

Degrees of Maps between Complex Grassmann Manifolds

By

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*A thesis submitted to the
Board of Studies in Mathematical Sciences*

In partial fulfillment of requirements

For the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE



Homi Bhabha National Institute

Recommendations of the Viva Voce Board

As members of the Viva Voce Board, we recommend that the dissertation prepared by Swagata Sarkar titled “Degrees of Maps between Complex Grassmann Manifolds” may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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DECLARATION

I, hereby declare that the investigation presented in this thesis has been carried out by me under the guidance of my supervisor P. Sankaran. My collaboration with him in our paper was necessitated by the difficulty and depth of the problem considered. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

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ABSTRACT

Let $f: \mathbb{G}_{n,k} \rightarrow \mathbb{G}_{m,l}$ be any continuous map between two *distinct* complex (resp. quaternionic) Grassmann manifolds of the same dimension. We show that the degree of f is zero provided n, m are sufficiently large and $l \geq 2$. If the degree of f is ± 1 , we show that $(m, l) = (n, k)$ and f is a homotopy equivalence. Also, we prove that the image under f^* of elements of a set of algebra generators of $H^*(\mathbb{G}_{m,l}; \mathbb{Q})$ is determined up to a sign, \pm , if the degree of f is non-zero.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Prof. P. Sankaran, for teaching and helping me so much, both in mathematics and in every day life. Each and every aspect of this thesis has taken its present shape under his persevering guidance and supervision. I thank him for the way he led me towards the completion of this thesis. And more than anything else, I am extremely grateful to him for his patience and understanding towards me.

I also thank all the Mathematics faculty at IMSc who have been responsible for imparting to me the knowledge of various aspects of Mathematics during my course work and through numerous seminars.

Profs. R. Balasubramanian and D. S. Nagaraj deserve my special thanks for their help with Proposition 4.3.1 and also for their valuable comments on the paper. I would also like to thank Prof. J. Oesterlé for pointing out Example 4.3.5 to us.

I thank my parents for their love and support, and all my teachers (right from high school through university) for their encouragement and for giving me the right flavour of Mathematics at different stages of my academic life, which was highly instrumental in motivating me to take up Mathematics at this level. I am also grateful to all my friends, both in IMSc and elsewhere, for their constant support.

The work leading to this thesis was carried out completely at the Institute of Mathematical Sciences, Chennai. I would like to thank the institute for its financial support and the wonderful infrastructure and working environment that it provides to its members. Last but not the least, I thank each and every

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member of IMSc for making my stay at IMSc a memorable and enjoyable one.

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LIST OF PUBLICATION(S)

- P. Sankaran and S. Sarkar, Degrees of maps between Grassmann Manifolds, To appear in the Osaka Journal of Mathematics.

CHAPTER 1

INTRODUCTION

A brief overview of the results presented in this thesis is given below.

Let M, N be compact, connected, oriented n -dimensional manifolds with orientation classes in top homology denoted by μ_M and μ_N respectively. Let $f : M \rightarrow N$ be a continuous map between them. Then the induced map in the top homology, $f_* : H_n(M; \mathbb{Z}) \rightarrow H_n(N; \mathbb{Z})$, is given by $f_*(\mu_M) = (\deg f) \cdot \mu_N$, where $\deg f \in \mathbb{Z}$. This integer, $\deg f$, is called the (Brouwer) degree of the map f .

Let \mathbb{F} denote the field of complex numbers, \mathbb{C} , or the skew field of quaternions, \mathbb{H} . Let $\mathbb{F}\mathbb{G}_{n,k}$ denote the Grassmann manifold of all k -dimensional left \mathbb{F} -vector subspaces of \mathbb{F}^n . Then $\dim_{\mathbb{F}} \mathbb{F}\mathbb{G}_{n,k} = k(n-k) =: N$. Now, given integers n, k, m, l such that $1 \leq k < n$, $1 \leq l < m$ and $(n, k) \neq (m, l)$, such that $\dim_{\mathbb{F}} \mathbb{F}\mathbb{G}_{n,k} = \dim_{\mathbb{F}} \mathbb{F}\mathbb{G}_{m,l}$, what can be the possible degrees of maps from $\mathbb{F}\mathbb{G}_{n,k}$ to $\mathbb{F}\mathbb{G}_{m,l}$? The existence of maps of arbitrarily large degrees when the target space is the projective space, $\mathbb{F}\mathbb{P}^N$, is well known.

Let $\widetilde{\mathbb{R}\mathbb{G}}_{n,k}$ be the oriented Grassmann manifold of oriented k -dimensional vector subspaces of \mathbb{R}^n . Then, in [19], V. Ramani and P. Sankaran prove that, for any continuous map $h : \widetilde{\mathbb{R}\mathbb{G}}_{n,k} \rightarrow \widetilde{\mathbb{R}\mathbb{G}}_{m,l}$, where $(n, k) \neq (m, l)$, $2 \leq l \leq m/2$, $1 \leq k \leq n/2$ and $\dim \widetilde{\mathbb{R}\mathbb{G}}_{n,k} = \dim \widetilde{\mathbb{R}\mathbb{G}}_{m,l}$, $\deg h = 0$.

In the same paper, they also prove that if, $f : \mathbb{C}\mathbb{G}_{n,k} \rightarrow \mathbb{C}\mathbb{G}_{m,l}$ (resp. $g :$

$\mathbb{H}\mathbb{G}_{n,k} \rightarrow \mathbb{H}\mathbb{G}_{m,l}$, $1 \leq k < l \leq [m/2]$, is any continuous map between two complex (resp. quaternionic) Grassmannians of the same dimension, then $\deg f = 0 = \deg g$.

In this thesis, we obtain the following results:

Let \mathbb{F} denote \mathbb{C} or \mathbb{H} and let $d = \dim_{\mathbb{R}} \mathbb{F}$. Then we have:

Theorem 1.0.1 *Let $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ be any continuous map between two \mathbb{F} -Grassmann manifolds of the same dimension. Then, there exist algebra generators $u_i \in H^{di}(\mathbb{F}\mathbb{G}_{m,l}; \mathbb{Q})$, $1 \leq i \leq l$, such that the image $f^*(u_i) \in H^{di}(\mathbb{F}\mathbb{G}_{n,k}; \mathbb{Q})$, $1 \leq i \leq l$, is determined upto a sign \pm by the degree of f , provided the degree of f is non-zero.*

Theorem 1.0.2 *Fix integers $2 \leq l < k$. Let $m, n \geq 2k$ be positive integers such that $k(n-k) = l(m-l) = N$ and $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ any continuous map. Then, degree of f is zero if $(l^2 - 1)(k^2 - 1)((m-l)^2 - 1)((n-k)^2 - 1)$ is not a perfect square. In particular, degree of f is zero for n sufficiently large.*

Theorem 1.0.3 *Suppose that $k(n-k) = l(m-l)$, and $1 \leq l \leq [m/2]$, $1 \leq k \leq [n/2]$. If $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ is a map of degree ± 1 , then $(m, l) = (n, k)$ and f is a homotopy equivalence.*

Since the proofs of the above theorems only involve studying algebra homomorphisms between the cohomology algebras of the concerned Grassmannians and the (integral) cohomology ring of the quaternionic Grassmannian is isomorphic to the cohomology ring of the corresponding complex Grassmannian, via a degree doubling isomorphism, *we need only consider the case of the complex Grassmann manifold, $\mathbb{C}\mathbb{G}_{n,k}$.*

The complex Grassmann manifold, $\mathbb{C}\mathbb{G}_{n,k}$, admits of a natural orientation, since it has a complex structure. Though $\mathbb{H}\mathbb{G}_{n,k}$ does not admit even an almost complex structure [14], it is oriented since it is simply connected. The orientation of $\mathbb{H}\mathbb{G}_{n,k}$ is chosen such that the image of the positive generator

of the top integral cohomology of $\mathbb{C}\mathbb{G}_{n,k}$, under the degree doubling isomorphism induced in the cohomology rings, is positive.

The Schubert cells give a cell decomposition of $\mathbb{C}\mathbb{G}_{n,k}$, with cells only in the even (real) dimensions. The closure of a Schubert cell is the corresponding Schubert cycle. The classes of the Schubert cycles form a \mathbb{Z} -basis for the integral homology of $\mathbb{C}\mathbb{G}_{n,k}$. The dual cohomology classes corresponding to the *special* Schubert cycles form a set of *algebra* generators of the cohomology algebra, $H^*(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Z})$.

The proofs of the above theorems use the concept of the degree of a Schubert class. Since the dual Schubert classes generate the cohomology of $\mathbb{C}\mathbb{G}_{n,k}$ in each dimension, it makes sense to talk of the degree of any cohomology class in $H^*(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Z})$. The degree of the whole space $\mathbb{C}\mathbb{G}_{n,k}$ is defined to be the degree of the dual cohomology class corresponding to the fundamental homology class $\mu_{n,k} \in H_{2N}(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Z})$.

The fact that $f^* : H^*(\mathbb{C}\mathbb{G}_{m,l}; \mathbb{Q}) \rightarrow H^*(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Q})$ is an algebra homomorphism leads to a system of diophantine constraints. In particular, using a well-known, closed form, numerical formula for the degree of a Schubert class, applied to a specific class in $H^4(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Q})$, we get: $\deg(f) = 0$ unless $Q(l, k, m-l, n-k) := (l^2 - 1)(k^2 - 1)((m-l)^2 - 1)((n-k)^2 - 1)$ is a perfect square. Now Siegel's Theorem on integer solutions of polynomial equations of the form $y^2 = F(x)$ allows us to complete the proof for Theorem 1.0.2.

Theorem 1.0.2 provides strong evidence in support of the following conjecture stated in [19]:

Conjecture: Let $\mathbb{F} = \mathbb{C}$ or \mathbb{H} and let $2 \leq l < k \leq n/2 < m/2$, where $k, l, m, n \in \mathbb{N}$. Assume that $k(n-k) = l(m-l)$. Let $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ be any continuous map. Then, the degree of f is zero.

In their paper [18], K. Paranjape and V. Srinivas prove that, for $l \geq 2$ and $\dim \mathbb{C}\mathbb{G}_{n,k} = \dim \mathbb{C}\mathbb{G}_{m,l}$, if there exists a non-constant morphism, $f : \mathbb{C}\mathbb{G}_{n,k} \rightarrow \mathbb{C}\mathbb{G}_{m,l}$, of projective varieties, then $(n, k) = (m, l)$ and f is an isomorphism. Theorem 1.0.3 is a topological analogue of Paranjape and Srinivas's result mentioned above. The proof of Theorem 1.0.3 uses Whitehead's

Theorem.

The proof of Theorem 1.0.1 uses various results from Hodge theory. The complex Grassmann manifold, $\mathbb{C}\mathbb{G}_{n,k}$, is a compact Kähler manifold. One can take the first Chern class of the universal quotient bundle as the Kähler class, ω . Using the Kähler class, it is possible to define a symmetric, non-degenerate bilinear form $(\cdot, \cdot)_\omega$ on $H^r(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{R})$, $1 \leq r \leq N$. In case of the complex Grassmann manifold, $\mathbb{C}\mathbb{G}_{n,k}$, this bilinear form is rational and we have a orthogonal decomposition of the rational cohomology groups $H^r(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Q})$, known as the *rational Lefschetz decomposition*.

Let $f: \mathbb{C}\mathbb{G}_{n,k} \rightarrow \mathbb{C}\mathbb{G}_{m,l}$ be a continuous map of *non-zero degree*. The induced map in cohomology, f^* , preserves the rational Lefschetz decomposition. Let \bar{c}_1 denote the first Chern class of the universal quotient bundle over $\mathbb{C}\mathbb{G}_{m,l}$. Then, using the rational Lefschetz decomposition, we can find algebra generators, $\{\bar{c}_1 =: u_1, u_2, \dots, u_l\}$, of the cohomology algebra, $H^*(\mathbb{C}\mathbb{G}_{m,l}; \mathbb{Q})$, such that the image $f^*(u_i)$ can be determined upto a sign \pm .

The quaternionic Grassmann manifolds are not Kähler, nor cohomologically Kähler. However, since the degree doubling isomorphism from $H^*(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Z})$ to $H^*(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Z})$ maps the i -th Chern class of the tautological complex k -plane bundle over $\mathbb{C}\mathbb{G}_{n,k}$ to the i -th symplectic Pontrjagin class of the tautological left \mathbb{H} -bundle over $\mathbb{H}\mathbb{G}_{n,k}$, the proof of Theorem 1.0.1 still carries through.

Theorem 1.0.1 generalises the results of M.Hoffman in [11], which classify the *endomorphisms* of non-zero degree of the cohomology algebra of $\mathbb{C}\mathbb{G}_{n,k}$.

The rest of the thesis is organised as follows:

Chapter 2 contains certain definitions and basic results needed for the proofs of the main results. None of the material in this chapter is original. The first section of this chapter recalls the concept of Brouwer degree of a map between two smooth manifolds, while the second section describes the main problem of this thesis and the previous work on related subjects. The third section gives two descriptions of the cohomology of the complex Grassmann

manifold and introduces the concept of degree of a Schubert variety.

Chapter 3 discusses maps between c -Hodge manifolds with second Betti number equal to one. The first section of this chapter discusses some well-known facts about the cohomology of compact Kähler manifolds, following §15 of [8]. The second section contains certain preliminary results required for the proofs of the main theorems.

Chapter 4 contains the proofs of the main theorems and some examples.

CHAPTER 2

PRELIMINARIES

In this chapter we will recall certain well-known definitions and results which will be used in the thesis. In the first section, we recall the definition of Brouwer degree of a map and give some of its properties. In the second section, we describe the main problem of the thesis and discuss related work done previously by others. The third section gives two descriptions of the cohomology of the complex Grassmann manifold - one in terms of the Chern classes of the “tautological” bundle and the other in terms of Schubert classes. It also states a well-known formula for the degree of a Schubert variety. In the fourth section we recall the proof given in [19] of a result related to one of the main theorems in this thesis.

2.1 BROUWER DEGREE

In this section, M, N will denote closed, connected, oriented n -dimensional smooth manifolds. The symbol ‘ \simeq ’ denotes the relation ‘is homotopic to’.

Let the orientation classes in the top homology of M and N be denoted by μ_M and μ_N respectively. Recall that, given a continuous map, $f : M \rightarrow N$, the induced map in the top homology, $f_* : \mathbb{Z} \cong H_n(M; \mathbb{Z}) \rightarrow H_n(N; \mathbb{Z}) \cong \mathbb{Z}$, is given by $f_*(\mu_M) = (\deg f) \cdot \mu_N$, where $\deg f \in \mathbb{Z}$. This integer, $\deg f$, is called the (Brouwer) degree of the map f . Given two continuous maps $f, g : M \rightarrow N$ such that $f \simeq g$ are homotopic, $\deg f = \deg g$. Intuitively, $\deg f$ is

the number of times f wraps M around N .

