LECTURES ON

SEMIGROUPS OF OPERATORS

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THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-20 (INDIA)
Lecture Notes
on
SEMIGROUPS OF OPERATORS

With particular reference to applications in Mathematical Physics

by

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## CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER 1</th>
<th>Introduction</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHAPTER 2</td>
<td>Basic Theory I: The Fundamental Operators of Semigroup Theory and Their Properties</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>2.1 Introduction</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>2.2 The Semigroup Operator</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>2.3 The Infinitesimal Operator</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>2.4 The Resolvent Operator</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>2.5 Representation of Semigroup Operators. The Exponential Formulae</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>References</td>
<td>62</td>
</tr>
<tr>
<td>CHAPTER 3</td>
<td>Basic Theory II: Generation of Semigroups of Operators.</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>3.1 Introduction</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>3.2 The Basic Generation Theorem</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>3.3 Generation of groups of Operators</td>
<td>69</td>
</tr>
<tr>
<td></td>
<td>3.4 Generation of Semigroups in Hilbert Space</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>3.5 Uniqueness of the Generation Problem</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>3.6 Other Studies on the Generation of Semigroups</td>
<td>76</td>
</tr>
<tr>
<td></td>
<td>References</td>
<td>77</td>
</tr>
</tbody>
</table>
CHAPTER 4 Some Additional Topics in Semigroup Theory

4.1 Introduction

4.2 Perturbation Theory

4.3 Semigroups of Contraction Operators in Hilbert Space

4.4 Equivalent Semigroups of Operators

References

CHAPTER 5 Semigroup Methods in Mathematical Physics

5.1 Introduction

5.2 Semigroups Associated with Some Differential Equations of Mathematical Physics

6.3 Semigroups of Operators in Quantum Mechanics

5.4 Semigroups as Solutions of the Boltzmann Equation of Transport Theory

References
CHAPTER 1
INTRODUCTION.

1. The theory of one-parameter semigroups of operators in Banach spaces is concerned with the study of the operator-valued exponential function, its representation and properties, in infinite-dimensional function spaces. The theory can also be considered as a generalization of Stone's theorem on one-parameter groups of unitary operators in Hilbert space. The theory of semigroups of operators had its origin in 1936, when E. Hille investigated certain concrete semigroups of operators. In 1948 Hille published his treatise on semigroups of operators. Hille's treatise stimulated research in the area of semigroups of operators; and during the 1950's a large number of papers were published on semigroup theory and its applications. In 1967 Hille and R.S. Phillips published an extensive revision of Hille's treatise, and their treatise is the bible of workers in semigroup theory. In addition to the Hille-Phillips treatise, semigroups of operators are the subject of a set of lecture notes by K.Yosida; and chapters on semigroups of operators can be found in the books of N.Dunford and J.T. Schwartz, K. Maurin and F. Nieß and B. Sz.-Nagy. The reader is also referred to the papers of R.S. Phillips and the first of which presents a survey of semigroup theory, while the second gives a very readable introduction to semigroup theory and its applications in the theory of partial differential equations.

2. Semigroups of operators are not only of interest in modern functional analysis, but are of great importance in
applications of functional analysis to differential equations, Markov processes, and mathematical physics. The concept of a semigroup also serves as a unifying concept for many problems in applied functional analysis. In this Introduction we wish to give some examples which point up the connection between the exponential function and the semigroup relation, and present a brief discussion of how semigroups of operators arise in the applied areas mentioned above.

Let \( f(t) \) be a real (or complex) function of \( t, \ t \geq 0 \). A classical problem is to find the most general continuous function \( f(t) \) which satisfies the functional equation

\[
f(s + t) = f(s) f(t),
\]

with \( f(0) = 1 \). It is well-known that the exponential function \( f(t) = e^{\lambda t} \), \( \lambda \) a constant, is the most general continuous real (or complex) function \( f(t) \) which satisfies (1.1).

Let us now consider the solution \( x(t), \ t \geq 0 \), of the scalar differential equation

\[
\frac{dx}{dt} = \lambda x
\]

satisfying the initial condition \( x(0) = 1 \). Again, it is well-known that the solution of (1.2) is \( x(t) = e^{\lambda t} \). Hence, the solution of (1.2) satisfies the functional equation \( x(s + t) = x(s) x(t) \), \( s, t \geq 0 \).

Suppose we now consider the operator analogue of equation (1.1). Let \( \mathcal{X} \) be a Banach space, \( T(t) \) a linear operator in \( \mathcal{X} \), and \( \mathcal{L}(\mathcal{X}) \) the Banach algebra of endomorphisms of \( \mathcal{X} \). In this case we wish to determine the most general continuous
operator-valued function $T(t) : [0, \infty) \rightarrow \mathcal{L}(\mathcal{H})$ which satisfies the functional equation

$$T(s + t) = T(s) T(t), \quad s, t > 0, \quad (1.3)$$

with $T(0) = I$ (the identity operator). A family of operators $\{T(t), t \geq 0\}$ satisfying (1.3) is called a one-parameter semigroup of operators in the Banach space $\mathcal{H}$. If $T(t)$ admits an inverse, $T^{-1}(t) = T(-t)$, so that $T(t) T(-t) = I$, we say that the family of operators $\{T(t), -\infty < t < \infty\}$ forms a one-parameter group of operators in the Banach space $\mathcal{H}$.

From the classical scalar case considered above, one might expect that the operator-valued function $T(t)$ satisfying (1.3) admits, in some sense, a representation as an exponential function. This is indeed the case; and we shall subsequently show the following: (i) if $T(t)$ is continuous in the uniform operator topology, then there exists a bounded operator $A$ on $\mathcal{H}$ such that $T(t) = e^{At}$, and (ii) if $T(t)$ is continuous in the strong operator topology, then $T(t)$ can be represented, in an appropriate sense, as $e^{At}$, but in this case $A$ is an unbounded operator. The operator $A$ is called the infinitesimal generator of the semigroup $\{T(t), t \geq 0\}$.

In order to motivate the study of semigroups of operators, we shall now briefly consider some examples which illustrate how semigroups of operators arise in various applications. In presenting these examples we make no attempt to be complete or precise.
(1). Matrix differential equations. As an example we consider the vector-matrix equation

\[ \frac{dx}{dt} = Ax, \]  

(1.4)

with \( x(0) = c \). In the above \( x(t) \) is an \( n \)-vector, and \( A \) is an \( n \times n \) constant matrix. In place of (1.4) we can consider the matrix equation

\[ \frac{dx}{dt} = AX, \]  

(1.5)

with \( x(0) = I \), the identity matrix. We can restrict our attention to (1.5) since \( x = x_c \). The Banach space in this example is \( E_n \). By analogy with equation (1.2) we consider a formal solution meaningful; we introduce the matrix series

\[ e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}, \quad A^0 = I, \]  

(1.6)

which defines the matrix function on the left. It can be shown that the above series converges, and that the sum function is a continuous function of \( t \) for all finite \( t \). Similarly, it can be shown that the differentiated series converges uniformly; and it is clear that \( e^{At} = I \) for \( t = 0 \). From uniqueness considerations it follows that \( X(t) = e^{At} \). Finally, using (1.6) it is easy to show that \( e^{As} e^{At} = e^{A(s + t)} \); hence

\[ X(s + t) = X(s) X(t), \quad s, \ t \geq 0. \]  

(1.7)

The above shows that the family of matrices \( \{X(t), \ t \geq 0\} \) forms a semigroup of operators in \( E_n \).

(2). Partial differential equations. The application of semi-group theory to partial differential equations is based on the fact that the solution of the Cauchy problem for a linear
partial differential equation, with initial values at \( t = 0 \) belonging to a concrete Banach space, is obtained by applying an operator \( T(t) \) to the initial values. The family of operators \( \{T(t), \ t \geq 0\} \) forms a semigroup, the associated infinitesimal generator being a differential operator.

As an example, we shall briefly consider the well-known diffusion, or heat, equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(t, x), \quad t > 0, \quad -\infty < x < \infty \quad (1.8)
\]

with the initial condition

\[
\lim_{t \to 0} u(t, x) = f(x) \quad (1.9)
\]

We shall take the Banach space in this case to be \( C[-\infty, \infty] \); hence \( f(x) \in C[-\infty, \infty] \). To stress the dependence of the solution \( u(t, x) \) on the particular initial value \( f(x) \), we will denote the solution of (1.8) associated with the initial condition (1.9) by \( u(t, x; f) \). We now assume that there exists at most one solution of (1.8) for a given function \( f \), and that \( u(t, x; f) \in C[-\infty, \infty] \) for each \( t > 0 \). It now follows from the linearity of (1.8) and (1.9), and uniqueness, that

\[
u(t, x; \alpha f + \beta g) = \alpha u(t, x; f) + \beta u(t, x; g). \quad (1.10)
\]

Let us now consider how we might obtain a solution \( u(s_1 + s_2, x; f) \). Such a solution can be obtained directly from \( f(x) \) at time \( t = s_1 + s_2 \) or from the solution \( u(s_2, x; f) \) after time \( t = s_1 \). Hence we can write

\[
u(s_1 + s_2, x; f) = u(s_1, x; u(s_2, x; f)). \quad (1.11)
\]

In view of the above, the solution \( u(t, x; f) \) defines a
linear operator \( T(t) \) on \( C \left[ -\infty, \infty \right] \) to itself as follows:

\[
( u(t,x;f) = T(t) f(x) ) \tag{1.12}
\]

From (1.11) and (1.12) we see that

\[
T(s_1 + s_2) f = T(s_1) T(s_2) f, \quad s_1, s_2 > 0. \tag{1.13}
\]

The initial condition (1.9) can be expressed in the form

\[
\lim_{t \to 0} T(t)f(x) = f(x), \quad f(x) \in C \left[ -\infty, \infty \right],
\]

or

\[
\lim_{t \to 0} T(t) = I.
\]

Hence we can conclude that the family of operators \( \{ T(t), t > 0 \} \), as defined by (1.12), forms a semigroup of operators in \( C \left[ -\infty, \infty \right] \).

It is of interest at this point to refer to Hadamard's analysis of Huyghen's principle in optics, and the semigroup property implied by (1.11) and (1.12). In 1903 J. Hadamard (cf. 2) noted that Cauchy's problem for the wave equation led to certain groups of transformations, and that the group property implied, and was implied by certain transcendental 'addition theorems' satisfied by the elementary solutions used in constructing the solution. We remark that the wave equation, which describes reversible phenomena, is of hyperbolic type and leads to groups of operators; while the diffusion equation, describing irreversible phenomena, is of parabolic type and leads to semigroups of operators.

Hadamard found that the group property of the transformations was a consequence of what he termed the Principle of Scientific Determinism. This Principle can be stated as follows: Given the state of a physical system at time \( t = t_0 \) (\( t_0 > 0 \)) one can deduce the state of the system at time \( t > t_0 \). An important
corollary of the above Principle was formulated by Hadamard, and referred to as the major premise of Huygen's principle. This corollary can be stated as follows: The state of a physical system at time \( t, t > 0 \), can be deduced from its state at an intermediate time \( t_1 \) by first computing the state at time \( t_1 \) and then from the latter the state at time \( t \), the result being the same as that obtained by direct computation from the original state at time \( t_0 \).

In mathematical form the major premise of Huygen's principle is expressed by (1.11), (1.12), and (1.13).

(3). Markov processes. Markov processes constitute an important class of stochastic processes. Until recently methods of classical analysis and measure theory were used to investigate the structure of Markov processes and their properties. It is now known, however, that from an abstract point of view a Markov process is a semigroup of operators in a concrete Banach space. To establish the relationship between Markov processes and semigroups of operators we can proceed as follows. Let \((\mathcal{X}, \mathcal{A}, \mathcal{P})\) be a probability space, \((E, \mathcal{C})\) a measurable space, and let \(X(t)\) be a random variable, that is, an \(\mathcal{A}\)-measurable function from \(\mathcal{X}\) to \((E, \mathcal{C})\). Let \(P(t, x, S)\) denote the transition probability function associated with the process \(\{X(t), t \geq 0\}\), that is \(P(t, x, S) = P\{X(t) \in S | X(0) = x\}\), \(S \in \mathcal{C}\). If the transition probability is stationary (i.e., it is invariant under translations in time), it satisfies the Chapman-Kolmogorov equation.

\[
P(s + t, x, S) = \int_E P(s, x, dy) P(t, y, S), s, t \geq 0 (1.14)
\]
and the initial condition
\[ \lim_{t \to 0} P(t, x, S) = \chi(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases} \quad (1.15) \]

Let \( \mathcal{K} \) denote the Banach space of all bounded Borel measurable functions \( f(x) \) on \( E \), with norm \( \sup_{x \in E} |f(x)| \). We define an operator \( T(t) \) on \( \mathcal{K} \) as follows:
\[ T(t) f(x) = \int_E P(t, x, dy) f(y), \quad f(y) \in \mathcal{K} \quad (1.16) \]
Using (1.14) and (1.15) it is easy to see that the family of operators \( T(t), t > 0 \) defined by (1.16) forms a semigroup of operators in \( \mathcal{K} \).

(4). Mathematical physics. Semigroup methods are introduced in mathematical physics via the many partial differential equations which are used to describe various physical phenomena. As an example, we consider the connection between semigroups of operators and the Schrödinger equation of quantum mechanics. The Schrödinger equation can be written as
\[ i\hbar \frac{\partial \psi}{\partial t} = H \psi, \quad \psi = \psi(t, x), t > 0, 0 < x < \infty \quad (1.17) \]
where \( H \) is the Hamiltonian operator, and \( \hbar = h/2\pi \) (\( h \) is Planck's constant). We take the Banach space \( \mathcal{K} \) to be \( L^2_{\mathbb{R}}(0, \infty) \); and we assume that
\[ \lim_{t \to 0} \psi(t, x) = f(x) \in L^2_{\mathbb{R}}(0, \infty). \quad (1.18) \]
Proceeding formally, the solution of (1.17) satisfying the initial condition (1.10) is
\[ \psi(t, x) = e^{\frac{Ht}{\hbar}} f(x). \quad (1.19) \]
Hence if we put \( T(t) = e^{\frac{Ht}{\hbar}} \), then we see that the family of
operators \( \{ T(t), t > 0 \} \) forms a semigroup of operators in \( L_2(0, \infty) \).

Proceeding another way, we can utilise the quantum mechanical kernel function \( K(t, x, y) \) (cf. \( \mathcal{L}_7 \)). The kernel function, or transition amplitude function, is the quantum mechanical analogue of the transition probability function in the theory of Markov processes, and it satisfies an equation analogous to the Chapman-Kolmogorov equation. Hence we can define an operator on \( L_2(0, \infty) \) to itself as follows:

\[
T(t)g(x) = \int_0^\infty K(t, x, dy) g(y), \quad g(y) \in L_2.
\]

It then follows that \( \{ T(t), t > 0 \} \) is a semigroup of operators.

We remark that, in view of the above examples, semigroups of operators arise in connection with the mathematical analysis of physical systems which are linear and have the property of being stationary or temporally-homogeneous.

3. In these notes we attempt to present a brief introduction to semigroup theory, and to point out some applications of this theory in mathematical physics. Chapters 2 and 3 are based on material found in Hille and Phillips; but we have restricted our attention to those concepts and results which we feel are rather basic, and should be known to the mathematical physicist interested in an introduction to semigroup theory, and in using semigroup methods in his own studies. In chapter 4 we present some additional topics in semigroups theory. We regret that time did not permit the inclusion of other theoretical topics which are important in certain fields of applications.
the near future; and in addition to giving more theory, we will also present some applications to differential equations and Markov processes, in particular we will consider the use of Markov processes, via semigroup theory, to study certain partial differential equations.

The notes* prepared on Banach spaces (which will be referred to in these notes as B.S.) hopefully contain an outline of all of the prerequisites required for the study of these notes. Throughout these notes references such as Theorem N 12, or Definition N 3, refer to Theorem 12 or Definition 3, as given in our notes on Banach spaces.

References


CHAPTER 2

BASIS THEORY I

THE FUNDAMENTAL OPERATORS OF SEMIGROUP THEORY AND THEIR PROPERTIES

2.1. Introduction

In this chapter we introduce and study the fundamental operators of semigroup theory. These operators, three in number, are the semigroup operator, the infinitesimal operator (or generator) of the semigroup, and the resolvent of the infinitesimal operator. These operators, their properties, and the relationships between them, are the subject of this chapter. Sections 2, 3 and 4 are devoted, respectively, to each of these fundamental operators, and in section 5 we consider the representation of semigroups and the exponential formula of semigroup theory.

2.2. The Semigroup Operator

2.2.1. Definition of a Semigroup of Operators. Some Examples.

Definition 2.1. A family \( \{T(t), t \geq 0\} \) of endomorphisms in a Banach space \( \mathcal{X} \) is called a one-parameter strongly continuous semigroup of operators if the following conditions are satisfied:

(i) \( T(s+t) = T(s)T(t), \quad s, t \in (0, \infty) \)

(ii) \( T(0) = I \)

(iii) For each \( x \in \mathcal{X} \), \( T(t)x \) is strongly continuous in \( t \) for \( t > 0 \).

\[ \text{Notes} \]

In these notes we restrict our attention to one-parameter semigroups. For a discussion of \( k \)-parameter semigroups we refer to (2.5).
If, in addition, the mapping \( t \rightarrow T(t) \) is continuous in the uniform operator topology, then \( \{ T(t), t \geq 0 \} \) is called a uniform \textit{continuous semigroup of operators}.

To the three conditions stated above we can also add the following:

(iv.) There exists a nonnegative real number \( \beta \) such that \( \| r(t) \| \leq e^{\beta t} \)

This fourth condition is a restriction on the order of \( \| r(t) \| \) near \( t = 0 \). In another section we shall show that conditions (i) and (iii) imply the following:

(a) \( \lim_{t \to 0} \frac{1}{t} \log \| r(t) \| = \omega_0 < \infty \) \( (\omega_0 \text{ may be} -\infty) \)

(b) \( \| r(t) \| \) is bounded in any bounded interval \( [a, b] \), \( 0 < a < b < \infty \)

Throughout these notes the term \textit{semigroup operator} will refer to an operator \( T(t), t > 0 \) which belongs to a family of operators \( \{ T(t), t > 0 \} \) with the \textit{semigroup property}. 

We now define two important classes of semigroups.

**Definition 2.** A semigroup of operators \( \{ T(t), t \geq 0 \} \) such that \( \| T(t) \| \leq 1 \) is called a **semigroup contraction operators** or, simply a **contraction semigroup**. A semigroup of positive contraction operators with the property that \( \| T(t) \| < 1 \) for all \( t \geq 0 \) is called a **semigroup of transition operators** or simply a **transition semigroup**.

Let \( \mathcal{X} = T(t) \mathcal{X}_0 \) denote the range of the semigroup operator \( T(t), t > 0 \). Clearly \( \mathcal{X}_{t_1} \supseteq \mathcal{X}_{t_2} \) if \( t_1 < t_2 \).

**Definition 3.** The set \( \mathcal{X}_0 = \mathcal{X}_{t} \) is called the **range space** of the semigroup \( \{ T(t), t > 0 \} \). Thus, \( \mathcal{X} \) is the least linear subspace containing the range spaces of the operators \( T(t), t > 0 \).

**Some Examples of Semigroups of Operators**

1. Let \( \mathcal{X} = \mathcal{C} \left[ 0, \infty \right) \) and define an operator \( T(t) \) by \( T(t) x(s) = x(t+s), \ x(s) \in \mathcal{C} \left[ 0, \infty \right) \)

   \[ (2.1) \]

   Clearly \( T(0) = I \) and since \( T(t_1 + t_2) x(s) = x(t_1 + t_2 + s) = T(t_1) x(t_2 + s) = T(t_1) T(t_2) x(s) \), the semigroup property is established. To establish strong continuity, we have only to note that strong continuity follows from the uniform continuity of \( x(s) \) since \( \| (T(t) - T(t_0)) x(s) \| \rightarrow 0 \) as \( \sup_{s \geq 0} \| x(t+s) - x(t_0 + s) \| \rightarrow 0 \)
Therefore \[ \lim_{t \to 0} \| T(t) x(t_0) \| = 0 \] for any \( x(t_0) \in C[0, \infty] \).

Since the maximum of \( | x(t + s) | \) cannot exceed the maximum of \( | x(s) | \) to have \( \| T(t) x(t_0) \| \times \| \| \times \| \) with equality holding if \( x(t_0) \) is constant-valued. Hence \( \| T(t) \| = 1 \).

Therefore \( \{ T(t), t \geq 0 \} \) with \( T(t) \) defined by (2.1) is a semigroup of contraction operators in \( C[0, \infty] \).

We will call this semigroup the translation semigroup.

2. Let \( \chi \in C[-\infty, \infty] \) and define an operator

\[
T(t) \chi(u) = \begin{cases} 
\int_{-\infty}^{\infty} \mathcal{L}(t, s-u) \chi(u) du, & t > 0 \\
\chi(u), & t = 0 
\end{cases}
\]  

(2.3)

where

\[
f(t, u) = \frac{1}{\sqrt{2\pi t}} e^{-u^2/2t}, \\
\mathcal{L}(t, u) \in C(-\infty, \infty), \quad t > 0
\]  

(2.3)

The function \( f(t, u) \) is the density function of the normal distribution of probability theory. From \( \mathcal{L}(0, u) = 1 \) and the semigroup property follows from the easily verified formula

\[
e^{-\frac{u^2}{2}(t_1 + t_2)} \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^{\infty} e^{-\frac{(u-\tau)^2}{2t_1}} e^{-\frac{\tau^2}{2t_2}} d\tau \]  

(2.4)
\[
\left. \frac{\partial}{\partial s} \left( \int_{s}^{\infty} f(t) \, dt \right) \right|_{s=\infty} = \lim_{s \to \infty} \frac{d}{ds} \left( \int_{s}^{\infty} f(t) \, dt \right)
\]

We can now rewrite (5.6) as

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sin \left( \frac{\pi}{2} n \right) \right) \left( \cos \left( \frac{\pi}{2} n \right) \right)
\]

where \( s > \frac{1}{2} \) that for any \( \epsilon > 0 \) there exists a \( N \) such that \( x < N \). Since \( x < \infty \) the uniformly convergent series

\[
\sum_{n=1}^{\infty} \left( \sin \left( \frac{n\pi}{2} \right) \right) \left( \cos \left( \frac{n\pi}{2} \right) \right)
\]

and \( \epsilon \) can be rewritten as

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sin \left( \frac{\pi}{2} n \right) \right) \left( \cos \left( \frac{\pi}{2} n \right) \right)
\]

and becomes

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sin \left( \frac{n\pi}{2} \right) \right) \left( \cos \left( \frac{n\pi}{2} \right) \right)
\]

Then the integral

\[
\int_{0}^{\infty} e^{-x^2} \cos(x) \, dx
\]

Let \( x = \sqrt{2} \theta \) with

\[
\int_{0}^{\infty} e^{-x^2} \cos(x) \, dx = \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\pi}}
\]

For \( f \neq 0 \), then the integral

\[
\int_{0}^{\infty} f(x) \cos(x) \, dx
\]

is obtained as follows.
\[ \leq c \int f(t, v) dv + 2 \left\| \int f(t, v) dv \right\|_{\sqrt{t}} \cdot \left\| \frac{1}{\sqrt{t} - \sqrt{t_0}} \right\|_{\sqrt{t}} \geq \delta \]

As \( |t_1 - t_0| \to 0 \), the second term tends to 0 since the integral \( \int_{-\infty}^{\infty} f(t, v) dv \) converges. Hence

\[ \lim_{t \to t_0} \sup_{S} \left| \left( T(t) - T(t_0) \right) x (3) \right| < \epsilon \]

Since \( \epsilon > 0 \) was arbitrary, we have established the strong continuity of \( T(t) \) at \( t = t_0 \).

Finally, we have

\[ ||T(t)x|| \leq \left| \int_{-\infty}^{\infty} f(t, \omega) d\omega \right| \|x\| \tag{2.7} \]

Since \( f(t, \omega) \) being a probability density, \( \int_{-\infty}^{\infty} f(t, \omega) d\omega = 1 \). Hence, the family \( \{ T(t), t \geq 0 \} \) is a semigroup of contraction operators in \( C[-\infty, \infty] \). We will call this semigroup the normal or Gaussian semigroup.
3. let $X = \mathbb{C}[-\infty, \infty]$ and define an operator $T(t)$ by

$$T(t)x(s) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} x(s - k \mu), \quad x(s) \in \mathbb{C}$$ (2.3)

where $\lambda$ and $\mu$ are positive constants. Clearly $T(s) : I$ to establish the semigroup property we simply carry out the following computation

$$T(t_1)T(t_2)x(s) = e^{-\lambda t_1} \sum_{k=0}^{\infty} \frac{(\lambda t_1)^k}{k!} \left[ e^{-\lambda t_2} \sum_{j=0}^{\infty} \frac{(\lambda t_2)^j}{j!} x(s - (k+j) \mu) \right]$$

$$= e^{-\lambda (t_1 + t_2)} \sum_{k=0}^{\infty} \frac{(\lambda t_1)^k (\lambda t_2)^j}{j!} \left[ \sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{\lambda t_1}{(i+j)!} \right)^i \right] x(s - j \mu)$$

$$= e^{-\lambda (t_1 + t_2)} \sum_{j=0}^{\infty} \frac{(\lambda t_1 + \lambda t_2)^j}{j!} x(s - j \mu)$$

$$= T(t_1 + t_2) x(3)$$

To show that $T(t)$ is strongly continuous we note that

$$|T(t)x(s) - T(t_0)x(s)| \leq ||x|| \left| e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} - e^{-\lambda t_0} \sum_{k=0}^{\infty} \frac{(\lambda t_0)^k}{k!} \right| = 0$$

as $t \to t_0$. It is clear that $||T(t)|| \leq 1$ hence the semigroup $\{T(t) : t \geq 0\}$ defined by (2.3) is a contraction semigroup. We will call this semigroup the Polisson semigroup.
E. Measurability and Continuity of Operators.

