24

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LECTURES ON
PARASTATISTICS

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PARASTATISTICS

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CONTENTS

PARASTATISTICS

I. INTRODUCTION

It was originally supposed that all particles must obey either the Bose-Einstein (B.E.) statistics or the Fermi-Dirac (F.D.) statistics. It was assumed that particles with integral spin obeyed B.E. Statistics= where the occupation number n_k of each mode with a definite value of the momentum $\underline{k}$ and spin projection μ could be zero or any positive integer $n = 0, 1, 2, \ldots \infty$, whereas particles with half-integral spin obey F.D. statistics, where the occupation number of each mode had to be 0 or 1.

Gentile was the first to point out that B.E. and F.D. statistics were not the only possible one, and that theoretically at least, several other types of statistics were possible which were neither Bose nor Fermi, and where the maximum occupation number of a state might be neither 1 nor ∞ but could be any finite (positive) integer.

Such generalized statistics were discussed from a classical point of view by Sommerfeld, Ter Haar and others.

In field theory, the statistics of particle enters through the method of quantization of the associated field. The usual commutation and anticommutation relations gives the ordinary B.E. and F.D. statistics, to obtain a more general statistics, the rules for quantization i.e. the commutation relations, have to be generalized. This, in turn, also requires a generalization of some of the fundamental concepts of quantum theory.

The different types of generalized statistics have been called by different names-- 'Intermediate Statistics', 'Strange Statistics' and 'Para-statistics'. We shall use the term para-statistics, and will distinguish two classes of parastatistics

(i) Para-fermi statistics, which may be considered as a generalisation starting from the normal Fermi statistics; here, the maximum occupation number may be any positive integer, depending on the 'order' of the para-fermi statistics, and

(ii) Para-bose statistics, which may be regarded as a generalization away from normal Bose statistics, the maximum occupation number is still infinity, but the symmetry property of the states is different from that in normal Bose statistics.

Parastatistics in field theory has been dealt with by various authors; the earlier work was done by Okayama, Green Volkov and others, more recently, work has been done by Greenberg and by

Sudarshan+ and others (4)

The earlier papers all considered only special cases, the first (unified) treatment of the problem (has recently been given) by Kamefuchi and Takahashi and myself (4)

In these lectures we shall develop a general formalism for parastatistics. We shall begin by showing from the simple example of the harmonic oscillator that the Heisenberg equations of motion are consistent with a set of commutation relations that are much more general than the usual ones.

We shall then consider how in a field theory the method of quantization has to be generalized in order to lead to parastatistics. We shall illustrate how the commutation relations for parastatistics of a given order may be explicitly derived, and point out how parastatistics goes over into normal statistics in the limit of infinite order. We shall discuss the symmetry properties of state vectors for fields obeying parastatistics and indicate how the fundamental concepts of quantum theory must be generalised in order to accommodate parastatistics .

We shall examine what further restrictions arise in a relativistic field theory and show that the usual spin-statistics connection has a general analogue: tensor fields must be quantized according to parabose statistics, and spinor fields according to parafermi statistics.

We shall then consider how the commutation properties of the field operators restrict the nature of the Hamiltonian for interacting fields; we shall derive a generalization of Nishijima's theorem and prove that in our formalism, the number of particles with the same type of statistics should be conserved. (as to the precise meaning of the statement he[illegible]) Finally, [illegible] we apply this theorem to determine the statistics of the known elementary particles from their interactions.

We shall conclude by briefly discussing other possible methods of generalizing field theory and the relation of these to the question whether parafermions and parabosons actually exist in Nature.

II: THE EQUATIONS OF MOTION AND COMMUTATION RELATIONS: THE HARMONIC OSCILLATOR

Wigner* first asked the question: How far do the equations of motion determine the commutation relations in quantum mechanics?

In this lecture we shall discuss this question with reference to the simple harmonic oscillator and show that the Heisenberg equations of motion are consistent with a more general set of commutation relations than the ordinary ones.

Consider the simple harmonic oscillator. In the Lagrangian and canonical (Hamiltonian) formalisms it is described by the following equations.

Classical description of a Simple Harmonic Oscillator

Lagrangian formalism	Canonical formalism
Variables $q, \dot q$	Variables q, p
Lagrangian: $L = \frac{1}{2}(\dot q^2 - q^2)$	Hamiltonian: $H = \frac{1}{2}(p^2 + q^2)$ [We normalise such that $\sqrt{\omega} = 1$]
Equation of motion: $\ddot q + q = 0$	Equation of motion: $p = \dot q$, $q = -\dot p$

(II.1)

In quantum mechanics, q and p become operators; by the correspondence principle, they may be assumed to obey the same equations of motion as the classical equations $p = \dot q$, $q = -\dot p$

In the usual formalism, one usually assumes the commutation relation

$$[q, p] = i$$

We do not assume this relation here

Instead we ask the following question:

If we assume the basic equations in the quantum theory to be the Heisenberg equations of motion for the dynamical variables A.

$$\dot{A} = i\,[H, A] \qquad \text{(II.2)}$$

and if we require that these be consistent with the classical equations of motion (i.e. if we require the correspondence principle to be valid) then what are the restrictions on the commutation relations of q and p?

Consistency between the classical (II.1) and Quantum Mechanical (II.2) equation gives

$$\dot{p} = i\,[H, p] = -q,$$

$$\dot{q} = i\,[H, q] = p$$

where $H = \frac{1}{2}(p^2 + q^2)$ (II.3)

Expanding eq. (II.3), we get

$$-q = \frac{i}{2}\left(q\,[q,p] + [q,p]\,q\right)$$

or

$$q\,[q,p] + [q,p]\,q - 2i\,q = 0 \qquad \text{(II.4)}$$

Define

$$S = [q, p] - i \qquad \text{(II.5)}$$

so that $\{S, q\} = 0$

and $\{S, p\} = 0$ (II.6)

Also, we have $[S^2, q] = 0$

and $[S^2, p] = 0$ (II.7)

Hence must be a C-number, say

$$S^2 = a \qquad \text{(II.8)}$$

Now we ask what is the most general form of the commutation relations of p and q.

Take the matrix elements A_{nm} of the dynamical variables A between states m and n.

For p and q this gives

$$p_{nm} = i\,(E_n - E_m)\,q_{nm}\,,$$

$$-\,q_{nm} = i\,(E_n - E_m)\,p_{nm}\,. \qquad \text{(II.9)}$$

Hence

$$q_{nm} = (E_n - E_m)^2\,q_{nm} \qquad \text{(II.10)}$$

Therefore, $(E_n - E_m) = \pm 1$ (II.11)

Let E_0 be the least value of E_n. Then, from the form of H we see $E_0 \geqslant 0$ and from (II.11)

$$E_n = E_0 + n . \tag{II.12}$$

As we shall see later E_0 depends on the type of commutation relation used. The phase of Ψ_n is to some extent arbitrary. If we choose the phase of Ψ_n to be real, since q is hermitian

$$q_{n+1,n} = q_{n,n+1} . \tag{II.13}$$

For p we get,

$$p_{n,n+1} = -i\, q_{n,n+1} = -i\, q_{n+1,n} , \tag{II.14}$$

$$p_{n+1,n} = i\, q_{n+1,n} = -p_{n,n+1} ,$$

$$E_n = \langle n | H | n \rangle$$

$$= \frac{1}{2}\left[\langle n | p^2 | n \rangle + \langle n | q^2 | n \rangle\right]$$

$$= \frac{1}{2}\Big\{\left[p_{n,n-1}\, p_{n-1,n} + p_{n,n+1}\, p_{n+1,n}\right] \tag{II.15}$$

$$+ \left[q_{n,n-1}\, q_{n-1,n} + q_{n,n+1}\, q_{n+1,n}\right]\Big\} \tag{II.15a}$$

by using (II.14)

$$= (q_{n-1,n})^2 + (q_{n,n+1})^2$$

for $n \geqslant 1$

and

$$E_0 = \frac{1}{2}\left(p_{01}\, p_{10} + q_{01}\, q_{10}\right) = q_{01}^2 \tag{II.16}$$

$$q_{01} = \sqrt{E_0} , \quad E_{0+1} = E_1 = E_0 + q_{12}^2 .$$

Thus, (II.12) gives $q_{12} = 1$, $q_{23} = \sqrt{E_0 + 1}$ (II.17)

In general,

$$q_{n,(n+1)} = \sqrt{E_0 + n/2} \quad \text{if } n \text{ is even}$$
$$= \sqrt{\tfrac{1}{2} + \tfrac{n}{2}} \quad \text{if } n \text{ is odd} \qquad \text{(II.18)}$$

Also

$$\langle 0 | S^2 | 0 \rangle = \sum_{n=0,2} \langle 0 | [q,p] - i | n \rangle \langle n | [q,p] - i | 0 \rangle$$
$$= \left(i q_{01}^2 + i q_{01}^2 - i \right)^2$$
$$= -(2E_0 - 1)^2 = a$$

(II.19)

In ordinary quantum mechanics,

$$S^2 = 0, \qquad \text{(II.20)}$$

corresponding to the relation $[q,p] = i$

and thus $E_0 = \frac{1}{2}$,

In general and $E_0 \neq \frac{1}{2}$, we have

$$\langle 0 | [q,p] | 0 \rangle = q_{01} p_{10} - p_{01} q_{10} = 2 i E_0$$
$$\langle 0 | [q,p] | 1 \rangle = 0 ;$$
$$\langle 0 | [q,p] | 2 \rangle = 0 ;$$
$$\text{- - - - - - - - - - - -}$$
$$\langle n | [q,p] | n \rangle = 2 i E_0 \quad \text{if } n \text{ is even}$$
$$= 2i(1 - E_0) \quad \text{if } n \text{ is odd}$$

(II.21a)

$$\langle n | [q,p] | n\pm 1\rangle = 0,$$
$$\langle n | [q,p] | n\pm 2\rangle = 0 \qquad \text{(II.21b)}$$

We may write these succinctly as

$$\langle n | [q,p] | m\rangle = 2iE_0\,\delta_{mn} \text{ if } n \text{ is even}$$
$$= 2i(1-E_0)\,\delta_{mn} \text{ if } n \text{ is odd} \qquad \text{(II.22)}$$

Therefore

$[q,p]$ is not a C-number;

but it is a diagonal matrix:

$$[q,p] = \begin{bmatrix} 2iE_0 & & & 0 \\ & 2i(1-E_0) & & \\ & & 2iE_0 & \\ 0 & & & \ddots \end{bmatrix} \qquad \text{(II.23)}$$

A special case is the ordinary quantummechanics, where $E_0 = \frac{1}{2}$ ( ) leading to $[q,p] = i$

The form (II.23) of the commutation relation is much more general than the particular case $[q,p] = i$. Now,

$$S = [q,p] - i = \begin{bmatrix} 2iE_0 - i & & 0 \\ & 2i(1-E_0) - i & \\ 0 & & \ddots \end{bmatrix} \qquad \text{(II.24)}$$

Hence $$S^2 = -(2E_0-1)^2 \qquad \text{(II.25)}$$

which checks (II.19).

In ordinary quantum mechanics,

$q_{n,n+1}$ is the same for both odd and even n,

$$q_{n,n+1} = \sqrt{\frac{n+1}{2}} \quad \text{for all } n \qquad \text{(II.26)}$$

For large n, $q_{n,n+1}$ tends to $\sqrt{n/2}$.

We note however that in the more general theory

$$q_{n,n+1} \rightarrow \sqrt{\frac{n}{2}} \quad \text{as} \quad n \longrightarrow \infty \qquad \text{(II.27)}$$

That is, for large n, both theories tend to the same classical limit.

Next, we shall make use of anticommutators to derive further informations. We have already noted that

$$\{S, q\} = 0 \; ; \; \{S, p\} = 0$$

We ask what is the most general expression for p in the q-diagonal representation.

We have

$$\begin{aligned} 0 &= \langle q' | \{S, q\} | q'' \rangle \\ &= (q' + q'') \langle q' | S | q'' \rangle \end{aligned} \qquad \text{(II.28)}$$

since $\{S, q\} = 0$

The solution to (II.23) is

$$\langle q' | S | q'' \rangle = i\, c(q')\, \delta(q' + q'') \qquad \text{(II.29)}$$

The relation

$$\langle q' | S | q'' \rangle = -\langle q'' | S | q' \rangle^* \qquad \text{(II.30)}$$

gives

$$c^*(q') = c(-q'). \qquad \text{(II.31)}$$

Define an operator R such that

$$R\,|q'\rangle = |-q'\rangle \tag{II.32}$$

This implies that $\langle q''|R|q'\rangle = \delta(q''+q')$ (II.33)

(II.29) may be written

$$S = i\,c(q)\,R \tag{II.34}$$

From (II.) it follows that (II.35)

$\{R, q\} = 0$; We see later that $\{R, p\} = 0$.

We assert that

$$p = -i\frac{d}{dq} + i\,\frac{c(q)\,R}{2q} \tag{II.36}$$

satisfies all the requirements

We have

$$\begin{aligned}[q, p] &= \left[q,\ -i\frac{d}{dq} + i\,\frac{c(q)}{2q}R\right] \\ &= i + \frac{i}{2}\,c(q)R - i\,\frac{c(q)}{2q}\,R\,q \\ &= i\left(1 + c(q)R\right)\end{aligned} \tag{II.37}$$

where we have used the relation $\{R, q\} = 0$

Equation (II. 5) and (II. 36) give

$$\begin{aligned}S &= [q, p] - i \\ &= \left[q,\ -i\frac{d}{dq} + i\,\frac{c(q)R}{q}\right] - i \\ &= i\,c(q)\,R\end{aligned}$$

Substituting the expression (II.36) for p in the relation

we obtain

$$0 = -i\{S, p\} = c(q) R \frac{d}{dq} + \frac{d}{dq}\left(c(q)R\right)$$

$$- c(q) R \frac{c(q) R}{2q} - \frac{d\, c(q)}{dq} R\, c(q) R$$

(II.39)

Expanding $\frac{d}{dq}\left(c(q)R\right) = \frac{d\, c(q)}{dq} R - c(q) R \frac{d}{dq}$

we obtain

$$\frac{d\, c(q)}{dq} = 0 \qquad \text{(II.40)}$$

which means that

$c(q) = c =$ a real constant

Thus

$$p = -i \frac{d}{dq} + \frac{ic}{2q} R \qquad \text{(II.41)}$$

This may be taken as the general form of the ~~momentum~~ p.

In the ordinary theory $c = 0$.

Consider again our Hamiltonian of the simple harmonic oscillator.

$$H = \frac{1}{2}\left(q^2 + p^2\right)$$

Define operators a, $a^\dagger$ (which are not hermitian) by

$$a = \frac{p - iq}{\sqrt{2}}, \qquad a^\dagger = \frac{p + iq}{\sqrt{2}} \qquad \text{(II.42)}$$

so that

$$p = \frac{a + a^{+}}{\sqrt{2}} \, ; \qquad q = \frac{a^{+} - a}{i\sqrt{2}} \qquad \text{(II.42a)}$$

In terms of a, a^{+} the Hamiltonian is given by

$$H = \frac{1}{2}\left(a^{+}a + a\,a^{+}\right) \qquad \text{(II.43)}$$

Define

$$N = H - E_0 \qquad \text{(II.44)}$$

where E_0 is the minimum eigenvalue of H.
Thus the minimum eigenvalue of N is 0.

$$H = N + E_0 \qquad \text{(II.44a)}$$

In the equations for p, q,

$$-q = \dot{p} = i[H, p] \, ; \qquad \dot{p} = i[H, q]$$

we substitute (II.42a) and (II.44a) to obtain the equations in terms of a, a^{+} and N

$$a = [a, N] \qquad \text{(II.45)}$$

and

$$a^{+} = -[a^{+}, N]$$

We shall assume only these euqations and not the relation $[a, a^{+}] = 1$ which gives back the ordinary theory.

Using (II.45), we obtain

$$N|n'\rangle = n'|n'\rangle \tag{II.46}$$

$$N\,a|n'\rangle = (-a + aN)|n'\rangle = (n-1)\,a|n'\rangle$$

Now define the vacuum state as the state of lowest eigenvalue, i.e.

$$N|0\rangle = 0 \qquad a|0\rangle = 0 \tag{II.47}$$

so that

$$\frac{1}{2}(a^{\dagger}a + aa^{\dagger})|0\rangle = E_0|0\rangle \tag{II.48}$$

and

$$a a^{\dagger}|0\rangle = 2E_0|0\rangle \tag{II.49}$$

The last relation gives

$$|\langle 1|a^{\dagger}|0\rangle|^2 = 2E_0 \tag{II.50}$$

In the ordinary case, this is equal to unity.

