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LECTURES ON
DUALITY THEORY IN LOCALLY CONVEX SPACES

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DUALITY THEORY IN LOCALLY CONVEX SPACES

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C O N T E N T S

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PRELIMINARIES

1.1 DEFINITION: A linear space E (over real or complex number field) is said to be <u>normed</u> if to every $x \in E$, we can associate a real number called the norm of x (denoted by $\|x\|$) such that

1. $\|x\| \geq 0$ for all x in E

2. $\|x\| = 0$ implies x = the zero element of the space.

3. $\|\lambda x\| = |\lambda| \|x\|$, for every scalar λ , and

4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$ (triangle inequality)

This norm introduces in E, a metric, with the distance function given by $d(x,y) = \|x-y\|$. We note that addition and scalar multiplication are continuous in the metric topology and also that the norm is a continuous function. Thus every normed linear space is a linear topological space.

1.2 DEFINITION: A <u>seminorm</u> on E is a real valued function $\| \ \|_E$ in E satisfying (1), (2) and (4) of 1.1 In this case, E is called a <u>semi-normed space</u>.

We observe that in a seminormed space E, $\|x\|_E = 0$ need not imply $x = 0$. Also if E and F are seminormed spaces, and T is a linear map from E to F, then T is continuous if and only if there exists a constant $M > 0$ such that $\|Tx\|_F \leq M\|x\|_E$ for every $x \in E$.

1.3 Example of a seminormed linear space which is not normed.

Let $\mathscr{C}(E)$ denote the linear space of all real valued continuous functions on a locally compact Hausdorff space E. For each compact subset K of E, define $p_K(f) = \sup_{x \in K} |f(x)|$ where $f \in \mathscr{C}(E)$. One easily verifies that $p_K(f)$ is a seminorm on E, for each compact set K. The fact that there exist non-zero continuous functions vanishing on K shows that $p_K(f)$ is not a norm in $\mathscr{C}(E)$.

1.4 Let $p_\alpha : \alpha \in \mathbb{A}$ be a family of seminorms on E. (E, p_α) is a seminormed space for each $\alpha \in \mathbb{A}$, but in general it need not be Hausdorff. Let us impose the following condition on the family of seminorms:-

> (*) For each $x \in E$, $x \neq 0$, there exists at least one $\alpha \in \mathbb{A}$ such that $p_\alpha(x) \neq 0$.

Take the l.u.b. of the seminormed topologies on E defined by the p_α's (i.e., the topology generated by their union). This is called the topology generated by the family

p_α of seminorms. Obviously, it is a linear topological space. The Hausdorff nature follows from the condition (*) which we have imposed on the p_α 's. A basis of neighborhoods of 0 is given by finite intersection of sets of the form $\{ x \in E \mid p_\alpha(x) < \epsilon \}$. Since each of these sets is convex, we have a basis of convex neighborhoods of zero and hence the space is a <u>locally convex linear Hausdorff space</u>(abbreviated as l.c.l.H.s. or simply an l.c. space). In fact, every l.c. space can be obtained from such a family of seminorms.

1.5 Let us go back to example 1.3. If $\mathcal{K}$ denotes the family of all compact subsets of E, then the topology defined by the family of seminorms p_K: $K \in \mathcal{K}$ makes $\mathcal{C}(E)$, a l.c. space. One easily sees that this is nothing but the topology of uniform convergence on compact subsets of E.

1.6 As in 1.2, the condition for a linear map to be continuous can be obtained as follows:

> Let $E(\tau)$, $F(\tau')$ be l.c. spaces, τ, τ' being respectively defined by families P, Q of seminorms. A linear map $T : E(\tau) \longrightarrow F(\tau')$ is continuous if and only if to every $q \in Q$, there exist $p_i \in P$, $i = 1,2, \ldots n$ and a constant $\rho > 0$ such that
>
> $$\rho\, q(Tx) \leq \sup_{1 \leq i \leq n} p_i(x) \quad \forall x \in E$$

Equivalently, T is continuous if and only if for each $q \in Q$, there exists a continuous seminorm p on E such that $q(Tx) \leq p(x)$ for all $x \in E$.

1.7 Let E be an infinite dimensional Banach space and let E' be its dual. For each element $x' \in E'$, $p_{x'}(x) = |x'(x)|$ for all $x \in E$ defines a seminorm on E. The family of seminorms $p_{x'}(x)$, as x' runs through the elements of E', defines a l.c. topology on E. That this topology is Hausdorff follows from Hahn-Banach theorem. This is called the <u>weak topology</u> on E, generated by E'. A representative element of a fundamental basis of neighborhood of 0 is $\{x \in E \mid |x_i'(x)| < \varepsilon, x_i' \in E', i = 1, 2, \cdots n\}$. Convergence in this topology means uniform convergence on finite subsets of E'. If we consider the element of E as functions defined on E' by the rule $x : x' \longrightarrow x'(x)$, the above is just pointwise convergence, also known as simple convergence.

