since it is known that the decay asymmetry α_Λ of Λ is near maximum[18] while the relative sign of these amplitudes is left unspecified since evidence is conflicting as to the sign of α_Λ[18]. Further let B be the amplitude for the nucleon absorbing a $\bar{K}$ -meson via a P -wave interaction. The decay of Ξ can then be written as follows assuming that the Λ -decay satisfies the $|\Delta\vec{I}| = \frac{1}{2}$ rule for which there is good experimental evidence[19]:

$$\Xi^- = (\Lambda^0 K^-)_S \rightarrow A K_S^- \left[-\sqrt{\tfrac{2}{3}}\,(\pi^- p) + \sqrt{\tfrac{1}{3}}\,(\pi^0 n)\right]_S$$

$$\pm A K_S^- \left[-\sqrt{\tfrac{2}{3}}\,(\pi^- p) + \sqrt{\tfrac{1}{3}}\,(\pi^0 n)\right]_P$$

$$\rightarrow -\sqrt{\tfrac{1}{3}}\,AB\,(\Lambda^0\pi^-)_P \mp \sqrt{\tfrac{1}{3}}\,AB\,(\Lambda^0\pi^-)_S \qquad (1)$$

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18) Boldt et. al., Phys. Rev. Letters, 1, 256 (1958); R. Birge and W. Fowler, Phys. Rev. Letters, 5, 254 (1960). These two papers contradict each other in their results on the sign of α_Λ. See however Leitner et al., Phys Rev Letters 7, 264 (1961); Beall et al., loc cit.

19) These are summarized by M. Schwartz, Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester, page 726.

Here the subscript S or P on two-particle states denotes the relative angular momentum of the two particles while the subscripts S for K^- denotes that it is in an S -state relative to the centre-of-mass of the other two particles. The decay clearly satisfies the $|\Delta \vec{I}| = \frac{1}{2}$ rule since the basic weak interaction viz., the decay of the Λ, satisfies it while the strong interaction as usual has $|\Delta \vec{I}| = 0$. (1) indicates that the asymmetry parameter α_{Ξ} in Ξ^- decay is equal to α_Λ . (Notice that by the $|\Delta \vec{I}| = \frac{1}{2}$ rule, the asymmetry parameters for Ξ^- and Ξ^0 decays are equal). Repeating the calculation for the other possible combinations of the relative parities of the particles involved, one finds that this result is independent of these relative parities. Tentative experimerimental evidence are at present available indicating that α_{Ξ} and α_Λ have opposing signs.[20] If this is confirmed, the model will be of no further practical interest.

It is to be noted that the decay mechanism of Ξ cannot be via the decay of the $\overline{K}$ into two pions since the Λ cannot subsequently absorb a pion and remain a Λ because of isotopic spin conservation. Also the process $\pi + \pi \to \pi$

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20) Fowler et. al., Phys. Rev. Letters, 6 , 134 (1961).

is forbidden because of G -conjugation invariance.[21] However the process $\overline{K} \to \pi + \pi + \pi$ with $\pi + \pi + \pi \to \overline{\pi}$ is possible. This will modify the S -wave amplitude in Ξ -decay.

It may also be verified that the proposed $\frac{\Delta Q}{\Delta S} = 1$ rule for strangeness-changing currents[5] is implied by this model for Ξ - decays if all the weak interactions are to proceed only through four-fermion couplings. Thus the decay $\Xi^0 \to \Sigma^- + e^+ + \nu$ which violates this rule has to happen through the process $\Lambda^0 \to n + \pi^- + e^+ + \nu$ with a subsequent strong interaction uniting all the strongly interacting particles (including the $\overline{K}$) into a Σ^- . The Λ however cannot decay into $n + \pi^- + e^+ + \nu$ if only four-fermion couplings exist. Thus the Ξ^0 cannot decay into a Σ^- and leptons.

Application of these methods to models such as that of Goldhaber and Gyorgyi[7] where the fundamental elementary constituents are the nucleon and the K -meson shows that at any rate with these simple arguments it will not yield any asymmetry for the decay of, for instance, the Λ- hyperon since the basic decay mechanism is that of the K -meson into spinless pions. This can therefore be regarded as a very tentative indication that such a model may not be correct.

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21) T.D.Lee and C.N.Yang, Nuovo Cimento, 3 , 749 (1956).

III. The Tests for Compound Models*

a) As we have shown in Section II, in a composite particle model for a hyperon it is possible to predict the asymmetry parameter for its decay products if the decay characteristics of its constituent particles are known. For instance, as was pointed out, the Goldhaber-Gyorgi model[7] is not capable of explaining the decay features of the composite particle in a simple fashion since the basic weak interaction is the decay of K into pions.

b) In the effective range theory, there is a relation between the binding energy of the compound particle and the scattering length, effective range as well as the S -wave scattering phase shift of its constituent particles. This fact may be used to test the validity of a compound particle model. Thus for instance if $\Lambda = \frac{1}{\sqrt{2}}(pK^- - n\bar{K}^0)$ and the K -meson parity $P_{K\Lambda}$ is even so that the bound state is an S -state, the S- wave K^-p scattering parameters may be related to the binding energy E_g of the Λ where E_g = -318.7 MeV. The K^-p elastic scattering is almost entirely through S -waves at laboratory energies upto 100 MeV[22), 23)] and takes place through the two isotopic spin channels $I = 0$ and $I = 1$. If δ_I denotes the S -wave phase-shift in the isotopic spin channel I and A_I and R_I are the corresponding scattering length and effective range respectively, we have the usual formula

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* G. Bhamathi, S.Indumathi, A.P.Balachandran and N.G. Deshpande, Nuovo Cimento, 19 , 190 (1961).

22) R.H.Dalitz and S.F.Tuan, Ann. Phys. 8 , 100 (1959).