(II.50) may be written

$$\frac{1}{2}(p - iq)(p + iq)|0\rangle = 2E_0|0\rangle \tag{II.51}$$

i.e.

$$\left(H - \tfrac{i}{2}(i + iCR)\right)|0\rangle = 2E_0|0\rangle \tag{II.52}$$

This gives

$$\langle 0|\,H + \frac{1}{2}(1 + CR)|0\rangle = \langle 0|2E_0|0\rangle = 2E_0. \tag{II.53}$$

or

$$\langle 0 | E_0 + \frac{1}{2}(1 + cR) | 0 \rangle = 2E_0 \tag{II.54}$$

We have

$$R|0\rangle = R_0|0\rangle \tag{II.55}$$

as R commutes with H

Therefore,

$$E_0 + \frac{1}{2}(1 + cR_0) = 2E_0$$

$$c = (2E_0 - 1)/R_0 \tag{II.56}$$

Since $E_0 \geqslant 0$

We have

$$cR_0 \geqslant -1 \tag{II.57}$$

. The vacuum was defined by

$$a|0\rangle = 0$$

i.e.

$$\frac{1}{\sqrt{2}}(p - iq)|0\rangle = 0$$

This gives

$$\langle q' | -i\frac{d}{dq} + i\frac{c}{2q}R - iq | 0 \rangle = 0 \tag{II.58}$$

Now introduce a complete set of intermediate states $|q'\rangle$ and write

$$\int dq'' \langle q' | -i\frac{d}{dq} + i\frac{c}{2q}R - iq | q'' \rangle \times \langle q'' | 0 \rangle = 0 \tag{II.59}$$

or

$$\int dq'' \left\{ \langle q' | \frac{d}{dq''} | q'' \rangle + \frac{c}{-2q''} | -q'' \rangle - q'' | q'' \rangle \right\} \langle q'' | 0 \rangle = 0$$

which leads to the following equation for the wave function $\psi(q)$ (for the Vac state)

$$\left(\frac{d}{dq} + q\right)\psi_0(q) = \frac{c}{2q}\,\psi_0(-q) \qquad \text{(II.61)}$$

The solution to equation (II.61) is

$$\psi_0^{\tau} = b_\tau\, q^{2}\, \exp\left(-\frac{q^2}{2}\right) \qquad \text{(II.62)}$$

where

$$\tau = \frac{c}{2}(-1)^{\tau}$$

$$c = 2\tau(-1)^{-\tau} \quad \text{and} \quad \tau \geqslant -\frac{1}{2} \qquad \text{(II.63)}$$

which follows from the requirement for the behaviour of $\psi_0(q)$ at the origin.
Before turning to field theory, we briefly mention some considerations put forward recently by C.Boulware and S.Deser(5).
Defining

$$a = \frac{1}{\sqrt{2}}(q + ip)$$

$$a^{+} = \frac{1}{\sqrt{2}}(q - ip)$$

(II.64)

the matrix elements of a are given by

$$a_{n,(n+1)} = \sqrt{2}\, q_{n,(n+1)} = \begin{cases} \sqrt{2E_0 + n}, & n \text{ even} \\ \sqrt{n+1}, & n \text{ odd} \end{cases}$$

(II.65)

all other matrix elements of a are $= 0$

OR

$$a_{n,m} = \delta_{m,(n+1)} \sqrt{\left(E_0 - \frac{1}{2}\right)\left(1 - (-1)^{n+1}\right) + n + 1} \qquad \text{(II.65a)}$$

Define

$$\overline{H} = H - E_0 + 1 \qquad \text{(II.66)}$$

$$U = \overline{H}^{\frac{1}{2}} \left(\left(E_0 - \frac{1}{2}\right)\left(1 - (-1)^{\overline{H}} + \overline{H}\right)\right)^{-1/2} \qquad \text{(II.67)}$$

U has the property

$$U^{\dagger} = U , \qquad \text{(II.68)}$$

and its matrix elements are given by

$$U_{nn} = \sqrt{n+1} \; \frac{1}{\left[\left(E_0 - \frac{1}{2}\right)\left(1 - (-1)^{n+1}\right) + n + 1\right]^{\frac{1}{2}}} \qquad \text{(II.69)}$$

Define

$$A = Ua ; \quad A^{\dagger} = a^{\dagger} U \qquad \text{(II.70)}$$

then non-vanishing matrix elements of A and $A^{\dagger}$ are

$$\left.\begin{aligned} A_{n,(n+1)} &= \sqrt{n+1} \\ \text{and} \quad A^{\dagger}_{(n+1),n} &= \sqrt{n+1} \end{aligned}\right\} \qquad \text{(II.71)}$$

A and $A^{\dagger}$ obey the relation

$$[A, A^{\dagger}] = 1 \qquad \text{(II.72)}$$

and H is given by

$$H = A^{\dagger}A + E_0 = \frac{1}{2}\left(A^{\dagger}A + AA^{\dagger}\right) + \left(E_0 - \frac{1}{2}\right) \qquad \text{(II.73)}$$

That is, if instead of $a, a^{\dagger}$, we use the operators $A, A^{\dagger}$ then the commutation relations are the same as those in the ordinary theory, but the zero-point-energy E_0 is not equal to 1/2, in general.

Various states are given by

$$|0\rangle$$

$$|1\rangle = \frac{1}{\sqrt{2E_0}} a^{\dagger}|0\rangle$$

$$|2\rangle = \frac{1}{[2(2E_0)]^{1/2}} (a^{\dagger})^2 |0\rangle,$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots \tag{II.74}$$

$$|n\rangle = \frac{1}{\left[2^n \frac{n}{2}!\ E_0(E_0+1)\cdots\cdots(E_0+\frac{n}{2}-1)\right]^{1/2}} \times (a^{\dagger})^n |0\rangle$$

if n is even

$$= \frac{1}{\left[2^n \frac{n-1}{2}!\ E_0(E_0+1)\cdots\cdots(E_0+\frac{n-1}{2})\right]^{1/2}} (a^{\dagger})^n |0\rangle$$

if n is odd

We shall now turn to the question of the corresponding generalization in field theory.

A simple commutation relation of the form $[a, a^{\dagger}]_+ = 1$ can hold in ordinary quantum mechanics if $a, a^{\dagger}$ are bounded operators such that a has the matrix representation

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

However, p and q are unbounded operators and cannot be expressed in terms of such $a, a^{\dagger}$.

III. GENERALIZED STATISTICS IN FIELD THEORY+

1. Preliminaries:

We shall now see how we can generalize the commutation relations in a field theory so as to obtain a generalization of Fermi and Bose statistics. Not many physicists seem to have interested in this question before[3].

The problem in field theory is the following

Consider for simplicity, a non-relativistic on Schrodinger field, with the Hamiltonian

$$H = \frac{1}{4m}\int\left[\nabla\psi^{+}\nabla\psi \mp \nabla\psi\,\nabla\psi^{+}\right]d^{3}x \qquad \text{(III.1)}$$

with field equations

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^{2}\psi \qquad \text{(III.2)}$$

The question is what are the most general commutation relations that are consistent with the Heisenberg equations of motion for the field:

$$i\frac{\partial\psi}{\partial t} = -[H,\psi] \qquad \text{(III.3)}$$

and a corresponding equation for ψ^{+} ?

As in the case of simple harmonic oscillators we require that (III.2) and (III.3) should be consistent with each other, so that

$$\frac{1}{2m}\nabla^{2}\psi = [H,\psi] \qquad \text{(III.4)}$$

To find the most general commutation relations for ψ and $\psi^{\dagger}$ that satisfy (III.4), we go over into mom. space and introduce the operators a_k, $a_k^{\dagger}$ and the N_k in the following way

Define a_k, $a_k^{\dagger}$, N_k by the following relations:

$$\psi(x,t) = \frac{1}{\sqrt{V}} \sum_k a_k \, e^{i(\vec{k}\cdot\vec{x} - E_k t)} \tag{III.5a}$$

$$\psi^{\dagger}(x,t) = \frac{1}{\sqrt{V}} \sum_k a_k^{\dagger} \, e^{-i(\vec{k}\cdot\vec{x} - E_k t)} \tag{III.5b}$$

Then

$$H = \sum_k E_k N_k \tag{III.6}$$

where

$$N_k = \frac{1}{2}\left(a_k^{\dagger} a_k \mp a_k a_k^{\dagger}\right) \tag{III.6a}$$

and

$$E_k = \frac{k^2}{2m} \geq 0 \tag{III.6b}$$

as we are using a non-relativistic theory

Eqn. (III.4) is equivalent to the following relations;

$$[a_k, N_l] = \delta_{kl} a_k , \tag{III.7a}$$

$$[a_k^{\dagger}, N_l] = -\delta_{kl} a_k^{\dagger} , \tag{III.7b}$$

$$[N_k, N_l] = 0 \tag{III.7c}$$

We see that a_k, $a_k^{\dagger}$ respectively decrease and increase the eigenvalues of N_k by 1. Thus a_k can be termed the annihilation operator, and $a_k^{\dagger}$ the creation operator for one particle. The eigenvalues of N_k are integers $n_0, n_0+1, \ldots$

We must find out what commutation relations for $a_k, a_k^\dagger$ will lead to an appropriate eigenvalue spectrum for N_k. Define the number operator n_k by

$$n_k = N_k - n_0 \tag{III.8}$$

Then n_k will have the eigenvalues $0, 1, 2, \ldots n_{max}$ and

$$H = \sum_k n_k E_k + n_0 \sum_k E_k \tag{III.9}$$

The second term in (III.9) corresponds to the zero-point energy

2. Group properties of $a_k, a_k^\dagger$:

We shall construct the commutation ruler for two cases: (We follow here the arguments of Kamefuchi and Takahashi[4)])

(i) Case R:

When a_k and $a_k^\dagger$ form a representation of the rotation group with dimension $= \infty$

(ii) Case S:

When a_k and $a_k^\dagger$ form a representation of the symplectic group with dimension $= \infty$

These two cases arise naturally in the following way. Consider an infinitesimal linear transformation on a_k and $a_k^\dagger$

$$a_k \to a_k' = a_k - i \sum_{m=1}^{\infty} a_m \varepsilon_{km} - i \sum_{m=1}^{\infty} a_m^\dagger \eta_{km},$$

$$a_k^\dagger \to a_k^{\dagger\prime} = a_k^\dagger + i \sum_{m=1}^{\infty} a_m^\dagger \varepsilon_{mk} + i \sum_{m=1}^{\infty} a_m \eta_{mk} \tag{III.10}$$

where $\xi^{*}_{km} = \xi_{mk}$; $\eta^{*}_{km} = \zeta_{mk}$

(Notation: $\dagger$ denotes hermitian conjugate) .
$*$ denotes complex conjugate)

Also define the hermitian operators

$$p_k \doteq \frac{(a_k^{\dagger} + a_k)}{\sqrt{2}} \quad ; \quad q_k = \frac{i}{\sqrt{2}}\left(a_k^{\dagger} - a_k\right), \tag{III.11}$$

so that

$$a_k = (p_k + i q_k)/\sqrt{2} \quad ; \quad a_k^{\dagger} = \frac{(p_k - i q_k)}{\sqrt{2}} \tag{III.11a}$$

What are the invariants of the transformation (III.10)?.

We have two particular cases:

(⁺) When we require that

$$\eta_{mk} = -\eta_{km} \; ; \; \left(\zeta_{mk} = -\zeta_{km} \right) \tag{III.12R}$$

the quantity

$$\Lambda_+ = \sum_{k=1}^{\infty}\left(a_k^{\dagger}a_k + a_k a_k^{\dagger}\right) = \sum_k \left(p_k^2 + q_k^2\right)/2 \tag{III.13 R}$$

is an invariant .

Thus (III.10) now defines the group of rotations in the (p, q) space. This is <u>case R</u>.

(ii) If, instead of (I), we require that

$$\eta_{km} = +\eta_{mk} \; ; \; \left(\zeta_{mk} = -\zeta_{km} \right) \tag{III.12)}$$

then the quantity

$$\Lambda_- = \sum_k \left(a_k^{\dagger}a_k - a_k a_k^{\dagger}\right) \tag{III.13S}$$

is an invariant. This is <u>Case S</u>.

We can prove in this case that the infinitesimal transformation (III.10) defines a symplecte group.

<u>Proof:</u>

Define the (infinite-dimensional) Vector

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_1^{+} \\ a_2^{+} \\ \vdots \end{bmatrix} \qquad \text{(III.14)}$$

and the matrix

$$S = \left[\begin{array}{c|c} \delta_{kl} - i\,\xi_{kl} & -i\,\eta_{kl} \\ \hline i\,\eta^{*}_{kl} & \delta_{kl} + i\,\xi^{*}_{kl} \end{array}\right] \qquad \text{(III.15)}$$

The transformation (III.10) is then given by

$$A \longrightarrow A' = S A \; ; \qquad \text{(III.16)}$$

The invariant

$$\Lambda_{-} = A^{t}\, I\, A \qquad \text{(III.17)}$$

where A_t denotes the transposed row vector,

and I is the matrix

$$I = \left[\begin{array}{c|c} 0 & -\delta_{kl} \\ \hline \delta_{kl} & 0 \end{array}\right] \qquad \text{(I}\ldots$$

Λ_- has been seen to be invariant under (III.10), i.e. it is invariant under (III.16)

$$A^t I A = \Lambda_- = A'^t I A' = A^t (S^t I S) A \; . \qquad \text{(III.19)}$$

This implies that (III.20)

$$S^t I S = I$$

which defines a symplectic transformation.

We have just seen that the infinitesimal transformation (III.10) defines a rotation group for Case R, and a symplectic group for Case S.

We now ask the following questions for each case:

1) What are the generators of the group?

2) What are the finite-dimensional representations of the group, if any?

We look for a unitary transformation G such that

$$a_k' = G_k^{-1} a_k G_k \, , \qquad \text{(III.21)}$$

where a_k' is related to a_k by (III.10)

As (III.10) is an infinitesimal transformation, write G as

$$\begin{aligned} G = 1 &- i \sum_{l,m=1}^{\infty} N_{lm} \xi_{lm} \\ &- \tfrac{1}{2} i \sum_{l,m=1}^{\infty} L_{lm} \eta_{lm} \\ &- \tfrac{1}{2} i \sum_{l,m=1}^{\infty} M_{lm} \zeta_{lm} \end{aligned} \qquad \text{(III.22)}$$

As G must be unitary, the quantity

$$\sum_{l,m=1}^{\infty}\left(N_{lm}\xi_{lm}+L_{lm}\eta_{lm}+M_{lm}\zeta_{lm}\right)$$ must be hermitian.

(Note: $\xi_{lm}, \eta_{lm}, \zeta_{lm}$ are C-numbers, whereas N, L, M are operators)

The property (III.10a) of ξ_{lm}, η_{lm}, ζ_{lm} then implies that

$$N_{lm}^{\dagger} = N_{ml}; \quad L_{ml}^{\dagger} = M_{ml}; \tag{III.23}$$

Further the antisymmetry (or symmetry) of η_{lm} for Case R (or Case S) leads to the conditions

$$L_{lm} = -L_{ml}, \quad M_{lm} = -M_{ml}; \quad \text{for Case R} \tag{III.24R}$$

$$L_{lm} = +L_{ml}; \quad M_{lm} = +M_{ml} \quad \text{for Case S} \tag{III.24S}$$

Now apply the condition that

$$a_k' = G_k^{-1} a_k G_k, \tag{III.21}$$

Ithe left-hand side being equated to (III.10)
This gives the conditions

$$\left.\begin{aligned} [a_k, N_{lm}] &= \delta_{kl} a_m; \\ [a_k, L_{lm}] &= \delta_{kl} a_m^{\dagger} - \delta_{km} a_l^{\dagger}; \\ [a_k, M_{lm}] &= 0 \end{aligned}\right\} \quad \text{for } \underline{\text{Case R}} \tag{III.25 R}$$

and
$$[a_k, N_{\ell m}] = \delta_{k\ell}\, a_m ,$$
$$[a_k, L_{\ell m}] = \delta_{k\ell}\, a_m^\dagger + \delta_{km}\, a_\ell^\dagger , \qquad \text{for } \underline{\text{case S}}$$
$$[a_k, M_{\ell m}] = 0 \qquad \text{(III.25 S)}$$

The conditions (III.23) - (III.25) are necessary conditions on L, M, N but are not sufficient to guarantee that G generates the transformation group (III.10)

We must impose, in addition, the 'integrability condition' namely, that if
$$\delta a_k = a_k' - a_k , \qquad \text{(III.26)}$$
is used to define an infinitesimal transformation δ, and if $\delta^{(1)}$, $\delta^{(2)}$ be two such transformations, then $(\delta^{(1)}\delta^{(2)} - \delta^{(2)}\delta^{(1)}) = \delta^{(12)}$ defined by
$$\delta^{(12)} a_k = \delta^{(1)}\delta^{(2)} a_k - \delta^{(2)}\delta^{(1)} a_k \qquad \text{(III.26a)}$$

is also a transformation of the same type.

This gives the integrability condition or the Lie group property of the generators:
$$[N_{k\ell}, N_{mn}] = \delta_{\ell m} N_{kn} - \delta_{kn} N_{m\ell} ,$$
$$[L_{k\ell}, L_{mn}] = 0 ,$$
$$[M_{k\ell}, M_{mn}] = 0 ,$$
$$[L_{k\ell}, M_{mn}] = -\delta_{kn} L_{m\ell} \pm \delta_{\ell n} L_{mk} ,$$

$$[M_{kl}, N_{mn}] = \delta_{km} M_{nl} \mp \delta_{lm} M_{nk},$$

$$[L_{kl}, M_{mn}] = -\delta_{km} N_{ln} \pm \delta_{kn} N_{lm} - \delta_{ln} N_{km} \pm \delta_{lm} N_{kn} \qquad \text{(III.27)}$$

where the upper sign is for Case R, and the lower sign is for Case S.

Equations (III.23) - (III.27) define the generators of the Lie group defined by the infinitesimal transformation (III.10)

Possible forms of the generators in terms of a_k and a_k^+ are the following:

$$\left.\begin{aligned} N_{kl} &= \kappa\,(a_k^+ a_l \mp a_l a_k^+), \\ L_{kl} &= \kappa\,(a_k^+ a_l^+ - a_l^+ a_k^+), \\ M_{kl} &= \kappa\,(a_k a_l \mp a_l a_k), \end{aligned}\right\} \begin{array}{l} \text{upper sign for Case R} \\ \text{lower sign for Case S} \end{array} \qquad \text{(III.28)}$$

These satisfy the conditions (III.23) - (III.27) (III.28) defines a special representation of the group, there are also other possible forms.

<u>Note.</u> The first equation in (III.25) is a generalization of the relations (III.7a) and (III.7b). The renormalization of a_k, a_k^+ corresponds to taking $\kappa = \frac{1}{2}$.

We now ask whether finite-dimensional representations exist.

We assume that N_k is of the form (III.28).

We can then prove the following statements:

a) <u>For Case R</u>, finite-dimensional matrix representations do exist, the eigenvalues of $N_k = N_{kk}$ are the following:

0 for a 1-dimensional representation

$-\frac{1}{2}, \frac{1}{2}$ for a 2-dimensional representation

$1 \quad 0 \quad -1$, for a 3-dimensional representation

(III.29)

b) <u>For Case S</u>, <u>no</u> finite-dimensional representation exists, the eigenvalues of N_k are

$n_0, n_0+1, n_0+2, \ldots\ldots \infty$ where $n_0 > 0$

(III.30)

Proof:
a) <u>Case R</u>

Note that all the N_k's commute with one another, and that they are all hermitian, they can all be therefore diagonalized simultaneously. Suppose finite-dimensional representations exist. (We here show this leads to no contradiction, we shall later prove that they exist by constructing them explicitly.
If N_k is an $(s \times s)$ matrix with eigenvalues n_0, $(n_0+1), \ldots (n_0+s-1)$ then (III.28) gives

$$\sum_{r=0}^{(s-1)} \langle n_0+r | N_k | n_0+r \rangle = \sum_{r=0}^{(s-1)} \langle n_0+r | \kappa (a_k^+ a_k - a_k a_k^+) | n_0+r \rangle \qquad \text{(III.31)}$$

where $|n_0+r\rangle$ is an eigenvector of N_k with eigenvalue (n_0+r).