In sections 20 and 21 of Part II, we considered Banach space-valued and operator-valued measurable functions. We now investigate the measurability and continuity of semigroups of operators, the semigroup operator \( T(t) \) being a function in \((\mathcal{L}(\mathcal{X}), \sigma)\) to \(\mathcal{E}(\mathcal{Y})\), the Banach algebra of endomorphisms of \(\mathcal{X}\).

Our first theorem establishes the connection between uniform measurability of an operator and continuity in the uniform operator topology.

**Theorem 1.** If \( T(t) \) is uniformly measurable, then \( Y(x) \) is continuous in the uniform operator topology.

**Proof:** By theorem 33, we know that uniform measurability of \( T(t) \) is equivalent to weak measurability and almost separable-valuedness of \( T(t) \) in \(\mathcal{E}(\mathcal{Y})\). Hence the statement of the theorem follows from theorems 38 and 39.

We now establish the relationship between the strong measurability and boundedness of \( T(t) \).

**Theorem 2.** If \( T(t) \) is strongly measurable, then

\[
\| T(t) \| \text{ is bounded in each interval } [a, b], 
\]

Proof: By theorem 16, it is sufficient to prove that

\[
\| T(t) x \| \text{ is bounded in } [a, b] \text{ for each } x \in \mathcal{X}. 
\]

We assume \( \| T(t) x \| \) is not bounded for some \( x \) and establish a contradiction. For assumption \( j \), we know the existence of \( t_j \in [a, b] \) and a sequence \( \{ t_k : k \in \mathcal{N}, t_k \in [a, b] \} \) such that \( t_k \to t_j \) and \( \| T(t_k) x \| \to \infty \) for all.
However, the measurability of \( \| T(t) \times \| \) implies the existence of a constant \( M \) and a measurable set \( E \subset [0,1] \) with \( \mu(E) > \frac{1}{2} \) such that \( \| T(t) \| \leq M \). Let \( E_n = \{ (t, \omega), \omega \in \Omega \cap E \} \). Then \( E_n \) is a measurable set, and for all sufficiently large \( m(T_n) > \frac{1}{2} \) and for \( \omega \in \Omega \cap E \),

\[
\| T(t_n) \| \leq \| T(t_n - \omega) \| \leq \| T(t_n - \omega) \| M
\]

and, therefore, \( \| T(t) \| > \frac{1}{M} \) for all \( t \in E \). If we put \( F = \lim_{n \to \infty} E_n \), then \( \| T(t) \| > \frac{1}{M} \) for all \( t \in E \), and \( \mu(F) > \frac{1}{2} \). However, this contradicts the fact that \( \| T(t) \| \) is finite-valued for all \( t \in (0, \infty) \) and thus implies the statement of the theorem.

Finally, we prove that strong measurability of a semigroup operator implies its strong continuity.

**Theorem 2:** If \( T(t) \) is strongly measurable, then \( T(t) \) is strongly continuous for \( t > 0 \).

**Proof:** Pick four integers \( \alpha, \beta, \gamma, \delta \) such that \( 0 < \alpha < \beta < \gamma < \delta \) and an \( \gamma \) such that \( \beta < \gamma \). Now the semigroup property enables us to write

\[
T(t) = T(t - \tau) \left[ T(t - \tau) \times \right] \]

the right-hand side being integrable on \( \tau \) is certainly integrable with respect to \( \tau \) that is

\[
\int_0^\infty \int_0^\infty \left[ T(t - \tau) \right] \times \left[ T(t - \tau) \right] d\tau
\]

for
Since $\tau(t)$ is strongly measurable, by theorem 2 there exists an $M$ such that $\|\tau(t)\| \leq M$ for $t \in [\alpha, \beta]$. Therefore the norm of the integrand does not exceed

$$M \| \int \tau(t) \eta \| \geq \int \tau(t) \eta \| = \int \tau(t) \eta \| dt$$

which is a measurable function of $\tau \in [\alpha, \beta]$, and according to theorem 3 uniformly bounded on $[\alpha, \beta]$ for $\eta$ sufficiently small. Hence we can write

$$\int_{\alpha}^{\beta} \int \tau(t) \eta \| dt \leq M \int \int \tau(t) \eta \| dt$$

By theorem 290 the right-hand term tends to zero with $\eta$. Hence it follows that $\tau(t)$ is strongly continuous for $t > 0$.

**Type of a Semi-group of Operators.** We now consider the behavior of $\|I(t)\|$ as $t \to \infty$ and introduce the notion of the type of a semi-group. We first state a lemma on subadditive functions, i.e., functions $\omega(t)$ such that

$$\omega(t_1 + t_2) \leq \omega(t_1) + \omega(t_2)$$

**Lemma 1.** Let $\omega(t)$ be a subadditive function on $[0, \infty]$ which is bounded on each finite subinterval. Then $\omega_c = \max_{t > 0} \omega(t)^t$ is finite or equal $-\infty$ and

$$\omega_c = \lim_{t \to \infty} \frac{\omega(t)}{t}$$

For a proof of this lemma, refer to (2, pp. 618-619).

We have the following corollary.

**Corollary:** The limit $\omega_c = \lim_{t \to \infty} \frac{1}{t} \log \|I(t)\|$ exists. For each $\beta > \omega_c$, there exists a constant $\lambda(\beta)$ such that

$$\|I(t)\| < \lambda(\beta) t^{-\lambda(\beta)} \quad \text{for} \quad t > 0.$$
Proof: Define \( \omega(t) = \log \| T(t) \| \), \( t > 0 \)

Since \( \omega(x+y) > \omega(x) + \omega(y) \), the function \( \omega(t) \) is subadditive.

The result now follows from the lemma and theorems 2 and 3.

We have the following definition:

**Definition 4:** A semigroup \( \{T(t), t > 0\} \) is said to be of type \( \omega_0 \) if

\[
\omega_0 = \lim_{t \to \infty} \frac{1}{t} \log \| T(t) \|
\]

The type of a semigroup plays an important role in connection with the Laplace transform of the Banach space-valued function \( T(t) \times \) when it exists.

In this case the abscissa of absolute convergence of the Laplace integral is given by

\[
\sigma_a(x) = \lim_{t \to \infty} \sup_{t \to 0} \frac{1}{t} \log \| T(t) \times \|
\]

Since we want the Laplace transform of \( T(t) \times \) to exist for all \( x \in X \), the pertinent abscissa of convergence is given by \( \sup_{x \in X} (\sigma_a(x)) \). The connection between the type of the semigroup \( \{T(t), t > 0\} \) and the abscissa of convergence of the Laplace transform of \( T(t) \times \) is given by the following theorem.

**Theorem 4:** \( \omega_0 = \sup_{x \in X} (\sigma_a(x)) \)

For a proof of this theorem we refer to (5, p.306)
Convergence of Semigroups as \( t \to 0 \).

The Basic Classes of Semigroups of Operators. Thus for we have not any-
thing about the convergence of a semigroup operator \( T(t) \)
as \( t \to 0 \). As it turns out, this convergence is of great im-
portance, and the sense of convergence is used as a basis for
classifying semigroups of operators. Throughout this section we
shall assume (i) that \( T(t) \) is strongly measurable, and there-
fore by theorem 3, strongly continuous for \( t > 0 \), and (ii)
that \( \mathcal{H} \) (the range space of the semigroup) is dense in \( \mathcal{H} \).

There are three kinds of convergence at \( t = 0 \), and these are
given by the following definition.

**Definition 5.** For \( x \in \mathcal{H} \), \( T(t)x \) is said to be
\((C_0)\) _summable to \( x \) if \( \lim_{t \to 0} T(t)x = x \). If
\[
\int_{0}^{\infty} \| T(t)x \| dt < \infty,
\]
and \( B(t)x \) is said to be \((C, 1)\) _summable to \( x \) if
\[
\lim_{t \to 0} B(t)x = x.
\]

If \( \int_{0}^{\infty} \| T(t)x \| dt < \infty \) then
\[
R(\lambda)x = \int_{0}^{\infty} e^{-\lambda t} T(t)x dt
\]
exists for all \( \lambda \) with \( R(\lambda) > 0 \), and \( T(t)x \) is said
to be _Abel summable to \( x \) if \( \lim_{\lambda \to \infty} \lambda R(\lambda)x = x \).

We remark that \((C_0)\) summability implies \((C, 1)\) summ-
ability, and that \((C, 1)\) summability implies Abel summability.

The basic classes of semigroups of operators are given by the following definition.
Definition 6. A semigroup $\{T(t), t \geq 0\}$ is said to be of class $C_0$ if it is Abel summable, of class $(A)$ if it is Abel summable, and of class $(1, C)$ if it is summable and

$$\int_0^\infty \|T(t)\| dt < \infty$$

for each $x \in X$ of class $(1, A)$ if it is Abel summable and

$$\int_0^\infty \|T(t)\| dt < \infty$$

and of class $(0, C)$ if it is Abel summable and

$$\int_0^\infty \|T(t)\| dt < \infty$$

for each $x \in X$.

We summarize the classification of semigroups and the relations between the basic classes in the following theorem.

Theorem 8: The inclusion relations between the six basic classes are given by the following diagram:

$$(0, A)$$

$$(A) \rightarrow (0, A) \rightarrow (1, C) \rightarrow C_0$$

For all of these classes except $(A)$, the linear bounded operator $R(x)$ is defined by (2.9) for all $x \in X$. The operator $R(x)$ is a holomorphic function of $\lambda$ for $R_0(\lambda) > \omega_0$ and $\|R(\lambda)\|$ is bounded for $R_0(\lambda) > \omega$ for any fixed $\omega > \omega_0$. For each semigroup belonging to class $(A)$, there is an $\omega' > \omega_0$ such that $R(\lambda)$ is holomorphic and bounded in norm for $R_0(\lambda) > \omega'$.
For a proof of this theorem, and a detailed discussion of the basic classes, we refer the interested reader to [5].

We remark that an application of theorem N33 shows that the operators $B(t), t > 0$ and $R(\lambda), Re(\lambda) > \omega_0$ belong to $E(\mathcal{I})$.

Throughout the remainder of these notes will restrict our attention to semigroups of class $(C_0)$ since the semigroups which we consider in our applications will in the main, belong to class $(C_0)$. Two results which give sufficient conditions for a semigroup to belong to class $(C_0)$ are given below.

These theorems can be employed in applied problems if the required properties of the semigroup operators are known.

**Theorem 6:** If $\{T(t), t > 0\}$ belongs to any of the six basic classes and if $\|T(t)\| = O(t)$ as $t \to 0$, then $\{T(t), t > 0\} \in (C_0)$.

**Theorem 7:** If the semigroup operator $T(t), t > 0$, is such that $\lim_{t \to 0} T(t) = I$ in the weak operator topology then $\{T(t), t > 0\} \in (C_0)$.

For proofs of these theorems we refer to (5, p. 324).
E. Continuity of Semigroups at the Origin. In the last subsection, we considered the convergence of semigroups as $t \to 0$. We now consider the continuity of semigroups at the origin. The theorems we present belong to a class of theorems known as ergodic theorems. The term ergodic theorem as used in semigroup theory refers to a theorem which asserts that a semigroup operator has a generalized limit in one sense or another as $t \to 0$ or $\infty$.

**Theorem 3:** Let $\tau(t)$ be a strongly continuous semigroup operator for $t > 0$ and suppose that

\begin{align*}
\text{(a) } & \int_0^1 \| \tau(tz) \| \, dz < \infty \quad \text{for each } x \in \mathcal{X} \\
\text{(b) } & \lim_{\tau \to 0} \int_0^2 \tau(tz) \, dz = \rho_x \quad \text{exists for each } x \in \mathcal{X}
\end{align*}

Then $\rho$ is a projection operator which maps all of $\mathcal{X}$ onto $\mathcal{X}_0$ and $\tau(t) \rho = \rho \tau(t) = \tau(t) \rho$. 

**Proof:** From theorem N38 it follows that

$$
\frac{1}{2} \int_0^2 \tau(tz) \, dz
$$

defines an endomorphism of $\mathcal{X}$ and by theorem N16 (uniform boundedness) we conclude that $\rho$ is also an endomorphism of $\mathcal{X}$. For $t > 0$, we have

$$
\tau(t) \left[ \frac{1}{2} \int_0^2 \tau(tz) \, dz \right] = \frac{1}{2} \int_0^2 \tau(tz) \, dz = \frac{1}{2} \int_0^2 \tau(tz) \left[ \tau(tz) \right] \, dz
$$
If we now pass to the limit as $\delta \to 0$, we have

$$\mathcal{T}(\delta)P = \mathcal{T}(\delta)\mathcal{T}(t) - \frac{1}{2} \int_0^\infty \mathcal{T}(\tau) \times d\tau$$

and as $\delta \to 0$, we obtain $P^2 = P$, hence $\mathcal{P}$ is a projection

From the definition of an integral it follows that

$$\frac{1}{2} \int_0^\infty \mathcal{T}(\tau) \times d\tau$$

lies in the closed linear hull of $\mathcal{H}_0$ (i.e., the smallest closed linear subset of the range space $\mathcal{H}_0$), and this implies that $P_x \in \mathcal{H}_0$. Conversely, if $x \in \mathcal{H}_0$, then $P_x = x$.

Let $x \in \mathcal{H}_0$. Then the strong continuity of $\mathcal{T}(t)$ for $t > 0$ implies that $\lim_{t \to 0} \mathcal{T}(t)x = x$. Since $P_x$ is the Cesàro mean of $\mathcal{T}(t)x$ near $t = 0$, it is clear that $P_x = x$ for all $x \in \mathcal{H}_0$. Now $\mathcal{H}_0$ is dense in $\mathcal{H}_0$, so employing theorem M20 (Banach-Stone) we have $P_x = x$ for all $x \in \mathcal{H}_0$. Hence $P(\mathcal{H}) = \mathcal{H}_0$ i.e. $P$ maps all of $\mathcal{H}$ onto $\mathcal{H}_0$.

**Theorem 2:** If $\mathcal{T}(t)$ is a semigroup operator for $t > 0$ and if (strong) $\lim_{t \to 0} \mathcal{T}(t) = P$ exists, then $P$ is a projection operator, with $P(\mathcal{H}) = \mathcal{H}_0$ and

$$\mathcal{T}(t) = \mathcal{T}(\delta)P, \quad P \mathcal{T}(t) = \mathcal{T}(\delta)P.$$
Necessary and sufficient conditions that \( \lim_{t \to 0} \tau(t) = P = I \) are that (i) \( \tau(t) \) be strongly measurable \( \{ t \to t > 0 \} \)
(ii) there exists a positive \( M \) such that \( ||\tau(t)|| \leq M \) for \( t \in (0, 1) \) and (iii) \( X_0 = X \).

Proof: If \( \tau(0) \) exists as a limit in the strong sense, we have
\[
\lim_{t \to 0} \tau(t) = P = \lim_{t \to 0} \tau(t + \varepsilon) - \lim_{\varepsilon \to 0} \tau(t) \tau(\varepsilon) = P^2
\]
similarly, \( \lim_{t \to 0} \tau(t) = P \) implies
\[
\lim_{\varepsilon \to 0} \tau(t + \varepsilon) x = \lim_{\varepsilon \to 0} \tau(t) \tau(\varepsilon) x = \tau(t) \tau(\varepsilon) x
\]
for each \( x \in X \). Hence it follows that \( \tau(t) \) is strongly (right) continuous and hence measurable. In fact, for each \( \varepsilon > 0 \) and any point \( t_0 \) at which the oscillation of \( \tau(t) \) is greater than or equal to \( \varepsilon \), there is an open set in \( (0, \infty) \) abutting \( t_0 \) (on the right) in which the oscillation is less than \( \varepsilon \). Since these open sets are disjoint, we have that the points of oscillation greater than or equal to \( \varepsilon \) are at most denumerable, and consequently the same is true of the points of discontinuity of \( \tau(t) \). It follows that \( \tau(t) \) is separately valued (i.e., Definition N29) and weakly measurable, and hence, by Theorem 7, strongly measurable. Now from Theorem 3 and the fact that \( \lim_{t \to 0} \tau(t) = P \) we have that the semigroup \( \tau(t) \) strongly continuous for \( t > 0 \). This proves \( \tau(t) = \rho \tau(t) \tau(t) = \tau(t) P \) and shows that (i) and (ii) are necessary for the existence.
of \( \lim_{t \to \infty} T(t) = \beta \). Since the hypothesis of Theorem 8 is satisfied, we have \( P[X] = \overline{\mathcal{X}}_0 \). Hence, it follows that (iii) is necessary in order that \( P = I \).

To establish the sufficiency of conditions (i)-(iii), we first observe that condition (i) implies that \( T(t) \) be strongly continuous for \( t > 0 \) and hence that \( \lim_{t \to 0} T(t)x = x \) for each \( x \in \mathcal{X}_0 \). Since by (iii) \( \mathcal{X}_0 \) is dense in \( \mathcal{X} \) and since by condition (ii) \( \| T(t)x \| \leq M \) for all \( t \in (0,1) \), we conclude that the limit exists and is equal to \( x \) for all \( x \in \mathcal{X} \). This completes the proof.
2.3 The Infinitesimal Operator

A. Introduction: In the last sections we studied semigroup operators and some of their properties. In this section we introduce and study the properties of an operator which plays a central role in semigroup theory. The operator we consider is called the infinitesimal operator, and, as we shall see, this operator reflects infinitesimal properties of the semigroup operator, and in terms of this operator the semigroup admits various representations.

We will assume throughout this section that the semigroup \( \{ \tau(t), t \geq 0 \} \) in a Banach space \( \mathcal{H} \) is strongly continuous for \( t > 0 \).

B. Some definitions and basic properties

Definition 7: Let
\[
A_2 = \frac{1}{2} \left[ \tau(t^2) - I \right], \quad t > 0
\]
The infinitesimal operator of the semigroup \( \{ \tau(t), t > 0 \} \) is the linear operator \( A_0 \) defined by
\[
A_0 x = \lim_{t \to 0} A_2 x, \quad x \in \mathcal{H}
\]
whenever the (strong) limit exists. The least closed extension of \( A_0 \), when it exists, is called the infinitesimal generator of the semigroup. We will denote the infinitesimal generator by \( \mathcal{A} \).

Definition 8: The set of elements \( x \in \mathcal{H} \) for which
\[
\lim_{t \to 0} A_2 x
\]
exists is the domain of \( \mathcal{A} \), which we
It is clear that $A_0$ (respectively $A$) is a linear operator, and that $D(A_0)$ (respectively $D(A)$) is a linear subspace of $X$.

We now prove a basic theorem concerning the differentiability of semigroups:

**Theorem 10:** If the semigroup operator $T(t)$ is strongly continuous for $t > 0$ then for $x \in D(A_0)$, the following differential equations are satisfied:

$$\frac{d}{dt} T(t)x = A_0 T(t)x = T(t) A_0 x , \quad t > 0$$

**Proof:** Using the semigroup property we have that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ T(t + \varepsilon) - T(t) \right] x = T(t) \frac{[T(\varepsilon) - I]}{\varepsilon} x = \frac{[T(\varepsilon) - I]}{\varepsilon} T(t) x$$

Now, the middle limit exists by hypothesis, and the above relation implies the existence of the other two limits. In particular, we have $T(t) x \in D(A_0)$ and we see that the right-hand derivative

$$\frac{d^+}{dt} T(t)x = T(t) A_0 x = A_0 T(t)x$$

exists. Finally, since $T(t) = T(\varepsilon) T(t - \varepsilon)$ we can write

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ T(t + \varepsilon) - T(t) \right] x = T(t - \varepsilon) \frac{[T(\varepsilon) - I]}{\varepsilon} x$$
Since $T(t)$ is strongly continuous (strong)
\[ \lim_{t \to 0} T(t - \frac{2}{2}) = T(0) \text{ and } \left[ T(\frac{2}{2}) - I \right] \chi = A_0 \chi \]
in norm. Hence
\[ \lim_{\epsilon \to 0} \frac{T(t - \frac{2}{2}) - T(t)}{\epsilon} = T(t) A_0 \chi \]
This establishes the existence of the derivative.

The relationship between $D(A_0)$ and the range space $\mathscr{R}_0$ of a semigroup is given by the following theorem.

**Theorem II:** $D(A_0)$ is dense in $\mathscr{X}_0$ and $\overline{D(A_0)} = \overline{\mathscr{X}_0}$

**Proof:** If $x \in \mathscr{X}_0$, then there exists a $y$ and $\alpha > 0$ such that $x = T(\epsilon) y$

Put
\[ y(\epsilon, \frac{2}{2}) = \int_{\frac{2}{2}}^{2} T(t)y \, dt \]

Then the following obtain:
\[ y(\epsilon, \frac{2}{2}) = T(\frac{2}{2}) \left( \int_{\frac{2}{2}}^{2} T(t)y \, dt \right) \in \mathscr{X}_0 \]

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} y(\epsilon, \frac{2}{2}) = \frac{1}{\frac{2}{2} - \frac{2}{2}} \int_{\frac{2}{2}}^{\frac{2}{2}} T(t)y \, dt = T(\frac{2}{2})y = x \]

(so that the elements $y(\epsilon, \frac{2}{2})$ are dense in $\mathscr{X}_0$)

\[ y(\epsilon, \frac{2}{2}) \in D(A_0) \]

To establish this we proceed as follows. Clearly
\[ A_{z}^{y}(z_{1}, z_{2}) = \frac{\int T(z_{1}) - T(z_{2}) y dt}{T(z_{1}) - T(z_{2})} \]

\[ = \frac{1}{2} \int_{z_{2}}^{z_{1}} \frac{\partial}{\partial t} T(z) y dt - \frac{1}{2} \int_{z_{2}}^{z_{1}} T(z) y dt \]

\[ = \frac{1}{2} \int_{z_{2}}^{z_{1}} T(z) y dt - \frac{1}{2} \int_{z_{2}}^{z_{1}} \frac{\partial}{\partial t} T(z) y dt \]

and

\[ \lim_{z_{2} \to 0} A_{z}^{y}(z_{1}, z_{2}) = \left[ T(z_{2}) - T(z_{1}) \right] y \]

Therefore, every element of type \( y(\bar{z}_{1}, \bar{z}_{2}) \in D(A_{0}) \) and \( A_{0} y(\bar{z}_{1}, \bar{z}_{2}) = \left[ T(z_{2}) - T(z_{1}) \right] y \).

Now (i) and (ii) show that \( D(A_{0}) \) is dense in \( X_{0} \) and hence \( X_{0} \subset D(A_{0}) \). For \( x \in D(A_{0}) \), we have \( \lim_{z \to 0} T(z) x = x \), so that \( x \in X_{0} \), and relation \( D(A_{0}) \subset X_{0} \) obtains. Their completes the proof of the theorem.

The next theorem, which is a representation theorem for semigroup operators, gives a relationship between the category of \( D(A_{0}) \) and the representation of the semigroup operator.

Before giving this theorem, we state a semigroup operator-theoretic interpretation of a result from the theory of Banach algebras which is required in the proof. [Cf. (5, pp. 287-288).]
Theorem 12: Let $T(t)$ be a semigroup operator on $(0, \infty)$ to $E(X)$. If $\lim_{t \to 0} \|T(t) - I\| = 0$, then $P$ is a projection operator. (ii) $T(t) = T(t)P = P T(t)$, and (iii) $T(t)$ is continuous in the uniform operator topology for $t > 0$ where $T(0) = P$ by definition. Further there exists an operator $U \in E(X)$ such that $U = uP = pu$ and $T(t) = e^{u^t}P = P \sum_{n=0}^{\infty} (u^t)^n$. If $P = I$ the series defines a group of operators on $(-\infty, \infty)$ which is continuous is the uniform operator topology.

We now prove

Theorem 13: If $D(A_0)$ is of second category in $X$ then (i) $\lim_{t \to 0} \|T(t) - I\| = 0$, (ii) $A_0 \in E(X)$ and $T(t) = e^{A_0 t}$.

Proof: By hypothesis $\lim_{t \to 0} A_{\frac{t}{2}} X$ exists on a set of second category in $X$. Therefore, the uniform boundedness theorem implies $\|A_{\frac{t}{2}}\| \leq M$ for some $M > 0$. Consequently $\|T(t) - I\| \leq M \frac{t}{2} \to 0$ as $\frac{t}{2} \to 0$.