23) J.D.Jackson, D.G.Ravenhall and H.W.Wyld, Jr., Nuovo Cimento, 9 , 834 (1958).

$$k \cot \delta_I = \frac{1}{A_I} + \frac{1}{2} R_I k^2 + \cdots \qquad (2)$$

where δ_I, A_I and R_I are complex because of the presence of absorption channels with productionof hyperons. Following Jackson, Ravenhall and Wyld[23], we may neglect all but the first term in the expansion (2) below laboratory energies of about 100 MeV so that we may write

$$k \cot \delta_I = \frac{1}{a_I + i b_I} \qquad (3)$$

where $A_I = a_I + i b_I$. The binding energy E_g may be related to the real part of the $I = 0$, S-wave scattering phase shift δ_0 through the formula

$$k \cot [\mathrm{Re}\, \delta_0] = k_g \qquad (4a)$$

where

$$k_g = \sqrt{\frac{-2\mu E_g}{\hbar^2}} = 2.324 \times 10^{-13}\ \mathrm{cm}^{-1} \qquad (4b)$$

and μ is the reduced mass of the $K^- p$ system. Thus we have, to a good approximation

$$a_0 = \frac{1}{k_g} = 0.43 \text{ fermis} \qquad (5)$$

Dalitz and Tuan[22] obtain the value $a_0 = 0.20$ fermis for their $(a+)$ solutions and 1.88 fermis for their $(b+)$ solutions. The agreement between (5) and the Dalitz-Tuan solutions is not good.

A similar analysis for Σ considered as a $\bar{K}$ and an N bound in an S-state (i.e. we assume that the Σ-K relative parity is even) gives $a_1 \simeq 0.50$ fermis[7] which is to be compared with the Dalitz-Tuan $(a+)$ and $(b+)$

solutions of 1.62 fermis and 0.40 fermis respectively[22]. For the model of Σ as the bound S-state of a Λ and π (i.e. the Σ-Λ relative parity is void)[16], we obtain $a_1 \simeq$ 1.55 fermis while for the model of Ξ as the bound S-state of a Λ and a $\bar{K}$ (i.e. the Ξ-Λ-K coupling is scalar)[17], $a_{1/2} \simeq$ 0.44 fermis. In the lowe energy region, there is no absorption channel in the last two cases and the scattering phase shifts are real. Thus the elastic scattering cross-sections are given by

$$\sigma = \frac{4\pi}{k^2 + k_g^2} \qquad (6)$$

Experimental results on Λ-π and Λ-$\bar{K}$ scattering cross-sections with which these numbers are to be compared are not at present available.

c) By requring that charge conjugation C and parity P be good operations, it is possible to derive selection rules for $K^+ K^-$ and $K^0 \bar{K}^0$ annihilation into pions[22]. Similarly the application of P, C and the charge symmetry operations will give selection rules for $K^+ \bar{K}^0$ and $K^- K^0$ annihilation into pions. We sketch the method below.

Let η denoter the space-parity and ω the charge parity of the systems under consideration. The $K^+ K^-$, $K^0 \bar{K}^0$ and $\pi^+ \pi^-$ states are composed of a boson and its own anti-boson which is distinct from it and are consequently eigenstates of charge conjugation with $\omega = (-1)^l$ where l

denotes the relative orbital angular momentum of the two particles[24]. For the $2\pi^0$ and $3\pi^0$ systems, since π^0 even under C, $\omega = +1$. A similar argument shows that $\omega = (-1)^y$ for $(\pi^+\pi^-\pi^0)$ system where y is the relative orbital angular momentum of π^+ and π^-. The space parities of the $K\bar{K}$ and 2π systems are clearly $(-1)^l$. For the $3\pi^0$ system, let us denote by y the relative orbital angular momentum of two of the pions and by η the orbital angular momentum of the third pion with respect to the centre-of-mass of the other two. Further for the $(\pi^+\pi^-\pi^0)$ system, let η denote the angular momentum of π^0 with respect to the centre-of-mass of the π^+ and π^-. φ for the 3π system now follows: it is $(-1)^{y+\eta+1}$ where we have taken into account the fact that the pion is pseudoscalar. The following table shows ω and φ for the $K\bar{K}$ (K^+K^- or $K^0\bar{K}^0$) and pion systems where now the relative orbital angular momentum is denoted by x for the $K\bar{K}$ and by y for the 2π systems.

	$K\bar{K}$	$\pi^0\pi^0$	$\pi^+\pi^-$	$\pi^0\pi^0\pi^0$	$\pi^+\pi^-\pi^0$
φ	$(-1)^x$	$(-1)^y$	$(-1)^y$	$(-1)^{y+\eta+1}$	$(-1)^{y+\eta+1}$
ω	$(-1)^x$	1	$(-1)^y$	1	$(-1)^y$

Table I

24) C.N.Yang, Phys. Rev., 77, 242 (1950); L. Wolfestam and D.G.Ravenhall, Phys. Rev., 88, 279 (1953); L. Michel, Nuovo Cimento, 10, 319 (1953). See also P. Roman, Theory of Elementary Particles, North-Holland Publishing Company (1960).

Consider now $(K\bar{K})$ decaying into $2\pi^0$. From ω conservation, we ~~xh~~ see that X must be even, i.e. the initial state is $S_0, D_2 \cdots$ (our notation for angular momentum states of two particles is L_J where L is the relative and J the total angular momentum. We also use the S, P, D, F, ... notation for $L = 0, 1, 2, 3, \ldots$. For three particles we write $Y y_J$ where the significance of Y and y has been explained before, J is the total angular momentum and an S, P, D, F, ... notation is again used). From $\mathcal{G}$ conservation, we find that Y too must be even i.e. the final state is S_0, D_2, ... These states are allowed by the Pauli principle also according to which the $2\pi^0$ state must be even under the exchange of the two particles. Finally, by angular momentum conservation, the allowed transitions are $S_0 \to S_0$, $D_2 \to D_2$ etc. Proceeding in this fashion, we have the following table of selection rules for the $K\bar{K}$ annihilation into pions :

K^+K^- or $K^0\bar{K}^0$	$\pi^0\pi^0$	$\pi^+\pi^-$	$\pi^0\pi^0\pi^0$	$\pi^+\pi^-\pi^0$
S_0	S_0	S_0	—	—
P_1	—	P_1	—	$P_{P_1}, F_{f_1}, H_{h_1}, \cdots$
D_2	D_2	D_2	$D_{P_2}, D_{f_2}, G_{f_2}, G_{h_2}, \cdots$	$D_{P_2}, D_{f_2}, G_{f_2}, G_{h_2} \cdots$