The left-hand side of (III.31) is

$$\sum_{r=0}^{(s-1)} (n_0+r) ,$$

while the righthand side $= 0$, since it is the trace of a commutator of finite matrices.

Thus we obtain

$$n_0+(n_0+1)+(n_0+2)+\ldots\ldots+(n_0+s-1) = 0 \qquad \text{(III.32)}$$

We then obtain the following eigenvalues for N_k and $n_k = (N_k - n_0)$,

s	n_0	Eigenvalues of N_k	Eigenvalues of $n_k = N_k - n_0$
1	0	0	0
2	$-\frac{1}{2}$	$-\frac{1}{2}, \frac{1}{2}$	$0, 1,$
3	-1	$-1, 0, 1$	$0, 1, 2$
4	$-3/2$	$-3/2, -1/2, 1/2, 3/2$	$0, 1, 2, 3$
5	-2	$-2, -1, 0, 1, 2$	$0, 1, 2, 3, 4$

Note:

In the above, we have assumed the same order for all N_k
We call the parameter $s-1$ $(= n_{max})$ as the "order of statistics"

b) Case S

N_k cannot have a finite-dimensional representation if it is taken to be of the form (III.28)

From equation (III.27), we obtain

$$N_k = N_{kk} = -\frac{1}{4}\left[L_{kk}, M_{kk}\right]. \qquad \text{(III.33)}$$

Thus if a finite-dimensional representation exists for N_k the sum of the eigenvalues must vanish

$$\sum_{r=1}^{s} (n_0 + r) = 0 \qquad \text{(III.34)}$$

However

$$\sum_{r=1}^{s} \langle n_0 + r | (a_k^+ a_k + a_k a_k^+) | n_0 + r \rangle \geqslant 0 \text{ is} \qquad \text{(III.35)}$$

as each term is $\geqslant 0$

This is a contradiction. Thus N_k has no finite-dimensional representation for Case S, any particular representation must be of infinite dimension. $n_k = (N_k - n_0)$ will have eigenvalues

$$0, 1, 2 \ldots . \infty$$

we have thus seen that

i) for case R, the occupation number of a mode with momentum k is limited from above, $n_k < (n_k)_{max}$ = a integer so that this looks like a generalization of the Fermi-Dirac statistics,

and ii) for Case S the maximum occupation number is infinite, this looks like a generalization of the Bose-Einstein statistics.

We shall later consider the symmetry and degeneracy of the state vectors for the two cases. Before that we shall show how to derive explicitly the commutation relations for a_k, a_k^{+} for representation of given dimension $\mathcal{S}$.

3) Explicit Derivation of the Commutation Relations:

(A) Case R

(i) $\mathcal{S} = 1$

The eigenvalue of N_k is zero

$$\therefore \quad N_k = 0 \; ; \tag{III.36}$$

$$a_k = [a_k, N_k] = 0 \; ; \tag{III.36a}$$

This is a trivial case.

(2) $\mathcal{S} = 2$

We have $N_k^2 = \frac{1}{4}$ (III.37)

Using the relation eqn. (III.25B)

$$[a_k, N_k] = [a_k, N_{kk}] = a_k ,$$

we obtain, taking the commutator of a_k with eqn. (III.37),

$$0 = [a_k, N_k^2] = a_k N_k - N_k a_k ; \qquad \text{(III.38)}$$

Again take the commutator of (III.38) with a_k

This gives

$$a_k a_k = 0 \qquad \text{(III.39)}$$

Now take the commutator of (III.39) with $N_{\ell k}$ to obtain

$$\{ a_k , a_\ell \} = 0 \qquad \text{(III.40)}$$

To find $\{ a_k , a_\ell^\dagger \}$, take the commutator of (III.38) with $a_\ell^\dagger$ and use the relations

$$[a_k \; N_{\ell m}] = \delta_{k\ell} a_m ,$$

$$N_{k\ell} = K (a_k^\dagger a_\ell - a_\ell a_k^\dagger) ,$$

and $$N_\ell^2 = \frac{1}{4} ,$$

This gives

$$a_\ell a_\ell^\dagger + a_\ell^\dagger a_\ell = \frac{1}{2K} \qquad \text{(III.41)}$$

Take the commutator of $N_{\ell k}$ with (III.41)

This gives

$$a_k a_\ell^\dagger + a_\ell^\dagger a_k - \delta_{k\ell} (a_\ell a_\ell^\dagger + a_\ell^\dagger a_\ell) = 0$$

or using (III.41),

$$(a_k a_\ell^\dagger + a_\ell^\dagger a_k) = \frac{1}{2K} \delta_{k\ell} \qquad \text{(III.42)}$$

We have thus obtained the commutation relations for $b = 2$

$$\{ a_k , a_\ell \} = 0 \qquad \text{(III.40)}$$

and
$$\{ a_k , a_\ell^\dagger \} = \frac{1}{2k} \delta_{k\ell} , \tag{III.42}$$

These are just the rules for Fermi statistics, if we choose the normalization constant k to be $\frac{1}{2}$. The possible values of the occupation number are 0 1-

$\underline{b = 3}$

The characteristic equation is
$$(N_\ell - 1) N_\ell (N_\ell + 1) = 0 \tag{III.43}$$

OR
$$N_\ell^3 = N_\ell ; \tag{III.43a}$$

Taking the commutator with a_ℓ gives, using (II.25 R),
$$a_\ell N_\ell^2 + N_\ell a_\ell N_\ell + N_\ell^2 a_\ell = a_\ell \tag{III.44}$$

Commutation with $a_\ell^\dagger$ gives, using (II.28)
$$\frac{3}{k} N_\ell^3 - a_\ell a_\ell^\dagger N_\ell - a_\ell N_\ell a_\ell^\dagger - a_\ell^\dagger a_\ell N_\ell - N_\ell a_\ell a_\ell^\dagger - a_\ell^\dagger N_\ell a_\ell - N_\ell a_\ell^\dagger a_\ell = \frac{1}{k} N_\ell \tag{III.45}$$

OR, using (III.43a),
$$a_\ell^\dagger a_\ell N_\ell + a_\ell a_\ell^\dagger N_\ell + a_\ell N_\ell a_\ell^\dagger + a_\ell^\dagger N_\ell a_\ell + N_\ell a_\ell a_\ell^\dagger + N_\ell a_\ell^\dagger a_\ell = \frac{2}{k} N_\ell . \tag{III.46}$$

Commutation with a_ℓ now gives, using (II.28),
$$a_\ell a_\ell^\dagger a_\ell + a_\ell^\dagger a_\ell a_\ell + a_\ell a_\ell a_\ell^\dagger - \frac{1}{k} a_\ell N_\ell^2 - \frac{1}{k} N_\ell a_\ell N_\ell - \frac{1}{k} a_\ell N_\ell^2 = \frac{1}{k} a_\ell$$

OR, using (III.44),

$$a_\ell a_\ell^\dagger a_\ell + a_\ell^\dagger a_\ell a_\ell + a_\ell a_\ell a_\ell^\dagger = \frac{2}{K} a_\ell ; \qquad \text{(III.47)}$$

Using (III.25 R), this gives

$$a_\ell a_\ell^\dagger a_\ell = \frac{1}{K} a_\ell , \qquad \text{(III.48)}$$

$$a_\ell^\dagger a_\ell a_\ell + a_\ell a_\ell a_\ell^\dagger = \frac{1}{K} a_\ell . \qquad \text{(III.49)}$$

Commutation of (III.48) with $N_{\ell k}$ gives, using (III.25R),

$$a_k a_\ell^\dagger a_\ell + a_\ell a_\ell^\dagger a_k - \delta_{k\ell} a_\ell a_\ell^\dagger a_\ell = \frac{1}{K} a_k \qquad \text{(III.50)}$$

OR, using (III.48)

$$a_k a_\ell^\dagger a_\ell + a_\ell a_\ell^\dagger a_k = \frac{1}{K} (1+\delta_{k\ell}) a_k , \qquad \text{(III.51)}$$

Commutation of $N_{\ell m}$ with (III.51) gives, using (III.25 R),

$$a_k a_\ell^\dagger a_m + a_m a_\ell^\dagger a_k + \delta_{k\ell} \left(a_m a_\ell^\dagger a_\ell + a_\ell a_\ell^\dagger a_m \right)$$
$$- \delta_{\ell m} \left(a_k a_\ell^\dagger a_\ell + a_\ell a_\ell^\dagger a_k \right) = \frac{2}{K} \delta_{k\ell} a_m .$$

OR, using (III.51),

$$a_k a_\ell^\dagger a_m + a_m a_\ell^\dagger a_k = \frac{1}{K} \delta_{\ell m} a_k + \frac{1}{K} \delta_{k\ell} a_m \qquad \text{(II52)}$$

The operation

$\left[\left[(\text{III}.49), N_{\ell k} \right], N_{\ell m} \right]$ gives, similarly,

$$a_\ell^\dagger a_k a_m + a_m a_k a_\ell^\dagger = \frac{1}{K} \delta_{k\ell} a_m \qquad \text{(II.53)}$$

Commutation of (III.44) with a_ℓ gives

$$a_\ell^2 N_\ell + a_\ell N_\ell a_\ell + N_\ell a_\ell^2 = 0 \qquad \text{(II.54)}$$

Commutation of (III.54) with a_ℓ gives

$$a_\ell\, a_\ell\, a_\ell = 0 \qquad \text{(II.55)}$$

Commutation of (III.55) with $N_{\ell k}$ gives

$$a_k\, a_\ell^2 + a_\ell\, a_k\, a_\ell + a_\ell^2\, a_k = 0 \qquad \text{(II.56)}$$

and commutation of (III.56) with $N_{\ell m}$ gives

$$a_k a_\ell a_m + a_k a_m a_\ell + a_\ell a_k a_m + a_\ell a_m a_k$$
$$+ a_m a_\ell a_k + a_m a_k a_\ell = 0 \qquad \text{(III.57)}$$

So far, we have used only the first of the three equations (III.25 R). (For $s = 2$ only the first equation was required). We now use the third equation in (III.25 R), which implies

$$\begin{aligned} a_k a_\ell a_m + a_m a_\ell a_k &= a_\ell a_k a_m + a_m a_k a_\ell \\ &= a_k a_m a_\ell + a_\ell a_m a_k \end{aligned} \qquad \text{(III.58)}$$

Using (III.58) in (III.57), we obtain

$$a_k a_\ell a_m + a_m a_\ell a_k = 0 \,. \qquad \text{(III.59)}$$

The basic commutation relations are given by (III.52), (III.53) and (III.59).

Operators obeying these commutation relations were first suggested by Green and were studied by Volkov.

N_k has the eigenvalues $-1, +1, 0$

(III.60)

$n_k = (N_k+1)$ has the eigenvalues 0, 1, 2, and satisfies .

$$[a_\ell , n_k] = \delta_{\ell k}\, a_k . \qquad \text{(III.61)}$$

By taking $K = \frac{1}{2}$, we can interpret) n_k as the occupation number operator, a_k as the annihilation. operator and a_k^+ as the creation operator. Thus for $p=3$, case R corresponds to a statistics in which the maximum occupation number of a state is 2. In general, the maximum occupation number is $(p-1)$ for case R.

There is a very interesting analogy between the commutation relations for parastatistics and those for the spin operators and matrices for higher spin fields. In particular, the above relations for $p=3$ (Case R) are very similar to those for the Duffin-Kemmer β_μ operators, if we put $a_k = (\beta_1 + i\beta_2)/\sqrt{2}$. The analogy is brought out inthe following table

Case R ($p = 3$)	Duffin-Kemmer field
$N_{k\ell} = K(a_k^+ a_\ell - a_\ell a_k^+)$,	$S_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu$.
$[a, N] = a$.	$[\beta_\mu , S_{\lambda\nu}] = \delta_{\lambda\mu}\beta_\nu - \delta_{\nu\mu}\beta_\lambda$

expression for N_{kl} (III.28R) (basic assumption *)

The corresponds to the

the spin operator in the Rao. Bhabha theory.

$s = 4$

The commutation relations may be obtained by the same method as we have used for $s = 3$

They are the following:

$$a_k (a_l a_m a_n + a_n a_m a_l) + (a_m a_l a_n + a_n a_l a_m) a_k = 0,$$

$$a_m (a_l a_k^{\dagger} + a_n a_k^{\dagger} a_l) + (a_n a_k^{\dagger} a_l + a_l a_k^{\dagger} a_n) a_m$$

$$= \frac{1}{2K} \delta_{km} \{a_n, a_l\} + \frac{3}{2k} \delta_{kl} \{a_m, a_n\}$$

$$+ \frac{3}{2k} \delta_{kn} \{ a_l , a_m \},$$

$$a_m (a_k^{\dagger} a_l a + a_n a_l a_k^{\dagger}) + (a_l a_k^{\dagger} a_n + a_n a_k^{\dagger} a_l) a_m$$

$$= \frac{1}{2K} \delta_{km} \{a_l, a_n\} + \frac{3}{2k} \delta_{kl} \{ a_n, a_m\} \qquad (\text{III}.62)$$

$$+ \frac{3}{2K} \delta_{kn} a_l a_m + \frac{1}{2k} \delta_{kn} a_m a_l$$

$$a_k^{\dagger} (a_l a_m a_n + a_n a_m a_l) + (a_m a_l a_n + a_n a_l a_m) a_k^{\dagger}$$

$$= \frac{1}{2k} \delta_{kl} \{a_n, a_m\} + \frac{1}{2k} \delta_{k\,} \{ a_l , a_n\} + \frac{1}{2k} \delta_{kn} \{a_m, a_l\}$$

$$a_k^{\dagger} (a_l a_n^{\dagger} a_m + a_m a_n^{\dagger} a_l) + (a_m a_n^{\dagger} a_l + a_l a_n^{\dagger} a_m) a_k^{\dagger}$$

$$+ a_m (a_n^{\dagger} a_l a_k^{\dagger} + a_k^{\dagger} a_l a_n^{\dagger}) + (a_n^{\dagger} a_k^{\dagger} a_l + a_l a_k^{\dagger} a_n^{\dagger}) a_m$$

$$= 2/_k\, \delta_{nm} \{ a_l, a_k^{\dagger}\} + \frac{2}{k} \delta_{kl} \{a_m, a_n^{\dagger} \} + \frac{1}{k} \delta_{km} \{a_n^{\dagger}, a_l\}$$

$$+ 3/_k\, \delta_{\,l}\, a_m a_k^{\dagger} + 2/_k\, \delta_{nl}\, a_k^{\dagger} a_m - 3/_{2k^2}(\delta_{km}\delta_{ln} + \delta_{kl}\delta_{n\,}$$

$$a_k^{+}\left(a_\ell a_m a_n^{+} + a_n^{+} a_m a_\ell\right) + \left(a_\ell a_n^{+} a_m + a_m a_n^{+} a_\ell\right) a_k^{+}$$
$$+\, a_m\left(a_k^{+} a_\ell a_n^{+} + a_n^{+} a_\ell a_k^{+}\right) + \left(a_\ell a_k^{+} a_n^{+} + a_n^{+} a_k^{+} a_\ell\right) a_m$$
$$= \frac{2}{k}\,\delta_{mn}\{a_k^{+}, a_\ell\} + 2/_k\,\delta_{k\ell}\{a_n^{+}, a_m\} + \frac{1}{k}\,\delta_{mk}\{a_\ell, a_n^{+}\}$$
$$+\, 3/_k\,\delta_{n\ell}\, a_m a_k^{+} + \frac{1}{k}\,\delta_{n\ell}\, a_k^{+} a_m - 3/_{2k^2}\left(\delta_{km}\delta_{\ell n} + \delta_{k\ell}\delta_{mn}\right)$$

The characteristic equation for N_k in the present case implies the existence of the operator N_k^{-1} and this enables us to derive (III.25R) from (III.62).

A difficulty that arises is that these commutation relations do not define only operators corresponding to $s = 4$; they have an $s = 2$ part mixed in them, as operators which satisfy the commutation relation for $s = 2$ automatically satisfy the commutation relations for $s = 4$ also, i.e. one solution of the commutation relations for $s = 4$ is

$$\{a_k, a_\ell^{+}\} = \frac{1}{2k}\,\delta_{k\ell}\,;\quad \{a_k, a_\ell\} = 0 \tag{III.63}$$

which are just the relations for $s = 2$.
To exclude such a solution, a restriction must be imposed on the vacuum. This restriction is obtained by noting that the occupation number operator n_k is different for different values of s and for each s, the vacuum state satisfies the condition

$$n_k|0\rangle = 0 \tag{III.64}$$

Imposing the condition with n_k chosen for $s = 4$

$$n_k = k\left(a_k^{+} a_k - a_k a_k^{+}\right) + 3/_2 \tag{III.65}$$

enables us to retain only states with $s = 4$
More explicitly, the condition

$$n_k(s=4)\,|0\rangle = 0 \tag{III.66}$$

together with the condition

$$a_k |0\rangle = 0$$

implies that

$$a_k a_k^\dagger |0\rangle = \frac{3}{2k} |0\rangle \qquad \text{(III.67)}$$

The vacuum for $s = 2$ satisfies a different condition, namely,

$$a_k a_k^\dagger |0\rangle = \frac{1}{2k} |0\rangle \qquad \text{(III.68)}$$

In general, for Case R with order s, we must impose the following conditions on the vacuum

$$a_k a_l^\dagger |0\rangle = -\delta_{kl} \frac{(s-1)}{} |0\rangle \qquad \text{(III.69a)}$$

in addition to

$$a_k |0\rangle = 0 . \qquad \text{(III.69b)}$$

if we are to eliminate admixtures of states with smaller values of s.