1.8 The situation described above is of a very particular nature. In order to obtain the weak topology, one need not have to start with a Banach space and its dual. It suffices to take any two linear spaces E_1 and E_2, provided they satisfy certain duality condition which we describe in the next section.

1.9 <u>Definition of a dual system</u>:

By a <u>bilinear form</u> on $E_1 \times E_2$ we mean a linear form $(u, x) \longrightarrow \langle u, x \rangle$ satisfying

$$\langle u, \alpha x_1 + \beta x_2 \rangle = \alpha \langle u, x_1 \rangle + \beta \langle u, x_2 \rangle \quad \text{and}$$

$$\langle \alpha u_1 + \beta u_2, x \rangle = \alpha \langle u_1, x \rangle + \beta \langle u_2, x \rangle .$$

DEFINITION: Two linear spaces E_1 and E_2 are said to be <u>in duality</u> if to every pair $(u, x) \in E_2 \times E_1$, we can associate a scalar denoted by $\langle u, x \rangle$ satisfying

1. $\langle u, x \rangle$ is a bilinear form on $E_2 \times E_1$
2. $u \in E_2$, $\langle u, x \rangle = 0$ for all $x \in E_1$ implies $u = 0$
3. $x \in E_1$, $\langle u, x \rangle = 0$ for all $u \in E_2$ implies $x = 0$

The above situation will be referred to as ' $\langle E_2, E_1 \rangle$ is a dual system'.

1.10 Let $\langle E_2, E_1 \rangle$ be a dual system. Consider the family of seminorms $p_u(x) = |u(x)| = |\langle u, x \rangle|$ for all $x \in E_1$ where $u \in E_2$. The topology defined on E_1 by this family is a l.c.l.H. space. This is called the <u>weak topology on E_1</u> defined by E_2. A sub-basis of neighborhoods of zero for this topology on E_1 consists of sets of the form $\{ x \in E_2 \mid p_u(x) < \varepsilon \}$

Every $u \in E_2$ is continuous for this topology. The weak topology the coarsest l.c. topology on E_1 for which all the functions $u \in E_2$ are continuous.

1.11 A bsis of neighborhoods of zero for the weak topology on E_1 defined by E_2 consists of sets of the form

$$\{ x \in E_1 \mid p_{u_i}(x) < \varepsilon ,\ u_i \in E_2 ,\ i = 1,2,\cdots n \}$$

1.12 By $E_1[\mathcal{T}_s(E_2)]$ we shall mean the space E_1 endowed with the weak topology generated by E_2 on it. Other notations in the literature for weak topology on E_1 defined E_2 are $\sigma(E_1, E_2)$ (Bourbaki), $\omega(E_1,E_2)$ (Kelley).

1.13 PROPOSITION: <u>If</u> E_2, E_1 <u>is a dual system, then</u> $E_1[\mathcal{T}_s(E_2)]$ <u>is a l.c.l.H. space and its dual is E_2.</u>

PROOF: The Hausdorff nature of the space follows from 1.9 (2) and (3). Since $u \in E_2$ implies u is weak-continuous on E_1, we have $E_2 \subset E_1[\mathcal{T}_s(E_2)]'$. Conversely, if $u \in E_1^*$ is continuous for $\mathcal{T}_s(E_2)$, then given $\varepsilon > 0$, there exists $U = U(u_1, u_2, \cdots, u_n, \varepsilon)$ such that $|u(x)| \leq \sup_{i=1,2,\cdots,n} |u_i(x)|$. This means that whenever $u_i(x) = 0$ for $i = 1,2....n$, then $u(x) = 0$. In other words, if x is in the orthogonal complement of $(u_1, \cdots, u_n)$, $u(x) = 0$ By a well-known theorem in Algebra (see Köthe, page 75), u is a linear combination of $u_1, u_2, \cdots, u_n$ and hence $u \in E_2$. Hence

$$E_2 = \left[E_1 \left[\mathcal{T}_s (E_2) \right] \right]'$$

1.14 Examples of l.c. spaces in duality

1) E any l.c. space and E', its dual

2) E any linear space and E^* its algebraic dual.