Table II

The $K^+\bar{K}^0$ or K^0K^- systems are not eigen-states of $\mathcal{C}$ the use of which will not therefore provide selection rules in this case. Instead, let us first define the charge symmetry operator V such that, under it, we have

$$K^+ \underset{\leftarrow}{\rightarrow} K^0 \quad , \quad K^- \underset{\leftarrow}{\rightarrow} \bar{K}^0,$$

$$\pi^+ \underset{\leftarrow}{\rightarrow} \pi^- \quad , \quad \pi^0 \rightarrow -\pi^0 \qquad (7)$$

Under $\mathcal{C}V$, we therefore have[25)]

$$\pi^+ \rightarrow \pi^+, \quad \pi^- \rightarrow \pi^-, \quad \pi^0 \rightarrow -\pi^0,$$

$$K^+ \underset{\leftarrow}{\rightarrow} \bar{K}^0 \quad , \quad K^- \underset{\leftarrow}{\rightarrow} K^0 \qquad (8)$$

Thus the states $\pi^\pm\pi^0$, $\pi^+\pi^-\pi^\pm$ and $\pi^0\pi^0\pi^\pm$ are eigen-states of $\mathcal{C}V$ as also the states $K^+\bar{K}^0$ and K^0K^-. For the 2π-state in question, the eigenvalue ρ is clearly equal to -1 while for the 3π-state, it is $+1$. To find the eigen value for the $K^+\bar{K}^0$ state, let us write the state vector in the form

$$\Phi = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\, a^{K^+\dagger}_{\vec{k}}\, b^{K^0\dagger}_{-\vec{k}}\, \Phi_0 \qquad (9)$$

Here Φ_0 is the vacuum state, $a^{K^+\dagger}_{\vec{k}}$ and $b^{K^0\dagger}_{-\vec{k}}$ are creation operators for K^+ and $\bar{K}^0$ of momenta $\vec{k}$ and $-\vec{k}$ respectively and $\vec{r}$ is the relative coordinate of K^+ and $\bar{K}^0$. Φ is written in the centre-of-mass system. Operating by $\mathcal{C}V$, we have

$$\mathcal{C}V\,\Phi = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\, b^{K^0\dagger}_{\vec{k}}\, a^{K^+\dagger}_{-\vec{k}}\, \Phi_0$$

$$= \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}}\, a^{K^+\dagger}_{\vec{k}}\, b^{K^0\dagger}_{-\vec{k}}\, \Phi_0 \qquad (10)$$

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25) P. Roman, loc. cit.

since $a^{+}_{\vec{k}}$ and $b^{+}_{\vec{k}}$ commute. (5.10) shows that the effect of $\mathcal{C}V$ on this system is equivalent to exchanging the coordinates of the particles. Thus a state of angular momentum x of $K^{+}\bar{K}^{0}$ or $K^{0}K^{-}$ has an eigen value $\rho = (-1)^{x}$. The following table summarizes all these results where the notation for angular momenta is as in Tables I and II. $\mathcal{S}$ is again the spatial parity.

	$K^{+}\bar{K}^{0}$ or $K^{0}K^{-}$	$\pi^{\pm}-\pi^{0}$	$\pi^{+}\pi^{-}\pi^{\pm}$ or $\pi^{0}\pi^{0}\pi^{\pm}$
$\mathcal{S}$	$(-1)^{x}$	$(-1)^{y}$	$(-1)^{y+y+1}$
ρ	$(-1)^{x}$	-1	1

Table III.

In the table, the states with the plus signs on π correspond to an initial state of $K^{+}-\bar{K}^{0}$ and those with the minus signs to an initial state of $K^{0}-K^{-}$. The table shows that the 2π mode occurs only when x is odd and the 3π mode when x is even. The selection rules implied by Table III are given in the following table.

$K^+ - \bar{K}^0$ or $K^- - K^0$	$\pi^\pm - \pi^0$	3π
S_0	–	–
P_1	P_1	–
D_2	–	P_{P_2} , D_{P_2} , ...
F_3	F_3	–

Table IV.

Notice that S_0 cannot decay into 3π for the following reason. $X = 0$ implies that $y+y+1$ is even by G conservation or $y+y$ is odd. This implies however that the total angular momentum of the 3π can never be zero so that J -conservation forbids S_0 decaying into 3π .

These selection rules may prove useful in testing some of the compound models. Thus, if for instance, information can be obtained on the K^+ or K^0 capture by Λ's and it is found that in the low energy region, uncharged states or states with one charged pion are absent in the final state, the result may be taken as an indication of the validity of the Goldhaber-Gyorgyi model. We assume

here that the mechanism of reaction is the capture of K by the $\overline{K}$ in Λ and neglect the distortion in the $\overline{K}$ -wave function due to its binding as a first approximation. Analogous results for Ξ will indicate that Ξ may be considered as the bound state of a Λ and a $\overline{K}$. The angular distribution of the pions in the final state in such reactions may also serve as a useful means to discriminate between such models. Similar arguments can be advanced for the annihilation of antiberyons with baryons with pions or K -mesons in the final state. Such considerations, while not conclusive, can be very suggestive in finding out the compound nature of a particle. Unfortunately, however the type of data required for such analyses seem very difficult to obtain.

PART III

CAUSALITY IN DETERMINISTIC, STOCHASTIC AND QUANTUM MECHANICAL PROCESSES

CHAPTER XIV

CAUSALITY IN DETERMINISTIC, STOCHASTIC AND QUANTUM MECHANICAL PROCESSES*

1. Introduction

It is almost axiomatic that quantum mechanics is based upon the twin principles of complementarity and superposition though by a suitable interpretation, one can be understood to imply the other. It is in the inclusion of these principles that quantum mechanics marks a departure from the classical picture of dynamical systems. The principle of relativity however is commonly applicable to both classical and quantum mechanical systems and is characterised by two important consequences:

1) The symmetry between space and time, and

2) The quadratic nature of the relation between energy and momentum, resulting in the introduction of negative energy states.

It is worthwhile distinguishing between these two since the latter is of no significance in classical theory but plays a fundamental role in quantum mechanics. The inclusion of negative energy states in the quantum mechanical description with suitable reinterpretation leads to a multiparticle formalism

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* Alladi Ramakrishnan and A.P.Balachandran (to be published)

necessitating a field-theoretic description. But there is something more implied than multiplicity since the postulate is made that the negative energy states "travel back in time".

