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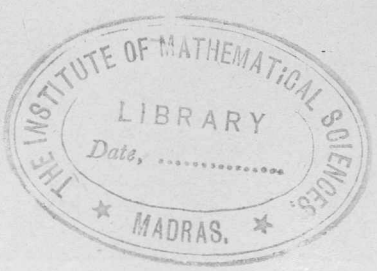
LECTURES ON

THE NON-LINEAR SPINOR THEORY OF ELEMENTARY PARTICLES

H. PETER DÜRR



THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-4, INDIA.

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THE NON-LINEAR SPINOR THEORY OF ELEMENTARY PARTICLES

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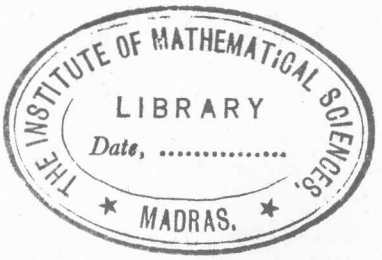

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NON-LINEAR SPINOR THEORY OF ELEMENTARY PARTICLES

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INTRODUCTION:

This series of lectures will be devoted to the non-linear spinor theory of elementary particles which was proposed and investigated in its earliest form by Heisenberg about ten years ago. This theory has had a dramatic history. Heisenberg was led to develop this theory involving local fields because of the following difficulty in S-matrix approach to elementary particles. Since the study of S-matrix involves to a large extent arguments of analyticity, treatment of problems like multiple production will necessarily require the analyticity properties of functions of an enormous number of complex variables which would be a hopelsss task and perhaps hard for physicists! Since the Pauli-Gursey transformation is known to be isomorphic to isotopic spin transformations in a three dimensional space, Heisenberg initially felt encouraged about the possibility of interpreting isotopic spin in a space-time formulation. However this did not come through as isotopic spin is in many respects a foreign element to the usual concepts in physics.

The theory essentially in its present form was investigated by Heisenberg, Mitter, Schlieder, Yamazaki and Dürr in a long paper in 1958, and was based on a proposal by Heisenberg and Pauli which was only circulated as a preprint early in 1958. Though there exist valuable contributions on various aspects of non-linear theory, we shall be mainly dealing with the work of the Munich group.

This course of lectures consists of five parts. In part I, a general review of the theory as a whole will be presented. We shall stress in this part the more fundamental ideas and the experimental facts which have led to the formulation of this particular theory and also state some of the essentially new features of the theory, however, without going into any details. This part may be considered as self-contained as this will give an idea of the general aspects of the theory without entering too much into the difficult and intricate investigations which are necessary and unfortunately unavoidable to establish the theory on safe grounds. These general considerations will also serve us to clearly spell out the goal of the theory which may be hard to trace in the subsequent detailed investigations.

In Part II we shall discuss the concrete formulation of the theory which is concerned with the selection of a fundamental field equation on the basis of group theoretical considerations.

In Part III we have to concern ourselves with approximation methods which are applicable to theories of the non-linear type to calculate physically interesting quantities, in particular masses and coupling constants of elementary particles. The discussion of the approximation methods comes at a fairly early stage of the theory since many properties of this theory, as in any other theory of elementary particles, can be stated rigorously only at every step of a certain approximation procedure. On the basis of such considerations for example, the necessity for the introduction of an indefinite metric in Hilbert space is clearly seen.

In part IV we discuss the difficult question of the compatibility of the indefinite metric in Hilbert space with the quantum mechanical probability interpretation. Special models like the Lee model will be investigated from this point of view.

In part V we utilize the material of the last parts to investigate the contraction function of the spinor field, and to calculate approximately the masses of the simplest non-strange fermions and bosons and their effective interactions.

I. GENERAL OUTLINE OF THE THEORY:

In this part we present a general review of the non-linear spinor theory without spending time on the detailed investigations which will be given in the subsequent parts.

In section A we collect all the relevant experimental facts which will be used for the formulation of the basis postulates which are stated in section B. The choice of these postulates are governed by the requirement that the theory based on them will be simple. In sections C and D we discuss in general terms two new and important aspects of the theory, the indefinite metric in Hilbert space and the unsymmetrical vacuum (also known as degenerate vacuum). In section E we finally make a brief comparison of our theory with other theories of elementary particles.

A. Basic Experimental facts:

Experiments at high energies which were only feasible at first with cosmic rays, but now are with much greater efficiency and accuracy, using high energy accelerators in the last 3 decades have led to a very fundamental change in our conception of the structure of matter.

First of all these experiments have led to the discovery of a tremendous number of new elementary particles. Their number, at present, is comparable with the number of different elements before the discovery of the periodic table by Mendeljeff. Like the earliest known particles, the proton and the electron, these new particles too, should only be considered as particles in the restricted sense of a quantum theory as demonstrated by the interference properties of particle beams. Although the "wave

character'' of these particles cannot be directly demonstrated experimentally for all elementary particles, (e.g. neutrino) this assumption is, nevertheless, a necessary condition for the self-consistency of a quantum theoretical description because of mutual interaction between various particles.

Commutability:- More important, than the increase in the number of elementary particles was the demonstration that all these elementary particles can be created out of energy and as a consequence of this, can be transformed into each other - at least if certain additional conditions for these transitions are fulfilled. Hence the classical concept of an elementary particle as the ultimate, indestructible smallest unit of matter is no longer valid. An elementary particle can always be transformed into other elementary particles at high energies. The additional mass necessary for the creation of the secondary particles is provided by the original energy of the system. Hence matter in principle can only be subdivided down to the level of elementary particles. Thus energy can be considdred to be more fundamental quantity than an elementary particle which is nothing but the coagulation of certain amount of energy as exhibited by its mass.

Causality:- The investigation of the processes of elementary particles, in particular with the help of coincidence measurements --have led to the conviction that all these reactions, like the reactions in the realm of classical physics, proceed in a casual fashion.

Causality in this connection means several things: First of all, it implies that there exists some connection between the

present events and the events at some arbitrary later time which can be expressed in terms of a unique or nonunique rule. (For a distinction of these two possibilities we may use the term deterministic or nondeterministic.) We say: the future events are 'caused' by some events at present. Causality in this general context is the basis of all scientific description. However, we shall use causality in a somewhat more precise sense: The past shall be connected with the future only through the present or all future events are fully caused by events at present and we shall call causality in this sense ''primitive causality''. (In classical mechanics primitive causality is assumed to be valid even for infinite time intervals as expressed by the existence of differential equations of motion with respect to time.)

As it is well known in relativistic dynamics the concept of simultaneity of two events which are separated spatially needs careful consideration. In order to establish the causality principle for all possible observers, causality must ~~ever~~ have a more precise meaning. All actions can only be propagated atmost with the velocity of light. We shall call this 'relativistic causality'. It implies that all effects have to lie within the future light cone of their cause. Spacelike events, i.e. events which cannot be connected by a light signal, cannot influence one another. (In relativistic theories this more precise form of causality leads to a prescription in terms of differential equations of motions for locally coupled fields.) Experimental evidence seems to be consistent with the assumption that all reactions of elementary particles proceed in accordance with the relativistic causality principle as defined above atleast for spacetime

regions which are accessible to actual measurement. We wish to point out that small deviations from relativistic causality in spacetime regions smaller than 10^{-13} certainly cannot be excluded experimentally.

As mentioned before, the actually observed reactions and transition process of elementary particles obey additional conditions, which can be expressed in terms of the conservation of certain quantities. In particular, in all reactions of isolated systems the total energy, the total linear and angular momentum and the centre of mass are conserved. According to the investigations by E.Noether these ten classical conservation laws are directly related to the relativistic space-time symmetries of the classical equations of motion. The classical conservation laws, however, are not enough to describe all the observed reactions, e.g. we find for all processes involving elementary particles the electrical and baryonic charges are also conserved. We also need in addition, certain properties of the elementary particles, like isospin, strangeness and parity, which are not conserved in all processes but nearly in all processes. Further all reactions in which the isotopic spin is not conserved, in general, occur about 10^{12} times less frequently and processes in which strangeness and parity conservation also is violated about 10^{14} less frequent than the reactions in which these quantities are conserved.

Phenomenologically the various transition rates of these processes are interpreted in terms of the existence of several distinct modes of interactions of elementary particle of various

strength: The 'strong' interaction is considered as that part of the interaction which conserve isotopic spin, strangeness and parity, the electromagnetic interactions are the interactions which violate isospin conservation and the weak interactions those which in addition do not conserve parity and strangeness. Finally the gravitational interaction is so weak that up to now it is considered without importance in the physics of elementary particles. Eventually, however, it should be included.

Let us now briefly remark on the mechanical properties of elementary particles, i.e. mass and spin. All elementary particles have either integral or half-integral spin (fermions) Halfintegral spin particles or spinor particles obey Fermi-statistics Integral spin particles or tensor-particles obey Bose-statistics (bosons). The masses of the particles are rather different. However, it is interesting to note that with the remarkable exception of the μ - meson (which is perhaps an exception) the mass of the particles appear to reflect their interaction: Strong interactions are accompanied by large masses, electromagnetic interactions by masses which are about 10^3 times smaller and similar ''fine structure splittings'' of the larger masses.

The experimental situation of elementary particle physics which is in particular, characterized by the Quantum properties of elementary particles and their commutability, the causal characters of their interactions and the existence of conservation laws in their reactions, and the specific properties of the particle with respect to mass and spin shall now be

confronted with a number of fundamental assumptions which form the basis of the non-linear spinor theory of elementary particles. These basic assumptions can hardly be considered a necessary consequence of the experimental facts but may be conceived as their most natural interpretation.

B. Fundamental Postulates of the theory:

The existence and the commutability of the elementary particles can easily be understood if all elementary particles were treated on the same footing. In our theory we do not use a local Lagrangian and canonical commutation relations between local fields as these would imply the existence of elementary particles according to Nishijima. We are certain that in our theory we are not dealing with elementary particles but rather with elementary fields. This point requires further investigation.

The masses of the elementary particles and their effective interaction have to follow once the fundamental dynamical law of the theory has been formulated. From our point of view no elementary particle can be really considered elementary as we believe that the elementary particles are certain bound state configurations of matter. This statement may appear today common-place but it was a novel idea more than a few years ago when it was first stressed by Heisenberg. Perhaps we may compare the situation to some extent with the situation in atomic physics. In case of the H-atom, the energy levels follow most simply from the Schrodinger equation though with perhaps great ingenuity, we can postulate some of the states, the first few low-lying states (say) to be elementary and attempt to derive the others in terms of these. This example suggests that we can expect that only the inclusion

of all elementary particles will render the theory simple and at the same time permit vigorous mathematical description.

In order to deal with the quantum properties of matter in connection with the relativistic causality principle (which we will discuss in some more detail later) our theory of matter is developed within the frame-work of a quantum field theory. In such a theory certain space-time dependent field operators $\psi(x)$ have to be introduced which are linear operators in a Hilbert space. The quantization is expressed by the algebraic properties of these operators (commutation or anticommutation rules). Though these field operators need not necessarily be observables, all elementary particles and their observable properties can be described in terms of them.

Experimentally we find that all elementary particles have either integral or half-integral spin. From a group theoretical point of view, this means that the fields associated with these particles have to transform according to the integral or half-integral representations of the Lorentz group. Since all representations, in principle, can be generated as direct products of the representation with spin 1/2, it is enough if we consider fields which transform according to the spin 1/2 representation of the Lorentz group (or rotation group). In other words for the description of states with arbitrary spin (integer or half-integer) the introduction of an ordinary spinor field is sufficient as it is possible to generate all the higher spin states ~~can be~~ by 'binding an appropriate number of these spin 1/2 fields' together according to the vector addition rule of angular momenta.

From this point of view it appears reasonable to consider the fundamental matter field $\psi(x)$ to be a spinor field.

We know from experiments spinor particles obey Fermi-statistics. This implies that the spinor fields have to be quantized with anticommutation rules. Hence our fundamental field operator $\psi(x)$ will obey an anticommutation rule. As in the usual quantum field theories, the empirically required connection between spin and statistics may also be derivable in this theory from other postulates like the positive definiteness of the energy and the CPT invariance. However we do not want to discuss this point here.

In this the theory formulation of the relativistic causality principle raises some basic question. In the framework of a quantized field theory, it is not clear to what extent we may weaken the causality principle for the operators, which are unobservable in our case, without losing the causal behaviour of the observables, i.e. quantum mechanical expectation values, the latter being true for experimentally accessible space-time domains. Such a weakening of causality should not be confused with the well-known indeterminism of quantized theories, but would rather find its expression in a nonlocal character of the theory. Many attempts were made in order to get rid of the divergent difficulty through the introduction of weaker concepts of relativistic causality which besides being in agreement with the causal character of observable processes would also admit deviations of local causality in the inaccessible small space-time regions. However many of these investigations seem to demonstrate that probably, the only

unambiguous and satisfactory way to formulate relativistic causality is to formulate it as a local property, at least in a formal way, where formula is used to distinguish it from actual. It is possible that in a formally local theory the expectation values need not necessarily possess the local property and hence will not strictly obey the local relativistic causality principle. Normally in the definition of quantum mechanical expectation values, the usual probability interpretation of quantum mechanics is presupposed which in turn is linked to certain properties of the Hilbert-space. It may turn out that the parameters x_μ which are interpreted as the position parameters and with respect to which relativistic causality in the local sense is formulated, are not identical with the measurable spacetime concepts but only coincides within the experimentally accessible, more macroscopic space-time domains.

The formally local formulation of relativistic causality in terms of the field operator $\psi(x)$ leads to two important conditions:-

1) The field operators $\psi(x)$ have to fulfil, according to Haag, an equation of motion which is quasidifferential in time, i.e. an integro differential equation which possibly still may depend on a finite but arbitrarily small timeinterval Δt . This differential equation of motion may only be formally meaningful. Unlike in the case of classical field theory, the equations for the field operators in quantum field theory, still leave the dynamics of our theory arbitrary if the Hilbertspace and the products of field operators which occur in the equation are not more precisely defined.

2) The anticommutator of the field operators must vanish for the space like distances, i.e.

$$\{\psi(x), \psi^*(x')\} = 0 \quad \text{for } (x-x')^2 < 0 \text{ (spacelike)} \quad (A)$$

Here $\psi^*(x)$ is defined as the hermitian conjugate operator to $\psi(x)$ where $\psi(x)$ and $\psi^*(x')$ are non-hermitian operators. The condition (A) is commonly known as the locality or mocrocausality condition and is intimately connected with a corresponding condition for the commutator of the observables, which can be constructed from $\psi(x)$ and $\psi^*(x)$.

In order to express interaction in the theory, we have to postulate that the differential equation for the spinor field is nonlinear, since linear differential equations under normal conditions only describe free fields because of the validity of the superposition principle.

Because of the empirical relation between the strength of the interaction and the masses of the elementary particles, it is rather suggestive to interpret the mass of the particles as purely due to their selfinteraction. Thus our differential equation will not contain any bare mass term.

As in classical theory, in quantized theories there is a close relationship between the symmetry properties of the equation of motion and the conservation laws. For every continuous symmetry transformation (with a countable number of parameters) which leaves the action function numerically invariant or the Lagrangian density form-invariant (upto a 4-divergence) there exists a conservation law which holds locally (existence of

continuity equation) in case of a local Lagrangian (Noether's theorem). In quantized theory there exists an alternative definition which actually supersedes the 'classical' one (which uses the correspondence principle) if a Lagrangian formalism is not used. There is a symmetry operation defined as a unitary transformation in Hilbert space which does not change the dynamics or the development in time. In other words the unitary operator corresponding to a symmetry operation commutes with the unitary transformation corresponding to a time-displacement and does not affect the commutation relations. As a consequence the hermitian generators of the symmetry transformation commute with the total Hamiltonian, the generator of the time displacement, at least if the generator of the symmetry transformation does not explicitely depend on the time. This argument is also valid for multiplicative quantum numbers. This approach yields less information than the Lagrangian approach since it only expresses the overall and not the local conservation of the quantities. To obtain local conservation in this case without making use of the local properties of a Lagrangian, we may consider the parameters in the transformation explicitly depend on space-time (transformation of the second kind), e.g. gauge transformation of the second kind). We will come back to this at a later point.

In our non-Lagrangian approach the symmetry of the dynamical law is expressed in terms of the symmetries of the differential equation for the field operators which expresses their properties under time displacement and their commutation

relations rather than, as usually by the symmetries of a formal Lagrangian from which the differential equation as well as the commutation relations, (e.g. in the Schwinger way,) may be derived. This subtle distinction actually is irrelevant for most symmetries except in the case of the scale transformation.

The differential equation for the field operator is expected to be invariant under the relativistic transformation to insure the validity of the ten classical conservation laws. Relativistic invariance has already been implied in our concept of causality. In addition to the relativistic invariance the theory shall be invariant under the two different gauge transformations of the first kind in order to express the conservation of electrical and baryonic charge. In addition we require invariance under certain discrete transformations like CPT and CP, which seem necessary empirically. Invariance under isotopic transformations will also be assumed in our theory. Finally we shall assume that the field equation is maximally symmetrical in the sense that the symmetry of the theory cannot be increased without increasing the number of components and that the field operator belongs to a representation. Also $\psi(x)$ will be irreducible with respect to the total symmetry group of the theory. In this sense we may speak of a unified theory of matter or a description of elementary particles in terms of a single field.

The postulates stated are certainly not sufficient to define the theory completely. Certainly we have to require that there exists a state of lowest energy which we identify with the vacuum state. In addition, the symmetry properties of this vacuum

state have to be defined as a necessary boundary condition of the theory. We shall discuss this point at length in section D. Beyond this we hope that the requirement of self-consistency and meaningfulness of all relevant expressions will determine the structure of the Hilbertspace and hence the dynamics completely. It is to be emphasized again that the differential equation for the fields is still without any dynamical content, without the specification of the Hilbert space.

C. Regularization and Probability interpretation:-

As the most simple example of an elementary partide theory which essentially incorporates the postulates of the last section, we may choose a theory which is based on the following non-linear differential equation for a 2-component non-hermitian field

$$-i\sigma_\nu \frac{\partial}{\partial x_\nu} \psi(x) + \ell^2 : \sigma_\mu \psi(x) \left(\psi^* \sigma^\mu \psi\right) : = 0$$

with the (2 x 2) pauli matrices $\sigma_\nu = (I, \vec{\sigma})$ and ℓ^2 a coupling constant which is usually given the dimension of a (length)^2. The notation: : in the non-linear product shall indicate a definite ordering procedure (Wick product) to give meaning to the product of field operators at the same space time point. The equation is invariant under the relativistic transformation (the Lorentz group).

$$\psi(x) \to e^{i\vec{\alpha}\cdot\vec{\sigma} + \vec{\beta}\cdot\vec{\sigma}} \psi\left(L^{-1}(\alpha, \beta_\mu, a_\mu)\, x\right)$$

a gauge transformation.

$$\psi(x) \to e^{i\alpha} \psi(x)$$

and the following discrete transformations:

CPT: $\psi(x) \rightarrow \psi^*(-x)$, $c \rightarrow c^*$

CP: $\psi(x) \rightarrow (-i\sigma_2)\ \psi^*(-\vec{r}, t)$

and also the scale transformation

$$\psi(x) \longrightarrow \eta^{1/2}\ \psi(\eta x)$$

The last symmetry under scale transformation excludes bare mass.

This theory is difficult to solve as it is unrenormalizable. This means that in the normal approach which uses perturbation expansion techniques, the physically interesting quantities are all divergent and hence meaningless, and cannot be made finite by means of new interpretation or subtraction procedures as it is possible in case of the renormalizable theories.

The renormalizability of a theory was often considered as a necessary criterion for every physical theory, and hence theories of the above type were not studied seriously (except for e.g. in connection with weak interactions). However, it appears very probable to us that the divergence difficulties are a feature of every theory with real interaction and reflect an inconsistency in the basic assumptions. Also the renormalizable theories contain these infinities in a hidden fashion which are indicated perhaps by the suspected infinite values of the renormalization constants However, in this case, one is fortunate enough that the behaviour at high energies and at high momentum transfer (small distances) influences only indirectly through the mass and the coupling strength etc., which are fixed empirically, the low energy behaviou

of the particles which is physically accessible. In the case of non-renormalizable theories, on the other hand the low energy behaviour is so intimately connected with the high energy behaviour due to the strong energy dependence of the matrix elements. This means that the difficulties of the theory at high-energy, high momentum transfer cannot be avoided by adjusting a finite number of parameters of experiment but have to be resolved, in principle, to establish a physically meaningful theory.

In our theory, however, it will be demonstrated that the divergence difficulties can be attributed to the additional postulate that all the quantized field theories have to be quantized canonically, i.e. that the anticommutator for equal times $\{\psi(x), \psi^*(x')\}_{t=t'}$ is proportionate to the 3-dimensional δ-function. It is remarkable, however, that in case of theories without interaction the canonical quantization rules appear as a consequence of the equation of motion and the locality condition rather than a new independent condition. In a free scalar field theory with the equation $(\Box + m^2)\varphi = 0$ the c-number character of the equation $[\varphi(x), \varphi(x')] = C(x-x')$ can be established. Then it immediately follows that $C(x-x')$ has to be proportional to the usual $\Delta(x-x')$ since it is the only solution of the classical equation $(\Box + m^2)C = 0$ which fulfils the boundary condition $C = 0$ for $z^2 < 0$ (space like) (locality condition). The singular function $\Delta(x-x') = \Delta(z)$ however, exhibits a $\delta(z^2)$ singularity on the light cone which directly leads to the $\delta^3(\vec{r}-\vec{r}')$ on the right-hand side of the likelike commutator in turn and hence the canonical character of the commutation rule. Similarly this can be shown also for a

spihor field. Here $\delta(z^2)$ and $\delta'(z^2)$ singularities appear in the singular function. This suggests strongly that the commutation rule should not be considered an additional postulate of the theory but should rather be chosen in connection with the particular form of the equation of motion, i.e. in such a way as to lead to a consistent and meaningful theory. In fact, one can quite easily demonstrate that in a nonlinear equation due to appearance of products of field operators at the same space-time point the assumption of the $\delta(z^2)$ and $\delta'(z^2)$ singularities on the light cone in the anticommutator relation is the source of the troublesome divergence difficulties, since products of these distributions occur. Instead of postulating the canonical commutation condition we shall postulate the finiteness and the physical interpretability of the theory at every step of our approximation procedure which will definitely exclude the possibility of canonical quantization. We are aware of the possibility that the postulate of finiteness of a theory at every step of an approximation procedure may be too strong a condition on the theory, since cancellations of akward terms may occur if we sum up the series which characterize our subsequent approximations. Also the opposite may happen, i.e. that the series do not converge. However, at the moment, we shall not concern ourselves with these possibilities because of our inability to tackle these complicated questions but hope that the actual calculations will give us some hints in this respect. We may also stress that the non-appearance of δ and δ'-singularities on the light cone of the commutator may be considered a necessary requirement to prevent the introduction of an elementary particle into the theory as the existence

of a local equation for a local field with canonical commutation relations was overy often considered as a necessary condition for an elementary particle associated with this field. Looking at it from this angle we may say the unphysical postulate of the existence of certain elementary particles in the common Lagrangian theories which introduce locally coupled fundamental fields together with canonical quantization rules for these fields is the source of the divergence difficulties. The introduction of elementary particles may be avoided by giving up the local Lagrangian or, as we want to do, the canonical commutation rules.

Investigation about the character of the singularities on the light cone go as far back as 1954 when Heisenberg studied the invariant classical solution of a nonlinear spinor equation near the light cone which vanish for space like distances. These investigations led to the result that the solutions of this type exhibit an oscillatory character near the light cone $\sim \cos \ln(z^2)$ which are undefined on the light cone itself, and do not give any contribution after integration. Under some plausible assumption he suggested at that time, that the singular behaviour of these solutions may very well coincide with the most singular part of the anticommutator. However such a connection, though trivial in linear theories, cannot be proved in nonlinear theories. Only recently Mitter could show that the behaviour of the anticommutator, in general or at least near the light cone may be rigorously derivable from the postulate of scale invariance of the theory. It is well-known that there exists a very strong connection between the scale invariance of a theory and their renormalizabilit

We shall see that the postulate of 'Weak scale invarianced of the theory, which is somewhat weaker than the scale transformation which will be defined as the proper place at least in the asymptotic region of high momentum transfer will suffice to secure the finiteness and hence the physical interpretability of the theory. Physically this means, that at very high energy + high momentum transfer particles behave not only kinematically but dynamically like particles of zero mass. Scale invariance gives us good arguments to limit the possible choices of the differential equations. It is very tempting to replace the postulate of ~~that the property of the theory to be~~ meaningfulness by scale invariance as we can guarantee the finiteness of all expressions. Lehmann has shown under the assumption of a positive metric in Hilbertspace and other rather general conditions which include the completeness of the eigenstates of energy and momentum that the singularities on the lightcone for the vacuum ~~experimental~~ stable value of the anticommutator with interaction must be δ and δ'-function which even might be multiplied by infinite constants In fact, the factors of the δ and δ'-functions turn out to be proportional to $\int m^2 \rho(m^2)\, dm^2$ and $\int \rho(m^2)\, dm^2$ respectively, where $\rho(m^2)$ represents the positive measure of a fermion 'state'with mass $p^2 = -m^2$. These integrals, in fact, diverge usually. The δ-functions in $\rho(m^2)$ represent discrete states of the spectrum, the rest (nonnormalizable) the continuum states. Hence theory which suppresses the appearance of the δ and δ'-function in the anticommutator necessarily requires the conditions

$$\int \rho(m^2)\, dm^2 = 0$$

$$\int m^2 \rho(m^2)\, dm^2 = 0$$

which can only be realized if an indefinite metric in Hilbert space is admitted. In this case, the $\rho(m^2)$ cannot be a positive definite function. Because of the intimate connection of the positive definiteness of the metric in Hilbert space with probability interpretation, it is clear there will be a deviation from the quantum mechanical probability interpretation.

Such a deviation from the probability interpretation may be a too high price. We do not believe that a deviation from the normal probability interpretation has fatal consequences if the S-matrix is shown still to be unitary. It is conceivable that one can construct from the general states in a theory of this kind in and outgoing states in such a manner that they have only nonvanishing projections in a subspace of the Hilbert space with positive semi-definite metric which includes all physically interpretable states and perhaps irrelevant states of norm zero. For such states then the usual quantum ~~statistical~~ mechanical probability interpretation is possible and the corresponding S-matrix will be unitary. These states, however, in general, will not be complete. Physically one may perhaps interpret the situation loosely by the statement, that there exists a new uncertainty principle between the effective interaction of a certain state, and its probability interpretation, in the sense that to the same degree to which a state under consideration can be considered a superposition of free particle states also its probability interpretation in the usual sense will be possible. This uncertainty is connected

with the non-linearity of the differential equation, and hence must be related to the "elementary" length ℓ . The S-matrix by definition only connects free particle-states with free particle-states, and hence should exhibit the usual properties.

The compatibility of the asymptotic probability interpretation with the indefinite metric in Hilbert space was investigated by Heisenberg in the Lee model. Here it could be proved under certain conditions (dipole case) for the lowest sectors. Investigations by Maksimov and Ferxetti even indicated that the same holds for the "hopeless" Pauli-Kallen case. Discussing a number of simple models Sudarshan went as far as to conclude that the compatibility of probability interpretation and indefinite metric can always be achieved by a proper definition of the physically realizable states, i.e. only the states with positive definite norm. Our general feeling with respect to indefinite metric is very much in accordance with Sudarshan's who states that in any theory with local interaction and local operators the introduction of the indefinite metric is unavoidable. However we may dispense with the indefinite metric and work only if we are willing to give up the requirement that the equation of motion is manifestly Lorentz invariant and admit certain apparently non-local interactions. A demonstration of this point of view is given by quantum electrodynamics which can be formulated in manifestly covariant way with indefinite metric using the Gupta-Blenler method or in the Schwinger non-manifestly covariant way with possible definite metric with apparently non-local

unretarded coulomb interaction term. Both formulations are completely equivalent. In our case, however, the situation is probably somewhat more complicated. In conclusion of this section we may state that we believe that the indefinite metric which is a central part of the present theory can be incorporated in the theory without any harm but a general proof of this is still lacking. Doubtless the introduction of the indefinite metric implies the elimination of elementary particles associated with the fundamental field $\psi(x)$ but, in addition, will probably lead to deviations from the strict locality concept. Local properties will appear to be smeared out over hopefully small space-time volumes due to the break down of the local probability interpretation.

D. APPROXIMATE CONSERVATION LAWS:

It was already stated that there exist certain quantities, which are not strictly conserved in the processes of elementary particles, but are only conserved in strong or in electromagnetic interactions. Of course, it is always possible to interprete the approximate conservation laws in terms of approximate symmetry requirements for the fundamental laws of nature. This can be most easily done and in fact, represents the common approach if we introduce into the equation of motion various interaction terms with appropriate symmetries to represent the strong, the electromagnetic and weak interactions. Theories of this type with their limited symmetry requirements are necessarily quite arbitrary as they are incapable of giving a satisfactory reason for the existence of the various types of interactions. We expect that this fact would be fundamentally built in the theory from the very beginning in terms of a number of non-calculable dimensionless coupling constants. This contradicts our point of view that all dimensionless constants in nature should be calculable. An indication for a different interpretation of the approximate conservation laws may be given by the observation, that even in completely symmetrical theories conservation laws are only strictly valid for isolated systems.

In a field theory the question whether a system can be isolated or not is related to some extent to the assumptions on the vacuum state and the local character of the interaction. In general, it is assumed, that there exists a state of lowest energy, which is invariant under all the symmetry groups considered

and hence non-degenerate. Such a ground state has all properties zero; it represents the "empty" and hence is identified with the vacuum state. A particle created into such a vacuum is necessarily isolated; a scattering experiment of two particles necessarily will lead to strict conservation laws if the dynamical law is fully symmetrical. For theories without interactions, one can show that the ground state, indeed, is such a vacuum state. For theories with interaction, however, this is not clear from the outset.

For a general orientation on this point, we may, at first, consider theories which involve only a finite number of dynamical degrees of freedom. There the nature of the ground state, is a dynamical question. In the case of the relativistic H-atom (without consideration of the protons spin) the ground-state has to carry angular momentum, since it is a doublet state. In the case of a finite but arbitrarily big ferromagnet the ground-state is N-fold degenerate if N electron spins are aligned. An interesting example in case of the nonrelativistic quantum field theiry, which according to Nambu, shows great resumblance to a non-linear spinor theory is the B C S model of superconductivity which can be solved exactly as shown by Bogoliubov. Also in this case the physically interacting groundstate which is separated by a gap from the first excited states is a nonsymmetrical, a degenerate state. All these theories have in common that they involve only finite degrees of freedom, or that the results are obtained in the limiting process of letting a finite number of degrees of freedom become arbitrarily large. This is also the case in theories which employ a cutoff which eventually

is increased beyond all bounds. It turns out, however, that in the exact limit of infinite degrees of freedom, the choice of the groundstate becomes an additional boundary condition on the theory which is independent of its dynamical content. Groundstates with different symmetry characters and subsequently the states we build up upon them are orthogonal to each other in the sense of Van Hove-Haag, i.e. they define inequivalent representations of the commutation relations in Hilbert space.