By theorem 12 there exists an operator $A_0 \in E(X)$ such that $T(t) = e^{A_0 t}$, and it clear from this representation of $T(t)$ that $A_0$ is the infinitesimal operator of the semigroup $\{T(t), t > 0\}$.
Finally, we prove the following theorem.

**Theorem 8.4:** If \( \{ T(t), t > 0 \} \) is a semigroup of class \((C_0)\), then \( A_0 \) is a closed linear operator.

**Proof:** If \( x \in D(A_0) \) then

\[
[T(t) - I]x = \int_0^t T(\tau) A_0 x \, d\tau, \quad t > 0
\]  

(2.10)

Consider a sequence \( \{ x_n \} \subset D(A_0) \) with \( x_n \to x_0 \) and \( A_0 x_n \to y_0 \). Then

\[
\| T(\tau) [A_0 x_n - y_0] \| \leq M(\tau) \| A_0 x_n - y_0 \|
\]

goes to zero uniformly with respect to \( \tau \in [0, t] \).

Hence if we replace \( x \) by \( x_n \) in (2.10) and let \( n \to \infty \) we obtain

\[
[T(t) - I]x_0 = \int_0^t T(\tau) y_0 \, d\tau
\]

Consequently

\[
A_0^2 x_0 = \left[ T(\tau) - I \right] x_0 = \frac{1}{2} \int_0^t \frac{\partial}{\partial \tau} T(\tau) y_0 \, d\tau
\]

and

\[
\lim_{\tau_0 \to 0} A_0^2 x_0 = y_0 \quad \text{Hence} \quad x_0 \in D(A_0) \quad \text{and} \quad A_0 x_0 \to y_0. \]

This proves that the infinitesimal operator \( A_0 \) is closed.

We remarked earlier that the least closed extension of \( A_0 \) if it exists, is called the infinitesimal generator. A semigroup of class \((C_0)\), \( A_0 = A \) is itself the infinitesimal generator of the semigroup. Since we are primarily concerned with semigroups of class \((C_0)\) from now on we will refer to
infinitesimal operator.

2.4 The Resolvent Operator

A. Introduction: The resolvent of an operator \( \tau \) has been defined as \( R(\lambda; \tau) = (\lambda - \tau)^{-1} \) and the values of \( \lambda \) for which \( R(\lambda; \tau) \) is defined, i.e. the values of \( \lambda \) for which \( \lambda - \tau \) has a bounded inverse, form the resolvent set \( \rho(\tau) \) of \( \tau \). In this section we consider the resolvent operator arising in semigroup theory, and investigate some the relationships between a semigroup operator \( \tau(t) \), its infinitesimal generator \( A \) and the resolvent operator \( R(\lambda; A) \).

In section B we will consider resolvent operators associated with uniformly continuous semigroups and in section C we consider resolvent operators associated with strongly continuous semigroups.

B. The resolvent operator associated with uniformly continuous semigroups. Let \( \tau(t) \) be a measurable semigroup operator on \((0, \infty)\) to \(E(X)\). By Theorem 3 uniform measurability of \( \tau(t) \) implies uniform continuity of \( \tau(t) \), and if, in addition \( \lim_{t \to 0} \tau(t) = I \) then by theorem 12 there exists an \( A \in E(X) \) such that \( \tau(t) \) admits the representation \( \tau(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} \). We now prove the following theorem.
Theorem 3: If $A$ is an infinitesimal generator of a uniformly continuous semigroup $\{T(t), t \geq 0\}$ so that, $T(t) = e^{tA}$ then $R(\lambda; A)$ the resolvent of $A$ is the Laplace transform of $T(t)$.

Proof: We first observe that
\[
\limsup_{t \to \infty} \frac{1}{t} \log \|T(t)\| = \limsup_{t \to \infty} \frac{1}{t} \log \|e^{At}\| \leq \|A\|
\]
so that if we form the Laplace integral
\[
R(\lambda) = \int_0^\infty e^{-\lambda t} T(t) dt
\]
\[
= \int_0^\infty e^{-\lambda t} e^{At} dt
\]
is absolutely convergent for $\Re \lambda > \|A\|$.

For those values of $\lambda$ such that $\Re \lambda > \|A\|$ we can simply compute the value of the integral by substituting the powers series representation of $T(t)$ and integrating termwise. We have
\[
R(\lambda) = \int_0^\infty e^{-\lambda t} \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} dt = \sum_{n=0}^{\infty} \frac{A^n \lambda^{-n-1}}{n!} t^n
\]
By theorem N66,
\[
R(\lambda; A) = \sum_{n=0}^{\infty} \frac{A^n \lambda^{-n-1}}{n!} = (\lambda - A)^{-1}
\]
This function is holomorphic for $|\lambda| > \|A\|$ and $\sigma(A)$ the spectrum of $A$, is located inside the circle $|\lambda| = \|A\|$.

*Since $A$ is bounded $R(\lambda; A)$ is also holomorphic at infinity.*
Since the function \( e^A \) is an entire function of order one and type \( \alpha \leq \| A \| \), the integral can be taken along an arbitrary ray \( \alpha \gamma \xi^2 = \phi \) instead of along the real axis in the \( \hat{\mathbb{C}} \)-plane. Hence the integral converges absolutely for \( |e^{\lambda \xi^2} > \| A \| \), and \( R(\lambda; A) \) represents the resolvent \( R(\lambda; A) \) in this half-plane.

Given a resolvent operator \( R(\lambda; A) \) associated with a uniformly continuous semigroup \( \{ T(t), t \geq 0 \} \), the semigroup can be attained by application of the following theorem.

**Theorem 1:** If \( C \) is a simple closed rectifiable curve surrounding \( \sigma(A) \) in the positive sense, then

\[
T(t) = e^{\lambda t} = \frac{1}{2\pi i} \int_C e^{\lambda z} R(\lambda; A) \, d\lambda
\]

(2.11)

**Proof:** We first observe that \( C \) can be deformed into a circle \( |\lambda| = r > \| A \| \). If we put \( R(\lambda; A) \prod_{n=0}^{\infty} A^{-n} \) and integrate term by term, we obtain the power series representation of \( e^A \).

We remarks that the integral represents \( e^A \) for all complex values of \( \beta \).

---

\( \star \) A function \( \chi(z) \) is of order \( \eta \) if \( n = \lim_{n \to \infty} \sup \frac{\log \log M(n; \chi)}{\log n} \), where \( M(n; \chi) = \max \| \chi(\gamma e^{\lambda z}) \| \) and a function of order \( \eta \) is \( \eta \)-type if \( \sigma = \lim_{n \to \infty} \sup \gamma^{-n} \frac{\log M(n; \chi)}{\log n} \)
The resolvent operator associated with strongly continuous semigroups of operators, since our interest is mainly in semigroups in class \( (C_0) \), we shall only consider the resolvents associated with semigroups of this class. We refer to (5, Chap. XI) for a discussion of resolvent operators associated with semigroups of other basic classes.

We now consider the case where the semigroup is strongly continuous for \( t > 0 \) and is of class \( (C_0) \). Let the semigroup be of type \( \omega_0 \), and form the Laplace integral

\[
R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t) x d\lambda
\]

(2.12)

We remark that since in the strong case the infinitesimal generator \( A \) is an unbounded operator, the resolvent operator cannot be holomorphic at infinity, and the powers series representation of \( R(\lambda; A) \) that obtains in the uniform case has no analogue in the strong case.

We prove the following theorem:

**Theorem 17:** The integral (2.12) converges for \( \Re(\lambda) > \omega_0 \) regardless of \( x \in \mathcal{H} \). For each such \( \lambda \) the operator \( R(\lambda; A) \) defined by the integral is an endomorphism of \( \mathcal{H} \) with the following properties.

(a) \((\lambda - A)K(\lambda; A)x = x \) for each \( x \in \mathcal{H} \).

(b) \(R(\lambda; A)(\lambda - A)x = x \) for each \( x \in D(A_0) \).
(2) For any \( \lambda \) the range of \( R(\lambda; A) \) is dense in \( \chi \).

(3) If \( \mathcal{R}(\lambda; A) \not= \emptyset \) for a fixed \( \lambda \) then \( \chi \cong \mathbb{C} \).

Proof: From the results of Section 2.2C we know that the absolute of absolute convergence of (2.12) is given by \( |\sigma(x)| \leq \frac{\omega}{\omega} \), which proves the first statement of the theorem, and the integral defines a holomorphic function of \( \lambda \) in the half-plane \( \Re(\lambda) > \omega \), regardless of \( \chi \in \chi \). Also, from the corollary of the lemma given in Section 2.2C if \( \beta \) is any fixed number such that \( \beta > \omega \), there is a constant \( M(\beta) \) such that \( \| T(t) \| \leq M(\beta) \| e^{\beta t} \| \) for \( t > 0 \). Hence from (2.11) we have

\[
\| R(\lambda; A)x \| \leq M(\beta) \left( \Re(\lambda) - \beta \right)^{-1} \| x \|
\]

for \( \Re(\lambda) > \beta > \omega \). Hence the integral defines an endomorphism of \( \chi \). For \( \Re(\lambda) > \omega \), and the norm is uniformly bounded with respect to \( \lambda \) for \( \Re(\lambda) > \omega + \epsilon \), \( \epsilon > 0 \). This proves the second statement of the theorem.

To prove the properties (1)-(3), we proceed as follows. We first show that \( R(\lambda; A)x \in D(A) \) for all \( x \in \chi \) if \( \Re(\lambda) > \omega \). We have

\[
A^{-1} R(\lambda; A)x = \left[ \frac{T(\lambda) - I}{\hat{\xi}} \right] \int_0^\infty e^{-\lambda \tau} T(\tau) x d\tau
\]
\[
\mu \lim_{\lambda \to 0} R(\lambda; A) x = \lambda R(\lambda; A) x - x
\]

Hence

\[
\mu \lim_{\lambda \to 0} R(\lambda; A) x = \lambda R(\lambda; A) x - x
\]

Since \( \lambda \downarrow 0 \), \( T(\lambda) x = x \). Therefore, \( (\lambda - A) R(\lambda; A) x = x \)

and we have shown that \( \lambda R(\lambda; A) x \) always exists, and that \( (\lambda - A) x \) obtained. To establish (1b) we use theorem 10,

which permits to write, for \( x \in D(A) \)

\[
R(\lambda; A) A x = \int_0^\infty e^{-\lambda \tau} T(\tau) A x d\tau = \int_0^\infty e^{-\lambda \tau} \frac{d}{d\tau} [T(\tau) x] d\tau
\]

\[
= \mu \lim_{\gamma \to 0} [e^{-\gamma \frac{d}{d\tau}} T(\tau) x + \gamma \int_0^\infty e^{-\lambda \tau} T(\tau) x d\tau - x + \lambda R(\lambda; A) x
\]

Hence \( R(\lambda; A) (\lambda - A) x = x \) for each \( x \in D(A) \)

Since these basic relations of a resolvent operator are satisfied (N.B., Sec.5E3), we can say that the operator \( R(\lambda; A) \)

defined by the integral (2.12) is indeed the resolvent of the infinitesimal generator \( A \).
To show that the range of \( R(\lambda; A) \) is dense in \( X \) we establish a contradiction utilizing Lerch's theorem. Let us assume, therefore, that the range of \( R(\lambda; A) \) is not dense in \( X \). In this case there exists a bounded linear functional \( x^* \) such that \( x^*(R(\lambda; A)x) = 0 \) for all \( x \). Now from theorem No. 78', we know that the resolvent of \( A \) being the resolvent of a closed unbounded linear operator, satisfies the first resolvent equation

\[
R(\lambda; A)x - R(\mu; A)x = (\mu - \lambda)R(\lambda; A)R(\mu; A)x
\]

for all \( \lambda \) and \( \mu \) whose real parts are greater than \( \omega_0 \). Hence from the above we can conclude that \( x^*(R(\mu; A)x) = 0 \) for all \( x \) and all \( \mu \) with \( \Re(\mu) > \omega_0 \). We then have

\[
\int_{\omega_0}^{\infty} e^{-t^2} x^*(I(t)x) d\frac{t}{2} = 0
\]

as a function \( \lambda \) for all \( x \in X \). Applying Lerch's theorem we obtain that \( x^*(I(t)x) = 0 \) since it is a continuous function if \( t \) must vanish identically for all \( t \). In particular, \( x^*(x) = 0 \) for all \( x \in X \). From this contradiction we conclude that (3) holds.

To prove (3) we have only to observe that if \( R(\lambda; A)x = 0 \) for a particular choice of \( \lambda \) and \( x \), then the first

Lerch's theorem states that if the Laplace transform \( \int f(\lambda) \) is holomorphic in \( \Re(\lambda) > \sigma_0 \), the abscissa of ordinary convergence and \( \int f(\lambda) = 0 \) for \( \lambda = \lambda_0 + n, n = 1, 2, 3, \ldots \).

\[
f(\lambda) = 0
\]
resolvent equation shows that \( k(\lambda; \Lambda) \) \( x = \Theta \) for all \( \mu \) with \( k(\lambda; \Lambda) > 0 \). By the inversion theorem (Theorem A.2), this would imply that \( T(0) \) \( x = \Theta \) for all \( \gamma > 0 \). In particular, \( x = \Theta \) since \( T(0) \) is not nilpotent, i.e., the null operator for any \( \gamma > 0 \).

We now give an integral formula, which gives a representation of \( T(t) \) in terms of the resolvent operator \( R(\lambda; \Lambda) \).

**Theorem 18.** For every \( x \in X \), \( t > 0 \) and for \( \gamma = \max(0, \omega) \),

\[
\int_0^t T(\zeta) x d\zeta = \lim_{\omega \to \infty} \frac{1}{i2\pi} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t} R(\lambda; \Lambda) x d\lambda \quad (2.13)
\]

with the limit existing uniformly with respect to \( t \) in any finite interval. For every \( x \in D(A), \quad \gamma > 0 \) the semigroup operator \( T(t) \) admits the representation

\[
T(t) x = \lim_{\omega \to \infty} \frac{1}{i2\pi} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t} R(\lambda; \Lambda) x d\lambda \quad (2.14)
\]

with the limit existing uniformly with respect to \( t \) in any interval \( (0, 1/\epsilon), \quad \epsilon > 0 \). For \( t = 0 \) the limit is \( \Theta \).

**Proof:** To establish the first part of the theorem we put \( x(t) = \int_0^t T(\zeta) x d\zeta \), which is an absolutely continuous function of \( t \) (2.13) now follows from Theorem N12.

To establish (2.14), put
To establish (2.24) we put \( y(t) = T(t)x \), and apply theorem 11. We first note the following relations

\[(a) \quad \int_0^T T(\tau) A x \, d\tau = \left[ T(T) - I \right] x\]

\[(b) \quad R(\lambda; A) A x = \left[ \lambda R(\lambda; A) - I \right] x\]

Relation (a) can be obtained utilizing theorem 10, and (b) was given in theorem 17. If we use these relations in (2.13) we obtain

\[
\left[ T(Ct) - I \right] x = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{C + \alpha \omega} \frac{e^{\lambda t} R(\lambda; A) x \, d\lambda}{\lambda} = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{C - \alpha \omega} \frac{e^{\lambda t} x \, d\lambda}{\lambda}
\]

The second limit is simply \( x e^{i t} \) but \( \frac{1}{2} x e^{i t} = 0 \) and the limit exists uniformly with respect to \( t \) in \((C, i\epsilon)\).
2.6 Representation of Semigroup Operators

The exponential formulas

1. Introduction. In Chapter 1 we pointed out that analysis of decay can be regarded as being concerned with the study of the solution of the operator functional equation

$$ T(s + t) = T(s)T(t), \quad s, t > 0 \text{ with } T(0) = I $$

where $T(t)$ is an operator-valued function on $(-\infty, \infty)$ to $\mathcal{L}(\mathcal{X})$.

Further, we remarked that the solution of the above equation was, in some sense, an exponential function. We showed in Section 2.4B that when $T(t)$ is continuous in the uniform operator topology, there exists a bounded operator $A$ on $\mathcal{X}$ such that

$$ T(t) = e^{At} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} $$

For the case of a semigroup operator continuous in the strong operator topology we have shown thus far that $T(t)$ admits the representation given by (2.14).

In this section we continue our study of the representation of semigroup operators, and we prove certain "exponential formulas" which give various interpretations of the semigroup operator as an exponential function. As it turns out, there are a rather large number of exponential formulas. For completeness we list all known exponential formulas.
Formulas (2.16), (2.16) and (2.17) with $\lambda = 0$ hold for semigroups of class $\mathcal{C}(\epsilon)$ and formulas (2.17)-(2.25) hold for semigroups of class $(\omega, A)$. Formula (2.16) is often referred to as the "first exponential formula". In general, the limit relations hold uniformly with respect to $\tau$ in each interval of the form $\left(\epsilon, \frac{1}{\epsilon}\right)$. In the case of formulas (2.16) and (2.20) the interval is $\left(0, \frac{1}{\epsilon}\right)$ for formula (2.16) the interval is $\left(0, \frac{1}{\epsilon}\right)$, and for formula (2.17) the interval is $\left(\alpha, \alpha + \frac{1}{\epsilon}\right)$. In formulas (2.20)-(2.22) $\gamma > m_{\omega}(0, \omega)$, where $\omega_0$ is the type of the semigroup.

In the last section (Theorem 18), we proved formulas (2.20) and (2.21), and in this section we will prove formulas (2.16), (2.16), (2.17) (2.23) and (2.24). In subsection B we give a proof of (2.15) due to N. Dunford and I. E. Segal (C) and in subsection C we use probabilistic methods to prove formulas (2.16), (2.17), (2.23) and (2.24). The interested reader will find proofs of the other remaining exponential formulas in (5).

* A recent paper on the first exponential formula is that of H. w. [5]
B. Representation of Contraction Semigroups.

We now prove a representation theorem for contraction semigroups of class \( (C_0) \) in an arbitrary Banach space. Because of the occurrence of contraction semigroups in many applications, this representation is of particular importance.

**Theorem 19:** If \( \{ T(t), t > 0 \} \) is a contraction semigroup with infinitesimal generator \( A \), then \( D(A) \) is dense in \( X \) and \( T(t) \) admits the representation

\[
T(t) = \lim_{\rho \to 0} \exp \left\{ A_2 t \right\} \rho X, \quad x \in X, \quad t > 0
\]

the limit existing uniformly with respect to \( t \) in every finite interval.

**Proof:** We first observe that the definition of \( A_2 \) enables us to write

\[
\exp \left\{ A_2 t^2 \right\} = \exp \left\{ c_{1/2} \right\} \exp \left\{ T(t) t^{1/2} \right\}
\]

Hence

\[
\| \exp \left\{ A_2 t^2 \right\} \| \leq e^{-t/2} \exp \left\{ \| T(t) \| t^{1/2} \right\} \leq e^{-t/2} e^{t/2} = 1,
\]

since \( T(t) \) is a contraction operator. Therefore

\[
\| \exp \left\{ A_2 \right\} \| = 1.
\]

(2.27)
which shows that \( \exp \{ A_2(t) \} \) is also a contraction operator for all \( t \geq 0 \).

Let \( x \in D(A) \), and consider the operator \( \exp \{ A_2(t) \} \).

For \( x \in D(A) \), it follows from theorem 10 that
\[
\exp \{ A_2(t) \} x = \text{a differentiable function of } t.
\]

Since both \( A_2 \) and \( A \) commute with the semigroup operator \( T(t) \), we can write
\[
\begin{align*}
\left[ T(t) - \exp \{ A_2(t) \} \right] x &= \int_0^t \frac{d}{d\tau} \left[ \exp \{ A_2(t-\tau) \} T(\tau) x \right] d\tau \\
&= \int_0^t \exp \{ A_2(t-\tau) \} T(\tau) \left[ A - A_2 \right] x d\tau
\end{align*}
\]

Since \( T(t) \) and \( \exp \{ A_2(t) \} \) are contraction operators, it follows from (2.28) that
\[
\left\| \left[ A - A_2 \right] x \right\| \leq \int_0^t \left\| \exp \{ A_2(t-\tau) \} \right\| \left\| \left[ A - A_2 \right] \right\| d\tau
\]

but the last expression tends to zero as \( t \to \infty \) uniformly with respect to \( x \) in any finite interval. Hence we have shown that the representation (2.16) is valid for \( x \in D(A) \).

To show that (2.15) is valid even if \( x \in A \), we proceed to follow. Since \( \lim_{t \to 0} T(t) x = x \) for each \( x \in A \),

the space \( X \) is dense in \( X \). However, theorem 11 asserts that \( D(A) \) is also dense in \( X \). Hence it follows from (2.30) that the representation (2.15) is valid for \( x \in A \).
interval.

C. **Probabilistic approach to the exponential formulas.**

The use of probabilistic methods in the study of exponential formulas was introduced by J. Riesz, who proved the first exponential formula using these methods (cf. (8), p. 312). Probabilistic methods can be employed are since the exponential formulas are essentially summability methods, and the connection between summability methods and probability concepts has long been known to probabilists*. Recently, K.I. Chung [5], using probabilistic methods, has provided a simple unified approach to the exponential formulas, and it is the work of Chung that is presented in this section. These methods do not require an extensive knowledge of probability theory, however in order to assist the reader. It is not familiar with certain notions of probability theory we first present some definitions and results that will be used in the text. For other notions and results we refer to the reader to the treatise of H. Loeve (8).

Let \((\Omega, \mathcal{F}, \mu)\) be a probability measure space, and let \((\mathcal{B}, \mathcal{M})\) be a measurable space where \(\mathcal{B}\) is a Banach space and \(\mathcal{M}\) is the \(\sigma\)-algebra of \(\mathcal{B}\) and \(\mathcal{M}\) is the \(\sigma\)-algebra of \(\mathcal{B}\) subsets of \(\mathcal{B}\).

---

*We refer to S. Berstein's proof of the Weierstrass approximation theorem [cf. (6)] and a forthcoming paper of L. Schmetterer in Monatsh. Math.
A mapping \( y(\omega) : \Omega \rightarrow \mathcal{K} \) is called a generalized random variable \( \{ \{ \omega : y(\omega) \in \mathcal{B} \} : \mathcal{B} \in \mathcal{B} \} \in \mathcal{A} \). If \( \mathcal{K} = \mathbb{R} \) (the real line) we call \( y(\omega) \) an ordinary random variable, or simply a random variable. The mean or expectation of \( y(\omega) \) is defined as

\[
\mu = E \{ y(\omega) \} = \int_{\Omega} y(\omega) \, d\mu(\omega)
\]

The variance of \( y(\omega) \) is given by

\[
\operatorname{var} \{ y(\omega) \} = E \{ [y(\omega) - \mu]^2 \} = E \{ y^2(\omega) \} - \mu^2
\]

Let \( \{ \omega(\lambda) \} \) be a family of random variables, indexed by \( \lambda \), which may depend on a real parameter \( \tau \), and let \( \omega(\xi), \xi \in (-\infty, \infty) \) be a strongly measurable function with values in a Banach space. We state the following lemma.

**Lemma 2.** Suppose that

1. as \( \lambda \to \infty \), \( \omega(\lambda) \) converges in probability (i.e. measure) to \( \omega(\xi) \),
2. for some \( r > 1 \), \( \sup_{\lambda} \| \omega(\xi(\lambda)) \|^r \) is bounded in \( \lambda \),
3. \( \omega(\xi) \) is strongly continuous at \( \xi = \tau \),

then

\[
\lim_{\lambda \to \infty} E \{ \| \omega(\tau(\lambda)) - \omega(\lambda) \| \} = 0
\]

(2.28)
Further more, if \( \{ z_{\lambda}(\omega) \} \) depends on the parameter \( \lambda \) and if (i), (ii) and (iii) hold uniformly for \( \lambda \) in a finite interval, then the limit relation (2.29) also holds uniformly in the same finite interval.

We remark that (2.29) implies \( \omega(\mathcal{A}) \sim \lim_{\lambda \to \infty} E \{ \omega(z_{\lambda}) \} \) and it is their result which will be used to prove the exponential formulas.

We will also need the following lemma.

**Lemma 3.** Let \( \{ y_{k}(\omega) \} \) be a sequence of independent and identically distributed random variables, introduce the sums

\[ \omega_{n}(\omega) = \sum_{k=1}^{n} y_{k}(\omega) \]

and let \( \nu \) be a nonnegative, integer-valued random variable which is independent of the \( y_{k}(\omega) \).

If \( y_{l}(\omega) \) and \( \nu \) have finite variables, then

\[
E \left\{ S_{\nu}(\omega) \right\} = E \left\{ \nu \right\} E \left\{ y_{l} \right\}
\]

\[
D^{2} \left\{ S_{\nu}(\omega) \right\} = E \left\{ \nu \right\} D^{2} \left\{ y_{l} \right\} + D^{2} \left\{ \nu \right\} E^{2} \left\{ y_{l} \right\}
\]

We refer the reader to [1] for the proofs of these lemmas.

We now prove the "first exponential formula", that is formula (2.17).