Thus if the concept of causality is closely tied with time, its role is more complicated in quantum mechanical processes than in classical theory due to the inclusion both of negative energy states and of the principle of complementarity.

The principle of causality in any physical phenomenon is usually stated thus:

Two events A and B occurring at space-time points x_1 and x_2 cannot be causally connected unless the interval $x_1 - x_2$ is time-like. Actually this definition is only residuary since what it defines is 'acausality'. The statement implies the trivial concept that in a frame of reference where the spatial coordinates of the two events are identical cause and effect should be separated by time. But the logical question remains - whether all events separated in time are causally connected and whether sequence implies consequence. To study these logical implications, the meaning of an "event" itself has to be compared in the classical and quantum mechanical descriptions. Even classical processes admit of a classification into deterministic and stochastic systems and it is only through the examination of stochastic systems that we may be able to understand the causal connection between various events in quantum mechanical processes. The point of view has often been emphasised that the evolutionary nature of the quantum mechanical and stochastic processes differ in

that we deal with positive definite amplitudes in the latter case and complex amplitudes in the former[1]. As long as this distinction is remembered, the analogy is found useful and the concepts of events in stochastic and quantum mechanical processes can be put into close correspondence[2].

We shall take up a systematic and comparative study of causality in deterministic, stochastic and quantum mechanical processes in this chapter. Such a study is considered important by us in view of the rather paradoxical situation that has arisen in collision theory in modern physics today. The early papers on the dispersion theoretic approach emphasised the causality principle as the cornerstone of scattering phenomena[3]. But very recently the extreme view has been taken that the matrix element of a scattering process is just a function with definite analytic properties postulated through the principle of "lack of sufficient reason" or "the principle of maximum smoothness"[4] rather than through the evolutionary nature of the process from the infinite past to the infinite future.

The fundamental question therefore is :

Are these two views so distinct as they are asserted to be or do they just emphasise different facets of the same structure of a collision phenomenon ?

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1) R.P.Feynman, Proc. Berkeley Symposium on Statistics and Probability (1957)
2) A.Ramakrishnan and N.R.Ranganathan, J.Math. Analysis and Applications, 3 , 261 (1961)
3) See for example N.N.Bogoliubov and D.V.Shirkov, Theory of Quantized Fields (Interscience, 1959) p. 557.
4) G.F.Chew and S.C.Frautschi, Phys. Rev. Letters, 7, 394 (1961)

We shall first introduce the concept of causality in a stochastic process and consider a deterministic process as a particular limiting case of stochastic evolution. We will then replace positive definite amplitudes by complex quantities and study collision phenomena in a quantum mechanical process.

II. Probability Distributions and Conditional Probabilities

The concepts of independence, dependence and correlation are as old as probability theory itself. But it is curious that the concept of causality, though of fundamental importance, has escaped attention of both mathematicians and statisticians. This is perhaps due to the fact that the study of the dynamical characteristics of probabilistic systems, that is, stochastic processes are of comparatively recent origin. In extending the concept of events to evolutionary phenomena, that is those which unfold with respect to a one-dimensional parameter (say time), we shall first show that while the concept of correlation is easily introduced for dynamical processes by just attributing to the events under consideration, different values of the parameter t it is inadequate to define the notion of causality. We are impelled by the conviction that the principle of causality is primitive while dependence and correlation are only concepts derivative therefrom.

In probability theory, two events A and B are considered independent if the joint probability $P(A,B)$ for the occurrence of A and B is equal to $P(A)\,P(B)$

where $P(A)$ and $P(B)$ are the probabilities for the occurrence of A and B respectively irrespective of the occurrence of the other. But the important question is, what is the causal relationship between A and B ? We shall first show that a simple-minded definition satisfying our "intuition" leads not only to difficulties, but to contradictions.

Let us try to express mathematically the statement that the event B is caused by the event A . Could we assume then that the probability $P(B|A)$ for B given $A = 1$? This of course means that if A occurs , B will occur, but does not imply that $P(B|\text{Non } A) = 0$ where Non A is the event complementary to A . In other words, besides A , there may be other events which may cause B though it is certain that A causes B . In a similar manner, there is the other question whether given B , A is the event which has caused it. $P(A|B)$ is not determined by the information that $P(B|A) = 1$, a statement which is just equivalent to there being causes other than A for the occurrence of B . In fact, the question of conditional probabilities and their logical structure has worried probabilists interested in establishing correspondence between theory and phenomena though from the point of view of set and measure theory, there is no difficulty as regards its definition

This is because the meaning of mutually exclusive events is just postulated in measure theory while why such exclusiveness results is not the concern of the mathematician Correlation exists between two events if these events can be interpreted as composed of more elementary events which are not mutually exclusive. The notion of causality gives some "life and substance" to this "non-exclusiveness" or "commonness" of some events which comprise two correlated events.

For this purpose, we find it convenient to attribute the value of a one-dimensional parameter t to any event which in the language of the probabilists merely implies that we are considering an evolutionary stochastic process and the events attributed to any point t are represented as the values of a random variable $X(t)$ or more generally of an aggregate $\{X(t)\}$. Thus in considering the causal connection between A and B, we assume that A belongs to $X(t_1)$ and B to $X(t_2)$. Since t can be varied, this brings in its wake two notions which are of great importance to the physicist, but which may not be significant from the point of miew of measure theory.

i) An event A can be realised at any time t though for measure theory, A at t_1 is a different event from A at t_2. To realise the significance of this difference in altitude, let us consider the energy of a particle as a random variable when it is passing through a thickness of matter. If by A, we imply the existence of a particle