The existence of an unsymmetrical vacuum state necessarily brings about a partial 'breakdown' of the symmetries in the one particle states. A corresponding violation in the conservation laws, however, will require, in addition, a somewhat nonlocal character of the interaction, which may be provided in our theory through the appearance of the indefinite metric or smeared out vacuum properties. Thus in the nonlinear spinor theory the breakdown of the symmetries in nature, in connection with the stepwise change in the form and strength of the interaction is not considered as a reflection of an asymmetrical dynamical law but is interpreted in terms of a rather complicated structure of the groundstate.

More specifically we shall now require for the theory that it shall be fully invariant under isospin and also probably under scale transformations. The groundstate however, shall have isospin and scale properties, in order to provide an understanding for the violation of isospin conservation and the ~~fact that~~ existence of the finite masses of the elementary particles which introduce a definite length scale. The isospin properties of the vacuum will necessarily be accompanied by hypercharge properties,

though we shall not discuss these in this lecture series.

In closing this section we may point out that the notion of "unsymmetrical vacuum" and "degenerate vacuum" have been used by us in a synonymous way. This, in fact, follows from a group theoretical consideration which reveals that states of a given energy must belong to a representation of the symmetry groups which leaves the Hamiltonian invariant. Only if the state is invariant under the symmetry group does it belong to one-dimension irreducible representation, i.e. that is non-degenerate. From the point of view of the Hilbert space and probability interpretation, however, the degeneracy of the vacuum states is without physical consequences because the degenerate states lie in different Hilbert spaces. In the case of the ferromagnet e.g. the degenerate ground state consists in the large number of states one gets by rotating the direction of the resulting spin of the aligned electrons. If the number of electrons becomes infinite, and hence also the spin vector becomes infinite, then all the different ground states become orthogonal to one another in the sense of Van Hove, since a transition from one state to the other would involve the rotation of an infinite number of electron spins which has zero probability. Hence in this case, the vacuum state effectively consists of a single, but unsymmetrical state. The direction of its spin, of course, can be chosen arbitrarily and is without consequences. Therefore, the notion of an "unsymmetric vacuum state" is more apt to describe the actual situation than the notion of the degenerate vacuum state.

E. COMPARISON WITH OTHER THEORIES:

A fair comparison of present theories of elementary particles is, perhaps, a very difficult and laborious task. We will only attempt a rough sketch of the more fundamental assumptions which form the basis of other elementary particle theories in order to indicate more closely our own position and the border lines which separate us from other theories. In view of the old and still existing severe mathematical difficulties of quantum field theories or theories of elementary particles, in general, even such a comparison may be valuable.

The existing theories may be cast into essentially three different groups in which each subsequent groups add additional restrictions in their formulation. The groups may be defined as follows:

I. The behaviour of elementary particles at large mutual distances as measured by their cross-sections in various collision processes, angular distributions, etc. shall be describable in terms of a unitary S-matrix which incorporates all the empirically required groups (Lorentz group, gauge group, etc.) and is analytic to such an extent, as is necessary for the expression of relativistic causality, as far as it can be verified experimentally (macroscopic causality).

II. In addition to I there is assumed the existence of local field operators $\psi(x)$ in an appropriate Hilbert space, which commute or anticommute.

III. In addition to I and II it is postulated that the metric in Hilbert space is positive definite and that the asymptotic

states (the free particle states) shall span the entire Hilbert space.

Generally it is assumed today that the requirements of group I can be simultaneously fulfilled mathematically. However, it seems that the frame it defines is still too wide to define a theory uniquely. This is mainly due to the fact that 'macroscopic' causality (e.g. Wanders) is too weak a requirement to have very restrictive consequences on the analytic structure of the S-matrix. Hence, in the current S-matrix theories much stronger assumptions for the analyticity properties of the S-matrix are introduced, e.g. the concept of 'maximum analyticity' in the theory of Chew and coworkers. These theories are already outside of Group I. Such stronger requirements, of course, immediately raise again the question of their mathematical consistency. In fact, many people have conjectured that such a mathematical consistency is not automatic but only holds for theories with very specific symmetry properties. Such proposals then attempt to derive certain symmetries, in particular, the "approximate" symmetries, by means of strong analyticity requirements (bootstrap calculations).

The conjectures of theories which belong to group III are most restrictive. They are closest to ordinary quantum mechanics and represent essentially the orthodox quantum field theories. They were formulated by Wightman and others in a very precise form and have led to many important specific and general results. In particular, quantum electrodynamics has led to predictions which are verified by experiment to an exceedingly great extent. General considerations have led to the derivation of

dispersion relations which are extensively used in S-matrix theories. Orthodox quantum field theories and their perturbative solutions are still used very frequently by all elementary particle theorists as training ground for new approaches. Nevertheless, orthodox quantum field theories, despite of tremendous efforts, have failed to produce a single mathematical example of a lorentz invariant theory with local interaction which fulfils all its postulates. Quantum electrodynamics certainly comes closest to this goal but does not perhaps seem to reach it (Baker and Johnson approach). This may be an indication that, in fact, the postulates of III are not mathematically consistent.

Theories of group II would constitute a theory which lies in between the theories which are probably too wide and the theories III which are likely to be too narrow. The non linear spinor theory discussed here belongs to this category. Certainly other postulates have to be added to make the theory unique. This holds for all theories I, II and III. It is hoped that the postulates of II are mathematically compatible. However, a mathematical proof of their consistency and a proof of the inconsistency of postulates III cannot be given.

Obviously there are other elementary particle theories which in their axiomatic structure lie in between I and III. We have already mentioned the S-matrix theory of Chew et al, which is certainly more limited than I, and utilizes many theoretical implications of III. In these cases, the interesting question arises whether such theories differ from a theory of group II, and if so in what respects. There is a good chance that they

may be equivalent and just express different ways of attacking the same problem. The assumption of a local field equation is really not so strange as most S-matrix-theorists believe; since it essentially only incorporates the symmetry and primitive causality requirements which must be necessarily contained in any meaningful S-matrix theory. The point of departure clearly involves the meaning of "essentially". In the most optimistic case we may believe that the differential equation is completely and uniquely dictated by the symmetry requirements. This possibility exists since the structure of the Hilbert space will be established on the basis of a choice of vacuum state and the requirement of self-consistency and meaningfulness of the theory. Under these most favorable circumstances, the field equation is merely an incorporation of causality and symmetry and is not related to any specific and arbitrary dynamical conjecture. This difficult question cannot be answered at present completely.

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II. CONCRETE FORMULATION OF THE THEORY.

This part consists of the following six sections:

A) Selection of symmetries

B) The Heisenberg-Pauli equation

C) Other representations

D) Vacuum expectation values of two and four field operators and their interrelation

E) Parity problem

F) Scale invariance and regularization.

A. Selection of symmetries:

For the concrete formulation of the theory we have to select the symmetries we want to incorporate in our theory. In general, the symmetries lead to certain conservation laws. In classical physics for continuous transformation this connection was established by Noether's theorem. This theorem can immediately be carried over in the same form to canonical quantum field theories because of the correspondence principle which relates the equal time commutator and the Poisson brackets. In noncanonical quantum field theories the construction of the "real" Lagrangian is not related to the classical Lagrangian by correspondence principle, In noncanonical theories, it is therefore more advantageous to express the symmetries without reference to the Lagrangian formalism. As in the usual canonical theories, the symmetries of the theory can equally well be expressed by the symmetries of the formal differential equation for the field operators and their commutation rules. These symmetries will also imply the time independence of the eigenvalues of the

generators of these symmetry transformations. The local character of the conservation laws, however, is less obvious in this approach, and, in fact, may not be true, in general.

In order to make our theory unique and compact, we shall also demand that the differential equation of motion for the ψ field operators in our theory, should not only possess some symmetries but also must be optimally symmetric, i.e., the symmetries cannot be increased without introduction of additional components of the fields.

As indicated in the first part, all observed deviations from strict symmetries will not be interpreted as a violation of the symmetry to an unsymmetrical character imbedded in the differential equation and commutation rules, but as arising from the unsymmetrical nature of a ''boundary condition'' which is reflected on the choice of the groundstate. We will assume the ground state to have definite isotopic spin and scale invariance and this will be used to explain the observed deviations from isospin and scale invariance. The strong interaction approximation would then be considered as an approximation in which the isospin properties of the vacuum can be neglected.

For the empirical description of elementary particles, the following quantum numbers are necessary.

1) Electrical charge number	Q (exactly conserved).
2) Baryonic number	B (exactly conserved).
3) Hypercharge number	Y (Conserved in strong interactions).
4) Isotopic spin I	$\vec{I}$ (conserved in strong interactions).
5) Lepton number	L (?)
6) Muon number	μ (?)

As is well known there is a relation between the first four quantum numbers given by the Gell-Mann-Nishijima rule.

For the differential equation (and the commutation relation) we require also the following symmetry properties,

1) Invariance under the inhomogeneous proper Lorentz group.
2) Isotopic rotation group.
3) Gauge group of the first kind.
4) Dilatation group (scale transformation).
5) Invariance under the CPT and CP reflection.

These symmetry requirements appear to be the bare minimum and, in fact, seem insufficient to account for the empirical facts. The invariance with respect to isotopic spin, and a gauge group can only take care, in principle, of isotopic spin, charge and baryonic number conservation. Hence hypercharge, lepton and muon number are still not provided for. In the discussion of the strange particles, however, we will see how hypercharge is automatically introduced with the spurion concept in the case of an unsymmetrical vacuum having isotopic spin properties. We do not at present have any clear notion regarding the origin of the lepton and muon number which both seem to be necessary according to the latest experiments. It may be pointed out that we can include further quantum numbers in our theory by increasing the number of independent components of the field operators. The main disadvantage of this procedure lies in the fact that with an increase in the number of components of the field operators the number of symmetries in the maximal case increase very fast, e.g. a doubling of components of the field operators makes possible not only the introduction of a new gauge group but also of at least a new 3-parameter rotation group. In particular inclusion of a lepton

number in the theory would automatically introduce a new symmetry group which may imply a complete symmetry between leptons and baryons. Since this symmetry is not observed in nature, we have to invoke a symmetry breaking device, i.e. an unsymmetrical vacuum. It is quite possible that the baryons and leptons have different scales and the breakdown of scale invariance may account for the observed differences between them. But we shall not pursue this matter any further.

B. The Heisenberg-Pauli equation:

The simplest non-linear differential equation for a 4-component non-hermitian spinor field consistent with the above discussion of minimum symmetry requirements is

$$-i\sigma_\nu \frac{\partial}{\partial x_\nu}\psi(x) + \ell^2 : \sigma_\mu \psi \left(\psi^* \sigma^\mu \psi\right): = 0$$

and the conjugate equation

$$i\partial_\nu \psi^* \sigma_\nu + \ell^2 : \sigma_\mu \left(\psi^* \sigma^\mu \psi\right): = 0 \qquad (2.1)$$

These equations were originally written down by Heisenberg and Pauli in a different form but the more transparent form given above was introduced by Dürr.

This equation resembles the one mentioned in the first part. However, the σ_ν here are supposed to be the 2 x 2 unit matrix and the 2 x 2 Pauli spin matrices (I, $\vec{\sigma}$) multiplied by a unit (2 x 2) matrix in isotopic spin space. Hence, they are 4 x 4 matrices; $\sigma^\nu = (-I, \vec{\sigma})$ in our metric. The equation (2.1) expresses the self-interaction of a 2-component spin (Weyl) - isospinor field. The coupling constant ℓ^2 is

usually given the dimension of a length. We shall adopt this but this corresponds to a specific definition of ψ . The notation : : denotes a Wick product which will be defined in the next part. The differential equation is invariant under the following substitutions (active transformation).

Relativistic transformations. L^{+}: $\psi(x) \to e^{i\frac{1}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{\beta}\cdot\vec{\sigma}} \psi\left(L^{-1}(\alpha,\beta,a_\mu)x\right)$ (2.2)

Isotopic spin rotation. $\vec{I}$: $\psi(\alpha) \to e^{i/2\,\vec{\alpha}\cdot\vec{I}}\,\psi(x)$ (2.3)

Baryon gauge transformations. B: $\psi(x) \to e^{i\alpha}\psi(x)$ (2.4)

Scale transformation. : $\psi(x) \to \eta^{1/2}\psi(\eta x)$ (2.5)

and the discrete transformations.

PC: $\psi(\vec{r},t) \to (-i\sigma_2)\,\psi^{*}(-\vec{r},t)$ (2.6)

PG: $\psi(\vec{r},t) \to (-i\tau_2)(-i\sigma_2)\,\psi^{*}(-\vec{r},t)$ (2.7)

TPC: $\psi(x) \to \psi^{*}(-x)\ , C \to C^{*}$ (2.8)

where C represents any C-number.

We assume the existence of certain pseudounitary operators in Hilbertspace by which these transformations can be expressed,

$$\psi(x) \longrightarrow T\,\psi(x)$$

$$U_T\,\psi(x)\,U_T^{-1} = T\,\psi(x) \tag{2.9}$$

The 'pseudo' shall indicate that in the case of indefinite metric we have to construct an operation involving the metric tensor

which will generalize the unitary operation of positive definite Hilbert spaces.

The pseudounitary operators can be related in the usual way to pseudohermitian operators. For the continuous transformations these are the infinitesimal generators of the transformations. In particular, we define pseudohermitian energy-momentum operators and a Baryon number operator B which fulfil the conditions

$$\left[\mathbb{P}^\mu, \psi(x)\right] = \frac{1}{i}\frac{\partial}{\partial x^\mu}\psi(x) \; ; \left[\mathbb{P}^\mu, \psi^*(x)\right] = \frac{1}{i}\frac{\partial}{\partial x^\mu}\psi(x) \tag{2.10}$$

$$\left[\mathbb{B}, \psi(x)\right] = +\psi(x) \; ; \left[\mathbb{B}, \psi^*(x)\right] = \psi^*(x) \tag{2.11}$$

The generators P^μ, B etc. in the case of non-canonical quantization cannot be expressed as integrals over 3-dimensional space for a fixed time i.e. the formal translation operator

$$\mathbb{P}^\mu_f = \int \psi^*(x)\,\frac{1}{i}\frac{\partial}{\partial x^\mu}\,\psi(x)\,d^3x \tag{2.12}$$

does not have the commutation properties of the physical $\mathbb{P}^\mu$ and hence cannot be identified with $\mathbb{P}^\mu$. Investigations in connection with the Lee model, however, indicate that there may be a close relationship between P^μ and P^μ_f of the form

$$\mathbb{P}^\mu = \lim_{\Delta t \to 0} C(\Delta t) \int_{-\frac{1}{2}\Delta t}^{\frac{1}{2}\Delta t} P^\mu_f \, dt \tag{2.13}$$

with $C(\Delta t)$ some appropriate function of the small time difference Δt which can be made arbitrarily small. If this supposition can be substantiated it would state that the observables can only be constructed from the field operators in a finite but arbitrarily small time slab Δt.

We may consider this differential equation (2.1) to arise from the variation of a formal action function $W(I_1, I_2)$, of the two invariants:

$$I_1 = \int d^4x_1 : \psi^*(-i)\sigma_\nu \psi :$$

$$I_2 = \int d^4x : (\psi^*\sigma_\mu\psi)(\psi^*\sigma^\mu\psi) \qquad (2.14)$$

where we define $\ell^2 = \partial W/\partial I_2 \,/\, \partial W/\partial I_1$. The Pauli-Heisenberg equation is uniquely determined by the symmetry requirements and the condition that it is maximally symmetric. First of all one can show by Fierz transformations that

$$:(\psi^*\sigma_\mu\psi)(\psi^*\sigma^\mu\psi): \equiv : (\psi^*\sigma_\mu\gamma_i\psi)(\psi^*\sigma^\mu\gamma^i\psi): \qquad (2.15)$$

where the Greek indices have values 0, 1, 2, 3 and the Latin indices only 1, 2, 3. Hence there exists only one possible 4-fermion interaction term without derivative coupling. There will be other interaction terms which involve higher products of 4 and

$$:(\psi^*\sigma_\mu\psi)(\psi^*\sigma^\mu\psi)(\psi^*\sigma_\nu\psi)(\psi^*\sigma^\nu\psi): \qquad (2.16)$$

However, such a term is not invariant under the scale transformati

If only such an interaction term exists, and not the 4-fermion term we may redefine the scale transformation by

$$\psi(x) \to \eta^{1/6} \psi(\eta x) \tag{2.17}$$

This scale transformation attributes to the spinor field the dimension of $(1/\ell)^6$ or a 'scalespin' of 1/6. It was shown, however, by Kastrup on rather general grounds that due to uniqueness requirements of the theory the scale spin of spinor fields must be half integer. This will also become apparent in our consideration of the commutation rules. This rules out all nonlinear non-derivative ψ terms except the 4-fermion interaction used in the Heisenberg-Pauli equation.

We may now consider nonlinear terms which involve derivativ couplings. Again such derivative coupling terms cannot exist with the 4-fermion terms because of the scale invariance but may replace it. Terms of this kind have to involve at least two derivatives, e.g.

$$(\psi^* \sigma_\mu \partial^\mu \psi)(\psi^* \sigma_\nu \partial^\nu \psi) \sim (\psi^* \sigma_\mu \partial_\nu \psi)(\psi^* \sigma^\mu \partial^\nu \psi) \tag{2.18}$$

The higher derivatives may be always eliminated by introducing new symbols for the derivative functions which will effectively increase the number of components. This may either incorporate higher symmetries into the equation, which we do not want, or leave the equation not maximally symmetric. We have already given arguments to exclude these possibilities. We may add some more reasons which are heuristic in nature to justify the choice of our equation.

1) Since the 'two-component nature' of the field equation does not allow a 'bare mass term', i.e. a term $m_0\psi$ all masses must arise from selfinteraction.

2) In the case of the weak interaction of fermions we are able to observe experimentally the form of the 4-fermion interaction, and find then basically of the V-A-type. Our nonlinear interaction term $(\psi^*\sigma_\nu\psi)(\psi^*\sigma^\nu\psi)$ is essentially identical with the V-A-interaction.

C. Other representations:

1. The Fierz symmetric representation.

We discuss initially some slight modification of equation (2.1) which suggests itself in connection with the Fierz identity (10). For the 4-component spinor-isospinor-Weyl fields ψ_a ..... ψ_d, the Fierz transformation yields

$$:(\psi_a^* O_1 \psi_b)(\psi_c^* O_2 \psi_d):$$

$$\to\to \frac{1}{4}\frac{1}{4}:(\psi^* O_1 \bar\sigma_\mu \bar\tau_\nu O_2 \psi_d)(\psi^*\sigma^\mu\tau^\nu\psi_b):_{(2.17)}$$

where the minus sign arises from the anticommutability of the operators. The barred matrices $\bar\sigma_\mu$ and similarly $\bar\tau_\nu$ are the common spinmatrices of charge conjugate representation (i.e. if $\sigma_\mu \equiv (\sigma_\mu)_{\alpha\beta}$ then $\bar\sigma_\mu \equiv (\sigma_\mu)^{\alpha\beta}$ and have components $\bar\sigma_\mu = (I, -\vec\sigma)$) where

$$\frac{1}{2}\frac{1}{4}\frac{1}{2}(\sigma_\mu\bar\sigma_\nu + \sigma_\nu\bar\sigma_\mu) = -g_{\mu\nu}$$

$$\sigma_\nu\bar\sigma_\mu\sigma^\nu = 2\sigma_\mu$$

$$\bar\sigma_\nu\sigma_\rho\bar\sigma_\tau\sigma^\nu = 4\,g_{\rho\tau}$$

$$\sigma_\nu\bar\sigma_\rho\sigma_\tau\bar\sigma_\kappa\sigma^\nu = 2\sigma_\kappa\bar\sigma_\tau\sigma_\rho \qquad (2.20)$$

We are interested in the case when all the ψ's are the same. We have now

$$:(\psi^*\sigma_\nu\psi)(\psi^*\sigma^\nu\psi): = -\frac{1}{2}(\psi^*\sigma_\nu\bar{\sigma}_\mu\tau_k\sigma^\nu\psi)(\psi^*\sigma^\mu\tau^k\psi)$$

$$= -\frac{1}{2}:(\psi^*\sigma_\mu\bar{\tau}_k\psi)(\psi^*\sigma^\mu\tau^k\psi)$$

$$= \frac{1}{2}[:(\psi^*\sigma_\mu\psi)(\psi^*\sigma^\mu\psi): + :(\psi^*\sigma_\mu\vec{\tau}\psi)(\psi^*\sigma_\mu\vec{\tau}\psi):]$$

$$\mp \frac{1}{2}(II+\vec{\tau}\vec{\tau})\sigma_\nu\sigma^\nu(\psi^*\psi)(\psi^*\psi) \qquad (2.21)$$

Similarly we establish:

$$:(\psi^*\sigma_\nu\vec{\tau}\psi)(\psi^*\sigma_\nu\vec{\tau}\psi): = \frac{1}{2}:(\psi^*\sigma_\mu\vec{\tau}\bar{\tau}_k\vec{\tau}\psi)(\psi^*\sigma^\mu\tau^k\psi):$$

$$\vec{\tau}\vec{\tau}\sigma_\nu\sigma^\nu:(\psi^*\psi)(\psi^*\psi): \equiv \frac{1}{2}[3(\psi^*\sigma_\mu\psi)(\psi^*\sigma^\mu\psi)$$

$$-(\psi^*\sigma_\mu\vec{\tau}\psi)(\psi^*\sigma_\mu\vec{\tau}\psi):]$$

$$\mp \frac{1}{2}(3II-\vec{\tau}\vec{\tau})\sigma_\nu\sigma^\nu(\psi^*\psi)(\psi^*\psi) \qquad (2.22)$$

We have also indicated above a shorthand notation which is useful. The linear combination of the two equivalent forms (2.21) and (2.22) which is invariant under the Fierz transformation, is of particular interest. We find such a linear combination in

$$(3II+\vec{\tau}\vec{\tau})\sigma_\nu\sigma^\nu:(\psi^*\psi)(\psi^*\psi):$$

Because of the identity of the $\sigma_\nu\sigma^\nu$ and the $\vec{\tau}\vec{\tau}\sigma_\nu\sigma^\nu$ interaction we can rewrite the Heisenberg-Pauli equation in the

Fierz ~~first~~ symmetric form:

$$-i\sigma_\nu \frac{\partial}{\partial x_\nu}\psi(x) + l^2 \frac{1}{4}(3II + \vec{\tau}\vec{\tau})\sigma_\nu \sigma^\nu : \psi(\psi^* \psi): = 0 \qquad (2.23)$$

and the hermitian conjugate equation.

This formulation has the algebraic advantage that in practical calculations the direct and the exchange interaction terms will always be identical, and hence formally only the direct term (multiplied by a factor 2)) has to be considered.

2. The Heisenberg-Pauli representation:

This is the representation in which the theory was first formulated by Heisenberg and Pauli and it is this representation which led to certain misunderstandings in the early days of the theory. This representation corresponds to a different labelling of the components of the field operator.

In our notation the field operators $\psi(x)$ carry two different indices α, β explicitly $\psi_{\alpha,\beta}(x)$ where both α and β assume only the values 1, 2. The first index is a spin-index and indicates the two possible spin orientations of a spinor field. In this 2-dimensional space we establish a representation of the Lorentz group (i.e. without space reflection). The second index refers to the isotopic spin in the usual manner. We may visualiz $\psi(x)$ as the field of a 2-component neutrino-electron doublet (where the mass of the electron is also considered to be zero and hence the electron has 2-components). The antiparticles are described by the PC-transformed field $\psi^{PC} = (-i\sigma_2)\psi^*(-\vec{r}, t)$ which have helicity opposite to that of particles. Instead of combining the fields which correspond to the same helicity (or

baryonic number) in a single field operator, i.e. neutrino and electron, and antineutrino and positron we may also combine fields of same I_z i.e. neutrino-positron and electron-antineutrino. Such a representation would essentially correspond to the Pauli-Gursey form. More precisely we define the following new 4-component field operator $\chi_\gamma(x)$ with $\gamma = 1, 2, 3, 4$.

$$\chi_1 = \psi_{11} \;,\; \chi_2 = \psi_{21} \;,\; \chi_3 = -\psi_{22}^* \;,\; \chi_4 = \psi_{12}^*$$

or

$$\chi(x) = \begin{pmatrix} \frac{1}{2}(1+\tau_3)\,\psi(x) \\ \frac{1}{2}(1+\tau_3)(+i\tau_2)(-i\sigma_2)\,\psi^*(x) \end{pmatrix} \tag{2.24}$$

where the $\psi(x)$ and $\psi^*(x)$ after the projection $\frac{1}{2}(1+\tau_3)$ are considered 2-component fields only. The couple of equations (2.1) can now be expressed for the field operator in the form:

$$\Gamma_\nu \frac{\partial \chi}{\partial x_\nu} + l^2 \Gamma_\mu \Gamma_5 :\chi(\bar{\chi}\Gamma^\mu\Gamma^5\chi):=0$$

$$\frac{\partial}{\partial x_\nu}\bar{\chi}\Gamma_\nu + l^2 :\bar{\chi}\Gamma_\mu\Gamma_5(\bar{\chi}\Gamma^\mu\Gamma^5\chi):=0 \tag{2.25}$$

These are the equations which were used by Heisenberg and Pauli in their original paper. The Γ's here have exactly the transformation properties of the Dirac γ's and, in fact, were so used in their formulation. We prefer to use the capital Γ's ($\Gamma \sim \vec{\tau} \times \vec{\sigma}$) in this connection, in order to prevent the confusion with the Dirac γ's ($\gamma \sim \vec{P} \times \vec{\sigma}$) which are connected with the general Lorentz transformations. With respect to earlier

publications (Dürr and DH) we have reversed the meaning of γ and Γ, and also of χ and ψ. The symmetry transformations for χ take the following form:

$$L_+: \chi(x) \to e^{i a_{\mu\nu} \Gamma^{\mu\nu}} \chi(L^{-1}x) \tag{2.26}$$

$$\vec{I}: \chi(x) \to a\,\chi(x) + b\,\Gamma_5 C^{-1} \bar{\chi}^T(x) \; ; \tag{2.27}$$

$$\chi^T = C^{-1} \bar{\chi}(x)$$

a, b are complex with $|a|^2 + |b|^2 = 1$. For infinitesimal isospin rotation with angles α_i, $a = 1 + \frac{i\alpha_3}{2}$; $b = 1/2\,(\alpha_2 + i\alpha_1)$. This is the Pauli-Gursey transformation.

$$B: \quad \chi(x) \to e^{i/2\,\alpha\,\Gamma_5} \chi(x) \tag{2.28}$$

$$PC: \quad \chi(x) \to \Gamma_4\, \chi(-\vec{r}, t) \tag{2.29}$$

$$CPT: \quad \chi(x) \to \Gamma_5\, \chi^*(-x) \,, C \to C^* \tag{2.30}$$

The isospin transformations in this form are the Pauli-Gursey transformations. Written with the Dirac γ these transformations seemed to indicate, according to Gursey, a possible connection between the space-time concepts connected with the Dirac γ's and baryonic number and isotopic spin: baryonic conservation connected with the chirality transformation of Touschek and isotopic spin with the Pauli-Gursey transformation, studied by Pauli in connection with the massless neutrino equation. Such a connection, however, does not exist. The non-commutability of the parity operation $\chi(x) \to \Gamma_4\, \chi(-\vec{r}, t)$ with the baryon-transformation B indicates that the Γ's are not identical with the Dirac γ's.

In fact, from the non-commutability of the Γ_{44} parity operation with B we see that this parity operation is PC and not P . However, in a theory without a pure parity operation the representation of the Lorentz group requires only 2-component spinors. If we consider four components we can incorporate isotopic spin in the usual manner. This is not clearly exhibited in the above representation. Also the practical computations are rather cumbersome in this form, though one can explicitly show that they lead to the same results, as the other formulation.

3. Gursey representation:

There exists another quite interesting representation which was used in some other context by Gursey and which allows a very compact expression of the non-linear equation.

For a 2-component field (i.e. without isitopic spin) the only covariants are $(\psi^* \sigma_\mu \psi)$, $(\psi^T (i\sigma_2) \psi)$ and $(\psi^* (-i\sigma_2) \psi^{*T})$. In particular the last two expressions are scalars. The occurrence of these expressions in our equation was ruled out on the ground that they are not invariant under gauge transformations (B). However this is not completely true, since the product of both, i.e. $:[\psi^* (-i\sigma_2) \psi^{*T}][\psi^T (i\sigma_2) \psi]$ is gauge invariant. Nevertheless a Fierz transformation which exchanges ψ^{*T} with ψ reveals that this term is also identical with $(\psi^* \sigma_\mu \psi)(\psi^* \sigma_\mu \psi)$. If we consider the 4-component spinor-isospinor field, we observe that $[\psi^* (-i\sigma_2) \psi^{*T}]$ and $[\psi^T (i\sigma_2) \psi]$ are not isospin invariant, but $[\psi^* (-i\sigma_2)(-i\tau_2)$ and $[\psi^T (i\sigma_2)(i\tau_2) \psi]$ are. But both expressions vanish identically due to the symmetry of the operator $\sigma_2 \tau_2$ and the antisymmetry of the ψ^* and ψ .