**Theorem 20.** Let \( \{ \tau(\mathcal{A}), \mathcal{A} \supseteq \circ \} \) be a semigroup in \( \mathcal{X} \) which is strongly continuous for \( \mathcal{X} \subseteq [0, \infty] \), and let \( \mathcal{A} \supseteq \alpha \supseteq \circ \). Then for every \( x \in \mathcal{X} \)

\[
\tau(\mathcal{A}) x = \lim_{\nu \to 0} \nu \supseteq \left\{ A_{\nu}(\mathcal{A} - \alpha) \right\} \tau(\alpha) x
\]  

(2.17)
uniformly for \( t \) in any finite interval.

**Proof:** Let \( z > 0 \) and put \( \lambda = (t - \alpha)/z \).

Let us now assume that \( Z_\lambda(\omega) \) has a Poisson distribution, that is

\[
P_{\lambda} \left\{ Z_\lambda(\omega) = \alpha + n \right\} = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2 \ldots
\]

In this case

\[
E \{ Z_\lambda(\omega) \} = \alpha + \lambda z = t
\]

and

\[
D^2 \{ Z_\lambda(\omega) \} = \lambda z^2 - (t - \alpha) z^2
\]

An application of Chebyshev's inequality gives

\[
P \left\{ \left| Z_\lambda(\omega) - \alpha \right| > \varepsilon \right\} \leq \frac{D^2 \{ Z_\lambda(\omega) \}}{\varepsilon^2}
\]

hence \( Z_\lambda(\omega) \) converges to \( t \) in probability as \( z \to 0 \)

For fixed \( \alpha \in X \) let

\[
f(\alpha) = \left[ T(\pi) - T(\alpha) \right] \alpha \to T \geq \alpha
\]

Since \( T(\pi) \) is strongly continuous for \( t \geq \alpha \), there exists a finite quantity \( M \) such that \( \| T(\pi) \| \leq M \) for \( \pi \in [\alpha, \max(\alpha+1, 2\alpha)] \); and we can, without loss of generality, take \( M > 1 \).

In this case

\[
\| T(\pi) \| \leq M^{1+t} \quad \text{for all } t \geq \alpha
\]

Hence

\[
\| f(\pi) \|^2 \leq 4 M^2 \| x \|^2 M^{-\alpha}
\]
we have from Lemma 2.8, the distribution of \( z \) is 
\[
x(x) = \sum_{n=1}^{\infty} \frac{1}{\pi} \sin \left( \frac{\pi x}{n} \right) \frac{1}{n^2} \] 
and the fact

\[
x(\chi \circ \phi) \leq \sum_{n=1}^{\infty} \frac{1}{\pi} \sin \left( \frac{\pi x}{n} \right) \frac{1}{n^2} \leq 1,
\]

we see that \( \phi \) is bounded in \( L^2 \). Since any function \( \phi \) is uniformly \( \ell^2 \) and uniformity of \( \ell^2 \) is in any finite interval, it follows that \( \phi \) is bounded in \( L^2 \).

Since

\[
x \left( \left( \chi \right) \right) \leq \sum_{n=1}^{\infty} \frac{1}{\pi} \sin \left( \frac{\pi x}{n} \right) \frac{1}{n^2} \leq 1,
\]

we have

\[
x \left( \left( \chi \right) \right) \leq \sum_{n=1}^{\infty} \frac{1}{\pi} \sin \left( \frac{\pi x}{n} \right) \frac{1}{n^2} \leq 1.
\]
\[
T(t) = \lim_{\lambda \to \infty} E \left\{ \frac{1}{\lambda} \right\} T(2\lambda(\omega) x) \\
= \lim_{\lambda \to \infty} \exp \left\{ C(\lambda - 1) \right\} \frac{T(\lambda) x}{\lambda} \\
= \lim_{\lambda \to \infty} \exp \left\{ \lambda \right\} A_\lambda (\lambda - 1) \frac{T(\lambda) x}{\lambda}
\]

The next exponential formula we prove is formula (2.16), which was first obtained by P. G. Kemeny (7).

Theorem 21. Let \( \{T(t), t \geq 0\} \) be a semigroup in \( X \) which is strongly continuous for \( t \in [0, 1] \)
and \( T(0) = I \). Then for every \( x \in X \)
\[
T(t) x = \lim_{n \to \infty} \left\{ (C - t I + t(K/n)) \right\}^n x \quad (2.16)
\]
uniformly for \( t \) in \( [0, 1] \).

Proof: In this case, let \( \{X_n(\omega), n \geq 1\} \) be a family of random variables with the binomial distribution, that is,
\[
p \left\{ X_n(\omega) = k \right\} = C \binom{\lambda}{k} (1 - x)^{\lambda - k}, \quad k \in \{0, 1, \ldots, \lambda\}
\]
We then have
\[
D \left\{ \frac{X_n(\omega)}{n} \right\} = C(1 - x) \quad /n
\]
From Chebyshev's inequality, we have that \( \frac{1}{n} \sum_{i=1}^{n} Y_i \) converges in probability to \( \lambda \) as \( n \to \infty \), uniformly for \( \lambda \in [\omega, \bar{\omega}] \). Furthermore, since \( \|T(t)\| \leq M \), the uniform boundedness theorem (Theorem M16) permits us to write

\[
E \left\{ \|T(\frac{1}{n} \sum_{i=1}^{n} Y_i)\|^2 \right\} \leq M^2
\]

We can now use Lemma 2. We have

\[
E \left\{ T(\frac{1}{n} \sum_{i=1}^{n} Y_i) \right\} = \sum_{k=0}^{\infty} \binom{n}{k} t^k (1-t)^{n-k} T(t^{1/n})^k
\]

\[
= \sum_{k=0}^{\infty} \binom{n}{k} (1-t)^n \int [t T(t^{1/n})]^k dx
\]

\[
= \left\{ (1-t) I + t T(\frac{1}{n}) \right\}^n
\]

Therefore

\[
T(\lambda) x = \lim_{n \to \infty} E \left\{ T(\frac{1}{n} \sum_{i=1}^{n} Y_i) x \right\}
\]

\[
= \lim_{n \to \infty} \left\{ (1-t) I + t T(\frac{1}{n}) \right\}^n x
\]

The next two exponential formuals give representations of a semigroup operator in terms of the associated resolvent operator. We shall now assume that \( \{T(\lambda), \lambda > 0\} \) is strongly continuous, and that \( \|T(\lambda)\| \) is bounded as \( \lambda \to 0 \).

In this case, as we have shown earlier, there exists positive constants \( M \) and \( \beta \) such that

\[
\|T(t)\| \leq M e^{\beta t}, \quad t \in [0, \infty].
\]
\[ D_{2n} \cap \{ \alpha \} \subseteq \{ \alpha \} \cap \{ \alpha \} \cap \{ \alpha \} \]

and

\[ \{ \alpha \} \cap \{ \alpha \} \cap \{ \alpha \} \]

\[ \{ \alpha \} \cap \{ \alpha \} \cap \{ \alpha \} \]

In this case, let \( \gamma (x) \) be a sequence of independent and identically distributed random variables with distribution function \( \gamma (x) \), and put \( \gamma (x) \) to be a sequence of independent and identically distributed random variables with distribution function \( \gamma (x) \).

\[ \gamma (x) = \gamma (x) \]

for \( x \in \mathbb{R} \) and \( \gamma (x) > 0 \). Let \( \gamma (x) \geq 0 \).

The negative exponential distribution is the distribution of the exponential distribution. Let \( \gamma (x) \geq 0 \),

\[ \gamma (x) = \gamma (x) \]

The negative exponential distribution is the representation

\[ \gamma (x) \geq 0 \]
We will need the following lemma.

**Lemma 4.** For \( \beta > \beta \), \( \mathbb{E} \left\{ \mathcal{T} \left( \sum_{k} \zeta_{k}(\lambda) \right) \right\} = \left[ \lambda \mathcal{R}(\lambda) \right]^n \)

**Proof:*** For each \( k \)

\[
\mathbb{E} \left\{ \mathcal{T} \left( \zeta_{k}(\lambda) \right) \right\} = \sum_{0}^{\infty} \lambda e^{-\lambda t} \mathcal{R}(\lambda) = \lambda \mathcal{R}(\lambda)
\]

Since the random variables \( \zeta_{k}(\lambda) \) are independent, the operator-valued random variables \( \mathcal{T}(\zeta_{k}(\lambda)) \) are also independent. Therefore we can write

\[
\mathbb{E} \left\{ \mathcal{T} \left( \sum_{k=1}^{n} \zeta_{k}(\lambda) \right) \right\} = \mathbb{E} \left\{ \prod_{k=1}^{n} \mathcal{T}(\zeta_{k}(\lambda)) \right\}
\]

\[
= \prod_{k=1}^{n} \mathbb{E} \left\{ \mathcal{T}(\zeta_{k}(\lambda)) \right\}
\]

\[
= \left[ \lambda \mathcal{R}(\lambda) \right]^n
\]

The next theorem is the semigroup formulation of the Fourier-Widder theorem [5], p. 234, and proves formula (2.24).

**Theorem 2.** Let \( \{ T(t) : t > 0 \} \) be a strongly continuous semigroup in \( \mathcal{H} \) and \( t > 0 \) for every \( t > 0 \) and \( x \in \mathcal{H} \), \( T(t) \). Then the representation

\[
T(t)x = \lim_{n \to \infty} \left[ \frac{1}{\lambda^n} \mathcal{R}(\frac{n}{t}; A) \right]^n x,
\]

where \( \mathcal{R}(\lambda; A) \) is the resolvent of the infinitesimal generator of \( T(t) \).
Proof: For each \( n \), let \( \xi_{nk} \), \( 1 \leq k \leq n \), be independent and identically distributed random variables, with distribution function \( F_{\lambda} \) with parameter \( \lambda = \sqrt{n} \); and let \( S_n = \sum_{k=1}^{n} \xi_{nk} \). In this case \( E \{ S_n \} = \lambda \) and \( \text{var} \{ S_n \} = \lambda^2 / n \). As a consequence of Chebychev's inequality, we have that \( S_n \) converges to \( \lambda \) in probability as \( n \to \infty \) uniformly over \( \lambda \) in any finite interval.

Now

\[
P_n \left\{ -\frac{1}{\sqrt{n}} S_n(\lambda) \geq \frac{1}{\sqrt{n}} \right\} = \int_{-\infty}^{\infty} \frac{\lambda^{n/2}}{(\sqrt{n})^n} e^{-\lambda t} \, dt \geq \frac{1}{n} \text{ for } n \to \infty.
\]

Using this result, together with \( \| \tau(\lambda) \| \leq c \sqrt{n} \) and \( \lambda \to \infty \), we have

\[
\varepsilon \geq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\lambda^{n/2}}{(\sqrt{n})^n} e^{-\lambda t} \, dt \, dt = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{\lambda^{n/2}}{(\sqrt{n})^n} e^{-\lambda t} \, dt \right) \, dt \leq \int_{-\infty}^{\infty} \frac{\lambda^{n/2}}{(\sqrt{n})^n} \, dt = \int_{-\infty}^{\infty} \frac{\lambda^{n/2}}{(\sqrt{n})^n} \, dt \leq \frac{C}{n}.
\]
Since this bounded in \( n \), uniformly for \( x \) in any finite interval Lemma 2 is applicable, and Lemmas 2 and 3 yield

\[
\tau(t) = \lim_{n \to \infty} \mathbb{E}\left\{ \tau(S_n) x \right\} = \lim_{n \to \infty} \left[ \frac{\alpha}{\lambda} R \left( \frac{\lambda}{\alpha}; \frac{n}{\alpha} \right) \right],
\]

Finally, we prove exponential formula (2.23), which is due to A.S. Phillips (9).

**Theorem 23.** Let \( \{ T(x), x > 0 \} \) be a strongly continuous semigroup in \( X \) for \( x > 0 \). For every \( t > 0 \) and \( x \in X \), \( T(t) \) admits the representation

\[
T(t) x = \lim_{\lambda \to \infty} \exp \left\{ \mathbb{E} R \left( x; \lambda \right) \lambda I \right\} t x
\]

uniformly for \( x \) in any finite interval.

**Proof:** Let \( \lambda > 0 \) be fixed, and let the random variable \( \nu = \nu(\lambda) \) have the Poisson distribution, i.e.,

\[
\mathbb{P} \left\{ \nu = n \right\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, \ldots
\]

In this case \( \mathbb{E} \left\{ \nu \right\} = \lambda t \). Let the random variables \( T_x \) and \( S_n \) be defined as in theorem 22, and let \( \nu \) be independent of them. Now, using \( \mathbb{E} \left\{ S_n \right\} = \frac{n}{\lambda} \) and \( \mathbb{V} \left\{ S_n \right\} = \frac{n}{\lambda^2} \), Lemma 3, we have

\[
\mathbb{E} \left\{ S_{\nu} \right\} = \lambda t \left( \frac{\lambda}{\nu} \right) = t
\]

\[
\mathbb{V} \left\{ S_{\nu} \right\} = \lambda t \left( \frac{\lambda^2}{\nu^2} \right) + \lambda t \left( \frac{\lambda}{\nu} \right)
\]

\[
= \frac{2 \lambda t}{\nu}
\]
\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} y^k
\]
and
\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x^k
\]
for any \( x, y \in \mathbb{R} \). Hence, we can apply Lemma 1, Theorems 2 and 4.

The integral on the right-hand side is finite. Hence, we can apply Lemma 1, Theorems 2 and 4.

This, however, is bounded in a uniform manner, for \( x \) in any

\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} y^k
\]

The expression is bounded by

\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} x^k
\]

Since

\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} y^k
\]

Any finite integral, we now have

in \( \gamma \) to the productivity as \( \gamma \) to the productivity, as \( \gamma \) to the productivity. So a consequence of Chebyshev's inequality, we have

61
REFERENCES


CHAPTER 3

BASIC THEORY II:

Generation of Semigroups of Operators.

3.1 Introduction

In Chapter 2 we pointed out the central role played by the infinitesimal generator in semigroup theory. It is clear, therefore, that a very important problem in semigroup theory is that of determining when a given closed linear operator is the infinitesimal generator of a semigroup of operators of class ($C_0$). This problem is of great importance in applied problems, for in these cases we often derive an 'infinitesimal generators' on the basis of a physical model, and it is, therefore, important to know what conditions that operator must satisfy in order that it be the infinitesimal generator of a semigroup of operators, for it is the semigroup operator which enters as a 'solution operator' in many applied problems.

There are five sections in this chapter. In section 3.2 we consider the generation of semigroups of operators of class ($C_0$) in an arbitrary Banach space. The Voller-Miyadera-Phillips generation theorem is stated and proved, and the celebrated Hille-Yosida theorem is stated as a corollary of the above theorem. Section 3.3 considers the generation of groups of operators. Because of the importance of Hilbert space as a setting for many problems in mathematical physics, we consider
in Section 3.4 the generation of semigroups in Hilbert space. In section 3.5 the uniqueness of the generation problem is considered; and finally, in section 3.6, we make brief reference to other studies on the generation of semigroups.
3.2 The Basic Generation Theorem

In this section we study the generation of semigroups of class \((C_0)\) in Banach spaces. In 1948, E. Hille [16] and K. Yosida [16] obtained a theorem which gave conditions under which a closed linear operator was the infinitesimal generator of a semigroup of operators of class \((C_0)\). For years their theorem was the basic generation theorem of semigroup theory. In 1953 W. Feller [1], I. Miyadera [8], and R. S. Phillips [19] obtained a generation theorem for semigroups which yields the Hille-Yosida theorem as a corollary.

We now state and prove the Feller-Miyadera-Phillips theorem, which we will refer to as the basic generation theorem.

**Theorem 1 (Feller-Miyadera-Phillips).** A necessary and sufficient condition that a closed linear operator \(A\) generate a semigroup \(\{T(t), t \geq 0\}\) of class \((C_0)\) such that \(\|T(t)\| \leq M\) is that \(\{A\}\) be dense in \(X\), and

\[
\|\int_0^t R(t, 0; \lambda; u)\| \leq M \lambda^{-n} \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad n = 1, 2, \ldots.
\]

In the proof of the theorem we recognize the following lemma, which we state without proof.

**Lemma 1.** If \(f(\lambda)\) is the Laplace transform of a function \(g(t)\), then \(\|g(t)\| \leq M\) for almost all \(t > 0\) if and only if for all \(\lambda > 0\) and \(n = 0, 1, 2, \ldots\) we have

\[
\lambda^{n+1} \|f^{(n)}(\lambda)\| \leq M_n.
\]

We remark that a semigroup of class \((C_0)\) will be bounded in norm, hence the above lemma is applicable; and the Laplace transform of \(T(t)\), namely \(R(\lambda; A)\), satisfies the
irregularity
\[ \lambda^{n+1} \| R^{(n)}(\lambda;u) \| \leq M_n \]
for \( \lambda > 0 \) and \( n = 0, 1, 2, \ldots \).

**Proof of Theorem.**

**Proof.** (i) We first observe that the irregularities
\[ \lambda^{n+1} \| R^{(n)}(\lambda;u) \| \leq M_n \]
and
\[ \| R^{(n)}(\lambda;u) \|^n \leq M_n \]
are equivalent by virtue of the following valid formula:
\[ R^{(n)}(\lambda;u) = (-1)^n n! \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left( \lambda^k \right) \]
(3.1)

Now, if the semigroup \( \{ T(t), t \geq 0 \} \) is of class \((C_{\gamma})\), and its
infinitesimal generator is \( \mathcal{A} \), then \( \mathcal{D}(\mathcal{A}) \) is dense in \( \mathcal{H} \).

Further, for \( \lambda > 0 \)
\[ R(\lambda;u) x = \int_0^\infty e^{-\lambda t} T(t) x \, dt; \]
and differentiation yields
\[ R^{(n)}(\lambda;u) x = \int_0^\infty (-t)^n e^{-\lambda t} T(t) x \, dt. \]

Since \( \| T(t) \| \leq M \), we have
\[ \| R^{(n)}(\lambda;A) \| \leq \int_0^\infty t^n e^{-\lambda t} M dt = \frac{Mn!}{\lambda^{n+1}}, \]
for \( n = 1, 2, \ldots \). But because of the equivalence formula (3.1), this
shows that \( \| R(\lambda;u) \|^n \leq M \lambda^{-n} \)

(ii) **Sufficiency:** The proof of sufficiency is based on
the exponential formula
\[ T(t) x = \lim_{\lambda \to \infty} \exp \left\{ \left[ R(\lambda;A) - \lambda I \right] t \right\} x, \]
which can be written as
\[ T(t) x = \lim_{\lambda \to \infty} \exp \left\{ \lambda^2 R(\lambda;A) - \lambda I \right\} x, \]
where we have put
\[ B_\lambda = \lambda^2 R(\lambda;A) - \lambda I. \]
Hence we introduce the operator $B_\lambda$, as defined above, with $U$ replacing $A$. It is clear that the operators $B_\lambda$ commute.

Next we note that for $x \in H^D(U)$, $B_\lambda x = \lambda R(\lambda; U) U x$, since for $x \in D(U)$, $R(\lambda; U) x = \lambda^{-1} R(\lambda; U) U x$.

We also have that $\lim_{\lambda \to \infty} B_\lambda x = U x$.

We now introduce the approximating semigroup operator

$$S_\lambda(t) = \exp \left\{ B_\lambda t \right\} = e^{-\lambda^2 t} \sum_{n=0}^\infty \frac{(\lambda^2 t)^n}{n!} \left[ R(\lambda; U) \right]^n$$

Using the inequality $\| R(\lambda; U) \|^n \leq M \lambda^{-n}$, the above yields

$$\| S_\lambda(t) \| \leq e^{-\lambda^2 t} \sum_{n=0}^\infty \frac{(\lambda^2 t)^n}{n!} M \lambda^{-n} = e^{-\lambda^2 t} \sum_{n=0}^\infty \frac{(\lambda^2 t)^n}{n!} \frac{M}{\lambda^n} = e^{-\lambda^2 t} M e^{\lambda^2 t}$$

for $\lambda > 0$. Hence $\| S_\lambda(t) \| \leq M$. Further, we have for $x \in H^D(U)$

$$[S_{\lambda_1}(t) - S_{\lambda_2}(t)] x = \int_0^t \left[ \partial_{\lambda_1} S_{\lambda_1}(\tau; \lambda_1) - \partial_{\lambda_2} S_{\lambda_2}(\tau; \lambda_2) \right] S_{\lambda_1}(\tau) d\tau$$

$$\quad = \int_0^t \left[ S_{\lambda_2}(\tau) - S_{\lambda_1}(\tau) \right] S_{\lambda_2}(\tau) \left[ B_{\lambda_1} - B_{\lambda_2} \right] x d\tau$$

Using $\| S_{\lambda}(t) \| \leq M$, the above expression can be rewritten as

$$\| [S_{\lambda_1}(t) - S_{\lambda_2}(t)] x \| \leq M^2 t \| [ B_{\lambda_1} - B_{\lambda_2} ] x \|$$

Now, for $x \in H^D(U)$, $\lim_{\lambda_1, \lambda_2 \to \infty} \| [ B_{\lambda_1} - B_{\lambda_2} ] x \| = 0$

hence $S_{\lambda}(t) x$ converges strongly to a limit as $\lambda \to \infty$, this convergence being uniform with respect to $t$ in any finite interval $(0, \beta)$, $\beta < \infty$. We denote the limit by $T(t) x$. If we now use the fact that $\mathcal{D}(U)$ is dense in $K$, the Banach-
Steinhaus's theorem gives \( \lim_{t \to \infty} S_{\lambda}(t) x = T(t) x \) for each \( x \in X \)
and the fact that \( || S_{\lambda}(t) || \leq M \) implies that this convergence is again uniform with respect to \( t \) in \((0, \beta)\). It follows that (i) \( T(t) \) is continuous in the strong operator topology for \( t \geq 0 \), (ii) \( || T(t) || \leq M \), and (iii) \( T(0) = I \).

Now \( S_{\lambda}(t) \) is clearly a semigroup operator, since \( S_{\lambda}(t) = \exp \{ B_{\lambda} t \} \). Hence as \( \lambda \to \infty \) we see that \( \{ T(t), t \geq 0 \} \) is a semigroup of class \((C_0)\), with \( || T(t) || \leq M \).

Let \( A \) be the infinitesimal generator of \( \{ T(t), t \geq 0 \} \).

The final step is to show that \( A \equiv \mathcal{U} \). We now use the relation
\[
\left[ S_{\lambda}(t) - I \right] x = \int_0^t S_{\lambda}(\tau) B_{\lambda} x d \tau
\]
for \( x \in \mathcal{D}(\mathcal{U}) \) the integrand converges boundedly to \( T(t) \mathcal{U} x \) since \( \lim_{\lambda \to \infty} S_{\lambda}(\tau) x = T(\tau) x \) and \( \lim_{\lambda \to \infty} B_{\lambda} x = \mathcal{U} x \);

and, therefore,
\[
\left[ T(t) - I \right] x = \int_0^t T(\tau) \mathcal{U} x d \tau, \quad x \in \mathcal{D}(\mathcal{U})
\]
Thus if \( A \) is the infinitesimal generator of \( \{ T(t), t \geq 0 \} \), differentiation of the above expression gives \( \mathcal{U} x = A x \) on \( \mathcal{D}(\mathcal{U}) \subseteq \mathcal{D}(A) \cdot \) Now, for \( \lambda > 0 \), \( R(\lambda; A) \) is an inverse operator, and \( R(\lambda; \mathcal{U}) \) is a right inverse of \((\lambda - A)^{-1}\).

Hence \( R(\lambda; A) = R(\lambda; \mathcal{U}) \), and \( A = \mathcal{U} \).

We observe that if \( M = 1 \) and \( || R(\lambda; A) \lambda^n || \leq \lambda^{-n} \) holds for \( n = 1 \), then it holds for all \( n \geq 1 \). This proves the Hille-Yosida theorem, which can be stated as follows.

**Theorem 2 (Hille-Yosida).** If \( A \) is a closed linear operator with dense domain, if the resolvent operator \( R(\lambda; A) \) exists for \( \lambda > 0 \), and if \( || R(\lambda; A) \lambda^n || \leq 1/\lambda, \lambda > 0 \), then \( A \) is the infinitesimal generator of a semigroup of contraction operators of class \((C_0)\).
The Feller-Miyadera-Phillips theorem gives a complete characterization of the infinitesimal generator for semigroups of class \( (C_c) \). We note, however, that in application of semigroup theory the Hille-Yosida theorem has the advantage of only requiring a bound on the first power of the resolvent operator, while the Feller-Miyadera-Phillips theorem requires a bound on all the positive integral powers of the resolvent operators.
3.3 Generation of Groups of Operators

It is also possible to apply the basic generation theorem for semigroups to groups of operators, and this result may be of interest in certain physical applications. By a strongly continuous group of operators of class \((C_0)\) in a Banach space we mean a one-parameter family of operators satisfying the group relation

\[
T(t_1 + t_2) = T(t_1) T(t_2), \quad t_1, t_2 \in (-\infty, \infty)
\]

\[
T(0) = I,
\]

and such that \(T(t)x\) is strongly continuous on \((-\infty, \infty)\) for each \(x \in X\). Since \(\{T_+(t) = T(t), \ t > 0\}\) and \(\{T_-(t) = T(-t), \ t > 0\}\) are both semigroups of class \((C_0)\), the theory developed in the last section is applicable to these semigroups. We remark that

\[
\lim_{t \to \infty} \frac{1}{t} \log \|T(t)\| = \omega_+,
\]

and

\[
\lim_{t \to -\infty} \frac{1}{t} \log \|T(t)\| = \omega_-
\]

exist, and hence for each \(\omega > \max(\omega_+, \omega_-)\) there exists an \(M(\omega)\) such that \(\|T(t)\| \leq M(\omega) e^{\omega t}\).