in the energy state E, this state can occur not only at any time t but can continue to do so over an interval. Thus to ask the question whether B is caused by A is incomplete and perhaps even destitute of meaning. The correct question would be whether the event B at t_2 is caused by the event A at t_1. Since this definition should be valid for all values of t_1 and t_2, it should be so even for t_2 tending to t_1. It is well-known in stochastic theory that the limiting form of $P(B|A; t_2, t_1)$, the probability that B occurs at t_2 given that A occurred at t_1 as $t_2 \to t_1$ has completely different mathematical properties from the case when $t_2 - t_1$ is finite. This is because $P(B|A; t_2, t_1) \to R(B|A)\Delta$ if $B \neq A$ and $P(A|A; t_2, t_1) = 1 - \Delta \sum_B R(B|A)$ as $t_2 - t_1 \to \Delta \to 0$. In fact, to determine $P(B|A; t_2, t_1)$ for finite $t_2 - t_1$, we must assume $R(B|A)$. The solution for $P(B|A; t_2, t_1)$ is formally obtained as an integral over the interval $t_2 - t_1$ of a function which is composed only of R's and τ's where the τ's are parametric values between the interval $t_2 - t_1$. A close examination of this gives us the necessary lead to introduce the notion of causality. Since the R function is fundamental, so is $R(B|A)\Delta$ and instead of interpreting $R(B|A)\Delta$ as the occurrence of B at $t + \Delta$ given that A has occurred at t, we shall assume that the event representing the transition from A to B has occurred in the interval Δ. Hereafter, we shall use the following nomenclature. Occurrence will always be attributed

to infinitesimal intervals Δ . Events attributed to a particular time point t , we shall call either "outcome" or "input" depending on whether we are considering it with reference to another event attributed to a time point earlier or later respectively. From the point of view of measure theory, this distinction is not important since even outcomes and inputs are events to which probability magnitudes are assigned. But to facilitate understanding, we shall stick to this notation to interpret causality. The outcome at any point t is thus the cumulative effect of the events occurring in the interval Δ as Δ varies from $-\infty$ to t . The events occurring in Δ we shall call the α events. Therefore the correlation between two outcomes is to be sought in the commonness of the α -events which comprise the two outcomes.

ii) It is well-known that the P function satisfies a differential equation with respect to t which in turn implies that the solution of P in terms of R is a sum of iterated integrals over τ's where the τ's have to be ordered. The integrand will therefore represent the ordered connection between the α -events occurring in the various infinitesimal intervals Δ . The connection between them may not be simple and it may be necessary to decompose the α- events in each interval Δ into more elementary Γ -events such that the Γ -events in various infinitesimal intervals can be simply connected. The simplicity gained however implies paying a heavy price in that the Γ -events belonging

to the same α may be connected, albeit simply, to Γ-events in different intervals. Hence in expressing the outcome as iterated integrals over the functions representing the connection between the Γ -events, the ordering of the time parameter may be lost.

Thus the choice between viewing the outcome as the cumulative effect of α- events or of Γ -events is a matter of taste and most often depends on the nature of the problem. When the emphasis is on causality or correlation and when the number of events comprising the outcome is small, the Γ- or the causative aproach is useful. Otherwise, the α- or the conventional approach is used.

III. The Postulate of Causal Connection

The starting point now is the postulate of a causal connection between α- events occurring in the various intervals. We shall not attempt to give a generalization to include all types of physical processes, but postulate a causal relation for a certain class which will satisfy our intuitive notions of causality. We shall assume that the α- events occur in the interval dt according to the following law :

a) The probability of one α- event occurring in dt is $O(dt)$. The probability of n α- events occurring in dt is $O(dt^n)$. In addition, we now postulate that an α- event itself can be treated to be the simultaneous occurrence of

Γ events which fall into two classes Γ^C and Γ^R the prescription for this will be given presently. The division to be Γ events therefore obey a law of occurrence different from that of the α since the simultaneous occurrence of more than one Γ in dt will be proportional to dt if such an occurrence describes an α.

b) We now make our fundamental postulate that a event in dt_1 causes a Γ^R event in dt_2 with probability $q(t_2, t_1)\, dt_2$ or if the process is homogeneous in time, with probability $q(t_2 - t_1)\, dt_2$. We shall call Γ^C a causative event and Γ^R a resultant event. We then state that alpha events occurring in the intervals $dt_1, dt_2, \ldots, dt_n$ are causally connected if for any two intervals dt_j, dt_k there exists a Γ^R in dt_j and a Γ^C in dt_k when $t_k < t_j$. This however does not mean that for every Γ^C in dt_k, there should be a Γ^R in dt_j; to understand this, we shall go back to the description of the system at a particular time. Any state at t is arrived at by assuming that α- events have occurred in the intervals $dt_1, dt_2, \ldots, dt_n$ and summing over n and all possible ranges of the dt_i. Since α's are composed of Γ's, the outcome at t can now be thought of as composed only of the Γ's which occur in the various intervals. In any stochastic problem, we are concerned with the probability $P(B|A; t_2, t_1)$ for the outcome B at t_2 given that A is the input at t_1 zero. It is clear that B can be composed of Γ^C and Γ^R events while A must be deemed to be composed only of Γ^C.

We first make the following <u>two</u> postulates :

1) A resultant event Γ^R is produced in an infinitesimal interval Δ by a causative event Γ^C which occurred in dt_g, i.e. at a point separated by an interval τ from Γ^R with a probability $\varphi(\tau)\Delta$. If the process is evolving in the direction t , then φ is defined only for $\tau > 0$. If on the other hand, the process is progressing with respect to $-t$ (this may be unrealistic if t is time, but we can find processes where t is a one-dimensional spatial parameter and it is meaningful to talk of processes in both directions of t) then φ is defined for $\tau < 0$.

2) A causative event Γ^C occurs in an interval Δ given that its resultant has occurred at a point separated by an interval τ from it, with probability $-\varphi(\tau)\Delta$ for $\tau < 0$ if the process is progressing with respect to $-t$ and for $\tau > 0$ if the process is progressing with respect to $-t$. That this probability is negative does not in any way contradict the fundamental postulates of probability theory since this magnitude is used only in an auxiliary manner inside the integrands whereas the probabilities of the outcomes are obtained as positive definite quantities after integration is performed over τ . There is a considerable discussion about this in an earlier paper by one of us on inverse probability. We recall here the result since the negative nature of the amplitude assigned to the causative event may seem surprising at first sight.

If we are concerned with the probability $P(x;t)$ that a random variable assumes the value x at t, it is well-known that P satisfies the integral equation

$$P(x;t) = \int_{x'} P(x';t')\, P(x \mid x';t,t') \quad , \quad t > t' \qquad (1)$$

This is just the Chapman-Kolmogoroff equation of consistency. It was shown by one of us [5] that this equation is valid even for $t < t'$, but in such a case $P(x \mid x';t,t')$ will have no probability significance and is only a functional operator which we shall denote by $\hat{P}$. This of course leads to the result that if we insist for positive definite solutions of the domain of $\tau = t - t'$ lies between $-\tau_0$ and $+\infty$ where $\tau_0 > 0$.