Let $g : M \rightarrow N$ be a smooth map. Then $\deg g$ can be computed as follows: Let $x \in M$ be a regular point. The induced map of the tangent spaces $dg_x : TM_x \rightarrow TN_{g(x)}$ is a linear isomorphism of oriented vector spaces. Define the *sign* of dg_x to be $+1$ or -1 according as dg_x preserves or reverses orientation. Then, for any regular value $y \in N$, the degree of g at y is defined to be:

$$\deg(g; y) := \sum_{x \in g^{-1}(y)} \text{sign}(dg_x)$$

This integer, $\deg(g; y)$, does not depend upon the choice of the regular value y . In fact, $\deg(g; y) = \deg g$. In particular, if $f : M \rightarrow N$ is any continuous map and $g : M \rightarrow N$ a smooth map such that $g \simeq f$ then $\deg f$ equals $\deg g$. Since the degree of a map is invariant under homotopy, the above formula is independent of the choice of g , as long as $f \simeq g$. (Refer to [7], [15].)

The degree is ‘multiplicative’, that is, if P is a closed, connected, oriented, n -dimensional manifold and $f : M \rightarrow N$ and $g : N \rightarrow P$ are continuous maps, then $\deg(g \circ f) = \deg g \cdot \deg f$. Suppose $p : \tilde{N} \rightarrow N$ is a covering projection, with \tilde{N} connected. Then there exists a unique orientation on \tilde{N} such that p is orientation preserving, and, $\deg p$ equals $[\pi_1(N) : p_{\#}\pi_1(\tilde{N})]$ if \tilde{N} is compact. If \tilde{N} is non-compact (equivalently, $[\pi_1(N) : p_{\#}\pi_1(\tilde{N})] = \infty$), then $H_n(\tilde{N}; \mathbb{Z}) = 0$ and it is convenient to set $\deg p = 0$.

Let S^n denote the n -sphere, $n \geq 1$. Let $f, g : M \rightarrow S^n$ denote continuous maps. Then $f \simeq g$ if and only if $\deg f = \deg g$ and there exist such maps of every degree $m \in \mathbb{Z}$.

We end this section by giving the proof of the following well-known lemma:

Lemma 2.1.1 *Let $f : M \rightarrow N$ be a continuous map.*

(i) *If $\deg f \neq 0$, the induced map in cohomology, $f^* : H^*(N; \mathbb{Q}) \rightarrow H^*(M; \mathbb{Q})$, is a monomorphism.*

(ii) *If $\deg f = 1$, the induced map of fundamental groups, $f_{\#} : \pi_1(M) \rightarrow \pi_1(N)$, is surjective.*

Proof:

(i) Let μ_N denote the orientation class in $H_n(N; \mathbb{Z}) \hookrightarrow H_n(N; \mathbb{Q}) \cong \mathbb{Q}$. One has the non-degenerate pairing :

$$\begin{aligned} H^p(N; \mathbb{Q}) \times H^{n-p}(N; \mathbb{Q}) &\rightarrow \mathbb{Q} \\ (\alpha, \beta) &\mapsto \langle \alpha \cup \beta, \mu_N \rangle \\ &= \langle \alpha, \beta \cap \mu_N \rangle \end{aligned}$$

Let $\deg f = \lambda \neq 0$, $\lambda \in \mathbb{Z}$. Thus $f_*(\mu_M) = \lambda\mu_N$. If $0 \neq \alpha \in H^p(N; \mathbb{Q})$, choose $\beta \in H^{n-p}(N; \mathbb{Q})$ such that $\langle \alpha, \beta \rangle = 1$. Then, $\langle f^*(\alpha) \cup f^*(\beta), \mu_M \rangle = \langle \alpha \cup \beta, f_*(\mu_M) \rangle = \langle \alpha \cup \beta, \lambda\mu_N \rangle = \lambda \neq 0$. Therefore f^* is a monomorphism.

(ii) Suppose $f_\# : \pi_1(M) \rightarrow \pi_1(N)$ is not surjective. Then, $H := \text{Im}(f_\#)$ is a proper subgroup of $\pi_1(N)$ and $1 < [\pi_1(N):H] \leq \infty$. Let $p : \tilde{N} \rightarrow N$ be the covering of N corresponding to this subgroup H , with suitable orientation on \tilde{N} such that p is orientation preserving. Then, since $f_\#(\pi_1(M)) = H = p_*(\pi_1(\tilde{N}))$, f lifts to a map $\tilde{f} : M \rightarrow \tilde{N}$ such that $f = p \circ \tilde{f}$. Hence, $\deg f = \deg \tilde{f} \cdot \deg p$. If $[\pi_1(N):H] = \infty$ then \tilde{N} is non-compact and $H_n(\tilde{N}; \mathbb{Z}) = 0$, and it follows that $\deg f = 0$. If $[\pi_1(N):H] < \infty$, then \tilde{N} is compact and $\deg p = [\pi_1(N):H] > 1$. It follows that $\deg f > 1$ since degree is multiplicative. \square

2.2 THE PROBLEM

Let \mathbb{F} denote the field of real numbers, \mathbb{R} , complex numbers, \mathbb{C} , or the skew field of quaternions, \mathbb{H} . Let $\mathbb{F}\mathbb{G}_{n,k}$ denote the Grassmann manifold of all k -dimensional left \mathbb{F} -vector subspaces of \mathbb{F}^n . Then $\mathbb{F}\mathbb{G}_{n,k}$ is a compact, connected manifold of dimension $\dim_{\mathbb{F}} \mathbb{F}\mathbb{G}_{n,k} = k(n-k)$.

The complex Grassmann manifold, $\mathbb{C}\mathbb{G}_{n,k}$, admits a complex structure, and hence, has a natural orientation. Though $\mathbb{H}\mathbb{G}_{n,k}$ does not admit even an almost complex structure [14], it is oriented since it is simply connected. The real Grassmann manifold, $\mathbb{R}\mathbb{G}_{n,k}$, is orientable if and only if n is even. Assuming that $n > 2$, its universal (double) cover, the space of all oriented

k -planes in \mathbb{R}^n , denoted by $\widetilde{\mathbb{R}G}_{n,k}$, is orientable, and is referred to as the oriented Grassmann manifold. Now, let f be a continuous map between two oriented Grassmann manifolds, or between two complex (quaternionic) Grassmann manifolds, of the same dimension. Then, what can be the possible values for $\deg f$? Note that, any map $f : \mathbb{R}G_{2n,k} \rightarrow \mathbb{R}G_{2m,l}$, (where $k(2n - k) = l(2m - l)$), lifts to a map $\widetilde{f} : \widetilde{\mathbb{R}G}_{2n,k} \rightarrow \widetilde{\mathbb{R}G}_{2m,l}$, such that $\deg \widetilde{f} = \deg f$. Hence it is enough to address the question for the oriented Grassmann manifold.

In the case when $l = 1$ and $k(n - k) = m - 1$, it is known (by Hopf-Whitney Theorem [16]) that there are maps $f_\lambda : \widetilde{\mathbb{R}G}_{n,k} \rightarrow \widetilde{\mathbb{R}G}_{m,1} = S^{m-1}$ such that $\deg f_\lambda = \lambda$, for every integer $\lambda \in \mathbb{Z}$.

Let $f : \mathbb{C}G_{n,k} \rightarrow \mathbb{C}G_{m,l}$ be a continuous map, $k(n - k) = l(m - l) =: N$. When $l = 1$, $N = m - 1$ and $\mathbb{C}G_{m,1} = \mathbb{C}P^N$. Let c_1, C_1 denote the first Chern classes of the tautological bundles over $\mathbb{C}G_{n,k}$ and $\mathbb{C}P^N$ respectively. If f^* is the induced map in cohomology, $f^*(C_1) = \lambda_f c_1$, for some integer λ_f . Then, as observed in [19], since $\mathbb{C}P^N$ is the $(2N + 1)$ -skeleton of $\mathbb{C}P^\infty$ and $H^2(\mathbb{C}G_{n,k}; \mathbb{Z}) \cong [\mathbb{C}G_{n,k}; \mathbb{C}P^\infty] \cong \mathbb{Z}$, any two maps $f, g : \mathbb{C}G_{n,k} \rightarrow \mathbb{C}P^N$ are homotopic if and only if $\lambda_f = \lambda_g$. Also, given $\lambda \in \mathbb{Z}$, there exists an $f : \mathbb{C}G_{n,k} \rightarrow \mathbb{C}P^N$ such that $\lambda = \lambda_f$. The degree of f can be determined in terms of λ_f and is non-zero if and only if λ_f is non-zero.

Similarly, the set of homotopy classes of maps $f : \mathbb{H}G_{n,k} \rightarrow \mathbb{H}P^N$ is in bijection with the set of homomorphisms of abelian groups $\mathbb{Z} \cong H^4(\mathbb{H}P^N; \mathbb{Z}) \rightarrow H^4(\mathbb{H}G_{n,k}; \mathbb{Z}) \cong \mathbb{Z}$. In this case, f^* is the induced map in cohomology, $f^*(\bar{c}_2) = \lambda_f c_2$, for some integer λ_f , where c_2, \bar{c}_2 denote the second Chern classes of the tautological bundles over $\mathbb{H}G_{n,k}$ and $\mathbb{H}P^N$ respectively. The degree of f is again determined by λ_f and is non-zero if and only if λ_f is non-zero.

It is known that there exist continuous *self-maps* of complex and quaternionic Grassmann manifolds, which have arbitrarily large positive degrees, cf. [1], [21].

In their paper [18], K. Paranjape and V. Srinivas prove the following theorem:

Theorem 2.2.1 *Let $k < n$ and let $2 \leq l < m$ be two integers. If there exists a finite surjective morphism, $f : \mathbb{C}\mathbb{G}_{n,k} \rightarrow \mathbb{C}\mathbb{G}_{m,l}$, of projective varieties, then $(n, k) = (m, l)$ and f is an isomorphism.*

The results of M.Hoffman in [11] classify the *endomorphisms* of non-zero degree of the cohomology algebra of $\mathbb{C}\mathbb{G}_{n,k}$. He proves the following theorem:

Theorem 2.2.2 *Let h be an endomorphism of $H^*(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Q})$ such that $h(c_1) = mc_1$, $m \neq 0$, where $c = 1 + c_1 + \dots + c_k$ denotes the total Chern class of $\gamma_{n,k}$, the canonical k -plane bundle over $\mathbb{C}\mathbb{G}_{n,k}$. Then, if $k < n/2$, $h(c_i) = m^i c_i$, $1 \leq i \leq k$. If $k = n/2$, there is the additional possibility that $h(c_i) = (-m)^i (c^{-1})_i$, $1 \leq i \leq k$, where $(c^{-1})_i$ is the $2i$ -dimensional part of the inverse of c in $H^*(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Q})$.*

He also conjectures that the only endomorphism of the cohomology algebra $H^*(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Q})$ such that $h(c_1) = 0$ is the zero endomorphism.

In their paper [19], V. Ramani and P. Sankaran prove the following two theorems:

Theorem 2.2.3 *If $h : \mathbb{R}\tilde{\mathbb{G}}_{n,k} \rightarrow \mathbb{R}\tilde{\mathbb{G}}_{m,l}$ is any continuous map, where $(n, k) \neq (m, l)$, $2 \leq l \leq m/2$, $1 \leq k \leq n/2$ and $\dim \mathbb{R}\tilde{\mathbb{G}}_{n,k} = \dim \mathbb{R}\tilde{\mathbb{G}}_{m,l}$, then, $\deg h = 0$.*

Theorem 2.2.4 *If $f : \mathbb{C}\mathbb{G}_{n,k} \rightarrow \mathbb{C}\mathbb{G}_{m,l}$ (resp. $g : \mathbb{H}\mathbb{G}_{n,k} \rightarrow \mathbb{H}\mathbb{G}_{m,l}$), $1 \leq k < l \leq [m/2]$, is any continuous map between two complex (resp. quaternionic) Grassmannians of the same dimension, then $\deg f = 0 = \deg g$.*

They also state the following conjecture in [19]:

Conjecture: Let $\mathbb{F} = \mathbb{C}$ or \mathbb{H} and let $2 \leq l < k \leq n/2 < m/2$ where $k, l, m, n \in \mathbb{N}$. Assume that $k(n - k) = l(m - l)$. Let $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ be any continuous map. The degree of f is zero.

We recall the proof of Theorem 2.2.4 in §4 of this chapter.

Note that, since it is not true in general that any given continuous map, $f: \mathbb{C}\mathbb{G}_{n,k} \rightarrow \mathbb{C}\mathbb{G}_{m,l}$, can be homotoped to a complex analytic map, it does not follow from the work of Paranjape and Srinivas in [18] that the degree of such a map f is zero.

In this thesis, we consider maps $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$, where \mathbb{F} denotes \mathbb{C} or \mathbb{H} , $d = \dim_{\mathbb{R}} \mathbb{F}$, $2 \leq l < k \leq \lfloor n/2 \rfloor$ and $\mathbb{F}\mathbb{G}_{n,k}$ and $\mathbb{F}\mathbb{G}_{m,l}$ are \mathbb{F} -Grassmann manifolds of the same dimension. We prove the following:

- (i) If f is such a map of degree ± 1 then $(m, l) = (n, k)$ and f is a homotopy equivalence.
- (ii) If we fix integers $2 \leq l < k$, there can exist at most finitely many positive integers $m, n \geq 2k$ with $k(n - k) = l(m - l) =: N$, such that there exists a continuous map $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ of non-zero degree.
- (iii) Given a continuous map $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ of non-zero degree, where l, k, n, m satisfy the above conditions, there exist algebra generators $u_i \in H^{di}(\mathbb{F}\mathbb{G}_{m,l}; \mathbb{Q})$, $1 \leq i \leq l$, such that the image $f^*(u_i) \in H^{di}(\mathbb{F}\mathbb{G}_{n,k}; \mathbb{Q})$, $1 \leq i \leq l$, is determined upto a sign \pm by the degree of f .

The proofs of the results discussed in this thesis only involve studying algebra homomorphisms between the cohomology algebras of the concerned Grassmannians. The (integral) cohomology ring of the quaternionic Grassmannian is isomorphic to the cohomology ring of the corresponding complex Grassmannian, via a degree doubling isomorphism. The orientation of $\mathbb{H}\mathbb{G}_{n,k}$ is chosen such that the image of the positive generator of the top integral cohomology of $\mathbb{C}\mathbb{G}_{n,k}$, under the degree doubling isomorphism induced in the cohomology rings, is positive. (In the course of our proof of Theorem 1.0.3, simply- connectedness of the complex Grassmann manifold will be used; the

same property also holds for the quaternionic Grassmann manifolds.) For this reason, we need only consider the case of the complex Grassmann manifold, $\mathbb{C}\mathbb{G}_{n,k}$. Further, since $\mathbb{C}\mathbb{G}_{n,k}$ is diffeomorphic to $\mathbb{C}\mathbb{G}_{n,n-k}$, we need to consider only the cases $1 \leq k \leq [n/2]$ and $1 \leq l \leq [m/2]$.