This is, however, another curious combination of the fields which makes use of the possibility that the contraction with spin and isospin has not to be "parallel". The following combination is an invariant

$$\begin{aligned} & Tr\left[\psi^*(-i\sigma_2)\psi^{*T}\tau_0\psi^T(i\sigma_2)\psi\tau_0\right] \\ &= \psi^*_{\alpha\alpha'}(-i\sigma_2)_{\alpha\beta}\psi^*_{\beta\beta'}\delta_{\beta'\gamma'}\psi_{\gamma\gamma'}(i\sigma_2)_{\gamma\delta}\psi_{\delta\delta'}\delta_{\delta'\alpha'} \\ &= Tr\left[\psi^*(-i\sigma_2)\psi^{*T}(i\tau_2)(-i\tau_2)\psi^T(-i\sigma_2)\psi\tau_0\right] \end{aligned} \tag{2.31}$$

One can easily check that this combination is not only Lorentz- and gauge-invariant as for the 2-component field but also isotopic spin invariant. To compare it with the usual expression we perform now a Fierz transformation with respect to the spin indices only, i.e. we exchange ψ^{*T} with ψ in the spin space. Then after this partial Fierz transformation the spin and isospin contractions will be parallel again:

$$\begin{aligned} & Tr\left[\psi^*(-i\sigma_2)\psi^{*T}\tau_0\,\psi^T(i\sigma_2)\psi\tau_0\right] \\ &= \tfrac{1}{2}\left[\psi^*(-i\sigma_2)\tau_0\bar{\sigma}_\mu(i\sigma_2)\psi\right]\left[\psi^T\sigma^\mu\gamma_0\psi^{*T}\right] \\ &= -\tfrac{1}{2}\left[\psi^*(-i\sigma_2)\bar{\sigma}_\mu^T(i\sigma_2)\psi\right]\left[\psi^*\sigma^\mu\psi\right] = -\tfrac{1}{2}(\psi^*\sigma_\mu\psi)(\psi^*\sigma^\mu\psi) \end{aligned} \tag{2.32}$$

This means that this "new invariant" is again just a different form of our interaction term.

In this new form all operators appear to have left and right sides. Instead of looking at $\psi_{\alpha\alpha'}$ as a four-component spinor, it is more apt in this form to consider $\psi_{\alpha\alpha'}$ to be of 2 x 2 form. Spin multiplications occur from left, isospin multiplication from right; for ψ^* it is the other way around. To express this more clearly, let us define according to Gursey:

$$\Phi_{\alpha\alpha'} = \psi_{\alpha,\alpha'} \; ; \; \Phi^*_{\alpha,\alpha'} = \psi^*_{\alpha,\alpha'}$$

$$(\bar{\Phi})_{\alpha',\alpha} = \left[\psi^T (i\sigma_2)(i\tau_2)\right]_{\alpha,\alpha'}$$

$$(\bar{\Phi}^*)_{\alpha\alpha'} = \left[(-i\tau_2)(-i\sigma_2)\,\psi^{*T}\right]_{\alpha,\alpha'} \tag{2.33}$$

The nonlinear term then can be written in the form:

$$(\psi^*\sigma_\mu\psi)(\psi^*\sigma^\mu\psi) \to -2\,\mathrm{Tr}\left[\Phi^*\bar{\Phi}^*\bar{\Phi}\Phi\right] \tag{2.34}$$

The $\bar{\Phi}$ operator has a rather simple meaning. This we can see if we construct Φ explicitly. If

$$\psi(x) = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix}$$

we have

$$\Phi(x) = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(\psi_{11}+\psi_{22}) + \frac{1}{2}(\psi_{11}-\psi_{22}) & \psi_{12} \\ \psi_{21} & \frac{1}{2}(\psi_{11}+\psi_{22}) - \frac{1}{2}(\psi_{11}-\psi_{22}) \end{pmatrix}$$

and

$$\bar{\Phi}(x) = \begin{pmatrix} \frac{1}{2}(\psi_{11}+\psi_{22}) - \frac{1}{2}(\psi_{11}-\psi_{22}) & -\psi_{12} \\ -\psi_{21} & \frac{1}{2}(\psi_{11}+\psi_{22}) + \frac{1}{2}(\psi_{11}-\psi_{22}) \end{pmatrix} \tag{2.35}$$

since $\psi^T(i\sigma_2)(i\tau_2) = (\psi_{22}, -\psi_{12}, -\psi_{21}, \psi_{11})$

From (2.35) we see that if we expand $\phi(x)$ as

$$\phi(x) = \varphi^{\mu} \sigma_{\mu}$$

then we have for

$$\bar{\phi} = \varphi^{\mu} \bar{\sigma}_{\mu}$$

If ϕ and $\bar{\phi}$ were normal commuting functions, they would be essentially reciprocal matrices.

We have for $\bar{\phi}\,\phi$

$$\bar{\phi}\,\phi = (\varphi^{\mu}\bar{\sigma}_{\mu})(\varphi^{\nu}\sigma_{\nu}) = \varphi^{\mu}\varphi^{\nu}\tfrac{1}{2}(\bar{\sigma}_{\mu}\sigma_{\nu} + \bar{\sigma}_{\nu}\sigma_{\mu}) \qquad (2.36)$$

If $\varphi_{\mu}\varphi_{\nu} = \varphi_{\nu}\varphi_{\mu}$ (2.36) becomes

$$-\varphi^{\mu}\varphi^{\nu} g_{\mu\nu} = -\varphi^{\mu}\varphi_{\nu} = \det \phi = \det \bar{\phi} = \phi^{T}\bar{\phi}^{T}$$

If we introduce a 2 x 2 differential matrix: $D = \sigma_{\mu}\dfrac{\partial}{\partial x_{\mu}}$ (where σ_{μ} really has the meaning of a spin) then our differential equation can be cast into the simple form

$$i D\phi + 2\ell^{2} :\bar{\phi}^{*}\phi\,\bar{\phi}: = 0 \qquad (2.37)$$

and a corresponding conjugate equation. All operators are now 2 x 2 matrices and multiplied in the normal fashion. However, the analogy between the c-number matrices D and the q-number matrices ϕ is not complete due to the anticommuting character of This form is most inviting for speculations about the isotopic spin. In practical calculations, it is not as easy to handle as the more familiar isospin representation, despite of its compact form.

4. Representation with Hermitian fields:

There exists another representation which leads to many formal simplifications of the theory and therefore is of advantage in many practical calculations. In this representation only Hermitian field operators are introduced. The advantage of using Hermitian field representations for spinor fields was pointed out earlier by Schwinger. Dürr found this representation specially suitable for discussion of strange particles in his recent work. The transition from non-hermitian to hermitian fields can be accomplished by a doubling of the components of the operators. The new space which we call λ- space will be connected with the particle-antiparticle space.

The non-hermitian field operators $\psi(x)$ and $\psi^*(x)$ can be written as linear combinations of two hermitian operators $\psi_1(x)$ and $\psi_2(x)$:

$$\psi(x) = \frac{1}{\sqrt{2}} \left(\psi_1(x) + i\psi_2(x)\right)$$
$$\psi^{*T}(x) = \frac{1}{\sqrt{2}} \left(\psi_1(x) - i\psi_2(x)\right) \qquad (2.38)$$

We introduce a 2-dimensional vector space with the matrices which shall have the same algebraic properties as the unit and Pauli spin matrices, I, λ_1, λ_2, λ_3. We choose for λ's the usual representation:

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (2.39)$$

$\psi_1(x)$ and $\psi_2(x)$ are projections with λ_3 = + 1 and -1, respectively. We can construct in this representation a general 8-component field operators which has the form:

$$\tilde{\psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{2.40}$$

In a representation where λ_2 is diagonal

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.41}$$

the 8-spinor has the form

$$\tilde{\psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_1(x) + i\psi_2(x) \\ \frac{1}{\sqrt{2}}(i\psi_1(x) + \psi_2(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi(x) \\ i\psi^{*T}(x) \end{pmatrix} \tag{2.42}$$

(2.42) is a non-hermitian representation.

In the λ_3 diagonal representation we have the condition

$$\tilde{\psi}^{*T} = \psi(x) \tag{2.43}$$

In the λ_2 diagonal representation we have the condition

$$\tilde{\psi}^{*T} = i\lambda_1 \psi(x) \tag{2.44}$$

In both cases $\psi^*(x)$ can be effectively eliminated, but the λ_3 diagonal representation is simpler. We can write how the field equations for $\psi(x)$ and $\psi^*(x)$ in a single field equation for $\tilde{\psi}(x)$

$$-i\tilde{\lambda}_2 \tilde{\sigma}_\nu \frac{\partial}{\partial x_\nu} \tilde{\psi}(x) + \ell^2 \tilde{\sigma}_\mu : \tilde{\psi}(x) \underbrace{[\tilde{\psi}^*(x)\sigma^\mu \psi(x)]}_{=0} : \tag{2.45}$$

where we introduce the new matrices

$$\begin{aligned} \tilde{\sigma}_\mu &= [i\lambda_2, i\lambda_2\sigma_1, -i\sigma_2, i\lambda_2\sigma_3] \\ \tilde{\tau}_i &= [i\lambda_2\tau_1, -i\tau_2, i\lambda_2\tau_3] \\ \tilde{\lambda}_i &= [\lambda_1\sigma_2\tau_2, i\lambda_2, \lambda_3\sigma_2\tau_2] \end{aligned} \tag{2.46}$$

In this definition all $\underset{\sim}{\sigma}_\mu$ again commute with $\underset{\sim}{\tau}_i$ and also with $\underset{\sim}{\lambda}_2$ (which is identical with $\underset{\sim}{\sigma}_0$). The spin matrices $\underset{\sim}{\sigma}_i$ and $\underset{\sim}{\tau}_i$ obey the commutation rules:

$$\underset{\sim}{\sigma}_1 \underset{\sim}{\sigma}_2 = \underset{\sim}{\sigma}_3 \qquad \text{and cyclic} \qquad (2.47)$$
$$\underset{\sim}{\tau}_1 \underset{\sim}{\tau}_2 = \underset{\sim}{\tau}_3 \qquad '' \quad ''$$
$$\underset{\sim}{\lambda}_1 \underset{\sim}{\lambda}_2 = \underset{\sim}{\lambda}_3 \qquad '' \quad ''$$

We also note that $\underset{\sim}{\lambda}_1$ and $\underset{\sim}{\lambda}_2$ cammute with $\underset{\sim}{\tau}_i$ but not $\underset{\sim}{\sigma}_i$. Furthermore in the $\underset{\sim}{\lambda}_3$ diagonal representation, $\underset{\sim}{\sigma}$, $\underset{\sim}{\tau}$, $\underset{\sim}{\lambda}$ matrices are all real and also antisymmetrical except $\underset{\sim}{\lambda}_1$ and $\underset{\sim}{\lambda}_3$. In the $\underset{\sim}{\lambda}_2$ diagonal representation the matrices are simply related to the ordinary representation. If we write out (2.46) explicitly

$$\underset{\sim}{\lambda}_2 = i\lambda_2 = i\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \; ; \; \underset{\sim}{\tau}_i = -i\begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i^T \end{pmatrix}$$
$$\underset{\sim}{\sigma}_\mu = -i\begin{pmatrix} \sigma_\mu & \\ & -\sigma_\mu^T \end{pmatrix} \qquad (2.48)$$

In this representation our above field equation can be written:

$$-i\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} \sigma_\mu & \\ & -\sigma_\mu^T \end{pmatrix}\frac{\partial}{\partial x_\nu}\begin{pmatrix} \psi \\ i\psi^{*T} \end{pmatrix} - \frac{l^2}{2}\begin{pmatrix} \sigma_\mu & \\ & -\sigma_\mu^T \end{pmatrix} i\begin{pmatrix} \psi \\ i\psi^{*T} \end{pmatrix}\Big[(\psi^* - i\psi^T)\begin{pmatrix} \sigma^\mu & 0 \\ 0 & -\sigma_\mu^T \end{pmatrix}\begin{pmatrix} \psi \\ i\psi^{*T} \end{pmatrix} \qquad (2.49)$$
$$= 0$$

which decomposes into the two equations:

$$i\sigma_\nu \frac{\partial}{\partial x_\nu}\psi - \frac{i}{2}l^2\sigma_\mu : \psi\left[(\psi^*\sigma^\mu\psi) - \psi^T\sigma^{\mu T}\psi^{*T}\right] : = 0$$
$$i\sigma_\nu^T \frac{\partial}{\partial x_\nu}\psi^{*T} + \frac{i}{2}l^2\sigma_\mu^T\psi^{*T}\left[\psi^*\sigma^\mu\psi - \psi^T\sigma^{\mu T}\psi^{*T}\right] : = 0 \qquad (2.50)$$

which are identical with the field equations in the original representation

$$-i\sigma_\nu \frac{\partial}{\partial x_\nu}\psi + \ell^2 \sigma_\mu : \psi[\psi^* \sigma^\mu \psi]: = 0$$

$$+i\sigma_\nu^T \frac{\partial}{\partial x_\nu}\psi^{*T} + \ell^2 \sigma_\mu^T : [\psi^* \sigma^\mu \psi]: = 0 \qquad (2.51)$$

The interesting representation is the λ_3 diagonal representation. Here the field equation can be written in the simple form:

$$\tilde{\lambda}_2 \tilde{\sigma}_\nu \left(\frac{1}{i}\frac{\partial}{\partial x_\nu}\right)\tilde{\psi} + \ell^2 \tilde{\sigma}_\mu : \psi(x)[\psi^T(x)\tilde{\sigma}^\mu \psi(x)]: = 0 \qquad (2.52)$$

which contains only the hermitian field operators $\psi(x)$ and the real matrices $\tilde{\sigma}, \tilde{\lambda}$. Hence the i due to the gauge group i.e. $e^{i\alpha}$ is absent in the above equation. The only i which still enters the equation is of quantum mechanical origin i.e. the i arises from the equation $[p^\mu, \psi] = \frac{\hbar}{i}\frac{\partial}{\partial x_\mu}\psi$ Thus the various transformations can be written without an i .

$$L_+: \tilde{\psi}(x) \to e^{-\frac{1}{2}\vec{\theta}\cdot\vec{\sigma} + \frac{1}{2}\vec{\varphi}\cdot\lambda_2\vec{\sigma}}\tilde{\psi}\left(L^{-1}(x)\right)$$

$$B: \tilde{\psi}(x) \to e^{-\frac{1}{2}\alpha_0\tilde{\lambda}_2}\psi(x)$$

$$\vec{I}: \tilde{\psi}(x) \to e^{-\frac{1}{2}\vec{\alpha}\vec{\tau}}\tilde{\psi}(x)$$

$$PC: \tilde{\psi}(x) \to \tilde{\lambda}_3\tilde{\psi}(-\vec{r}, t_x)$$

$$CPT: \tilde{\psi}(x) \to \tilde{\psi}(-x), \quad c_\nu \to c^*$$

(2.53)

The transformations are simply related to the transformations for $\psi(x)$ by the replacement $\tilde{\lambda}_2 \to -i, \vec{\tilde{\sigma}} \to -i\vec{\sigma}, \vec{\tilde{\tau}} \to -i\vec{\tau}$. except for the CPT-transformation which depends in its form on the particular

representation. The CPT transformation is nearly trivial in this form, since i occurs only as factor of χ in our new representation. It should be emphasized again that all the transformations are real transformations.

5. The Fierz-symmetrical Hermitian Representation:

In the hermitian representation stated in section 4 we may again go one step further and symmetrize the interaction term with respect to exchange of any two ψ . Consider

$$A=(\bar{\psi}^T\tilde{\sigma}\psi)(\bar{\psi}^T\tilde{\sigma}^\mu\psi)\equiv\tilde{\sigma}_\mu\tilde{\sigma}^\mu(\bar{\psi}\psi)(\bar{\psi}\psi) \quad (2.54)$$

Because of the antisymmetry of $\tilde{\sigma}_\mu$ this expression is already symmetrical under exchange of $1\to 2$ or $3\to 4$. If we can make it also symmetrical urder $2\to 4$ it will be fully symmetrical. We define the following other invariant expressions, which are equivalent to A :

$$B\equiv\tilde{\lambda}_2\tilde{\lambda}_2\vec{\gamma}\vec{\gamma}\tilde{\sigma}_\mu\tilde{\sigma}_\mu(\bar{\psi}\psi)(\bar{\psi}\psi) \quad (2.55)$$

$$C\equiv-(\tilde{\lambda}_1\tilde{\lambda}_1+\tilde{\lambda}_3\tilde{\lambda}_3)\vec{\gamma}\vec{\gamma}(\bar{\psi}\psi)(\bar{\psi}\psi) \quad (2.56)$$

Also these expressions are symmetrical under exchange $1\leftrightarrow 2$, $3\leftrightarrow 4$ because of the antisymmetry of $\tilde{\gamma}$, $\tilde{\lambda}_2$ and the symmetry of $\tilde{\lambda}_1$, $\tilde{\lambda}_3$. Under Fierz transformation

$$O_1O_2(\psi_1\psi_2)(\psi_3\psi_4)\to-\frac{1}{8}(O_1\tilde{\lambda}_k\vec{\gamma}\tilde{\sigma}_\nu O_2)(\tilde{\lambda}_k\vec{\gamma}^\mu\tilde{\sigma}_\nu)(\psi_1\psi_4)(\psi_3\psi_2) \quad (2.57)$$

$$=O_1'O_2'(\psi_1\psi_4)(\psi_3\psi_2)$$

with

$$(\tilde{\lambda}_k)(\tilde{\lambda}_k)=II+\tilde{\lambda}_1\tilde{\lambda}_1+\tilde{\lambda}_3\tilde{\lambda}_3-\tilde{\lambda}_2\tilde{\lambda}_2. \quad (2.58)$$

If all the ψ's are the same as in our case, then many terms on the right hand side will vanish due to the antisymmetry properties

of i.e. only antisymmetrical operators O_1' and O_2' will survive, i.e.

$$O_1 O_2 (\psi\psi)(\psi\psi) \to -\frac{1}{8}\frac{1}{8}\sum_A (O_1 \Gamma_A O_2)(\Gamma_A)(\psi\psi)(\psi\psi) \tag{2.59}$$

with

$$\sum_A (\Gamma_A)(\Gamma_A) = \bar{\sigma}_\mu \sigma^\mu + \lambda_2 \lambda_2 \vec{\tau}\vec{\tau}\bar{\sigma}_\mu\sigma^\mu - (\lambda_1\lambda_1 + \lambda_3\lambda_3)\vec{\tau}\vec{\tau} \tag{2.60}$$

Using the rules:

$$\bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu = 2g_{\mu\nu} \quad \text{or} \quad \bar{\sigma}_\mu \sigma^\mu = 4$$
$$\sigma_\nu \bar{\sigma}_\mu \sigma^\nu = -2\sigma_\mu$$
$$\lambda_1 \bar{\sigma}_\mu \lambda_1 = \lambda_3 \bar{\sigma}_\mu \lambda_3 = -\sigma_\mu \tag{2.61}$$

we find

$$A \to 1/4\ (A + B) + 1/2\ C$$

$$B \to 1/4\ (3A - B) + 1/2\ C$$

$$C \to 1/4\ (3A + B) \tag{2.62}$$

There is one special combination of A, B, C which transforms into itself under Fierz transformation:

$$3A + B + 2C \to 1/4\ A\ \ 3 + 3 + 6 + 1/4 B\ \ 3 - 1 + 1?$$
$$+ 1/2\ C\ \ 3 + 1 = 3A + B + 2C \tag{2.63}$$

When all ψ_A' are same, from (2.63) we have

$$A = B = C \tag{2.64}$$

i.e., A, B, C are just 3 different forms of the same interaction term. Hence we can also write

$$A = 1/6\ \ 3A + B + 2C \tag{2.65}$$

This implies that our differential equation may be written in the form:

$$-i\bar{\lambda}_2\bar{\sigma}_\nu\frac{\partial}{\partial x_\nu}\bar{\psi}+\ell^2\left[\left(3II+\bar{\lambda}_2\bar{\lambda}_2\vec{\tau}\vec{\tau}\right)\bar{\sigma}_\mu\bar{\sigma}^\mu - (2\lambda_1\lambda_1+\lambda_2\lambda_3)\vec{\tau}\vec{\tau}\right]:\bar{\psi}(\bar{\psi}\bar{\psi}):=0 \tag{2.66}$$

or symbolically

$$\Gamma^\mu_{\alpha\beta}\left(\frac{\partial}{i\partial x^\mu}\right)\bar{\psi}_\beta+\ell^2\left[\Gamma\right)_{\alpha\beta\gamma\delta}\psi_\beta\psi_\gamma\psi_\delta=0. \tag{2.67}$$

In this representation the interaction term is completely symmetric under the exchange of any of the ψ i.e. the operator part : $\psi_\beta(\psi_\gamma\psi_\delta)$: as well as the matrix-operator part $(\Gamma)_{\alpha\beta\gamma\delta}$ are both completely antisymmetrical under exchange of any two indices. Although this way of writing the equation may look more complicated, it is for many considerations simpler because the nature of the interaction is transparent. Here the exchange term is identical to the direct term, and hence can be simply taken care of by a numerical factor. Such a simplicity is not present in non-Fierz symmetrical representation.

D. Vacuum expectation values of Field operators.

For a spinor theory only vacuum expectation values for an even number of spinor operators are non zero, if we assume that the vacuum does not carry spin and baryon number properties. We shall investigate the analytic form of the vacuum expectation values of 2 and 4 operators.

1. Vacuum expectation values for two field operators:- For the present discussion we shall consider a vacuum which shares all the symmetry of the theory except scale invariance. In such

a case the v.e.v. of 2 operators has the form

$$\langle 0 | \psi(x)\, \psi^*(x') | 0 \rangle = \bar{\sigma}_\mu \frac{\partial}{\partial z_\mu} f(z^2, \varepsilon(z)) \qquad (2.68)$$

with $z = x - x'$, where $f(z^2, \varepsilon(z))$ is a function invariant under all symmetry transformations (except scale transformations) and depends on the Lorentz-invariants z^2 with $\varepsilon(z) = \pm 1$ depending on whether z_μ is positive or negative time like. The form of $f(z^2)$ will also depend on the particular product of the field operators which is considered on the left hand side. We denote here by $| \psi(x), \psi^*(x) |$ any of the usual products i.e. anticommutator, time-ordered, ordinary, etc.. In general $f^2(z^2, \varepsilon(z))$ will be singular. Let us assume that it is a tempered distribution so that a Fourier transform exists. We assume that function can be given a Lehmann-Kallen spectral representation

$$f(z^2, \varepsilon(z)) = \int dm^2 \rho(m^2) \int d^4p \, \frac{e^{ipz}}{p^2 + m^2} \qquad (2.69)$$

with $\rho(m^2)$ an invariant function which fulfils certain conditions in particular

$$\rho(m^2) \equiv 0 \quad \text{for } m^2 < 0 \qquad (2.70)$$

Such a representation was originally derived under the assumptions

1) Existence of a vacuum state of lowest energy;
2) Completeness of the eigenstates of energy and momentum;
3) Positive definite metric in Hilbert space.

Can we still have such a representation when we relax some of the assumptions? Since our theory requires indefinite metric, we have to allow $\rho(m^2)$ to be negative. It is possible the eigenstates of P_μ do not form a complete set though they are sufficient to describe the physical phenomena. As we shall see later this means that $\rho(m^2)$ will contain higher derivatives of δ functions, unlike the Lehmann case where only δ function singularities occur and the higher derivatives of δ function are connected with dipole and multipole ghost states.

If such a generalized Lehmann-Kallen representation is possible, then all the singular functions, representing vacuum expectation values of the various products of the field operators, can be expressed in the usual fashion by appropriate choice of the path of integration around the poles at $p_0 = \pm\sqrt{m^2+p^2}$ in the p_0 plane. All the vacuum expectation values are uniquely defined by the invariant function $\rho(m^2)$. Since our theory is essentially a 2-component spinor theory, a single invariant function suffices instead of two function as in a Dirac spinor theory. From this point of view, a 2-component theory is no more complicated than a scalar field theory.

The occurrence of $\delta(m^2-k^2)$ functions in the mass spectral function $\rho(m^2)$ will indicate the existence of discrete one-particle states $|1_k\rangle$ of mass K. The corresponding single particle matrix elements, e.g. $\varphi(x) = \langle 0|\psi(x)|1_k\rangle$ which is the wave function of a particle, will either obey a neutrino-equation

$$\sigma_\mu \partial^\mu \psi(x) = 0 \qquad (2.71)$$

if the mass K = 0, or a Weyl equation (Klein-Gordon equation

for spinors)

$$(\Box - K^2)\,\varphi(x) = 0 \tag{2.72}$$

if the mass K is finite. In both cases, due to the definition of the matrix element, we are only concerned with the positive energy solutions. It is important to point out the difference between the Weyl equation valid for a spinor field, and a Klein-Gordon equation which is commonly used for a boson field. This difference is not obvious from the form of the differential equation, but the Lagrangians for the respective case are different.

To demonstrate the connection between δ-function in our mass spectral function and solutions of the Weyl equation, we shall show that the quantization of a free Weyl field ψ_{in} (or ψ_{out}) of mass K leads to

$$\langle 0|\psi_{in}(x)\,\psi^{*}_{in}(x')|0\rangle = \frac{1}{(2\pi)^3}\int d^3p\,\frac{\bar{\sigma}_\mu p^\mu}{2E_p}\,e^{ip_\mu(x-x')^\mu}$$

$$= \frac{-i}{(2\pi)^4}\int d^4p\,\frac{\bar{\sigma}_\mu p^\mu}{(p^2+K^2)}\,e^{ipz} = \frac{-i}{(2\pi)^4}\int dm^2\,\delta(m^2-K^2)\int_{C_+} d^4p\,\bar{\sigma}_\mu p^\mu/p^2+m^2\cdot e^{ipz} \tag{2.73}$$

where the L.H.S. of (2.73) stands for the ordinary operator pdoduct and we define the integration path C_+ in the p_0 - plane to encircle the positive ole at $p_0 = +E_p$ with $E_p = \sqrt{p^2+K^2}$. In the first integral p_0 is meant to be E_p.

Because of the equality for single particle matrix elements we have

$$\varphi(x) = \langle 0|\psi(x)||_K\rangle = \langle 0|\psi_{in}(x)||_K\rangle = \langle 0|\psi_{out}(x)||_K\rangle \tag{2.74}$$

We consider the matrix element $\varphi(x) = \langle 0 | \psi_{1\dot{m}}(x) | 1_K \rangle \equiv \langle 0 | \psi_{1\dot{m}}(x) | (\varphi_1)^K \rangle$ where $\varphi(x)$ is any arbitrary normalizable solution with positive frequencies and baryon number +1 of the Weyl equation. We will use the standard procedure to replace the normalizable wave packets by the plane wave states of fixed momentum defined in a finite volume V, taking the $V \to \infty$ limit later on.

In the rest frame of the particle where $\vec{p} = 0$ the solutions are

$$\varphi_0^{(1,2)}(x) = \frac{1}{\sqrt{V}} e^{-ikt} u^{(1,2)} \tag{2.75}$$

where $u^{(1,2)}$ are two orthonormal unit vectors in spin space, as we have

$$\int \varphi_0^{(i)*} \varphi_0^{(r)} d^3x = u^{(i)*} u^{(r)} = \delta_{ir}. \tag{2.76}$$

Making a Lorentz transformation to the frame moving with velocity $\vec{\beta} = \vec{p} / E_p$, where $E_p = K^2 + p^2$ and $\vec{p} = |p| \vec{n}$ the eigenfunctions

$$\varphi(x) \to e^{-\frac{1}{2}\vec{\alpha}\cdot\vec{\sigma}} \varphi(L^{-1}(\alpha)x) = e^{-\frac{1}{2}\alpha(\vec{n}\cdot\vec{\sigma})} \varphi(L^{-1}x) \tag{2.77}$$

where α is related to the velocity $\vec{\beta} = \beta \vec{n}$ by $\tanh \alpha = \beta = |p|/E_p$. Introducing $\cosh \alpha = E_p/K = \gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\sinh \alpha = \frac{|p|}{K} = \gamma\beta$ we have

$$L^{-1}t = \gamma t - \gamma\vec{\beta}\cdot\vec{x} = \frac{1}{K}(E_p t - \vec{p}\cdot\vec{x}) = -\frac{1}{K}(px)$$

$$e^{-\frac{1}{2}\vec{\alpha}\cdot\vec{\sigma}} = \cosh\frac{\alpha}{2} - \vec{\sigma}\cdot\vec{n}\sinh\frac{\alpha}{2} = \sqrt{\frac{\gamma+1}{2}} - \sqrt{\frac{\gamma-1}{2}}\,\vec{\sigma}\cdot\vec{n}$$

$$= \frac{1}{\sqrt{2(\gamma+1)}}\left[(\gamma+1) - \gamma\vec{\sigma}\cdot\vec{\beta}\right]$$

$$= \frac{1}{\sqrt{2K(E_p+K)}}\left[\bar{\sigma}_\mu p^\mu + K\right] \tag{2.78}$$

To preserve the normalization $\int \varphi^* \varphi \, d^3x = 1$ the normalization volume has to be changed

$$\psi \to \gamma \psi \qquad V \to \gamma V = \frac{E_p}{K} V \tag{2.79}$$

Thus the solution valid in any frame is

$$\varphi^{(1,2)}(x) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_p(E_p+K)}} \left[\bar{\sigma}_\mu p^\mu + K\right] e^{ipx} u^{(1,2)} \tag{2.80}$$

Using this solution we calculate now

$$\begin{aligned} \langle 0 | \varphi_{\dot{m}}(x) \varphi^*_{\dot{m}}(x') | 0 \rangle &= \sum_{p,i,\gamma} \langle 0 | \psi(x) | 1_K \rangle \eta \langle 1_K | \psi^*(x') | 0 \rangle \\ &= \sum_{p,i,\gamma} \eta \langle 0 | \psi(x) | 1_K \rangle \langle 0 | \psi(x') | 1_K \rangle^* \\ &= \sum_{p,i,\gamma} \eta \, \varphi(x) \varphi^*(x') \end{aligned} \tag{2.81}$$

with η is the metric tensor in Hilbert space. We will take here η to be + 1 i.e. positive definite metric. Now

$$\begin{aligned} \varphi(x)\varphi^*(x') &= \frac{1}{V} \sum_{p,i,\gamma} \frac{1}{2E_p(E_p+K)} \left[\bar{\sigma}_\mu p^\mu + K\right]\left[\bar{\sigma}^\mu p_\mu + K\right] \\ &\quad \times e^{ip(x-x')} u^{(i)} u^{(i)*} \\ &= \frac{1}{(2\pi)^3} \int d^3p \, e^{ipz} \frac{1}{2E_p(E_p+K)} \left[E_p^2 - 2E_p \vec{\sigma}\vec{p} + p^2 + 2K\bar{\sigma}_\mu p^\mu + K^2\right] \\ &= \frac{1}{(2\pi)^3} \int d^3p \, e^{ipz} \, \bar{\sigma}_\mu p^\mu / 2E_p. \end{aligned} \tag{2.82}$$

In this derivation K could also be assumed to be zero.