In our problem we notice that the function φ plays the same role as $\hat{P}$ in the above discussion.

Now we outline the procedure for obtaining the probability amplitude for an outcome A to occur at t given that B has occurred at 0, as a sum of iterated integrals over products of φ. We shall assume that the initial event B is composed only of causative events Γ^c. Our task is merely to go from causes to resultants and resultants to causes attaching the appropriate amplitude φ or $-\varphi$ as the case may be. The final event A which is the outcome consists of both resultants and causative events which have no corresponding

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5) Alladi Ramakrishnan, Proc. Ind. Acad. Sc., XLI, 145 (1955)

resultants. A method of studying all the causally connected events starting from an initial Γ^C belonging to B would be as follows. We first find out the resultant event Γ^R corresponding to Γ^C. This event Γ^R will be a part of an α-event which will contain both $\Gamma^{R'}\Delta$ and $\Gamma^{C'}\Delta$. Any one of the Γ^C in Δ will in turn give rise to another event Γ^R belonging to an α at a later interval. On the other hand, for an event Γ^R in Δ, we trace back to a Γ^C associated with it to an earlier interval. In this fashion, we can locate all the causally connected events evolving from the time $t=0$ and establish the connection which determines a typical pattern. Summing over all the possible patterns, we then ~~xxxx~~ arrive at an expression for the probability for the transition of the system from the initial to the final state.

Thus the essential departure ~~for~~ from the conventional mode of viewing the evolution of a stochastic process consists in the dichotomous division of events occurring in the infinitesimal intervals into causative and resultant, the former we follow up to their resultants at later times and the latter we trace back to the causative at earlier times. We shall illustrate this with respect to very simple stochastic processes and shall discuss it in detail in the quantum mechanical case.

IV. Application to Simple Stochastic Processes.

1) The Poisson Process:

We first take the simplest of all stochastic processes, the Poisson. From the first point of view, we study the probability $\Pi(n,t)$ that n events occur in time t. Π obeys the equation

$$\frac{\partial \Pi(n,t)}{\partial t} = \left[\Pi(n-1,t) - \Pi(n,t)\right]\lambda \qquad (2)$$

where λdt is the probability that an event occurs in dt. From the first point of view, this event is an α- event and the α -event is the transition $(n-1)$ to n as n takes all possible values. In this case, the α -event occurs and the Γ- event are identical. The $\varphi(\tau)$ corresponding to the Γ -events here is to be identified with $e^{-\lambda\tau}\lambda$ since $e^{-\lambda\tau}\lambda\, dt$ is the probability that an event happens in dt given that the previous event happened at an interval τ before. It is important to emphasize that though events are supposed to be independent in the Poisson case, we have causally connected two successive events by the function $\varphi(\tau)$. That is, the fact that they are successive amounts in a sense to causation. Therefore in this point of view, $\Pi(n,t)$ is written as

$$\Pi(n,t) = \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1\, \varphi(\tau_1)\varphi(\tau_2-\tau_1)\cdots\varphi(\tau_n-\tau_{n-1})\, e^{-\lambda(t-\tau_n)} \qquad (3)$$

where the last factor represents the factthat the causative events are counted in the outcome, but do not produce resultants.

2) The Furry Process :

If there are n individuals at a time t, the probability that a new individual is born is $n\lambda\,dt$. This is the probability for the α- event. The equation obeyed by $\Pi(n,t)$ is therefore

$$\frac{\partial \Pi(n,t)}{\partial t} = \left[(n-1)\,\Pi(n-1,t) - n\,\Pi(n,t)\right]\lambda \qquad (4)$$

The probability however of the resultant Γ^R event which is the birth in dt relating to a particular parent born at a time τ before is defined by the fundamental function $e^{-\lambda\tau}\lambda$. When a particular parent produces an "offspring", both these are causative with respect to future resultants and we have to follow both of them. Thus, $\Pi(n,t)$ can be written as an integral assuming the α- events to occur in the intervals $d\tau_1$, $d\tau_2$, $\cdots$, $d\tau_n$.

$$\Pi(n,t) = \int_{\tau} \prod_{j,k} \varphi(\tau_j - \tau_k)\, d\tau_1\, d\tau_2 \cdots d\tau_n \qquad (5)$$

with the prescription that j and k must run over all the indices with $\tau_j > \tau_k$, the integration is performed over the range 0 to t.

However in the conventional approach, this may be written

$$\Pi(n,t) = \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1\, e^{-\lambda\tau_1}\lambda\, e^{-2\lambda(\tau_2-\tau_1)}\,2\lambda \cdots e^{-(n+1)\lambda(t-\tau_n)} \qquad (6)$$

The difference between the two approaches lies in the non-occurrence and occurrence of n respectively.

It becomes even more striking if the α- events are split up into Γ^C and Γ^R in which case the integrals in terms of τ will not be ordered with respect to the τ -parameter, but will be so in the α -approach.

We shall not however 'devise' processes* in classical stochastic theory to illustrate it since quantum electrodynamics is by far the best example of such a mathematical structure. We shall therefore deal with the electrodynamic problem directly.

Before doing so, we shall just refer to the deterministic processes. They can in principle be considered as the limiting cases of stochastic processes by introducing δ -functions. This has the corresponding mathematical consequence that the Γ- events which are causally connected relate to 'adjacent' infinitesimal intervals. This is precisely what is meant when we say a function satisfies only a differential equation and not an integro differential equation as in stochastic theory where the integration is over the values of the random variables.