2.3 THE COMPLEX GRASSMANN MANIFOLDS

In this section we discuss the complex Grassmann manifold and give two descriptions of its cohomology algebra.

Let $\mathbb{C}\mathbb{G}_{n,k}$ denote the Grassmann manifold of all k -dimensional linear subspaces in the n -dimensional complex space, \mathbb{C}^n . Recall that it is a closed, connected, oriented complex manifold of complex dimension $k(n-k)$. We will refer to $\mathbb{C}\mathbb{G}_{n,k}$ as $\mathbb{G}_{n,k}$ for the rest of the section.

The space of all unitary linear transformations $U(n)$ of \mathbb{C}^n acts transitively on $\mathbb{G}_{n,k}$. Under the action of $U(n)$, the stabiliser of \mathbb{C}^k , the subspace spanned by the first k standard basis vectors $\{e_1, \dots, e_k\}$, is $U(k) \times U(n-k)$. Hence, $\mathbb{G}_{n,k}$ is homeomorphic to the homogeneous space $U(n)/U(k) \times U(n-k)$.

The Plücker imbedding, $p : \mathbb{G}_{n,k} \hookrightarrow \mathbb{P}(\Lambda^k(\mathbb{C}^n)) = \mathbb{C}\mathbb{P}^{\binom{n}{k}-1}$, is given by $U \mapsto \Lambda^k(U)$, where $\Lambda^k(U)$ denotes the k -th exterior power of the vector space U . This gives $\mathbb{G}_{n,k}$ a projective variety structure.

2.3.1 COHOMOLOGY OF THE GRASSMANNIAN

The ‘tautological’ bundle over $\mathbb{G}_{n,k}$, $\gamma_{n,k}$, is constructed as follows: The total space of $\gamma_{n,k}$ is $E = E(\gamma_{n,k}) := \{(V, x) | x \in V \in \mathbb{G}_{n,k}\} \subset \mathbb{G}_{n,k} \times \mathbb{C}^n$ with the subspace topology. The projection map $\pi : E \rightarrow \mathbb{G}_{n,k}$ is given by $(V, x) \mapsto V$ and the fibre over a point $V \in \mathbb{G}_{n,k}$ is the k -dimensional complex vector space V . Evidently, $\gamma_{n,k}$ is rank k -subbundle of the rank n trivial bundle \mathcal{E}^n with projection $pr_1 : \mathbb{G}_{n,k} \times \mathbb{C}^n \rightarrow \mathbb{G}_{n,k}$. The quotient bundle $\mathcal{E}^n/\gamma_{n,k}$ is isomorphic to the orthogonal complement $\gamma_{n,k}^\perp$ in \mathcal{E}^n (with respect to a hermitian metric

on \mathbb{C}^n) of the bundle $\gamma_{n,k}$. Let $c_i(\gamma_{n,k}) \in H^{2i}(\mathbb{G}_{n,k}; \mathbb{Z})$, be the i -th Chern class of $\gamma_{n,k}$, $1 \leq i \leq k$. Denoting the total Chern class of a vector bundle η by $c(\eta)$ we see that $c(\gamma_{n,k}) \cdot c(\gamma_{n,k}^\perp) = 1$.

Let c_1, \dots, c_k denote the elementary symmetric polynomials in k indeterminates x_1, \dots, x_k . Define $h_j = h_j(c_1, \dots, c_k)$ by the identity:

$$\prod_{1 \leq i \leq k} (1 + x_i t)^{-1} = \sum_{j \geq 0} h_j t^j \quad (2.3.1.1)$$

Thus $c_j(\gamma_{n,k}^\perp) = h_j(c_1(\gamma_{n,k}), c_2(\gamma_{n,k}), \dots, c_k(\gamma_{n,k}))$, $1 \leq j \leq n - k$. (See [17].) In particular,

$$(1 + c_1 + \dots + c_k)(1 + h_1 + \dots + h_{n-k}) = 1 \quad (2.3.1.2)$$

Consider the ring $\mathbb{Z}[c_1, \dots, c_k]/\mathcal{I}_{n,k}$ where degree of $c_i = 2i$, and $\mathcal{I}_{n,k}$ is the ideal $\langle h_j \mid j > n - k \rangle$. Equating the coefficient of t^{n+1} in equation(2.3.1.1) to zero, we can express h_{n+1} in terms of a polynomial expression in h_{n-k+1}, \dots, h_n . Now, if we assume that $h_{n+1}, \dots, h_{n+m-1}$; $1 < m$ can be expressed in terms of h_{n-k+1}, \dots, h_n , then, by equating the coefficient of t^{n+m} to zero, we see that h_{n+m} can be expressed in terms of h_{n-k+1}, \dots, h_n . Hence, by induction, we see that the elements h_j , $n - k + 1 \leq j \leq n$, generate $\mathcal{I}_{n,k}$.

The homomorphism of graded rings $\mathbb{Z}[c_1, \dots, c_k] \longrightarrow H^*(\mathbb{G}_{n,k}; \mathbb{Z})$ defined by $c_i \mapsto c_i(\gamma_{n,k})$ is surjective and has kernel $\mathcal{I}_{n,k}$ and hence we have an isomorphism $H^*(\mathbb{G}_{n,k}; \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_k]/\mathcal{I}_{n,k}$. Henceforth we shall write c_i to mean $c_i(\gamma_{n,k}) \in H^*(\mathbb{G}_{n,k}; \mathbb{Z})$. We shall denote by \bar{c}_j the element $c_j(\gamma_{n,k}^\perp) = h_j \in H^{2j}(\mathbb{G}_{n,k}; \mathbb{Z})$.

As an abelian group, $H^*(\mathbb{G}_{n,k}; \mathbb{Z})$ is free of rank $\binom{n}{k}$. A \mathbb{Q} -basis for $H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$ is the set \mathcal{C}_r of all monomials $c_1^{j_1} \dots c_k^{j_k}$ where $j_i \leq n - k$ for $1 \leq i \leq k$ and $\sum_{1 \leq i \leq k} i j_i = r$. In particular, c_k^{n-k} generates $H^{2N}(\mathbb{G}_{n,k}; \mathbb{Q}) \cong \mathbb{Q}$. If \mathbf{j} denotes the sequence j_1, \dots, j_k , we shall denote by $c^\mathbf{j}$ the monomial $c_1^{j_1} \dots c_k^{j_k}$. If $k \leq n/2$, the set $\bar{\mathcal{C}}_r := \{\bar{c}^\mathbf{j} \mid c^\mathbf{j} \in \mathcal{C}_r\}$ is also a basis for $H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$, where

$\bar{c}^{\mathbf{j}} := \bar{c}_1^{j_1} \cdots \bar{c}_k^{j_k}$. (See [17] for details.)

2.3.2 SCHUBERT CALCULUS

Another, more classical description of the cohomology ring of the Grassmann manifold $\mathbb{G}_{n,k}$ is via the Schubert calculus. Recall that $\mathbb{G}_{n,k} = \mathrm{SL}(n, \mathbb{C})/P_k$, for the parabolic subgroup $P_k \subset \mathrm{SL}(n, \mathbb{C})$ which stabilizes $\mathbb{C}^k \subset \mathbb{C}^n$ spanned by e_1, \dots, e_k ; here $e_i, 1 \leq i \leq n$, are the standard basis elements of \mathbb{C}^n . Denote by $B \subset \mathrm{SL}(n, \mathbb{C})$ the Borel subgroup of $\mathrm{SL}(n, \mathbb{C})$ which preserves the flag $\mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n$ and by $T \subset B$ the maximal torus which preserves the coordinate axes $\mathbb{C}e_j, 1 \leq j \leq n$.

Let $I(n, k)$ denote the set of all k element subsets of $\{1, 2, \dots, n\}$; we regard elements of $I(n, k)$ as increasing sequences of positive integers $\mathbf{i} := i_1 < \cdots < i_k$ where $i_k \leq n$. One has a partial order on $I(n, k)$ where, by definition, $\mathbf{i} \leq \mathbf{j}$ if $i_p \leq j_p$ for all $p, 1 \leq p \leq k$. Let $\mathbf{i} \in I(n, k)$ and let $E_{\mathbf{i}} \in \mathbb{G}_{n,k}$ denote the vector subspace of \mathbb{C}^n spanned by $\{e_j \mid j \in \mathbf{i}\}$. The fixed points for the action of $T \subset \mathrm{SL}(n)$ on $\mathbb{G}_{n,k}$ are precisely the $E_{\mathbf{i}}, \mathbf{i} \in I(n, k)$.

Schubert varieties in $\mathbb{G}_{n,k}$ are in bijection with the set $I(n, k)$. The B -orbit of the T -fixed point $E_{\mathbf{i}}$ is the Schubert cell corresponding to \mathbf{i} and is isomorphic to the affine space of (complex) dimension $\sum_j (i_j - j) =: |\mathbf{i}|$; its closure, denoted $\Omega_{\mathbf{i}}$, is the Schubert variety corresponding to $\mathbf{i} \in I(n, k)$. It is the union of all Schubert cells corresponding to those $\mathbf{j} \in I(n, k)$ such that $\mathbf{j} \leq \mathbf{i}$.

Schubert cells yield a cell decomposition of $\mathbb{G}_{n,k}$. Since the cells have even (real) dimension, the class of Schubert varieties form a \mathbb{Z} -basis for the integral homology of $\mathbb{G}_{n,k}$. Denote by $[\Omega_{\mathbf{i}}] \in H^{2(N-|\mathbf{i}|)}(\mathbb{G}_{n,k}; \mathbb{Z})$ the fundamental *dual* cohomology class determined by $\Omega_{\mathbf{i}}$. (Thus $[\mathbb{G}_{n,k}] \in H^0(\mathbb{G}_{n,k}; \mathbb{Z})$ is the identity element of the cohomology ring.) We shall denote the fundamental homology class of $\mathbb{G}_{n,k}$ by $\mu_{n,k} \in H_{2N}(\mathbb{G}_{n,k}; \mathbb{Z})$.

Schubert varieties corresponding to $(n-k+1-i, n-k+2, \dots, n) \in I(n, k), 0 \leq i \leq n-k$, are called *special* and will be denoted Ω_i . More generally, if $\nu = \nu_1 \geq \cdots \geq \nu_k \geq 0$ is a partition of an integer $r, 0 \leq r \leq N$, with $\nu_1 \leq n-k$,

we obtain an element $(n - k + 1 - \nu_1, n - k + 2 - \nu_2, \dots, n - \nu_k) \in I(n, k)$ with $|\mathbf{i}| = N - r$. This association establishes a bijection between such partitions and $I(n, k)$, or, equivalently, the Schubert varieties $\Omega_{\mathbf{i}}$ in $\mathbb{G}_{n,k}$. It is sometimes convenient to denote the Schubert variety $\Omega_{\mathbf{i}}$ by Ω_{ν} where ν corresponds to \mathbf{i} . This is consistent with our notation for a special Schubert variety.

The special Schubert classes form a set of algebra generators of $H^*(\mathbb{G}_{n,k}; \mathbb{Z})$. Indeed, $[\Omega_i] = c_i(\gamma_{n,k}^\perp) = \bar{c}_i, 1 \leq i \leq n - k$.

Taking the special Schubert classes $[\Omega_i], 1 \leq i \leq n - k$, as algebra generators of $H^*(\mathbb{G}_{n,k}; \mathbb{Z})$, the structure constants are determined by (i) the Pieri formula, which expresses the cup-product of an arbitrary Schubert class with a special Schubert class as a linear combination of with non-negative integral linear combination of Schubert classes, and, (ii) the Giambelli formula, which expresses an arbitrary Schubert class as a determinant in the special Schubert classes. These two formulae can be stated as follows [2]:

Let Ω_a denote a special Schubert class and Ω_ν denote any Schubert class.

(i) The Pieri formula: $[\Omega_a] \cdot [\Omega_\nu] = \sum [\Omega_\lambda]$, where the sum is taken over all λ such that $\sum_{i=1}^k \lambda_i = \sum_{i=1}^k \nu_i + n - k + 1 - a$ and $n - k \geq \lambda_1 \geq \nu_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq \nu_k \geq 0$.

(ii) The Giambelli formula: $[\Omega_\nu] = |[\Omega_{\nu_i + j - i}]|$, where $|x_{i,j}|$ denotes the determinant of the matrix $(x_{i,j})$.

The basis $\{[\Omega_{\mathbf{i}}] \mid \mathbf{i} \in I(n, k)\}$ is ‘self-dual’ under the Poincaré duality. That is, assume that $\mathbf{i}, \mathbf{j} \in I(n, k)$ are such that $|\mathbf{i}| + |\mathbf{j}| = N$. Then $\langle [\Omega_{\mathbf{i}}][\Omega_{\mathbf{j}}], \mu_{n,k} \rangle = \delta_{\lambda, \mathbf{j}}$, where $\lambda = (n + 1 - i_k, \dots, n + 1 - i_1) \in I(n, k)$.

2.3.3 DEGREE OF A SCHUBERT VARIETY

The *degree* of a Schubert variety $\Omega_{\mathbf{i}}$ of (complex) dimension r is defined as the integer $\langle [\Omega_{\mathbf{i}}] \bar{c}_1^r, \mu_{n,k} \rangle \in \mathbb{Z}$. We recall the following well-known numerical formula for the degree of a Schubert Variety, [10],[2]:

Lemma 2.3.1 *Let $\Omega_{\mathbf{i}}$ be a Schubert variety of (complex) dimension r . Then the degree of $\Omega_{\mathbf{i}}$ is given by:*

$$\deg(\Omega_{\mathbf{i}}) = \frac{r! \prod_{1 \leq t < s \leq k} (i_s - i_t)}{(i_1 - 1)! \cdots (i_k - 1)!} \quad (2.3.3.1)$$

In particular,

$$\deg(\mathbb{G}_{n,k}) = \langle \bar{c}_1^N, \mu_{n,k} \rangle = \frac{N! 1! \cdots (k-1)!}{(n-k)! \cdots (n-1)!}. \quad (2.3.3.2)$$

For the proof of the above formula we refer to [10, Chapter XIV, §7].