(III.69a) follows, as $n_k(s) = \frac{1}{2}\left(a_k^\dagger a_k \mp a_k a_k^\dagger\right) + \frac{(s-1)}{2}$ (III.70)

We now turn to <u>Case S</u>

<u>Case S</u>

The commutation relations for Case S can be derived from the corresponding ones for Case R by the following procedure.

We notice that

a) Apart from the signs of certain terms, the commutation relations for Case S are of the same form as those for Case R.

b) The relative signs of terms on the left-hand-side of each commutation relation are positive or negative depending on the parity of the permutation which changes one term into the other.

c) The terms on the right-hand side are the result of contracting terms on the left-hand-side i.e. replacing a pair such as: $a_k^+ a_l$ by δ_{kl} the sign of such a contracted term is given by (the parity of the permutation shifting a_k^+ immediately to the left of a_l) x (the sign of the term before the contraction is made) x (-1)

In this way we can write down the commutation relations corresponding to the cases $s = 1, 2, 3, 4$.

(1) <u>s = 1</u>

In this case we have the trivial result

$$a_k - a_k^+ = 0 , \tag{III.71}$$

(2) <u>s = 2</u>

In this case we have

$$\left[a_k , a_l^+\right] = \frac{1}{2k'} \delta_{kl} ; \quad \left[a_k , a_l\right] = 0 .$$

This is the well-known Bose Einstein case (III.72)

(3) <u>s = 3</u>

Here we find

$$\begin{aligned} a_k a_l^+ a_m - a_m a_l^+ a_k &= \frac{1}{k'} \delta_{kl} a_m - \frac{1}{k'} \delta_{lm} a_k , \\ a_k a_l a_m^+ - a_m^+ a_l a_k &= \frac{1}{k'} \delta_{lm} a_k , \\ a_k a_l a_m - a_m a_l a_k &= 0 \end{aligned}$$

(III.73)

(4) s = 4

We now find

$$a_k\,(a_\ell a_m a_n - a_n a_m a_\ell) + (a_m a_\ell a_n - a_n a_\ell a_m)\,a_k = 0,$$

$$a_m\,(a_\ell a_k^\dagger a_n - a_n a_k^\dagger a_\ell) + (a_n a_k^\dagger a_\ell - a_\ell a_k^\dagger a_n)\,a_m$$

$$= \frac{1}{2k'}\,\delta_{km}\,[a_n, a_\ell] + \frac{3}{2k'}\,\delta_{k\ell}\,[a_m, a_n] + {}^{3}/_{2k'}\,\delta_{kn}\,[a_\ell, a_m] \qquad \text{(III.74)}$$

$$a_m\,(a_k^\dagger a_\ell a_n - a_n a_\ell a_k^\dagger) + (a_\ell a_k^\dagger a_n - a_n a_k^\dagger a_\ell)\,a_m$$

$$= \frac{1}{2k'}\,\delta_{km}\,[a_\ell, a_n] + \frac{3}{2k'}\,\delta_{k\ell}\,[a_n, a_m] - \frac{3}{2k'}\,\delta_{kn}\,a_\ell a_m + \frac{1}{2k'}\,\delta_{kn}\,a_m a_\ell,$$

$$a_k^\dagger\,(a_\ell a_m a_n - a_n a_m a_\ell) + (a_m a_\ell a_n - a_n a_\ell a_m)\,a_k^\dagger$$

$$= \frac{1}{2k'}\,\delta_{k\ell}\,[a_n, a_m] + \frac{1}{2k'}\,\delta_{km}\,[a_\ell, a_n] - \frac{1}{2k'}\,\delta_{kn}\,[a_m, a_\ell],$$

$$a_k^\dagger\,(a_\ell a_n^\dagger a_m - a_m a_n^\dagger a_\ell) + (a_m a_n^\dagger a_\ell - a_\ell a_n^\dagger a_m)\,a_k^\dagger$$

$$+ a_m\,(a_k^\dagger a_\ell a_n^\dagger - a_n^\dagger a_\ell a_k^\dagger) + (a_n^\dagger a_k^\dagger a_\ell - a_\ell a_k^\dagger a_n^\dagger)\,a_m$$

$$= \frac{2}{k'}\,\delta_{nm}\,[a_\ell, a_k^\dagger] + \frac{2}{k'}\,\delta_{k\ell}\,[a_m, a_n^\dagger] + \frac{1}{k'}\,\delta_{km}\,[a_n^\dagger, a_\ell]$$

$$- \frac{3}{k'}\,\delta_{n\ell}\,a_m a_k^\dagger + \frac{2}{k'}\,\delta_{n\ell}\,a_k^\dagger a_m + \frac{3}{2k'^2}\,(\delta_{km}\delta_{\ell n} - \delta_{k\ell}\delta_{mn})$$

$$a_k^\dagger\,(a_\ell a_m a_n^\dagger - a_n^\dagger a_m a_\ell) + (a_\ell a_n^\dagger a_m - a_m a_n^\dagger a_\ell)\,a_k^\dagger$$

$$+ a_m\,(a_k^\dagger a_\ell a_n^\dagger - a_n^\dagger a_\ell a_k^\dagger) + (a_\ell a_k^\dagger a_n^\dagger - a_n^\dagger a_k^\dagger a_\ell)\,a_m$$

$$= 2/_{k'}\,\delta_{nm}\,[a_k^\dagger, a_\ell] + \frac{2}{k'}\,\delta_{k\ell}\,[a_n^\dagger, a_m] + \frac{1}{k'}\,\delta_{mk}\,[a_\ell, a_n^\dagger] \qquad \text{(III.75)}$$

$$+ {}^{3}/_{k'}\,\delta_{n\ell}\,a_m a_k^\dagger - \frac{1}{k'}\,\delta_{n\ell}\,a_k^\dagger a_m - {}^{3}/_{2k'^2}\,(\delta_{km}\delta_{\ell n} - \delta_{k\ell}\delta_{mn})$$

Throughout all cases of the s-type except for the trivial case s = 1, the minimum eigenvalue of the operator N_k is greater than zero. This means that N_k^{-1} exists. This enables us to derive (III.258) from (III.74) in a way similar to case R.

We can also see that (III.75) can be satisfied by the operators for s = 2. By using the explicit representations one can prove that n_0 in Case S is $n_0 = \frac{s-1}{2}$.
Admixture of lower values of s can be eliminated as before by imposing two restrictions on the vacuum:

$$n_k(s)|0\rangle = 0$$

$$a_k|0\rangle = 0 \qquad \text{(III.76)}$$

where

$$n_k = K\left(a_k^\dagger a_k + a_k a_k^\dagger\right) - \frac{s-1}{2}. \qquad \text{(III.77)}$$

Thus, in general,

$$n_k = K\{a_k^\dagger, a_k\}_\mp \mp \frac{(s-1)}{2} \qquad \text{(III.78)}$$

(upper sign for Case R, lower for Case S)

so that we have the conditions

$$a_k a_k^\dagger|0\rangle = \mp \frac{s-1}{2K}|0\rangle$$

$$= \mp(s-1)|0\rangle \quad (\text{when } K = 1/2)$$

$$a_k|0\rangle = 0 \qquad \text{(III.79)}$$

From (III.78) we see that the zero-point energy is $\mp \frac{s-1}{2} E_k$ which corresponds to the parameter E_0 introduced in Sec. II. Compare also (III.79) with (II.49).

We can easily show, that in order to evaluate vacuum expectation values

$$\langle a_k a_l^\dagger \ldots\ldots a_n^\dagger \ldots\ldots \rangle_0 \tag{III.80}$$

we do not need the explicit commutation relations for a_k and $a_k^\dagger$ (such as III.75), provided we are given the restriction on the vacuum,

$$a_k a_l^\dagger |0\rangle = \mp \frac{(s-1)}{2k} \delta_{kl} |0\rangle \tag{III.81}$$

which fixes the value of s,

and the commutation relations between a_k, (or $a_k^\dagger$) and the generators $N_{l,m}$, L_{lm}, M_{lm}.

4. Normal Statistics as the limiting cases of parabose and parafermi statistics of infinite order; the parasymmetry clock*

In the following, we follow the argument Greenberg and Messiah[14]. They introduce the parameter

$$p = (\lambda - 1) \tag{III.82}$$

so that

$$a_k a_\ell^\dagger |0\rangle = \frac{p}{2\kappa} |0\rangle = p|0\rangle \quad (\text{when } \kappa = \tfrac{1}{2}) \tag{III.83}$$

The p - dependence does not enter in the relations (III.5 f.)

$$\left[a_k, [a_\ell^\dagger, a_m]_\mp\right]_- = 2\delta_{k\ell} a_m, \quad \kappa = \tfrac{1}{2} \tag{III.84}$$

$$\left[a_k, [a_\ell^\dagger, a_m^\dagger]_\mp\right]_- = 2\left(\delta_{k\ell} a_m^\dagger \mp \delta_{km} a_\ell^\dagger\right) \tag{III.85}$$

$$\left[a_k, [a_\ell, a_m]_\mp\right]_- = 0 \tag{III.86}$$

To eliminate the p-dependence from the vacuum condition we make the following change of variables

$$a_k \to c_k = \frac{1}{\sqrt{p}} a_k, \quad a_k^\dagger \to c_k^\dagger = \frac{1}{\sqrt{p}} a_k^\dagger,$$

$$n_k(p) = \frac{p}{2}\left([c_k^\dagger, c_k]_\mp \pm 1\right) \tag{III.87}$$

where the upper sign is for parafermi and the lower for para-Bose. In terms of $c, c^\dagger$ (III.83) becomes

$$c_k c_\ell^\dagger |0\rangle = |0\rangle . \tag{III.88}$$

We have also

$$c_k |0\rangle = 0$$

and (III.84) - (III.86) become

$$\left[c_k, [c_\ell^+, c_m]_\mp\right]_- = \frac{2}{p}\delta_{k\ell} c_m ,$$

$$\left[c_k, [c_\ell^+, c_m^+]_\mp\right]_- = \frac{2}{p}\left(\delta_{k\ell} c_m^+ \mp \delta_{km} c_\ell^+\right), \qquad \text{(III.89)}$$

$$\left[c_k, [c_\ell, c_m]_\mp\right]_- = 0$$

We indidcate the p-dependence of c, c^+, Φ by writing them as $c_k(p)$, $c_k^+(p)$, $\Phi_0(p)$.

We now consider the limit $p \longrightarrow \infty$. Since the Hilbert space can be completely determined by the vacuum expectation values, the limits $\Phi_0(\infty)$, $c_k(\infty)$, $c_k^+(\infty)$ are defined in such a way that for each vacuum expectation value we have

$$\lim_{p\to\infty} \left(\Phi_0(p), \mathcal{P}\left(c_k(p), c_k^+(p)\right)\Phi_0(p)\right)$$

$$= \left(\Phi_0(\infty)\ \mathcal{P}\left(c_k(\infty), c_k^+(\infty)\right)\Phi_0(\infty)\right)$$

(III.90)

from which one can reconstruct a Hilbert space for $\Phi_0(\infty)$, $c(\infty)$, $c^+(\infty)$ by virtue of the 'Inverse theorem' of Wightman and others.

(Note: $\Phi_0(p)$ need not converge strongly to $\Phi_0(\infty)$)

Then one can show

$$\left.\begin{aligned} \lim_{p\to\infty} \mathcal{H}^{b}(p) &= \mathcal{H}^{f}(p=1) \\ \lim_{p\to\infty} \mathcal{H}^{f}(p) &= \mathcal{H}^{b}(p=1) \end{aligned}\right\}, \qquad \begin{aligned} b &= \text{Bose} \\ f &= \text{Fermi} \end{aligned}$$

(III.

That is, the limit of parabose is fermi.
and
the limit of parafermi is bose.

Proof:

We have to consider the limit of expressions like

$$\left(\Phi_0(p),\ \mathcal{P}\left(c_k(p), c_k^{+}(p)\right)\Phi_0(p)\right) \qquad \text{(III.92)}$$

The evaluation of this matrix element can be done by moving $c_k(p)$ from left to right as many times as possible so that it reaches $\Phi_0(p)$ after which one can use the vacuum condition (2)(III.88) to make it vanish. Such a change of the position can be done by using our commutation property (III.89), for example*

$$c_\ell c_m^{+} c_k^{+} = c_k^{+} c_\ell c_m^{+} \mp c_k^{+} c_m^{+} c_\ell \pm c_m^{+} c_\ell c_k^{+} \mp 2/p\, \delta_{k\ell} c_m^{+}$$

Now, when using (III.89) we can put the right hand side equal to zero if we want to get the terms which do not vanish at $p \to \infty$. Now in the case of para fermi (III.89) with zero right hand side is satisfied by bose operators because for such operators the commutator [ ] in $[\,,[\ ]_-]$ is always a c-number; and so

$$\Phi_0^{f+}(p)\, \mathcal{P}\left(c_k^{f}(p),\ c_k^{f+}(p)\right) \Phi_0^{f}(p)$$
$$= \left(\Phi_0^{b}(1),\ \mathcal{P}\left(c_k^{b}(1),\ c_k^{b+}(1)\right) \Phi_0^{b}(1)\right)$$

(III.93)

*Throughout, the upper sign refers to the parafermi case and the lower to the parabose case.

In the above, we have, considered the creation and annihilation operators $c_k^{\dagger}$, c_k which do not depend explicitly on p. Some operators depend explicitly on p the operator $n_k(p)$

$$n_k(p) = \frac{p}{2}\left([c_k^{\dagger}, c_k]_{\mp}\right) \pm 1$$

(III.94)

If the product of operators contain the occupation number operator then we use the property

$$[c_k, n_\ell] = \delta_{k\ell}\, c_\ell \qquad \text{(III.95)}$$

and the condition on the vacuum

$$n\, \Phi_0(p) = 0 \qquad \text{(III.96)}$$

The limit $p \to \infty$ exists for $n_k(p)$

$$\lim_{p\to\infty} n_k(p) = \frac{1}{2}\, [c_k^{\dagger}, c_k]_{\pm} \mp 1, \qquad \text{(III.97)}$$

which gives the familiar expressions in the bose or fermi case. We thus see that both for the number operator and for the commutation relations, parafermi with $p \to \infty$ is identical with Bose, and parabose with $p \to \infty$ with fermi (For the commutation relations, putting $p \to \infty$ on the R.H.S. of (III.89) gives these limiting cases).

<u>Example</u>: We illustrate the limit $p \to \infty$ for a simple vacuum expectation value.

In the parabose theory

$$\left(\Phi_0(p),\ c_k(p)\, c_\ell(p)\, c_m^{\dagger}(p)\, c_n^{\dagger}(p)\, \Phi_0(p)\right)$$

$$= \delta_{kn}\,\delta_{\ell m} + \left(\frac{2}{p} - 1\right)\delta_{km}\,\delta_{\ell n} \qquad \text{(III.98)}$$

Using the relation

$$c_\ell c_m^\dagger c_n^\dagger = c_n^\dagger c_\ell c_m^\dagger - c_m^\dagger c_\ell c_n^\dagger + \frac{2}{p}\delta_{n\ell} c_m^\dagger$$

we obtain

$$c_\ell c_m^\dagger c_n^\dagger |0\rangle = \left(\delta_{\ell m} c_k^\dagger - c_m^\dagger \delta_{\ell n} + \frac{1}{p}\delta_{n\ell} c_m^\dagger\right)|0\rangle$$

and therefore

$$\langle 0| c_k c_\ell c_m^\dagger c_k^\dagger |0\rangle = \delta_{\ell m}\delta_{kn} - \delta_{km}\delta_{\ell n} + \frac{2}{p}\delta_{n\ell}\delta_{km}$$
$$= \delta_{kn}\delta_{\ell m} + \left(\frac{2}{p} - 1\right)\delta_{km}\delta_{n\ell}$$

$p = 1$ gives the bose matrix element which is a symmetrized expression. However, if we put $2/p = 0$ the expression is antisymmetrized and is the one we get for fermi operators.

In the parafermi theory

$$\left(\Phi_0(p)\ c_k(p)\ c_\ell(p)\ c_m^\dagger(p)\ c_n^\dagger(p)\ \Phi_0(p)\right)$$
$$= \delta_{kn}\delta_{\ell m} + \left(1 - \frac{2}{p}\right)\delta_{km}\delta_{\ell n}. \qquad \text{(III.99)}$$

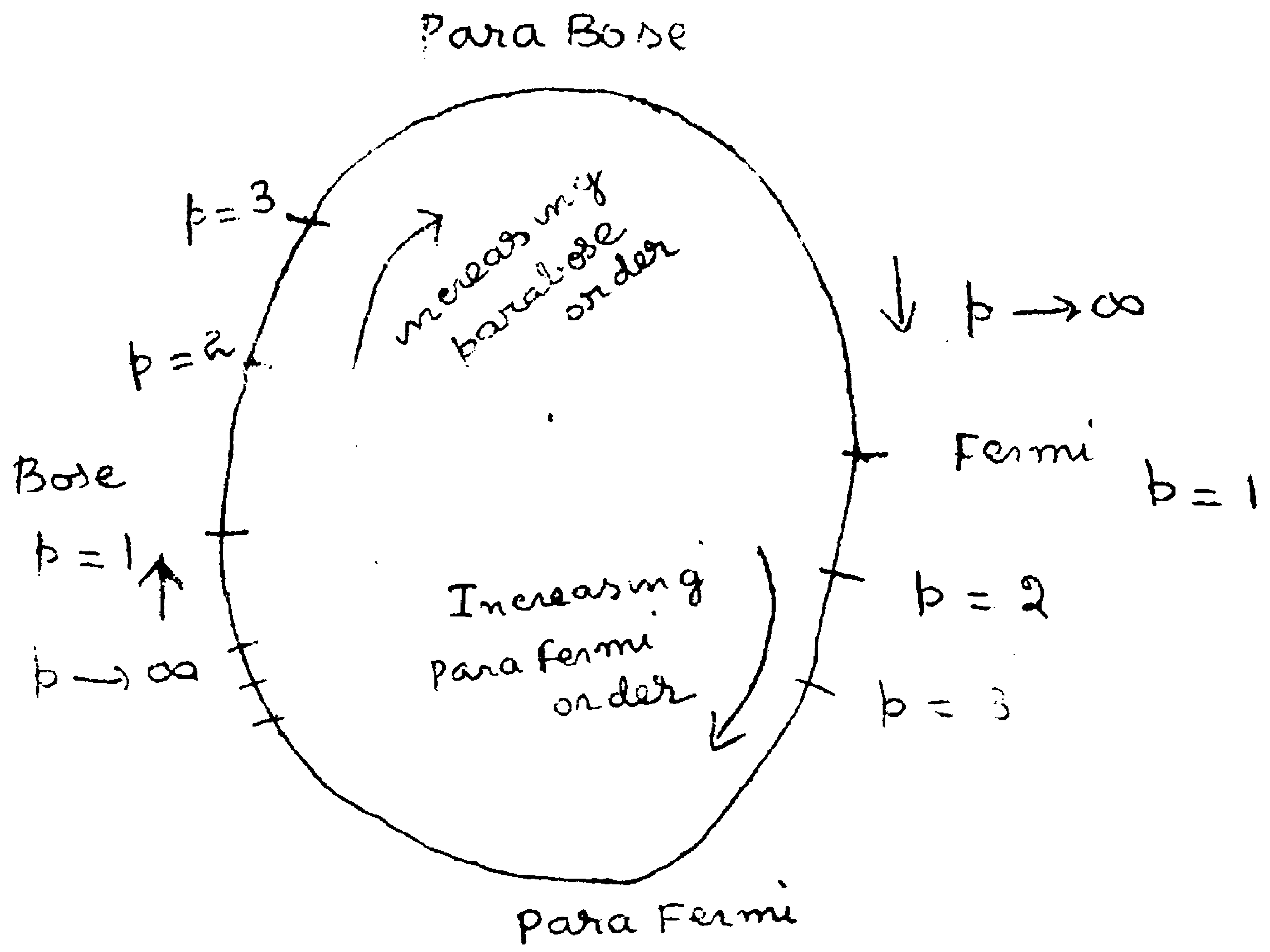
$p = 1$ fermi case if we take $p \longrightarrow 0$ it tends to the symmetrized form which is the bose case.