The meaning of the Weyl equation becomes more transparent by the observation that the second order Weyl equation is equivalent to the first order Dirac equation. We now find that the Weyl equation

(and the subsidiary condition) is invariant under the new transformation:

$$\varphi(r,t) \rightarrow \frac{\bar{\sigma}_\mu p^\mu}{k} \varphi(-\vec{r},t) \qquad \text{with } p_\mu = \frac{1}{i}\frac{\partial}{\partial x_\mu} \tag{2.83}$$

This is a parity transformation which commutes with the baryonic charge and isotopic spin transformation. We may define a new wave function

$$\varphi'(r,t) \equiv \sigma_\mu p^\mu / k \;\; \varphi(r,t) \tag{2.84}$$

For this wave function, we find

$$\bar{\sigma}_\nu p^\nu \varphi'(r,t) = \frac{1}{k}(\bar{\sigma}_\nu p^\nu)(\sigma_\mu p^\mu)\varphi(r,t) = \frac{1}{k}\Box\varphi(r,t)$$
$$= k\,\varphi(r,t) \tag{2.85}$$

Using the fact that $\varphi(r,t)$ is a solution of the Weyl equation, and $k\,\varphi'(r,t) = \sigma_\mu p^\mu \varphi(r,t)$ we can define

$$\Phi = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \tag{2.86}$$

and write the equation for Φ as

$$\begin{pmatrix} \sigma_\mu p^\mu & 0 \\ 0 & \bar{\sigma}_\mu p^\mu \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} = \mp k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \tag{2.87}$$

We see that the Weyl equation can be replaced by a differential equation of first order for a spinor with the double number of components. If we introduce the following notation:

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.88}$$

then the equation (2.87) can be written as

$$\left[I p_0 + \rho_3 \vec{\sigma} \cdot \vec{p} \right] \phi = \kappa \rho_1 \phi \tag{2.89}$$

which can be restated if we introduce the matrices

$$\gamma_0 = i \rho_1 \sigma_0 = i \gamma_4 \,, \quad \vec{\gamma} = \rho_2 \vec{\sigma} = i \gamma_4 \rho_3 \vec{\sigma} \tag{2.90}$$

in the familiar Dirac form

$$\left(\gamma_\mu p^\mu - i \kappa \right) \phi = 0 \tag{2.91}$$

The parity operation (P) in this notation is of the common form:

$$\phi(r,t) \longrightarrow \gamma_4 \phi(-r\vec{r}, t) \tag{2.92}$$

It is interesting that we are able to define a parity operation for the free particle, when we construct the Dirac equation using the solutions of Weyl equation for finite mass. But then these solutions must necessarily occur as helicity doublets. However it is to be remembered that the Weyl equation is not invariant under the above parity transformation and hence the theory cannot lead to a conservation law for this parity. We may attempt a generalization of the above parity operation such as e.g.

$$\varphi(\vec{r}, t) \rightarrow \sigma_\mu p^\mu / \sqrt{-p^2} \quad \varphi(-\vec{r}, t) \tag{2.93}$$

But due to the occurrence of $1/\sqrt{-p^2}$ (2.93) is non-local and hence does not leave the equation invariant. Thus parity cannot be an exact symmetry. However this need not bother us as parity itself is not strictly conserved in nature as evidenced in process involving weak interactions. The parity problem will be discussed in greater detail in the next section.

The vacuum expectation value for the anticommutator is obtained by encircling clockwise the positive and negative pole in the p_0 integration. We find

$$\langle 0 | \{ \psi(x), \psi^*(x') \} | 0 \rangle = \int dm^2 \rho(m^2) \bar{\sigma}_\mu \frac{\partial}{\partial z_\mu} \Delta(z;m)$$

$$= -\frac{\bar{\sigma}_\mu z^\mu}{2\pi} \varepsilon(z) \int_0^\infty dm^2 \rho(m^2) \left[\delta'(z^2) + \frac{1}{4} m^2 \delta(z^2) - \frac{1}{4} \theta(-z^2) \frac{m^2}{z^2} J_2\left(\sqrt{m^2 z^2}\right) \right] \tag{2.94}$$

$\Delta(z,m)$ is the usual invariant functions for a boson of mass m; $J_2(y)$ is the Bessel function.

We see in (2.94) the explicit appearance of δ and δ' functions on the light cone. They are the source for divergent integrals as will be seen later. In order to remove them, we must demand the following two conditions for the mass spectral function $\rho(m^2)$:

$$\int \rho(m^2)\, dm^2 = 0$$

$$\int m^2 \rho(m^2)\, dm^2 = 0 \tag{2.95}$$

These necessarily introduce an indefinite metric in Hilbert space and destroy the canonical form of the quantization since for the equal time anticommutator, we have

$$\langle 0 | \{ \psi(x), \psi(x') \} | 0 \rangle = \int \rho(m^2) dm^2 \delta(x - x') \tag{2.96}$$

When (2.95) hold

$$\int dm^2 \rho(m^2) \frac{1}{p^2 + m^2} = \int dm^2 \rho(m^2) \left[\frac{1}{p^2 + m^2} - \frac{1}{p^2} + \frac{m^2}{p^2} \right]$$

$$= \int dm^2 \rho(m^2) \frac{m^4}{(p^2)^2 (p^2 + m^2)} \tag{2.97}$$

From (2.97) we see that every 'state' (whether discrete or continuous) is regularized by a 'dipole ghost state' at $m^2=0$ (double pole). The properties of dipole ghost states will be discussed later in Part III. The above regularization is not the most general one. We can as well, regularize with the help of a fixed pole at mass K_1 and K_2 i.e. we write

$$\int dm^2 \rho(m^2) \frac{1}{p^2+m^2} = \int dm^2 \rho(m^2) \left[\frac{1}{p^2+m^2} - \frac{(m^2-K_1^2)}{(K_1^2-K_2^2)} \frac{1}{p^2+K_1^2} + \frac{(m^2-K_1^2)}{(K_1^2-K_2^2)} \frac{1}{p^2+K_2^2} \right]$$

$$= \int dm^2 \rho(m^2) \frac{(m^2-K_1^2)(m^2-K_2^2)}{(p^2+m^2)(p^2+K_1^2)(p^2+K_2^2)} \qquad (2.98)$$

Always the pole, the mass of which lies between the others, is the ghost pole, i.e., e.g. the pole at K_1 if $m > K_1 > K_2$ or $K_2 > K_1 > m$. In obtaining (2.98) we have used a general formula for partial fractions:

$$\frac{1}{\prod_{k=1}^{n} (x-a_k)} = \sum_{i=1}^{n} \frac{1}{\prod_{\ell \neq i} (a_i - a_\ell)} \cdot \frac{1}{(x-a_i)} \qquad (2.99)$$

From (2,99) it is seen that to increase the number of factors (p^2+K^2) in the denominator, poles have to be added the residues of which alternates with increasing mass.

In section F of this part we shall make a few more remarks about the mass spectral function due to the peculiar behaviour of the anticommutator near the light cone as a consequence of scale invariance of the theory.

In our later considerations we will be mainly concerned with the vacuum expectation value of time ordered product.

$$\langle 0|T(\psi(x),\psi^*(x'))|0\rangle = F(x-x') \tag{2.100}$$

which is the 'causal' function or the Feynman propagator or the contraction function. We obtain this function by the $(-i\delta)$ prescription for the poles, i.e.

$$F(x-x') = -\frac{i}{(2\pi)^4}\int dm^2\,\rho(m^2)\int d^4p\, e^{ip(x-x')}\frac{\bar{\sigma}_\mu p^\mu}{p^2+m^2-i\delta}$$

$$= \frac{-i}{(2\pi)^4}\int dm^2\,\rho(m^2)\int d^4p\, e^{ip(x-x')}\frac{m^4\bar{\sigma}_\mu p^\mu}{(p^2-i\delta)^2(p^2+m^2-i\delta)} \tag{2.101}$$

(In the Heisenberg-Pauli form, $F = -1/2\ S_F$ (Dyson) $= S_+$ (Schwinger)).

In concluding this subsection on the 2-point function, we collect here the form of these functions for the other representations discussed in section C. The only change occurs with respect to the matrices $\bar{\sigma}_\mu p^\mu$.

Representation	Isospin	Heisenberg-Pauli	Gursey	Hermitian
V.E.V. of	$\{\psi(x),\psi^*(x')\}$	$\{\chi(x),\bar{\chi}(x)\}$	$\{\phi(x)\phi^*(x)\}$	$\{\tilde{\psi}(x)\ \tilde{\psi}^*(x')\}$
Terms acting on $\int dm^2\rho(m^2)\Delta(z,m)$	$\bar{\sigma}_\mu\frac{\partial}{\partial z_\mu}$	$-i\Gamma_\mu\frac{\partial}{\partial z_\mu}$	$\bar{D}$	$-\tilde{\lambda}_2\bar{\sigma}_\mu\frac{\partial}{\partial z_\mu}$

For example in the hermitian representation

$$\langle 0|\{\tilde{\psi}(x)\tilde{\psi}^*(x')\}|0\rangle = -\int dm^2\,\rho(m^2)\,\tilde{\lambda}_2\,\bar{\sigma}_\mu\frac{\partial}{\partial z_\mu}\Delta(z,m) \tag{2.102}$$

The left-hand side is symmetrical under exchange of both $\tilde{\psi}$. The right-hand side is also symmetrical since it is the product of 4 anti-symmetrical objects, $\hat{\lambda}_2 \tilde{\sigma}_\mu \frac{\partial}{\partial z_\mu}$ and $\mathcal{E}(z)$ contained in $\Delta(z,m)$. The CPT-invariance is here due to the invariance of the product $\frac{\partial}{\partial z_\mu}\mathcal{E}(z)$ under $z \to -z$ since all quantities are real.

2. Vacuum expectation values for 4-point functions:- We shall follow, in this section, closely the work of Montaldi who has used the Heisenberg-Pauli representation which is cumbersome in this respect. It is hoped that the use of the isospin and Hermitian operator representations will simplify the calculations though this work is yet to be done.

Montaldi in his investigations has not included isotopic spin and hence his treatment involves only two component spinors. However, inclusion of isotopic spin is straight forward, which has not been carried out. Montaldi attempted to find a Kallen-Lehmann representation for the 4-point function. He also investigated possible restrictions on the generalized 'mass' spectral functions which appear in the 4-point function, due to the regularisation condition of the 2-point function in the non-linear theory.

We consider a 2-component field $\xi(x)$ which obeys the equation

$$-i\sigma_\nu \frac{\partial}{\partial x_\nu}\xi(x) + l^2 \sigma_\mu : \left[\xi^*(x)\,\sigma^\mu\,\xi(x)\right] : \xi(x) = 0 \qquad (2.103)$$

We shall be interested in the v.e.v. of the 4-point function

$$\tau\begin{pmatrix} x_1 & x_2 \\ \alpha_1 & \alpha_2 \end{pmatrix}\begin{pmatrix} x_3 & x_4 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \langle 0 | T\left(\xi_{\alpha_1}(x_1)\,\xi_{\alpha_2}(x_2)\,\xi^*_{\alpha_3}(x_3)\,\xi^*_{\alpha_4}(x_4)\right) | 0 \rangle \qquad (2.104)$$

Since the τ_0 function is seen to be antisymmetric in all coordinates, we have

$$\tau_0\begin{pmatrix} x_1 x_2 \\ \alpha_1 \alpha_2 \end{vmatrix} \begin{matrix} x_3 x_4 \\ \alpha_3 \alpha_4 \end{pmatrix} = \bar{\sigma}^{\lambda}_{\alpha_1\alpha_3} \bar{\sigma}^{\mu}_{\alpha_2\alpha_4} A_{\lambda\mu}(x_1, x_2 | x_3, x_4) \tag{2.105}$$

where the sixteen functions $A_{\lambda\mu}$ transform according to a tensor of 2nd rank. This is the most general form, since terms $\bar{\sigma}^{\lambda}_{\alpha_1\alpha_4} \bar{\sigma}^{\mu}_{\alpha_2\alpha_3}$ can be included above if we use the Fierz transformation.

$$\bar{\sigma}^{\lambda}_{\alpha_1\alpha_4} \bar{\sigma}^{\mu}_{\alpha_2\alpha_3} = \frac{1}{2} C^{\lambda\mu\nu k} \bar{\sigma}_{\nu,\alpha_1\alpha_3} \bar{\sigma}_{k,\alpha_2\alpha_4} \tag{2.106}$$

where

$$C^{\lambda\mu\nu k} = -i\varepsilon^{\lambda\mu\nu k} - g^{\lambda\mu}g^{\nu k} + g^{\lambda\nu}g^{\mu k} + g^{\lambda k}g^{\mu\nu}; \quad \varepsilon_{0123} = +1 \tag{2.107}$$

The tensor $A_{\lambda\mu}$ may be decomposed into its symmetrical and antisymmetrical part:

$$A_{\lambda\mu} = A_{(\lambda\mu)} + A_{[\lambda\mu]}$$

$$A_{(\lambda\mu)} = \frac{1}{2}(A_{\lambda\mu} + A_{\mu\lambda}); \quad A_{[\lambda\mu]} = \frac{1}{2}(A_{\lambda\mu} - A_{\mu\lambda}) \tag{2.108}$$

Due to the antisymmetry of τ_0 in all fields, $A_{\lambda\mu}$ will exhibit certain symmetry properties:

1) Antisymmetry $1 \leftrightarrow 2$

$$A_{\lambda\mu}(x_1 x_2 | x_3 x_4) = -\frac{1}{2} C_{\lambda\mu\nu k} A^{\nu k}(x_2 x_1 | x_3 x_4) \tag{2.109}$$

or $A_{(\lambda\mu)}(x_1 x_2 | x_3 x_4) = \frac{1}{2} g_{\lambda\mu} A^{k}_{k}(x_2 x_1 | x_3 x_4) - A_{\lambda\mu}(x_2 x_1 | x_3 x_4)$

$$A_{[\lambda\mu]}(x_1 x_2 | x_3 x_4) = -\frac{1}{2} i \varepsilon_{\lambda\mu\nu k} A^{[\nu k]}(x_2 x_1 | x_3 x_4) \tag{2.110}$$

2) Antisymmetry $3 \longleftrightarrow 4$

$$A_{\lambda\mu}(x_1 x_2 | x_3 x_4) = -\frac{1}{2}\varepsilon_{\lambda\mu\nu k} A^{\nu k}(x_1 x_2 | x_3 x_4)$$

or $$A_{(\lambda\mu)}(x_1 x_2 | x_3 x_4) = \frac{1}{2} g_{\lambda\mu} A^{k}_{k}(x_1 x_2 | x_4 x_3) - A_{(\lambda\mu)}(x_1 x_2 | x_4 x_3)$$

$$A_{[\lambda\mu]}(x_1 x_2 | x_3 x_4) = \frac{1}{2}[\varepsilon_{\lambda\mu\nu k} A^{[\nu k]}(x_1 x_2 | x_3 x_4) \tag{2.111}$$

3) Symmetry $1 \longleftrightarrow 2$, $3 \longleftrightarrow 4$

$$A_{(\lambda\mu)}(x_1 x_2 | x_3 x_4) = A_{(\lambda\mu)}(x_2 x_1 | x_4 x_3)$$

$$A_{[\lambda\mu]}(x_1 x_2 | x_3 x_4) = -A_{[\lambda\mu]}(x_2 x_1 | x_4 x_3) \tag{2.112}$$

To obtain a spectral representation for the 4-point function $T_0(x_1 x_2 | x_3 x_4)$ let us study first the ordinary product of field operators:

$$\langle 0 | \xi_{\alpha_1}(x_1) \xi_{\alpha_2}(x_2) \xi_{\alpha_3}(x_3) \xi_{\alpha_4}(x_4) | 0 \rangle$$
$$= \overline{\sigma}^{\lambda}_{\alpha_1\alpha_3} \overline{\sigma}^{\mu}_{\alpha_2\alpha_4} G_{\lambda\mu}(x_1 x_2 | x_3 x_4) \tag{2.113}$$

In the case of the two point function we assumed that it was a tempered distribution which guaranteed the existence of the momentum transform. As there was only one coordinate difference, there was only one spectral function $\rho(m^2)$. In the case of 4-point function, we have six coordinate differences and hence for the Fourier transform we require six momenta and consequently our spectral weight functions will be functions of six different masses. Montaldi shows that

can be written in the following form:

$$G_{\lambda\mu}(x_1 x_2 | x_3 x_4) = \sum_{i=1}^{3} G_{\lambda\mu}^{(i)}(x_1 x_2 | x_3 x_4)$$

$$G_{\lambda\mu}^{(1)}(x_1 x_2 | x_3 x_4) = \int (dm^2) \prod_{\ell=1}^{6} \Delta^{(+)}(z_\ell; m_\ell) \times M(1 \cdots 6) \tag{2.114}$$

$$G_{\lambda\mu}^{(2)}(x_1 x_2 | x_3 x_4) = \sum_{k=1}^{6} \int (dm^2) M_k(1 \cdots 6) \times \prod_{\ell=1}^{6}{}^{(k)} \Delta^{(+)}(z_\ell; m_\ell) \frac{i}{(2\pi)^3} \int d^4 p_k (p_k)_\lambda (p_k)_\mu \delta(p_k^2 + m_k^2) \times e^{i p_k z_k} \theta(p_k^0) \tag{2.115}$$

$$G_{\lambda\mu}^{(3)}(x_1 x_2 | x_3 x_4) = \sum_{k \neq \pi = 1}^{6} \int (dm)^2 M_{k\pi}(1 \cdots 6) \prod_{\ell=1}^{6}{}^{(k\pi)} \Delta^{(+)}(z_\ell; m_\ell) \times \frac{1}{(2\pi)^6} \int d^4 p_k \, d^4 p_\pi \, (p_k)_\lambda (p_\pi)_\mu \, \delta(p_k^2 + m_k^2) \delta(p_\pi^2 + m_\pi^2) \times e^{i(p_k z_k + p_\pi z_\pi)} \theta(p_k^0) \theta(p_\pi^0) \tag{2.116}$$

(2.117)

with $(dm^2) = dm_1^2 \cdots dm_6^2$

z_ℓ = the six coordinate differences; $\Pi^{k,\pi} = \prod_{\substack{\ell=1 \\ \ell \neq k,\pi}}^{6}$

$M(1 \cdots 6) = M(m_1^2 \cdots m_6^2)$

We notice that $G_{\lambda\mu}^{(2)}$ and $G_{\lambda\mu}^{(3)}$ contain six and fifteen spectral functions respectively which arise when constructing a second rank tensor from the available momentum vectors; we can simplify these formulas. Define

$$K^+_{\alpha_1 \alpha_3}(z; m) = \bar{\sigma}^{\lambda}_{\alpha_1 \alpha_3} \frac{\partial}{\partial x_\lambda} \Delta^+(z; m)$$

$$K^+_{\alpha_1 \alpha_3; \alpha_2 \alpha_4}(z; m) = \bar{\sigma}^{\lambda}_{\alpha_1 \alpha_3} \frac{\partial}{\partial x_\lambda} K^+_{\alpha_2 \alpha_4} = \bar{\sigma}^{\lambda}_{\alpha_1 \alpha_3} \bar{\sigma}^{\mu}_{\alpha_2 \alpha_4} \frac{\partial}{\partial x_\lambda} \frac{\partial}{\partial x_\mu} \Delta^{(+)}(z; m) \tag{2.118}$$

Then it follows that

$$\langle 0|\xi_{\alpha_1}(x_1)\xi_{\alpha_2}(x_2)\xi^*_{\alpha_3}(x_3)\xi^*_{\alpha_4}(x_4)|0\rangle = \bar{\sigma}^{\lambda}_{\alpha_1\alpha_3}\bar{\sigma}_{\lambda x_2\alpha_4}\int(dm^2)\prod_{\ell=1}^{6}\Delta^{(+)}(z_\ell;m_\ell)\times M(1\cdots 6)$$
$$+\sum_{k=1}^{6}\int(dm^2)M_k(1\cdots 6)\prod_{\ell=1}^{6}\Delta^{(+)}(z_\ell;m_\ell)\,K^{(+)}_{\alpha_1\alpha_2;\alpha_3\alpha_4}(z_k,m_k)$$
$$+\sum_{k,\pi=1}^{6}\int(dm^2)M_{k\pi}(1\cdots 6)\prod_{\substack{\ell=1\\ \ell\neq k,\pi}}^{6}\Delta^{(+)}(z_\ell;m_\ell)\,K^{+}_{\alpha_1\alpha_3}(z_k;m_k)K^{+}_{\alpha_2\alpha_4}(z_\pi;m_\pi).$$

(2.119)

Hence all products contain six 2-point functions of which only one or two can be fermion functions, $K^{(+)}$.

In this representation the transition to the propagator function is achieved by the replacement $\Delta^{(+)}\rightarrow\Delta_F$ and $K^{(+)}\longrightarrow K^{\#}_F$.

For many purposes it is convenient to write the K-dependent part in a symmetrical form:

$$\tau_0\left(\begin{matrix}x_1 x_2\\ \alpha_1\alpha_2\end{matrix}\middle|\begin{matrix}x_3 x_4\\ \alpha_3\alpha_4\end{matrix}\right) = \overleftarrow{\sigma}^{\lambda}_{\alpha_1\alpha_3}\bar{\sigma}_{\lambda\alpha_2\alpha_4}\int(dm^2)\prod_{\ell=1}^{6}\Delta_F(z_\ell;m_\ell)\,M(1\cdots 6)$$
$$+\sum_{k=1}^{6}\int(dm^2)M^{(1)}_k(1\cdots 6)\prod_{\ell=1}^{6}{}^{(k)}\Delta_F(z_\ell;m_\ell)\times K_{F\alpha_1\alpha_3\alpha_2\alpha_4}(z_k;m_k)$$
$$+\sum_{k=1}^{6}\int(dm^2)M^{(2)}_k(1\cdots 6)\cdots\cdots K_{F\alpha_1\alpha_4;\alpha_2\alpha_3}(z_k;m_k)$$
$$+\sum_{k\neq\pi=1}^{6}\int(dm^2)M^{(1)}_{k\pi}(1\cdots 6)\prod_{\ell=1}^{6}{}^{(k\pi)}\Delta_F(z_\ell;m_\ell)\,K_{F\alpha_1\alpha_3}(z_k;m_k)K_{F\alpha_2\alpha_4}(z_\pi;m_\pi)$$
$$+\sum\cdots\cdots M^{(2)}_{k\pi}(1\cdots 6)\cdots\cdots K_{F\alpha_1\alpha_4}(z_k;m_k)K_{F\alpha_2\alpha_3}(z_\pi;m_\pi)$$

(2.120)

We have used two spectral functions $M^{(1)}$ and $M^{(2)}$ in (2.120) in order to avoid the unpleasant δ-functions which arise under the exchange $\alpha_2 \longleftrightarrow \alpha_4$.

Consider the 2-point function

$$T_0\begin{pmatrix} x & | & x' \\ \alpha & | & \alpha' \end{pmatrix} = \langle 0 | T\left(\xi_\alpha(x)\,\xi^*_{\alpha'}(x')\right) | 0 \rangle = \bar{\sigma}^\mu_{\alpha\alpha'} \frac{\partial}{\partial z_\mu} \int (dm^2)\, \rho(m^2)\, \Delta_F(z; m) \tag{2.121}$$

with

$$\int (dm^2)\, \rho(m^2) = 0$$

$$\int (dm^2)\, m^2 \rho(m^2) = 0$$

If we use the differential equation for $\xi^*(x')$ we get

$$-\ell^2 \bar{\sigma}^\mu_{\alpha\delta} \sigma_{\mu,\beta\gamma} \langle | T\, \xi_\alpha(x) : \xi^*_{\alpha'}(x') \xi^*_\beta(x') \xi_\gamma(x') : | \rangle = i \delta_{\alpha\delta} \int_0^\infty dm^2\, m^2 \rho(m^2) \Delta_F(z; m) \tag{2.122}$$

since

$$i \sigma^\lambda_{\alpha\delta} \frac{\partial}{\partial x'_\lambda} \sigma_{\mu,\alpha\alpha'} \frac{\partial}{\partial z_\mu} = -i (\bar{\sigma}^\mu \sigma^\lambda)_{\alpha\delta} \frac{\partial^2}{\partial x^\lambda \partial x_\mu} = i \delta_{\alpha\delta} \Box^2 \tag{2.123}$$

Using the identity:

$$\bar{\sigma}^\mu_{\alpha'\delta} \sigma_{\mu,\beta\gamma} = 2 \left[I_{\alpha'\gamma} I_{\beta\delta} - I_{\alpha'\delta} I_{\beta\gamma} \right] \tag{2.124}$$

and setting

$$T\left(\xi(x')\,\xi^*(x')\,\xi(x')\right) = : \xi(x')\, \xi^*(x')\, \xi(x') : \tag{2.125}$$

(which is true if the contraction function

$$F(x - x') = 0 \quad \text{for } x = x'$$

This can be recast in the form

$$T_0\begin{pmatrix} x\,x' & | & x'\,x' \\ \alpha\;\gamma & | & \gamma\;\delta \end{pmatrix} = \frac{-i}{4e^2} \delta_{\alpha\delta} \int (dm^2)\, m^2 \rho(m^2)\, \Delta_F(x - x'; m) \tag{2.126}$$

Similarly one finds on using the differential equation for $\xi(x)$

$$\tau_0\left(\begin{matrix} x & x \\ \alpha & \gamma \end{matrix}\middle|\begin{matrix} x & x' \\ \gamma & \delta \end{matrix}\right) = -\frac{i}{4\ell^2}\delta_{\alpha\delta}\int (dm^2)\rho(m^2)\,\Delta_F(x-x';m) \tag{2.127}$$

These 4-point functions are special cases of the general 4-point function discussed before, and we can now discuss the possible limitations on the functions appearing in the spectral representation of the 4-point function due to the above conditions.

The following relationship can be proved:

$$\Delta_F(x;m_1)\,\Delta_F(x;m_2) = \int_{(m_1+m_2)^2}^{\infty} d\mu^2\,\phi(m_1,m_2;\mu)\,\Delta_F(x;\mu) \tag{2.128}$$

with

$$\phi(m_1,m_2;\mu) = \frac{-i}{(4\pi)^2}\sqrt{\left[1-\frac{(m_1+m_2)^2}{\mu^2}\right]\left[1-\frac{(m_1-m_2)^2}{\mu^2}\right]} \tag{2.129}$$

This relationship essentially expresses the fact that the propagation of two bosons with mass m_1 and m_2 at the same space-time point x can be considered as the propagation of a single particle with a variable mass μ ranging from $\mu = m_1+m_2$ to $\mu = \infty$. The weight $\phi(m_1,m_2;\mu)$ is given by the phase space vector.

The invariant phase space density of a 2-particle system of total momentum $\vec{p}$ with total energy between $E = E_1 + E_2$ and $E + dE$ is given by

$$\rho(E) = \frac{1}{(2\pi)^3}\int \frac{d^3p_1}{2E_1}\int \frac{d^3p_2}{2E_2}\,\delta^4(p_1^\mu + p_2^\mu - p^\mu) = \frac{4\pi}{(2\pi)^3}\frac{d\bar{p}}{4\mu} = \rho'(\mu)$$

where $p^\mu = E, \vec{p}$; $E = \sqrt{\mu^2 + \vec{p}^2}$; $\bar{p}$ = momentum of particles in c.m.s.

and μ = energy E in c.m.s.

(2.130)

$\bar{p}$ is an invariant and can be easily expressed in terms of the masses m_1, m_2, μ only. In c. m. s.

$E_1 + E_2 = \mu$; $E_1^2 - E_2^2 = m_1^2 + m_2^2 + 2(\bar{p}/2)^2$; $E_1^2 - E_2^2 = m_1 - m_2^2$

with $\mathcal{E}_1 = \frac{E_1}{\mu}$; $\mathcal{E}_2 = \frac{E_2}{\mu}$, $\mathcal{E}_1 + \mathcal{E}_2 = 1$, $A = \frac{m_1 + m_2}{2}$; $B = \frac{m_1 - m_2}{\mu}$

We have $\frac{\bar{p}^2}{\mu^2} = \left[\mathcal{E}_1^2 + \mathcal{E}_2^2 - \frac{1}{2}(A^2 + B^2)\right]$

where $\mathcal{E}_1 - \mathcal{E}_2 = AB$. (2.131)

Thus $\bar{p}/\mu = \left[(1-A^2)(1-B^2)\right]^{1/2}$ and hence

$$\phi = -\frac{i}{2}\rho'(\mu) \qquad (2.132)$$

We can reexpress the product of three propagation functions in a similar way.

$$\Delta_F(x;m_1)\,\Delta_F(x;m_2)\,\Delta_F(x;m_3) = \int_{(m_1+m_2+m_3)^2}^{\infty} d\nu^2\, \varphi(m_1,m_2,m_3;\nu)\,\Delta_F(x;\nu)$$

(2.133)

with

$$\varphi(m_1,m_2,m_3;\nu) = \int_{(m_1+m_2)^2}^{(\nu-m_3)^2} d\mu^2\, \phi(m_1,m_2;\mu)\,\phi(\mu,m_3;\nu)$$

(2.134)

which is related to the phase space volume of three particles.

Before we go back to the general representation, let us study the effect of the regularisation conditions

$$\int dm^2\ \rho(m^2) = 0$$

$$\int dm^2\ m^2\ \rho(m^2) = 0 \tag{2.135}$$

on the propagator functions. For the v.e.v. of the anticommutator we have already observed that it leads to the elimination of the δ and δ'-functions. For the propagation function we find:

$$\int_0^\infty \rho(m^2)\, \Delta_F(x;m)\ dm^2\Big|_{x\to 0} = \frac{-i}{8\pi^2}\int_0^\infty m^2 \ln m\ \rho(m^2)\ dm^2$$

$$\int_0^\infty \rho(m^2)\,(K_F)_{\alpha_1\alpha_2}(x;m)\,dm^2\Big|_{x\to 0} = 0 \tag{2.136}$$

This can be immediately seem if we bear in mind the explicit form of Δ_F and K_F i.e.

$$\Delta_F(x;m) = S(\lambda) + R(\lambda) \tag{2.137}$$

S=Singular part R = Regular part near light cone

with $S(\lambda) = -\frac{1}{4\pi}\delta(\lambda) + \frac{i}{4\pi^2\lambda} + m^2\left\{\frac{1}{16}\theta(\lambda) - \frac{i}{8\pi^2}\left[\theta(\lambda)\ln\sqrt{\lambda} + \theta(-\lambda)\ln\sqrt{-\lambda}\right.\right.$

$$R(\lambda) = \frac{-im^2}{(8\pi)^2}\ln\left(\tfrac{1}{2}m\right) + \frac{im^2}{16\pi^2}\left[\psi(1)+\psi(2)\right] + \text{terms}$$

which vanish for $x=0$

ψ = logarithmic derivative of the Γ-function

(2.138)

with $S'(\lambda) = \frac{1}{2\pi}\delta'(\lambda) - \frac{m^2}{8\pi}\delta(\lambda) + \frac{m^4}{64\pi}\theta(\lambda) + \frac{i}{2\pi^2\lambda^2} + \frac{im^2}{8\pi^2\lambda}$

$$-\frac{im^4}{32\pi^2}\left[\theta(\lambda)\ln\sqrt{\lambda} + \theta(-\lambda)\ln\sqrt{-\lambda}\right] \tag{2.139}$$

$R'(\lambda) = 0$ for $x = 0$.