More generally, $\deg([\Omega_{\mathbf{i}}][\Omega_{\mathbf{j}}]) := \langle [\Omega_{\mathbf{i}}][\Omega_{\mathbf{j}}]\bar{c}_1^q, \mu_{n,k} \rangle = q! |1/(i_r + j_{k+1-j} - n - 1)!|$ where $q = \dim(\Omega_{\mathbf{i}}) + \dim(\Omega_{\mathbf{j}}) - \dim \mathbb{G}_{n,k}$ (See [2, p.274]. We caution the reader that our notations for Grassmann manifolds and Schubert varieties are different from those used in Fulton's book [2].)

One has the following geometric interpretation for the degree of a Schubert variety. More generally, given any algebraic imbedding $X \hookrightarrow \mathbb{C}\mathbb{P}^m$ of a projective variety X of dimension d in the complex projective space $\mathbb{C}\mathbb{P}^m$, the *degree* of X is the number of points in the intersection of X with d hyperplanes in general position. The degree of a Schubert variety defined above is the degree of the Plücker imbedding $\Omega_{\mathbf{j}} \subset \mathbb{G}_{n,k} \hookrightarrow \mathbb{P}(\Lambda^k(\mathbb{C}^n))$, defined as $U \mapsto \Lambda^k(U)$, where $\Lambda^k(U)$ denotes the k -th exterior power of the vector space U .

2.4 PROOF OF THEOREM 2.2.4

In this section we reproduce the proof of Theorem 2.2.4 as given in [19, §2], with a slight change of notation.

Let $1 \leq k \leq [n/2]$, $n, k \in \mathbb{Z}$. Let c_1, \dots, c_k denote the elementary symmetric

polynomials in k indeterminates and define $h_j = h_j(c_1, \dots, c_k)$ by the identity (2.3.1.1). Let $H_{n,k}$ denote the graded \mathbb{Q} -algebra $\mathbb{Q}[c_1, \dots, c_k]/\mathcal{I}_{n,k}$ where degree of $c_i = 2i$, and $\mathcal{I}_{n,k}$ is the ideal $\langle h_j \mid j > n - k \rangle$. Recall from §3.1 of this chapter that $\mathbb{Q}[c_1, \dots, c_k]/\mathcal{I}_{n,k}$ is isomorphic to $H^*(\mathbb{C}G_{n,k}; \mathbb{Q})$. There are no algebraic relations among c_1, \dots, c_k upto degree $n - k$. We have the following lemma:

Lemma 2.4.1 *If $1 \leq l < k \leq n/2$, and $m \in \mathbb{Z}$ is such that $k(n - k) \leq l(m - l)$, then for any homomorphism $\varphi: H_{n,k} \rightarrow H_{m,l}$ of graded \mathbb{Q} -algebras, $\ker \varphi \neq 0$.*

Proof: Let z_1, \dots, z_l denote the defining algebra generators of $H_{m,l}$, with degree of $z_j = 2j$, $1 \leq j \leq l$. Let, if possible, $\varphi: H_{n,k} \rightarrow H_{m,l}$ be a monomorphism of \mathbb{Q} -algebras. Since $k > l$, there exists an $i \leq l + 1$ such that $\varphi(c_i)$ is in the subalgebra generated by z_1, \dots, z_{i-1} . We may assume $1 \leq i$ is the smallest such integer. If $i = 1$, then $\varphi = 0$. Therefore, $i \geq 2$ and

$$\varphi(c_i) = \lambda_1 z_1, \lambda_1 \neq 0, \quad (2.4.1)$$

$$\varphi(c_j) = \lambda_j z_j + P_j(z_1, z_2, \dots, z_{j-1}) \lambda_j \neq 0, \quad (2.4.2)$$

and

$$\varphi(c_i) = P_i(z_1, z_2, \dots, z_{i-1}).$$

for suitable polynomials P_j , $1 \leq j \leq i$. In view of (2.4.1) and (2.4.2), one can express the z_j as a polynomial in $\varphi(c_1), \varphi(c_2), \dots, \varphi(c_j)$ for $1 \leq j \leq i - 1$. Thus

$$\mathbb{Q}[z_1, z_2, \dots, z_{i-1}] = \mathbb{Q}[\varphi(c_1), \varphi(c_2), \dots, \varphi(c_{i-1})].$$

In particular, for a suitable polynomial Q , one has

$$P_i(z_1, z_2, \dots, z_{i-1}) = Q(\varphi(c_1), \varphi(c_2), \dots, \varphi(c_{i-1})),$$

and hence

$$\varphi(c_i) = Q(\varphi(c_1), \varphi(c_2), \dots, \varphi(c_{i-1})) = \varphi(Q(c_1, c_2, \dots, c_{i-1})).$$

Therefore, $c_i - Q(c_1, c_2, \dots, c_{i-1}) \in \ker \varphi = 0$. But this contradicts the fact that there are no algebraic relations among c_1, \dots, c_k upto degree $n - k$.

Therefore, we must have $\ker \varphi \neq 0$. □

Theorem 2.2.4 *If $f: \mathbb{C}G_{n,k} \rightarrow \mathbb{C}G_{m,l}$ (resp. $g: \mathbb{H}G_{n,k} \rightarrow \mathbb{H}G_{m,l}$), $1 \leq k < l \leq [m/2]$, is any continuous map between two complex (resp. quaternionic) Grassmannians of the same dimension, then $\deg f = 0 = \deg g$.*

Proof: Let $1 \leq l < k \leq [n/2]$, $k(n-k) = l(m-l)$. Let $f: \mathbb{C}G_{n,k} \rightarrow \mathbb{C}G_{m,l}$ be any continuous map. Then, f induces an algebra homomorphism $f^*: H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \rightarrow H^*(\mathbb{C}G_{m,l}; \mathbb{Q})$. As $H^*(\mathbb{C}G_{n,k}; \mathbb{Q}) \cong H_{n,k}$, it is immediate from Lemma 2.4.1 that $\ker(f^*) \neq 0$. Hence, $\deg f = 0$. Proof for $g: \mathbb{H}G_{n,k} \rightarrow \mathbb{H}G_{m,l}$ is similar. □

CHAPTER 3

MAPS BETWEEN c -HODGE MANIFOLDS WITH $b_2 = 1$

3.1 COHOMOLOGY OF COMPACT KÄHLER MANIFOLDS

In this section we discuss the cohomology of compact Kähler manifolds. We will follow §15 of [8].

Let X denote a compact, complex manifold of complex dimension N . Let $A^{p,q}$ denote the complex vector space of global forms of type (p, q) . The operator d on the differential forms can be written as $d = \partial + \bar{\partial}$, where ∂ denotes differentiation with respect to z -variables and $\bar{\partial}$ denotes differentiation with respect to \bar{z} -variables. Let Ω^p denote the sheaf of holomorphic p -forms on X . Then, $H^{p,q}(X)$ denotes $H^q(X; \Omega^p)$, the q -th cohomology group of X with coefficients in Ω^p . By Dolbeault - Serre Theorem we have: $H^{p,q}(X) \cong Z^{p,q}/\bar{\partial}(A^{p,q-1})$, where $Z^{p,q}$ is the module of all those global forms of type (p, q) which vanish under $\bar{\partial}$.

Let $\#: A^{p,q} \rightarrow A^{n-p, n-q}$ denote the duality operator given by $\# = \bar{\cdot} \circ *$, where $*$ denotes the Hodge $*$ -operation with orientation obtained from the complex structure, and $\bar{\cdot}$ is the complex conjugation. Then we can define a natural hermitian scalar product on $A^{p,q}$ given by: $(\cdot, \cdot)_*: A^{p,q} \times A^{p,q} \rightarrow \mathbb{C}$, $(\alpha, \beta)_*$

We use the symbol $(\cdot, \cdot)_*$ as (\cdot, \cdot) will be reserved for another inner product, to be introduced later.

$:= \int_X \alpha \wedge \# \beta$. Let $\vartheta: A^{p,q} \rightarrow A^{p,q-1}$ be the adjoint of $\bar{\partial}$ with respect to this scalar product, that is, for $\alpha \in A^{p,q}$, $\beta \in A^{p,q+1}$, $(\alpha, \vartheta\beta)_* = (\bar{\partial}\alpha, \beta)_*$. Then, we define the complex Laplace-Beltrami operator $\square: A^{p,q} \rightarrow A^{p,q}$ by $\square = \vartheta\bar{\partial} + \bar{\partial}\vartheta$. An element $\alpha \in A^{p,q}$ such that $\square\alpha = 0$ is called ‘‘complex harmonic’’. The subspace of all complex harmonic forms of type (p, q) will be denoted by $B^{p,q}(X)$ or, simply, $B^{p,q}$.

It is known, due to the work of Dolbeault, that $H^{p,q}(X)$ can be identified with $B^{p,q}$. Kodaira proved that $B^{p,q}$ is finite-dimensional (cf. [12]).

Let (X, Ω) be a Kähler manifold. Then, with respect to local coordinates z^α ($\alpha = 1, \dots, N$), Ω has the following local description:

$$\Omega = 2i \sum_{\alpha, \beta=1}^N g_{\alpha, \beta} dz^\alpha \wedge d\bar{z}^\beta$$

where Ω is the Kähler form associated to the Hermitian metric: $ds^2 = 2 \sum g_{\alpha, \beta}(z, \bar{z}) dz^\alpha d\bar{z}^\beta$ with $\overline{g_{\alpha, \beta}} = g_{\beta, \alpha}$. The form Ω represents an element ω of the cohomology group $H^2(X, \mathbb{R})$, and is called the Kähler class. Note that $\omega^n \neq 0$.

For a Kähler manifold the Laplace-Beltrami operator \square equals $\frac{\Delta}{2}$, where Δ is the real Laplace operator, that is, $\Delta = d\delta + \delta d$, δ being the adjoint operator to d . Therefore, \square commutes with conjugation. In this case, $\alpha \mapsto \bar{\alpha}$ defines a conjugate-isomorphism between $B^{p,q}$ and $B^{q,p}$. By the theory of de Rham and Hodge, we have:

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r} B^{p,q} \quad (3.1.1)$$

where $H^r(X, \mathbb{C})$ denotes the usual singular cohomology. Therefore, elements of the image of $B^{p,q}$ in $H^{p+q}(X, \mathbb{C})$, under this isomorphism, can be represented by forms α of type (p, q) such that $d\alpha = 0$.

For any compact Kähler manifold (X, Ω) with $\dim_{\mathbb{C}} X = N$, even Betti numbers upto dimension $2N$ are positive and odd Betti numbers are even.

Any complex analytic submanifold of a Kähler manifold is Kähler. The complex projective spaces, $\mathbb{C}\mathbb{P}^n$, $n \geq 1$, are well-known to be Kähler, and hence, any smooth projective variety over \mathbb{C} is Kähler. In particular, the complex Grassmann manifolds are Kähler. In fact, $H^{p,q}(\mathbb{C}\mathbb{G}_{n,k}) = 0$ if $p \neq q$, and, $H^{2p}(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{C}) = B^{p,p}$, since $\mathbb{C}\mathbb{G}_{n,k}$ has an algebraic cell decomposition. See [2, p. 23].

Suppose $\dim_{\mathbb{C}} X = N = 2s$ is even, where (X, Ω) is a compact Kähler manifold. Let $z_j = x_{2j-1} + \sqrt{-1}x_{2j}$ be local complex coordinates. Then, we will use the orientation on X given by $dx_1 \wedge dx_2 \wedge \dots \wedge dx_{2N}$.

There exist \mathbb{C} -linear operators L and Λ on $H^*(X; \mathbb{C})$, which are given by $L: H^{p,q}(X) \rightarrow H^{p+1,q+1}(X)$, $\alpha \mapsto \omega \wedge \alpha$ and $\Lambda: H^{p,q}(X) \rightarrow H^{p-1,q-1}(X)$, $\alpha \mapsto (-1)^{p+q} \#L\#\alpha$ for $\alpha \in B^{p,q}$. Since ω is real, $\overline{L\alpha} = L\bar{\alpha}$.

The operator Λ is dual to L with respect to the hermitian scalar product, $(\cdot, \cdot)_*$, on $H^r(X; \mathbb{C}) = \bigoplus_{p+q=r} B^{p,q}$. The kernel of Λ is denoted by $B_0^{p,q}$ and is called the subspace of *effective harmonic forms of type (p, q)* . Then, for $p+q \leq N$ and $k \geq 1$ we have the following:

- (a) $\Lambda L^k: B_0^{p-k, q-k} \rightarrow B^{p-1, q-1}$ is a non-zero scalar multiple of L^{k-1} .
- (b) $L^k: B_0^{p-k, q-k} \rightarrow B^{p, q}$ is a monomorphism.
- (c) There is a direct sum decomposition

$$B^{p, q} = \bigoplus_{0 \leq k \leq \min\{p, q\}} L^k B_0^{p-k, q-k} \quad (3.1.2)$$

where, for $k = 0$, L^k equals the identity operator.

For $1 \leq k \leq \min\{p, q\}$, define the subspace $B_k^{p, q}$ of elements of *harmonic forms of type (p, q) and class k* to be equal to $L^k B_0^{p-k, q-k}$. Then we have:

(d) $\#\varphi = (-1)^{q+k}\overline{\varphi}$, where $\varphi \in B_k^{p,q}$, $\overline{\varphi} \in B_k^{q,p}$ and $p+q \leq N$.

Therefore, the Hodge decomposition (3.1.1) of the cohomology group $H^N(X, \mathbb{C})$ can be rewritten as:

$$H^N(X, \mathbb{C}) = \bigoplus_{p+q=N, k \leq \min\{p,q\}} B_k^{p,q} \quad (3.1.3)$$

Since the scalar product of two forms $(\alpha, \beta)_*$ is non-zero only if $\alpha \wedge \#\beta$ is of type (N, N) , $B_k^{p,q}$ and $B_{k'}^{p',q'}$ are mutually orthogonal if $(p, q) \neq (p', q')$. Also, if $\alpha \in B_k^{p,q}$ and $\beta \in B_{k'}^{p',q'}$, where $p+q = N$ and $k > k'$, there exist $\alpha_0 \in B_0^{p-k, q-k}$ and $\beta_0 \in B_0^{p-k', q-k'}$, such that $(\alpha, \beta)_* = (L^k \alpha_0, L^{k'} \beta_0)_* = (\alpha_0, \Lambda^k L^{k'} \beta_0)_* = 0$. Therefore, the distinct summands in the Hodge decomposition (3.1.3) are pairwise orthogonal with respect to the scalar product $(\cdot, \cdot)_*$.