1.15 PROPOSITION: If E has dimension 'd'. then $E^*[\tau_s(E)]$ is homeomorphic to $\omega_d(K)$, which is the product of d copies of scalar field K.

PROOF: An element $v \in E^*$ is uniquely determined when we know the values of v on the basis element of E, say, $e_\alpha : \alpha \in A$, having cardinality d. This means $V \longrightarrow (V e_\alpha)$: $\alpha \in A$ is a (1-1) correspondence and is also a linear isomorphism, But $(V_\alpha) = (V e_\alpha) \in \omega_d(K)$. To show that this isomorphism is topological as well, we observe that the topology on $E^*[\tau_s(E)]$ is given by the basis element $\{ v \in E^* \mid |\langle v, x_i \rangle| < \epsilon, i = 1, 2, \cdots, n, x_i \in E \}$. This is just a set of the form $\{ (v_\alpha) \in \omega_d(K) \}$, whose finite number of coordinates are bounded by ϵ, which we know, defines the product topology on $\omega_d(K)$.

1.16 PROPOSITION: Let $\langle E_2, E_1 \rangle$ be a dual system. Then $E_2^*[\tau_s(E_2)]$ is the completion of $E_1[\tau_s(E_2)]$ and is therefore homeomorphic to $\omega_d(K)$, where d is the dimension of E_2.

PROOF: Clearly $E_2^* \left[\mathcal{T}_s(E_2) \right]$ is homeomorphic to $\omega_d(K)$ by 1.15, and since the uniform structures are also preserved, here by the homeomorphism, it follows that $E_2^* \left[\mathcal{T}_s (E_2) \right]$ is complete. It remains to show that E_1 is dense in $E_2{}^* \left[\mathcal{T}_s (E_2) \right]$. If not, an application of Hahn-Banach theorem shows that there exists $u \in E_2$ vanishing on E_1. This contradicts that $\langle E_2, E_1 \rangle$ is a dual system.

CHAPTER II

BASIC THEOREMS OF DUALITY THEORY

2.1 DEFINITION A set $A \subset E$ absorbs a point $x \in E$ if there exists $\rho > 0$ such that $x \in \rho A$. A set $A \subset E$ absorbs a set B if there exists $\rho > 0$ such that $B \subset \rho A$. A set A in a linear topological space is said to be bounded if every neighborhood of zero absorbs A.

A set A in a l.c. space E is therefore bounded, if A is bounded in every seminorm. In $E_1[\tau_s(E_2)]$, A is bounded if $\sup_{x \in A} |\langle u, x \rangle| < \infty$ for every $u \in E_2$.

2.2 DEFINITION: A set A in a linear topological space E is precompact if its completion is compact.

One verifies (how!) that a set is precompact if and only if it is totally bounded i.e., for every zero neighborhood there exists a finite set $x_1, x_2, \cdots, x_n \in E$ such that $\bigcup_{i=1,2,\cdots,n} (x_i + U) \supset A$. Total boundedness (and hence precompactness) implies boundedness, but converse is not in general true. However, the situation is pleasant in the weak topology.

2.3 PROPOSITION: The weakly bounded sets of a l.c. space $E(\tau)$ and the weakly precompact sets are the same.

PROOF: Let $M \subset E$ be bounded in $\tau_s(E')$. It is also bounded in the completion $E'^*[\tau_s(E')$ which is homeomorphic to $\omega_d(K)$, d being dimension of E'. But in the latter, every bounded set is just of the form $\prod B_\alpha$, where B_α is a bounded set of K; an application of Tychonoff's theorem yields the result.

2.4 DEFINITION: If $\langle E_2, E_1 \rangle$ is a dual system, and $A \subset E_1$ the set $A^\circ = \{ u \in E_2 \mid \mathcal{R}(ux) \leq 1 \}$ is called the <u>polar</u> of A in E_2.

If A is circled [viz., $x \in A$ implies $\lambda x \in A$ for all λ such that $|\lambda| \leq 1$]. A° is then the set $\{ u \in E_2 / |u(x)| \leq 1$ for all $x \in A \}$. $A^\circ_\circ$ is then the proof of A° in E_1. The following facts about the polar are easily verified.

(1) A° is convex, circled and weakly closed
(2) $A \subset B$ implies $A^\circ \supset B^\circ$
(3) $(\lambda A)^\circ = \frac{1}{\lambda} A^\circ$ for every scalar $\lambda \neq 0$
(4) $A^\circ{}_\circ$ is the weakly closed convex hull of A and the zero element of E_1.
(5) $A^\circ = A^\circ{}_\circ{}^\circ$

2.5 ALAOGLU BOURBAKI THEOREM:

<u>If</u> U <u>is a neighborhood of 0 in an l.c. space</u> E(τ) <u>then</u> U° <u>is</u> τ_s(E) - <u>compact in</u> E^*.