The Parasymmetry Clock:

To illustrate the statement

Greenberg and Messiah draws the following diagram which they call the 'parasymmetry clock

The Parasymmetry Clock



Note: Although the limits, as $p \to \infty$ exist for $c_k(p)$, $c^\dagger_k(p)$ and $n_k(p)$ and therefore for the total Hamiltonian H limit does not exist for the Hamiltonian density $\mathcal{H}(x)$

For example, the matrix element $\langle \lim_{p\to\infty} \mathcal{H}(x)\rangle$ for a scalar field is not equal to the matrix element of the Hamiltonian density of the scalar field

$$\langle \lim_{p\to\infty} \mathcal{H}(x), \text{parabose}\rangle \neq \langle \mathcal{H}(x), \text{Fermi}\rangle \qquad \text{(III.100)}$$

5. State Vectors and Degeneracy

We have already derived the generalized commutation relations. Let us here discuss some properties of state vectors spanned by the operators a_k and $a_k^\dagger$. For simplicity, let us take the cases of the simplest possible generalization of the ordinary Fermi-Dirac and Bose-Einstein commutation relations, viz. those for $p = 3$. More specifically, consider the following commutation relations for a_k and $a_k^\dagger$ (with $\kappa = 1$ for convenience)

$$
\begin{aligned}
(a_k a_l^\dagger a_m \pm a_m a_l^\dagger a_k) &= \delta_{kl} a_m \pm \delta_{lm} a_k ; \\
(a_k a_l a_m^\dagger \pm a_m^\dagger a_l a_k) &= \delta_{lm} a_k ; \qquad \text{(III.101)} \\
a_k a_l a_m &= \mp a_m a_l a_k ;
\end{aligned}
$$

where the upper (lower) sign refers to Case R(Case S)
The number n_k is defined by

$$n_k = (a_k^\dagger a_k \mp a_k a_k^\dagger) \pm 1$$

which has the eigenvalues 0, 1, 2 for Case R and 0, 1, 2 . . . ∞ for Case S.

The vacuum state is defined as the state for which the eigenvalues of all n_k are zero and which we assume to be non-degenerate

$$n_k|0\rangle = 0 \quad \text{for all } k \tag{III.101}$$

The above definition of the vacuum must be supplemented by the condition

$$a_k|0\rangle = 0 \tag{101'}$$

All other state vectors are generated by applying a certain product of creation and annihilation operators onto $|0\rangle$ and are thus, in general, of the form

$$a_k \cdots\cdots a_l^{\dagger} \cdots\cdots a_k^{\dagger} \cdots\cdots a_m^{\dagger} |0\rangle \tag{III.102}$$

The vectors like $a_k|0\rangle$, $a_k a_l^{\dagger}|0\rangle$ $(k \neq l)$ are zero, since otherwise for these states n_k would have the eigenvalue -1, and this contradicts the above-mentioned property of n_k. In general (III.102) is a simultaneous eigenvector for $n_k, n_l, \ldots\ldots$ with eigenvalues

$$n_k' = (\text{number of } a_k^{\dagger} \text{ in the product}) - (\text{number of } a_k \text{ in the product})$$

unless the state vector is zero. The vector (III.102) becomes zero if at least one $n_k' < 0$ or $n_k' \geq 3$ in the case of R-type and $n_k' < 0$ in the case of S-type. This condition is however, not a necessary condition for the vector to be zero For example.

$$a_k a_l^{\dagger} a_k^{\dagger} |0\rangle = \mp a_k^{\dagger} a_l^{\dagger} a_k |0\rangle = 0$$

We can show, however, that a general vector (III.102) can be expressed as a linear combination of state vectors with 'normal form', that is, vectors generated from the vacuum by a product of creation operators only.

The proof of this statement is quite simple: consider the annihilation operator nearest to the vacuum state vector in the above product, a_m, say. By using the hermitian conjugate of the second relation in (III.101) i.e.

$$a_m a_{m'}^{+} a_{m''}^{+} = \mp a_{m''}^{+} a_{m'}^{+} a_m + \delta_{mm'} a_{m''}^{+}$$

we can move a_m to the right of two preceding creation operators $a_{m'}^{+}$, $a_{m''}^{+}$ thereby getting an extra (contracted) term. We can repeat the process until we get $a_m a_n^{+}|0\rangle = \delta_{mn}|0\rangle$ (the equality follows from (III.101')) or to $a_m|0\rangle = 0$ whereby a_m diappears. By the same technique, finally we get a linear combination of vectors with normal forms.

A one-particle state $|k\rangle$ is given by

$$|k\rangle = a_k^{+}|0\rangle \tag{III.103}$$

which is correctly normalized. The two-particle state $|kk\rangle$ with correct normalization is

$$|kk\rangle = a_k^{+} a_k^{+}|0\rangle \tag{III.104}$$

For a system of twoparticles with different momenta k_1, k_2 $(k_1 \neq k_2)$ we have, however, two possible states.

$$|12\rangle = a^+_{k_1} a^+_{k_2} |0\rangle \; ; \qquad |21\rangle = a^+_{k_2} a^+_{k_1} |0\rangle \qquad \text{(III.105)}$$

which are orthogonal to each other.

This can be proved by taking a scalar product and using commutation relations ~~and~~ III 101 (III.101). Thus the system has a two-fold degenracy. The linear combination of the states (III.105).

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left\{ |12\rangle \pm |21\rangle \right\} \qquad \text{(III.106)}$$

is symmetric or anti-symmetric, with respect to the exchange of two particles.

For three particles with different momenta k_1, k_2, k_3 we have $3! = 6$ states $|123\rangle = a^+_{k_1} a^+_{k_2} a^+_{k_3} |0\rangle$ etc However, as a direct consequence of the $\underbrace{\qquad\qquad}_{}$ last relation in III 101

$$a_k a_\ell a_m = \mp a_m a_\ell a_k$$

only three of them are independent. In fact, we have the following relations.

$$e_1 \equiv |123\rangle = \mp |321\rangle ,$$
$$e_2 \equiv |231\rangle = \mp |132\rangle ;$$
$$e_3 \equiv |312\rangle = \mp |213\rangle ,$$

(III

where the vectors e_1, e_2 and e_3 form an orthogonal set. Thus the three-particle state has a threefold degeneracy.

Now let us consider the symmetric group of degree 3, Π_3, which consists of 6 elements (1), (12), (13), (23), (123) = (13) (12) and (132) = (12) (13). Under these transformations the three vectors get transformed as follows:

$$(12)\, e_1 = \mp e_3 ,$$

$$(12)\, e_2 = \mp e_2 ,$$

$$(13)\, e_1 = \mp e_1 ,$$

$$(13)\, e_2 = \mp e_3 ,$$

$$(23)\, e_1 = \mp e_2 ,$$

$$(23)\, e_2 = \mp e_3$$

The 3-dimensional representation of the group thus obtained is reducible, however, and can be reduced to two irreducible ones, the bases of which are, respectively,

$$e_0 = \frac{1}{\sqrt{3}} (e_1 + e_2 + e_3) , \qquad \text{(III.108)}$$

and

$$e_1' = \frac{1}{\sqrt{2}} (e_2 - e_3) ,$$

$$e_2' = \sqrt{2/3} \left[e_1 - \frac{1}{2} (e_2 + e_3) \right] , \qquad \text{(III.109)}$$

Thus, in the case of the R-type we get the irreducible representations, Corresponding to the Young diagrams of Fig. 1(a), and in the case of the S-type those corresponding to the fig. 1(b)

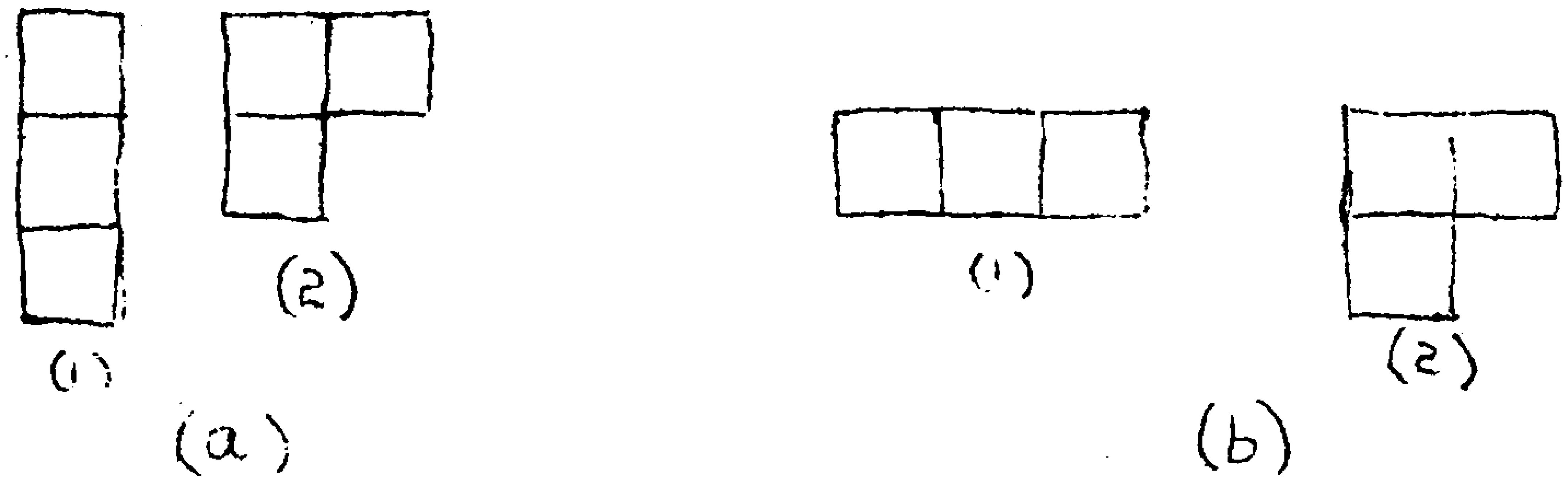


Fig. 1.

For the 4-particle states with different momenta, although we have $4! = 24$ states, only six of them are independent. In the same way as above, the 6-dimensional space is reduced to three invariant sub-spaces the dimensions of which are 1, 2 and 3 respectively and which lead to the irreducible representations of π_4. The corresponding Young diagrams are given in Fig. 2(a) for the R-type and in Fig. 2(b) for the S-type.

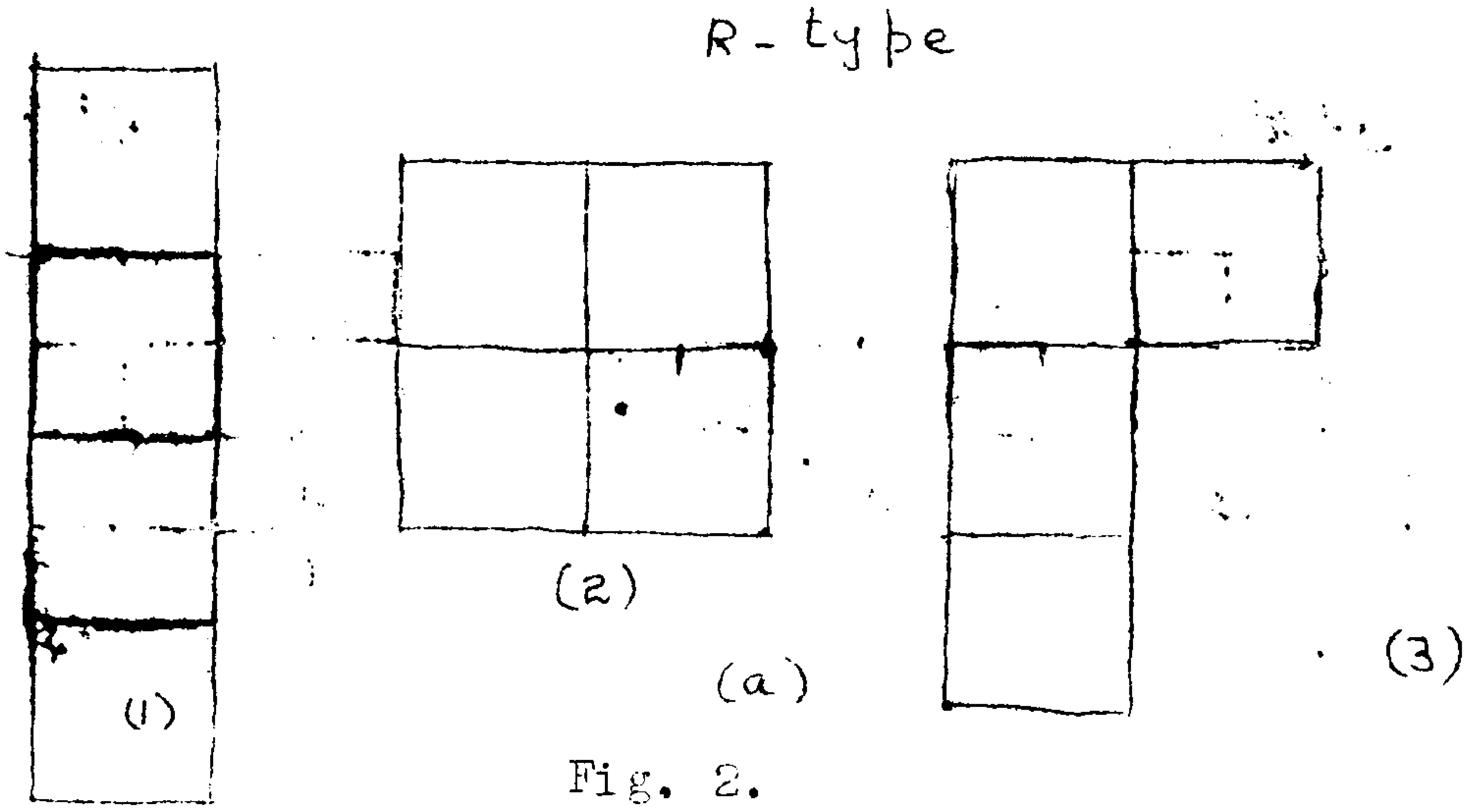
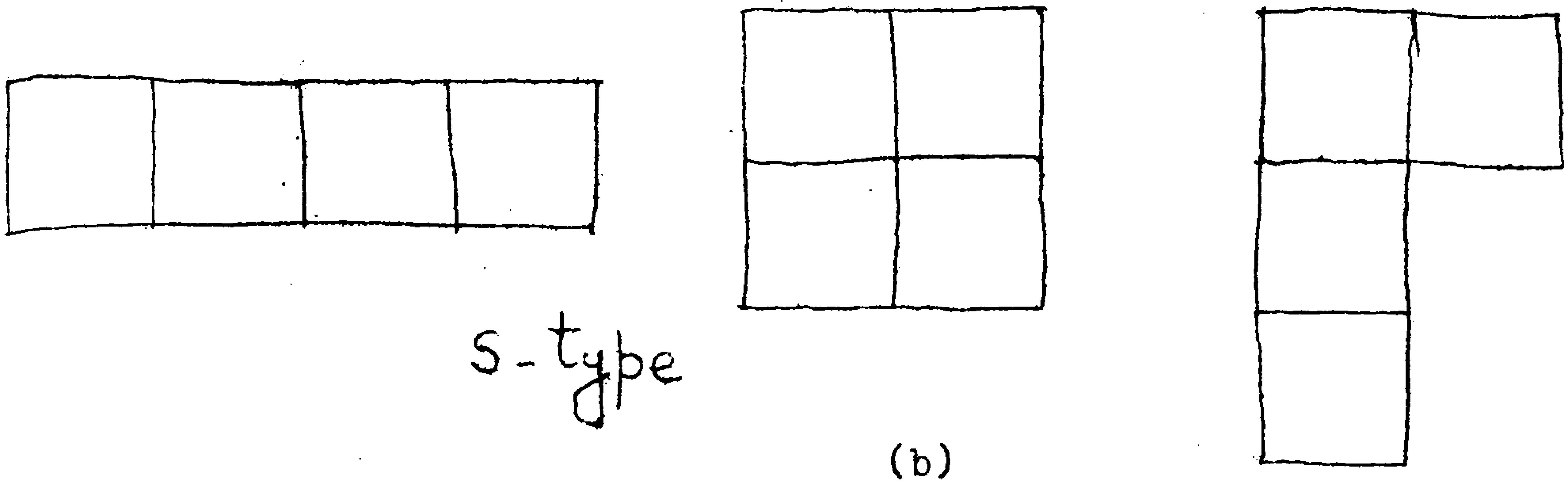


Fig. 2.