In case of the real cohomology group $H^N(X; \mathbb{R})$, we can define a quadratic form $Q(\cdot, \cdot)$ by: $Q(\alpha, \beta) = \int_X \alpha \wedge \beta$, where $\alpha, \beta \in H^N(X; \mathbb{R})$. Now, let $E_k^{p,q}$ be the real vector space of real harmonic forms α , which can be written in the form $\alpha = \varphi + \overline{\varphi}$, where $\varphi \in B_k^{p,q}$, $\overline{\varphi} \in B_k^{q,p}$ and $p+q = N$. Then we have direct sum decomposition of the real cohomology group, $H^N(X; \mathbb{R})$, given by:

$$H^N(X, \mathbb{R}) = \bigoplus_{p+q=N, k \leq \min\{p,q\}} E_k^{p,q} \quad (3.1.4)$$

where, $p+q \leq N$ and the summands in the above decomposition are mutually orthogonal with respect to the quadratic form $Q(\cdot, \cdot)$.

Since our proofs require only the cohomological properties of compact Kähler manifolds, in the subsequent discussions we will mainly work with c -Kähler manifolds.

A compact connected orientable smooth manifold X is called c -symplectic (or cohomologically symplectic) if there exists an element $\omega \in H^2(X; \mathbb{R})$, called a c -symplectic class, such that $\omega^N \in H^{2N}(X; \mathbb{R}) \cong \mathbb{R}$ is non-zero where $N = (1/2) \dim_{\mathbb{R}} X$. If ω is a c -symplectic class in X , then (X, ω) is said to satisfy the *weak Lefschetz* (respectively *hard Lefschetz*) condition if $\cup \omega^{N-1}: H^1(X; \mathbb{R}) \longrightarrow H^{2N-1}(X; \mathbb{R})$

(respectively $\cup\omega^i: H^{N-i}(X; \mathbb{R}) \longrightarrow H^{N+i}(X; \mathbb{R})$, $1 \leq i \leq N$),

is an isomorphism. If (X, ω) satisfies the hard Lefschetz condition, then X is called c -Kähler or cohomologically Kähler. If (X, ω) is c -Kähler, and if ω is in the image of the natural map in $H^2(X; \mathbb{Z}) \longrightarrow H^2(X; \mathbb{R})$, we call X c -Hodge. Note that if (X, ω) is c -Kähler and if $H^2(X; \mathbb{R}) \cong \mathbb{R}$, then $(X, t\omega)$ is c -Hodge for some $t \in \mathbb{R}$.

Clearly Kähler manifolds are c -Kähler and smooth projective varieties over \mathbb{C} are c -Hodge. It is known that $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ is c -symplectic but not symplectic (hence not Kähler) since it is known that it does not admit even an almost complex structure. It is also c -Kähler. Examples of c -symplectic manifolds which satisfy the weak Lefschetz condition but not c -Kähler are also known (cf. [13]).

Any c -symplectic manifold (X, ω) is naturally oriented; the fundamental class of X will be denoted by $\mu_X \in H_{2N}(X; \mathbb{Z}) \cong \mathbb{Z}$.

Let (X, ω) be a c -Kähler manifold of real dimension $2N$. Let $1 \leq r \leq N$. One has a bilinear form $(\cdot, \cdot)_\omega$ (or simply (\cdot, \cdot) when there is no danger of confusion) on $H^r(X; \mathbb{R})$ defined as $(\alpha, \beta)_\omega = \langle \alpha\beta\omega^{N-r}, \mu_X \rangle$, $\alpha, \beta \in H^r(X; \mathbb{R})$. When (X, ω) is c -Hodge, the above form is rational, that is, it restricts to a bilinear form $H^r(X; \mathbb{Q}) \times H^r(X; \mathbb{Q}) \longrightarrow \mathbb{Q}$. It will be important for us to consider the bilinear form on the rational vector space $H^r(X; \mathbb{Q})$ rather than on the real vector space $H^r(X; \mathbb{R})$.

The bilinear form (\cdot, \cdot) is symmetric (resp. skew symmetric) if r is even (resp. odd). Note that the above form is non-degenerate for all r . This follows from Poincaré duality and the hard Lefschetz condition that $\beta \mapsto \beta \cup \omega^{N-r}$ is an isomorphism $H^r(X; \mathbb{Q}) \longrightarrow H^{2N-r}(X; \mathbb{Q})$. Further, if $r \leq N$, the monomorphism $\cup\omega: H^{r-2}(X; \mathbb{Q}) \longrightarrow H^r(X; \mathbb{Q})$ is an isometric imbedding, i.e., $(\alpha, \beta) = (\alpha\omega, \beta\omega)$ for all $\alpha, \beta \in H^{r-2}(X; \mathbb{R})$.

As in the case of Kähler manifolds (cf. [9],[22],[8]), one obtains an orthogonal decomposition of the real cohomology groups of a c -Kähler manifold (X, ω) . The decomposition, which preserves the rational structure when (X, ω) is c -Hodge, is obtained as follows: Let $1 \leq r \leq N$. Let \mathcal{V}_ω^r , or more briefly

\mathcal{V}^r when ω is clear from the context, be the kernel of the homomorphism $\cup \omega^{N-r+1}: H^r(X; \mathbb{R}) \rightarrow H^{2N-r+2}(X; \mathbb{R})$. An element of \mathcal{V}^r will be called a *primitive class*. One has the *Lefschetz decomposition*

$$H^r(X; \mathbb{R}) = \bigoplus_{0 \leq q \leq [r/2]} \omega^q \mathcal{V}^{r-2q}. \quad (3.1.5)$$

3.2 MAPS BETWEEN c -HODGE MANIFOLDS WITH $b_2 = 1$

It is important for the proofs of our main results that we work with rational cohomology. Recall that a c -Kähler manifold (X, ω) is said to be c -Hodge if ω is in the image of the natural map in $H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{R})$. We had defined, in the previous section, the bilinear form $(\cdot, \cdot)_\omega$ (or simply (\cdot, \cdot)) on $H^r(X; \mathbb{R})$ as $(\alpha, \beta)_\omega = \langle \alpha \beta \omega^{N-r}, \mu_X \rangle$, $\alpha, \beta \in H^r(X; \mathbb{R})$. We prove the following lemma for c -Hodge manifolds with *second Betti number*, $b_2 = 1$.

Lemma 3.2.1 *Suppose that (X, ω) is a c -Hodge manifold of dimension $2N$ with second Betti number equal to 1. Let $f: X \rightarrow Y$ be any continuous map of non-zero degree where Y is a compact manifold with non-vanishing second Betti number. Then:*

- (i) $(\cdot, \cdot)_{t\omega} = t^{N-r}(\cdot, \cdot)_\omega$ on $H^r(X; \mathbb{Q})$ for $t \in \mathbb{Q}$, $t \neq 0$.
- (ii) (Y, φ) is c -Hodge where $\varphi \in H^2(Y; \mathbb{Q})$ is the unique class such that $f^*(\varphi) = \omega$. Furthermore, f^* preserves the Lefschetz decomposition (3.1.5), that is, $f^*(\mathcal{V}_\varphi^r) \subset \mathcal{V}_\omega^r$ for $r \leq N$.
- (iii) If $\alpha, \beta \in H^r(Y; \mathbb{Q})$, then $(f^*(\alpha), f^*(\beta))_\omega = \deg(f)(\alpha, \beta)_\varphi$. In particular, degree of f equals $\frac{\langle \omega^N, \mu_X \rangle}{\langle \varphi^N, \mu_Y \rangle}$.

Proof. (i) This is trivial.

(ii) Let $\dim(X) = 2N$. Since $\deg(f) \neq 0$, $f^*: H^i(Y; \mathbb{Q}) \rightarrow H^i(X; \mathbb{Q})$ is a monomorphism for all $i \leq 2N$. Comparing the second Betti numbers of X and Y we conclude that $f^*: H^2(Y; \mathbb{Q}) \rightarrow H^2(X; \mathbb{Q}) \cong \mathbb{Q}$ is an isomorphism. Let $\varphi \in H^2(Y; \mathbb{Q})$ be the unique class such that $f^*(\varphi) = \omega$. Since f^* is

a homomorphism of *rings*, we have $0 \neq \omega^N = (f^*(\varphi))^N = f^*(\varphi^N)$ and so $\varphi^N \neq 0$.

Let $r \leq N$ be a positive integer. One has a commuting diagram:

$$\begin{array}{ccc} H^r(Y; \mathbb{Q}) & \xrightarrow{\cup \varphi^{N-r}} & H^{2N-r}(Y; \mathbb{Q}) \\ f^* \downarrow & & \downarrow f^* \\ H^r(X; \mathbb{Q}) & \xrightarrow{\cup \omega^{N-r}} & H^{2N-r}(X; \mathbb{Q}) \end{array}$$

The vertical maps are monomorphisms since $\deg(f) \neq 0$. By our hypothesis on X , the homomorphism $\cup \omega^{N-r}$ in the above diagram is an isomorphism. This implies that $\cup \varphi^{N-r}$ is a monomorphism. Since, by Poincaré duality, the vector spaces $H^r(Y; \mathbb{Q})$ and $H^{2N-r}(Y; \mathbb{Q})$ have the same dimension, $\cup \varphi^{N-r}$ is an *isomorphism* and so $(Y; \varphi)$ is c -Hodge. It is clear that $f^*(\mathcal{V}_\varphi^r) \subset \mathcal{V}_\omega^r$.

(iii) Suppose that $\alpha, \beta \in H^r(Y; \mathbb{R})$. Then

$$\begin{aligned} (f^*(\alpha), f^*(\beta))_\omega &= \langle f^*(\alpha) f^*(\beta) \omega^{N-r}; \mu_X \rangle \\ &= \langle f^*(\alpha\beta) f^*(\varphi^{N-r}); \mu_X \rangle \\ &= \langle f^*(\alpha\beta \varphi^{N-r}); \mu_X \rangle \\ &= \langle \alpha\beta \varphi^{N-r}, f_*(\mu_X) \rangle \\ &= \deg(f) \langle \alpha\beta \varphi^{N-r}; \mu_Y \rangle \\ &= \deg(f) (\alpha, \beta)_\varphi. \end{aligned}$$

The formula for the degree of f follows from what has just been established by taking $\alpha = \beta = \varphi$. \square

Observe that the summands in the Lefschetz decomposition (3.1.5) are mutually orthogonal with respect to the bilinear form (\cdot, \cdot) . Indeed, let $\alpha \in \mathcal{V}^{r-2p}, \beta \in \mathcal{V}^{r-2q}, p < q$. Thus $\alpha\omega^{N-r+2p+1} = 0$ and so $\alpha\omega^{N-r+p+q} = 0$. Therefore $(\omega^p\alpha, \omega^q\beta) = \langle \alpha\beta\omega^{N-r+p+q}, \mu_X \rangle = 0$. As observed earlier the form (\cdot, \cdot) is non-degenerate. It follows that the form restricted to each summand in (3.1.5) is non-degenerate. In favourable situations, the form is either positive or negative definite as we shall see in Proposition 3.2.1 below.

Proposition 3.2.1 *Suppose that (X, Ω) is a compact connected Kähler man-*

ifold such that $H^{p,q}(X) = 0$ for $p \neq q$. Let ω be the cohomology class corresponding to Ω . Then the form $(-1)^{q+r}(\cdot, \cdot)_\omega$ restricted to $\omega^q \mathcal{V}^{2r-2q} \subset H^{2r}(X; \mathbb{R})$ is positive definite for $0 \leq q \leq r, 1 \leq r \leq [N/2]$.

Proof. First assume that $N = \dim_{\mathbb{C}} X$ is even, say $N = 2s$. In view of our hypothesis, all odd Betti numbers of X vanish and we have $B_k^{p,q} = 0$ for all $p \neq q, k \geq 0$, so that

$$H^{2r}(X; \mathbb{C}) = H^{r,r}(X) = \bigoplus_{0 \leq k \leq r} B_k^{r,r}. \quad (3.2.1)$$

Recall that the real cohomology group $H^{2r}(X; \mathbb{R}) \subset H^{2r}(X; \mathbb{C}) = H^{r,r}(X)$ has an orthogonal decomposition with respect to $(\cdot, \cdot)_\omega$ induced from (3.1.4):

$$H^{2r}(X; \mathbb{R}) = \bigoplus_{0 \leq k \leq s} E_k^{r,r} \quad (3.2.2)$$

where $E_k^{p,p} = \{\alpha \in B_k^{p,p} \mid \alpha = \bar{\alpha}\}$. Now taking $r = s = N/2$ one has $\#\alpha = (-1)^{s+k} \alpha$ for $\alpha \in E_k^{s,s}$. In particular the bilinear form $(\cdot, \cdot)_*$ equals $(-1)^{s+k} Q(\cdot, \cdot)$, which in turn equals $(-1)^{s+k}(\cdot, \cdot)_\omega$. Therefore $(-1)^{s+k} Q(\cdot, \cdot)$ restricted to each $E_k^{s,s}$ is positive definite.

We shall show in Lemma 3.2.2 below that $\omega^k \mathcal{V}^{N-2k} = E_k^{s,s}$. The proposition follows immediately from this since $(\alpha, \beta)_\omega = (\omega^{s-r} \alpha, \omega^{s-r} \beta)_\omega$ for $\alpha, \beta \in \omega^k \mathcal{V}^{2r-2k}$ as $N = 2s$, completing the proof in this case.

Now suppose that N is odd. Consider the Kähler manifold $Y = X \times \mathbb{C}\mathbb{P}^1$ where we put the Fubini-Study metric on $\mathbb{C}\mathbb{P}^1$ with Kähler class η being the ‘positive’ generator of $H^2(\mathbb{C}\mathbb{P}^1; \mathbb{Z}) \subset H^2(\mathbb{C}\mathbb{P}^1; \mathbb{R})$ and the product structure on Y so that the Kähler class of Y equals $\omega + \eta =: \varphi$. By Künneth theorem $H^*(Y; \mathbb{R}) = H^*(X; \mathbb{R}) \otimes H^*(\mathbb{C}\mathbb{P}^1; \mathbb{R})$. We shall identify the cohomology groups of X and $\mathbb{C}\mathbb{P}^1$ with their images in $H^*(Y; \mathbb{R})$ via the monomorphisms induced by the first and second projection respectively. Under these identifications, $H^{p,q}(Y) = H^{p,q}(X) \oplus H^{p-1,q-1}(X) \otimes H^{1,1}(\mathbb{C}\mathbb{P}^1)$. In particular, $H^{p,q}(Y) = 0$ unless $p = q$. By what has been proven already, the form $(-1)^{r+k}(\cdot, \cdot)$ is positive definite on $\varphi^k \mathcal{V}_\varphi^{2r-2k} \subset H^{2r}(Y; \mathbb{R})$.