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* A model process for example can be as follows:
We postulate the probability of spontaneous creation of a pair of individuals to be $\lambda\, dt$ and the probability for the annihilation of two when there are n individuals to be $\frac{n(n-1)}{2}\,\mu\, dt$. This is a departure from the birth and death process considered by Kendall and others since as regards 'death', the particles do not behave independently. In the conventional approach $\Pi(n,t)$ obeys the equation

$$\frac{\partial \Pi(n,t)}{\partial t} \left[\Pi(n-2,t)-\Pi(n,t)\right]\lambda + \left[\frac{(n+2)(n+1)}{2}\Pi(n+2,t) - \frac{n(n-1)}{2}\Pi(n,t)\right]\mu$$

V. Application to Quantum Mechanics

It has been stressed in a series of papers [1) 6)] that many of the concepts of stochastic theory can be carried over to quantum mechanics provided we remember that we have to work with complex amplitudes instead of positive definite quantities and the probabilistic interpretation can be applied to the squares of a certain class of complex amplitudes. Here we shall accept this and consider any electrodynamic collision process as one evolving from the infinite past $t = -\infty$ to the infinite future $t = +\infty$, the final state being the result of events happening at n space-time points and none elsewhere, n taking all possible values from 0 to ∞ and the amplitude being integrated over all space and time. In the scattering of a_r electrons by a_λ photon, the events that occur at any space time point are :

1) Scattering of an electron with emission or absorption of a photon,

2) Scattering of a positron with emission or absorption of a photon,

3) Pair creation with emission or absorption of a photon,

4) Pair annihilation with emission or absorption of a photon.

These ~~xxxx~~ are α -events from our point of view and the amplitude for their occurrence is $H'(t)\,dt$ where $H' = e\bar{\psi}\gamma_\mu\psi A_\mu$, ψ and A_μ being the field variables of the electron and photon. But from the point of view of causality, annihilation should be treated as the resultant and creation as causative. Thus when

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1) 6) A. Ramakrishnan, and N.R.Ranganathan, loc. cit., A. Ramakrishnan, T.K.Radha and R.Thunga, J.Math. Analysis and Applications (in press).

we pursue an electron which destroys itself with a positron, the destruction of the positron is a Γ -event which must be traced back to the point where it was born. The minus sign which we have prescribed is just the one which occurs in the Feynman propagator for negative time and negative energies. The equivalence between the Feynman and field theoretic points of view was considered quite exciting at the time when it was first demonstrated and is now accepted so much as an established fact that it has become almost stale to discuss the details of this equivalence. However in comparing Feynman diagrams with their corresponding field theoretic analogues, a combinatorial (or topological) problem has arisen the solution of which may have some bearing on the intrinsic meaning of causality. To simplify our discussion, we shall ignore the emission and absorption of photons and confine ourselves to the electron events.

The combinatorial problem in question may be formulated as follows. It has been shown by one of us [6] that one can split the Feynman propagator so that one part of it refers to the electron travelling forward in time and the other to the electron travelling backward in time. Thus in this picture, the relative time ordering of any two neighbouring vertices becomes important. We shall call such a diagram a "pattern". Clearly, for a Feynman diagram with n vertices, there are λ^{n-1} patterns. However in the field theoretic picture, the relative time ordering of any two vertices (neighbouring or not) is significant. We shall call these field theory diagrams Γ' diagrams. In this case, for an

n th order Feynman diagram, there are $n!$ FT diagrams. Now one may pose the following question : What are the number of FT diagrams corresponding to any one particular pattern ?

To solve this problem we notice that any one particular pattern consisting of $\alpha_1 + \alpha_2 + \cdots + \alpha_n = m$ vertices is uniquely specified by the relative sequence of its neighbouring vertices. Let us label the vertices as $1, 2, \ldots, m$. Then the pattern is completely characterised by the equations

$$1 < 2 < \cdots < \alpha_1,$$

$$\alpha_1 > \alpha_1 + 1 > \alpha_1 + 2 > \cdots > \alpha_1 + \alpha_2,$$

$$\vdots$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} > \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + 1 > \cdots > \alpha_1 + \alpha_2 + \cdots + \alpha_n \quad (7)$$

where the $>$ symbol denotes the fact that the vertex (event) on the left is in the future of the vertex (event) on the right while the $<$ symbol denotes the fact that it is in the past of the event on the right. Now the number of ways of arranging the $(\alpha_1 + 1)^{th}$ vertex such that in each of these arrangements, it is in a different time ordering with respect to any one of the previous vertices is $\sum_{k=0}^{\alpha_1 - 1} P_k$ where $P_k = 1$ for all k. Hence the number of FT diagrams for the sequence

$$1 < 2 < \cdots < \alpha_1,$$

$$\alpha_1 > \alpha_1 + 1 > \cdots > \alpha_1 + \alpha_2 \quad (8)$$

is

$$N(\alpha_1, \alpha_2) = \sum_{R_1=0}^{\alpha_1 - 1} \sum_{R_2=0}^{R_1} \cdots \sum_{R_{\alpha_2}=0}^{R_{\alpha_2 - 1}} P_{R_1} P_{R_2} \cdots P_{R_{\alpha_2}} \quad (9)$$

We immediately see that the number of FT diagrams for the pattern given by

$$1 < 2 < \cdots < \alpha_1 ,$$
$$\alpha_1 > \alpha_1 + 1 > \cdots > \alpha_1 + \alpha_2 ,$$
$$\alpha_1 + \alpha_2 > \alpha_1 + \alpha_2 + 1 > \cdots > \alpha_1 + \alpha_2 + \alpha_3 \qquad (10)$$

is

$$N(\alpha_1, \alpha_2, \alpha_3) = \sum_{\beta=1}^{\alpha_1} N(\beta, \alpha_2)\, N(\alpha_2 + \beta, \alpha_3) \qquad (11)$$

Generally for the pattern characterized by

$$1 < 2 < \cdots < \alpha_1 ,$$
$$\alpha_1 > \alpha_1 + 1 > \cdots > \alpha_1 + \alpha_2 ,$$
$$\vdots$$
$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} > \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + 1 > \cdots > \alpha_1 + \alpha_2 + \cdots + \alpha_n \qquad (12)$$

the number of FT diagrams are

$$N(\alpha_1, \alpha_2, \cdots, \alpha_n) = \sum_{\beta_{n-2}=1}^{\alpha_{n-2}} \sum_{\beta_{n-1}=1}^{\alpha_{n-1}} \cdots \sum_{\beta_1=1}^{\alpha_1} N(\beta_1, \beta_2)\, N(\beta_1 + \beta_2, \beta_3) \cdots$$
$$\times N(\beta_1 + \beta_2 + \cdots + \beta_{n-2} + \alpha_{n-1}, \alpha_n) \qquad (13)$$

We have thus given a unique prescription for obtaining the number of field theoretic diagrams corresponding to a particular pattern. We now wish to emphasize that it is the patterns that bring out the causal connection between the various events occurring along the time axis. The α_1- events are followed from the causative to the resultant while the α_2 - events are traced back from the resultant to the causative.