For the positive half of the group the infinitesimal generator is defined as in that case of a semigroup; that is,

\[
Ax = \lim_{t \to 0} \frac{1}{t} [T(t) - I] x
\]

It can also be shown that \(A\) is the infinitesimal generator of the negative half of the group.
Similarly, we see that the infinitesimal generator of 
\[ \{ T_-(t), t \geq 0 \} \text{ is } -A. \] Therefore, the resolvent operator
\[ R(\lambda; -A) = \int_0^\infty e^{-\lambda \tau} T_-(\tau) \mathcal{X} d\tau \]
extists for \( \lambda > \omega_- \). Now \( \lambda - (A \tau) = (\lambda - A) \) and it follows 
that \( R(\lambda; -A) = -R(-\lambda; A) \). We see, therefore, that \( R(\lambda; A) \) exists for all real \( \lambda \) with \( |\lambda| > \max(\omega_+, \omega_-) \).

The generation theorem for groups of operators is as follows.

**Theorem 3.** A necessary and sufficient condition for a closed linear operator \( \mathcal{U} \) with dense domain to generate a strongly continuous group of operators of class \( (\mathcal{C}_0) \) in \( \mathcal{X} \) is that there exist constants \( M \) and \( \omega > 0 \) such that
\[ \| \Gamma R(\lambda; \omega) \|^n \leq \frac{M}{(|\lambda| - \omega)^n} \]
for all real \( \lambda \), \( |\lambda| > \omega \) and all integers \( n \geq 1 \).

**Proof.** The necessity follows as in the case of Theorem 1
and from our remarks concerning the positive and negative halves of the group. To prove sufficiency, suppose that \( \mathcal{U} \) is such that the condition on \( \| \Gamma R(\lambda; \omega) \|^n \) is satisfied. Then it is clear that \( \mathcal{U} \) generates a semigroup, say \( \{ T_+(t) = T(t), t \geq 0 \} \). However, because of the relation 
\( -R(\lambda; -\mathcal{U}) = \mathcal{R}(-\lambda; \mathcal{U}) \), it follows that \( -\mathcal{U} \) also generates a semigroup, say \( \{ T_-(t) = T(-t), t \geq 0 \} \). Now \( T_+(t) \) admits the representation
\[ T_+(t) x = \lim_{\lambda \to \infty} \exp \left\{ [\lambda^2 R(\lambda; \mathcal{U}) - \lambda I] t \right\} x \]
and \( T_-(t) \) admits the representation
\[ T_-(t) x = \lim_{\lambda \to \infty} \exp \left\{ [-\lambda^2 R(-\lambda; \mathcal{U}) - \lambda I] t \right\} x; \]
and it follows that the operators $T_+(t)$ and $T_-(t)$ commute for all $t \geq 0$. Therefore $S(t) = T_-(t) \circ T_+(t)$ defines a semigroup operator of class $(C_0)_r$. For $x \in \mathcal{D}(\mathcal{U})$, $S(t)x$ is strongly differentiable for $t > 0$, and
\[
\frac{d}{dt} S(t)x = \frac{d}{dt} T_-(t) \left[ T_+(t)x \right] + T_-(t) \frac{d}{dt} T_+(t)x = \left[ T_-(t)(-\mathcal{U}) \right] T_+(t)x + T_-(t) \left[ \mathcal{U} \circ T_+(t)x \right] = 0.
\]
Consequently $S(t)x = x$ for all $t \geq 0$ and $x \in \mathcal{D}(\mathcal{U})$. Since $\mathcal{D}(\mathcal{U})$ is dense in $\mathcal{X}$, we see that $S(t) = T_-(t) \circ T_+(t) \equiv I$, $t \geq 0$. A similar argument shows that $T_+(t) \circ T_-(t) \equiv I$. Therefore, we have $T_-(t) = \left[ T_+(t) \right]^{-1}$. If we now put
\[
T(t) = \begin{cases} T_+(t), & \text{for } t > 0 \\ T_-(t), & \text{for } t < 0 \end{cases}
\]
it is clear that $T(t)$, $t \in (-\infty, \infty)$, is a strongly continuous group of operators of class $(C_0)$ with infinitesimal generator $\mathcal{U}$. 
3.4 Generation of Semigroups of Operators in Hilbert Space.

The results given in the last section hold when the space \( \mathcal{X} \) is an arbitrary Banach space, so it might be expected that a simpler criterion hold in case \( \mathcal{X} \) is a Hilbert space. This indeed the case, and we now give two theorems, due to R.S. Phillips [14], on the generation of semigroups of contraction operators in a Hilbert space \( \mathcal{X} \).

We first introduce the following definition.

**Definition 1.** An operator \( A \) on a Hilbert space \( \mathcal{H} \) is said to be **dissipative** if

\[
(\langle Ax_{1}, x \rangle - \langle x_{1}, Ax \rangle) \leq 0, \quad x \in \mathcal{D}(A);
\]

and \( A \) is said to be **maximal dissipative** if it is not the proper restriction of any other dissipative operator.

A necessary and sufficient condition that an operator \( A \) on a Hilbert space \( \mathcal{H} \) is maximal dissipative is given by the following result: Let \( \lambda > 0 \), and suppose that \( A \) is a dissipative operator with dense domain. Then \( A \) is maximal dissipative if and only if \( R(\lambda - A) = \mathcal{H} \). Another result concerning dissipative operators which we will require is the following: Let \( A \) be a dissipative operator, and let \( \lambda > 0 \), put \( y = (\lambda - A)x, \quad x \in \mathcal{D}(A) \), then

\[
\| y \| \geq \lambda \| x \|.
\]  

We now state and prove the following generation theorem.

**Theorem 4.** A necessary and sufficient condition for an operator \( A \) to generate a strongly continuous semigroup of contraction operators \( \{T(t), t \geq 0\} \) in a Hilbert space \( \mathcal{H} \) is that \( A \) be a maximal dissipative operator with dense domain.
Proof. If $A$ is the infinitesimal generator of a strongly continuous semigroup of contraction operators, say $\{T(t), t \geq 0\}$ then $||T(t)x|| \leq ||x||$, and for $x \in \mathcal{D}(A)$

$$(x_1 Ax) + (Ax_1 x) = \frac{d}{dt} \left( T(t)x, T(t)x \right) \leq 0.$$  

It follows that $A$ is a dissipative operator. We know from the general theory that $\mathcal{D}(A)$ is dense in $H$, and that $\rho(A)$ contains all real $\lambda > 0$. This implies, in particular, that $\mathcal{R}(\lambda - A) = H$ for $\lambda > 0$, so that $A$ is maximal dissipative. Conversely, if $A$ is maximal dissipative in the dense domain and $\lambda > 0$, then we have that $\mathcal{R}(\lambda - A) = H$. This together with (3.2), where $y = (\lambda - A)x$, $x \in \mathcal{D}(A)$, shows that $\mathcal{R}(\lambda : A) = (\lambda - A)^{-1}$ exists and satisfies the inequality $||\mathcal{R}(\lambda ; A)|| \leq 1$, $\lambda > 0$. Hence the hypothesis of the Hille-Yosida Theorem is satisfied, and we can conclude that $A$ is the infinitesimal generator of a strongly continuous semigroup of contraction operators in $H$.

A particular subclass of contraction operators that are of interest in quantum mechanics is the class of isometric operators or isometries. We now give a generation theorem for semigroups of isometric operator. In dealing with isometric operators we need the notion of a conservative operator, which is defined as follows.

Definition 2. An operator $A$ on a Hilbert space $H$ is called conservative if $(Ax_1 x) - (x, Ax) = 0, x \in \mathcal{D}(A)$.

We will need the following result: An operator $A$ is conservative if and only if $iA$ is symmetric. If $A$ is conservative and maximal dissipative with dense domain then $iA$ is maximal symmetric.
Theorem 5. A necessary and sufficient condition that an operator $A$ generate a semigroup of isometric operators, say $\{T(t), t \geq 0\}$, in a Hilbert space $H$ is that $A$ be conservative and maximal dissipative with dense domain.

Proof. If $\{T(t), t \geq 0\}$ is a semigroup of isometric operators, then $||T(t)x|| = ||x||$ for all $t \geq 0$. Then for $x \in \mathcal{D}(A)$

$$(Ax_1, x) + (X, Ax) = \frac{d}{dt} \left[ T(t)x_1, T(t)x \right]_{t=0} = 0,$$

and we see that $A$ is a conservative operator. In addition, it follows from theorem 4 that $A$ is also maximal dissipative with dense domain. Conversely, suppose $A$ is a conservative and maximal dissipative operator with dense domain. Then theorem 4 also asserts that $A$ is the infinitesimal generator of a strongly continuous semigroup of contraction operators, say $\{T(t), t \geq 0\}$.

For $x \in \mathcal{D}(A)$, $T(t)x \in \mathcal{D}(A)$ for all $t > 0$, and if we use the fact that $A$ is conservative, we get

$$\frac{d}{dt} \left[ T(t)x, T(t)x \right] = \left[ AT(t)x, T(t)x \right] + A \left( T(t)x, AT(t)x \right) = 0.$$ 

Consequently $||T(t)x|| = ||x||$, $t > 0$, for each $x \in \mathcal{D}(A)$, and since $\mathcal{D}(A)$ is dense in $H$, this relation holds for all $x \in H$. Thus $T(t)$ is an isometric operator.
3.5 Uniqueness of the Generation Problem

In this final section we establish the uniqueness of the generation problem. Because the resolvent operator is the Laplace transform of a semigroup operator, it is easily seen that this problem can be reduced to that of the uniqueness of Laplace transforms.

Theorem 6. A closed linear operator \( \mathcal{A} \) can be the infinitesimal generator of at most semigroup of class \((C_0)\).

Proof. Suppose \( \mathcal{A} \) is the infinitesimal generator of two semigroups \( \{ T_1(t), t > 0 \} \) and \( \{ T_2(t), t > 0 \} \), each of class \((C_0)\). Let \( x \in \mathcal{D}(\mathcal{A}) \). Then \( \lim_{t \to 0} T_1(t)x = x, i = 1, 2; \) and

\[
\mathbb{R}(\lambda; \mathcal{U}) = \int_0^\infty e^{-\lambda t} T_i(t) x \, dt, \quad i = 1, 2, \ldots
\]

for \( \lambda > 0 \). Since \( T_i(t)x, i = 1, 2, \) is strongly continuous for \( t > 0 \), the uniqueness theorem for Laplace transforms (Theorem N 11) implies that \( T_1(t)x = T_2(t)x \) for all \( t > 0 \) and \( x \in \mathcal{D}(\mathcal{A}) \). Since bounded linear operators are uniquely determined by their values on a dense set, and since \( \mathcal{D}(\mathcal{A}) \) is dense in \( \mathcal{X} \), it follows that the two semigroups coincide.
3.6 Other studies on the Generation of Semigroups

In this section we mention some additional studies on the generation of semigroups which were not referred to in the previous section. For studies of a somewhat general nature we refer to \cite{3, 4, 5, 9, 12}; for a treatment of the generation of semigroups in Banach lattices we refer to \cite{15}. Dissipative operators in a Banach space are studied* in \cite{6}; and in \cite{7, 10} the generation of semigroups in Fréchet spaces is considered.

* Dissipative operators and the generation of contraction semigroups are studied in a forthcoming paper by E. Nelson entitled 'Feynman integrals and the Schrödinger equation.'
REFERENCES.


CHAPTER 4.
SOME ADDITIONAL TOPICS IN SEMIGROUP THEORY.

4.1 Introduction

In this chapter, consisting of three sections, we consider some additional topics in semigroup theory which we feel are of interest to workers in mathematical physics. In section 4.2 we consider perturbation theory for semigroups of class \( C_0 \) and give an application to differential equations in Banach spaces. Section 4.3 is devoted to an exposition of Sz.-Nagy's representation theory of contraction semigroups in Hilbert space. Finally, in section 4.4 we introduce the notion of equivalent semigroups of operators, and discuss some of their properties.

4.2 Perturbation Theory for Semigroups of Operators.

A. Introduction. Perturbation theory has long been a subject of interest to analysts and applied mathematicians, and in certain branches of mathematical physics perturbation techniques are employed as a standard tool in obtaining solutions of many problems. In particular, perturbation theory is used to determine the state of a system which is, in some sense, close to a known system. In our studies the state of the known system will be expressed by a semigroup of operators with an infinitesimal generator \( A \). What we want to know is what happens to the system when \( A \) is perturbed by a bounded linear operator \( B \).
The state of the new system will now be expressed by a semigroup whose infinitesimal generator is $A + B$.

A problem of theoretical and practical importance is that of determining the properties of semigroups which are stable. In this connection we introduce the following definition.

**Definition 1.** A semigroup property is said to be stable if it holds for all semigroups with infinitesimal generator $A$, with $A$ sufficiently close to the infinitesimal generator $A'$, whenever it holds for the semigroup generated by $A'$.

This section has two subsections. In section B we consider perturbation theory for semigroup of class $(C_0)$ and in section C we consider an application to differential equations.

For additional studies on perturbation theory we refer to \cite{4, Chap. VIII}, \cite{5, Chap. XIII} and \cite{8}.
B. Perturbation Theory For Semigroups Of Operators of Class \((C_0)\). We consider a semigroup \(\{T(t), t \geq 0\}\) of class \((C_0)\) with infinitesimal generator \(A\) in a Banach space \(\mathcal{X}\). The perturbed infinitesimal generator will be denoted by \(A'\). We assume \(A\) and \(A'\) have a common domain, and that the operator \(B = A' - A\) is bounded on \(D(B) = D(A) = D(A')\). In this case it is known that the operator \(B\) has a unique bounded linear extension on \(\overline{D(B)} = \mathcal{X}\).

Our first two theorems prove that the property of being a semigroup and being a semigroup of class \((C_0)\) are stable properties.

**Theorem 1.** Let \(A\) be the infinitesimal generator of a semigroup of class \((C_0)\) in \(\mathcal{X}\); and let \(E \in \mathcal{E}(\mathcal{X})\). Then \(A + B\) defined on \(D(A)\) is also the infinitesimal generator of a semigroup of class \((C_0)\).

**Proof.** Since \(A\) is the infinitesimal generator of a semigroup of class \((C_0)\), the basic generation theorem (Theorem 3, is applicable, and we have that there exist real constants \(M > 0\) and \(\omega\) such that \(\|e^{\lambda(t + t)}A\|^n \leq M (\lambda - \omega)^n\) for all \(\lambda > \omega\) and integers \(n = 1, 2, \ldots\). Hence for \(\lambda > \omega\),

\[= \omega + M\|B\|,\]

we have \(\|e^{\lambda(t + t)}(A + B)\|^n \leq M\|B\|^n (\lambda - \omega)^n < 1\).

We now wish to determine \(R(\lambda, A + B)\), the resolvent operator of \(A + B\). In order to do this we need the following lemma, which we state without proof.
Lemma 1. Let \( T \in \mathcal{C}(X) \), and let \( S \) be a linear operator with \( D(S) = D(T) \). Suppose further that \( \lambda \in \sigma(T) \). Then \( S \in \mathcal{C}(X) \) and \( \lambda \in \sigma(S) \) implies \( I - [(S - T)] \) regular in \( \mathcal{C}(X) \). In this case

\[
R(\lambda; S) = R(\lambda; T)\left[I - (S - T)R(\lambda; T)\right]^{-1}
\]

In particular, if \( \| (S - T)R(\lambda; T) \| = \kappa < 1 \), then

\[
S \in \mathcal{C}(X), \lambda \in \sigma(S),
\]

\[
R(\lambda; S) = R(\lambda; T)\left[\sum_{n=0}^{\infty} (S - T)R(\lambda; T)^n\right]
\]

and

\[
\| R(\lambda; S) - R(\lambda; T) \| \leq \| R(\lambda; T) \| \kappa (1 - \kappa)^{-1}
\]

Returning to the proof of the theorem, we see by putting \( A = T \) and \( A + B = S \) that the existence of the resolvent follows from the lemma, and that from (4.1)

\[
R(\lambda; A + B) = \sum_{n=0}^{\infty} R(\lambda; A)\left[B R(\lambda; A)\right]^n, \quad \lambda > \omega,
\]

with the series converging absolutely.

Now, in order to show that \( R(\lambda; A + B) \) is the resolvent of the semigroup of class \( (C_o) \) with infinitesimal generator \( A + B \), we must show, by the basic generation theorem, that

\[
\| [R(\lambda; A + B)]^n \| = \| \sum_{\gamma=0}^{\infty} R(\lambda; A)\left[B R(\lambda; A)\right]^\gamma \| \leq \kappa \| R(\lambda; A) \| \kappa (1 - \kappa)^{-1}
\]

for all \( \lambda > \omega \), and integers \( n = 1, 2, \ldots \). In order to utilize the estimate \( \| [R(\lambda; A)]^n \| \leq M(\lambda - \omega)^{-n} \), we
resort to the following combinatorial analysis. We first regroup the terms of \( \left[ R(\lambda, A+B) \right]^n \) according to powers of B. Now, each term containing \( K \) of the B's must also contain \( \gamma + \kappa \) of the \( R(\lambda, A) \), since each B introduces another \( R(\lambda, A) \). Further, the \( R(\lambda, A) \) will be grouped in \( (k+1) \) nonempty sets separated from each other by the \( A \). Hence, the typical term containing \( \kappa \) of \( B \) will be of the form

\[
\left[ R(\lambda, A) \right]^{n_1} \left[ R(\lambda, A) \right]^{n_2} \cdots \left[ R(\lambda, A) \right]^{n_k} \left[ R(\lambda, A) \right]^{n_{k+1}},
\]

where \( n_i > 0 \) and \( \sum n_i = \gamma + \kappa \). Now using the estimate for \( \left\| \left[ R(\lambda, A) \right]^n \right\| \), we see that the above term will be bounded in norm by

\[
\left\| \left[ R(\lambda, A) \right]^{n_1} \left[ R(\lambda, A) \right]^{n_2} \cdots \left[ R(\lambda, A) \right]^{n_k} \left[ R(\lambda, A) \right]^{n_{k+1}} \right\| \leq \sum_{K=1}^{\infty} \left\| R(\lambda, A) \right\|^{n_k} \left\| R(\lambda, A) \right\|^{n_{k+1}} = M^{\kappa+1} \left\| B \right\|^{K} \left( \lambda - \omega \right)^{-\left( \gamma + \kappa \right)}
\]

The number of terms containing \( \kappa \) of the \( B^n \) is the coefficient of \( x^k \) in \( (1-x)^{\gamma} = \sum_{k=0}^{\infty} \binom{\gamma}{k} x^k \). Hence we obtain the estimate

\[
\left\| \left[ R(\lambda, A+B) \right]^n \right\| \leq \sum_{K=1}^{\infty} \left\| R(\lambda, A) \right\|^{n_k} \left\| R(\lambda, A) \right\|^{n_{k+1}} = M^{\kappa+1} \left\| B \right\|^{K} \left( \lambda - \omega \right)^{-\left( \gamma + \kappa \right)}
\]

The result now follows from the basic generation theorem.
Theorem 2. Let \( \{ T(t, A), t \geq 0 \} \) be a semigroup of class \( (C_0) \), and let \( B \in \mathcal{L}(X) \). If
\[
\| T(t, A) \| \leq e^{\alpha t},
\]
then \( \{ T(t, A + B), t \geq 0 \} \) is also a semigroup of class \( (C_0) \), and
\[
\| T(t, A + B) \| \leq e^{\beta t},
\]
where \( \alpha = \omega_1 + M \| B \| \).

Proof. If \( \| T(t, A) \| \leq e^{\alpha t} \), then
\[
\| R(\lambda; A) \| \leq \int_0^{\infty} e^{-\lambda t} e^{\alpha t} e^{-\alpha t} = (\lambda - \alpha)^{-1}, \lambda > 0,
\]
so that
\[
\| [R(\lambda; A)]^n \| \leq M (\lambda - \omega_1)^{-n}
\]
is satisfied for \( M = 1 \). The Hille-Yosida theorem, together with the estimate
\[
\| [R(\lambda; A + B)]^n \| \leq M (\lambda - \omega_1)^{-n},
\]
now imply that \( A + B \) generates a semigroup of class \( (C_0) \) such that
\[
\| T(t, A + B) \| \leq e^{\omega_1 t},
\]
where, as before,
\[
\omega_1 = \omega + M \| B \|.
\]

Our last theorem in this subsection gives a series representation for the perturbed semigroup.

Theorem 3. If \( A \) is the infinitesimal generator of a semigroup \( \{ T(t), t \geq 0 \} \) of class \( (C_0) \) and if \( B \in \mathcal{L}(X) \), then the semigroup \( \{ S(t), t > 0 \} \) generated by \( A + B \) with \( D(A + B) = D(A) \), can be represented by the series expansion
\[
S(t) = \sum_{n=0}^{\infty} S_n(t),
\]
where
\[
S_n(t) = \frac{t^n}{n!} T(t).
\]

For \( \chi \in D(A + B) \),
\[
S_n(t) \chi = \int_0^t T(t - \tau) B \xi_n(t) \chi(\tau) d\tau,
\]
where
\[
\xi_n(t) = \frac{t^n}{n!} T(t).
\]
Proof. Since \( \mathcal{S}_n(t) = T(t) \), it is clear that \( \mathcal{S}_n(t) \) is strongly continuous on the interval \( [0, \infty) \), and that \( |\mathcal{S}_n(t)| \leq M \). Let us now assume that \( \mathcal{S}_n(t) \) is strongly continuous on \( [0, \infty) \), and that

\[
\| T(e^{it}) \| \leq M \left( \frac{M\|B\|}{\rho} \right)^{\rho/\rho + 1} t^{\rho/\rho + 1} / \| A \| \quad (4.2)
\]

Then, \( T(e^{i(t - \infty)} \mathcal{S}_n(t) \) will be strongly continuous on \( [0, \infty) \), so that the integral defining \( \mathcal{S}_n(t) \) exists in the strong topology. Further

\[
\mathcal{S}_n(t) = \mathcal{S}_n(0) + \int_0^t e^{i(t - \tau)} \mathcal{S}_n(\tau) d\tau \leq \frac{M \left( \frac{M\|B\|}{\rho} \right)^{\rho/\rho + 1} t^{\rho/\rho + 1}}{\| A \|}.
\]

Now, for \( t_1 < t_2 \)

\[
\| \mathcal{S}_n(t_2) - \mathcal{S}_n(t_1) \| \leq \int_{t_1}^{t_2} \| T(e^{i(t - \tau)}) - T(e^{i(t - \tau)}) \| B \mathcal{S}_n(\tau) d\tau + \int_{t_1}^{t_2} \| T(e^{i(t - \tau)}) \| B \| \mathcal{S}_n(\tau) \| d\tau.
\]

As \( t_1, t_2 \to t_0 \), say, the integrand in the first term on the right converges to zero boundedly, and the integrand of the second term is bounded. Hence, it follows that \( \mathcal{S}_n(t) \) is strongly continuous on \( [0, \infty) \). Therefore, by induction, \( \mathcal{S}_n(t) \) is well-defined, strongly continuous, and satisfies (4.2) for all integers \( n \). Thus the series defining \( \mathcal{S}(t) \) is majorised by the series expansion of \( \sum \mathcal{S}_n(t) \), where, as

...
before, \( \omega = \omega + N \), \( \mathbb{B} \subset \mathbb{B} \). Hence for \( \lambda > \omega \), we can write
\[
\int_0^\infty e^{-\lambda t} \left[ \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{(t-s)^{\alpha-1}}{s^{\alpha-n}} \right] dt = \sum_{n=0}^{\infty} \int_0^\infty e^{-\lambda t} \mathbb{B} \mathbb{S}_{\alpha-1}(t) x \, dt.
\]

Let us now consider a linear functional \( \mathcal{F} \in \mathcal{A}^* \). It is a consequence of the strong convergence of the integral and of Fubini's theorem, that
\[
\mathcal{F} \left[ \int_0^\infty e^{-\lambda t} \mathbb{S}_\alpha(t) x \, dt \right] = \int_0^\infty e^{-\lambda t} \mathcal{F} \left[ \mathbb{S}_\alpha(t) x \right] dt
\]
\[
= \int_0^\infty \int_0^t e^{-\lambda \tau} \mathcal{F} \left[ \mathbb{S}_\alpha(t-\tau) \mathbb{B} \mathbb{S}_{\alpha-1}(\tau) x \right] d\tau dt
\]
\[
= \int_0^\infty \int_0^\infty e^{-\lambda \tau} \mathcal{F} \left[ \mathbb{R}(\lambda; A) \mathbb{B} \mathbb{S}_{\alpha-1}(\tau) x \right] d\tau dt
\]
\[
= \mathcal{F} \left[ \mathbb{R}(\lambda; A) \mathbb{B} \mathbb{S}_{\alpha-1}(t) x \right] dt
\]
\[
= \mathcal{F} \left[ \mathbb{R}(\lambda; A) \mathbb{B} \mathbb{S}_{\alpha-1}(t) x \right].
\]

Hence, by induction,
\[
\int_0^\infty e^{-\lambda t} \mathbb{S}_\alpha(t) x \, dt = \mathbb{P}(\lambda; A) \left[ \mathbb{B} \mathbb{R}(\lambda; A) \right] x.
\]
From the proof of Theorem 1, it is known that
\[ R(\lambda; A+B) x = \int_0^\infty e^{-\lambda t} \mathcal{S}(t) x dt = \sum_{n=1}^\infty D(\lambda; A) B^n B(\lambda; A) x. \]

Thus, for \( \lambda \gg \epsilon \), the Laplace transforms of \( \mathcal{S}(t) \) and \( \sum_{n=1}^\infty \mathcal{S}_n(t) \) are equal. We now use the uniqueness theorem for Laplace transforms (Theorem N.11), and this completes the proof of the theorem.