Choose a base point in $\mathbb{C}\mathbb{P}^1$ and consider the inclusion map $j: X \hookrightarrow Y$. The imbedding j is dual to η . Also $j^*(\varphi) = \omega$. It follows that $j^*(\varphi^k \mathcal{V}_\varphi^{2r-2k}) \subset \omega^k \mathcal{V}_\omega^{2r-2k}$ for $0 \leq k < r$, $1 \leq r < N$. Since the kernel of $j^*: H^{2r}(Y; \mathbb{R}) \rightarrow H^{2r}(X; \mathbb{R})$ equals $H^{2r-2}(X; \mathbb{R}) \otimes H^2(\mathbb{C}\mathbb{P}^1; \mathbb{R})$, and maps $H^{2r}(X; \mathbb{R}) \subset H^{2r}(Y; \mathbb{R})$ isomorphically onto $H^{2r}(X; \mathbb{R})$, we must have $j^*(\varphi^k \mathcal{V}_\varphi^{2r-2k}) = \omega^k \mathcal{V}_\omega^{2r-2k}$.

Let $\alpha, \beta \in H^{2r}(X; \mathbb{R}) \subset H^{2r}(Y; \mathbb{R})$. Since $j: X \hookrightarrow Y$ is dual to η , we have $j_*(\mu_X) = \eta \cap \mu_Y$. Therefore,

$$\begin{aligned} (j^*(\alpha), j^*(\beta))_\omega &= \langle j^*(\alpha\beta) j^*(\omega)^{N-2r}; \mu_X \rangle \\ &= \langle \alpha\beta \omega^{N-2r}, j_*(\mu_X) \rangle \\ &= \langle \alpha\beta \omega^{N-2r}, \eta \cap \mu_Y \rangle \\ &= \langle \alpha\beta \omega^{N-2r} \eta, \mu_Y \rangle \end{aligned}$$

Since $\eta^2 = 0$ we have $\varphi^{N-2r+1} = \omega^{N-2r+1} + (N-2r+1)\omega^{N-2r}\eta$. Furthermore, $\alpha\beta\omega^{N-2r+1} \in H^{2N+2}(X; \mathbb{R}) = 0$. Therefore, we conclude that $(j^*(\alpha), j^*(\beta))_\omega = \frac{1}{N-2r+1} \langle \alpha\beta\varphi^{N-2r+1}, \mu_Y \rangle = \frac{1}{N-2r+1} (\alpha, \beta)_\varphi$. This shows that the bilinear form $(\cdot, \cdot)_\omega$ on $H^{2r}(X; \mathbb{R})$ is a positive multiple of the form $(\cdot, \cdot)_\varphi$ on $H^{2r}(Y; \mathbb{R})$ restricted to $H^{2r}(X; \mathbb{R})$. It follows that the bilinear form $(-1)^{r+k}(\cdot, \cdot)$ on $H^{2r}(X; \mathbb{R})$ restricted to $\omega^k \mathcal{V}^{2r-2k}(X)$ is positive definite. \square

We must now establish the following

Lemma 3.2.2 *With notations as above, assume that $N = 2s$ is even. Under the hypothesis of the above proposition, $E_k^{s-k, s-k}$ equals $\omega^k \mathcal{V}^{N-2k}$, $0 \leq k \leq s$.*

Proof. Since L preserves real forms, it suffices to show that $E_0^{r,r} = \mathcal{V}^{2r}$ when $r \leq s$. By definition $E_0^{r,r} = B_0^{r,r} \cap H^{2r}(X; \mathbb{R}) = \{\alpha \in H^{r,r}(X; \mathbb{C}) \mid \Lambda(\alpha) = 0, \alpha = \bar{\alpha}\}$.

Let $\alpha \in E_0^{r,r}$. Suppose that $p \geq 1$ is the largest integer such that $\omega^{N-2r+p}\alpha =: \theta$ is a *non-zero* real harmonic form of type $(N-r+p, N-r+p)$. Since $L^{N-2r+2p}: H^{r-p, r-p}(X) \rightarrow H^{N-r+p, N-r+p}(X)$ is an isomorphism, and since ω is real there must be a real form $\beta \in H^{r-p, r-p}(X)$ such that $L^{N-2r+2p}(\beta) = \theta = L^{N-2r+p}(\alpha)$. Since p is the largest, using the decomposition (3.2.1) we see that $\beta \in B_0^{r-p, r-p}$. Applying Λ^{N-2r+p} both sides and (repeatedly) using

$\Lambda L^q \beta$ is a *non-zero* multiple of $L^{q-1} \beta$ when $r - p + q < N$ we see that β is a non-zero multiple of $\Lambda^p \alpha = 0$. Thus $\beta = 0$ and hence $\theta = 0$, which contradicts our assumption. Therefore $L^{N-2r+1}(\alpha) = 0$ and so $\alpha \in \mathcal{V}_0^r$. On the other hand Λ maps $H^{2r}(X; \mathbb{C})$ onto $H^{2r-2}(X; \mathbb{C})$. A dimension argument shows that $E_0^{r,r} = \mathcal{V}^{2r}$. \square

Example 3.2.2 (*The Case of the Complex Grassmann Manifold*)

The Grassmann manifold $\mathbb{C}\mathbb{G}_{n,k}$, which has complex dimension $N = k(n-k)$, has the structure of a Kähler manifold with Kähler class $\omega := \bar{c}_1 = [\Omega_1] \in H^2(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Z})$. (This fact follows, for example, from the Plücker imbedding $\mathbb{C}\mathbb{G}_{n,k} \hookrightarrow \mathbb{C}\mathbb{P}^{\binom{n}{k}-1}$.) The bilinear form (\cdot, \cdot) is understood to be defined with respect to ω .

Example 3.2.3 (*The Case of the Quaternionic Grassmannian*)

Although quaternionic Grassmann manifolds are not c -Kähler, one could use the symplectic Pontrjagin class $\eta := e_1(\gamma_{n,k}) \in H^4(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Z})$ in the place of $\bar{c}_1 \in H^2(\mathbb{C}\mathbb{G}_{n,k}; \mathbb{Z})$ to define a pairing $(\cdot, \cdot)_\eta$ on $H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$ and the primitive classes $v_j \in H^{4j}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$. We define $\mathcal{V}^{4r} \subset H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$ to be the kernel of

$$\cup \eta^{N-2r+1}: H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q}) \longrightarrow H^{4N-4r+4}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q}).$$

The form $(\cdot, \cdot)_\eta$ is definite when restricted to the space $\eta^q \mathcal{V}^{4r-4q} \subset H^{4r}(\mathbb{H}\mathbb{G}_{n,k}; \mathbb{Q})$. The degree doubling isomorphism from the cohomology algebra of $\mathbb{C}\mathbb{G}_{n,k}$ to that of $\mathbb{H}\mathbb{G}_{n,k}$ maps the i -th Chern class of the tautological complex k -plane bundle over $\mathbb{C}\mathbb{G}_{n,k}$ to the i -th symplectic Pontrjagin class of the tautological left \mathbb{H} -bundle over $\mathbb{H}\mathbb{G}_{n,k}$.

CHAPTER 4

PROOFS OF MAIN RESULTS

In this chapter we prove the main results of the thesis, namely Theorems 1.0.1, 1.0.2 and 1.0.3. *We will only consider the case of complex Grassmann manifold $\mathbb{C}\mathbb{G}_{n,k}$, which we will refer to as $\mathbb{G}_{n,k}$ in this chapter.* The proofs in the case of quaternionic Grassmann manifold follows in view of the fact that the cohomology algebra of $\mathbb{H}\mathbb{G}_{n,k}$ is isomorphic to that of $\mathbb{G}_{n,k}$ via an isomorphism that doubles the degree.

Recall that complex Grassmann manifold $\mathbb{G}_{n,k}$ is a smooth projective variety and that Schubert subvarieties yield an *algebraic* cell decomposition. In particular its Chow ring is isomorphic to singular cohomology ring (with \mathbb{Z} -coefficients) via an isomorphism that doubles the degree. It follows that $H^{p,q}(\mathbb{G}_{n,k}; \mathbb{C}) = 0$ for $p \neq q$. Therefore results of the previous chapter hold for $\mathbb{G}_{n,k}$. The bilinear form $(\cdot, \cdot): H^r(\mathbb{G}_{n,k}; \mathbb{R}) \times H^r(\mathbb{G}_{n,k}; \mathbb{R}) \rightarrow \mathbb{R}$ is given by $(\alpha, \beta) = \langle \alpha\beta\omega^{N-r}, \mu_X \rangle$, $\alpha, \beta \in H^r(\mathbb{G}_{n,k}; \mathbb{R})$, is understood to be defined with respect to $\omega = \bar{c}_1 \in H^2(\mathbb{G}_{n,k}; \mathbb{Z}) \cong \mathbb{Z}$.

We have the following lemma whose proof follows immediately from Lemma 3.2.1(i) and (iii).

Lemma 4.0.3 *Let $f: \mathbb{G}_{n,k} \rightarrow \mathbb{G}_{m,l}$ be any continuous map where $k(n-k) = l(m-l)$. Suppose that $f^*(c_1(\gamma_{m,l}^\perp)) = \lambda c_1(\gamma_{n,k}^\perp)$ where $\lambda \in \mathbb{Z}$. Then*

$$\deg(f) = \lambda^N \frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}}.$$

4.1 THEOREM 1.0.3

We are now in a position to prove Theorem 1.0.3. We first recall the statement of Whitehead's theorem, which will be used in the proof of Theorem 1.0.3:

Whitehead's Theorem: *A map $f: N \rightarrow M$ between simply connected CW-complexes is a homotopy equivalence if $f_*: H_q(N; \mathbb{Z}) \rightarrow H_q(M; \mathbb{Z})$ is an isomorphism for each q .*

Theorem 1.0.3 *Let $\mathbb{F} = \mathbb{C}$ or \mathbb{H} . Suppose that $k(n - k) = l(m - l)$, and $1 \leq l \leq [m/2], 1 \leq k \leq [n/2]$. If $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ is a map of degree ± 1 , then $(m, l) = (n, k)$ and f is a homotopy equivalence.*

Proof of Theorem 1.0.3. We may suppose that $\mathbb{F} = \mathbb{C}$ and that $l \leq k$; otherwise $k < l \leq [m/2]$ in which case $\deg(f) = 0$ for any f by [19, Theorem 2] (see also Theorem 2.2.4 in Chapter 2 of this thesis).

Suppose that $\deg(f) = \pm 1$, and $l < k$. We have

$$\begin{aligned} \frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}} &= \frac{1! \cdots (k-1)! (m-l)! \cdots (m-1)!}{1! \cdots (l-1)! (n-k)! \cdots (n-1)!} \\ &= \frac{l! \cdots (k-1)! (m-l)! \cdots (m-1)!}{(n-k)! \cdots (n-1)!} \\ &= \left(\prod_{1 \leq j \leq k-l} \frac{(l-1+j)!}{(n-k+j-1)!} \right) \left(\prod_{1 \leq j \leq l} \frac{(m-j)!}{(n-j)!} \right) \end{aligned}$$

Note that after simplifying $(l + j - 1)! / (n - k + j - 1)!$ for each j in the first product, we are left with product of $(k - l)$ blocks of $(n - k - l)$ consecutive positive integers in the denominator, the *largest* to occur being $(n - l - 1)$. Similar simplification in the second product yields a product of l blocks of $(m - n)$ consecutive integers, the *smallest* to occur being $(n - l + 1)$. Since $(k - l)(n - k - l) = l(m - n)$ we conclude that $\deg(\mathbb{G}_{n,k}) > \deg(\mathbb{G}_{m,l})$.

In the notation of Lemma 4.0.3 above, we see that either $\deg(f) = 0$ or $|\deg(f)| > |\lambda|^N \geq 1$ —a contradiction. Therefore $(m, l) = (n, k)$ if $\deg(f) = \pm 1$. Now $f^*: H^*(\mathbb{G}_{n,k}; \mathbb{Z}) \rightarrow H^*(\mathbb{G}_{m,l}; \mathbb{Z})$ induces an isomorphism. Since $\mathbb{G}_{n,k}$ is a simply connected CW complex, by Whitehead's theorem, f is a homotopy equivalence. \square

Theorem 1.0.3 is a topological analogue of the result of Paranjape and Srinivas [18] that any non-constant morphism $f: \mathbb{G}_{n,k} \rightarrow \mathbb{G}_{m,l}$ is an *isomorphism* of varieties provided the $\mathbb{G}_{m,l}$ is not the projective space. Our conclusion in the topological realm is weaker. Indeed it is known that there exist continuous self-maps of any complex and quaternionic Grassmann manifold which have large positive degrees. See [1] and also [21].

4.2 THEOREM 1.0.1

The proof of Theorem 1.0.1 requires the construction of an orthogonal basis for $\mathcal{V}_{n,k}^{2r} \subset H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$, which can be obtained inductively using Gram-Schmidt orthogonalization process as follows. Recall from §2 the basis $\bar{\mathcal{C}}_r$ for $H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$. Clearly $\omega \cdot \bar{\mathcal{C}}_{r-1} = \bar{c}_1 \cdot \bar{\mathcal{C}}_{r-1} = \{\bar{c}^{\mathbf{j}} \in \bar{\mathcal{C}}_r \mid j_1 > 0\}$ is a basis for $\omega H^{2r-2}(\mathbb{G}_{n,k}; \mathbb{Q})$. Therefore we see that the subspace spanned by $\bar{\mathcal{C}}_{r,0} := \{\bar{c}^{\mathbf{j}} \in \bar{\mathcal{C}}_r \mid j_1 = 0\}$ is complementary to $\bigoplus_{q>0} B_q^{r-q, r-q} \subset H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$. The required basis is obtained by taking the orthogonal projection of $\bar{\mathcal{C}}_{r,0}$ onto \mathcal{V}^{2r} . Indeed, inductively assume that an orthogonal basis $\{v_{\mathbf{j}}\}$ for $\omega H^{2r-2}(\mathbb{G}_{n,k}; \mathbb{Q})$ that is compatible with the direct sum decomposition $\bigoplus_{q>0} B_q^{r-q, r-q}$ has been constructed. We need only apply the orthogonalization process to the (ordered) set $\{v_{\mathbf{j}}\} \cup \{\bar{c}^{\mathbf{j}} \in \bar{\mathcal{C}}_r \mid j_1 = 0\}$ where the elements $\bar{c}^{\mathbf{j}}$ are ordered, say, according to lexicographic order of the exponents. For example, taking $n = 12, k = 6, r = 6$, the elements of $\bar{\mathcal{C}}_{6,0}$ are ordered as $\bar{c}_2^3, \bar{c}_2 \bar{c}_4, \bar{c}_3^2, \bar{c}_6$. We denote the basis element of \mathcal{V}^{2r} obtained from $\bar{c}^{\mathbf{j}} \in \bar{\mathcal{C}}_r$ by $v_{\mathbf{j}}$. Note that when $r \leq k$, the span of the set $\{v_{\mathbf{j}} \mid j_r = 0\} \subset H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$ equals space of the decomposable elements in $H^{2r}(\mathbb{G}_{n,k}; \mathbb{Q})$ since, according to our assumption on the ordering of elements $\bar{c}^{\mathbf{j}}$, the element \bar{c}_r is the greatest and so \bar{c}_r does not occur in any other $v_{\mathbf{j}}$. Thus $v_r - \bar{c}_r$ belongs to the ideal $\mathcal{D} \subset H^*(\mathbb{G}_{n,k}; \mathbb{Q})$ of decomposable elements, and $v_{\mathbf{j}} \in \mathcal{D}$ for all other \mathbf{j} .