PROOF: Without loss of generality, take U to be absolutely convex. Otherwise, it contains an absolutely convex V, and hence $U^0 \subset V^0$. Since U^0 is weakly closed, the compatness of U^0 follows from that of V^0.

Let U be absolutely convex. Note that U has the same polar U^0 in $E^{\wedge}$ as well as E^*. Since U absorbs each point of E, U^0 is weakly bounded in E^*. Then U^0 is a bounded subset of $E^* \left[\tau_s(E) \right]$. But the latter is homeomorphic to $\omega_d(K)$, in which every bounded closed set is relatively compact. U^0 being weakly closed, the result follows.

2.6 DEFINITION: Let $\langle E_2, E_1 \rangle$ be a dual system. Any l.c. topology τ on E_1 for which the topological dual is E_2 is said to be an <u>admissible topology</u> for the dual system $\langle E_2, E_1 \rangle$.

<u>Examples of admissible topologies are:-</u>

(1) For the dual system $\langle E_2, E_1 \rangle$, $\tau_s(E_2, E_1)$ on E_1 is admissible.

(2) If $E(\tau)$ has dual E', τ is admissible for $\langle E', E \rangle$.

As a consequence of the separation theorem for convex compact sets, we get the following theorem:

The closure of a convex set is the same for all admissible topologies.

2.7 DEFINITION: Let $\langle E_2, E_1\rangle$ be a dual system. Let $M \subset E_1$. M is said to be E_2-<u>bounded</u> (or simply E_2-bounded or simply bounded for the dual system) if

$$\sup_{x\in M} |\langle u, x\rangle| < \infty \quad \text{whatever be } u \text{ in } E_2$$

In other words, this is equivalent to saying $M \subset E_1$ is $\mathcal{T}_s(E_2)$-bounded i.e. M is weakoy bounded.

The class of all simply E_2-bounded subsets of E_1, is denoted by $\mathcal{M}(E_2,E_1)$. The meaning of $\mathcal{M}(E_1\ E_2)$ is now clear.

DEFINITION: A set $B \subset E_1$, is <u>strongly</u> E_2-<u>bounded</u> if it is bounded uniformly on each simply bounded subset of E_2 (i.e.)

$$\sup_{\substack{x\in B\\ u\in M}} |\langle u, x\rangle| = \mu(M) < \infty \text{ for all } M \in \mathcal{M}(E_1\ E_2).$$

$\mathcal{B}(E_2, E_1)$ denotes the class of all strongly E_2-bounded subsets of E_1. A similar meaning holds for $\mathcal{B}(E_1\ E_2)$.

Obviously $\mathcal{M}(E_2,E_1) \supset \mathcal{B}(E_2,E_1)$ (*)

2.8 In General, the inclusion (*) in 2.7 cannot be reversed. For example consider the dual system $\langle\phi, \phi\rangle$, ϕ = space of all sequences with finitely many non-zero coordinates under the duality $\langle u, x\rangle = \sum_1^\infty u_i x_i$ where $u = (u_i)$, $x = (x_i)$ are elements of ϕ. A simply bounded set of ϕ is just a set of the form $x = (x_i)$ such that $|x_i| \leq M_i$, $i = 1, 2, \ldots$ (i.e. coordinatewise bounded). A strongly bounded set is of the form

$x = (x_i)$, with $|x_i| \leq M_i$, $i = 1, 2, \ldots, M_i = 0, \forall\, i \geq i_0$ (after a certain stage).

2.9 The question arises when the inclusion (*) can be reversed. We know certain situations where this is possible. The uniform boundedness principle for Banach spaces asserts just this in the dual of a Banach space.

2.10 Let $\langle E_2, E_1 \rangle$ be a dual system. Let M be a weakly bounded, absolutely convex, set of E_1. Consider the linear space generated by M in E_1. Call it E_{1M}. (This is equal to $\bigcup_{n>0} nM$). $p_M(x) = \inf\{\rho > 0 \mid x \in \rho M\}$ is a norm in E_{IM} and E_{IM} is a normed linear space under this norm. On E_{1M}, the space $E_1[\mathcal{T}_s(E_2)]$ induces a certain l.c. topology. Since M is weakly bounded, M is absorbed by every weak neighborhood U of 0, in $\mathcal{T}_s(E_2)$. This means for some $\lambda > 0$, $\lambda M \subset U$ In other words, every weak neighborhood of 0, in the induced topology which is of the form $U \cap E_{1M}$ contains a multiple of the unit ball in the n.l. space E_{1M}, which is a neighborhood in the norm topology. Hence the induced weak topology is coarser than the norm topology.