We can now investigate the conditions on the general spectral functions M(1.....6), of the 4-point function which ensure that the above special cases are contained in it.

$$T_0\begin{pmatrix} x_1 & x_2 \\ \alpha_1 & \alpha_2 \end{vmatrix} \begin{matrix} x_3 & x_4 \\ \alpha_3 & \alpha_4 \end{matrix}\Big) = \bar{\sigma}^{\lambda}_{\alpha_1\alpha_3}\bar{\sigma}_{\lambda,\alpha_2\alpha_4}\int (dm^2)\prod_{\ell} \Delta_F(z_\ell;m_\ell)\, M(1\cdots 6) + \cdots\cdots \tag{2.140}$$

In particular for $x_1 = x$, $x_2 = x_3 = x_4 = x'$: and contraction of α_2 α_3 , we have

$$T_0\begin{pmatrix} x & x' \\ \alpha_1 & \gamma \end{vmatrix} \begin{matrix} x' & x' \\ \gamma & \alpha_4 \end{matrix}\Big) = 2\delta_{\alpha_1\alpha_4}\int (dm^2)\prod_{\ell=1}^{3} \Delta_F(z;m_\ell)\prod_{\ell=4}^{6} \Delta_F(0;m_\ell)\, M(1\cdots 6) \tag{2.141}$$

If we now impose the conditions:

$$\int (dm^2)\, M(1\cdots 6) = 0, \qquad \ell = 4,5,6$$

$$\int (dm^2)\, m_\ell^2\, M(1\cdots 6) = 0 \qquad \ell = 4,5,6 \tag{2.142}$$

we get an especially simple form

$$T_0\begin{pmatrix} x & x' \\ \alpha & \gamma \end{vmatrix} \begin{matrix} x' & x' \\ \gamma & \delta \end{matrix}\Big) = 2\delta_{\alpha\delta}\int (dm_1^2)(dm_2^2)(dm_3^2)\, \bar{M}(123)\prod_{\ell=1}^{3} \Delta_F(z;m_\ell) + \cdots\cdots \tag{2.143}$$

with

$$\bar{M}(123) = \left(\frac{-i}{8\pi^2}\right)^3 \int M(1\cdots 6)\prod_{\ell=4}^{6} m_\ell^2 \ln m_\ell\, (dm_\ell^2) \tag{2.144}$$

If we assume that the terms (+ .....) are negligible we can identify (2.143) with (2.127), provided the following relation holds ~~the 2-point~~

$$\int_0^{\nu^2} dm_1^2 \int_0^{(\nu-m_1)^2} dm_2^2 \int_0^{(\nu-m_1-m_2)^2} dm_3^2\, \bar{M}(123)\,\varphi(m_1 m_2 m_3;\nu) = \frac{-i}{8\ell^2}\,\nu^2\rho(\nu^2) \tag{2.145}$$

Similarly we can discuss $T_0(x\,x|\,x\,x')$ which leads to a similar formula which can be expressed by a certain symmetry property of M(1, 2, ... 6).

We have now to investigate whether the forms containing the K-functions really do not contribute. Considerations of the terms which contain $(K_F)_{\alpha_1\alpha_3\alpha_2\alpha_4}$ and five 4-functions lead indeed to the result that such terms cannot be present if the differential equations hold. Hence we have

$$M_k(1 \cdots 6) = 0 \quad \text{for } k = 1 \cdots 6. \tag{2.146}$$

Finally we have to consider the terms containing two K_F - functions and four Δ_F functions. We can show that

$$(K_F)_{\alpha_1\alpha_3}(z;m)\,K_{F\,\alpha_2\alpha_4}(z;m_2) = \frac{-i}{(4\pi)^2}\int_{(m_1+m_2)^2}^{\infty} d\mu^2\, F_1(m_1,m_2;\mu)\,k^F_{\alpha_1\alpha_3\alpha_2\alpha_4}(z;\mu)$$

$$\frac{-i}{(4\pi)^2}\,\bar{\sigma}^{\lambda}_{\alpha_1\alpha_3}\bar{\sigma}_{\lambda\alpha_2\alpha_4}\int_{(m_1+m_2)^2}^{\infty} d\mu^2\,\Delta_F(z;m)\,F_2(m_1,m_2;\mu) \tag{2.147}$$

where $F_1(m_1,m_2;\mu)$ and $F_2(m_1,m_2;\mu)$ are phase space factors similar to $\Phi(m_1,m_2;\mu)$

We observe that for $T_0(x\,x'|x'\,x')$ only terms of the form $\Delta_F(0;m_1)\,\Delta_F(0;m_2)\,\Delta_F(0;m_3)\,\Delta_F(z;m_4)$

$\times\, K_F(z;m_5)\,K_F(z;m_6)$

survive since $K_F(0;m) = 0$. We observe that, in order to reduce the above terms to the form (2.127), we must impose two

integral conditions of the form (2.142) on each of the spectral functions occurring in these terms. However, these conditions together with (2.145) introduce into the theory a high degree of arbitrariness. If we wish to have a criterion of 'uniqueness' in the establishment of the connection between four-point and two point function, the only available way seems to be the requirement that the following spectral functions:

$$M^{(1,2)}_{k\pi}(1\cdots 6) = 0 \qquad \text{except for } (k,\pi) = (16,25,34) \tag{2.148}$$

Thus the final spectral formula subject to these conditions has the form.

$$\begin{aligned} T_0\left(\begin{matrix} x_1 x_2 \\ \alpha_1 \alpha_2 \end{matrix} \middle| \begin{matrix} x_3 x_4 \\ \alpha_3 \alpha_4 \end{matrix}\right) &= \bar{\sigma}^{\lambda}_{\alpha_1\alpha_3} \bar{\sigma}_{\lambda\alpha_2\alpha_4} \int (dm^2) \prod_{\ell=1}^{6} \Delta_F(z_\ell; m_\ell)\, M(1\cdots 6) \\ &+ \sum_{k,\pi=1}^{3} \int (dm^2)\, M^{(1)}_{k\pi}(1\cdots 6) \prod_{\ell}{}^{(k\pi)} \Delta_F(z_\ell; m_\ell)\, K_F(z_k, m_k)_{\alpha_1\alpha_3} K_F(z_\pi, m_\pi)_{\alpha_2\alpha_4} \\ &+ \sum_{k,\pi=1}^{3} \int (dm^2)\, M^{(2)}_{k\pi}(1\cdots 6) \prod_{\ell}{}^{(k\pi)} \Delta_F(z_\ell; m_\ell)\, K_F(z_k, m_k)_{\alpha_1\alpha_4} K_F(z_\pi, m_\pi)_{\alpha_2\alpha_3} \end{aligned} \tag{2.149}$$

Due to the antisymmetry of the operators in the T-product, we have symmetry relationships between the various spectral functions, e.g.

$$M(123456) = M(145236)$$

and also relationships between $M^{(1)}_{k\pi}$ and $M^{(2)}_{k\pi}$. It will turn out, that only 4 of the nonvanishing functions $M^{(1,2)}_{k\pi}$ are linearly independent.

We may now use this rather general representation of $T_0\begin{pmatrix} x_1 x_2 \\ \alpha_1 \alpha_2 \end{pmatrix}\begin{pmatrix} x_3 x_4 \\ \alpha_3 \alpha_4 \end{pmatrix}$ to derive spectral representations for the special 4-point functions;

$$T_0\begin{pmatrix} x x \\ \alpha_1 \alpha_2 \end{pmatrix}\begin{pmatrix} x' x' \\ \alpha_3 \alpha_4 \end{pmatrix} \quad \text{and} \quad T_0\begin{pmatrix} x x' \\ \alpha_1 \alpha_2 \end{pmatrix}\begin{pmatrix} x x' \\ \alpha_3 \alpha_4 \end{pmatrix}$$

These functions are of interest, since they contain expressions which can be interpreted as the propagators of bosons like the deuteron and pion respectively. We quote the final result of Montaldi

$$T_0\begin{pmatrix} x x \\ \alpha_1 \alpha_2 \end{pmatrix}\begin{pmatrix} x' x' \\ \alpha_3 \alpha_4 \end{pmatrix} = \bar{\sigma}^{\lambda}_{\alpha_1\alpha_3}\bar{\sigma}_{\lambda\alpha_2\alpha_4}\int (dm^2) A^{(1)}(\mu^2) \Delta_F(z;\mu)$$
$$+ \int_0^\infty d\mu^2 \int_0^\infty d\nu^2 B^{(1)}(\mu^2,\nu^2)\left[K^{*}_{F\,\alpha_1\alpha_3}(z;\mu) K^{*}_{F\,\alpha_2\alpha_4}(z;\mu) - K_{F\,\alpha_1\alpha_4}(z;\mu) K_{F\,\alpha_2\alpha_3}(z;\mu)\right] \tag{2.150}$$

$$T_0\begin{pmatrix} x x' \\ \alpha_1 \alpha_2 \end{pmatrix}\begin{pmatrix} x x' \\ \alpha_3 \alpha_4 \end{pmatrix}$$
$$= \bar{\sigma}^{\lambda}_{\alpha_1\alpha_3}\bar{\sigma}_{\lambda\alpha_2\alpha_4}\int (d\mu^2) A^{(2)}(\mu^2) \Delta_F(z;\mu)$$
$$+ \int_0^\infty d\mu^2 \int_0^\infty d\nu^2 \left[B^{(2)}(\mu^2,\nu^2) K^{*}_{F\,\alpha_1\alpha_4}(z;\mu) K_{F\,\alpha_2\alpha_3}(z;\mu)\right.$$
$$\left. + C^{(2)}(\mu^2,\nu^2) K_{F\,\alpha_1\alpha_3}(z;\mu) K_{F\,\alpha_2\alpha_4}(z;\mu)\right] \tag{2.151}$$

Here $A^{(i)}(\mu^2)$, $B^{(i)}(\mu^2,\nu^2)$ and $C^{(2)}(\mu^2,\nu^2)$ are complicated integrals involving the spectral functions $M^{(1)},\ldots,M^{(6)}$ and the phase space functions $\phi(m_1,m_2;\mu)$. The formulae given above have, in fact, the structure one would expect. If we can show that $A^{(i)}(\mu^2)$ contain δ-function-like contributions then we have established the existence of a stable boson. We see from this general relationship that there exists a connection

between all the spectral functions and in particular, between the fermion and boson spectral functions.

Physically this is plausible, since e.g. the fermion spectral function $\rho(m^2)$ must contain not only δ-function terms e.g. due to a stable nucleon, but also a continuous spectrum due to the nucleon-2π scattering states. Hence the π-meson which occurs as a pole in the $A^{(2)}(\mu^2)$ spectrum is already contained in the function form of $\rho(m^2)$.

Since the relationship between $A^{(i)}$ and ρ is rather complicated, it is not clear at the moment whether this relationship can be used for actual computations. Perhaps questions such as the conditions on $\rho(m^2)$ necessary to produce a δ-function in $A^{(2)}(\mu^2)$ can be considered and successfully tackled as also questions regarding the high-energy behaviour of fermion-fermion scattering.

We should bear in mind that these formulae were derived under certain simplifying assumptions. Some people consider the assumptions to be too restrictive. Further investigations regarding the physical meaning of these assumptions are needed.

E. The Parity Problem:

We have emphasized already that our fundamental equation of motion is invariant only under a PC-reflection and not under P and C separately. Though the observed interactions between the elementary particles seem to be only PC invariant, it is to be noted that the derivations from strict conservation under P and C operation separately are only very small as the violations are mainly due to weak interaction. In a theory where separate P

C conservation do not hold at initio, as in ours, we have therefore to understand why these symmetries should be so exact in nature, although only the symmetry of the product is enough. We do not know the complete answer at present lent we shall make few remarks on this 'parity problem'.

Whenever discrete fermion solutions of finite mass exist, a parity operation can be defined for these one particle states as we have already shown that there is a complete degeneracy between a positive and negative helicity solutions. We can show that such a symmetry must also exist for even continuous states of finite m. We shall now refer to a difficulty associated with above definition of parity. Since mass is generated by interaction, it would be hard to understand the occurrence of helicity doublets for finite m, when the interactions are parity non-conserving. In fact we may conclude that particles of finite mass should not occur for interactions violating parity. However we can introduce a vacuum which contains a mass K (say) in which case we can obtain a helicity doublet for finite m. From our point of view, the most important question is to understand the simultaneous presence of parity violating weak interactions and the Dirac particle of finite mass.

To obtain a form of the theory which is explicitly parity invariant, which can be used in all 'strong and electromagnetic' interaction approximations, we proceed in the following way.

We start with the original four component equation

$$-i\sigma_\nu \frac{\partial}{\partial x_\nu}\psi + l^2 :\sigma_\mu \psi(\psi^* \sigma_\mu \psi): = 0 \qquad (2.152)$$

We require that $\langle 0|\, \psi(x)\, \psi^*(x')\,|0\rangle$ is the form

$$\langle 0|\psi(x)\,\psi^*(x')|0\rangle = \text{Const}\int (dm^2)\,\rho(m^2)\,\frac{\bar{\sigma}_\mu p^\mu}{p^2+m^2}\,e^{ip(x-x')} \tag{2.153}$$

and assume that with an appropriate choice of the vacuum $\rho(m^2)$ contains a 'nucleon' solution at $m = K$ i.e. $a\,\delta(m^2-K^2)$. Let us construct now a four component field operator $\psi'(x)$ which may be complicated functional of the operator functions $\psi(x)$ and $\psi^*(x)$ with the following conditions.

1) $\psi'(-\vec{r},t)$ shall transform like $\psi(\vec{r},t)$ under all transformations considered.

2) $\psi'(-\vec{r},t)$ shall obey the same differential equation as

3) The Fourier transform of $\psi'(x)$ shall coincide on the mass shell of the nucleons i.e. $p^2 = -K^2$ with the Fourier transform of

We can formally introduce a 8-component spinor

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} \tag{2.154}$$

In the new '2-component' space which we call ρ space, we introduce Pauli-like matrices $\vec{\rho}$. The parity operation will be defined by

$$\psi(r,t) \longrightarrow \psi'(-r,t) \quad \text{or} \quad \Psi(x) \Longrightarrow \gamma_4\,\Psi(-\vec{r},t) \tag{2.155}$$

where

$$\gamma_4 = \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.156}$$

For the matrix elements of the nucleons $\langle 0|\,\psi(x)\,|1_K\rangle$ the

above procedure agrees exactly with the parity operation. Due to the requirement (2), $\Psi(x)$ will obey now a Dirac-type equation

$$\gamma_\nu \frac{\partial}{\partial x_\nu}\Psi + \frac{l^2}{2}\left[:\gamma_\mu \Psi(\bar{\Psi}\gamma^\mu\Psi):+:\gamma_5\gamma_\mu\Psi(\bar{\Psi}\gamma_5\gamma_\mu\Psi):\right]=0 \tag{2.157}$$

The v.e.v. of two operators $\Psi, \bar{\Psi}$ can be written as

$$\langle 0|\Psi(x)\bar{\Psi}(x')|0\rangle$$
$$= \text{Const}\int(dm^2)\left[\frac{\gamma_\mu p^\mu + im}{p^2+m^2}\rho_1(m^2)+\frac{\gamma_\mu p^\mu - im}{p^2+m^2}\rho_2(m^2)\right]d^4p\, e^{ip(x-x')} \tag{2.158}$$

where due to condition(3) only $\rho_1(m^2)$ contains a δ-function at the nucleon mass κ^2 but not $\rho_2(m^2)$. The assumption $\rho_1(m^2) \neq \rho_2(m^2)$ implies that $\psi(x)$ and $\psi'(x)$ do not anticommute. The $\rho_1(m^2)$gives the mass spectrum of the positive parity states, e.g. N, N + 2π , N + 4π, ...$\rho_2(m^2)$the mass spectrum of the negative parity states (Mπ , N + 3π, ...).

We cannot guarantee the existence of $\psi'(x)$ under conditions 1 to 3. We shall however show later on using Tamm-Dancoff-approximation method, that the functional dependence of $\psi'(x)$ on $\psi(x)$ and $\psi^*(x)$ can be established in every step of that approximation if the conditions above are postulated from the outset. This functional dependence can be only expressed in an implicit form, and hence would establish a parity symmetrical approximation in which only the original four component $\psi(x)$ are used.

In concluding this section, it is interesting to remark that our theory in the parity symmetric form allows a completely different interpretation. We may think of Ψ as a genuine

8-component field with $\psi(x)$ and $\psi'(x)$ as independent components. The equation of motion is Touschek-invariant, i.e. invariant under the transformation

$$\Psi(x) \rightarrow e^{i\alpha\gamma_5}\,\Psi(x) \tag{2.159}$$

However the v.e.v. given (2.158) for $\rho_1 \neq \rho_2$ is not Touschek-invariant because of the appearance of a mass term. This deviation may well be attributed to a non-Touschek invariant nature of the vacuum state, as noted by Nambu and Dürr. Though the 'unsymmetrical vacuum' and the 'functional dependence between components of field operators' seem to agree in this case, further investigations are needed to bear out this observation.

The parity symmetric approximation of the theory which is enough for non-weak interactions, was originally derived as an exact form of the theory using the symmetry property of ℓ- reversal i.e. $\ell \rightarrow -\ell$. In this derivation the fact was used that the equation of motion only contains ℓ^2 whereas space and time are linear expressions of ℓ . The sign of ℓ i.e. $\epsilon_\ell = \mathrm{sign}\,\ell$ hence may be used as an additional parameter. If one defines the physical space-time coordinates as $\vec{x}_{phys} = \epsilon_\ell \vec{x}$; $t_{phys} = t$ then ℓ -inversion implies a parity operation which can be used in the normal physical sense and leaves the equation strictly invariant. We however consider the PC invariance obtained in our theory as an important result.

F. Scale Invariance and Regularization:

Earlier in our discussion of the 2-point function, we introduced the two conditions on $\rho(m^2)$

$$\int \rho(m^2)\, dm^2 = 0; \quad \int m^2 \rho(m^2)\, dm^2 = 0$$

as a regularization procedure. In the study of the non-linear equation for classical anharmonic oscillator, Heisenberg used these conditions to obtain solutions which vanish on space like surfaces but are oscillatory near the light cone. We can however determine these conditions by means of scale invariance of the theory. To understand this we observe that mass of the particles can be neglected in extreme relativistic regions. Since only the presence of mass renders the theory non scale invariant, it is clear we shall have scale invariant theory on and near light cone. Further the light cone in momentum space msps into light cone in ordinary space and hence we have scale invariance also in ordinary space, near the light cone. It is to be noted that the v.e.v. of the two field operators is still not invariant under scale transformation $\psi(x) \to \eta^{1/2} \psi(\eta x)$ even though the equation of motion is invariant. However we may argue that this deviation is caused by the fact that the vacuum is not invariant, under scale transformation.

In order to have a clear distinction between the behaviour of the anticommutator and its vacuum expectation value we will assume that the anticommutator on or in the immediate neighbourhood of the light cone behaves like a C-number. However it is still possible that the behaviour of the anticommutator on the light cone is not well defined small proximity. Hence we assume

that the symmetry properties of our commutation rule can be expressed in terms of the symmetry properties of the expression

$$\lim_{z^2\to 0}\left\{\psi(x),\psi^*(x')\right\}=\lim_{z^2\to 0}\varepsilon(z)\bar{\sigma}_\nu\frac{\partial}{\partial z_\nu}f(z^2) \tag{2.160}$$

In particular $f(z^2)$ has to be an invariant with respect to all symmetry groups considered in the asymptotic region $z^2\to 0$ (the relative light cone in x-space), and hence with the new hypothesis also under scale invariance at least up to an additive constant.

We find

$$\begin{aligned}\lim_{z^2\to 0}\left\{\psi(x),\psi^*(x')\right\}&=\lim_{z^2\to 0}\text{Const}\frac{1}{\ell^2}\varepsilon(z)\,\bar{\sigma}_\nu\frac{\partial}{\partial z_\nu}\,\mathrm{Im}\ln(z^2+i\delta)/\ell^2\\&=\lim_{z^2\to 0}\text{Const}\frac{1}{\ell^2}\varepsilon(z)(\bar{\sigma}_\nu z^\nu)\,\mathrm{Im}\frac{1}{z^2+i\delta}\\&=-i\pi\,\text{Const}/\ell^2\,\varepsilon(z)(\bar{\sigma}_\nu z^\nu)\,\delta(z^2)\end{aligned} \tag{2.161}$$

with
$$\ln(z^2+i\delta)/\ell^2=+\ln|z^2/\ell^2|+i\pi\,\Theta(-z^2/\ell^2)$$

and
$$(z^2+i\varepsilon)^{-1}=P\frac{1}{z^2}-i\pi\,\delta(z^2)$$

If the scale transformation for spinor field is

$$\psi(x)\to\eta^\alpha\psi(\eta x) \tag{2.162}$$

with α not half-integer, (on the right hand side (2.161) would have non-integral power which would introduce cuts in the plane and hence destroy the uniqueness of the commutation condition. Since the physical meaning of the scale invariance of the theory is an overall change of all lengths, it is clear that this can be the case only, if the theory contains no scale except ℓ ,

which we can put equal to one arbitrarily. This implies that there are no restmasses in the theory. From (2.161) we however obtain the interesting conclusion that $\int d m^2 \rho(m^2) = 0$. since there are no δ' -function. Also we have

$$\frac{\text{Const}}{\ell^2} \sim \int d m^2 \rho(m^2) \frac{1}{m^2} \tag{2.163}$$

The scale invariance seems to make a non-renormalizable theory, logarithmic singularities are left when (2.163) does not vanish. renormalizable in the sense only, the theory is not yet made finite. The requirement of scale invariance therefore necessarily leads to a noncanonical quantization rule and necessitates the introduction of an indefinite metric. However we would prefer a condition which would make even this const = 0 in order to avoid divergences at all steps. To achieve this purpose we define a 'weak scale transformation'

$$\psi(x) \longrightarrow \eta_n^{1/2} \psi(\eta_n x) \tag{2.164}$$

which holds only for a discrete set of scale factors with ϵ some constant and $n = \ldots\ldots -1, 0, +1 \ldots\ldots$ all integers, and demand for the anticommutator only weak scale invariance on and in the immediate neighbourhood of the light cone. Under this more general condition, one finds

$$\lim_{z^2 \to 0} \{\psi(x), \psi^*(x)\} = \lim_{z^2 \to 0} \text{Const} \frac{1}{\ell^2} \varepsilon(z) \bar{\sigma}_\nu \frac{\partial}{\partial z_\nu} \left(\frac{-i}{2}\right) \left[P(\tau) - P(\tau^*)\right] \tag{2.165}$$

where $P(\tau)$, is a periodic function of the logarithmic function with a period, lets say $2\pi/\omega$

$$P\left(\tau + \frac{2\pi}{\omega}\right) = P(\tau) \tag{2.166}$$

The invariance condition then reads:

$$P(\ln z^2) = P\left(\ln \frac{z^2}{\eta^2}\right) = P(\ln z^2 - \ln \eta_n^2) = P\left(\ln z^2 + \frac{2\pi}{\omega} n\right) \tag{2.167}$$

if $$\eta_n = e^{n\pi/\omega} \quad \text{or} \quad \varepsilon = \pi/\omega \tag{2.168}$$

We may also write the anticommutator condition as

$$\lim_{z^2 \to 0} \left\{ \psi(x), \psi^*(x') \right\} = \lim_{z^2 \to 0} (-i) \frac{\text{const}}{\ell^2} \left[\frac{P'(\tau)}{z^2 + i\delta} - \frac{P'(\tau^*)}{z^2 - i\delta} \right] \tag{2.169}$$

where the derivative function $P'(\tau)$ has the same period as $P(\tau)$. This form is preferable since now the transition to the strong scale invariance can be easily made by $\lim \omega \to 0$ since then $P'(\tau) = P'(\tau^*) = \text{const}'$. This means that in order to exclude the appearance of δ-functions in the anticommutator connected with the condition $\int dm^2 \rho(m^2)\, m^2 = 0$ we have to require $\omega \neq 0$. For $\omega \neq 0$, however, we have an essential singularity on the light cone; if we approach the light cone from finite z^2 the anticommutator as a function of z^2 will oscillate with increasing frequency $\omega \ln \ell^2/z^2$. The function is ill defined on the light cone itself. In a certain way the length parameter ℓ enters here for the first time in a non-trivial way. It may be that the scale of physical theories is already established here i.e., by the transition from strong to only weak scale invariance requirements. This has to be further investigated. We may expand $P'(\tau)$ in a Fourier series

$$P'(\tau) = \sum_n b_n e^{i n \omega \tau}$$

and hence

$$\lim_{z^2\to 0}\{\psi(x),\psi^*(x')\} = \lim_{z^2\to 0}(-i)\frac{\text{Const}}{\ell^2}\varepsilon(z)(\bar{\sigma}_\nu z^\nu)\sum_n b_n\left[\frac{e^{in\omega\tau}}{z^2+i\delta} - \frac{e^{in\omega\tau}}{z^2-i\delta}\right]$$

$$= \lim_{z^2\to 0}(-2)\frac{\text{Const}}{\ell^2}\varepsilon(z)(\bar{\sigma}_\nu z^\nu)\theta(-z^2)\sum_n b_n e^{in\omega\ln|z|^2}$$

$$\times\left[\pi\,\delta(z^2)\cosh n\omega\pi - i\,P\left(\frac{1}{z^2}\right)\sinh(n\omega\pi)\right] \qquad (2.170)$$

We see that for $\omega \neq 0$ the $\delta(z^2)$ will be without effect $P(\tau)$ should not contain a constant part since it is multiplied by the oscillating factor $e^{in\omega\ln|z^2|}$. We may write

$$\lim_{z^2\to 0}\delta(z^2)\frac{\text{Const}}{\ell^2}\sum_n b_n\cosh(n\omega\pi)e^{in\omega\ln|z^2|}$$

$$= b_0/\ell^2\,\delta(z^2) = 0 \qquad (2.171)$$

However, the zero is not quite an 'honest zero' due to the infinite oscillations. In fact, the behaviour near the light cone is exactly of the type as derived by Heisenberg earlier in connection with the solutions of the classical nonlinear spinor equations. For all practical calculations in which we integrate over the z-coordinates in some fashion we may essentially state

$$\int (dm^2)\,\rho(m^2)\,m^2 = 0. \qquad (2.172)$$

A closer investigation by Mitter has shown that the above functions are still tempered distributions. The Fourier transforms are, in fact, readily given in the book by Gelfand-Schilov. Since $\rho(m^2)$ is essentially the Fourier transform of the Δ function, i.e. $\rho(-p^2) = \Delta(-p^2)$ needed the

behaviour near the light cone in $\mathbb{C}$- space can be easily derived from

$$\lim_{z^2 \to 0} \Delta_1(z^2) = \frac{\text{Const}}{12} \left[P(\tau) + P(\tau^*) \right]$$

The $\rho(m^2)$ turns out to be also a periodic function in such a way that the condition $\int dm^2 \rho(m^2) = 0$ is only fulfilled in some 'Abel' sense. The details of these considerations will be given in a forthcoming paper by Mitter.

Scale invariance of the theory restricts the functional form of vacuum expectation values of field operators whenever two coordinates approach each other. It is possible that the scale invariance may be connected with Reggeisation as both of these methods attempt to eliminate elementary particles.

III Approximation Methods.

In this part we shall develop some methods which can be used to derive approximate linear equations for certain matrix elements. From the solutions of these equations the mass eigenvalues of the simplest systems can be deduced. We shall discuss only one method i.e. the 'New Tamm-Dancoff Method'. The validity of the NTD-method is not restricted to weak coupling as it does not involve any expansion in terms of coupling parameter. The NTD-method has been applied successfully to the pure anharmonic oscillator, the Leemodel and the BCS-model of superconductivity, leading to exact solutions in the last two cases. It is known that for the BCS theory perturbation theory fails completely, and the Bethe-Salpeter approximation gives the wrong result. A closer investigation reveals that the Leemodel and the BCS-model are especially suited for the NTD-approximation. The anharmonic oscillator case is perhaps a better example to serve as a guide to the more complicated field theoretical situations.

Sekine has developed the first step which may eventually lead to a variational approximation method. Yamazaki is looking into the possibility of using generalisations of the 'one level approximation method and variational method which work very well in the case of the anharmonic oscillator. In addition efforts are made at present to cultivate the NTD-method further with the use of generalized contraction functions, the - functions, introduced by Symanzik and Zimmermann.

In section A we shall show that the differential equation for the field operator $\psi(x)$. leads to an infinite linear system of coupled differential or integral equations for matrix elements which up to a certain degree may be called wave-functions in configuration space. In section B we shall discuss how this infinite system may be approximated by a finite system using the New-Tamm-Dancoff method which defines for us an eigenvalue problem. In section C we shall apply this method to the anharmonic oscillator.

A. The differential and integral equations for the 'Wave functions'

The differential equation for the field operator

$$-i\sigma_\nu \frac{\partial}{\partial x_\nu}\psi(x) + l^2 : (\sigma)(\sigma)\psi\left(\psi^*\psi\right) = 0 \qquad (3.1)$$

where $(\sigma)\,(\sigma)$ may be either interpreted as $\sigma_\mu \sigma^\mu$ or $\frac{1}{4}(3II + \vec{\tau}\vec{\tau})\sigma_\mu\sigma^\mu$. On using (3.1) we can obtain a connection between the matrix element $\langle A|\psi(x_1)\cdots\psi(x_n)\psi^*(x_{n+1})\cdots\psi^*(x_{n+m})|B\rangle$ of $n\psi$ and $m\psi^*$ field operators between the Heisenberg states A, B and matrix element with $(n+1)\psi$ and $(m+1)\psi^*$ operators between the same states. In fact, the differential equation can be considered simply as the generator of this linear system of differential equations for the infinite number of matrix elements $\langle A|\cdots\cdots|B\rangle$ of various numbers of operators.

For our purpose, we shall consider in particular the matrix elements where A refers to the physical vacuum state and B any physical state. Matrix elements of this type are called τ_B -functions. If B is also the physical vacuum state, the corresponding -function will be called τ_0 -functions.

$$\tau_B\left(\begin{matrix} x_1 x_2 \dots x_n \\ \alpha_1 \alpha_2 \dots \alpha_n \end{matrix} \middle| \begin{matrix} x_1' \dots x_m' \\ \beta_1 \dots \beta_m \end{matrix}\right)$$

$$= \left\langle 0 \middle| T\, \psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\dots\psi_{\alpha_n}(x_n) \middle| \psi^*_{\beta_1}(x_1') \dots \psi^*_{\beta_m}(x_m') \middle| B \right\rangle \quad (3.2)$$

The index B will be often dropped, except if B corresponds to the vacuum. The τ-functions are completely antisymmetrized in ψ and ψ^* due to the T-product. The T-product is relativistically invariant and implies that the τ -functions become equal to each other. The total set of τ -functions $\{\tau_B\}$ built from arbitrary numbers of field operators characterises the state B completely.