We illustrate this for $r = 2, 3$. (When $r = 1, \mathcal{V}^1 = 0$.) The element $v_2 = \bar{c}_2 - \frac{(\bar{c}_2, \omega^2)}{(\omega, \omega)} \omega^2 = \bar{c}_2 - \frac{\deg \bar{c}_2}{\deg \mathbb{G}_{n,k}} \omega^2 \in H^4(\mathbb{G}_{n,k}; \mathbb{Q})$ is a basis for the one-dimensional space \mathcal{V}^4 .

Similarly, v_3 is a basis for \mathcal{V}^6 where

$$\begin{aligned} v_3 &:= \bar{c}_3 - \frac{(\bar{c}_3, v_2 \omega)}{(v_2 \omega, v_2 \omega)} v_2 \omega - \frac{(\bar{c}_3, \omega^3)}{(\omega^3, \omega^3)} \omega^3 \\ &= \bar{c}_3 - \frac{\deg \bar{c}_3}{\deg \mathbb{G}_{n,k}} \omega^3 - \frac{\deg \mathbb{G}_{n,k} \deg(\bar{c}_3 \bar{c}_2) - \deg \bar{c}_2 \deg \bar{c}_3}{\deg \mathbb{G}_{n,k} \deg(\bar{c}_2^2) - (\deg \bar{c}_2)^2} v_2 \omega. \end{aligned}$$

This leads to

$$(v_3, v_3) = (v_3, \bar{c}_3) = \deg(\bar{c}_3^2) - \frac{(\deg \bar{c}_3)^2}{\deg \mathbb{G}_{n,k}} - \frac{\deg(\bar{c}_3 \bar{c}_2) \deg \mathbb{G}_{n,k} - \deg \bar{c}_2 \deg \bar{c}_3}{\deg \mathbb{G}_{n,k} \deg(\bar{c}_2^2) - (\deg \bar{c}_2)^2} \deg(\bar{c}_3 v_2).$$

To avoid possible confusion, we shall denote the primitive classes in $H^{2j}(\mathbb{G}_{m,l}; \mathbb{Q})$ corresponding to $j = 2, \dots, l$ by u_j . Also $\mathcal{V}_{m,l}^{2r} \subset H^{2r}(\mathbb{G}_{m,l}; \mathbb{Q})$ will denote the space of primitive classes. The following lemma is crucial for the proof of Theorem 1.0.1.

Lemma 4.2.1 *Suppose that $f: \mathbb{G}_{n,k} \rightarrow \mathbb{G}_{m,l}$ is a continuous map such that $f^*(c_1(\gamma_{m,l}^\perp)) = \lambda c_1(\gamma_{n,k}^\perp) = \lambda \bar{c}_1$ with $\lambda \neq 0$. Let $k(n-k) = l(m-l)$, $2 \leq k \leq n/2$, $2 \leq l \leq m/2$ and $l \leq k$. Then, with the above notations, $f^*(u_j) = \lambda_j v_j$ where $\lambda_j \in \mathbb{Q}$ is such that*

$$\lambda_j^2 = \lambda^{2j} \frac{\deg \mathbb{G}_{n,k} (u_j, u_j)}{\deg \mathbb{G}_{m,l} (v_j, v_j)}$$

for $2 \leq j \leq l$.

Proof. The degree of f equals $\lambda^N \deg \mathbb{G}_{n,k} / \deg \mathbb{G}_{m,l} \neq 0$ by Lemma 4.0.3.

Therefore $f^*: H^{2j}(\mathbb{G}_{m,l}; \mathbb{Q}) \rightarrow H^{2j}(\mathbb{G}_{n,k}; \mathbb{Q})$ is an isomorphism and $f^*(\mathcal{V}_{m,l}^{2j}) = \mathcal{V}_{n,k}^{2j}$, since f^* is a monomorphism and the dimensions are equal as $j \leq l \leq k$. Note that f^* maps the space of decomposable elements $\mathcal{D}_{m,l}^{2j} \subset H^{2j}(\mathbb{G}_{m,l}; \mathbb{Q})$ isomorphically onto $\mathcal{D}_{n,k}^{2j}$. Since $u_j \perp \mathcal{D}_{m,l}^{2j} \cap \mathcal{V}_{m,l}^{2j}$ we see that, by Lemma 3.2.1 (ii), $f^*(u_j) \perp \mathcal{D}_{n,k}^{2j} \cap \mathcal{V}_{n,k}^{2j}$. As the form (\cdot, \cdot) on $\mathcal{V}_{n,k}^{2j}$ is definite by Proposition 3.2.1 and $\mathcal{V}_{n,k}^{2j} = \mathbb{Q}v_j \oplus (\mathcal{V}_{n,k}^{2j} \cap \mathcal{D}_{n,k}^{2j})$ is an orthogonal decomposition, we must have $f^*(u_j) = \lambda_j v_j$ for some $\lambda_j \in \mathbb{Q}$.

Recall that $\deg(f) = \lambda^N \deg \mathbb{G}_{n,k} / \deg \mathbb{G}_{m,l}$. Note that

$$\begin{aligned} \lambda^{N-2j}(f^*(u_j), f^*(u_j)) &= (f^*(u_j), f^*(u_j))_{\lambda \bar{c}_1} \\ &= \deg(f)(u_j, u_j)_\omega \\ &= \lambda^N \frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}}(u_j, u_j) \end{aligned}$$

by Lemma 3.2.1. Thus $\lambda_j^2(v_j, v_j) = (f^*(u_j), f^*(u_j)) = \lambda^{2j} \frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}}(u_j, u_j)$. \square

We are now ready to prove Theorem 1.0.1.

Theorem 1.0.1 *Let $\mathbb{F} = \mathbb{C}$ or \mathbb{H} and let $d = \dim_{\mathbb{R}} \mathbb{F}$. Let $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ be any continuous map between two \mathbb{F} -Grassmann manifolds of the same dimension. Then, there exist algebra generators $u_i \in H^{di}(\mathbb{F}\mathbb{G}_{m,l}; \mathbb{Q})$, $1 \leq i \leq l$, such that the image $f^*(u_i) \in H^{di}(\mathbb{F}\mathbb{G}_{n,k}; \mathbb{Q})$, $1 \leq i \leq l$, is determined upto a sign \pm , provided degree of f is non-zero.*

Proof of Theorem 1.0.1: We need only consider the case $\mathbb{F} = \mathbb{C}$. Recall that the cohomology algebra $H^*(\mathbb{G}_{m,l}; \mathbb{Z})$ is generated by $\bar{c}_1, \dots, \bar{c}_l$ where $\bar{c}_j = c_j(\gamma_{m,l}^\perp)$. Therefore $f^*: H^*(\mathbb{G}_{m,l}; \mathbb{Z}) \rightarrow H^*(\mathbb{G}_{n,k}; \mathbb{Z})$ is determined by the images of \bar{c}_j , $1 \leq j \leq l$.

As observed in Example 3.2.2, one has $u_j - \bar{c}_j \in \mathcal{D}_{m,l}^{2j}$, $2 \leq j \leq l$. It follows easily by induction that each \bar{c}_j , $1 \leq j \leq l$, can be expressed as a polynomial with rational coefficients in $\bar{c}_1, u_2, \dots, u_l$. Therefore $\bar{c}_1 =: u_1, u_2, \dots, u_l$ generate $H^*(\mathbb{G}_{m,l}; \mathbb{Q})$.

Lemma 4.0.3 implies that $f^*(u_1) = \lambda c_1(\gamma_{n,k}^\perp)$ where λ^N —and hence λ upto a sign—is determined by the degree of f . Now by Lemma 4.2.1, the image of $f^*(u_j) = \lambda_j v_j$ where λ_j is determined upto a sign. \square

Note that, since the degree doubling isomorphism from the cohomology algebra of $\mathbb{G}_{n,k}$ to that of $\mathbb{H}\mathbb{G}_{n,k}$ maps the i -th Chern class of the tautological complex k -plane bundle over $\mathbb{G}_{n,k}$ to the i -th symplectic Pontrjagin class of the tautological left \mathbb{H} -bundle over $\mathbb{H}\mathbb{G}_{n,k}$, the formula given in Lemma 4.3.1 holds without any change for the quaternionic Grassmannian. Hence, the proof of Theorem 1.0.1 also carries through for the quaternionic Grassmannian.

Endomorphisms of the cohomology algebra of $\mathbb{G}_{n,k}$ have been classified by M. Hoffman [11]. They are either ‘grading homomorphisms’ defined by $c_i \mapsto \lambda^i c_i$, $1 \leq i \leq k$ for some λ or when $n = 2k$, the composition of a grading homomorphism with the homomorphism induced by the diffeomorphism $\perp: \mathbb{G}_{n,k} \longrightarrow \mathbb{G}_{n,k}$ defined as $U \mapsto U^\perp$.

4.3 THEOREM 1.0.2

The following calculation will be used in the course of the proof of Theorem 1.0.2.

Lemma 4.3.1 *Let $v_2 \in H^4(\mathbb{G}_{n,k}; \mathbb{Q})$ denote the primitive basis element obtained from \bar{c}_2 . Then, $(v_2, v_2) = \deg \mathbb{G}_{n,k} \frac{(k^2-1)((n-k)^2-1)}{2(N-1)^2(N-2)(N-3)}$.*

Proof. The proof involves straightforward but lengthy calculation which we work out below.

Since $(v_2, \bar{c}_1^2) = 0$, we get $(v_2, v_2) = (v_2, c_2) = (\bar{c}_2, \bar{c}_2) - \frac{\deg \bar{c}_2}{\deg \mathbb{G}_{n,k}} (\bar{c}_2, \omega^2) = \deg \mathbb{G}_{n,k} \left(\frac{\deg(\bar{c}_2^2)}{\deg \mathbb{G}_{n,k}} - \left(\frac{\deg \bar{c}_2}{\deg \mathbb{G}_{n,k}} \right)^2 \right)$.

Since $\bar{c}_2^2 = [\Omega_2]^2 = [\Omega_4] + [\Omega_{3,1}] + [\Omega_{2,2}]$, we see that $\frac{\deg \bar{c}_2^2}{\deg \mathbb{G}_{n,k}} = \frac{\deg \bar{c}_4}{\deg \mathbb{G}_{n,k}} + \frac{\deg \Omega_{3,1}}{\deg \mathbb{G}_{n,k}} + \frac{\deg \Omega_{2,2}}{\deg \mathbb{G}_{n,k}}$.

Now an explicit calculation yields, upon using $N = k(n - k)$:

$$\begin{aligned} \frac{\deg \bar{c}_4}{\deg \mathbb{G}_{n,k}} &= \frac{(n-k-1)(n-k-2)(n-k-3)(k+1)(k+2)(k+3)}{4!(N-1)(N-2)(N-3)}, \\ \frac{\deg \Omega_{3,1}}{\deg \mathbb{G}_{n,k}} &= \frac{(n-k+1)(n-k-1)(n-k-2)(k+2)(k+1)(k-1)}{2!4(N-1)(N-2)(N-3)}, \\ \frac{\deg \Omega_{2,2}}{\deg \mathbb{G}_{n,k}} &= \frac{N(k-1)(k+1)(n-k+1)(n-k-1)}{2!3 \cdot 2(N-1)(N-2)(N-3)}, \\ \frac{\deg \bar{c}_2}{\deg \mathbb{G}_{n,k}} &= \frac{(k+1)(n-k-1)}{2!(N-1)}. \end{aligned}$$

Substituting these in the above expression for (v_2, v_2) we get $(v_2, v_2) = \deg \mathbb{G}_{n,k} \frac{(k+1)(n-k-1)}{4!(N-1)^2(N-2)(N-3)} A$ where, again using $N = k(n - k)$ repeatedly,

$$\begin{aligned}
A &:= (N-1)\{(n-k-2)(k+2)(n-k-3)(k+3) \\
&\quad +3(n-k-2)(k+2)(n-k+1)(k-1) + 2N(k-1)(n-k+1)\} \\
&\quad -6(N-2)(N-3)((n-k-1)(k+1)) \\
&= (N-1)\{(N+2(n-2k)-4)(N+3(n-2k)-9) \\
&\quad +3(N+2(n-2k)-4)(N-(n-2k)-1) + 2N(N-(n-2k)-1)\} \\
&\quad -6(N-2)(N-3)(N+(n-2k)-1) \\
&= 12(N-(n-2k)-1) \\
&= 12(k-1)(n-k+1).
\end{aligned}$$

Therefore, $(v_2, v_2) = \deg \mathbb{G}_{n,k} \frac{(k^2-1)((n-k)^2-1)}{2(N-1)^2(N-2)(N-3)}$. \square

Next, we give the proof of Theorem 1.0.2:

Theorem 1.0.2 *Let $\mathbb{F} = \mathbb{C}$ or \mathbb{H} . Fix integers $2 \leq l < k$. Let $m, n \geq 2k$ be positive integers such that $k(n-k) = l(m-l)$ and $f: \mathbb{F}\mathbb{G}_{n,k} \rightarrow \mathbb{F}\mathbb{G}_{m,l}$ any continuous map. Then, degree of f is zero if $(l^2-1)(k^2-1)((m-l)^2-1)((n-k)^2-1)$ is not a perfect square. In particular, degree of f is zero for n sufficiently large.*

Proof of Theorem 1.0.2: We assume, as we may, that $\mathbb{F} = \mathbb{C}$. We preserve the notations used in the previous section. From Lemma 4.3.1 we have $(v_2, v_2) = \deg \mathbb{G}_{n,k} \frac{(k^2-1)((n-k)^2-1)}{2(N-1)^2(N-2)(N-3)}$. Therefore, by Lemma 4.2.1 we have

$$\begin{aligned}
\lambda_2^2 &= \lambda^4 \frac{\deg \mathbb{G}_{n,k} (v_2, v_2)}{\deg \mathbb{G}_{m,l} (u_2, u_2)} \\
&= \lambda^4 \left(\frac{\deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l}} \right)^2 \frac{(k^2-1)((n-k)^2-1)}{(l^2-1)((m-l)^2-1)} \\
&= B^2 (k^2-1)(l^2-1)((n-k)^2-1)((m-l)^2-1)
\end{aligned}$$

where $B := \frac{\lambda^2 \deg \mathbb{G}_{n,k}}{\deg \mathbb{G}_{m,l} (l^2-1)((m-l)^2-1)} \in \mathbb{Q}$. It follows that $\deg(f) = 0$ unless $Q := (l^2-1)(k^2-1)((m-l)^2-1)((n-k)^2-1)$ is a perfect square. It remains to show that there are at most finitely many values for m, n for which the Q is a perfect square. Before proving this in Proposition 4.3.1, we recall the statement of the following theorem of Siegel, which will be used for the proof of the proposition:

Siegel's Theorem: [6, Theorem D.8.3, p. 349] Let K/\mathbb{Q} be a number field, let $S \subset M_K$ be a finite set of absolute values on K that includes all the archimedean absolute values, and let R_S be the ring of S -integers of K . Let $f(X) \in K[X]$ be a polynomial of degree at least 3 with distinct roots (in \bar{K}). Then the equation $Y^2 = f(X)$ has only finitely many solutions $X, Y \in R_S$.