Before closing this section we add a few remarks concerning the series representation just obtained. Firstly, the zero-th term is, of course, the unperturbed semigroup. The first term
\[ \mathcal{S}_1(t) = \int_0^t \mathcal{T}(t - \tau) B \mathcal{S}_0(\tau) d \tau \]
\[ = \int_0^t \mathcal{T}(t - \tau) B \mathcal{T}(\tau) d \tau \]
can be thought of as resulting from a perturbation of the unperturbed semigroup after a time \( \tau \), followed by the action of the unperturbed semigroup over the remaining time \( t - \tau \), and this averaged over all \( \tau \in (0, t) \). The second term \( \mathcal{S}_2(t) \) consists of the unperturbed semigroup perturbed at times \( \tau_1 \) and \( \tau_2 \), and then averaged over all \( \tau_1 \), with \( 0 < \epsilon < \tau_2 < t \). A similar interpretation of the other terms is possible.
C. Perturbation Theory and a Linear Differential Equation in Banach Space. R. S. Phillips [8] has considered the following problem: Given a one-parameter family of closed linear operators \( A(t) \) with domains dense in a Banach space \( \mathcal{H} \), find a one-parameter family of bounded linear operators \( \{T(t), \ t \geq 0 \} \) strongly continuous for \( t \geq 0 \), such that

\[
\frac{d}{dt} T(t)x = A(t)T(t)x, \quad T(0) = I \tag{4.3}
\]

for all \( x \) in a given dense domain. Phillips did not give a solution to the general problem posed above, but gave a solution in the case where \( \Lambda(t) = \Lambda + \varepsilon(t) \); hence we consider differential equations of the form

\[
\frac{d}{dt} T(t)x = \left[ -\Lambda + \varepsilon(t) \right] T(t)x \tag{4.4}
\]

Because of the importance of equations of the form (4.4) in mathematical physics, we state, as a theorem, Phillips' result for equation (4.4).

**Theorem 4.** Let \( A \) be the infinitesimal generator of a semigroup \( \{\psi(t), t \geq 0\} \) of class \( (C_0) \). Let \( \varepsilon(t) \) be a strongly continuously differentiable function on \( [0, \infty) \) to \( \mathcal{E}(\mathcal{H}) \). Then there exists a unique one-parameter family of bounded linear operators \( \{T(t), t \geq 0\} \), strongly continuous on \( [0, \infty) \), such that \( T(0) = I \) and for \( x \in \psi(t) \),

\[
\text{is approximately} \]
\[ T(t) \times \text{is strongly continuously differentiable, and} \]

\[ \frac{d}{dt} T(t) \times = \left[ A + B(t) \right] T(t) \times 
\]

(4.4)

The solution of (4.4) can be represented on

\[ T(t) = \sum_{n=0}^{\infty} \mathcal{L}_n(t) \]

where

\[ \mathcal{L}_0(t) = \varphi(t) \]

\[ \mathcal{L}_n(t) = \int_0^t \mathcal{L}_{n-1}(t - \tau) B(\tau) \mathcal{L}_{n-1}(\tau) d \tau. \]

4.3 Semigroups Of Contraction Operators

In Hilbert Space

A. Introduction. In Chapters 2 and 3 we gave various results for semigroups of contraction operators in Banach Spaces; in particular, we gave a representation theorem for contraction operators (Theorem 2.19), and in chapter 3 we gave some generation theorems for contraction semigroups. In this section we will consider a representation theorem, due to B. Sz.-Nagy §10.7, for semigroups of contraction operators in Hilbert space. This theorem is of great interest to us in connection with various applications, for many problems in mathematical physics lead to the study of contraction semigroups in Hilbert spaces.

In section D we state and prove Sz.-Nagy's representation theorem, and in section C we state and prove a representation theorem, due to C.L.Dolph §3.7 for the resolvent of a maximal dissipative operator.
B. Representation Theorem for Contraction Semigroups in Hilbert Space. Sz.-Nagy's theorem can be stated as follows:

Theorem 5. Let \( \{ T(t), \ t \geq 0 \} \) be a strongly continuous semigroup of contraction operators on a Hilbert space \( \mathcal{H} \). Then there exists a group of unitary operators
\[
\{ U(t), \ t \in (\mathbb{R}, \infty) \}
\]
on a larger Hilbert space \( \mathcal{H}' \), containing \( \mathcal{H} \), as a subspace such that
\[
T(t) \cdot x = P U(t) \cdot x, \quad x \in \mathcal{H}, \quad t \geq 0.
\]
(4.5)

where \( P \) is a projection operator with range \( \mathcal{H} \). The space \( \mathcal{H}' \) can be constructed in a minimal fashion so that it is spanned by \( \{ U(t) \cdot x, \ x \in \mathcal{H}, \ t \in (\mathbb{R}, \infty) \} \). In this case the structure \( \{ \mathcal{H}, U(t), \mathcal{H}' \} \) is determined to within an isomorphism.

Before proving the above theorem we give a definition, due to Naimark [17], and state two theorems due to Naimark [17] and Stone [9].

Definition 2. A one-parameter family of bounded self-adjoint operators \( F(\omega) \) in a Hilbert space \( \mathcal{H} \) is said to be a generalized resolution of the identity (in the sense of Naimark) if

(i) \( F(\omega_1) = F(\omega_2) \Rightarrow \omega_1 = \omega_2 \)

(ii) \( F(-\omega) = F(\omega) \equiv \mathbb{I} \)

(iii) \( F(\omega + \omega) = F(\omega) \)


Theorem 6. (Naimark). If \( f(t) \) is an arbitrary resolution of the identity in a Hilbert space \( \mathcal{H} \), then there exists a larger Hilbert space \( \mathcal{H}' \) which contains \( \mathcal{H} \) as a subspace, and there exists an orthogonal resolution of the identity \( \mathcal{E}(t) \) such that for each \( x \in \mathcal{H}' \), one has

\[
F(t) x = P \mathcal{E}(t) x,
\]

where \( P \) is projection operator from \( \mathcal{H} \) to \( \mathcal{H}' \).

Theorem 7 (Stone). All continuous groups of unitary operators in a Hilbert space \( \mathcal{H} \) admit the spectral representation

\[
\mathcal{U}(t) = \int_{-\infty}^{\infty} e^{-i\lambda t} \mathcal{E}(\lambda) d(\lambda),
\]

where \( \mathcal{E}(\lambda) \) is a uniquely determined resolution of the identity.

Hence it follows from (4.5) and Stone's theorem that the contraction operator \( T(t) \) admits the representation.

\[
T(t) x = F \int_{-\infty}^{\infty} e^{-i\lambda t} \mathcal{E}(\lambda) x.
\] (4.6)

The space \( \mathcal{K} \) is often referred to as the dilation space, and \( \mathcal{U}(t) \) is called the dilation of the contraction operator \( T(t) \).

We now prove Sz.-Nagy's theorem.

Proof. For \( x \) fixed, consider the inner product

\[
(T \infty x, x)
\]

which is a continuous function of \( t \). We form the function

\[
f(t) = \int_{-\infty}^{\infty} e^{-i\lambda t} (-\lambda^2 x, x) d\lambda
\]

which is bounded and continuous with respect to \( t \) for \( \lambda^2 > 0 \).

Note that
We note that
\[ f(x) = \left( (I + A)^{-1}x, x \right) \]
where \( A \) is the infinitesimal generator of \( \{ T(t), t > 0 \} \).
Let \( \gamma = (x I - A)^{-1}x \), then we have
\[
\begin{align*}
  f(x) &= \left( \gamma, (I + A)\gamma \right) \\
       &= \gamma^T \gamma - \gamma^T A \gamma \\
       &= \| \gamma \|^2 + \frac{1}{t} \left( \gamma, (I + A \gamma) \gamma \right).
\end{align*}
\]
Therefore
\[
\Re f(x) = \Re \left( \| \gamma \|^2 + \frac{1}{t} \left( \gamma, (I + A \gamma) \gamma \right) \right)
\]
and
\[
|\Re f(x)| \leq \| \gamma \|^2 + \frac{1}{t} \| \gamma \|^2 \leq \| \gamma \|^2.
\]
Hence \( f(x) > 0 \).

We will now employ the following theorem due to R. Nevanlinna (cf. [12, 7]).

**Theorem 3.** Let \( k(t) \) be a bounded, measurable function defined for \( t \in [a, b] \) and let \( k(t) \) be its Laplace transform, \( \hat{k}(\alpha) = \mathcal{L}^{-1}(k(t)) \). Define \( \hat{k}(\alpha) \) for \( t < 0 \) by \( h_{\alpha}(t) = \hat{k}(\alpha t) \). If \( \hat{k}(\alpha) \geq 0 \) for \( \alpha > \rho \), then
\[
h(t) = \int_{-\alpha}^{\infty} 2e^{-\alpha t} \hat{\gamma}(\alpha) d\alpha.
\]
almost everywhere, so that
\[ h(\omega) = \int_{\mathcal{A}} \frac{1}{\mathbb{Z} - \omega} d\mathbb{P}(\omega) \]

In the above, \( \gamma(\omega) \) is an increasing function of \( \omega \) so normalized that \( \gamma(-\infty) = 0 \) and \( \gamma(\infty) = \gamma(\omega) \). Moreover, \( \gamma(\omega) \) is real-valued, bounded, and unique.

Now, the function \( f(t) \) satisfies the conditions of the above theorem, hence using the uniqueness of the Laplace transform, and inverting, we have

\[ \left( T(t) \right)(x, y) = \int_{\mathbb{R}} e^{-it} \gamma(t) dt \]

We should use the notation \( \gamma(\omega; x, y) \) to show that \( \gamma \) depends on \( x \). Using this quadratic form we can obtain a bilinear form, namely

\[
\gamma(\omega; x, y) = \frac{1}{2} \left[ \gamma(\omega; x + y) - \gamma(\omega; x - y) + iy \gamma(\omega; x + iy) - iy \gamma(\omega; x - iy) \right]
\]

Hence we can write

\[
\left( T(t) \right)(x, y) = \int_{\mathbb{R}} e^{-it} \gamma(t) dt \quad (4.7)
\]

Of course \( \gamma(\omega; x, y) = \gamma(\omega; x) \), and in (4.7) the left-hand side is symmetric in \( x \) and \( y \), so the same is true of \( \gamma(\omega; x, y) \) for fixed \( \omega \).

Now, it is well-known (cf. [11]) that if \( \gamma(\omega; x, y) \) is a symmetric bilinear form, then for each \( \omega \) we can find a self-adjoint operator \( F(\omega) \) such that \( \gamma(\omega; x, y) = (F(\omega)x, y) \).
where \( \chi \in \mathfrak{H} \), \( \omega \in (-\infty, \infty) \), and \( F(\omega) \) is a generalized resolution of the identity. Hence (4.7) becomes

\[
( T(t)x, y) = \int_{-\infty}^{\infty} e^{i\omega t} d( F(\omega)x, y) \tag{4.8}
\]

If we now use Theorem 6, we have

\[
( T(t)x, y) = \int_{-\infty}^{\infty} e^{i\omega t} d( \tilde{F}(\omega)x, y) \tag{4.9}
\]

C. Representation of the Resolvent of a Maximal Dissipative Operator. Maximal dissipative operators were defined in Chapter 3, and at that time we proved that a necessary and sufficient condition for a closed linear operator to be the infinitesimal generator of a contraction semigroup in Hilbert space was that it be maximal dissipative with dense domain. We now prove Dolph's theorem which gives a representation of the resolvent of a maximal dissipative operator.

Theorem 3. Let \( A \) be a maximal dissipative and closed operator in a Hilbert space \( \mathfrak{H} \). Then for any \( \lambda \), \( \lambda > 0 \), the resolvent operator \( R(\lambda; A) \) exists and can be represented uniquely as

\[
R(\lambda; A) = \int_{0}^{\infty} \frac{dF(\omega)}{e^{i\omega t} - \lambda} \tag{4.10}
\]

where \( F(\omega) \) is a generalized resolution of the identity, and hence the projection of an orthogonal resolution of the identity in a Hilbert space \( \mathfrak{H} \) containing \( \mathfrak{H} \) as a subspace. Further, \( \mathfrak{H} \) can be constructed in a minimal fashion, and
\[(R(\lambda; A)\psi, \psi) + (\psi, R(\lambda; A^*\psi)) \geq 0 \quad \text{for} \quad \Re \lambda > 0\]

**Proof.** \(A\) is the infinitesimal generator of a contraction semigroup \(\{T(t) : t \geq 0\}\), the associated resolvent operator being given by

\[R(\lambda; A)\psi = \int_{0}^{\infty} e^{-\lambda t} T(t)\psi \, dt, \quad \forall \psi \in X.\]

From Sz.-Nagy's theorem it follows that

\[R(\lambda; A)\psi = \int_{0}^{\infty} e^{-\lambda t} D(t)\psi \, dt.\]

Applying Stone's theorem we have

\[R(\lambda, A)\psi = \int_{0}^{\infty} e^{-\lambda t} \int_{X} e^{-\lambda t^*} d(P_{X}(\psi)) \, dt.\]

Finally, an application of Fubini's theorem gives

\[R(\lambda, A)\psi = \int_{0}^{\infty} e^{-\lambda t^*} \int_{X} e^{-\lambda t} d(P_{X}(\psi)) \, dt.\]

Since \(R(\lambda; A)\) is a resolvent operator, it is defined for all \(\lambda\) in the region \(\Re \lambda > c^*\), and hence its range is dense in \(X\). Also, \(R(\lambda; A)\) satisfies the first resolvent equation. For the uniqueness argument, we refer the reader to the article of C.L. Dolph*, and other references given therein.

4.4 Equivalent Semigroups of Operators.

A. Introduction. In this section we define equivalent semigroups of operators, and study some of their properties. However, before taking up equivalent semigroups, we briefly recall the notions of equivalence and similarity of matrix theory.* Let $A$ be the matrix of a linear transformation $T \in \mathcal{M}(\mathbb{R}, \mathbb{R}_2)$ relative to the bases $(e_1, \ldots, e_n)$ and $(f_1, f_2, \ldots, f_m)$, where $\mathcal{M}(\mathbb{R}, \mathbb{R}_2)$ is the set of linear transformations of the vector space $\mathbb{R}_2$ into $\mathbb{R}_2$. We now change the bases in $\mathbb{R}_2$ and $\mathbb{R}_2$, and calculate $A$ relative to the new bases. Thus, let $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ be bases in $\mathbb{R}_2$ and $\mathbb{R}_2$, respectively, where $u_i = e_i \cdot e_2$ and $v_i = e_i \cdot e_2$.

The matrices $B$ and $C$ are nonsingular, and we denote their inverses by $\beta = [\beta_{ij}]$ and $\gamma = [\gamma_{ij}]$. We now have

$$u_{\gamma}A = (\Sigma^t b_{ij} \alpha_{ij})A = \Sigma^t b_{ij} \alpha_{ij} A = \Sigma^t b_{ij} \alpha_{ij} A = \Sigma^t b_{ij} \gamma_{ij} e_i \otimes e_j = \Sigma^t a'_{ij} e_i \otimes e_j$$

where

$$a'_{ij} = \Sigma b_{ik} \gamma_{kj}$$

Hence the new matrix $A'$ is

$$A' = B \cdot \gamma^{-1},$$

where $B$ gives the change in basis in $\mathbb{R}_2$, and $C$ gives the change in basis in $\mathbb{R}_2$.

* We refer, for example, to the book of N. Jacobson [6].
Let $\mathcal{M}(\Delta, \mathbb{M})$ denote the set of matrices of order $n$ over $\Delta$, where $\Delta$ may be the field of real numbers, the field of complex numbers, etc.

**Definition 3.** Two $n \times n$ matrices $A$ and $A'$ are said to be equivalent if there exist matrices $B \in \mathcal{M}(\Delta, \Delta)$ and $C \in \mathcal{M}(\Delta, \mathbb{M})$ such that $A' = BAC$.

We now assume that $R_1, R_2, \ldots, R_n$ and that $(e_1, e_2, \ldots, e_n)$ is a basis. Let $A$ be the matrix of $\tau$ relative to $(e_1, e_2, \ldots, e_n)$. Then $A = \sum_{i=1}^{n} e_i^* \sigma_i e_i$ and the computation shows that the matrix of $\tau$ relative to $(u_1, u_2, \ldots, u_n)$ where $u_i = \sum_{j=1}^{n} \sigma_j^* e_i e_j$ is

$$A' = BAB^{-1}.$$  \hspace{1cm} (4.11)

**Definition 4.** Two matrices $A$ and $A'$ in the set of all $n \times n$ matrices are said to be similar if they are related in the manner expressed by (4.11).

To turn now to semigroups of operators, let

$$(H) = \left\{ T(t), t \geq 0 \right\}$$

be a semigroup of class $(C_0)$ in a Banach space $\mathcal{B}$. If $H$ is a linear homeomorphism of $\mathcal{B}$ onto another Banach space $\mathcal{F}$, then the semigroup $S = \left\{ S(t), t \geq 0 \right\}$, where $S(t) = H^{-1} T(t) H$, is a semigroup of class $(C_0)$ in $\mathcal{F}$. The semigroups are said to be homeomorphically equivalent or similar.

**Definition 5.** Let $\Sigma$ and $\psi$ be semigroups of operators of class $(C_0)$ in Banach spaces $\mathcal{B}$ and $\mathcal{F}$ respectively. Then $\Sigma$ and $\psi$ are said to be equivalent, written $\Sigma \sim \psi$, or $\Sigma(t) \sim \psi(t)$, if there exist
constants \( \omega \) and \( \alpha \), with \( \alpha \) real and positive, such that

\[
S(t) = \mathbf{H} \left( e^{\alpha t} T \omega t \right) \mathbf{H}^{-1}
\]

are homeomorphically equivalent or similar, i.e.

\[
(4.19)
\]

B. Some Properties of Equivalent Semigroups.

We will simply state the first result, since the proof is elementary.

Theorem 10. The relation \( \sim \) is an equivalence relation, that is, it is reflexive, symmetric, and transitive.

Now we consider the relations between the infinitesimal generators, their domains, and resolvent operators of equivalent semigroups. To prove the next theorem we will need two results from Hille and Phillips (2). Theorems 12.2.2 and 13.6.1.7.

Theorem 11. Let \( \frac{d}{dt}(t) \left( t \gg 0 \right) \) be a semigroup continuous in the strong operator topology for \( t > 0 \). Then

\[
T(t) = e^{-\alpha t} T(t)
\]

defines a semigroup continuous in the strong operator topology for \( t > 0 \). Also, the relationship between the types of the semigroups is given by \( C_{10}(T(t)) = \omega \lambda \). The range space of \( T(t), t \gg 0 \)

coincides with that of \( \frac{d}{dt}(t), t \gg 0 \)\). If \( A \) and \( B \)

are the infinitesimal generators of \( \frac{d}{dt}(t), t \gg 0 \)\) and

\( T(t), t \gg 0 \)\) respectively, then \( B = A - \alpha I \) and

\( D(B) = D(A) \).
Theorem 12. Let \( \lambda \in \mathbb{C}_0 \), \( t > 0 \) be a semigroup of class \((C_0)\). Then \( \alpha A \) is the infinitesimal generator of the semigroup \( \{ e^{t \alpha A}, t > 0 \} = \mathcal{S}(\alpha; A), t > 0 \)

Moreover, the two semigroups belong to the same basic class.

We can now state and prove the following result.

Theorem 13. Let \( H \) and \( \Sigma \) be equivalent semigroups of class \((C_0)\) in Banach spaces \( \Phi \) and \( \Psi \) respectively. If \( A \) and \( B \) are the infinitesimal generators of \( H \) and \( \Sigma \) respectively, then

\[
H (t) = e^{tA} = \alpha H D(A), \quad D(H) = H D(A)
\]

(4.13)

If \( R(A) \) is the resolvent of \( A \), then

\[
R(A)^{-1} = \Phi (\delta, \alpha A)^{-1}
\]

(4.14)

Proof. In our case we have

\[
S(t) = \Phi (e^{tA} - T(\alpha t)) H^{-1}
\]

(4.15)

By Theorem 11, we have \( D \) (infinitesimal generator of \( e^{tT(\alpha t)} = D(\alpha t) \)). Therefore, from (4.12) it follows that

\( D(B) = H D(A) \). The infinitesimal generator of \( \{ e^{tT(\alpha t)}, t > 0 \} \) is \( \alpha A \). Also, from Theorem 11, the infinitesimal generator of \( \{ e^{tT(\alpha t)} \} \) is \( \alpha \). Finally,

\[
\Phi (x, y) H^{-1} = R(A)^{-1}
\]
$R(\lambda, \beta) = \int_{-\infty}^{\infty} e^{-\lambda t} \mathcal{S}(t) dt$

$= \int_{-\infty}^{\infty} e^{-\lambda t} \left[ \mathcal{H} \left( e^{\omega t} T(\omega) \right) \mathcal{H}^{-1} \right] dt$

$= \mathcal{H} \left( \int_{-\infty}^{\infty} e^{-(\lambda-\omega) t} \frac{1}{T(\omega)} \mathcal{H}^{-1} \right)$

$= \mathcal{H} \left( \lambda - \omega ; \omega \right)^{-1}$

Our next result concerns the sets of fixed points of equivalent semigroups.

**Theorem 14.** Let $\{ S(t) \}, \; t \geq 0,$ and $\{ \tau(t) \}, \; t \geq 0,$ be equivalent semigroups of operators of class $(C_o)$ in Banach spaces $\mathcal{Y}$ and $\mathcal{Y}$ respectively. For $t$ fixed, let $\mathcal{M}$ denote the set of fixed points of $e^{t \mathcal{T}(\omega)}$ and $\mathcal{H}$ denote the set of fixed points of $S(t)$. Then $\mathcal{M} = \mathcal{H}.$

**Proof.** Let $x \in \mathcal{M}$ be a fixed point of $S(t)$ i.e., $x \in \mathcal{M}$. Then

$S(t)x = x = \int_{-\infty}^{\infty} H \left( e^{i \omega t} \mathcal{T}(\omega) \mathcal{H}^{-1} \right) x$

$= e^{i \omega t} \mathcal{T}(\omega) \mathcal{H}^{-1} x$

Hence $H^{-1} x = e^{i \omega t} \mathcal{T}(\omega) \mathcal{H}^{-1} x$. Hence $\mathcal{M} \subseteq \mathcal{H}.$

Now, let $y \in \mathcal{H}$ be a fixed point of $e^{i \omega t} \mathcal{T}(\omega)$ i.e., $y \in \mathcal{H}.$ Then

$e^{i \omega t} \mathcal{T}(\omega)y = y = (H^{-1} \mathcal{T}(\omega) + \cdot \cdot \cdot )y$

Hence $\mathcal{H} \subseteq \mathcal{M}.$
Therefore \( H \psi \in \mathcal{N} \), or \( \psi \in \mathcal{H}^{-1} \mathcal{N} \). But \( \psi \in \mathcal{N} \), so we have \( \mathcal{N} \subset H^{-1} \mathcal{N} \) or \( \mathcal{N} \subset \mathcal{H}^{-1} \mathcal{N} \). This, together with the inclusion relation \( \mathcal{H} \subset \mathcal{N} \), establishes the identity \( \mathcal{N} = \mathcal{H} \).

Our final result concerns equivalent semigroups of contraction operators in Hilbert space.

**Theorem 15.** Let \( \{ S(t), \ t \geq 0 \} \) and \( \{ T(t), \ t \geq 0 \} \) be equivalent semigroups of contraction operators in Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively. If \( A \) and \( B \) are the infinitesimal generators of \( \{ S(t), \ t \geq 0 \} \) and \( \{ T(t), \ t \geq 0 \} \) respectively, then

\[
R(\lambda; B) = H R(\lambda; A) H^{-1} = P \int_0^\infty e^{-\lambda t} \frac{d}{dt} \left( H E(\lambda t) H^{-1} \right) dt
\]

where \( P \) is a projection of the dilation space \( \mathcal{H}_A \) onto \( \mathcal{H}_B \) and \( E(\lambda t) \) is the resolution of the identity in the Sz.-Nagy representation of \( T(t) \).