The τ -functions can be related to another set of functions, the φ -functions $\{\varphi_B\}$ by the formal application of Wick's rule

$$\tau\left(\begin{matrix} x_1 \dots x_n \\ \alpha_1 \dots \alpha_n \end{matrix} \middle| \begin{matrix} x_1' \dots x_m' \\ \beta_1 \dots \beta_m \end{matrix}\right) = \varphi\left(\begin{matrix} x_1 \dots x_n \\ \alpha_1 \dots \alpha_n \end{matrix} \middle| \begin{matrix} x_1' \dots x_m' \\ \beta_1' \dots \beta_m \end{matrix}\right)$$

$$+\sum(-)_{iR}\,\tau_0\begin{pmatrix}x_i\\ \alpha_i\end{pmatrix}\begin{pmatrix}x_k'\\ \beta_k\end{pmatrix}\varphi\begin{pmatrix}x_1 \cdots \overset{i}{|} \cdots x_n\\ \alpha_1 \quad\; i \quad\; \alpha_n\end{pmatrix}\begin{pmatrix}x_1' \cdots \overset{k}{|} \cdots x_m'\\ \beta_1' \cdots r \cdots \beta_m\end{pmatrix}$$

$$+\sum\sum(-)_{ik,rs}\,\tau_0\begin{pmatrix}x_i\\ \alpha_i\end{pmatrix}\begin{pmatrix}x_k'\\ \beta_k\end{pmatrix}\tau_0\begin{pmatrix}\alpha_r\\ \alpha_r\end{pmatrix}\begin{pmatrix}x_s'\\ \beta_s\end{pmatrix}\varphi\begin{pmatrix}x_1 \cdots i \cdots r \cdots x_n\\ \alpha_1 \cdots k \cdots s \cdots \alpha_n\end{pmatrix}\begin{pmatrix}x_1' \cdots k \cdots s \cdots x_m'\\ \beta_1 \cdots r \cdots \beta_m\end{pmatrix}$$

$$+ \cdots\cdots \qquad (3.3)$$

where $\tau_0(x_i|x_i')$ are the contraction functions, i.e. the physical vacuum expectation values for the T-ordered field operatofs

$$\tau_0(x_i|x_i') = \langle 0|T\left(\psi(x_i)\,\psi^*(x_i')\right)|0\rangle \qquad (3.4)$$

The φ -functions are again antisymmetrical in all ψ's and ψ^*'s.

$\varphi(x_1 \cdots \overset{i}{|} \cdots x_n | x_1' \cdots \overset{k}{|} \cdots x_n')$ shall indicate that the x_i and x_k' coordinate do not appear any more. The sums go over all possible pairings of $\psi\psi^*$ and double pairings etc., until all the operators are contracted. The sign $(-)_{iR}$ in (3.3) is + or - depending whether it take an even or odd perumation to bring x_i and x_k' to the middle bar. Also the set of φ -functions $\{\varphi_B\}$ characterizes the state B completely.

Unlike the φ_0 functions which occur in the interaction representation, the φ_0 functions here do not vanish. In the following we will denote φ -functions as matrix elements of a generalized Wick product which we indicate by

$$\varphi_B\begin{pmatrix} x_1 \cdots x_n \\ \alpha_1 \quad \alpha_n \end{pmatrix}\begin{matrix} x_1' \cdots x_{n'}' \\ \beta_1 \cdots \beta_m \end{matrix}$$

$$= \langle 0 | : \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_n}(x_n)\, \psi_{\beta}(x_1') \cdots \psi_{\beta_m}(x_{n'}') : | B \rangle \qquad (3.5)$$

The $:\ :$ product in the differential equation shall be ~~exactly~~ understood in this sense, and we shall call these the wave functions. The φ -functions are less singular than the τ -functions, since the singularities which arise when two points coincide have been removed in the form 2-point functions $\tau_0(x|x')$. The φ's however will still be singular when 4 or more coordinates coincide. We may also define another set of functions $\lambda(x_1 \cdots x_n | x_1' \cdots x_{n'}')$ which are related to the τ -functions by suing not only 2-point contractions, but 4-point, 6-point etc. contractions in such a way that the vacuum λ -functions all vanish. The generalised contraction functions are called η -functions by Symanzik and Zimmermann. The λ -functions produced in this way may be wave atply called wave functions.

The differential equation for the field operator $\psi(x)$ allows us to establish a relationship between τ -functions, and also between φ -functions. We have e.g.

$$-i(\sigma_\nu)_{\alpha_n\alpha_n}\frac{\partial}{\partial x_n}\Upsilon\left(\begin{matrix}x_1\cdots x_n\\ \alpha_1\cdots\alpha_n\end{matrix}\middle|\begin{matrix}x_1'\cdots x_m'\\ \beta_1\cdots\beta_m\end{matrix}\right)$$

$$= l^2(\sigma)_{\alpha_n\alpha_2}(\sigma)_{\varepsilon\delta}\Upsilon\left(\begin{matrix}x_1\cdots x_n x_n\\ \alpha_1\cdots\alpha_n\ \delta\end{matrix}\middle|\begin{matrix}x_n\ x_1'\cdots x_m'\\ \varepsilon\ \beta_1\cdots\beta_m\end{matrix}\right)$$

$$+\text{Const}\ Z_2^{-1}\sum_{i=1}^{m}\delta(x_n-x_i')\Upsilon\left(\begin{matrix}x_1\cdots x_{n-1}\\ \end{matrix}\middle|\begin{matrix}x_1'\cdots\overset{i}{\not}\cdots x_m'\\ \end{matrix}\right)$$

(3.6)

where the 4-dimensional, δ -functions on the right result from the differentiation of the $\theta(t)$ functions in the definition of the time-ordered product. This derivative will give a $\delta(t_n-t_i')$ however multiplied by the equal time anti-commutator $\{\psi(x_n)\psi(x_i')\}_{t_n=t_i}$ where

$$\{\psi(x_n)\psi(x_i')\}_{t_n=t_i} = Z_2^{-1}\delta(\vec{r}_n-\vec{r}_i')$$

$$\text{with}\quad Z_2^{-1} = \int dm^2\,\rho(m^2). \tag{3.7}$$

Since $Z_2^{-1}=0$ in our theory, these terms disappear. In the Υ -function on the r.h.s. it should be remarked that the ψ's at the same space time point x_n are Wick ordered (see : : in the differential equation) which we have not indicated. We will incorporate this property in the rule that in the transition from the Υ -functions to be G -functions all contractions between the same coordinates (here x_n) shall be omitted.

(3.7) defines an infinite system of coupled differential equations which is linear in the Γ -functions. In order to terminate this infinite system and to obtain an eigenvalue problem, we have to impose same boundary condition on this system e.g. that $\varphi_B(N)$ are small for $N > \hat{N}$

It may well be that the ambiguity in closing the infinite system by a boundary condition is reflected in the ambiguity in selecting and defining the physical vacuum state.

We shall choose the φ -functions as defined by Wick's rule as the functions which become small for higher number of variables. This approximation method is known as the New-Tamm-Dancoff-approximation. (The 'New' indicates that all our functions are built with the physical, and not the bare, vacuum state) The φ -functions which can be deduced from the Γ-functions again obey an infinite system of coupled differential equations which however will involve the contraction functions $\Gamma_0(x|x')$ To deal with bound state configurations, it is more convenient to consider the system of coupled integral equations for the Γ -functions or the φ -functions which are obtained by integrating the differential equations with the Green's function $G(x-y)$ obeying

$$\sigma_\nu \frac{\partial}{\partial x_\nu} G(x-y) = -\delta(x-y) \qquad (3,8)$$

With Feynman boundary conditions. Then we can write

$$\tau_B\left(\begin{matrix} x_1 \cdots x_n \\ \alpha_1 \cdots \alpha_n \end{matrix} \middle| \begin{matrix} x_1' \cdots x_m' \\ \beta_1 \cdots \beta_m \end{matrix}\right)$$

$$= (-il^2)\int d^4y\, G_{\alpha_n \gamma}(x_n - y)(\sigma)_{\gamma k}(\sigma)_{\delta \varepsilon} \times$$

$$\tau_B\left(\begin{matrix} x_1 \cdots x_{n-1}\, y\, y \\ \alpha_1 \quad \alpha_{n-1}\, k\, \varepsilon \end{matrix} \middle| \begin{matrix} y\, x_1' \cdots x_m' \\ \delta\, \beta_1 \cdots \beta_m' \end{matrix}\right)$$

(3.9)

for arbitrary states B which are not neutrino states. For firmion states of mass zero, there is an inhomogeneous term All particles, except the neutrino have the character of "bound state solutions" and follow as solutions of homogeneous integral equations. The integral equation approach has the advantage that our mathematically ill defined functions, occur always under the integral sign and hence they become manageable. e.g. oscillatory functions of our type with infinite frequency near the lightcone are effectively equal to zero.

B) The New-Tamm-Dancoff-approximation method

The NTD-approximation is defined by the following procedure:

The nonlinear differential equations for the field lead to a differential connection between a τ-function of N operator variables, i.e. $\tau(N)$ and a τ-function of $N+2$ variables. This relationship can also be expressed in integral form (symbolically)

$$\tau(N) = \int G\, \tau(N+2)$$

An additional application of the differential equation - we will call this procedure 'iteration' - leads to Υ -function with $N+4$, $N+6 \cdots N+2n$ variables.

We express all Υ -functions with Wick's rule in terms of φ -functions, i.e.

$$\Upsilon(N) \longrightarrow \text{linear}(\varphi_N, \varphi(N-2) \cdots \varphi(0))$$
$$\Upsilon(N+2) \rightarrow \text{linear}(\varphi(N+2), \cdots \varphi(0))$$
$$\vdots$$
$$\Upsilon(N+2n) \rightarrow \text{linear}(\varphi(N+2n) \cdots \varphi(0))$$

(3.10)

and close the system by the conditions $\varphi(N+2n) = 0$. We then obtain a finite linear system of coupled integral equations for the wave function $\{\varphi(N)\}$ which defines the eigenvalue problem. The integral kernels are constructed from G and $F(x, x') = \Upsilon_0(x|x')$ (renormalized) contraction function. For actually solving the equations we have therefore to know $F(x-x')$.

The first step in our actual calculations hence will be to derive the approximate form of $F = \Upsilon_0$ itself using the NTD-method for Υ_0 itself. This problem however, will be non-linear since $F(x-x')$ occurs as "wave functions" as well as kernel. This procedure will be discussed at a later point. We will assume for the present discussion that $F(x-x')$ is a known function.

The above prescription can be also stated in graphical form. We introduce the notation

$G(x-x') \equiv$  ; $F(x-x') \equiv$ 

The differential equation leads to vertices of the form  and φ -functions with the n variables written like   Let us for the sake of illustration investigate the system we obtain for a fermion state for which we set $\varphi(5) = 0$ and higher.

We now have,

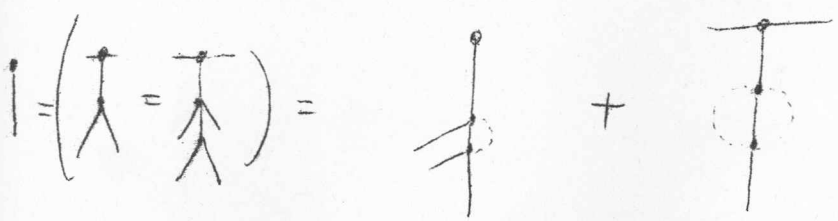
$$T(1) = \varphi(1) = \int G\, T(3) = \iint GG\, T(5) = \iint GG\, [\varphi_0^{(5)} + F\varphi(3) + FF\varphi(1)] \tag{3.11}$$

$$T(3) = \varphi(3) + F\varphi(1) = \int G\, T(5) = \int G[\varphi_0^{(5)} + F\varphi(3) + FF\varphi(1)] \tag{3.12}$$

$$\varphi(1) = \iint [GGF\varphi(3) + GGFF\varphi(1)] \tag{3.13}$$

$$\varphi(3) + F\varphi(1) = \int [GF\varphi(3) + GFF\varphi(1)] \tag{3.14}$$

We can represent (3.13) graphically



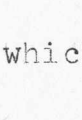
In the graphs in brackets the line $|$ denotes a T -function part instead of a φ -function part and are only used here to demonstrate the intermediate steps.

The system of equations established by this approximation method in most cases is still too complicated and further simplifications are used to get a manageable system.

To simplyfy further we neglect not only the function $\varphi(N+2n)$ but also all lower φ -functions down to $\varphi(N+2)$ if one has started the system with $T(N)$. Such an approximation, of course is only meaningful if the function at the starting point $T_B(N)$ at least contains as many variables as particles which constitute the state B. We shall call this approximation the modified NTD-approximation, or MNTD-approximation. Doubtlessly this approximation is much less accurate but may still be sufficient for a more qualitative investigation of the solutions.

For the Fermion state discussed above, in this approximation we have only to consider the equation:

$$\varphi(1) = \int\int G\,G\,F\,F\,\varphi(1) \qquad (3.15)$$

graphically  which is a linear integral equation for the fermion wave function with a Kernel CGFF. The solutions of this equation will lead to the nucleon mass eigenvalue.

In the MNTD-approximation we have the following simple rules for a general graph.

1) At every point of a graph, except the end-points, two lines enter and two lines leave. 

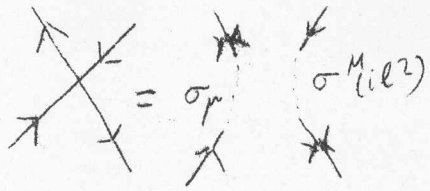
In the Hermitean representation we have the simpler rule that four lines meet in every point (except the endpoints)

2) There is an equal number of G- and F-lines connecting all points.

3) Each point is connected to one of the endpoints by one (and only one) G-line

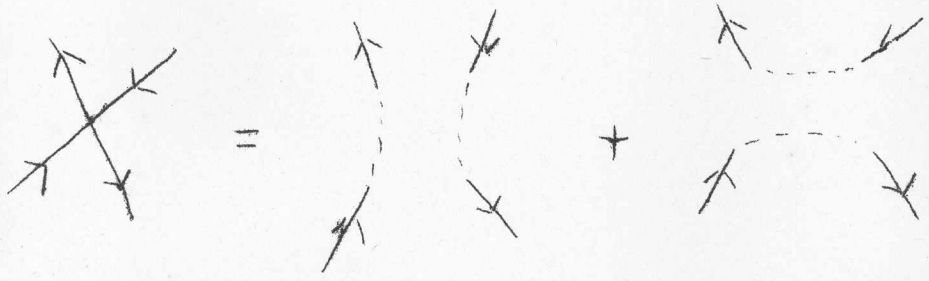
4) The number of points (excluding endpoints) indicate the number of iteractions.

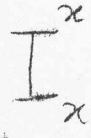
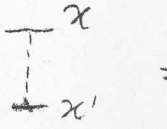
5) For each point the factor $-i\ell^2(\sigma_\mu)(\sigma^\mu)$ has to to be included in the integral kernels.



. However when we use the Fierz symmetric interaction we have instead

$(-i\ell^2)\frac{1}{4}(3II+\vec{\gamma}\vec{\gamma})\sigma_\mu\sigma^\mu$ and has the freedom to connect the fermion lines in either way, i.e.



6) Each G-line  represents $G_1(x-x')$, each F-line  $= F(x-x')$.

7) The Kernel of the integral equation is determined, if the points of the graphs, the directions of the lines at the endpoints, and the connecting G-lines with their directions are

given. The F-lines then have to be inserted in all possible ways. The integral Kernel is the sum of all the resulting graphs.

In our derivation of the fermion and boson eigenvalue equation we will consider special examples of these rules.

In concluding this section it should be remarked that the above graphs should not be confused with Feynman graphs which are related to S-matrix elements.

C. The anharmonic Oscillator.

The anharmonic oscillator will be discussed as a very simple example in which the NTD-method was applied for the derivation of the lowest eigenvalues. Since also the exact solutions can be derived by numerical methods the error of the approximation can be judged. The calculations given here are due to Heisenberg in 1953. We consider the following equation of motion for an anharmonic oscillator

$$\ddot{q} = -\omega_0^2 q - \lambda q^3 \tag{3.15}$$

This can be solved by perturbation theory (Heisenberg 1925) if λ is sufficiently small, i.e. if the dimensionless constant

$$\varepsilon = \frac{2\lambda a^2}{\omega_0^2} \ll 1 \tag{3.16}$$

where $$a^2 = \langle 0 | q^2 | 0 \rangle = \frac{1}{2\omega_0 m} \tag{3.17}$$

for the harmonic oscillator, and we find for $\varepsilon^2 = 0$

$$\langle n|q(t)|n-1\rangle = a\sqrt{n}\left(1 - \frac{3\varepsilon}{8}n - \cdots\right) e^{i\omega_{n,n-1}t}$$

$$\langle n|q(t)|n-3\rangle = \frac{1}{16}a\varepsilon\sqrt{n(n-1)(n-2)}\left(1 - \frac{3q\varepsilon}{16}(n-1) + \cdots\right) \times e^{i\omega_{n,n-3}t}.$$

$$E_n = (n+\tfrac{1}{2})\omega_0 + \frac{3}{8}\varepsilon\omega_0\left(n^2+n+\tfrac{1}{2}\right) \tag{3.18}$$

$$\omega_{n,n-1} = \omega_0\left(1 + \frac{3\varepsilon}{4}n\right) \tag{3.19}$$

(3.20)

We will try now the NTD-method, and introduce the τ -functions

$$\tau(t_1 \cdots t_n | t_1' \cdots t_n') = \langle 0|T\left(q(t_1)\cdots q(t_n)p(t_1')\cdots p(t_m')\right)|\beta\rangle \tag{3.21}$$

If two times become exactly equal we will define it as the symmetrized product i.e.

$$T\left(q(t_1)q(t_2)\right) = \theta(t_1-t_2)q(t_1)q(t_2) + \theta(t_2-t_1)q(t_2)q(t_1) \tag{3.22}$$

with

$$\theta(t) = \begin{cases} +1 & t>0 \\ \tfrac{1}{2} & t=0 \\ 0 & t<0 \end{cases} \tag{3.23}$$

Consider

$$\tau_0(t_1,t_2|\,) = \langle 0|T\left(q(t_1)q(t_2)\right)|0\rangle \tag{3.24}$$

$$\frac{d^2}{dt_1^2}\tau_0(t_1,t_2|) = -\omega_0^2\tau_0(t_1,t_2|) - \lambda\tau_0(t_1,t_1,t_1,t_2|) - \frac{i}{m}\delta(t_1-t_2) \tag{3.25}$$

The latter δ -function arises from the discontinuity

$$\frac{d}{dt_1}[\theta(t_1-t_2)q(t_1)q(t_2)+\theta(t_2-t_1)q(t_2)q(t_1)]$$
$$=\delta(t_1-t_2)q(t_1)q(t_2)-\delta(t_1-t_2)q(t_2)q(t_1)$$
$$+\theta(t_1-t_2)\dot{q}(t_1)q(t_2)+\theta(t_2-t_1)q(t_2)\dot{q}(t_1)$$

$$\frac{d^2}{dt_1^2}T(q(t_1)q(t_2)) = \delta(t_1-t_2)[\dot{q}(t_1)q(t_2)-\dot{q}(t_2)q(t_1)] + \ddot{q}(t_1)q(t_2)+q(t_2)\ddot{q}(t_1). \tag{3.26}$$

$$= \frac{1}{m}\delta(t_1-t_2)[p,q] = -\frac{\hbar}{i} \tag{3.27}$$

Since $p = m\dot{q}$

We define now the φ -functions by the Wick's rule:

$$T(t_1\cdots t_n|t_1'\cdots t_m') = \varphi(t_1\cdots t_n|t_1'\cdots t_m')$$
$$+\tau_0(t_1t_2|)\varphi(t_3\cdots t_n|t_1'\cdots t_m')+\cdots+$$
$$+\tau_0(t_1|t_1')\varphi(t_2\cdots t_n|t_2'\cdots t_m')+\cdots$$
$$+\tau_0(|t_1't_2')\varphi(t_1\cdots t_n|t_3'\cdots t_m')+\cdots$$

$$+ \quad \text{double contractions etc.} \tag{3.28}$$

We can also express the φ -functions in terms of the τ -functions

$$\varphi(t_1 \cdots t_n | t_1' \cdots t_m') = \tau(t_1 \cdots t_n | t_1' \cdots t_m')$$
$$- \tau_0(t_1 t_2 |)\, \tau(t_3 \cdots t_n | t_1' \cdots t_m')$$
$$- \tau_0(t_1 | t_1')\, \tau(t_2 \cdots t_n | t_2' \cdots t_m')$$
$$- \tau_0(| t_1' t_2')\, \tau(t_1 \cdots t_n | t_3' \cdots t_m')$$
$$+ \text{double Contractions etc.}$$

double contractions etc. (3,29)

The only difference is that the contraction functions are replaced by their negative (Compare Freese and Wick).

We have here all the contraction functions, i.e. also between two q's and two p's . The invariant functions $\tau_0(t_1 t_2 |) = \frac{1}{2} \Delta_F(t_1 - t_2)$, $\frac{1}{2}\Delta_1$, Δ, and $\bar{\Delta}$ can be approximately calculated in perturbation theory

$$\tau_0(t_1 t_2 |) = \tfrac{1}{2}\Delta_F(t_1 - t_2) = \langle 0 | T(q(t_1) q(t_2)) | 0 \rangle = a^2 \left(1 - \tfrac{3\varepsilon}{4}\right) e^{-i\omega_{10}|t_1 - t_2|} \tag{3.30}$$

$$\tfrac{1}{2}\Delta_1(t_1 - t_2) = \langle 0 | \tfrac{1}{2}\{q(t_1) q(t_2)\} | 0 \rangle = a^2 \left(1 - \tfrac{3\varepsilon}{4}\right) \cos \omega_{10}(t_1 - t_2) \tag{3.31}$$

$$\tfrac{1}{2}\Delta(t_1 - t_2) = \langle 0 | \tfrac{1}{2}[q(t_1), q(t_2)] | 0 \rangle = -a^2 \left(1 - \tfrac{3\varepsilon}{4}\right) \sin \omega_{10}(t_1 - t_2) \tag{3.32}$$

$$\tfrac{1}{2}\bar{\Delta}(t_1 - t_2) = a^2 \left(1 - \tfrac{3\varepsilon}{4}\right) \sin \omega_0 (t_1 - t_2) \tag{3.33}$$

The $\tau_0(t_1|t_2)$ and $\tau_0(|t_1\ t_2)$ can be obtained by differentiation with respect to t_2 and using the condition $\dot{q}(t_2) = \frac{1}{m} p(t_2)$. For the actual calculation it is practical to investigate the matrix elements only for equal time since this is sufficient for the determination of the eigenvalue

For equal-time we may write

$$\tau_B(t_1 \cdots t_k \,|\, t_1' \cdots t_\ell') = \tau_B(k|\ell) e^{i\omega t} \quad \text{with } t_1 = t_2 = \cdots = t \tag{3.34}$$

where $\tau(k|n-k)$ is now simply a constant and ω the frequen difference between the state B and the groundstate.

In a differentiation with respect to time now the derivatives of all the various q and p have to be taken, i.e. $\frac{d}{dt} = \lim_{t_1 = t_2 = \cdots = t} \left(\frac{d}{dt_1} + \frac{d}{dt_2} + \cdots\right)$. If we write the equation of motio in the form

$$\dot{p} = -m\omega_0^2 q - \lambda m q^3 \tag{3.35}$$

with

$$\dot{q} = \frac{1}{m} p \tag{3.36}$$

we find

$$i\omega\tau(k|\ell) = \frac{1}{m} k\tau(k-1|\ell+1) - m\omega_0^2 \ell \tau(k+1|\ell-1)$$
$$-m\lambda\ell\tau(k+3|\ell-1) + \frac{m\lambda}{4}\ell(\ell-1)(\ell-2)\tau(k+1|\ell-3) \tag{3.37}$$

The NTD-approximation is now characterized by the condition

$$\varphi(k|\ell) \cong 0 \quad \text{for } k+\ell < N.$$

The first approximation for N = 1 functions will be $\varphi(3) = 0$

$$i\omega\, \tau(1|\,) = \frac{1}{m}\, \tau(1|\,) \tag{3.38}$$

$$\frac{i\omega}{m}\, \tau(\,|1) = -\omega_0^2 \tau(1|\,) - \lambda \tau(3|\,) \tag{3.39}$$

$$\varphi(3|\,) = \tau(3|\,) - 3\Delta\, \tau(1|\,) = 0 \tag{3.40}$$

The factor 3 occurs in the latter since we can contract the q^3 in three different ways. If we introduce the time contraction function:

$$\tau_0(t t'|\,) = \Delta;\quad \tau_0(t'|t t) = \Gamma;\quad \tau_0(t|t) = 0 \tag{3.41}$$

Hence we can write

$$\begin{pmatrix} -i\omega & -\frac{1}{m} \\ m\omega_0^2 + 3\lambda m\Delta & i\omega \end{pmatrix} \begin{pmatrix} \tau(1|\,) \\ \tau(\,|1) \end{pmatrix} = 0 \tag{3.42}$$

determinant $= 0$ leads to $\omega^2 = \omega_0^2 + 3\Delta\lambda$ (3.43)

If we are only interested in the lowest approximation in λ, we may simply take for Δ (and Γ) the values for the harmonic oscillator i.e. the unperturbed propagators. (In quantum field theory the approximation method in which the renormalized contraction function is replaced by the unrenormalized substraction function, based on the bare vacuum, is the old and original Tamm-Dancoff-method).

i.e.

$$\Delta \simeq \Delta_{free} = \frac{1}{2m\omega_0} \; ; \; \Gamma = \Gamma_{free} = \frac{1}{2} m \omega_0 \tag{3.44}$$

and then gets the perturbation result in lowest order:

$$\omega = \pm \omega_0 \left(1 + \frac{3}{4} \varepsilon \right) \tag{3.45}$$

In the next higher approximation $\varphi(k|\ell) = 0$, $k+\ell > 5$ we have the following conditions

$$\varphi(5|\,) = \tau(5|\,) - 10\Delta\, \tau(3|\,) + 15\Delta^2\, \tau(1|\,) = 0 \tag{3.46}$$

$$\varphi(4|1) = \tau(4|\,) - 6\Delta\, \tau(2|1) + 3\Delta^2\, \tau(1|\,) = 0 \tag{3.47}$$

$$\varphi(3|2) = \tau(3|2) - 3\Delta\, \tau(1|2) - \Gamma\, \tau(3|\,) + 3\Delta\Gamma\, \tau(1|\,) = 0 \tag{3.48}$$

We use these equations to eliminate $\tau(5|\,)$, $\tau(4|1)$, $\tau(3|2)$
We
One obtains the following system:

$$\begin{pmatrix}
-i\omega & -1 & & & & \\
\omega_0^2 & +i\omega & \lambda & & & \\
0 & 0 & i\omega & -3 & & \\
-15\lambda\Delta^2 & 0 & \omega_0^2 + 10\lambda\Delta & i\omega & -2 & \\
0 & -6\lambda\Delta^2 & 0 & 2\omega_0^2 + 12\lambda\Delta & i\omega & -1 \\
\frac{-3\lambda}{2m^2} - \frac{9\lambda\Delta\Gamma}{m^2} & 0 & \frac{3\lambda\Gamma}{m^2} & 0 & 3\omega_0^2 + 9\lambda\Delta & i\omega
\end{pmatrix}
\begin{pmatrix}
\tau(1|\,) \\
\frac{1}{m}\tau(1|\,) \\
\tau(3|\,) \\
\frac{1}{m}\tau(2|1) \\
\frac{1}{m}\tau(1|2) \\
\frac{1}{m^3}\tau(1|3)
\end{pmatrix} = 0 \tag{3.49}$$

If we are again only interested in lowest order in λ , we can solve this system easily by replacing $\Delta = \Delta_{free}$; $\Gamma = \Gamma_{free}$ we then obtain the six roots:

$$\omega = \pm \omega_0 \left(1 + \frac{3}{4}\varepsilon \right)$$

$$\omega = \pm \omega_0 \left(1 + \frac{3}{2}\varepsilon \right)$$

$$\omega = \pm 3\omega_0 \left(1 + \frac{3}{2}\varepsilon \right)$$

(3.50)

which correspond to the frequencies ω_{10} , ω_{21} and ω_{30} in the perturbation treatment.

If we wish to obtain the results in higher order of λ (or ε) then probably the replacement $\Delta \simeq \Delta_{free}$, $\Gamma \simeq \Gamma_{free}$ is not valid. In the example given here we immediately realize that a boundary condition $\Upsilon(k|\ell) = 0$ for $k+\ell \geq N$ would never lead to a valid approximation scheme since the $\Upsilon(k|\ell)$ represents the expectation value of $q^k q^\ell$ which never becomes small.

In the case where we have only q -coordinates i.e. $\Upsilon(k|\,)$ the transition from the Υ -system to the φ -system essentially means the transition from expectation values of q^k to the Hermite polynomials which may lead to an approximation of the lowest eigenstates.

The examples given up to now still essentially resemble the ordinary perturbation theory since we heavily relied on the fact that was small. If we, however, now make the transition to the extreme anharmonic case

$$\ddot{q} = -\lambda q^3 \tag{3.51}$$

We cannot, however, in this
which resembles more the situation in the nonlinear theory as there does not exist a small parameter any more which could be used for an expansion in perturbation treatment. A change of λ would mean a rescaling of the equation which is without consequences.