DEFINITION: If a Cauchy sequence in M for the norm topology has a limit in M for this norm topology, then M is said to be <u>complete-in-itself</u>. In such a case E_{1M} is a Banach space with M as its closed ball.

2.11 PROPOSITION: If M is absolutely convex bounded, closed, sequentially complete subset of l.c. $E(\tau)$, then E_M is a Banach space with M as closed unit ball and on which τ induces a coarser topology than the norm topology on E_{1M}.

2.12 LEMMA: (used in the proof of 2.11 and also having intrinsic importance).

On l.t.s. $E(\tau)$, let a finer topology τ' be given. Let τ' possess a zero neighborhood basis of τ-closed sets. Then every τ'-Cauchy filter which converges as τ-Cauchy filter to $x_i \in E$ also converges in τ'. Consequently every τ-complete (τ-sequentially complete subset of E) is τ'-complete (τ'-sequentially complete).

2.13 PROOF OF 2.11: Complete this !

2.14 BANACH-MACKEY THEOREM: Let $\langle E_2, E_1 \rangle$ be a dual system. Every absolutely convex convex complete-in-itself weakly bounded subset B of E_1 is strongly E_2-bounded. Also every absolutely convex closed sequentially complete bounded subset of $E(\tau)$ (a.l.c. space) is strongly E'-bounded.

PROOF: E_{1B} is Banach space with B as its closed unit ball.

Let $M \in \mathcal{M}(E_1, E_2)$

This means $\sup_{u \in M} |\langle u, x\rangle| < \infty$ for all $x \in E$. Therefore $\{ u|_{E_{1B}}, u \in M \}$ is a pointwise bounded set of linear functionals on E_{1B}. The unifom bounded principle says that this is uniformly bounded in the unit ball B of E_{1B}. Therefore

$$\sup_{\substack{u \in M \\ x \in B}} |\langle u, x \rangle| < \infty$$

This says that B is uniformly bounded on M. M being arbitrary in $\mathcal{M}(E_1, E_2)$, we see that B is strongly E_2-bounded.

For the second part, consider $\langle E', E \rangle$ and use 2.11.

2.15 THEOREM: <u>The same sets are bounded in every admissible topology of the dual pair $\langle E_2, E_1 \rangle$. In particular, if $E(\tau)$ is l.c. with dual E' then the τ-bounded sets are precisely the weakly bounded sets of E.</u>

PROOF: It suffices to prove the last part. Clearly τ-bounded sets are weakly bounded. Suppose a set $M \subset E$ is $\tau_s(E')$-bounded. Let U be an absolutely convex closed neighborhood of zero in E. By Alaoglu Bourbaki theorem U° is $\tau_s(E)$-compact. By Banach-Mackey theorem, then U° is strongly E-bounded. Hence

$$\sup_{\substack{u \in U^{\circ} \\ x \in M}} |\langle u, x \rangle| < \infty \quad (= \mu \text{ say})$$

i.e.,
$$\left| \langle \frac{u}{\mu}, x \rangle \right| \leq 1 \text{ for } u \in U^{\circ}, x \in M$$

Hence
$$M \subset \left(\frac{U^{\circ}}{\mu} \right)_{\circ} = \mu U^{\circ}_{\circ} = \mu U$$

So M is τ bounded.

2.16 PROPOSITION: If E(τ) is sequentially complete, every weak bounded subset of E and of E' is strongly bounded for the dual system $\langle E', E \rangle$.

PROOF: By 2.15, τ-bounded sets are precisely the weak bounded sets. Again by Banach Mackey theorem τ-bounded closed sets are strongly E'-bounded. Hence weak boundedness implies closure is strongly E'-bounded and hence the set itself is strongly E'-bounded subset of E.

This means again $\sup_{\substack{x \in M \\ u \in B'}} |\langle u, x \rangle| < \infty$ for all $B' \in \mathcal{M}(E, E')$

So B' is uniformly bounded on M.

i.e. Every simply bounded set B' of E' is strongly E.bounded.

CHAPTER III

TOPOLOGIES ON E.