**Proof.** By theorem 5, \( T(t) \) admits the representation

\[
T(t) \psi = P \int_0^\infty e^{-\lambda t} \frac{d}{dt} E(\lambda t) \psi, \quad \psi \in \mathcal{H}_A.
\]

Then

\[
T(x(t)) \psi = P \int_0^\infty e^{x(t) \lambda} \frac{d}{dt} E(\lambda t) \psi,
\]

since the infinitesimal generator of \( \{ T(x(t)), \ t \geq 0 \} \) is \( x A \),
we have

\[ R(\lambda - \omega, \lambda A) = \int_0^\infty e^{-\lambda t} + (\lambda - \omega) t \, dt = \int_0^\infty e^{-\lambda t} + (\lambda - \omega) t \, dt \]

\[ = \int_0^\infty e^{-\lambda t} \left\{ \int_0^\infty e^{\lambda t} \right\} d(P \in (\xi)) \, dt \]

\[ = \int_0^\infty d(P \in (\xi)) \int_0^\infty (-\lambda t + \lambda t - \omega t) \, dt \]

Now

\[ R(\lambda - \omega, \lambda A) = \int_0^\infty d(P \in (\xi)) \frac{1}{\omega - \lambda} \]

Finally

\[ \mathcal{R}(\lambda; \rho) = \mathcal{R}(\lambda - \omega, \lambda A) R^{-1} \]

\[ = \mathcal{P} \int_0^\infty d(P \in (\xi) R^{-1}) \frac{1}{\omega - \lambda + \omega} \]

We close this section by pointing out that equivalent semigroups have found applications to parabolic differential equations \( \mathcal{L} \), and Markov processes \( \mathcal{L} \), and may be of interest in quantum mechanics (cf. section 5.30).
REFERENCES.


CHAPTER 5

Semigroup Methods In Mathematical Physics

5.1 Introduction

In Chapter 1, we pointed out that semigroup methods enter mathematical physics through the initial-value problems that arise in the description of the evolution of physical systems. It is of interest in this connection to quote from the recent book of G. W. Mackey (11): "Our fundamental viewpoint is that the change in time of a physical system may be described by a one-parameter semi-group \( U \) acting on a set \( S \) and that the laws of physics make assertions about the structure of \( S \) and the "infinitesimal generator" of \( U \)."

This chapter is devoted to some applications of semigroup methods in mathematical physics. In Section 2 we consider the semigroup associated with two well-known partial differential equations of mathematical physics. Section 3 is devoted to semigroups in quantum mechanics; and, finally, in Section 4, we consider the use of semigroup methods in obtaining the solutions of the Boltzmann equation of transport theory.
5.2 Semigroups Associated With Some Differential Equations of Mathematical Physics

A. Introduction: In this section we study the semigroups of operators associated with two of the well-known differential equations of mathematical physics. In section B, we study the equation of diffusion and heat conduction, and in Section C we study the equation of the vibrating string. For additional material on the semigroups considered in these sections we refer the reader to E. Hille [4].

For other studies utilizing semigroup theory to study differential equations of mathematical physics, we refer to the papers of W. Feller [3], and K. Yosida [23, 20, 21, 22].

B. The equation of diffusion and heat conduction.

We will consider the problem of diffusion or heat conduction on the real line. As is well-known, the equation in this case is the parabolic equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(t, x) \quad (t, u) \in f(x)
\]  

(5.1)

The solution of (5.1) is

\[
u(t, x) = \int_{-\infty}^{\infty} k(x - \xi, t) \phi(\xi) d\xi
\]

(5.2)

where

\[
k(x, t) = (\sqrt{\pi t})^{-1/2} e^{-x^2/4t}.
\]

(5.3)
We can write (5.2) in the form
\[
\mathcal{U}(t, \mathcal{X}) = \mathcal{T}(t) \mathcal{F}(\mathcal{X})
\]  
(5.4)

where \( \mathcal{T}(t) \) is an integral operator with kernel given by (5.3) (cf. Example 2, Section 2.2). It is easily verified that \( \mathcal{T}(t) \) is a semigroup operator, and, therefore \( \{ \mathcal{T}(t), t \geq 0 \} \) is a semigroup of operators. We now investigate this semigroup.

In this case we can take as the Banach space \( \mathcal{X} \) the space of continuous functions \( C(-\infty, \infty) \) or the Lebesgue space \( L^p(-\infty, \infty) \) where \( p \) is fixed, \( 1 \leq p < \infty \). In either case \( \{ \mathcal{T}(t), t \geq 0 \} \) is a strongly continuous semigroup of operators in \( \mathcal{X} \). Further, the initial condition is satisfied in the sense that \( \| \mathcal{T}(t)f - f_0 \| \leq \epsilon \) for every \( f \in \mathcal{X} \). We can easily show that \( \| \mathcal{T}(t)f \| \leq |t| \) so that \( \{ \mathcal{T}(t), t \geq 0 \} \) is a contraction semigroup of class \( (C_0) \).

Let us now determine the infinitesimal generator of \( \{ \mathcal{T}(t), t \geq 0 \} \). Equation (5.1) can be written in the form
\[
\frac{\partial u}{\partial t} = A u
\]
where
\[
A = \frac{\partial^2}{\partial x^2}
\]  
(5.5)

We now prove the following theorem.

**Theorem 1.** If \( f'(\gamma) \), \( f'(x) \) and \( f''(\gamma) \) are elements of \( \mathcal{X} \) in either of the two cases considered above, then
\[
\lim_{t \to 0} \left\| \left( \mathcal{T}(t) - I \right) f(x) - \left(f'(x)\right) - \int_0^t \left( \frac{\partial f}{\partial t} \right) dt \right\| = 0
\]
**Proof:** We have to show that the (righthand) difference quotient of \( T(t) \sum f(x) \) converges strongly to \( f''(x) \) as \( t \to 0 \). Under the assumptions of the theorem the second difference quotient of \( f(x) \) converges strongly to \( f''(x) \), that is, if

\[
\frac{f(x, \xi) + f(x-\xi) - 2f(x)}{\xi^2} \quad \xi \neq 0
\]

then \( \| f(x, \xi) \| \) is a continuous function of \( \xi \).

We have, therefore, that,

\[
\frac{\sum T(t) \{ f(x) \} - f(x)}{t} = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\xi^2/t} f(x, \xi) d\xi
\]

and

\[
\| \frac{\sum T(t) \{ f(x) \} - f''(x)}{t} \| \leq \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\xi^2/t} \| f(x, \xi) \| d\xi.
\]

Now, \( \| f(x, \xi) \| \) is a bounded function of \( \xi \) which is continuous, and by (5.6) is equal to zero when \( \xi = 0 \). It follows, therefore, that the singular integral tends to zero as \( t \to 0 \).

Finally, among the functions \( f(x) \) satisfying the required conditions, we have \( e^{-xy} \) and its derivatives. But, for case \( \mathcal{H} = \mathbb{R}(-\infty, \infty) \), linear combinations of these functions are dense in \( \mathcal{H} \), and for the case \( \mathcal{H} = C([-\infty, 0]) \) we add the functions \( f(x) = 1 \) and \( f(x) = \int_{-\infty}^{x} e^{-i} d\xi \).
As an exercise, we ask the reader to show that for
\[ \mathcal{H} = L^2 (-\infty, \infty) \], the operator \( A = \frac{d^2}{dx^2} \) is maximal dissipative with dense domain, and hence the infinitesimal generator of a contraction semigroup.

We close this section by considering a semigroup of operators equivalent to \( \{ T(t) \mid t \geq 0 \} \). For \( f(x) \in L^2 (-\infty, \infty) \) let
\[ f(\lambda) = \int_{\|x\| \rightarrow \infty} \frac{1}{\sqrt{2\pi}} e^{ix\lambda} f(x) \, dx \]
be the Fourier-Plancherel transform of \( f(x) \). In terms of \( f(\lambda) \) we can write
\[ \tau_t (x, x) = \frac{1}{\sqrt{2\pi}} \int_{\|x\| \rightarrow \infty} e^{-it\lambda} f(-x - \lambda t) \, d\lambda \quad (5.7) \]

Let \( \mathcal{F} \) be the space of all such Fourier-Plancherel transforms of elements of \( L^2 (-\infty, \infty) \). We know that the mapping
\[ \mathbb{H} : L^2 (-\infty, \infty) \to \mathcal{F} \]
defined by
\[ \mathbb{H} \left[ \{ f(x) \} \right] = \{ f(\lambda) \} \quad (5.8) \]
is an isometric (and thus homeomorphic) isomorphism. The inverse mapping \( \mathbb{H}^{-1} : \mathcal{F} \to L^2 (-\infty, \infty) \) is, of course,
defined by the inverse transformation
\[ \mathbb{H} \left[ \{ f(\lambda) \} \right] = f(x) = \int_{\|x\| \rightarrow \infty} \frac{1}{\sqrt{2\pi}} e^{ix\lambda} f(\lambda) \, d\lambda \quad (5.9) \].
On the space $Y$ let $\{S(t), t \geq 0\}$ be the semigroup defined by

$$S(t) \left[ f(x) \right] = e^{-\lambda^2 t} f(x)$$  \hspace{1cm} (5.10)

Now, for $f(x) \in L^2(-\infty, \infty)$

$$\left[ H^{-1} S(t) H \right] f(x) = \left[ H^{-1} S(t) \right] H f(x) = \left[ H^{-1} S(t) \right] f(x)$$

$$= H^{-1} \left[ e^{-\lambda^2 t} f(x) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda x^2 - \lambda^2 t} f(x) \, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda^2 x^2 - \lambda^2 t} f(x) \, d\lambda$$

$$= \pi(t, x) \cdot t_y (5.7)$$

$$= T(t) \left[ f(x) \right] \cdot t_y (5.4)$$

Thus

$$T(t) = H^{-1} S(t) H$$  \hspace{1cm} (5.11)

or

$$S(t) = H T(t) H^{-1}$$  \hspace{1cm} (5.12)
Therefore, we have shown that the semigroups \( \{ T(t) \} \), \( t > 0 \)
and \( \{ S(t) \}, t > 0 \)
are equivalent with \( r = 0 \), \( \alpha = 1 \),
and \( H \) as defined by (5.8).

C. The equation of the vibrating string.

We consider the initial-value problem given by the hyperbolic equation

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial u}{\partial x}, \quad \Omega = \Omega(t, x) \\
\left. \frac{\partial u}{\partial t} \right|_{t=0} &= f_2(x) \\
u(t, x) &= f_1(x)
\end{align*}
\]

(5.13)

The solution of Equation (5.13) is

\[
u(t, x) = \frac{1}{2} \int_{x-t}^{x+t} f_1(x + \xi) \, d\xi - f_1(x) + \frac{1}{2} \int_{x-t}^{x+t} f_2(\xi) \, d\xi
\]

(5.14)

We assume that the derivative of \( f_1(x) \) is absolutely continuous, and that \( f_2(x) \) is absolutely continuous. Then

\( \nu(t, x) \) as given (5.14) satisfies Equation (5.13) for almost all \( (t, x) \).

In this case let \( X \) be the space of vector functions

\( F(x) = (f_1(x), f_2(x)) \), where \( f_1(x) \) and \( f_2(x) \) are bounded and absolutely continuous together with their first derivatives

on \( (-\infty, \infty) \). We define the norm \( \| F \| \) as the supremum

of the absolute values of the functions
and \( f_\nu(x) \). With the norm so defined \( \mathcal{H} \) is a Banach space. If we put \( F(t, x) = (u(t, x), \varphi(t)/\partial t) \), we can write

\[
F(t, x) = T(t) \left[ F(x) \right]
\]

where

\[
T(t) = \begin{pmatrix}
\frac{\partial}{\partial t} & \varphi(t)
\\
\frac{\partial^2}{\partial t^2} & \frac{\partial}{\partial t}
\end{pmatrix}
\]

(5.16)

with

\[
\varphi(t)[f] = \frac{1}{2} \int_{x-t}^{x+t} f(\xi) d\xi
\]

(5.17)

Clearly \( T(0) = I \), and a rather involved calculation shows that \( T^{-1}(t) \) exists. \( T(t) \) is a group of operators.

Since the operator \( T(t) \) is a matrix, its infinitesimal generator will also be a matrix, but the identification of this operator with the right-hand side of Equation (5.13) will be obvious. We have, by definition,

\[
A_\varepsilon = \frac{\partial^2}{\partial t^2} \left[ T(t) - T(\varepsilon^2) \right] = \begin{pmatrix}
\frac{\partial^2}{\partial t^2} & \frac{\partial}{\partial t}
\\
\frac{\partial^2}{\partial t^2} & \frac{\partial^2}{\partial t^2}
\end{pmatrix}
\]

(5.18)
Now, if \( f'(x) \) is continuous, the diagonal elements

\[
\lim_{\xi \to 0} \frac{g(\xi) - I}{2\xi} \int_{-\xi}^{\xi} f(x) \, dx = \frac{1}{2\xi} \left\{ f(x + \xi) + f(x - \xi) - 2f(x) \right\}.
\]

Also, if \( f''(x) \) is continuous,

\[
\lim_{\xi \to 0} \frac{g(\xi)}{2\xi} \int_{-\xi}^{\xi} f'(x) \, dx = \frac{1}{2\xi} \int_{-\xi}^{\xi} f'(x) \, dx \to f'(x) \quad \text{as} \quad \xi \to 0.
\]

Finally,

\[
\lim_{\xi \to 0} \frac{g'(\xi)}{2\xi} \int_{-\xi}^{\xi} f''(x) \, dx = \frac{1}{2\xi} \left[ f''(x + \xi) - f''(x - \xi) \right] \to f''(x)
\]

whenever the limit exists. Hence it follows that

\[
A = \lim_{\xi \to 0} A_{\xi} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}
\] (5.10)

is the infinitesimal generator of the group \( \{ T(t), t \in (-\infty, \infty) \} \).
5.3 Semigroups Of Operators In Quantum Mechanics

A. Introduction. In this section we consider the use of semigroup methods in quantum mechanics. Section B is devoted to a brief discussion of the Schrödinger equation of quantum mechanics from the point of view of semigroup theory. We also give as a simple example the semigroup associated with the classical harmonic oscillator. We regret that time did not permit us to include a discussion of two very interesting applications of semigroup theory. We refer to the work of J. Feldman [2] on the Schrödinger equation for non-negative potentials, and the work of E. Nelson [13] on Feynman integrals and the Schrödinger equation. A discussion of their work will be given in the revised edition of these notes. In Section C we study semigroups and groups of dynamical mappings in quantum mechanics, and in Section D we make a few remarks about equivalent quantum mechanical systems.

B. The Schrödinger Equation From the Point of View of Semigroup Theory.

In the Schrödinger representation, the wave function $\Psi$ satisfies the differential equation

$$\frac{\partial \Psi}{\partial t} = -\hat{H} \Psi$$

(5.20)

where $\hat{H}$ is the Hamiltonian operator, and $\Psi = \Psi(t,x)$ in the one-dimensional case. Equation (5.20) is the abstract form of Schrödinger's equation, and can be regarded as a differential equation in a concrete Hilbert space $\mathcal{H}$. In Chapter 1 we...
in the form
\[ \psi(t, x) = \mathcal{S}(t) \psi(0, x) \]  \hspace{1cm} (5.21)

where \( \psi(0, x) \) is some suitable chosen function in \( \mathcal{H} \). If we introduce the operator-valued function \( \mathcal{S}(t) = e^{-\frac{i}{\hbar} H t} \), then (5.21) can be written as

\[ \psi(t, x) = \mathcal{S}(t) \left[ \psi(0, x) \right] \]  \hspace{1cm} (5.22)

Because of the exponential character of \( \mathcal{S}(t) \), it is clear that \( \mathcal{S}(0) = \mathbb{I} \) and that for all \( t > 0 \),

\[ \mathcal{S}(t + s) = \mathcal{S}(s) \mathcal{S}(t); \text{\forall} t > 0, \mathcal{S}(t) \text{ is a semigroup of operators in } \mathcal{H} \]. We will call this semigroup the Schrödinger semigroup.

The problem now is that of when a given Hamiltonian operator is the infinitesimal generator of a Schrödinger semigroup. This problem can be solved by the theory given in Section 3.4. We recall that an operator \( H \) on a Hilbert space \( \mathcal{H} \) is said to be conservative if \( (Hx, x) + (x, Hx) = 0 \), and that \( H \) is conservative if and only if \( iH \) is symmetric. Further, if \( H \) is conservative and maximal dissipative (\( i.H \). Definition 3.1) with dense domain, then \( iH \) is maximal symmetric. Theorem 3.5 then states that a necessary and sufficient condition that a Hamiltonian \( H \) generates a Schrödinger semigroup of isometric operators is that \( iH \) be maximal symmetric (i.e. \( H \) be conservative and maximal dissipative with dense domain.)
Since \( S(t) \) is an isometric operator, we can use Sz-Nagy's representation theorem (Theorem 4.5) to obtain the representation

\[
S(t) = D_n(v)
\]  

(5.23).

If \( \mathcal{H} \) is already its dilation space, then \( D_n \) and the Schrödinger semigroup is given by a semigroup of unitary operators. From Stone's theorem, we obtain

\[
S(t) = \int_{\mathcal{H}} e^{i A(t) \lambda} \mathcal{E}(\lambda)
\]

(5.24)

In order to obtain \( S(t) \) explicitly, the resolution of the identity \( \int_{\mathcal{H}} \mathcal{E}(\lambda) \frac{d\lambda}{2\pi} \) must be determined. At this stage Drolph's theorem (Theorem 4.9) together with an inversion operation can be used to determine \( \mathcal{E}(\lambda) \).

We close this section by considering the semigroup of operators associated with the classical harmonic oscillator. In classical mechanics the harmonic oscillator of unit mass and frequency \( \nu \) is described by the Hamilton equations

\[
\frac{d\varphi}{dt} = -\nu^2 \varphi
\]

\[
\frac{d\varphi}{dt} = \nu^2 \varphi
\]

where \( \varphi \) and \( p \) are the position coordinate and momentum, respectively, and \( \nu^2 = \pi \nu \). The solution of (5.25) is

\[
\varphi(t) = \nu(t(v) \cos \nu t + \frac{1}{2} \nu \sin \nu t)
\]

\[=-\nu(v) \varphi \Delta v x t + \frac{1}{2} \nu \cos \nu t
\]

(5.26)
(5.26) can be written in matrix form as

\[
\begin{pmatrix}
q(t) \\
p(t)
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha t & \frac{1}{\alpha} \sin \alpha t \\
-\frac{1}{\alpha} \sin \alpha t & \cos \alpha t
\end{pmatrix}
\begin{pmatrix}
q(0) \\
p(0)
\end{pmatrix}
\] (5.27)

Let us put

\[
T(t) =
\begin{pmatrix}
\cos \alpha t & \frac{1}{\alpha} \sin \alpha t \\
-\frac{1}{\alpha} \sin \alpha t & \cos \alpha t
\end{pmatrix}
\] (6.28)

We will show that \( \{ T(t), t \geq 0 \} \) is a semigroup of operators. Clearly \( T(0) = I \). For \( t_1, t_2 \geq 0 \), we have

\[
q(t_1 + t_2) = q(0) \cos \alpha (t_1 + t_2) + \frac{p(0)}{\alpha} \sin \alpha (t_1 + t_2)
\]

\[
p(t_1 + t_2) = p(0) \cos \alpha (t_1 + t_2) - \alpha q(0) \sin \alpha (t_1 + t_2)
\]

which can be rewritten as

\[
q(t_1 + t_2) = \left( q(0) \cos \alpha t_2 + \frac{p(0)}{\alpha} \sin \alpha t_2 \right) \cos \alpha t_1 +
\]

\[
\left( \frac{p(0) \cos \alpha t_2 - \alpha q(0) \sin \alpha t_2}{\alpha} \right) \sin \alpha t_1,
\]

\[
p(t_1 + t_2) = \left( p(0) \cos \alpha t_2 - \alpha q(0) \sin \alpha t_2 \right) \cos \alpha t_1
\]

\[- \times \left( q(0) \cos \alpha t_2 + \frac{p(0)}{\alpha} \sin \alpha t_2 \right) \sin \alpha t_1.\]
Hence, \( \psi(t) \) is the solution of (5.28) and the above calculations, we obtain

\[
T(t_1 + t_2) = T(t_1) T(t_2), \quad t_1, t_2 \geq 0.
\]

Let us now consider the infinitesimal generator \( A \) of \( \{ T(t), \ t \geq 0 \} \). From equation (5.25), and from (5.28), it is clear that

\[
A = \begin{pmatrix}
0 & 1 \\
-\alpha^2 & 0
\end{pmatrix}
\]

Hence equation (5.25) could have been written as

\[
\frac{d}{dt} \begin{pmatrix}
\psi(t) \\
\phi(t)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-\alpha^2 & 0
\end{pmatrix} \begin{pmatrix}
\psi(t) \\
\phi(t)
\end{pmatrix}
\]

(5.30)

We close this section with some problems that the reader may wish to consider.

1. What is the space on which \( T(t) \) and \( A \) operate?
2. Is \( \{ T(t) \} \) also a group of operators?
3. Compute \( \| T(t) \| \)
4. Compute \( \mathbb{R}(\chi, \lambda) \)
5. For the quantum harmonic oscillator, the Schrödinger equation is given by

\[
\frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + \frac{1}{2} m \alpha^2 \psi^* \psi.
\]

(5.31)

Using the matrix approach used in the study of the vibrating string (Section 5.2 C), write down the infinitesimal generator of the Schrödinger semigroup associated with the harmonic oscillator.
C. Semigroups of Dynamical Mappings of Density Operators.

It is well-known that the quantum mechanical state of a physical system can be specified by a density operator \( \mathcal{D} \) which satisfies the following conditions:

(a) \( (\Phi, \mathcal{D}\Psi) = (\mathcal{D}\Phi, \Psi) \) (Hermiticity)

(b) \( (\Phi, \mathcal{D}\Phi) \geq 0 \) (Positive-definiteness)

(c) \( T_n \mathcal{D} = 1 \) (Normalization)

where \( \Phi \) and \( \Psi \) are any elements of the Hilbert space on which the operator \( \mathcal{D} \) is defined. The density operators form a convex set, the extremal elements of which are the pure state density operators, and these elements are projection operators onto one-dimensional subspaces of \( \mathcal{H} \). It has been shown that the density operators belong to the Hilbert space \( \mathcal{H} \) of operators \( \mathcal{D} \) on \( \mathcal{H} \) for which \( T_n (\mathcal{D}^* D) \) is finite, the inner product in \( \mathcal{H} \) being defined by \( \langle \mathcal{D}_1, \mathcal{D}_2 \rangle = T_n (\mathcal{D}_1^* \mathcal{D}_2) \).

In a recent series of papers, T.F. Jordan and E.C.G. Sudarshan \([6,7,17]\) have studied dynamical mappings of density operators. Let \( T \) be a linear operator on \( \mathcal{H} \) such that, if \( \mathcal{D} \) is a density operator, then

\[
\mathcal{D}' = T \mathcal{D} \tag{5.32}
\]

is also a density operator. Such an operator \( T \) is called a dynamical mapping. It is clear that in order to represent dynamics in the usual sense, that is as a continuous time-dependent evolution of the state of a physical system, we must
\[ \mathcal{D} \rightarrow \mathcal{D}(t) = \mathcal{T}(t) \mathcal{D}. \quad (5.33) \]

where the dynamical mapping \( \mathcal{T} \) is a function of the real parameter \( t \). From stationarity considerations we require that \( \mathcal{T}(s+t) = \mathcal{T}(s), \mathcal{T}(t) \) and that \( \mathcal{T}(0) = \mathcal{I} \); that is to say, we require that the family of dynamical mappings \( \{ \mathcal{T}(t), t \geq 0 \} \) form a one-parameter semigroup. Let \( \mathcal{F} \) be a self-adjoint operator belonging to \( \mathcal{G} \). We also require that the expectation value

\[ E_T \left( \mathcal{F} \mathcal{D}(t) \right) = (\mathcal{F}, \mathcal{T}(t) \mathcal{D}). \quad (5.34) \]

be a continuous function of \( t \). Since the trace of the product is expressed as the inner product in \( \mathcal{G} \), the above requirement is equivalent to requiring that the semigroup operator \( \mathcal{T}(t) \) be weakly continuous as a function of \( t \).

Let us now consider various "models" for the evolution of density operators.

1. The mathematical model for the evolution of density operators as described above is that

(A) \( \{ \mathcal{T}(t), t \geq 0 \} \) be a weakly continuous semigroup of dynamical mappings in \( \mathcal{G} \).

(B) \( \{ \mathcal{T}(t), t \in (-\infty, \infty) \} \) is a weakly continuous group of dynamical mappings.
(3) A dynamical mapping is said to be a Hamiltonian dynamical mapping if it admits a representation as a unitary operator. In order to represent Hamiltonian dynamics we require the following model:

(C) There exists a (strongly or weakly) continuous group of unitary operators \( \mathcal{U}(t), t \in (-\infty, \infty) \) on \( \mathcal{H} \) such that \( T(t) \mathcal{D} = \mathcal{U}(t) \mathcal{D} \mathcal{U}^*(t) \) for each \( \mathcal{D} \in \mathcal{F} \).