If we use the first approximation we find

$$\omega^2 = 3\lambda \Delta \tag{3.52}$$

We cannot, however, in this case replace $\Delta \simeq \Delta_{free}$ but have to find another method in determining the contraction function. The situation is here very similar to the nonlinear spinor theory. We obtain an estimate for Δ in the following manner by introducing the full set of intermediate states.

$$\Delta = T_0(t\,t|\,) = \sum_k |T_k(t)|^2 \tag{3.53}$$

Using $[q, p] = i$ we have

$$\sum_k 2\omega_{k0} |T_k(t)|^2 = \frac{1}{m} = \left(\frac{\hbar}{m}\right) \tag{3.54}$$

If we assume that there exists a lowest intermediate state and that the transition $0 \to 1$ will dominate all others, (one level approximation), we can set

$$\Delta \simeq |T_1(t|\,)|^2 \tag{3.55}$$

and $$2\omega_{10}\Delta \simeq \frac{1}{m} \quad \text{or} \quad \Delta \simeq \frac{1}{2m\omega} \tag{3.56}$$

where $\omega = \omega_{10}$ must be the solution of the first approximation. It follows from the condition $\omega^2 = 3\lambda\Delta$ that

$$\omega = \omega_{10} \simeq \sqrt[3]{\frac{3}{2}\frac{\lambda}{m}} = 1.14\sqrt[3]{\frac{\lambda}{m}} \tag{3.57}$$

and

$$\Delta = \sqrt[3]{\frac{1}{12m^2\lambda}} \; ; \; \Gamma = m^2\omega^2\Delta \tag{3.58}$$

In the next higher approximation where we again neglect $\varphi(15)$ etc., we get the same determinant as before but with $\omega_0 = 0$ In the evaluation of this determinant, however, we are not allowed to use any expansion procedure, nor to replace Δ and Γ by the free values. A more elaborate procedure has to be used which starts from the Δ, Γ values of the first approximation and attempts an estimate for the states $T_3(t|)$ and ω_{03} to get a better approximation for Δ and Γ using again the sum rule. The correction is only of the order of a few percent, and hence is irrelevant. If we simply use the values of Δ and Γ of the first approximation, the vanishing of the determinant leads to the equation

$$\eta^6 - 21\eta^4 + 45\eta^2 - 25 = 0 \tag{3.59}$$

with

$$\omega = \eta\sqrt[3]{\frac{3}{2}\frac{\lambda}{m}} \tag{3.60}$$

The solutions are

$$\eta^2 = 1, \quad \eta^2 = 10 \pm \sqrt{75} \tag{3.61}$$

which corresponds to the following three ω values

$$\omega_{10} = 1.14\sqrt[3]{\frac{\lambda}{m}} \; ; \; \omega_{21} = 1.32\sqrt[3]{\frac{\lambda}{m}} \; ; \; \omega_{30} = 4.95\sqrt[3]{\frac{\lambda}{m}} \quad (3.62)$$

The corrected values of Δ and Γ are now obtained by using the sum rule:

$$\Delta = \frac{1}{2m\omega_{10}}\left(1 - 0.0071\right) \quad (3.63)$$

$$\Gamma = \frac{1}{2}m\omega_{10}\left(1 + 0.0292\right) \quad (3.64)$$

We may use the improved values and solve the determinant equation again which gives

$$\omega_{10} = 1.15\sqrt[3]{\frac{\lambda}{m}} \; ; \; \omega_{21} = 1.31\sqrt[3]{\frac{\lambda}{m}} \; ; \; \omega_{30} = 4.89\sqrt[3]{\frac{\lambda}{m}} \quad (3.65)$$

and again recalculate Δ and Γ

$$\Delta = 0.431\,\frac{1}{\sqrt[3]{\lambda m^2}} \; ; \; \Gamma = 0.582\sqrt[3]{\lambda m^2} \quad (3.66)$$

The exact solutions which can be obtained by a numerical calculation are given for $\lambda = m = 1$ which, exact up to one unit

$$\omega_{10} = 1.0871 \; ; \; \omega_{21} = 1.4508 \; , \; \omega_{30} = 4.2002 \quad (3.67)$$

$$\Delta = 0.4561, \quad \Gamma = 0.5611. \quad (3.68)$$

The error in our $N=3$ approximation is about 10-15% while for the lowest eigenvalue about 7%. In the $N-1$ approximation the lowest eigenvalue is even obtained with only a 6% error.

From these calculations it still cannot be deduced whether our approximation scheme is a convergent scheme, i.e. that the correct values are approached with increasing N. There exists, in fact, the possibility that our boundary condition for closing the system is not a good one, and should be replaced by something else (e.g. vanishing of the λ-functions mentioned above.) Symanzik in his doctoral thesis has investigated this problem more closely. For determination of lowest eigenvalues, the approximation procedure discussed above will be of practical value.

IV. INDEFINITE METRIC IN HILBERT SPACE AND PROBABILITY INTERPRETATION

In this part we will study some consequences of the introduction of indefinite metric in Hilbert space. We have emphasized before that such an introduction in our theory seems to be a basic necessity but the question arises whether the indefinite metric will not affect the important properties like the unitarity of S-matrix. It is quite certain that introduction of indefinite metric will immediately lead to negative probability but this need not deter us if we can still preserve the unitarity of the S-matrix as the usual experiments concern themselves with the transitions between asymptotic states. The indefinite metric was first introduced by Dirac (1942) to study the quantization of Bose fields. Soon afterwards Pauli referred to the difficulties in the physical interpretation of the new formalism. In 1950 Bleuler and Gupta used this metric again but this time in configuration space in their study of quantization of electromagnetic field. It is now known that the Feynman cut off is related to ghost states. In nonrenormalizable theory, the indefinite metric enters the theory in a more basic way.

In section A, we discuss without going into the mathematical niceties, the general aspects of indefinite metric in Hilbert space. Elaborate studies have been made by Bogoliubov, Schlider and others but we shall not refer to their work here. In section B we discuss the work of Pauli and Kallen on Lee model, in particular then use of indefinite metric to overcome certain difficulties connected with renormalization. The ghost states which arose in

this problem, were shown to be compatible with probability interpretation by Heisenberg. Other models have also been discussed by Sudarshan, Nagy and others. Sudarshan hopes that it is always possible to restore probability interpretation by confining to the subspace in Hilbert space which spans physically realisable states. In section we will briefly discuss the indefinite metric in our non-linear theory.

A. General Aspects of an Indefinite Metric in Hilbert space.

In ordinary quantum mechanics (with positive definite metric) a Hermitian or scalar product is defined for two vectors in Hilbert space.

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* \qquad (4.1)$$

where one may consider the free-vector $\langle\psi|$ the conjugate complex of the ket vector $|\psi\rangle$. These inner products are used to describe the probability of finding a given system in a certain configuration. If the system is described by a state vector ψ , then the probability to find it in a certain state ϕ_n a state corresponding to the energy E_n is given by

$$\omega_n = |\langle\phi_n|\psi\rangle|^2 \qquad (4.2)$$

If the vectors form a complete set of eigen-vectors we can use them for the decomposition of the unit operator. In particular

$$\|\psi\| = \langle\psi|\psi\rangle = \sum_k \langle\psi|\phi_k\rangle\langle\phi_k|\psi\rangle$$
$$= \sum_k \langle\psi P_k \psi\rangle = \sum_k |\langle\phi_k|\psi\rangle|^2 = \sum_k \omega_k \qquad (4.3)$$

$\|\psi\| = 0$ only if $\psi = 0$

where $\|\psi\|$ is the norm of the state which is equal to the total probability $\sum_k \omega_k$ (usually normalized to one). P_k are projection operators ($P_k = |\phi_k\rangle\langle\phi_k|$ on to the particular state with the familiar property $P_k^2 = P_k$, $P_k^* = P_k$

The norm $\|\psi\| = 0$ only if $\psi = 0$

The transformations of the vectors in Hilbertspace which are induced by a change in the coordinate system must be those which leave all probabilities the same

$$\omega_A = \omega_B \text{ or } |\langle\psi_A|\phi_A\rangle|^2 = |\langle\psi_B|\phi_B\rangle|^2 \qquad (4.4)$$

They can be represented by a unitary mapping

$$\langle\psi_A|\phi_A\rangle \rightarrow \langle\psi_B|\phi_B\rangle = \langle U\psi_A|U\phi_A\rangle = \langle\psi_A|\phi_A\rangle \qquad (4.5)$$

or an antiunitary mapping

$$\langle\psi_A|\phi_A\rangle \rightarrow \langle\psi_B|\phi_B\rangle = \langle U\psi_A|U\phi_A\rangle^* = \langle\phi_A|\psi_A\rangle \qquad (4.6)$$

of the Hilbertspace onto itself.

In particular all symmetry groupshave to induce unitary or antiunitary representation in Hilbert space. Can we generalize to obtain a non-unitary transformation? Such a generalization will be in analogy with the transition from a 4-dimensional Euclidean space to a 4-dimensional Minkowski space with metric tensor $g_{\mu\nu}$. In the first only unitary transformations leave the scalar product, and in particular the line element invariant, whereas in the Muikowski space with indefinite metric also non-unitary transformations can be admitted. e.g. the Lorentz transformations. We define now a linear space with the elements $\{\psi\}$. A system of basic vectors $\{|\phi_e\rangle\}$. will form a complete set, in which case, any ψ can be expressed as a linear combination of these basic vectors:

$$|\psi\rangle = \psi^e |\phi_e\rangle \tag{4.7}$$

We call $|\phi_e\rangle$ the covariant vector system, $|\psi^e\rangle$ the contravariant components of ψ

We consider certain symmetry groups and study their representations $V(G)$ in our linear space. If G_1 and G_2 are two group elements we have

$$V(G_1 . G_2) = V(G_1)\, V(G_2) \tag{4.8}$$

Schlieder has classified the representations V as follows

$V(G) = [V^*(G)]^{-1}$; V^* = hermitian Conjugate V

V = irreducible.

These are familiar unitary representations.
The non-unitary representations can be decomposed into irreducible and reducible representations. Among the irreducible representation we have

2) $V(G)$ Equivalent to $(V^*)^{-1}$

3) $V(G)$ non-equivalent to $(V^*)^{-1}$

The only interesting reducible representations are representations which cannot be decomposed.

4) $W(G) = \begin{pmatrix} V_1(G) & V(G) \\ 0 & V_2(G) \end{pmatrix}$ V_1; V_2 irreducible

Schlieder shows that in all four cases a (generalized) scalar product can be defined which is left invariant under the group transformations except for the first type. The bra vector however is not identical with the complex conjugate of the vector in these cases. We do not wish to go into any details of the discussion of the various types.

We denote the generalized product

$$(x|\psi) = x^{*l} g_{lm} \psi^m = \langle x|g|\psi\rangle \qquad (4.9)$$

where we use the round brackets for the new product, keeping bra and ket brackets for the old definition.

The g_{lm} is in general a constant nondegenerate, infinite dimensional hermitean matrix. The new scalar product has the properties.

$$(x|\psi) = (\psi|x)$$
$$(x|\psi_1+\psi_2) = (x|\psi_1)+(x|\psi_2)$$
$$(x|\alpha\psi) = \alpha(x|\psi)\ ;\ \alpha \text{ scalar}$$
$$(\alpha x|\psi) = \alpha^*(x|\psi) \quad " \qquad (4.10)$$

If we introduce $(\phi_i|\phi_k) = g_{ik}$ (4.11)

The norm of any vectory ψ is defined as usual as

$$\|\psi\| = (\psi|\psi) = \psi^{*l} g_{lm} \psi^m = \langle\psi|g|\psi\rangle \qquad (4.12)$$

which, however, in general is not positive definite, since is not positive definite

Since g_{ik} is nondegenerate, one can define an inverse

$$g_{ik} g^{kl} = \delta_i^{\ l} \qquad (4.13)$$

We may use the g_{ik} and g^{ik} to obtain the contravariant ψ^k and the covariant components ψ_k of a state vector .

The decomposition of the unit operator now has the form:

$$I = \sum_k |\phi_k\rangle\langle\phi_k| = |\phi_k\rangle\langle\phi^k| = |\phi^l\rangle\langle\phi_e|$$
$$= |\phi^k\rangle g_{ke} \langle\phi^l| = |\phi_k\rangle g^{kl}\langle\phi_e| \qquad (4.14)$$

Note that if g^{kl} is not diagonal the simple notion of "summing over intermediate states" is not preserved. A matrix operator M may be represented in four different forms

$$ {}^{k}M_{l}, \; {}_{k}M_{l}, \; {}^{kl}M, \; {}_{kl}M; \text{ i.e. } {}^{k}M_{l} = \langle \phi_k | M | \phi_l \rangle \qquad (4.15) $$

An operator A is called pseudohermitian if for arbitrary states ψ and χ,

$$ (\psi | A | \chi)^* = (\chi | A | \psi) \qquad (4.16) $$

or in the old form

$$ \langle \psi | g A | \chi \rangle^* = \langle \chi | g A | \psi \rangle $$

$$ \text{i.e.} \quad A = g^{-1} A^* g \equiv A^\dagger \qquad (4.17) $$

This implies for the matrix elements:

$$ {}^{k}A_{l}^{*} = {}_{l}A^{k} \; ; \quad {}_{l}A^{k*} = {}^{k}A_{l} \; ; \quad ({}^{k}A^{l})^* = {}_{l}A_{k} \qquad (4.18) $$

In particular it follows that the diagonal elements are real for a pseudohermitian operator. We now state and prove two important theorems concerning pseudohermitian operators.

Theorem 1. A is pseudo hermitian and ψ an eigenvector of A with eigenvalue a i.e., $A|\psi\rangle = a|\psi\rangle$ Then either a is real or $\|\psi\| = 0$

Proof: $(\psi|A|\psi) = (\psi|A|\psi) =$ real

$$= a(\psi|\psi) = a^*(\psi|\psi) \text{ or } (a-a^*)\|\psi\| = 0 \quad (4.19)$$

Theorem 2. If ψ_1 and ψ_2 are two eigenvectors of a pseudo hermitian operator A with eigenvalues a_1 and a_2 respectively, then either $a_1 = a_2$ or ψ_1 is orthogonal to ψ_2

Proof: $(\psi_2|A|\psi_1) = a(\psi_2|\psi_1) = (\psi_1|A|\psi_2)^*$

$$= a_2^*(\psi_2|\psi_1) \; ; (a_1 - a_2^*)(\psi_2|\psi_1) = 0$$

(4.20)

Pandit has shown that the eigenstates of a pseudohermitian operator do not necessarily provide a complete basic vector system. The Lee model will provide us an example to illustrate this point.

B. The Lee model:

1) General features of the Lee model and its renormalization.

The Lee model as discussed here was first proposed by Lee in 1954, but as Kallen pointed out, the paper by Wigner and Weisskopf on the natural line with in 1930 already contains this model. The Lee model consists of three different particles, two fermions N and V of spin $\frac{1}{2}$ and a spin less boson θ transitions.

$$V \rightleftharpoons N + \theta \quad (4.21)$$

occur but not the reactions $N \rightleftarrows V + \theta$. This leads to the important simplification that only systems with a finite number of particles have to be considered. However due to the asymmetry in emission and absorption processes the interaction will become necessarily non-local. Fermions have no kinetic energy, since they behave like infinitely heavy particles which are fixed in space. The θ boson however is treated as a relativistic particles.

The Hamiltonian describing this situation can be written as (4.22)

$$H = H_0 + H_I$$

with the free Hamiltonian

$$H_0 = m_V \psi_V^* \psi_V + m_N \psi_N^* \psi_N + \int \omega(k)\, a^*(\vec{k})\, a(\vec{k})\, d^3k \qquad (4.23)$$

and the interaction term

$$H_I = -\frac{g_0}{\sqrt{4\pi}} \int \frac{f(\omega)}{\sqrt{2\omega}} \left[\psi_V^* \psi_N a(\vec{k}) + a^*(\vec{k})\, \psi_N^*\, \psi_V \right] d^3k \qquad (4.24)$$

and ψ_V, ψ_N, $a(k)$ are annihilation operators; ψ_V^*, ψ_N^* and $a^*(\vec{k})$ are creation operators.

$$n_V = \psi_V^* \psi_V, \quad N_N = \psi_N^* \psi_N, \quad \text{and} \quad N_\theta(\vec{k}) = a^*(\vec{k})\, a(\vec{k}) \qquad (4.25)$$

are the number operators of the V, N and θ particles. The function $f(\omega)$ is a cut off function which goes to zero for large values of $\vec{k}$ or ω

$$f(\omega) = \theta(\hat{\omega}-\omega) = \begin{cases} 1 & \text{for } \omega < \hat{\omega} \\ 0 & \text{for } \omega > \hat{\omega} \end{cases} \qquad (4.26)$$

where $\hat{\omega}$ is the cutoff frequency. The interaction in this case is smeared out over a radius $1/\hat{\omega}$. The local interaction is obtained for $f(\omega) = 1$ for all ω. It is still not local in the usual sense of the word because of the asymmetry in creation and annihilation operators. The factor $1/\sqrt{2\omega}$ arises from the normalization of boson θ – field in the usual fashion.

The g_0 is a real (bare) coupling constant. The operators obey the commutation rules.

$$\{\psi_V, \psi_V^*\} = 1 \; ; \; \{\psi_N, \psi_N^*\} = 1$$
$$[a(\vec{k}), a^*(\vec{k})] = \delta(\vec{k}-\vec{k}') \qquad (4.27)$$

All other commutators and anticommutators vanish. We have the condition on the renormalized masses $m_V^2 < m_N^2 + m_\theta^2$ to make the V-particle a stable particle (discrete state). Since all number operators $n_V, n_N, \hat{n}_\theta$ commute with H_0 : the eigenstates of the free Hamiltonian can be characterized by n_V, n_N, n_θ. The eigenstates $|n_V, n_N, n_\theta\rangle$ may be used as basic vectors for the description of the "physical" states $|n\rangle$ the eigenstates of the total Hamiltonian

$$|n\rangle = \sum_{n_V, n_N, n_\theta} C_n(n_V, n_N, n_\theta) |n_V, n_N, n_\theta\rangle \qquad (4.28)$$

The basic vectors $|n_V; n_N; n_\theta\rangle$ can be constructed from the state vector $|0;0;0\rangle$ by application of the creation operators ψ_V^*, ψ_N^*, $a^*(\vec{k})$ in the usual fashion. The investigation of the solutions of the total Hamiltonian is extremely simplified by the observation that the combinations

$$n_1 = n_V + n_\theta$$
$$n_2 = n_V + n_N. \tag{4.29}$$

commute also with the interaction term, and hence are constants of motion.

Hence the Hilbert space will split up in various sectors which can be characterized by the numbers (n_1, n_2) and contain the following free particle (H_0) states:

$$|0\rangle;\ |\theta\rangle;\ |N\rangle;\ \begin{cases} |N,\theta\rangle \\ |V\rangle \end{cases},\ \begin{cases} |N,2\theta\rangle \\ |V\theta\rangle \end{cases} \cdots \begin{cases} |N,z\theta\rangle \\ |V,\rangle|z-1,\theta\rangle \end{cases}$$

$$(0,0);\ (1,0),\ (0,1),\ (1,1),\ (2,1),\ \cdots$$

(4.30)

The lowest sector contains only the vacuum state of the free Hamiltonian. Hence the physical vacuum is identical with the bare vacuum

$$H_0|0\rangle = H|0\rangle = 0 \tag{4.31}$$

since

$$H_I|0\rangle = 0 \tag{4.32}$$

A similar situation also holds for the $(1,0)$ and $(0,1)$ sectors. It implies the identity of physical and bare $|0\rangle$ respectively and $|N\rangle$ states. In particular this means that the renormalized masses of and are identical with their original mass:

$$m_0^r = m_0 \; ; \; m_N^r = m_N \tag{4.33}$$

Before we discuss the higher sectors we remark that due to the constancy of n , the free Hamiltonian H_0 may be written

$$H_0 = m_N n_1 + (m_V - m_N)\psi_V^* \psi_V + \int \omega(\vec{k})\, a^*(\vec{k})\, a(\vec{k})\, d^3k \tag{4.34}$$

where the term $m_N n_1$ gives a constant energy shift depending on the sectors. This energy shift is irrelevant, if we only discuss energy differences. Then the masses of the fermions only enter as $(m_V - m_N)$ Hence we do not loose anything in the generality of the problem if we arbitrarily set

$$m_N = 0 \tag{4.35}$$

The generalization from this case $m_N = 0$ to $m_N \neq 0$ can always be inferred later on by the rule

$$\begin{aligned} m_V &\rightarrow m_V + m_N \\ E &\rightarrow E + n_1 m_N . \end{aligned} \tag{4.36}$$

Hence the Hamiltonian we shall be interested in is

$$H = H_0 + H_I \tag{4.37}$$

with

$$H_0 = m_V \psi_V^* \psi_V + \int w(\vec{k})\, a^*(\vec{k})\, a(\vec{k})\, d^3k$$

$$H_I = -\frac{g_0}{\sqrt{4\pi}} \int \frac{\hat{\omega}}{\sqrt{2w}} \left[\psi_V^* \psi_N\, a(\vec{k}) + a^*(\vec{k})\, \psi_N^* \psi_N \right] d^3k \tag{4.38}$$

where we have also used (4.26)

The first nontrivial sector is the (1,1) sector. The most general state vector $|X\rangle$ in this sector will be a superposition of $|V\rangle$ and $|N,\theta\rangle$

$$|X\rangle = C\psi_V^* + \psi_N^* \int \chi(\vec{k})\, a^*(\vec{k})\, d^3k\, |0\rangle \tag{4.39}$$

where

$$|V\rangle = \psi_V^* |0\rangle$$

$$|N\rangle = \psi_N^* |0\rangle$$

$$|\theta(\vec{k})\rangle = a^*(\vec{k}) |0\rangle \tag{4.40}$$

where we shall assume C and $\chi(\vec{k})$ to be limited by a condition to make $|X\rangle$ a normalized state in case $|X\rangle$ is discrete (normalizable)

or

$$P_X |X\rangle = \begin{cases} C \\ \chi(\vec{k}) \end{cases} \tag{4.41}$$

If we apply the total Hamiltonian on this state we get in general

$$H|X\rangle = |X'\rangle = \begin{cases} c' \\ \chi'(\vec{k}) \end{cases} \tag{4.42}$$

with
$$c' = m_V c - \frac{g_0}{\sqrt{4\pi}} \int^{\hat{\omega}} \frac{\chi(\vec{k})}{\sqrt{2\omega}} d^3k$$
$$\chi'(\vec{k}) = \omega(\vec{k})\,\chi(\vec{k}) - \frac{g_0}{\sqrt{4\pi}} \frac{c}{\sqrt{2\omega}} \qquad (\text{for } \omega < \hat{\omega}) \tag{4.43}$$

The eigenstates are characterized by

$$H|X\rangle = E|X\rangle \tag{4.44}$$

i.e.
$$c' = E\,c$$
$$\chi'(\vec{k}) = E\,\chi(\vec{k}) \tag{4.45}$$

We observe immediately that in the integral $\int \frac{\chi(\vec{k})}{\sqrt{2\omega}} d^3k$ only the rotational symmetrical part of $\chi(\vec{k})$ gives a contribution. This means that we have only S-wave interaction. Hence we write

$$\chi(\vec{k}) = \chi(|\vec{k}|) = \frac{\eta(k)}{\sqrt{4\pi k}} \quad \text{with} \quad k = |\vec{k}| \tag{4.46}$$

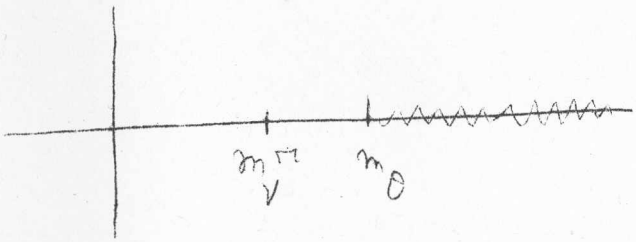
The eigen-value equation then reads

$$(m_V - E)\,c = g_0 \int^{\hat{\omega}} \frac{k\,dk}{\sqrt{2\omega}}\,\eta(k)$$
$$(\omega - E)\,\eta(k) = \frac{g_0 k}{\sqrt{2\omega}}\,c \qquad (\omega < \hat{\omega}) \tag{4.47}$$

There are two types of solutions

1) For $E \geq m_\theta$ there exists a continuous spectrum of scattering states

2) For $E < m_\theta$ one expects at least one bound state, namely the physical V-particle with mass



We investigate first the scattering states.

The scattering solution of

$$(\omega - E)\,\eta(k) = \frac{g_0 k}{\sqrt{2\omega}}\, c \tag{4.48}$$

have the familiar form for S-wave scattering

$$\eta(k) = \delta(k-k_0) - \lim_{\delta \to 0} \frac{g_0}{\sqrt{2\omega}} \frac{kc}{\sqrt{E-\omega+i\delta}} \tag{4.49}$$

with k_0 defined by $E = \sqrt{m_\theta^2 + k_0^2}$

The first part describes an unperturbed wave the and second part corresponds to an outgoing spherical wave. If we insert this into the first equation we obtain the condition

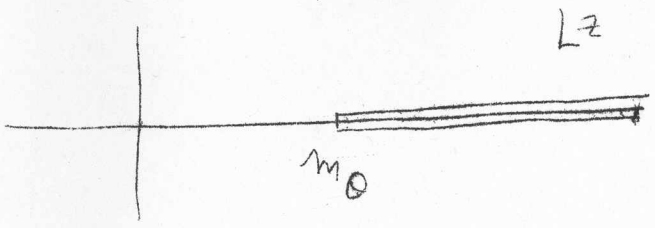
$$c\left[m_V - E + g_0^2 \int_0^{\hat{\omega}} \frac{k^2\,dk}{2\omega(E-\omega+i\delta)}\right] = \frac{g_0 k_0}{\sqrt{2E}} \tag{4.50}$$

which can be considered an equation for C.

For the following investigations it is convenient to introduce the complex function $h(z)$ of the complex variable z

$$h(z) = \frac{z - m_v}{g_0^2} + \int_0^{\hat{\omega}} \frac{k^2 dk}{2\omega(\omega - z)} \qquad (4.51)$$

This function is analytical in the cut complex z plane, where the cut starts from $z = m_0$ and extends up to infinity.



In the above equations we need $h(E + i\delta)$ for $E < m_0$ i.e. the function $h(z)$ just above the cut. We call it $h^+(E)$ Similarly we call the function below the cut $h^-(E)$

The discontinuity across the cut is pure imaginary because

$$h^+(E) - h^-(E) = \int \frac{k^2 dk}{2\omega(\omega - E - i\delta)} - \int \frac{k^2 dk}{2\omega(\omega - E + i\delta)}$$

$$= \int \frac{k^2 dk}{2\omega} \left\{ \left[P\frac{1}{\omega - E} + i\pi\, \delta(\omega - E) \right] - P\left[\frac{1}{\omega - E} - i\pi\, \delta(\omega - E) \right] \right\}$$

$$= i\pi \int \frac{k^2 dk}{\omega} \delta(\omega - E) = i\pi \int \frac{k^2}{\omega} \frac{dk}{\omega} d\omega\, \delta(\omega - E)$$

$$= i\pi\, h_0 \qquad \text{for } E < \hat{\omega} \qquad (4.52)$$

Hence

$$h^{\pm}(E) = \bar{h}(E) \pm i\pi \frac{k_0}{2} \tag{4.53}$$

where $h^{-}(E)$ is a real function defined with $P\frac{1}{\omega - E}$ under the integral. If we insert C in our equation for $\eta(k)$ we get the final solution

$$\eta(k) = \delta(k-k_0) + \frac{k k_0}{2\sqrt{\omega\omega_0}(\omega_0 - \omega + i\delta)\, h^{+}(\omega_0)} \tag{4.54}$$

$$C = \frac{k_0}{g_0\sqrt{2\omega_0}\, h^{+}(\omega_0)} \tag{4.55}$$

for the eigenvalue

$$E = \omega_0 = \sqrt{m_0^2 + k^2} \geq m_0^2 \qquad \text{for } E < \hat{\omega}$$

We now study the discrete states $E < m_0$

Here $h(z)$ is real. The solution of the second equation leads to

$$\eta(k) = \frac{g_0 k}{\sqrt{2\omega}\,(\omega - E)}\, C \tag{4.56}$$

and after insertion into the first equation yields

$$C\left(\frac{E - m_V}{g_0^2} + \int^{\hat{\omega}} \frac{k^2 dk}{2\omega(\omega - E)}\right) = C\, h(E) = 0, \tag{4.57}$$

i.e. the solutions of the equation

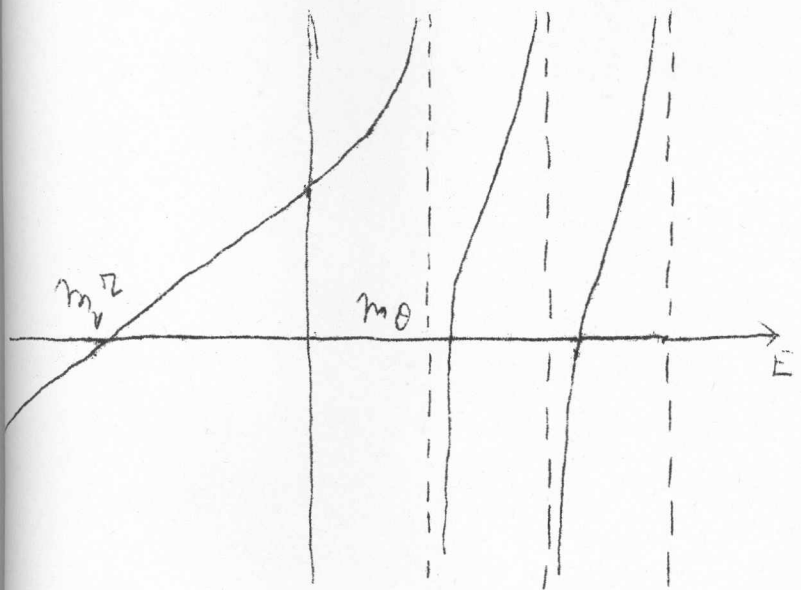
$$h(E) = 0 . \tag{4.58}$$

give the stationery bound states or explicitly

$$h(E) = \frac{E}{g_0^2} - \int^{\hat{\omega}} \frac{k^2\,dk}{2\omega(E-\omega)} \qquad (4.58)$$

We can easily infer that for a finite $\hat{\omega}$ for which only the integral is finite (it diverges linearly with $\hat{\omega}$) there is only one solution to the equation, since $h(E)$ has the approximate the form for $-E \gg \omega$ and $\hat{\omega}$

$$h(E) \simeq \frac{E}{g_0^2} + \text{const}\,\frac{1}{E}$$



This solution $E = m_V^2$ has to be identified with the physical V-particle state. The mass m_V^2 is the renormalized mass.

We have also indicated in the picture the continuous states as they would appear, if we (through the introduction of a finitte normalization volume) have dealt with an discrete k -spectrum (the k integrates being replaced by h sums). In this case, also the continuous states are normalizable and are represented in the figure by the intersections with the real axis

of an infinite number of hyperbolae. In the limit of normalization volume $V \longrightarrow \infty$ the hyperbolae get infinitely dense.