3.1 We know that for a dual system $\langle E_2, E_1 \rangle$, the weak topology $E_1 [\mathcal{T}_s (E_2)]$ is defined by the seminorms $p_F(x) = \sup_{u \in F} |\langle u, x \rangle|$ where F is a finite-subset of E_2. Let us by anology define $p_M(x) = \sup_{u \in M} |\langle u, x \rangle|$ for any $M \subset E_2$. One easily sees that $p_M(x)$ is a seminorm if and only if M is a bounded set of E_2 for the dual system $\langle E_2, E_1 \rangle$.

DEFINITION: If $\mathcal{M}$ is a family of bounded sets for the dual system $\langle E_2, E_1 \rangle$ of E_2 then $\mathcal{M}$ is said to be <u>total in</u> E_2 <u>over</u> E_1 if $\bigcup_{M \in \mathcal{M}} M$ is total in $E_2 [\mathcal{T}_s (E_1)]$ (i.e. the linear hull of $\bigcup_{M \in \mathcal{M}} M$ is weakly dense in E_2).

It is easy to see that the topology defined by the family $p_M(x)$, $M \in \mathcal{M}$ is a l.c.H. topology on E_1 if and only if $\mathcal{M}$ is total in E_2 over E_1. Starting with such a family $\mathcal{M}$, we define a l.c.l.H. topology on E_1. It is called the topology of uniform convergence on sets $M \in \mathcal{M}$, denoted by $\mathcal{T}_{\mathcal{M}}$. A neighborhood basis for this topology is

$\{M^o \mid M \in \mathcal{M}\}$ and a defining family of semi-norms for this is $\{p_M(x) \mid M \in \mathcal{M}\}$.

We observe that by widening the family $\mathcal{M}$ by adding to $\mathcal{M}$, any of the following sets, $\tau_{\mathcal{M}}$ would not be altered: i) all subsets of any $M \in \mathcal{M}$

ii) ρM, $\rho > 0$, $M \in \mathcal{M}$

iii) Weakly closed conves hull of $M_1 \cup M_2$, $M_1, M_2 \in \mathcal{M}$

The class $\mathcal{M}$ is said to be saturated when $\mathcal{M}$ contains all the families i, ii, iii mentioned above. $\mathcal{M}$ is then a saturated topologizing family.

For example, if $\mathcal{F}$ is the class of all finite subsets of E_2 (which defines $\tau_s(E_2)$ on E_1), its satuated class consists of all finite dimensional bounded sets of E_2; this also defines the same topology $\tau_s(E_2)$ on E_1:

3.2 Take $\mathcal{M}$ to be the family $\mathcal{M}(E_1, E_2)$ of all bounded sets of E_2. The topology of uniform convergence on this class is called the strong topology on E_1, and is denoted by $\tau_b(E_2)$. The family $p_M(x)$: $M \in \mathcal{M}(E_1, E_2)$ is the family of semi-norms defining the topology. A set $A \subset E_1$ is bounded in the strong topology if and only if it is bounded in every semi-norm. In other words, if and only if

$$\sup_{\substack{u \in M \\ x \in A}} |\langle u, x \rangle| < \infty \qquad \text{for all } M \in \mathcal{M}_b(E_1 E_2)$$

This just means A is strongly E_2-bounded.

It is to be noted that the strong topology $\tau_b(E_2)$ on E_1 is not in general an admissible topology. For, if it were admissible, then $\tau_b(E_2)$-bounded sets will have to be simply bounded (2.15). But we know that for dual pair $\langle \phi\ \phi \rangle$, the family of weakly bounded sets and the family of strongly bounded sets are not the same.

3.3 We now describe the neighborhoods in the strong topology. Let $E(\tau)$ be l.c. with dual E . A neighborhood basis of $\tau_b(E')$ on E is then the family $\{M^\circ \mid M$ is a bounded set of E' for $\langle E', E \rangle\}$. Since M is bounded in $'$, M° is absolutely convex, weakly closed, absorbing subset of E.

DEFINITION: A barrel in a linear topological space is an absolutely convex, weakly closed, absorbing set.

If B is bar l in $E(\tau)$, B° is weakly bounded since B is absorbling. Hence $B^\circ{}_\circ = B$ is a neighborhood for $\tau_b(E')$ on E. Thus in $E(\tau_b(E^1))$ the barrels in E form a basis neighborhoods of zero.

DEFINITION: A l.c. space is said to be barrelled if every barrel in it is a neighborhood. In other words, $E(\mathcal{T})$, with dual E', is barrelled if and only if $\mathcal{T} = \mathcal{T}_b(E')$. Also for such spaces, the topology is topology of u.c. on bounded sets of E'.