(b)

More generally, for a system of n particles with different momenta, we can show that among $n!$ possible state vectors, only $N(n)$ states are independent. where

$$
N(n) = \begin{cases} \dfrac{n!}{\left(\frac{n}{2}!\right)^2} & \text{for } n \text{ even} \\[2ex] \dfrac{n!}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} & \text{for } n \text{ odd} \end{cases} \qquad \text{(III.110)}
$$

We can also show that for $n \geqslant 3$ the antisymmetric representation (corresponding to the Young diagram with one column and n rows) always appears in Case R but not in Case S whereas the symmetric representation (corresponding to the Young diagram with n columns and one row always appear in Case S, but not in Case R. This can be said more generally for all young diagrams appearing in the theory. If the representations in Case R consist of Young diagrams A, B, C.......... then those in Case S consist of A_c, B_c, C_c where the lower script c corresponds to conjugate Young diagrams (A_c can be obtained from A by interchanging rows and colums). Only self-conjugate diagrams (square Young diagrams) could appear in both cases.

The existence of the new kind of degeneracy which is specific of many-particle system can be interpreted as the existence of a new degree of freedom for such systems. Hence we need to introduce some new variables which are essentially permutation operators. However, a permutation operator P is unitary, and therefore cannot be taken as an observable. There are however, operators which can be physical variables.

(1) the operators P such that $P^{-1} = P (= P^\dagger)$, have this property.

2) class functions $\chi_c = \sum_c P/n_c$

where the summation is taken over all n_c elements belonging to the class. Since a class contains P^{-1} $(= P^\dagger)$ as well as P, one has

$$\chi_c^\dagger = \chi_c .$$

Now, in our case, to distinguish various representations we can use χ_c since different values χ_c' of χ_c correspond to different classes, and so to different irreducible representations. However, to specify states within a given representation, we have to use operators of the kind (1) and any measurable quantity should be given in a form averaged over this kind of variables.

6. Generation of the Fundamental Concepts of Quantum Mechanics

As we have introduced symmetries more general than the usual symmetries or antisymmetric representations with respect to permutations, we have also to re-examine some of the fundamentals of quantum mechanics.

There is a requirement of indistinguishability of identical particles. Any physical measurement can be expressed by means of expectation values. We can express the above requirement by the condition that

$$\Big(\Psi(i_1, i_2, \ldots\ldots i_N),\ A\ \Psi(i_1, i_2, \ldots\ldots i_N)\Big)$$
$$\equiv \Big(\Psi(1,2,\ldots N)\ A\ \Psi(1,2\ldots N)\Big), \qquad \text{(III.111)}$$

where $i_1, i_2, \ldots\ i_N$ is obtained by permuting - 1, 2, 3, ... N. We assume that permutations can be represented by unitary transformations (not ray).

According to Dirac,

$$P(1,2,\ldots N) = (i_1, i_2, \ldots i_N),$$
$$\big|\Psi(i_1, i_2, \cdots i_N)\big\rangle = P^{-1}\big|\Psi(1,2,\ldots\ldots N)\big\rangle,$$
$$\vec{P} \doteq P^{-1}; \qquad \text{(III.113)}$$

Since (III.111) must hold for any vector, we substitute $|\Psi + \alpha\Phi\rangle$, and $|\Psi + i\alpha\Phi\rangle$

Then we get

(a) $$\langle\Psi A \Psi\rangle + \alpha\langle\Phi A \Psi\rangle + \alpha\langle\Psi A \Phi\rangle + \alpha^2\langle\Phi A \Phi\rangle$$

$$= \langle\Psi PAP^{-1}\Psi\rangle + \alpha\langle\Phi\ PAP^{-1}\Psi\rangle + \alpha\langle\Psi\ PAP^{-1}\Phi\rangle$$
$$+ \alpha^2\langle\Phi PAP^{-1}\Phi\rangle \qquad \text{(III.114)}$$

Since $\langle\Psi A \Psi\rangle = \langle\Psi PAP^{-1}\Psi\rangle$ by (III.113) and (b)

$$\langle \Psi A \Psi \rangle - i\alpha \langle \Phi A \Psi \rangle + i\alpha \langle \Psi A \Phi \rangle + \alpha^2 \langle \Phi A \Phi \rangle$$

$$= \langle \Psi PAP^{-1} \Psi \rangle - i\alpha \langle \Phi PAP^{-1} \Psi \rangle + i\alpha \langle \Psi PAP^{-1} \Phi \rangle + \alpha^2 \langle \Phi PAP^{-1} \Phi \rangle ; \qquad \text{(III.114b)}$$

From III.114a,b

$$\alpha \langle \Phi A \Psi \rangle + \alpha \langle \Psi A \Phi \rangle = \alpha \langle \Phi PAP^{-1} \Psi \rangle + \alpha \langle \Psi PAP^{-1} \Phi \rangle ,$$

$$i\alpha \langle \Phi A \Psi \rangle + i\alpha \langle \Psi A \Phi \rangle = i\alpha \langle \Phi PAP^{-1} \Psi \rangle + i\alpha \langle \Psi PAP^{-1} \Phi \rangle , \qquad \text{(III.115)}$$

which gives

$$\langle \Phi A \Psi \rangle = \langle \Phi PAP^{-1} \Psi \rangle$$

Then we conclude that

$$A = PAP^{-1} \text{ (or) } PA = AP \qquad \text{(III.116)}$$

We call such operators 'physical observable'.

In the usual literature there is the following kind of argument against the introduction of a general symmetry into the theory. It says that indistinguishability is expressed by

$$P \Psi (1, \dots N) = C \Psi (1, 2 \dots N) ; \quad |C| = 1$$

i.e. the result of permutation must be only a change of phase of the state vector. However, as we have seen above, this is too strong a condition for the indistinguishability. Moreover, such a restriction automatically leads to one-dimensional symmetric or antisymmetric representations with respect to permutation, that is, it allows only Bose or Fermi symmetries.

Another argument that only Bose or Fermi symmetries occur is from the complete set of commuting observables. This assumption is that there are a set of commuting observables whose simultaneous eigenvalues suffice to define a complete set of orthogonal one-dimensional subspace in Hilbert space. Sincethese observables must commute with all P such an eigen-subspace is invariant under and since this set is complete, the subspace is one-dimensional. This subspace is therefore, a symmetric or anti-symmetric representation of P. Thus, in order to allow a general symmetry we must relax this assumption. Instead of this we introduce (following Greenberg and Messiah[4]) an assumption of a maximal set of commuting observables. By a maximal set we mean a set whose eigenvalues define subspace which are irreducible with respect to P. Such sets give the maximum amount of information compatible with indistinguishability. We introduce new variable-the class function χ_C. as mentioned before which satisfies the condition for observables. Now, in general, such an eigen-subspace has dimension greater than one. So we have an indeterminancy, that is larger than in the usual theory. In the usual theory the st te vector can be determined up to a phase (ray). Here it can be done only up to this eigen-subspace (generalized ray). Even when we fix all the eigenvalues of observables, this is not enough to fix a vector in the subspace. Any vector in this subspace can represent our physical system. Now let us consider the problem of reproducibility of experiments. In the usual uantum Mechanics, the indeterminancy of phase does not lead to any difficulty (since only $|M|^2$ is measure). Thus, in the present case, if the physical results depend on what is undermined here, namely, on which vector in the eigen-subspace we choose to represent our physical system, then we no longer have reproducibility of experimental results. However, we can show that this is not the case, i.e., any vector in a given eigensubspace or generalized ray gives the same physical result. If Ψ, Φ lie in an irreducible eigenspace Υ then, for any observable A we have

$$\langle \Phi A \Phi \rangle = \langle \Psi A \Psi \rangle,$$
$$|\Phi\rangle, |\Psi\rangle \in \Upsilon \qquad \text{(III.117)}$$

because the class function commutes with all P, $P\chi_c P^{-1} = \chi_c$

We can show that any subsequent observation will not depend on the choice of the vector either. The time development of the state is described by

$$\hat{u}(t) = e^{-iHt}, \qquad \text{(III.118)}$$

Since H is a physical observable, both H and $u(t)$ commute with P. So, if we put

$$\Phi(t) = u(t)\,\Phi(0)$$
$$\Psi(t) = u(t)\,\Psi(0) \qquad \text{(III.119)}$$
$$\Phi(0), \Psi(0) \in \mathcal{E}^{\Upsilon},$$

then

$$\langle \Phi(t)\, A\, \Phi(t) \rangle = \langle \Psi(t) A\, \Psi(t) \rangle, \qquad \text{(III.120)}$$

i.e. $\langle \Phi(0) \left(u^{+}(t) A\, u(t)\right) \Phi(0) \rangle = \langle \Psi(0) \left(u^{+}(t) A\, u(t)\right) \Psi(0) \rangle$

Another consequence of $[u(t), P] = 0$ is that a vector in a given irreducible representation remains in the same representation in the course of time.

Comment (1)

Consider two spin 1/2 particles: assume para fermi with $p = 2$

We then have two states $|\pm\rangle$

$$|+\rangle = \text{triplet} \times S \text{ wave} + \text{singlet} \times p \text{ wave}$$

$$|-\rangle = \text{singlet} \times S \text{ wave} + \text{triplet} \times p \text{ wave}$$

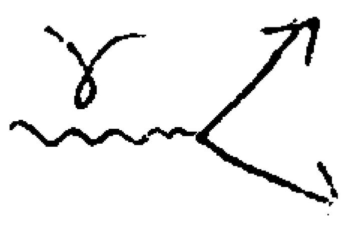
If the situation is such that only S waves come out then the measurement of $\chi(\pm)$ is equivalent to that of total spin ~~m~~

Comment (2)

We shall see later that in ~~QED~~ quantum electrodynamics for example, we have to assume the interaction term of the form

$$H_{int} = + J_\mu A_\mu \ , \ J_\mu = ie\left[\bar{\Psi}, \gamma_\mu \Psi\right]_-$$

In such a case, if we know how the particles concerned are created we can uniquely fix the state vector. For example= a a pair produced by a γ ray is in the state $|-\rangle$ (Pair creation in a Coulomb field).



7. Relativistic Field Theory; General Analogue of the Spin-Statistics Connection

Our method of field quantization can be easily applied to a relativistic field theory. From the view-point of quantization, the only new feature we encounter is that we have to take into account annihilation and creation operators for anti-particles, b_k and $b_k^\dagger$.

It is almost obvious that for the commutation relations between a_k and $a_k^\dagger$ and for those between b_k and $b_k^\dagger$ we can adopt the same commutation relations as we have in the non-relativistic case. In order to obtain the commutation relations involving both a_k, $a_k^\dagger$ and b_k, $b_k^\dagger$ we have only to reinterpret b_k (or $b_k^\dagger$) as one of the a_k (or $a_k^\dagger$) but with a suffix different from any of those of the a_k and then to apply the commutation relations for a_k and $a_k^\dagger$. In this way, we can get all necessary relations involving a_k, $a_k^\dagger$, b_k and $b_k^\dagger$.

In the non-relativistic case we saw that both the parafermi and parabose statistics can be applied to one and the same Schrodinger field. However, this is no longer true in the relativistic case, and in fact there exists a generalized theorem concerning spin and statistics. We can see this in the following way. First let us consider a complex field with half-integral spin. If we substitute its Fourier series expansion for this field operator into its unsymmetrized Hamiltonian we get

$$H = \sum_k |E_k| \left(a_k^\dagger a_k - b_k b_k^\dagger \right). \qquad \text{(III.1)}$$

In order to apply the S-type quantization method we have to symmetrize the Hamiltonian, which then reads

$$H_{sym} = \frac{1}{2} \sum_k |E_k| \left(a_k^\dagger a_k + a_k a_k^\dagger\right) - \frac{1}{2} \sum_k |E_k| \left(b_k^\dagger b_k + b_k b_k^\dagger\right), \tag{III.1222}$$

This operator, when quantized according to the S-type method, has, however, non-definite eigenvalues. Thus, we cannot apply this to spinor fields. On the other hand, if we antisymmetrize (III.121), we get

$$H_{antisym} = \frac{1}{2} \sum_k |E_k| \left(a_k^\dagger a_k - a_k a_k^\dagger\right) + \frac{1}{2} \sum_k |E_k| \left(b_k^\dagger b_k - b_k b_k^\dagger\right), \tag{III.123}$$

which, when quantized according to the S-type method, has definite (zero or positive) eigenvalues.

Exactly the opposite is the case with fields with integral spin. The unsymmetrized Hamiltonian for a tensor field is

$$H = \sum_k |E_k| \left(a_k^\dagger a_k + b_k b_k^\dagger\right) \tag{III.124}$$

On the other hand its symmetrized form

$$H_{Sym} = \frac{1}{2}\sum_k |E_k| (a_k^\dagger a_k + a_k a_k^\dagger) + \frac{1}{2}\sum_k |E_k| (b_k^\dagger b_k + b_k b_k^\dagger) \tag{III.126}$$

has positive definite eigenvalues when quantized according to the S-type method. For real fields the same argument applies, except that symmetrization is unnecessary. We thus arrive at the following theorem:

<u>Theorem</u>: (Generalization of the Fierz-Pauli Theorem).

Parafermi statistics, (R-type quantization) in which at most n_{max} particles can occupy one and the same state, can only be applied to spinor fields, and para-bose statistics (S-type quantization) can only be applied to tensor fields.

As mentioned above the causality condition in this theory is different from the ordinary one: We do not have a strong locality. Instead, we have the relation

$$\left[\left[\phi^\dagger(x)\circ\phi(x)\right]_\mp, \phi(y)\right]_- = 0$$

for space-like separation $x-y$ where $\phi^\dagger(x)\circ\phi(x)$ is a density of a physical quantity. Moreover, for vacuum expectation values we have

$$\langle\phi(x_1)\phi(x_2)\cdots\cdots\phi(x_n)\rangle = \mp\langle\phi(x_n)\cdots\cdots\phi(x_2)\phi(x_1)\rangle_0$$

for equal-time quantities, which guarantees the validity of CPT theorem in our theory.

8. Interacting Fields, Generalization of Nishijima's Theorem:

In the ordinary field theory (with the normal spin-statistics relationship), the condition that the Euler-Lagrange equations be consistent with the Heisenberg equations of motion leads to a restriction on the nature of the interaction Hamiltonian which was expressed first by Nishijima, and later elaborated by Umezawa, Araki and others[9].

In the conventional field theory, we usually assum anti-commutation rules between different fermion fields and commutation rules between different boson fields. (This will be called here the normal case). It is possible however to assume that different fermion fields commute and different boson fields anticommute (This will be called as the anomalous case) The same can be said of the relation between fermion and boson fields. Between a fermion field and a boson field one can assume an anti-commutation relation (anomalous) as well as commutation relation (normal).

However, it is not possible to assume the anomalous commutation relations with complete freedom once the interaction Hamiltonian has been given. In fact, the theorem due to Nishijima expresses the following restriction on the interaction Hamiltonian.

Theorem: In each term of the interaction Hamiltonian there must always occur an even number of field operators which anti-commute with any given field operator at relatively space-like points:

Proofs

Consider an interacting system of a certain number of fermion and boson fields, quantized as in the normal case, with the interaction Hamiltonian

$$H'_{int} = \int H_{int}(x)\, d^3x , \qquad \text{(III.127)}$$

$$H_{int}(x) = g\, \psi^{A\dagger} \psi^{B} \psi^{c\dagger} \cdots \varphi^{a} \varphi^{b} \varphi^{c} \qquad \text{(III.127a)}$$

$$= g \left(\psi^{A}\right)^{\dagger} \chi \quad (\text{say}) \qquad \text{(III.127b)}$$

where χ represents the product of all the operators to the right of $\psi^{A\dagger}$ in (III.127). The ψ's are fermion fields and the φ's boson fields.

The Euler-Lagrange equations for the field operator will be

$$i\, \dot{\psi}^{A}(x) = \left(\frac{1}{i}\, \vec{\alpha}\cdot\vec{\nabla} + m^{A}\beta\right)\psi^{A} + g\, \psi^{B} \psi^{c\dagger} \cdots \varphi^{a} \varphi^{b} \varphi \qquad \text{(III.128)}$$

whereas the Heisenberg equations of motion are

$$i\, \dot{\psi}^{A}(x) = \left[\psi^{A}(x), H_0^{A}\right] + \left[\psi^{A}(x), H_{int}(x)\right] \qquad \text{(III.129)}$$

$$= \left(\frac{1}{i}\, \vec{\alpha}\cdot\vec{\nabla} + m^{A}\beta\right)\psi^{A} + \left[\psi^{A}(x), H^{\dagger}(x)\right], \qquad \text{(III.129a)}$$

where

$$H_0^{A} = \psi^{A\dagger}(x) \left(\frac{1}{i}\, \vec{\alpha}\cdot\vec{\nabla} + m^{A}\beta\right) \psi^{A}(x) \qquad \text{(III.130)}$$

The requirement that these two equations , (III.128) and (III.129a) should be consistent, gives the condition

$$\left[\psi^A(x), H_{int}(x)\right] = g\, \psi^B(x)\, \psi^{d\dagger}(x) \ldots \varphi^a \varphi^b \varphi^c$$
$$= g\, \chi(x) \qquad \text{(III.131)}$$

Using eqn. (III.127)

$$\left[\psi^A(x), H_{int}(x')\right] = \left[\psi^A(x), g\, \psi^{A\dagger}(x')\, \chi(x')\right]$$
$$= g\left(\psi^A(x)\, \psi^{A\dagger}(x')\, \chi(x') - \psi^{A\dagger}(x')\, \chi(x')\, \psi^A(x)\right)$$
$$= g\left(\psi^A(x)\, \psi^{A\dagger}(x') + (-1)^{n_A+1}\, \psi^{A\dagger}(x')\, \psi^A(x)\right)$$
$$\times\, \chi(x') \qquad \text{(III.132)}$$

and $[\psi^A$, the second (h.c) term$] = 0$ III.132′

where n_A is the number of opprators in $\chi(x')$ which anticommute with ψ^A

(III.132) will be equal to $g\,\chi(x)$ as required by (III.131), provided that

$$\left(\psi^A(x)\, \psi^{A\dagger}(x') + (-1)^{n_A+1}\, \psi^{A\dagger}(x')\, \psi^A(x)\right)$$
$$= \delta(x - x') \qquad \text{(III.133)}$$

However, as $\psi^A(x)$ is a fermion field, we have

$$\left(\psi^A(x)\, \psi^{A\dagger}(x') + \psi^{A\dagger}(x')\, \psi^A(x)\right)$$
$$= \delta(x - x') \qquad \text{(III.1..)}$$

Therefore, the condition for (III.133) to hold is that

$$(-1)^{n_A+1} = +1 \tag{III.135}$$

i.e.