By virtue of the interaction the bare V-particles with mass m_V which is coupled to the (bare or physical) N and Θ particles becomes a physical particle V^r with mass $m_V{}^r$ We may now ask for the coupling strength of this physical V-particle to the physical N, Θ particles, i.e. for the renormalized coupling constant $g_0{}^r$. For this purpose we introduce the renormalized field operator for the V-particle by

$$\langle 0 | \psi_V{}^r | V^r \rangle = 1 \qquad (4.60)$$

with $|V^r\rangle$ the normalized physical V-particle state. Note that $\psi_V{}^r$ is not an operator which when applied to $|0\rangle$ gives just the $|V^r\rangle$ i.e. $\psi_V{}^{r*}|0\rangle \neq |V^r\rangle$. On the other hand we have

$$\langle 0 | \psi_V | V^r \rangle = C \qquad (4.62)$$

hence we have $$\psi_V{}^r = \frac{1}{C} \psi_V$$

The constant C is defined by the normalization condition

$$\langle V^r | V^r \rangle = 1 = |c|^2 + \int^{\hat{\omega}} |\chi(\vec{k})| \, d^3k$$

$$= |c|^2 + \int^{\hat{\omega}} |\eta(k)|^2 \, d^3k$$

$$= |c|^2 + g_0^2 \int^{\hat{\omega}} \frac{k^2 dk}{2\omega (\omega - E)^2} |c|^2$$

$$= g_0^2 |c|^2 \left(\frac{1}{g_0^2} + \int^{\hat{\omega}} \frac{k^2 dk}{2\omega (\omega - E)^2} \right)$$

$$= g_0^2 |c|^2 h'(E)\Big|_{E = m_V^r} = 1 \qquad (4.63)$$

$$\text{i.e.} \quad |c|^2 = \frac{1}{g_0^2 h'(E)} \Big|_{E = m_V^r} \qquad (4.64)$$

Hence we find that the field renormalization of the V-particle or any discrete state is connected with the slope of the curve for the eigenvalue E at the intersection point with the real axis. All the intersections have to occur with positive slope - as is also borne out in the above picture, including the 'continuous' states in order to insure that the renormalized state are again physically interpretable states. If $h'(E)$ would be negative a contradiction would be introduced into the theory since g_0^2 then must be negative (i.e. is imaginary) incontradiction to the original assumption. Since the other fields are not renormalized by the interaction. i.e.

$$\psi_N^r = \psi_N \quad \text{and} \quad a^r = a \qquad (4.65)$$

the renormalized coupling of constant is simply defined by

$$g_0 \psi_V = g \psi_V^r$$

$$g = g_0 e \quad \text{or} \quad g^2 = |g_0 e|^2 = \frac{1}{h'(E)} \tag{4.66}$$

The quantities m_V^r and g would correspond to the experimentally observable V-particle mass and its coupling strength ("charges") In order to give them certain observed numerical values, the bare mass m_N and bare coupling g_0 . have to be chosen

$$\frac{1}{g_0^2} = \frac{1}{g^2} - \int^{\hat{\omega}} \frac{k^2 dk}{2\omega(\omega - m_V)^2} = \frac{1}{g^2} - \frac{1}{g_{cut}^2(m_V^r)} \tag{4.67}$$

with

$$\frac{1}{g_{cut}^2(m_V^r)} = \int^{\hat{\omega}} \frac{k^2 dk}{2\omega(\omega - m_V^r)^2} \tag{4.68}$$

and

$$\frac{m_V}{g_0^2} = m_V^r \left(\frac{1}{g^2} - \frac{1}{g_{cut}^2} \right) + \int \frac{k^2 dk}{2\omega(\omega - m_V^r)} \tag{4.69}$$

If we consider now the theory in the limit $\hat{\omega} \longrightarrow \infty$ i.e. the "local" theory, the integrals

$$\int^{\hat{\omega}} \frac{k^2 dk}{2\omega(\omega - m_V^r)} \backsim \ln \hat{\omega}$$

diverges logarithmically (4.70)

$$\int^{\hat{\omega}} \frac{k^2 dk}{2\omega(\omega - m_V^r)} \backsim \hat{\omega}$$

diverges linearly (4.71)

The renormalization hence becomes infinite

$$\lim_{\hat{\omega}\to\infty} g^2_{cut} \backsim \frac{1}{\ln \hat{\omega}} \longrightarrow 0 \qquad (4.72)$$

We also note that whenever $\frac{1}{g^2_{cut}} > \frac{1}{g^2}$ the bare coupling has to be chosen pure imaginary. But this means that we have to start with a nonhermitean Hamiltonian. We shall show however that this difficulty can be circumvented if we introduce an indefinite metric in Hilbert space, as it was first suggested by Kallen and Pauli.

2. The Lee model with indefinite metric.

We have seen that in the limit $\hat{\omega} \longrightarrow \infty$ the unrenormalized coupling constant g_0 becomes pure imaginary and hence the original Hamiltonian non-hermitian. We shall now redefine the theory allowing an indefinite metric in Hilbert space.

If we consider an indefinite metric the condition for an operator to have real eigenvalues is pseudo hermiticity rather than hermiticity. In order to secure the reality of the eigenvalues of our Hamiltonian the interaction part

$$H_I = -\frac{g_0}{\sqrt{4\pi}} \int \frac{\hat{\omega} d^3k}{\sqrt{2\omega}} \left(\psi_V^* \psi_N a(\vec{k}) + a^*(\vec{k}) \psi_N^* \psi_V \right) \qquad (4.13)$$

which is non-hermitian due to g_0 should be made pseudo hermitian $H_I = H_I^\dagger = g^{-1} H_I^* g$ (4.74) through the introduction of an appropriate metric tensor.

We can achieve this purpose if we introduce a metric tensor diagonal in the n_V, n_N, n_θ representation which assigns negative norms to the bare V-particle states, i.e,

$$\langle n_V', n_N', n_\theta' | g | n_V, n_N, n_\theta \rangle = (-1)^{n_V} \delta_{n_V n_V'} \delta_{n_N n_N'} \delta_{n_\theta} \delta_{n_\theta'} \quad (4.75)$$

In this case $g = g^{-1}$ since $g^2 = 1$.
If we introduce the pseddo hermitian operators ψ_V^+, ψ_N^+, a^+ which have due to the special form of g, the properties

$$\begin{aligned} \psi_V^+ &= g^{-1} \psi_V^* g = -\psi_V^* \\ \psi_N^+ &= g^{-1} \psi_N^* g = \psi_N^* \\ a^+ &= g^{-1} a^* g = a^* \end{aligned} \quad (4.76)$$

we can write the interaction term now in the form

$$H_I = -\frac{g_0}{\sqrt{4\pi}} \int^{\hat{\omega}} \frac{d^3k}{\sqrt{2\omega}} \left(-\psi_V^+ \psi_N a(\vec{k}) + a^+(k) \psi_N^+ \psi_V \right) \quad (4.77)$$

which is obviously pseudohermitian. The free Hamiltonian can be written in the form

$$H_0 = -m_V \psi_V^+ \psi_V + \int^{\hat{\omega}} d^3k \, \omega(\vec{k}) a^+(k) a(\vec{k}) \quad (4.78)$$

The ψ_V^+, ψ_N^+, a^+ have now to be considered as the creation operators since the creation and annihilation operators are defined by the decomposition into positive and negative energy parts of operators $\psi_V, \psi_N(x), \psi(x)$ the matrix elements of which have to obey definite reality condition.

In particular from $\{\psi_V, \psi_V^*\} = 1$ we now deduce

$$\{\psi_V, \psi_V^+\} = -1 \tag{4.79}$$

from which, since $\psi_V|0\rangle = 0$ we derive $\langle 0|\psi_V\psi_V^+|0\rangle = -1$ which expresses the negative norm of the bare V-particles as defined in the beginning.

We now consider the sector (1,1). The formulae look exactly the same as before except we replace $\psi_V^* \longrightarrow -\psi_V^+$

In particular the most general state can be written.

$$|X\rangle = \left\{\begin{matrix} c \\ \chi(\vec{k}) \end{matrix}\right. = \left(-c\,\psi_V^+ + \psi_N^+ \int \chi(\vec{k})\, a^+(\vec{k})\, d^3k\,|0\right) \tag{4.80}$$

The function $h(z)$ has again the form.

$$h(z) = \frac{z - m_V}{g_0^2} + \int^{\hat{\omega}} \frac{k^2\,dk}{2\omega(\omega - z)} \tag{4.81}$$

Using the identity

$$\frac{1}{\omega - z} = \frac{1}{\omega} + \frac{z}{\omega^2} + \frac{z^2}{\omega^2(\omega - z)} \tag{4.82}$$

we can write

$$h(z) = a + bz + z^2 g(z) \tag{4.83}$$

with

$$a = \frac{m_V}{g_0^2} + \int^{\hat{\omega}} \frac{k^2 dk}{2\omega^2} \tag{4.84}$$

$$b = \frac{1}{g_0^2} + \int^{\hat{\omega}} \frac{k^2 dk}{2\omega^3} \tag{4.85}$$

$$g(z) = \int^{\hat{\omega}} \frac{k^2 dk}{2\omega^3(\omega - z)} \tag{4.86}$$

We can now easily carry out the limiting process $\hat{\omega} \to \infty$ since $g(z)$ will be finite, and a and b can be kept finite if only $-g_0^2 \to +\infty$ (g_0 is according to our assumption pure imaginary. i.e. $g_0^2 < 0$) and $m_V \to \infty$. There is no restriction on the sign of the bare mass m_V

$$-g_0^2 \simeq \frac{2}{\ln \hat{\omega}} \quad , \quad m_V \simeq -\frac{\hat{\omega}}{\ln \hat{\omega}} \tag{4.87}$$

The convergent integral can be readily evaluated ($\hat{\omega} \to \infty$)

We get for the principal value

$$\bar{g}(z) = P g(z) = \frac{1}{2}\left[g_+(z) + g_-(z)\right] \tag{4.88}$$

$$2m_0\bar{g}(z) = \frac{\pi}{2\zeta} + \frac{1}{\zeta} + \begin{cases} \frac{\sqrt{\zeta^2-1}}{\zeta^2}\ln\left(-\zeta+\sqrt{\zeta^2-1}\right) & \zeta \le -1 \\ -\frac{\sqrt{1-\zeta^2}}{\zeta^2}\left(\pi + \arcsin\zeta\right) & -1 \le \zeta < +1 \\ -\frac{\sqrt{1-\zeta^2}}{\zeta^2}\ln\left(\zeta+\sqrt{\zeta^2-1}\right) & \text{for } \zeta > 1 \end{cases}$$

with $\zeta = \frac{z}{m_0}$ (4.89)

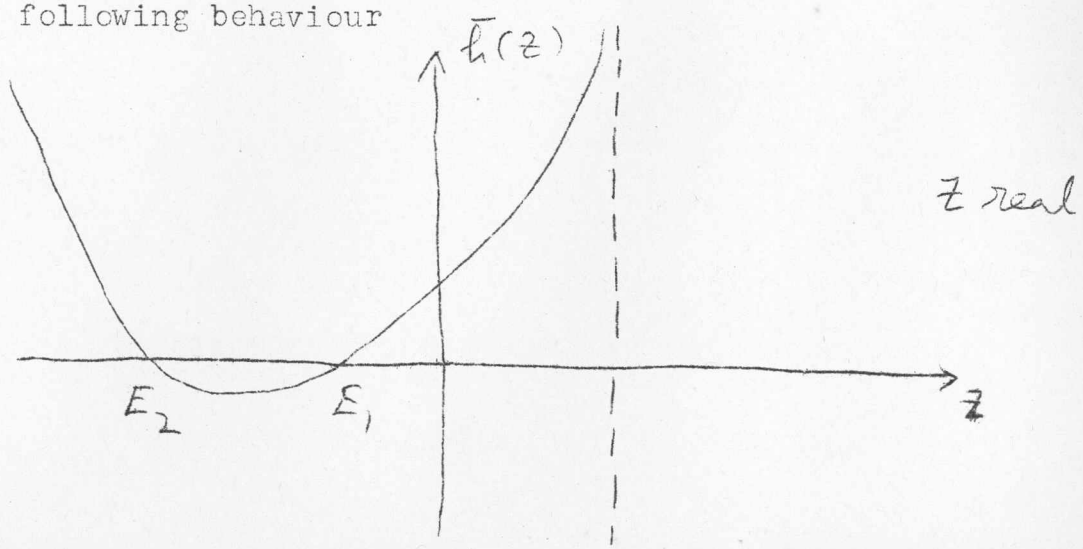
The discontinuity across the logarithmic cut which starts from $\pm m_0$ is given by

$$m_0\left[g_+(z) - g_-(z)\right] = \begin{cases} 0 & \zeta < 1 \\ \pm i\pi\,\frac{\sqrt{1-\zeta^2}}{\zeta^2} & ; \zeta > 1 \end{cases}$$

(4.90)

We see that $\pm i\pi\sqrt{1-\zeta^2}/\zeta^2$ is just the imaginary part of the function $\frac{\sqrt{1-\zeta^2}}{\zeta^2}\ln\left(\zeta+\sqrt{\zeta^2-1}\right)$

The function $\bar{h}(z) = a + bz + z^2\bar{g}(z)$ has approximately the following behaviour



For $z \to -\infty$, $\bar{g}(z) = -\frac{1}{2|z|}\left(\frac{\pi}{2} + 1 - \frac{\ln|z|}{m_0}\right)$

$$\to \frac{1}{2|z|}\,\frac{\ln|z|}{m_0}$$

$$i.e.\ \bar{h}(z) > a + bz + z^2 g(z) \to z^2 g(z) \to \infty \qquad (4.91)$$

further

$$\frac{d^2h}{dz^2} > 0 \quad \text{for } z < m_0 \qquad (4.92)$$

$$\left.\frac{dh}{dz}\right|_{z=-\infty} = -\infty\,; \quad \left.\frac{dh}{dz}\right|_{z=m_0} = +\infty \qquad (4.93)$$

i.e. the curve must have a minimum between $-\infty < z < m_0$ and hence we have to distinguish three cases depending on the specific values of a and b.

1. There are no real solutions, i.e., the curve above has no intersection for real z . There exists only two complex roots
2. There are two real solutions, i.e. the curve intersects twice as shown above for real z (Pauli-Kallen case)
3. There is only one real solution, i.e. the curve just touches the line $h(z) = 0$ for real z , or the two real solutions coincide (Heisenberg case).

The first case we discard since it does not lead to a stable V-particle. For the second case we have two stationary states. The norm of these stationary states is given by (the term $|c|^2$ is negative due to the negative norm of bare V-particle).

$$(\chi_{st}|\chi_{st}) = -|c|^2 + \int \chi(\vec{k})\, d^3k = -|c|^2\left[1 - |g_0|^2 \int \frac{d^3k\, k^2}{2w(w-E)^2}\right]$$

$$= \left[1 - |c^2 g_0^2|\, h'(E)\right] \quad \text{for } E = E_1 \text{ or } E_2$$

(4.94)

This means that the two stationary states corresponding to intersection E_1 and E_2 have opposite norm since $h'(E)$ has opposite sign at the intersections. In fact, the higher value of E corresponds to a state with positive norm, and hence we identify it with the physical V-particle V^π with mass m_V^π), the lower one has a negative norm. It is an unphysical particle and we will call it the ghost state. This ghost state is not allowed to appear in physical process since it will destroy the probability interpretation.

We introduce again the renormalized field operator and the renormalized coupling constant:

$$\langle 0|\, \psi_V^\pi\, |V^\pi\rangle = 1 \qquad (4.95)$$

$$g = c\, g_0 \qquad (4.96)$$

with $$\psi_V^r = \frac{1}{c}\psi_V = \frac{g_0}{g}\psi_V$$

and $$|c|^2 = \frac{1}{|g_0^2|\, h'(m_V^r)} \qquad (4.97)$$

For the anticommutator of the renormalized field operators we find:

$$\left\{\psi_V^r, \psi_V^{*r}\right\} = \frac{|g_0|^2}{g^2}\left\{\psi_V, \psi_V^+\right\} = \frac{-|g_0|^2}{g^2} \xrightarrow{\omega \to \infty} 0 \qquad (4.98)$$

which expresses the noncanonical quantization of the renormalized field. We may also define a space-time representation of the renormalized operator by

$$\psi_V^r(t) = \sum\!\!\!\!\!\!\int dE\ \psi_V^r(E)\, e^{iEt} \qquad (4.99)$$

where the $\sum\!\!\!\!\!\!\int$ goes over the total energy spectrum including discrete and continuous parts. Note that $\psi_V^r(E)$ is nonzero not only for the discrete value $E = m_V^r$ but also in other regions in particular for $E > m_0$ the $N\theta$ scattering states, in contrast $\psi_V(E)$ which is nonzero only for $E = m_V$ Due to the absence of a 3-momentum, there is no $\vec{r}$ dependence of the operators. The particles are fixed to a certain space point (here assumed to be the origin). The above result for the anticommutator not immediately leads to the same conclusion for the equal time anticommutator i.e.

$$\left\{\psi_V^r(t), \psi_V^{r+}(t)\right\}_{t=t'} \xrightarrow[\omega \to 0]{} 0 \qquad (4.100)$$

We do not have a classical theory, as can be seen from (4.100). If we calculate the anticommutator for different times by introducing the complete set of intermediate states (Ghost, V-particles, cont. spectrum) we find

$$\begin{aligned}\left\{ \psi_V^{r}(t), \psi_V^{r+}(t') \right\} &= (0 | \psi_V^{r}(t) \psi_V^{r+}(t') | 0) \\ &= \sum \int dE\, (0 | \psi_V^{r}(t) | \chi_E)(\chi_E | \psi_V^{r+}(t') | 0) \\ &= \frac{1}{g^2}\left[\sum_{n=1,2} \frac{e^{iE_n(t-t')}}{h'(E_n)} + \int_0^\infty \frac{k^2 dk\, e^{i\omega(t-t')}}{2\omega h^+(\omega) h^-(\omega)} \right] \\ &\neq 0\end{aligned} \tag{4.101}$$

This indeed vanishes for $t = t'$ as can be seen in the following way

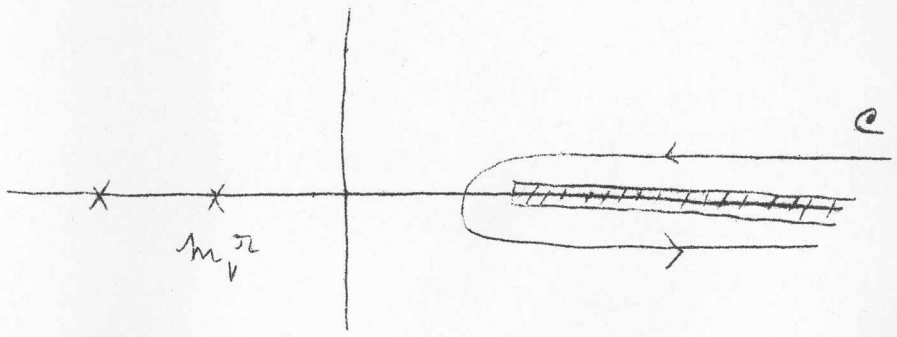
$$\int_0^\infty \frac{k^2 dk}{2\omega h^+(\omega) h^-(\omega)} = \int_{m_0}^\infty \frac{k^2 \frac{dk}{d\omega} d\omega}{2\omega h^+(\omega) h^-(\omega)} - \int_{m_0}^\infty \frac{k\, d\omega}{2 h^+(\omega) h^-(\omega)} \tag{4.102}$$

Since $h^+(\omega) - h^-(\omega) = i\pi k.$ (4.103)

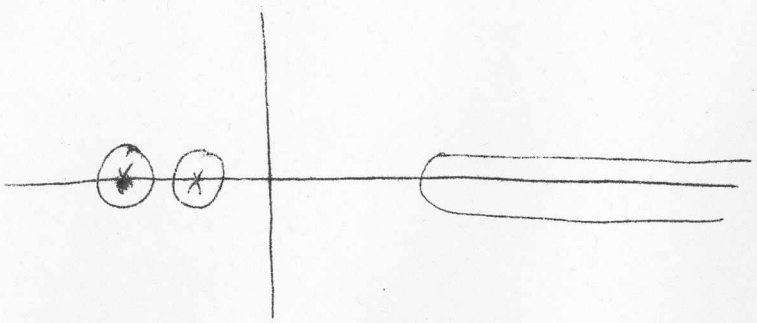
$$\int_{m_0}^{\infty} \frac{k\,dw}{2h^+(w)h^-(w)} = \frac{1}{i\pi}\int_{m_0}^{\infty} \frac{h^+(w)-h^-(w)}{2h^+h^-}\,dw$$

$$= \frac{1}{2\pi i}\int_{m_0}^{\infty} \frac{1}{h^{(-)}} - \frac{1}{h^{(+)}}$$

$$= \frac{1}{2\pi i}\int_{C} \frac{dw}{h(w)}$$



$$= \frac{1}{2\pi i}\oint_{\text{poles}} \frac{dw}{h(w)} = \sum_{n=1,2} \frac{1}{h'(\omega_n)} \qquad (4.104)$$



Hence that the right hand side of the equal time commutator, vanishes is due to the fact that the ghost state contribution $\frac{1}{R'(m_g)}$ is negative and just compensates the contribution from the physical V-particle and the $N-\theta$ scattering states.

Thus the description of sector (1,1) is complete and does not lead to any mathematical inconsistencies or difficulties in the physical interpretation. We may consider the ghost particle V^g as physically non existent, i.e. we define the states V_+ and N,θ as the physically realizable states. Because of energy conservation there are no transitions possible to the V^g state. The question however is whether in higher sectors e.g. the (2,1) sector which contains $N+2\theta$ and $V_+ + \theta$ transitions to V^g are possible which then would destroy the probability interpretation. Before we briefly turn to these higher sectors, we should remark that in the renormalized theory described above, the limit $g \to 0$ does not lead to the free V, N, θ fields since the theory is nonanalytic in g^2. Therefore one cannot expect that a perturbation approach can be successful. The appearance of the ghost is a typical nonperturbative result.

We will now consider the next higher sector (2,1) to see whether we run into trouble with the probability interpretation and/or also to see whether the S-matrix for scattering processes may become non-unitary. The general state in this sector consists of a superpositions of states $\begin{cases} N+2\theta \\ V+\theta \end{cases}$ where V may mean V_+ (physical) or V_- (ghost). In particular we must consider the

transition $V_+ + 0 \rightarrow V_- + 0$. We introduce the following amplitudes.

$$|X\rangle = \begin{cases} V_+ + 0 \\ V_- + 0 \\ N + 20 \end{cases} = \begin{cases} \varphi^+(\vec{k}) \\ \varphi^-(\vec{k}) \\ \varphi(\vec{k}_1, \vec{k}_2) \end{cases}$$

(4.105)

We may express an incoming wave of an N and 20 particles, by the outgoing states, which will be a superposition of all three possible states

$$|N+20\rangle_{in} = \int d^3k\, b^+(k) \,|V_+ + 0(\vec{k})\rangle_{out} + \int d^3k\, b^-(\vec{k}) \,|V_- + 0(\vec{k})\rangle_{out} + \int d^3k_1 d^3k_2\, c(k_1, k_2) \,|N + 0(k_1) + 0(k_2)\rangle_{out}$$

(4.106)

The normalization condition for the in states leads on the r.h.s. to the transition probabilities,

$$\| (N+20)_{in} \| = \int_{\hat{\omega}_{(V_+ + 0)}} d^3k \,|b^+(k)|^2 - \int_{\hat{\omega}_{(V_- + 0)}} d^3k \,|b^-(k)|^2 + \int_{\hat{\omega}_{N+20}} d^3k_1 d^3k_2 \,|c(k_1, k_2)|^2$$

(4.107)

but this integration now breaks down in the second term due to the negative sign. Hence (4.107) cannot be interpreted as probability. However there exists a possibility to avoid this difficulty if we adopt the situation of case 3 stated above of coinciding roots. In this case both the eigenvectors of the Hamiltonian $|V_+)$ and $|V_-)$ converge towards one vector which because of

$$(V_0|V_0) = |c^2 g_0^2| h'(E) = 0. \tag{4.108}$$

has the norm zero. We have "lost" in this procedure one basic vector and we suspect that the missing state vector which is necessary for the complete basis, will not be an eigenstate of the Hamiltonian. However, the eigenvector space spanned by the scattering states $N_+\Theta$ and V_0 is invariant under application of the Hamiltonian (time translation) and its metric is positive semidefinite.

In order to find the missing state vector we consider the case 3, in the limit of the two real roots approachingeach other. The solutions shall be at E_0 and $E_0+\Delta E$ i.e.

$$h(E_0) = 0 \quad , \quad h(E_0+\Delta E) = 0 \tag{4.109}$$

The corresponding state vectors will only differ slightly from each other $|(\Delta X)$ if ΔE is small

$$H|X_0) = E_0|X_0)$$
$$H|X_0+\Delta X) = E_0+\Delta E|X_0+\Delta X) \tag{4.110}$$

The second equation can be written with the help of the first

$$H|\Delta X) = \Delta E\,|X_0) + E_0\,|\Delta X) + \Delta E\,|\Delta X) \qquad (4.111)$$

If we take the transition $|\Delta X) \to 0$ and $\Delta E \to 0$ both the equations, go over to

$$H|X_0) = E_0\,|X_0) \qquad (4.112)$$

but from the last equation we get an additional relationship

$$H\left|\frac{\Delta X}{\Delta E}\right) = E_0\left|\frac{\Delta X}{\Delta E}\right) + |X_0) \quad ,(\Delta E \to 0) \qquad (4.113)$$

which introduces a new state $|X_{dip}) = \lim_{\Delta E \to 0}\left|\frac{\Delta X}{\Delta E}\right)$ which is proportional to the difference between a state of positive and negative norm and hence is called "dipole" ghost. It is the dipole ghost state which is missing.

The dipole ghost obeys the inhomogeneous Schrodinger equation

$$H|X_{dip}) = E\,|X_{dip}) + |X_0) \qquad (4.114)$$

hence it is not an eigenstate of the Hamiltonian. $|X_{dip})$ is not uniquely defined by above equation since before the roots coincide we may multiply $|X)$ by an arbitrary function of $f(E)$ which then would lead to the replacement

$$\frac{dX}{dE} \to \frac{dX}{dE} + \frac{f'(E)}{f(E)}\Big|_{E=E_0} X \qquad (4.115)$$

i.e. just add a constant X_0. We may use this arbitrariness to make the norm of X_{dip} also zero. If the norm of X_{dip} is originally

$$\| X_{dip} \| = (X_{dip} | X_{dip}) = N^2 \tag{4.116}$$

then the new vector $|X'_{dip}) = \{ |X_{dip}) + |c X_0) \}$ will have the norm

$$\| X'_{dip} \| = N^2 + c (X_{dip} | X_0) + c^* (X_0 | X_{dip}) \tag{4.117}$$

which can be made zero if

$$2 Re \{ c (X_{dip} | X_0) \} = - N^2 . \tag{4.118}$$

This can always be achieved since $(X_{dip} | X_0) \neq 0$ although $|X_0)$ is orthogonal to itself. We start out with a metric tensor for the $|V_+), |V_-)$ states of the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.119}$$

If V_+ and V_- approach each other the combination $(V_+ + V_-) = V_0$ becomes the only eigenvector of the Hamiltonian, and $V_+ - V_- = \Delta E \, V_{dip}$ becomes essentially the dipole ghost. In this representation

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.120}$$

which expresses that both vectors have the norm zero, and are not orthogonal to each other. In fact they are contravariant states with respect to each other. We can now define the space of the physically meaningful states as the subspace of Hilbert-space which contains V_0, N, θ states but no V_{dip} This space is invariant undertime translation i.e. if the particles before the interaction do not contain V_{dip} then cannot be created by the interaction. The space of the physical states has positive semidefinite metric and hence the normal probability interpretation is possible. However the physical states do not form a complete set in the (1,1) sector e.g. the physical states are $\begin{cases} N+\theta \\ V_0 \end{cases}$ but V_0 has the norm zero and hence does not contribute to the matrix elements, only the scattering states are measurable.
We can also consider the higher sectors $\begin{cases} N+2\theta \\ V+(2-1)\theta \end{cases}$ in a similar way and shall find that the model can be made consistent. In higher sectors involving more V or N particles, the consistency of this model has not been proved.

We may now remark that the study of representations of the symmetry group of type (4) of the form $\begin{pmatrix} \{V_1\} & \{V\} \\ 0 & \{V_2\} \end{pmatrix}$ i.e. reducible but not factorizable representations are connected with the dipole case of the Lee model (Note that the multiplication of two of such matrices again reproduces a matrix of this form). We find that $\{V_1\}$ and $\{V_2\}$ must

belong to the same irreducible representation. Invariance under the inhomogeneous Lorentz group implies that $\{V_1\}$ and $\{V_2\}$ belong to the same eigenvalue of p_μ^2 i.e. the same mass. Hence one may consider the more special case

$$\begin{pmatrix} \{V_1\} & \{V\} \\ 0 & \{V_2\} \end{pmatrix} \tag{4.121}$$

The metric tensor which is left invariant under this symmetry operations $V^{-1} g V = g$ must be of the form

$$g = \begin{pmatrix} 0 & g_2 \\ g_3 & g_4 \end{pmatrix} \; ; \; g_3 = g_2^{+} \tag{4.122}$$

or more trivially

$$g' = \begin{pmatrix} 0 & 0 \\ 0 & g_4 \end{pmatrix} \tag{4.123}$$

According to the work by Ferretti and Maksimov, also the Pauli-Kallen case of two real roots (V_+ and V_- different masses) can be made physically interpretable like the dipole case. The argument is more or less similar. Due to the pseudo unitarity of the S-matrix, $S^+ S = 1$ the norm of a system composed by particle (P) and ghost (G) remains unchanged

$$\psi = \begin{pmatrix} P \\ G \end{pmatrix}, \quad \|\psi\|^2 = |P|^2 - |G|^2 = \text{Const.} \tag{4.124}$$

but the partial probabilities are not positive. If the initial state contains only states with either positive or negative norm, the final state will be a mixture of both, ile.

$$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_P\\a_G\end{pmatrix} \text{ with } |a_P|^2 - |a_G|^2 = 1$$

$$S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b_P\\b_G\end{pmatrix} \text{ with } |b_P|^2 - |b_G|^2 = 1. \tag{4.125}$$

If we make the initial state a mixture

$$S\left[\begin{pmatrix}1\\0\end{pmatrix} + \xi\begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}a_P + \xi b_P\\ a_G + \xi b_a\end{pmatrix} \tag{4.126}$$

$$\text{with } 1 - |\xi|^2 = |a_P + \xi b_P|^2 - |a_G + \xi b_G|^2 \tag{4.127}$$

We may always choose ξ such that

$$|\xi|^2 = |a_G + \xi b_G|^2 \tag{4.128}$$

and hence

$$1 = |a_P + \xi b_P|^2. \tag{4.129}$$