3.4 DEFINITION: Let E' be the dual of $E(\mathcal{T})$, A set $M \subset E'$ is equicontinuous if to every $\varepsilon > 0$, there exists a neighborhood U of 0, in $E(\mathcal{T})$ such that

$$\sup_{\substack{u \in M \\ x \in U}} |\langle u, x \rangle| < \varepsilon$$

It follows that M is equicontinuous if and only if $M \subset U^o$ for some neighborhood U of 0 in E.

So the topology $\mathcal{T}$ is the topology of u.c. on the family U^o of equicontinuous subsets of E'. In fact, whenever we have a u.c. topology $\mathcal{T}_{\mathcal{M}}$ the class of equicontinuous sets for this topology is the saturated class of $\mathcal{M}$.

3.5 Let $\langle E_2, E_1 \rangle$ be a dual system. Let $\mathcal{K}$ consist of all absolute convex weakly compact subsets of E_2 and their subsets. The $\mathcal{K}$-topology on E_1 is called the Mackey topology on E_1 and is denoted by $\mathcal{T}_{\mathcal{K}}(E_2)$.

Mackey-Arens Theorem: The Mackey topology $\tau_k(E_2)$ is the finest admissible l.c. topology on E_1 for the dual system $\langle E_2, E_1 \rangle$.

(For proof see Köthe)

3.6 If $E(\tau)$ is l.c. space with dual E' then,

$\tau_s(E') \subset \tau \subset \tau_k(E') \subset \tau_b(E')$. In particular if $E(\tau)$ is a barrelled space, $\tau = \tau_b(E')$ and so for such spaces, $\tau_k(E') = \tau_b(E')$. This means for barrelled spaces, every bounded set of E' is relatively weakly compact (and also τ-equicontinuous).

3.7 DEFINITION: Let $\langle E_2, E_1 \rangle$ be a dual system. $\tau_b*(E_2)$ is the topology of u.c. on strongly bounded sets of E_2.

PROPOSITION: If $E(\tau)$ is metrizable, then $\tau = \tau_b*(E')$

(for proof see Köthe)

Consequently by Banach Mackey Theorem, if $E(\tau)$ is complete, metrizable then $\tau = \tau_b (E')$.

3.8 DEFINITION: If $\tau = \tau_b*(E')$, E is said to be quasibarrelled. In other words, quasibarrelled if and only if each barrel which absorbs bounded set is a neighborhood if and only if, in the dual, every strongly bounded set is equicontinuous.

3.9 Example of a normed linear space, which is not barrelled:-

Take E to be the space ϕ of finite sequences equipped with ℓ^2-norm. It is not complete. To prove that it is not barrelled, it is enough to show that $\tau_b*(E') \neq \tau_b(E')$. Clearly E' is ℓ^2. Simply bounded sets of ℓ^2 are coordinatewise bounded i.e. $x \in \ell^2$, $x = (x_i)$ / $|x_i| \leq M_i$, $i = 1, 2, \ldots$. But strongly bounded subsets of ℓ^2 with respect ϕ are norm-bounded sets. Thus the defining families for the topologies $\tau_b*(E')$ and $\tau_b(E')$ are different and hence $\tau_b*(E') \neq \tau_b(E')$.

3.10 We define the <u>bidual</u> E'' of E as the set $E'' = [E'[\tau_b(E)]]'$. Usually E'' is equipped with the topology $\tau_b(E', E'')$ and then it is called the strong bidual of E.

3.11 On E', there are two classes of simply bounded sets namely, E-bounded and E''-bounded. Since $E \subset E''$, E''-bounded sets are also E-bounded. We assert that for finite dimensional sets Eboundedness implies E''-bounded. For, start with a class of finite sets of E', saturate this class by taking closures with respect to $\tau_s(E'')$ and $\tau_s(E)$ The latter topology being weaker gives bigger closure so the saturated hull of this class for $\langle E', E\rangle$ should be bigger than saturated hull for $\langle E', E''\rangle$. But the saturated hull being finite dimensional bounded sets with respect to the duality, the latter hull should be bigger and hence the two hulls coincide.

3.12 Consider $\langle E'', E'\rangle$. Give the strong topology on E''-defined by all E''-bounded subsets of E', i.e., $\tau_s(E'')$-bounded sets of E'. These are of course $\tau_s(E)$-bounded. But $\tau_b(E)$-bounded sets of E' and $\tau_s(E'')$-bounded sets of E' are same. For, duals of $\tau_b(E)$ and $\tau_s(E'')$ are same, namely E'' and hence $\tau_b(E'')$ and $\tau_s(E'')$ are admissible topologies for the duality $\langle E'', E'\rangle$ and by Mackey's theorem, they have the same bounded sets.