The situation becomes obviously more complicated when we have more than one particle in the initial and final states and we also include emission and absorption of bosons at each vertex. In fact, this leads to the problem of defining sequences of intermediate states from the causative point of view and the concept of Feynman paths has to be generalized when we have more than one particle in the initial and final states.

With the first flush of success of the theory of dispersion relations when the details of the interaction were considered unimportant, the Feynman diagrams were considered as useful but not essential auxiliaries. However it is well-known now that great difficulties are encountered in the rigorous derivation of the analytic properties of matrix elements from the general axioms of quantum field theory and attention has been revived in the corresponding problem in perturbation theory. Also the topological aspects of the internal and external lines of Feyhman graphs are receiving considerable attention. This in the opinion of the authors is closely connected with the causal description of events in space and time.

APPENDIX

ON A POSSIBLE Ξ - π RESONANCE*

Recently, anumber of papers[1)-4)] have appeared suggesting that strong interactions approximately satisfy the symmetry of the unitary group in three-dimensions. These suggestions have been particularly successful in predicting the number of meson states and in some instances, also their masses. For instance, Gell-Mann[2)] has derived the following approximate relation between the masses of the members of a unitary octet:

$$\frac{M_2 + M_2'}{2} = \frac{3M_1 + M_3}{4} \qquad \text{..... (1)}$$

Here M_2 and M_2' are the masses of the isodoublets belonging to the unitary octet while M_1 and M_3 are those of the isosinglet and isotriplet respectively. If now we were to assume that the newly observed ~~xx~~ ρ and ω resonances belong to an octet, we can predict the mass of the companion doublets via equation (1):

$$M_2 \cong 780 \text{ Mev.} \qquad \text{--- (2)}$$

A $K\pi$ resonance with this mass seems to have been recently observed by Miller at Berkeley[5)]. Similarly, if we group the 880 MeV resonance and the ρ meson into the same octet, we can predict the existence of an isosinglet meson of zero strangeness with a mass of about 1 BeV for which also there seems to be some evidence.[5)]

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* A.P.Balachandran and H. Pietschmann, Physics Letters (in press).

1) J.E.Wess, Nuovo Cim, 15, 52 (1960)
2) M.Gell-Mann, Phys. Rev. 125, 1667 (1962)
3) Y.Ne'emann, Nucl. Phys. 26, 222 (1961)
4) A. Salam and J.C.Ward, Nuovo Cim. 20, 419 (1961)
A. Salam, Nuovo Cim. 23, 448 (1962)
5) Private communication from Dr.P.G.O.Freund to Prof.W.Thirring.

In view of these successes of the unitary theory in predicting the masses and quantum numbers of possible resonant states, we wish to point out the possibility of a low energy $\Xi\pi$ resonance in the 8-fold way[2]. This will arise if we classify Y_1^*, N^{**} and Y_0^{**} into the same octet, assuming, tentatively, that Y_1^* is a $D_{3/2}$ resonance. (Note that we cannot as yet classify the (3/2, 3/2) isobar into an octet since its companion strange particle isobars have not to-date been experimentally observed.) We would then expect a $\Xi\pi$ resonance in the $T = 1/2$ state with a mass

$$M_2' \simeq 1460 \text{ Mev} \qquad \text{---} (3)$$

as given by equation (1) and an odd parity with respect to the nucleon.* The $\Xi\pi$ threshold energy is $\simeq$ 1458 Mev so that this "resonance" can very well turn out to be a bound state. It should show up in such reactions as

$$\bar{K} + N \rightarrow \Xi^* + K \rightarrow \begin{matrix} \Xi + \pi + K \\ \Xi + \gamma + K \end{matrix} \qquad \text{---} (4)$$

Since the intermediate step, this is a pure two-particle interaction, the kinematics is completely determined by the energy and angle of the emitted K and hence the mass of Ξ^* can be established even in the case of the radiative decay mode which might be important because of the low Q-value and relatively high orbital momentum of the resonant state. The angular distribution of the K-meson with respect to the $\Xi\pi$ resonance in the centre-of-mass system of these reactions can be easily evaluated when only the S- and P-waves of the system are taken into account, an approximation which is reasonable for a CMS momentum of about 1 BeV/C (which is to be compared with thxe threshold momentum of 644 MeV/C.) The result is

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* A $\Xi\pi$ resonance with $T = 1/2$ and $J > 1/2$, but with a mass of 1535 MeV has been reported at the CERN Conference (1962).

$$\frac{d\sigma}{d\Omega} \sim \alpha + \beta \cos\theta + \gamma \cos^2\theta - \beta \cos^3\theta \qquad \text{---- (5)}$$

where

$$\alpha = 2|f_{1,0,1}|^2 + 3|f_{1,2,3}|^2 + |\sqrt{2} f_{0,1,1} - f_{1,2,3} + 2 f_{1,0,3}|^2 ,$$

$$\beta = 6\sqrt{2}\ \mathrm{Re}\,(f^*_{0,1,1}\, f_{1,2,3}) ,$$

and

$$\gamma = \beta + 12\ \mathrm{Re}\,(f^*_{1,2,3}\, f_{1,0,3}) \qquad \text{---- (6)}$$

Here $f_{L,l,2J}$ denotes the partial wave amplitude characterized by a total angular momentum J and orbital angular momenta L and l in the entrance and exit channels respectively. The equality of the coefficients of $\cos\theta$ and $\cos^3\theta$ in equation (5) is particularly to be noted since it may be testable with comparative ease experimentally. Since the conjectured resonance has $T = 1/2$ the only final states in which the resonance can make its appearance are

$$(\Xi^- \pi^+) + K^0$$
$$(\Xi^- \pi^0) + K^+$$

and

$$(\Xi^0 \pi^-) + K^+ \qquad \text{---- (7)}$$

Another posible experiment in which this resonance can show up is

$$K^- + p \rightarrow \Lambda + K^- + K^+ \qquad \text{---- (8)}$$

which however has a higher threshold.

A low energy Ξ-π resonance has been predicted by Sakurai[6] in another context, but in the $P_{3/2}$ state with isobaric spin 3/2.

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6) J.J.Sakurai, Phys. Rev. Letters, 1 , 426 (1961)