Proposition 4.3.1 *Let $1 < a < b$ be positive integers. Then there are at most finitely many solutions in \mathbb{Z} for the system of equations*

$$y^2 = Q(a, b, x, z), \quad az = bx, \quad (4.3.1)$$

where $Q(a, b, x, z) := (a^2 - 1)(b^2 - 1)(x^2 - 1)(z^2 - 1)$.

Proof. Let $r = \gcd(a, b)$ and write $a = rs, b = rt$ so that $tx = sz$. Then the system of equations (8) can be rewritten as $y^2 = F(x)$ where $F(x) := (1/s^2)(a^2 - 1)(b^2 - 1)(x^2 - 1)(t^2x^2 - s^2)$. Note that $F(x) \in \mathbb{Q}[x]$ has distinct zeros in \mathbb{Q} . By Siegel's Theorem it follows that the equation $y^2 = F(x)$ has only finitely many solutions in the ring $R_S \subset K$ of S -integers where K is any number field and S any finite set of valuations of K , including all archimedean valuations. In particular, taking $K = \mathbb{Q}$ and S the usual (archimedean) absolute value, we see that there are only finitely many integral solutions of (4.3.1). \square

For the rest of the chapter we shall only be concerned with the number theoretic question of $Q(a, b, c, d)$ being a perfect square.

Remark 4.3.2 (i) We observe that there are *infinitely* many integers $1 < a < b < c < d$ such that $Q(a, b, c, d)$ is a perfect square. Indeed given a, b , let c be any positive integer such that $(a^2 - 1)(b^2 - 1)(c^2 - 1) = Pu^2$ where $P > 1$ is square free. Let (x, y) be any solution with $x \neq 0$ of the so called Pell's equation $y^2 = 1 + Px^2$. Then $d = |y|$ is a solution whenever $d > c$. Since the Pell's equation has infinitely many solutions, there are infinitely many such d .

(ii) Suppose that $(l^2 - 1)(k^2 - 1)(c^2 - 1) = x^2$ is a perfect square. (There exists such positive integers c —in fact infinitely many of them— for which this happens if and only if $(l^2 - 1)(k^2 - 1)$ is *not* a perfect square.) Then

there does *not* exist any $d > 1$ such that $Q(l, k, c, d)$ is a perfect square. Assume further that $l|(kc)$ —this can be arranged, for example, taking k to be a multiple of l —and set $n := c + k$, $m := kc/l$ so that $k(n - k) = l(m - l)$. Then $Q(l, k, n - k, m - l)$ is not a perfect square.

(iii) We illustrate below situations $Q(l, k, n - k, m - l)$ is not a perfect square (assuming that $k(n - k) = l(m - l)$) depending on congruence classes modulo a suitable prime power of the parameters involved.

(1) For an odd prime p , suppose that $k \equiv p^{2r-1} \pm 1 \pmod{p^{2r}}$ and none of the numbers $l, m - l, n - k$ is congruent to $\pm 1 \pmod{p}$. Then $p^{2r-1}|Q$ but $p^{2r} \nmid Q$.

(2) Suppose that $m \equiv l \equiv 5 \pmod{8}$, and $k \equiv 7 \pmod{16}$. Then $(m - l)^2 - 1$ is odd, $l^2 - 1 \equiv 8 \pmod{16}$, $k^2 - 1 \equiv 16 \pmod{32}$ and $l(m - l) = k(n - k)$ implies $(n - k)$ is even and so $(n - k)^2 - 1$ is odd. Thus $Q \equiv 2^7 \pmod{2^8}$.

(3) Suppose that $l \equiv 0 \pmod{8}$, $m \equiv l \pmod{2}$, $k \equiv 3 \pmod{8}$, then $Q \equiv 8 \pmod{16}$.

Proposition 4.3.3 *Let $c > 1$ and let $k = 3$ or 7 . Suppose that $Q(2, k, 2c, kc)$ is a perfect square. Then there exists integers $\xi, \eta, v > 1$ such that $c = \frac{1}{2}(\xi^2\eta^2 + 1)$, $\xi^2\eta^2 - 3v^2 = -2$ and (i) $\xi^2 - 3\eta^2 = -2$ when $k = 3$ and (ii) $\xi^2 - 7\eta^2 = -6$ when $k = 7$.*

Proof. Assume that $k = 7$ and that $Q := Q(2, 7, 2c, 7c) = 3^2 2^4 (2c - 1)(2c + 1)(7c - 1)(7c + 1)$ is a perfect square. There are several cases to consider depending on the gcd of the pairs of numbers involved. Write $(2c - 1) = \alpha u^2$, $2c + 1 = \beta v^2$, $7c - 1 = \gamma x^2$, $7c + 1 = \delta y^2$, where $\alpha, \beta, \gamma, \delta$ are square free integers. Since Q is a perfect square and since $\gcd(2c - 1, 2c + 1) = 1$, $\gcd(7c - 1, 7c + 1) = 1$ or 2 , $\gcd(2c \pm 1, 7c \pm 1) = 1$, or 5 , $\gcd(2c \pm 1, 7c \mp 1) = 1, 3$, or 9 , the possible values for (α, β) are:

$(1, 1), (1, 5), (1, 3), (3, 1), (5, 1), (1, 15), (15, 1), (5, 3), (3, 5)$. The possible values for (γ, δ) are the same as for (α, β) as well as $(2\alpha, 2\beta)$.

Suppose $(\alpha, \beta) = (1, 1)$. Since $(2c - 1) + 2 = (2c + 1)$, we obtain $u^2 + 2 = v^2$ which has no solution. If $(\alpha, \beta) = (3, 1)$, then $3u^2 + 2 = v^2$. This equation has no solution mod 3. Similar arguments show that if $(\alpha, \beta) = (5, 1), (1, 5), (1, 15), (15, 1), (5, 3)$, there are no solutions for u, v . If $(\alpha, \beta) = (3, 5)$, then $(\gamma, \delta) = (5, 3)$ or $(10, 6)$. If $(\gamma, \delta) = (5, 3)$ again there is no solution mod 3 for the equation $5x^2 + 2 = 3y^2$. When $(\gamma, \delta) = (10, 6)$ we obtain

$10x^2 + 2 = 6y^2$. This has no solution mod 5.

It remains to consider the case $(\alpha, \beta) = (1, 3)$. In this case obtain the equation $u^2 + 2 = 3v^2$ which has solutions, for example, $(u, v) = (5, 3)$. Now $(\alpha, \beta) = (1, 3)$ implies $(\gamma, \delta) = (3, 1)$ or $(6, 2)$. If $(\gamma, \delta) = (3, 1)$ then we obtain the equation $3x^2 + 2 = y^2$ which has no solution mod 3. So assume that $(\gamma, \delta) = (6, 2)$. As $(\alpha, \delta) = (1, 2)$ we obtain $4y^2 - 7u^2 = 9$, that is, $4y^2 - 7u^2 = 9$. Thus $(2y - 3)(2y + 3) = 7u^2$. Either $7|(2y - 3)$ or $7|(2y + 3)$. Say $7|(2y - 3)$ and write $(2y - 3) = 7z$. Now $z(7z + 6) = u^2$. Observe that $\gcd(z, 7z + 6)$ divides 6.

Since $\beta = 3$, $2c - 1 = u^2$ is not divisible by 3. Also, u being odd, we must have $\gcd(z, 7z + 6) = 1$. It follows that both $z, 7z + 6$ are perfect squares. This forces that 6 is a square mod 7—a contradiction.

Finally, suppose that $7|(2y + 3)$. Then repeating the above argument we see that both $(2y - 3) =: \eta^2$ and $(2y + 3)/7 =: \xi^2$ are perfect squares. It follows that $7\xi^2 - 6 = \eta^2$ is a perfect square. Hence $2c - 1 = u^2 = \xi^2\eta^2$. Since $2c + 1 = 3v^2$, the proposition follows.

We now consider the case $k = 3$. We merely sketch the proof in this case. Let, if possible, $Q = 2^3 3(2c - 1)(2c + 1)(3c - 1)(3c + 1)$ be a perfect square. Write $2c - 1 = \alpha u^2$, $2c + 1 = \beta v^2$, $3c - 1 = \gamma x^2$, $3c + 1 = \delta y^2$, where $\alpha, \beta, \gamma, \delta$ are square free integers and u, v, x, y are positive integers. Arguing as in the case $k = 7$, following are the only possible values for $\alpha, \beta, \gamma, \delta$: $(\alpha, \beta) = (1, 3), (3, 1), (3, 5), (5, 3), (1, 15), (15, 1)$, and $(\gamma, \delta) = (1, 2), (2, 1), (2, 5), (5, 2), (1, 10), (10, 1)$. It can be seen that only the case $(\alpha, \beta, \gamma, \delta) = (1, 3, 2, 1)$ remains to be considered, the remaining possibilities leading to contradictions. Thus we have $2c - 1 = u^2$, $2c + 1 = 3v^2$, $3c - 1 = 2x^2$ and $3c + 1 = y^2$. Therefore, we have $4x^2 - 1 = 3u^2$, i.e., $(2x - 1)(2x + 1) = 3u^2$. Hence, $3|(2x - 1)$ or $3|(2x + 1)$.

Suppose that $3|(2x - 1)$. Write $3z = 2x - 1$, $z \in \mathbb{Z}$. Since z is odd, we have $\gcd(z, 3z + 2) = 1$. As $z(3z + 2) = u^2$ we conclude that z and $3z + 2$ have to be perfect squares. This implies that 2 is a quadratic residue mod 3—a contradiction. Therefore $3 \nmid (2x - 1)$ and we must have $3|(2x + 1)$ and both

z and $3z - 2$ will have to be perfect squares. Write $z = \eta^2$ and $3z - 2 = \xi^2$ so that $\xi^2 - 3\eta^2 = -2$ and $v^2 = u^2 + 2 = \xi^2\eta^2 + 2$. This completes the proof. \square

Remark 4.3.4 (i) Let $K = \mathbb{Q}[\sqrt{7}]$ and let R be the ring of integers in K . If $\xi + \eta\sqrt{7} \in R$, then $\xi, \eta \in \mathbb{Z}$. Denote the multiplicative ring of units in R by U . Note that any element of U has norm 1. (This is because -1 is a quadratic non-residue mod 7.) Using Dirichlet Unit theorem U has rank 1; indeed U is generated by $\nu := (8 + 3\sqrt{7})$ and ± 1 . The integers ξ, η as in the above proposition yield an element $\xi + \eta\sqrt{7}$ of norm -6 and the set $S \subset R$ of all elements of norm -6 is stable under the multiplication action by U . An easy argument shows that S is the union of orbits through $\lambda := 1 + \sqrt{7}, \bar{\lambda} = 1 - \sqrt{7}$. Thus $S = \{\pm\lambda\nu^k, \pm\bar{\lambda}\nu^k \mid k \in \mathbb{Z}\}$.

Observe that if ξ, η are as in Proposition 4.3.3(ii), then $\xi + \sqrt{7}\eta \in S$. Listing elements $\xi + \eta\sqrt{7} \in S$ with $\xi, \eta > 1$ in increasing order of η , the first three elements are $13 + 5\sqrt{7}, 29 + 11\sqrt{7}, 209 + 79\sqrt{7}$. Straightforward verification shows that when $\xi + \eta\sqrt{7}$ is equals any of these, then there does not exist an integer v such that $\xi^2\eta^2 + 2 = 3v^2$. Since the next term is $463 + 175\sqrt{7}$, we have the lower bound $2c > 175^2 \times 463^2 = 6565050625$ in order that $Q(2, 7, 2c, 7c)$ be a perfect square (assuming $c > 1$).

(ii) Now, let $K = \mathbb{Q}[\sqrt{3}]$ and let R be the ring of integers in K . Note that $\xi + \eta\sqrt{3} \in R$, then $\xi, \eta \in \mathbb{Z}$. Denote the multiplicative ring of units in R by U , which is generated by generated by $(2 + \sqrt{3})$ and ± 1 .

Suppose that $Q(2, 3, 2c, 3c)$ is a perfect square, $c > 1$. Then the integers ξ, η , as in the above proposition, yield an element $\xi + \eta\sqrt{3}$ of norm -2 . The set $S \subset R$ of all elements of norm -2 is stable under the multiplication action by U . In fact it can be verified easily that $S = \{\pm(1 + \sqrt{3})(2 + \sqrt{3})^m \mid m \in \mathbb{Z}\}$.

Listing these with $\xi, \eta > 1$, in increasing order of η , the first five elements are $5 + 3\sqrt{3}, 19 + 11\sqrt{3}, 71 + 41\sqrt{3}, 265 + 153\sqrt{3}, 989 + 571\sqrt{3}$. If $\xi + \eta\sqrt{3}$ equals any of these, direct verification shows that there is no integer v satisfying the equation $\xi^2\eta^2 + 2 = 3v^2$. The next term of the sequence being $3691 + 2131\sqrt{3}$ we obtain the lower bound $2c > 2131^2 \times 3691^2 = 61866420601441$.

Perhaps, the above arguments can be applied for other values of k and l to obtain lower bounds for n , particularly when $k + l$ and $k - l$ are primes or prime powers.

The above discussion might tempt one to conjecture that, for integers a, b, c, d such that $1 < a < b < c < d$ and $ad = bc$, $Q(a, b, c, d)$ cannot be a perfect square. However, J. Oesterlé gave the following counterexample:

Example 4.3.5 For $a = 23$, $b = 69$, $c = 1121$ and $d = 3363$,
 $1 < a < b < c < d$ and $ad = bc$ but

$$\begin{aligned} Q(a, b, c, d) &= (23^2 - 1)(69^2 - 1)(1121^2 - 1)(3363^2 - 1) \\ &= 22 \cdot 24 \cdot 68 \cdot 70 \cdot 1120 \cdot 1122 \cdot 3362 \cdot 3364 \\ &= 2^{16} \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 17^2 \cdot 29^2 \cdot 41^2 \end{aligned}$$

is a perfect square.

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