The relationship between the above models can be expressed as follows:

\[
C \Rightarrow B \Rightarrow A,
\]

\[
P \Rightarrow C \quad \text{and} \quad A \not\Rightarrow B
\]

The main theorem given in [6] can be stated as follows:

Theorem 2. A sufficient condition that a family of dynamical mappings \( \{T(t)\} \) represent Hamiltonian dynamics is that \( \{T(t)\} \) be a weakly continuous group.

We close this section by remarking that these are several open problems which seem to be of interest. Some of these are as follows:

(1) What operator \( A \) is the infinitesimal generator of the group of operators, and how is this related to the Hamiltonian operator in the Schrödinger representation? We refer to Equation (5.35) for a differential equation relationship between \( T(t) \) and \( H \).

(2) What generation theorems are required in the study of dynamical mappings, and how do the conditions of these theorems factor into the problem?
(3) What, if any, is the relationship between the Schrödinger semigroup \( \mathcal{S}(t) \), \( t \geq 0 \) and the group of dynamical mapping? It is known, for example, that if the Hamiltonian is self-adjoint, then the differential equation for \( \mathcal{D}(t) \) is given by

\[
\frac{\partial \mathcal{D}}{\partial t} = -\frac{i}{\hbar} \left( -\mathcal{D} - \mathcal{D} \mathcal{H} \right)
\]  

(5.35)

and in this case

\[
\mathcal{D}(t) = e^{-\frac{\mathcal{H} t}{\hbar}} \mathcal{D}(0) e^{\frac{\mathcal{H} t}{\hbar}}
\]

\[
= \mathcal{S}(t) \mathcal{D}(0) \mathcal{S}^*(t) \mathcal{Q}
\]

(5.36)

D. Equivalent Quantum-Mechanical Systems. In Section 4.4, we introduced the notion of equivalent semigroups of operators, and discussed some of their properties. In this section we consider what will be called equivalent quantum-mechanical systems and make a few brief remarks about such systems.

We will define a **quantum-mechanical system** to be a triple \( \mathcal{H}, \mathcal{S}(t), \mathcal{H} \) where \( \mathcal{H} \) is a concrete Hilbert space, \( \mathcal{S}(t) \) is a Schrödinger semigroup operator, and \( \mathcal{H} \) is the Hamiltonian. Let \( \mathcal{H}_1, \mathcal{S}_1(t), \mu_1 \) and \( \mathcal{H}_2, \mathcal{S}_2(t), \mu_2 \) be two quantum-mechanical systems. We will say that the two systems are **equivalent** if the Schrödinger operators are equivalent in the sense of Definition
\[ \psi_2(t) = M \left( e^{-iH\tau} \psi_1(x) \right) M^{-1} \]  

(5.37)

where \( M \) is a linear homeomorphism of \( H_1 \) onto \( H_2 \) and \( \alpha \) are constants, with \( \alpha \) real and positive.

The first example of equivalent quantum-mechanical systems is given by the well-known equivalence of the Heisenberg and Schrödinger representations (see [5, 12, 14]). Let \( \psi_2 \) and \( \gamma_2 \) represent the wave functions in the Heisenberg and Schrödinger representations, respectively. Then equivalence, with \( \omega = \alpha \) and \( \chi = 1 \), follows from the following relations:

\[ \psi_2 = \psi_2 \left( t \right) \left[ \gamma_2 \left( \delta, x \right) \right] = e^{-i\frac{\xi}{\alpha}} \left[ \psi_2 \left( t, x \right) \right] \]

\[ \psi_2 = \gamma_2 \left( \delta, x \right) \]

\[ \gamma_2 = \gamma_2 \left( t, x \right) \]

\[ \psi_2 = \gamma_2 \left( t \right) \left[ \gamma_2 \right] \]

\[ \gamma_2 = \gamma_2 \left( t \right) \left[ \gamma_2 \right] = \gamma_2 \left( t \right) \left[ \gamma_2 \right] \]

The question of interest to us is how this notion of equivalence might be used in the solution of problems in quantum mechanics. It is known that all Hilbert spaces are isomorphic, hence this implies the existence of a linear homeomorphism \( M \) between any two Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), which in turn means that all quantum-mechanical systems are equivalent. This however does not preclude the usefulness of the notion of equivalence, for one problem of interest would be to use this notion in deducing certain
properties of a complex system from known properties of an equivalent simpler system.

5.4 Semigroups As Solutions Of The Boltzmann Equation Of Transport Theory

A. Introduction In this section we utilize semigroup theory to obtain the solution of the linearized Boltzmann equation of transport theory. We will consider the case of an infinite plane slab of transport material extending from 

\[ -\infty < x < 0. \]

Let \( \sigma \) denote the probability of a collision occurring between a fixed nucleus and a particle moving between \( z \) and \( z + d(z) \) in either direction; \( \sigma \) is called the cross section, and we assume that it is constant. We also assume that the production of particles is isotropic, i.e. the direction of particles arising from a collision is independent of the colliding particle. The slab is assumed to be surrounded by a vacuum so that no particles may enter the slab from the outside.

The problem may be formulated as follows:

\[
\frac{1}{c} \frac{\partial N}{\partial t} + \mu \frac{\partial N}{\partial z} + \sigma N = \frac{\gamma}{2} \int_{-1}^{1} N(z, \xi, t) d\xi \quad (5.38)
\]

where \( N = N(z, \mu, t) \), denotes the density of the neutron beam in directions with \( \mu \)-direction - \( \xi = \mu \), and \( t \) denotes time. We have the boundary and initial conditions

\[
\begin{align*}
N(z, \mu, t) &= 0, & \mu < 0, & t > 0 \\
N(z, \mu, t) &= 0, & \mu > 0, & t > 0 \end{align*} \quad (5.39)
\]
\[ N(\tau, \mu, \sigma) = f(\tau, \mu), \quad -a \leq \tau \leq a, \quad -1 \leq \mu \leq 1 \quad (5.40) \]

Equation (5.38) can be simplified by putting
\[ n(\tau, \mu, \sigma) = e^{\tau \sigma} N(\tau, \mu, \sigma) \]
and then putting \( \sigma = 1 \) and \( \tau = 1 \). Then (1) becomes
\[ \frac{\partial n}{\partial \tau} = -\mu \frac{\partial n}{\partial \mu} + \frac{\rho}{2} \int_{-1}^{1} n(\tau, \xi, \sigma) d\xi, \quad (5.41) \]

with the boundary and conditions are not changed, but in the new notation we write
\[ n(\mu, \mu, \sigma) = 0, \quad \mu < 0, \quad \mu > 0 \quad (5.42) \]
\[ n(-\mu, \mu, \sigma) = 0, \quad \mu > 0, \quad \sigma > 0 \]
\[ n(\tau, \mu, \sigma) = f(\tau, \mu), \quad -a \leq \tau \leq a, \quad -1 \leq \mu \leq 1 \quad (5.38) \]

**B. Solution of the Boltzmann equation.** Let us now write (5.41) as the operator equation
\[ \frac{\partial n}{\partial \tau} = A \cdot n \quad (5.44) \]
where
\[
A = -\lambda \frac{\partial}{\partial \xi} + \frac{x}{2} \int_{-1}^{1} d\xi \frac{d}{dx} \tag{5.45}
\]

We wish to solve the operator equation (5.44) using semigroup theory. Hence the first thing to do is select a concrete Banach space, the elements of which are the functions \( \psi(r, \mu, \xi) \), and on which \( A \) operates. As our space we choose the Hilbert space \( \mathcal{H} \) of complex-valued functions \( \psi(r, \mu) \) defined and Lebesgue-square-integrable over the rectangle \( b > 1, a < 1 \)

that is,
\[
\int_{-1}^{1} \int_{-1}^{1} |\psi(r, \mu)|^2 \, dz \, d\mu < \infty.
\]

For functions \( \psi, \phi \) in \( \mathcal{H} \), the inner product, as usual, is given by
\[
(\psi, \phi) = \int_{-1}^{1} \int_{-1}^{1} \psi(r, \mu) \overline{\phi(r, \mu)} \, dz \, d\mu,
\]

and the norm of an element \( \psi \in \mathcal{H} \) is given by
\[
||\psi|| = \sqrt{(\psi, \psi)} = \sqrt{\int_{-1}^{1} \int_{-1}^{1} |\psi(r, \mu)|^2 \, dz \, d\mu}\]

Hence
\[
\mathcal{H} = \{ \psi(r, \mu) : |r| < a, ||\psi|| < \infty \}.
\]

Let us now rewrite the operator as
\[
A = -D + \frac{x}{2} \frac{d}{dx} \tag{5.46}
\]
where
\[ D = \mu \frac{\partial}{\partial z} \]
(5.47)
and
\[ J = \frac{1}{2} \int_{-1}^{1} \cdot \cdot \cdot d \xi \]
(5.48)

hence \( A \) can be regarded as the operator obtained by the perturbation of the operator \(-D\) by the operator \( J \). Let \( D(D) \) be the set of functions \( g \in \mathcal{H} \) such that \( g \) is absolutely continuous in \( z \) for each \( \mu \), such that \( \| g \|_{L^2} \leq 1 \), and such that \( \mathcal{E} \) exists for each \( z \) such that \( |z| \leq \omega \), and \( Jg \in \mathcal{H} \). Finally, \( D(A) \) is defined as the set of functions \( g \in D(D) \cap D(J) \), and such that

\[ g(a, \mu) = 0 \quad -1 \leq \mu < 0 \]
\[ g(-a, \mu) = 0 \quad 0 < \mu \leq 1 \]
(5.49)

Hence \( A \) is a linear operator on \( \mathcal{H} \) with domain \( D(A) \).

We remark that the operator adjoint to \( A \) is
\[ A^* = \mu \frac{\partial}{\partial z} + \frac{\gamma}{2} \int_{-1}^{1} d \xi \]
(5.50)

Thus, unfortunately, \( A \) is not a self-adjoint operator.

As an initial-value problem, we seek a solution of equation (5.44) of the form
\[ v(z, \mu, t) = T(t) \left[ f(z, \mu) \right] \]
(5.51)
where \( T(t), t \geq 0 \) is a semigroup of operators in \( \mathcal{H} \).

In order to obtain such a solution there are two things that need to be done: (a) We must characterize the spectrum and resolvent...
set of $A$, and (b) we must show that $A$ is the infinitesimal generator of a semigroup of operators in $\mathcal{B}_F$. We refrain from a complete discussion of (a) since it is rather involved and lengthy, and the interested reader can find this discussion in $[9, 18]$. We simply state as a theorem a summary of the known results.

Theorem 3. The spectrum and resolvent set of the linear operator $A$ defined by (5.45) are as follows:

$\mathcal{R}_\sigma(A)$: a finite nonempty point-set lying on $\lambda > 0$

$\mathcal{R}(A)$: empty

$\mathcal{C}_\sigma(A) = \{ \lambda : R_\sigma(\lambda) \leq 0 \}$

$\mathcal{R}(\lambda) = \{ \lambda : R_\sigma(\lambda) > 0 \}$, det $\sigma(1 - P_\lambda(\lambda)) > 0$.

We will denote by $\beta_1 > \beta_2 > \ldots > \beta_n > \lambda > 1$

the points of the point spectrum, and denote by

$\varphi_1(\beta_1), \varphi_1(\beta_2), \ldots, \varphi_1(\beta_n)$

the associated eigenfunctions.

To turn now to the generation problem, we first observe that since $A$ is a linear operator on a Hilbert space there are two methods available for us to use in determining that $A$ is the infinitesimal generator of a semigroup: firstly, we could employ the Hille-Yosida theorem (Theorem 3.2), or secondly, we could show that $A$ is a maximal dissipative operator and use Phillips' theorem (Theorem 3.4). We will utilise the Hille-Yosida theorem. Hence we must show that (1) $D(A)$ is dense in $M$,
(ii) \( A \) is a closed operator, and (iii) \( \| k(\lambda, A) \| < \frac{1}{\lambda}, \lambda \neq 0 \)

To prove (i) we proceed as follows. Let \( P(z, \mu) \) be an arbitrary polynomial in \((z, \mu)\), where \( |P| \leq M \), \( |z| \leq a \), \( |\mu| \leq 1 \). These polynomials are dense in \( \mathcal{A} \). Now, let \( q_v(z, \mu) \) be a function such that for fixed \( \mu, \mu \in [0,1] \),
\[
q_v(z) = q_v(z, \mu) = -a + \frac{\epsilon}{m^2} \leq z \leq a,
\]
and such that \( q_v(z, \mu) = 0 \) and \( q_v(z, \mu) \) is linear in \( z \) for \(-a \leq z \leq -a + \frac{\epsilon}{m^2}\). For \( \mu < 0 \) the function \( q_v(z, \mu) \) is defined in an analogous manner.

Charly \( q_v(z, \mu) \in D(A) \) and a routine estimation yields \( \| q_v - P \| \leq 8 \epsilon \). Hence \( D(A) \) is dense in \( \mathcal{A} \).

To prove (ii) we will show that \( R(\lambda; A) = (\lambda - A)^{-1} \) is bounded, and that \( D(R(\lambda; A)) \subseteq \mathcal{A} \). From this it follows that \( R(\lambda; A) \) is closed, and this in turn implies that \( A \) is closed.

Let \( M \) be the set of points in the right half of the \( \lambda \)-plane exclusive of the points of \( \mathcal{P}_0(A) \) and let
\[
\beta = R_2(\lambda).
\]
We consider the solution of the equation
\[
(\lambda - A)\nu = q, \quad q \in \mathcal{A} \quad (5.2)
\]
Hence we seek \( \nu = (\lambda - A)^{-1}q = R(\lambda; A)q \). The formal
solution of equation (5.52) is
\[
\psi(z, \mu) = R(\lambda, \mu) g = \int_1^\infty \exp \left\{ -\frac{\lambda(z-\xi)}{\mu} \right\} \left[ \frac{1}{2} \psi(\xi) + g(\xi, \mu) \right] d\xi, \quad \mu > 0
\]
where
\[
\psi(z) = \int_1^\infty \psi(z, \mu) d\mu
\]
It can be shown that \( \psi(z) \) satisfies the equation
\[
\frac{2}{\gamma} \psi(z) = \int_{-\infty}^\infty E(\lambda | z - \xi |) \psi(\xi) d\xi + G(\lambda, z)
\]
where
\[
E(\omega) = \int e^{-\frac{i\omega t}{\gamma}} e^{\frac{t}{\gamma}}, \quad Re \omega \geq 0, \quad \omega \neq 0.
\]
We can rewrite (5.54) as
\[
\left( \frac{2}{\gamma} \right) \psi = L \lambda \left[ \psi \right] + G(\lambda, z) \quad (5.55)
\]
or
\[
\left( \frac{2}{\gamma} - L \lambda \right) \psi = G(\lambda, z)
\]
where
\[
G(\lambda, z) = \int_1^\infty \int_{-\infty}^{\infty} \exp \left\{ -\frac{\lambda(z-\xi)}{\mu} \right\} g(\xi, \mu) d\xi d\mu \quad (5.56)
\]
Let $\mathcal{H}$ denote the Hilbert space of functions square-integrable on $(-a, a)$. Since $\frac{2}{\gamma}$ is not an eigenvalue of the kernel of (5.56) when $\lambda \not\in \Gamma$, it follows that $\phi(z)$ is uniquely determined for any choice of $\mathcal{G}(\lambda, z)$ in $\mathcal{H}$. That is, the inverse operator $\left(\frac{2}{\gamma} - L\lambda\right)^{-1}$ exists when $\lambda \in \mathbb{M}$ and $\mathcal{G} \left(\frac{2}{\gamma} - L\lambda\right) \in \mathcal{H}$.

Now the operator $\frac{2}{\gamma} - L\lambda$ is bounded by virtue of the finiteness of the norm of $\mathbb{M}$, and it maps $\mathcal{H}$ onto itself. Hence, it follows from theorem N 23 that $\left(\frac{2}{\gamma} - L\lambda\right)^{-1}$, for $\lambda \in \Gamma$, is bounded.

Let us now estimate $\mathcal{G}(\lambda, z)$. From (5.56) we have:

$$|\mathcal{G}(\lambda, z)| \leq \int_{-a}^{a} \int_{-a}^{a} e^{\beta t / \mu} \left| \mathcal{G}(z, t, \mu) \right| dt d\mu \leq \int_{-a}^{a} \int_{-a}^{a} e^{\beta t / \mu} \left| \mathcal{G}(z, t, \mu) \right| dt d\mu = \int_{-1}^{1} k_1(z, \mu) d\mu + \int_{-a}^{a} k_2(z, \mu) d\mu = \mathcal{G}(1, z) + \mathcal{G}_2(\lambda, z).$$

Now,

$$k_1(z, \mu) = \int_{0}^{\infty} e^{-\beta t / \mu} \left| \mathcal{G}(z, t, \mu) \right| dt \leq \int_{0}^{\infty} e^{-\beta t / \mu} \left| \mathcal{G}(z, t, \mu) \right| dt \leq \int_{-a}^{a} e^{-\beta t / \mu} \left| \mathcal{G}(z, t, \mu) \right| dt.$$

where we have defined $\mathcal{G}(z, \mu) = 0$ for $|z| > a$. 

--- 130 ---
An application of Schwartz's inequality yields

\[ |k_1(z, \mu)|^2 \leq \frac{1}{|\beta|} \int_0^\infty \frac{1}{\sqrt{\pi}} e^{-\beta \tau/\mu} \left| \frac{g(z - \tau, \mu)}{1 + \tau} \right|^2 d\tau \]

Similarly, the use of Schwartz's inequality also yields

\[ |g_1(\lambda, z)|^2 \leq \int_0^1 |k_1(z, \mu)|^2 d\mu \]

\[ \leq \frac{1}{|\beta|} \int_0^1 \int_0^\infty \frac{1}{\sqrt{\pi}} e^{-\beta \tau/\mu} \left| \frac{g(z - \tau, \mu)}{1 + \tau} \right|^2 d\tau d\mu \]

and

\[ \int_{-a}^a \left| G_1(\lambda, z) \right|^2 dz \leq \frac{1}{|\beta|} \int_0^\infty \int_0^\infty \frac{1}{\sqrt{\pi}} e^{-\beta \tau/\mu} \left| \frac{g(z - \tau, \mu)}{1 + \tau} \right|^2 dz d\tau d\mu \]

\[ \leq \frac{1}{|\beta|} \int_0^1 \int_0^\infty \int_0^\infty \frac{1}{\sqrt{\pi}} e^{-\beta \tau/\mu} \left| \frac{g(z - \tau, \mu)}{1 + \tau} \right|^2 dz d\tau d\mu \]

\[ = \frac{1}{|\beta|^2} \int_0^\infty \int_{-\alpha}^\alpha \left| g(z, \mu) \right|^2 dz d\mu = \frac{1}{|\beta|^2} \int_0^\infty \int_{-\alpha}^\alpha \left| g(z, \mu) \right|^2 dz d\mu \]

We can obtain a similar estimate for \( G_2(\lambda, x) \) which permits us to conclude that

\[ \left\| G \right\| = \left\| G_1 + G_2 \right\| \leq \frac{1}{|\beta|^2} \left\| g \right\| \quad \text{(5.57)} \]

Hence \( \left\| G \right\| < \infty \), and this shows that \( G \) is finite for almost all \( z \).
Because of the boundedness of \((\frac{1}{\gamma} - L_\lambda)^{-1}\), from (5.55) and (5.57) we can write
\[
||q|| = ||(\frac{1}{\gamma} - L_\lambda)^{-1}q|| \leq C_1(\lambda) ||q|| \tag{5.58}
\]
Finally, we estimate \(||\omega||\). In (5.53) we put
\[
h(\xi, \mu) = \frac{\gamma}{2} \varphi(\xi) + \varphi(\xi, \mu)
\]
By (5.58), \(h \in \mathcal{H}\), and
\[
||h|| \leq (C_2(\lambda) + 1) ||q|| = C_3(\lambda) ||q|| \tag{5.59}
\]
Now \(\mathcal{U}(z, \mu)\) is of the same form as \(K_1\) and \(K_2\) for \(\mu > 0\), and \(\mu \leq 0\), respectively; hence we can use the estimate (5.57) to estimate the norm of \(\mathcal{U}(z, \mu)\). We have
\[
||\mathcal{U}(z, \mu)|| \leq \frac{1}{\mu} ||h(z, \mu)||
\]
hence (5.58) enables us to write
\[
||\mathcal{U}(z, \mu)|| = ||R(\lambda; A)q|| \leq ||R(\lambda; A)|| ||q||
\]
This proves that for \(\lambda \in \Gamma\)
\[
||R(\lambda; A) || \leq C_4(\lambda) ||q|| \frac{||R(\lambda; A)||}{||R(\lambda; A)||}
\]
but it also shows that \(R(\lambda; A)\) is bounded \(\int_\Gamma \lambda \in \Gamma\). Since a bounded operator is closed, \(R(\lambda; A)\) is closed, and this implies that \(A\) is closed.

It remains to prove (iii). Let \(q \in \mathcal{E}(A)\).

We have
\[
2 \text{Re} \left( Dg, g \right) = \int \int \left( \mu \frac{\partial g}{\partial z} \overline{\gamma} + \mu \frac{\partial \overline{q}}{\partial z} \overline{g} \right) dz d\mu.
\]

\[
= \int\int \left[ g \overline{\overline{g}} \right] d\mu
\]
In establishing the above, we have used the boundary conditions (5.49). We now carry out the following computation:

\[
\Re \left( (\lambda - \Lambda) g, g \right) = \Re \left( (\lambda + D - \delta I) g, g \right) \\
= \lambda \| g \|^2 + \Re \left[ (D g, g) \right] - \delta \Re \left[ (J g, g) \right] \\
\geq \lambda \| g \|^2 - \delta \| (J g, g) \|
\geq (\lambda - \delta) \| g \|^2.
\]

Since \( \| J \| = 1 \).

Put \( (\lambda - \Lambda) g = f, f \in \mathcal{H} \) so that

\[ g = \mathcal{R} (\lambda, \Lambda) f, \quad \] Then

\[
\| f \| \| g \| \geq \| (f, g) \| \geq \Re \left( f, g \right) \\
= \Re \left( (\lambda - \Lambda) g, g \right) \geq (\lambda - \delta) \| g \|^2
\]

and

\[
\| g \| = \| \mathcal{R} (\lambda, \Lambda) f \| \leq \frac{\| f \|}{\lambda - \delta}, \quad \lambda > \gamma
\]

Since \( f \) was an arbitrary element of \( \mathcal{H} \), this proves (iii).

Since the hypotheses of the Hille-Yosida theorem are satisfied, we can conclude that \( \Lambda \) is the infinitesimal generator of a strongly continuous semigroup of operators of class \( C_0 \) in \( \mathcal{H} \), say \( \{ T(t), t \geq 0 \} \). Moreover, by theorem 2.17, \( T(t) \) is strongly differentiable, and for \( f \in \mathcal{D}(\Lambda) \)

\[
\frac{d}{dt} T(t) f = \Lambda T(t) f = T(t) \Lambda f \tag{5.60}
\]
If we now put
\[ n(z, \lambda, t) = T(\epsilon) \left[ f(z, \lambda) \right] \mathcal{D}(\lambda), \]

We see that equation (7) is satisfied, and that the initial condition is satisfied in the form
\[ \lim_{t \to 0} \| n(z, \lambda, t) - f(z, \lambda) \| = 0 \]

and the boundary conditions are satisfied because from (5.60)
\[ n = T(\epsilon) \mathcal{E}_\mathcal{D}(\beta). \]

Finally, the uniqueness of the solution follows from theorem 4.4 with \( R_\mathcal{D}(\beta) = 0. \)

Since the semigroup is of class \( (C_0) \), the solution

\[ \mathcal{U}(z, \lambda, t) \]

can be represented as a Laplace integral, that is
\[ \mathcal{U}(z, \lambda, t) = \frac{i}{2\pi} \int_{c-i\infty}^{c+i\infty} e^{-\lambda t} \mathcal{L}(\lambda, f) + c i \lambda \]

for \( t > 0, \ c > \beta, \ f \in \mathcal{D}(\mathcal{A}). \)

We summarize the result of Lehner and Wing \( \mathcal{L} \), \( 18 \) by stating that the solution of equation (5.44) is given by
\[ \mathcal{U}(z, \lambda, t) = \sum_{i}^{N} \left( f, \psi^i_\lambda \right) \psi^i_\lambda(z, \lambda, t) e^{i \lambda t} + \gamma(z, \lambda, t) \]

where
\[ \gamma(z, \lambda, t) = \left. \frac{d}{dr} \right|_{\lambda = \lambda_0} \left( e^{-\lambda t} \mathcal{L}(\lambda, f) + c i \lambda \right), \]

and \( \left( f, \psi^i_\lambda \right) \psi^i_\lambda \) is the residue of the pole of \( \mathcal{R}_{\mathcal{D}}(\lambda, \mu) f \) at \( \lambda = \beta \), \( \beta \) is the adjoint eigenfunction.

C. Other Studies on the Boltzmann Equation: We refer to \( \mathcal{L} \) for a summary of recent studies using semigroup methods in neutron transport theory.
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