(n_A+1) must be an even number. (III.135a)

n_A is the number of field operators in $\chi(x)$ that anti-commute with $\psi^{A\dagger}(x)$. The number of field operators in $H_{int}(x)$ that anticommute with $\psi^{A}(x)$ is n_A+1 if we include the additional factor $\psi^{A\dagger}(x)$ in

$$H_{int} = g\,\psi^{A\dagger}(x)\,\chi(x)$$

(III.132') is automatically satisfied with the condition (III.135). A similar argument applies also tothe boson fields ϕ^{a}'s.

Thus, the total number of field operators in $H_{int}(x)$ that anticommute with a given field operator $\psi^{A\dagger}(x)$ must be even, which demonstrates our theorem for the normal case.

<u>Example</u>

Suppose, we have an interaction including two different classes of fermions, which are considered to be independent, so that we require the field operators of one class to <u>commute</u> with those of the other class. For instance, we may have an interacting system of baryons and leptons, where the different baryons field operators ψ^B are assumed to anticommute with one another, the different lepton field operators ψ^L are assumed to anticommute with one another, while the baryon field operators ψ^B are assumed to commute with the lepton field operators ψ^L

$$\{\psi^{B_1}, \psi^{B_2}\}_+ = 0$$

$$\{\psi^{L_1}, \psi^{L_2}\}_+ = 0$$

$$[\psi^{B}, \psi^{L}]_- = 0$$

(III.136)

Then the number of baryon field operators occurring in an interaction Hamiltonian must be even; similarly, the number of lepton field operators occurring in an interaction must be even. This immediately leads to separate conservations of baryon and lepton numbers.

We could similarly get a separate conservation of muon number and electron number if we required the muon and ν_μ field operators to commute with the electron and ν_e field operators. The Nishijima theorem expresses reciprocity between the interaction Hamiltonian and the commutation relation. Given the commutation relations between all the field operators occuring in an interaction, the form of the interaction Hamiltonian is restricted. Conversely, given the Interaction Hamiltonian, the commutation relations between the different field operators are restricted.

We now ask the following question:
How can we generalize this theorem when one or more of the fields involved in an interaction obeys parastatistics? We shall try to answer this question by following arguments developed in a paper by Strathdee and myself[4]

We will now show that for parastatistics of order greater than one we obtain a superselection rule, namely, that the number of particles obeying a given type of statistics is conserved.

We first show that in order to accommodate in our theory fields obeyings parastatistics we have also to generalize the usual <u>bilinear</u> commutation relations between different field operators i.e $[\psi^A, \psi^B]_+ = 0 \quad A \neq B$, etc.

To illustrate consider an interaction between various spinor fields, $\psi^A, \psi^B, \psi^C, \ldots$. These fields must be quantized according to rules corresponding to Case R. Such a quantization is characterized by a number s, which is essentially, the dimension of the representation of the rotation group introduced there. Let the fields $\psi^A, \psi^B, \psi^C, \ldots$ have s-values $s_A, s_B, s_C, \ldots$ respectively.

Consider an interaction Hamiltonian

$$H'_{int}(x) = g\Big\{ [\psi^{A\dagger}, O_1 \psi^B]\, [\psi^{C\dagger}, O_2 \psi^D] \cdots\cdots$$

$$\cdots\; [\psi^{F\dagger}, O_i \psi^G] \cdots + h.c \Big\} \qquad \text{(III.3[illegible])}$$

where each factor has been antisymmetrized.

With our quantization, a field ψ^A must satisfy the trilinear commutation relations given below (in configuration space).

$$\left[\psi^a_\alpha(x), \left[\psi^{a\dagger}_\beta(y), \psi^{a\dagger}_\gamma(z)\right]\right]$$
$$= 2\delta_{\alpha\beta}\,\delta^{(3)}(x-y)\,\psi^{a\dagger}_\gamma(z) - 2\delta_{\alpha\gamma}\,\delta^{(3)}(x-z)\,\psi^{a\dagger}_\beta(y)$$

$$\left[\psi^a_\alpha(x), \left[\psi^{a\dagger}_\beta(y), \psi^a_\gamma(z)\right]\right] = 2\delta_{\alpha\beta}\,\delta^{(3)}(x-y)\,\psi^a_\gamma(z)$$

$$\left[\psi^a_\alpha(x), \left[\psi^a_\beta(y), \psi^a_\gamma(z)\right]\right] = 0 \qquad \text{(III.138)}$$

where we have taken $k = \frac{1}{2}$

(para-[illegible])

Suppose the fields appearing here are all different, and atleast one of them say ψ^A [illegible]. We impose the requirement that, as for the free fields, the Euler Lagrange variational equations shall be consistent with the Heisenberg equations of motion, which may be written in a covariant form as

$$-i\partial_\mu \psi^A(x) = \left[\psi^A(x), P_\mu\right], \qquad \text{(III.139)}$$

for each [illegible] ψ^a

where P_μ is the energy momentum 4-vector

The fourth component of this equation, namely,

$$-i\,\partial_4 \psi^A(x) = \left[\psi^A(x), P_4\right], \qquad \text{(III.139')}$$

must correspond to the Euler-Lagrange equations, which will be of the form

$$i\dot{\psi}_A = \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + m^a\beta\right)\psi^A + g\,O_1\,\psi^B\left[\psi^{C\dagger}, O_2\,\psi^D\right]\ldots$$
$$\ldots\left[\psi^{F\dagger}, O_i\,\psi^G\right] \qquad \text{(III.140)}$$

If we assume that the different fermion fields obey bilinear commutation or anticommutation relations $\left[\psi^A, \psi^B\right]_\pm = 0 \; (A \neq B)$ then the Heisenberg equation may be written as

$$i\dot{\psi}^A = \left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + m^A\beta\right) + g\int d^3y \left[\left[\psi^A(x), \psi^{A\dagger}(y)\right], O_1\psi^B(y)\right]$$

$$\cdot\left[\psi^{C\dagger}(y), O_2\psi^D(y)\right] \cdots\cdots \Big] \qquad \text{(III.141)}$$

This Heisenberg equation will be consistent with the Euler-Lagrange equations only if

$$\left[\psi^A(x), \psi^{A\dagger}(y)\right]_+ = 0$$

However for ψ^A with $s_a > 2$ we do not have any such bilinear anticommutation relations. We thus have a contradiction. Therefore, we cannot assume in such a case that the different fermion fields obey bilinear commutation relations. We must also generalize the commutation relations between different fields.

We shall first consider the normal case. The simplest generalization (which contains the ordinary case as a particular case) is to assume trilinear commutation relations for different field operators also

$$\left[\psi^A_\alpha(x), \left[\psi^{A'\dagger}_\beta(y), \psi^{A''}_\gamma(z)\right]\right]$$

$$= 2\,\delta_{\alpha\beta}\,\delta_{AA'}\,\delta^{(3)}(x-y)\,\psi^{A''\dagger}_\gamma(z)$$

$$-2\,\delta_{\alpha\gamma}\,\delta_{AA''}\,\delta^{(3)}(x-z)\,\psi^{A'\dagger}_\beta(y), \text{ etc}$$

$$\text{(III.142)}$$

In fact, when we require that the commutation relations be trilinear and local and that the resulting theory satisfy the ordinary requirements of field theory such as Lorentz invariance, CPT theorem, positive definiteness of the Hilbert space, and the existence of the vacuum, then we can show that (III.142) is the most general form for such commutation relations. We shall prove this later.

When we allow ~~an~~ as interaction bilinear forms such as $g[\psi^{a\dagger}, O\psi^{b}]$ then the trilinear form (III 142) is the unique choice.

We have earlier seen that to ensure that a field obeys parastatistics of a given order $p = s-1$, we must not only require the appropriate commutation relations between the field operators, but also impose a restriction on the vacuum, which eliminates fields with lower values of p obeying the same commutation relations:

$$a_k a_l^\dagger |0\rangle = \cancel{-}(s-1)\,\delta_{kl}|0\rangle, \tag{III.69'}$$

(assuming $K = \frac{1}{2}$ in eqn. (III. [illegible])

in addition to the usual condition

$$a_k |0\rangle = 0$$

Using this restriction, we can prove that except for the case $A \neq A', A'', \cdots$, the relations (III.142) leads to an inconsistency unless $s_{a'} = s_{a''}$. In other words, we can assume the trilinear commutation relations, with non-zero right-hand sides, only when all the fields satisfy the same statistics (same s-values).

Between operators belonging to different pairs in eqn. (III.137) it is consistent to assume bilinear anticommutation relations with vanishing right-hand side like

$$\{\psi^{A}\ \psi^{F}\}_{+} = 0$$

Thus in our formalism with the tri-linear commutation relations we obtain the following restrictions on the form of the interaction Hamiltonian (III.137).

Two members of the same factor must obey the same statistics. Two fields belonging to different pairs may or may not be of the same statistical type

This result continues to be valid when we consider interactions in which a field operator belonging to the same field occurs more than once.

For tensor fields $\varphi^{b}, \varphi^{a}, \varphi^{c}, \ldots$ we can argue in a similar way. These fields must be quantized according to the rules for Case S, characterized by the parameters

As for the spinor fields we can prove that for different boson fields, we can impose trilinear. commutation relations of a form similar to the relations for the same field. Again, we can assume the right-hand sides to be non-zero only when all fields in the commutator $[\varphi^{a}, \{\varphi^{b}\ \varphi^{c}\}_{+}]_{-}$ satisfy the same statistics, i.e. have the same s-value. Thus any interaction Hamiltonian conisting of only tensor fields should be of a form

$$H_{int} = g\,\{\varphi_a^{+}\,o_1\,\varphi_b\}\,\{\varphi_c^{+}\,o_2\,\varphi_d\}\ldots \qquad \text{(III.143)}$$

where the 2 members of each factors, obeying the same statistics, should be symmetrized.

Ordinary Bose fields, with $p = 2$, are an exception, they are not subject to the above restriction. They need not always appear in pairs, products with odd power are also allwed, as in the usual Yukawa interaction $(\psi^{+}\psi\phi)$

We shall now proceed to prove that relations of the form of (III.142) and the corresponding relations for tensor fields are the most general trilinear, local commutation relations that lead to a field theory with the usual requirements. In the following we consider free fields, since we expect that the equal-time commutation relations may be of the same form for both free and interacting fields.

First consider Hermitian scalar-fields.

$$\phi^{a}, \phi^{b}, \phi^{c}, \ldots \qquad \text{(III.144)}$$

with masses $m^{a}, m^{b}, m^{c}, \ldots$

obeying parastatistics with

$p_a, p_b, p_c, \ldots$ respectively

What is the most general expression to which we can equate the following trilinear commutator?

$$\left[\phi^{a}(x), \left[\phi^{b}(y), \phi^{c}(z)\right]_{+}\right]_{-} \qquad \text{(III.145)}$$

In a commutation relation with the expression (III.145) on the left-hand side, the right-hand side must be at most bilinear in the field operators if the relation is to be useful. By covariance it must be linear.

The most general form of the right-hand side may be written

$$F^{ab}(x,y)\,\varphi^{c}(z) + G^{bc}(y,z)\,\varphi^{a}(x) + H^{ca}(z,x)\,\varphi^{b}(y) \tag{III.146}$$

where F, G, and H are required to be local functions of their arguments.

We now require that the commutation relation

$$\left[\varphi^{a}(x),\,\{\varphi^{b}(y),\,\varphi^{c}(z)\}_{+}\right]_{-} \equiv F^{ab}(x,y)\,\varphi^{c}(z) + G^{bc}(y,z)\,\varphi^{a}(x) + H^{ca}(z,x)\,\varphi^{b}(y) \tag{III.147}$$

shall lead to a field theory that is Lorentz-invariant and obeys the CPT theorem, and that the creation and annihilation operators in terms of which the field operators may be expanded should obey the associative law. Lorentz invariance requires, on the one hand, that F, G and H be a function only of the Lorentz invariants $(x-y)^2$, $(y-z)^2$, $(z-x)^2$,

$$\text{i.e.}\quad F^{ab}(x,y) = F^{ab}\left((x-y)^2\right),\ \text{etc} \tag{III.148}$$

and on the other hand that, since the operators $(\Box_x^2 - m_a^2)$, $(\Box_y^2 - m_b^2)$, $(\Box_z^2 - m_c^2)$, acting on the left-hand side of the relation (III.147) gives zero because of the Klein-Gordon condition

$$(\Box^2 - m_i^2)\varphi_i = 0$$

the operator $(\Box^2 - m_a^2)$ etc, acting on the right-hand side of (III.147) must also give zero. This gives equations of the form

$$(\Box_x^2 - m_a^2)(\Box_y^2 - m_b^2) F^{ab}(x,y) = 0 \quad , \text{etc} \qquad \text{(III.149)}$$

the solutions of which are the following:

$$F^{ab}(x,y) = \delta_{m_a,m_b}\left[\rho^{ab}\Delta((x-y);m_a) + \sigma^{ab}\Delta^{(1)}(x-y;m_a)\right],$$

$$G^{bc}(y,z) = \delta_{m_b,m_c}\left[\rho^{bc}\Delta(y-z;m_b) + \sigma^{bc}\Delta^{(1)}(y-z;m_b)\right];$$

$$H^{ca}(z,x) = \delta_{m_c,m_a}\left[\rho^{ca}\Delta(z-x;m_c) + \sigma^{ca}\Delta^{(1)}(z-x;m_c)\right]$$

where ρ, σ are c-numbers. (III.149)

(note: Translational invariance requires that all these masses must be equal, i.e. only then is the Klein-Gordon condition satisfied for both x and y (corresponding to the form of the bracket in III.145)

F, G and H obey the symmetry relations

$$G^{bc}(y,z) = G^{cb}(z,y), \qquad \text{(III.150a)}$$

and $$F^{ab}(x,y) = H^{ba}(y,x), \qquad \text{(III.150b)}$$

and the 'jacobi identity'

$$F^{ab}(x,y) + G^{ab}(x,y) + H^{ab}(x,y) = 0 \qquad \text{(III.151)}$$

The requirement of hermiticity of the field operators implies that iF, iH and iG must be real

(III.152)

while the CPT theorem implies the following symmetry properties:

$$F^{ab}(x,y) = -F^{ab}(y,x),$$

$$G^{ab}(x,y) = -G^{ab}(y,x),$$

$$H^{ab}(x,y) = -H^{ab}(y,x). \qquad \text{(III.153)}$$

All these restrictions on F, G and H imply that (III.147) can be written as

$$\left[\phi^{a}(x),\ \left[\phi^{b}(y),\ \psi^{c}(z)\right]_{+}\right]_{-}$$

$$\equiv\ i\lambda^{ab}\Delta(x-y)\,\phi^{c}(z) + i\lambda^{ac}\Delta(x-y)\,\phi^{a}(x)$$

$$-\frac{i}{2}\left(\lambda^{bc}-\lambda^{cb}\right)\Delta(y-z)\,\phi^{a}(x)$$

(III.154)

where $$\lambda^{ab} = -\delta_{m^a m^b}\, f^{ab}. \qquad \text{(III.155)}$$

Note that λ^{ab} and λ^{ba} are given by

$$\langle a_k^a a_k^a a_k^{b+} a_k^{b+}\rangle_0 = \lambda^{ab}\langle a_k^a a_k^{b+}\rangle_0,$$

$$\langle a_k^b a_k^b a_k^{a+} a_k^{a+}\rangle_0 = \lambda^{ba}\langle a_k^b a_k^{a+}\rangle_0$$

(III.156)

where a_k^a, a_k^{b+}, ..., etc. are annihilation and creation operators respectively. From the above relations one can deduce that λ^{ab} is real and symmetric, provided $\langle a_k^a a_k^b \rangle \neq 0$ (If $\langle a_k^a a_k^b \rangle = 0$, then one can show that $\lambda^{ab} = 0$.) Therefore the last term on the R.H.S. of equation (III.154) drops out.

Make a similarity transformation on the field operators ϕ

$$\phi^{a'} = \sum_b C_{ab} \phi^b \tag{III.157}$$

so as to bring the λ's into diagonal form. In this new representation, equation (III.154) becomes

$$\begin{aligned} &\left[\phi^{a'}(x), \left[\phi^{b'}(y), \phi^{c'}(z)\right]_+\right]_- \\ &= 2i \left\{ \delta_{ab} \Delta(x-y) \phi^{c'}(z) \right. \\ &\qquad \left. + \delta_{ac} \Delta(x-y) \phi^{a'}(z) \right\} \end{aligned} \tag{III.158}$$

which, for $a' = b' = c'$ and $t_x = t_y = t_z$ leads to (III.255).

We have to impose one more restriction on the operators ϕ's that satisfy (III.158). Denote the creation or annihilation operators of ϕ^a, ϕ^b and ϕ^c by a, b and c respectively, and first take $c = b$. Then we have from (III.158)

$$\left[a, \left[a^+, b\right]_+\right]_- = 2b, \tag{III.159}$$

and

$$\left[b^{+}, [b, a^{+}]_{+}\right]_{-} = -2a^{+}, \qquad \text{(III.160)}$$

We may evaluate the vacuum expectation value

$$\langle a\, b\; a^{+} b^{+} \rangle_0 \qquad \text{(III.161)}$$

in two different ways,

(i) $\langle a b a^{+} b^{+} \rangle_0 = \langle (a b a^{+}) b^{+} \rangle_0$

$$= 2\,(s^{b}-1) - (s^{a}-1)\,(s^{b}-1) \qquad \text{(III.162)}$$

by using (III.159) and (III.69') for $(a b a^{+})$

<u>or</u>

(ii) $\langle a\, (b a^{+} b^{+}) \rangle_0$

$$= 2\,(s^{a}-1) - (s^{a}-1)\,(s^{b}-1) \qquad \text{(III.163)}$$

by using (III.160) and (III.69') for $(b a^{+} b^{+})$.

If we require the associative law for the algebra of these field operators, then (III.162) and (III.163) must be identical and so

$$s_a = s_b \,(= s_c). \qquad \text{(III.164)}$$

If $\varphi^a, \varphi^b, \varphi^c$ are all different we have no trouble, as the right-hand side of (III.158) is zero. However, if $\varphi^a, \varphi^b, \varphi^c$ are not all different s^a, s^b and s^c must be the same.

For fields of different ~~orders~~ s-values, $s_b \neq s_c$ we must assume $[\varphi^b, \varphi^c] = 0$, since this is the only possibility

Summarizing ~~i.e.~~ Trilinear commutation relations can be assumed only for fields with the same s-value; for fields with different s, we must assume bilinear commutation relations).

We have thus proved the result we stated earlier, for tensor fields.

For spinor fields, we proceed in the same way. Here we obtain the S-functions instead of the Δ-functions. Again the trilinear commutation relations will be consistent with the associative law only if $s_a = s_b = s_c$.

When we consider interactions containing both spinor fields ψ and tensor fields φ, we need commutation relations containing both kinds of operators. We can have non-vanishing commutators.

$$[\varphi^a(x), \{\psi^b(y), \psi^c(z)\}_+]_- = 2i\,\delta_{ab}\,\Delta_b(x-y)\,\psi^c(z),$$

$$\{\psi^a(x), \{\psi^b(y), \varphi^c(z)\}_+\}_+ = \frac{2}{i}\,\delta_{ab}\,S_b(x-y)\,\varphi^c(z)$$

(III.165)

only when all the fields occurring have the same s-value, regardless of whether they are quantized according to the rules for Case R or Case S.

In all the above cases we have obtained a common result, viz. that in every term of the interaction Hamiltonian, field operators with the same s-value always occur an even number of times no matter whether they are quantized according to Case A or according to Case B.

For quantization with commutation relations of the 'anomalous case', we have similar results. The generalization of the usual commutation relations are then given by the following

The generalization of $\{ \varphi^a , \varphi^b \}_+ = 0$ is

$$\left\{ \varphi^a(x), \left[\varphi^b(y), \varphi^b(z) \right]_- \right\}_+ = 2i \Delta_a(x-y) \varphi^b(z), \quad a \neq b;$$

The generalization of $[\psi^a, \psi^b]_- = 0$ is

$$\left\{ \psi^a(x), \left\{ \bar{\psi}^a(y), \psi^b(z) \right\}_+ \right\}_+ = \frac{2}{i} S_a(x-y) \psi^b(z)$$

and the generalization of $\{ \varphi, \psi \}_+ = 0$ is

$$\left\{ \varphi(x), \left[\varphi(y), \psi(z) \right]_- \right\}_+ = 2i \Delta(x-y) \psi(z),$$

$$\left[\psi(x), \left[\bar{\psi}(y), \varphi(z) \right]_- \right]_- = \frac{2}{i} S(x-y) \varphi(z).$$

We summarize below the left-hand sides of the trilinear commutation relations for the normal and anomalous cases:

The fields occurring in H_{int}	Normal Case	Anomalous case
Tensor fields	$[\varphi, \{\varphi, \varphi\}_+]_-$	$\{\varphi, [\varphi, \varphi']_-\}_+$ $\varphi \neq \varphi'$
Spinor Fields.	$[\psi, [\psi, \psi]_-]_-$	$\{\psi, \{\psi, \psi'\}_+\}_-$ $\psi \neq \psi'$
Mixed (Both tensor and spinor fields)	$[\varphi, \{\varphi, \psi\}_+]_-$ $\{\psi, \{\psi, \varphi\}_+\}_+$	$\{\varphi, [\varphi, \psi]_-\}_+$ $[\psi, [\psi, \varphi]_-]_-$

Commutation relations like these can be obtained by group-theoretical methods also.

Scharfstein has obtained them from the Action principle.(4,5)

We summarise all the above considerations on the restrictions on the interaction Hamiltonian following from the commutation properties of the fields (1) the following generalization of Nishijima's theorem:

Theorem:

Except for the ordinary fermions and bosons (quantized by the R-or S-method with $s = 2$) the numbers of particles obeying statistics with the same value of the parameter s are conserved (modulo 2, if hermitian fields are present)

This is a superselection rule in the sense of Wick, Wightman and Wigner, and so this conservation is rigorous for all interactions.

We shall next apply this theorem to determine the statistics of the known elementary particles.

IV. DO PARAFERMIONS AND PARABOSONS EXIST IN NATURE?

In the previous lectures we have developed a formalism for introducing a more general method of quantization into field theory. We showed that two classes of generalized commutation relations may be obtained, according as the field operators of a_k, a_k^{+} assumed to form a representation of the rotation group or of the symplectic group (in an infinite number of dimensions). The two cases were known as Case R and Case S respectively, and the corresponding statistics were termed the parafermi and parabose statistics. We showed how to derive the commutation relations explicitly for parastatistics of different orders and proved that in the limit of infinite order $p \to \infty$ the parafermi statistics tended to the normal Bose statistics and the parabose statistics to the normal Fermi statistics.

We saw how the fundamental concepts of quantum theory must be generalized in order to admit the possiblity of parastatistics, and we showed that in a relativistic field theory, there exists a generalization of the usual connection between spin and statistics, tensor fields must be quantized according to parabose statistics and spinor fields according to parafermi statistics.

Finally, we proved in the last lecture that the requirement that the Euler-Lagrange equations shall be consistent with the Heisenberg equations of motion leads to a general superselection rule which states that in every term of the interaction Hamiltonian, the field with the same s-value must occur an even number of times (except in the normal fermi and Bose case), or that the number of particles with the same type of statistics ($p > 2$) is conserved.

We shall now apply this last conservation law to determine the statistics of the known elementary particles.

The particles whose statistics are established directly are the following.

1) The electron is known to obey F.D. statistics (Case R, $p = 2$) from the shell structure of atoms,

2) The proton and neutron are known to obey F.D. statistics (Case R, $p = 2$) from the shell structure of nuclei, and

3) The photon is known to obey B.E. statistics (Case s, $p = 2$) from the observed validity of Planck's law for black-body radiation.

(A. parabose photon would lead to a different radiation law, as the partition function would be different.

The statistics of the following particles may be deduced indirectly from the interactions that they enter into

1) <u>The photon</u>: If the electromagnetic interaction of a fermion is linear in the electromagnetic field and bilinear in the fermion field, e.g. $(\bar{\psi} \gamma_\mu \psi) A_\mu$ then as the photon occurs here only once, this implies (from our conservation law) that the photon must obey normal B.E. statistics.

2) <u>The electron neutrino ν_e</u> (occurring in the β- decay of the neutron). Since the n, p and e obey normal F.D. statistics, the reaction

$$n \to p + e^- + \bar{\nu}_e$$

implies that the electron neutrino (ν_e) also obeys normal F.D. statistics.

3) <u>The π^0 meson:</u>

The decay mode $\pi^0 \to 2\gamma$ together with the fact that the photon is an ordinary boson tells us that the π^0 obeys ordinary B.E. statistics

4) <u>The $\pi^\pm$ mesons:</u>

The decay modes

$$\pi^- \to e^- + \bar{\nu}_e ,$$

$$\pi^+ \to e^+ + \nu_e$$

indicate that the $\pi^\pm$ are also ordinary bosons

5) <u>The K meson:</u>

The decay modes

$$K \to 2\pi ,$$

$$K \to 3\pi ,$$

imply that the K meson obeys ordinary B.E. statistics

6. Hyperons (Λ^0 , Σ) and the Cascade particle Ξ

The decay modes $Y \to N + \pi$ show that the Λ^0 and Σ are ordinary fermions, while the decay $\Xi \to Y + \pi$ shows that the Ξ is also an ordinary fermion.

7. The resonant states

The statistics of these can be seen to be normal by examining their decay modes.

The only particles whose statistics is not thus determined are the μ -meson and the muon neutrino ν_μ , as they always occur together. If the muon neutrino is the same as the electron neutrino, then the decay mode $\pi \to \mu + \nu$ implies that the μ is also a normal fermion. However, the muon neutrino is actually different from the electron neutrino and so we can only conclude that the μ and ν_μ have the same statistics.

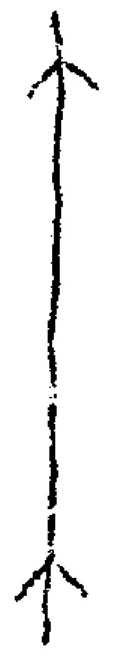
To establish the statistics of these (in the absence of systems containing several μ mesons, e.g. double μ -mesonic atoms), we must work out the quantitative consequences of the statistics on observable processes.

McCarthy and Volkov [3] have analysed the properties of Feynman diagrams for particles obeying parastatistics.

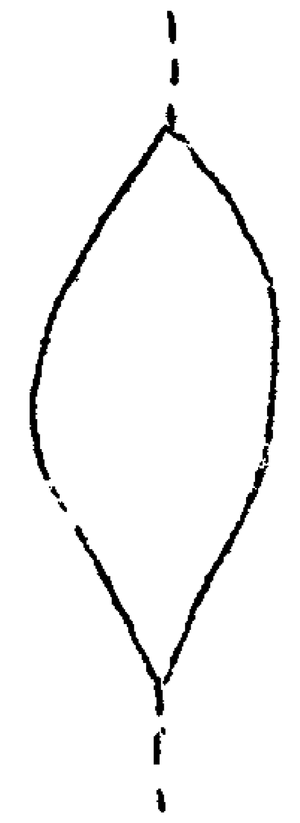
We [illegible]arize their results as follows:

To obtain quantities that are different for different statistics, we must consider processes that are essentially two-body in nature:

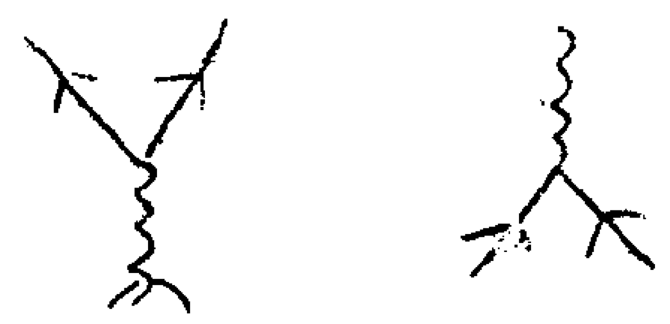
The Feynman diagrams consisting of only one open spinor line running from past to future does not depend on the statistics used, as it is essentially a one-body problem.



A closed spinor loop of a spinor field with the parameter δ) has a matrix element that is $(\delta-1)$ times that for a normal fermion. For a diagram with m spinor loops, there is an additional factor $(\delta-1)^m$



A spinor line with both ends incoming or outgoing has an extra factor $\sqrt{\delta-1}$



For 2-body scattering diagrams, the differences are more pronounced,† these have been studied by Volkov.

<u>The μ meson</u>: The most accurate available data on the μ meson are its g-factor and Michel parameter. However, these can depend on the statistics of the μ, only through the contribution of a closed μ loop, for instance as shown in the figure, which is, in any case negligible compared to the contribution

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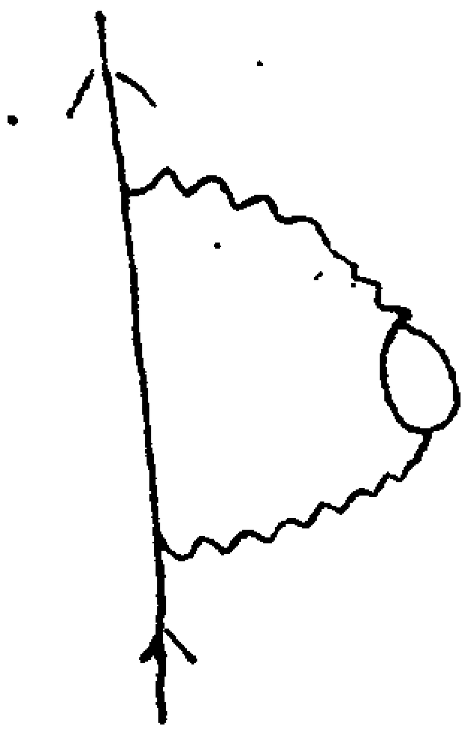
† D. C. [illegible]
There are some mistakes in the paper by Mc Carthy, concerning this statement.

of a closed electron loop (the ratio is $\sim \frac{m_e^2}{m_\mu^2}$).



These data alone would not therefore determine the statistics of the μ -meson.

To determine the statistics of the μ meson, one may measure $\mu - \mu$ scattering, or what is more practicable, μ -pair production[10].

The cross-section σ_p for the production of a μ pair by a γ -ray in a Coulomb field is given, for μ -statistics of the order $(p-1)$, by

$$\sigma_p = (p-1)\ \sigma_{(p=2)} \qquad \text{(IV.1)}$$

where $\sigma_{(p=2)}$ is the Bethe-Heitler cross-section.

Recently, this cross-section has been measured accurately by the Frascati group[11], they find that $(p-1) \approx 1$

This is the first definite piece of evidence for the statistics of the μ meson from which we may conclude that the μ and ν_μ are normal fermions.

We summarize the results of various experiments on μ pair production

1956	Panofsky[12]	$(\lambda - 1) = 1.93 \pm 0.68$)
		(seems to indicate that the μ is [illegible]
Later Experiment	Panofsky:[12]	$(\lambda - 1) = 1.42 \pm 0.34$
1962	Frascati group	$(\lambda - 1) = 1.00 \pm .52$

It thus appears from our analysis that the particles occurring in Nature are all normal fermions or normal bosons.

However, we must point out that our analysis is largely based on the theorem on the conservation of the number of particles with the same statistics, which was proved by assuming special commutation relations.

The formalism that we have developed is only the simplest method of generalizing the ordinary field theory so as to accommodate parastatistics, but is is not the only method of doing this. It is quite possible that other generalizations of the rules of quantization in field theory may lead to different selection rules, which may allow some of the particles occurring in Nature to obey parastatisics. We shall conclude by briefly mentioning the recent work of Greenberg and Messiah[(4)] and of Feshbach+.

Greenberg and Messiah have made a detailed phenomenological analysis. They ask what kind of symmetry properties are possessed by final state wave functions for various production processes if the particles concerned obey parastatistics, and show when one should look for direct experimental evidence for statistics of the known particles. For example, if the π-meson were a paraboson, the Dalitz analysis of $\omega - 3\pi$ decay would show a new feature which however is not big enough to distinguish the statistics.

However, in processes like

$$\pi^+ p \rightarrow n + \pi^+ + \pi^+$$
$$\rightarrow p + \pi^+ + \pi^0$$

the existence or absence of S-wave dipion states with $T = 1$ will give a direct test.

Various processes are analysed by Greenberg and Messiah, we refer to their paper for details.

Feshbach, in his paper proceeds in a different direction. He starts with a weaker assumptions than the ones we used. The most important assumption made by him is that

1) $$[H_{int}(x_1), H_{int}(x_2)] = 0 \quad \text{if} \quad (x_1 - x_2) \text{ is space-like} \qquad \text{(IV.2)}$$

Remember that our arguments were based on equations like $\dot{A} = i[H, A]$ where A is a single operator. Here the condition is given in terms of the products of several operators and hence the restriction is weaker.

For instance, for a $pn\pi$ -system, the commutation relations allowed in our theory are

(i)
$$[p, p']_+ = 0, \ [n, n']_+ = 0, \ [\pi, \pi']_- = 0,$$
$$[p, n]_+ = 0, \ [p, \pi]_- = 0, \ [n, \pi]_- = 0, \qquad \text{(IV.3)}$$

or

(ii)
$$[p, p']_+ = 0, \ [n, n']_+ = 0, \ [\pi, \pi']_- = 0$$
$$[p, n]_- = 0, \ [p, \pi]_+ = 0, \ [n, \pi]_+ = 0 \qquad \text{(IV.4)}$$

However, in Feshbech's theory there is a further possibility:

(iii)
$$[p, n]_- = 0, \ [p, \pi]_- = 0$$
$$[n, \pi]_- = 0, \ [\pi, \pi']_- = 0$$
$$[p, p']_+ = 0, \ [n, n']_+ = 0 \qquad \text{(IV.5)}$$

The other assumptions made by Feshbach are:

(2) Interactions are of local Yukawa Type (involving only three fields).

(3) Λ^0, Ξ are parafermions with $\lambda = 3$.

(4) The interactions $N\bar{\Lambda}^0 K$, $\bar{\Xi}\Lambda^0 K$ exist;

(5) Different spinor fields anticommute, while different tensor fields commute.

With these assumptions Feshbach proves that the allowed interactions are those that conserve strangeness. His result is similar to that given by the conservation law that particles obeying parastatistics must occur in pairs.

Possible criticism of his theory is that the assumption(1) does not guarantee that the theory is complete, sometimes the theory may become non-local.

Also a difficulty arises with regard to weak interactions where strangeness is not conserved. The structure of the Hamiltonian can then be different and we may have, for example

$$\left[H'_{weak}(x_1), H_{weak}(x_2) \right] \neq 0 \quad \text{even for spacelike } (x_1 - x_2)$$

which would lead to an S-matrix that is not Lorentz-covariant.

It is, of course, possible that weak interactions are really non-local.

We shall conclude this series of lectures with two remarks.

(i) It is desirable to prove our selection rule in some wider framework such as those considered by Jauch and Misra+(13).

(ii) In view of the possibility that other methods of generalizing field theory may exist that lead to different selection rules, it is important to have a direct experimental confirmation of the statistics of all known elementary particles.

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