3.13 Consider $\langle E'', E'\rangle$. Since $E'' = [E'[\tau_b(E)]]'$, $\tau_b(E)$ is admissible. But $\tau_k(E'')$ is the strongest admissible topology for the duality. Hence $\tau_b(E)$ is coarser than $\tau_k(E'')$.

3.14 At this stage, we refer the reader to two tables at the end. These are intended to make it easier to understand definitions, involving various topologies on l.c. spaces and their dual. Table 1 shows different topologies on E and E'' by means of certain classes of bounded sets of E'. Table 2 shows various topologies on E' defined by means of classes of bounded sets of E and E''. In each table, a topology mentioned in any compartment is defined by the class of sets which occur vertically below or above it. In each table, the topologies in any vertical column are identical, whereas the topologies in any vertical column are coarser than those to the right of them. Similarly the class of sets that occur in the same vertical column are identical whereas the clases that occur in any column are bigger than those to the left of them. An arrow at the bottom indicates that all the entries-all sets and all topologies within the arrow coalesce. For example in Table 1, column 3 to 6 coalesce for barrelled space. These arrows are to be interpreted as definitions of the types of spaces indicated by them.

3.15 DEFINITION: A l.c. space $E(\mathcal{T})$ is said to be <u>semireflexive</u> if and only if $E = E''$. From Table 2 we get the following criterion:

THEOREM: <u>$E(\mathcal{T})$ semireflexive if and only if</u> $\mathcal{T}_k(E) = \mathcal{T}_b(E)$ <u>on E, if and only if every bounded closed set is relatively</u> $\mathcal{T}_s(E')$<u>-compact in E</u>.

3.16 DEFINITION: A l.c. space $E(\mathcal{T})$ is <u>reflexive</u> if and only if

1) $E = E''$ i.e. $E(\mathcal{T})$ is semireflexive

and 2) $\mathcal{T}_b(E') = \mathcal{T}$

We have $\mathcal{T}_b(E')$ on E'' induces always $\mathcal{T}_{b^*}(E')$ on E (prove this). So $\mathcal{T}_b(E')$ on E'' induces $\mathcal{T}$ on E if and only if $\mathcal{T} = \mathcal{T}_{b^*}(E')$ on E. In other words $E(\mathcal{T})$ is quasi-barrelled.

Then E is reflexive if and only if it is semireflexive and quasibarrelled. For a quasibarrelled space, reflexivity and semireflexivity coincide. In particular this happens for Banach spaces, and for Banach spaces, we have

Reflexivity $\Longleftrightarrow$ semireflexivity $\Longleftrightarrow$ bounded closed sets are weakly compact $\Longleftrightarrow$ unit ball is weakly compact.

3.17 DEFINITION: E is said to be <u>quasicomplete</u> if every bounded closed set is complete

PROPOSITION: <u>E is semireflexive</u> $\Longleftrightarrow$ <u>E is weakly quasicomplete</u>.

(use 2.2 and prove this assertion)

TABLE-I

4	E''	$\tau_s(E',E'')$		$\tau_n(E',E'')$		$\tau_b(E',E'')$ strong bidual	
3	E'	all finite dimensional $\tau_s(E'')$-bounded sets and their subsets	The topology $\tau_K(E',E'')$ on E'' lies somewhere between columns 1 and 4	all τ-equicontinuous sets		all absolute convex $\tau_s(E'')$ bounded sets and their subsets	
2		all finite dimensional $\tau_s(E')$ bounded sets their subsets	If E is a Mackey space it also lies to the left of column 3. Otherwise nothing definite can be stated.	all τ-equicontinuous sets	all absolute convex $\tau_s(E)$ compact sets and their subsets	all absolute convex $\tau_b(E)$ bounded $\tau_s(E)$ closed sets and their subsets	all absolute convex $\tau_s(E)$-bounded sets and their subsets
1	E	$\tau_s(E',E)$ the weak topology		τ the intial topology	$\tau_K(E',E)$ Mackey topology	$\tau_b^*(E',E)$	$\tau_b(E'E)$ strong topology
		1	2	3	4	5	6

Note: For Normed linear spaces Column 3 to 5 coalesce. for Banach spaces columns 3 to 6 coalesce

<———————E barrelled ———————>

<————E Quasibarrelled————>

<———E Mackey spaces——>

<———E Quasi-M-barrelled——>

25 (b)

TABLE 2

		1	2	3	4	5	6	7
4	E″					all absolute convex $\tau_s(E')$ compact sets and their subsets	all absolute convex $\tau_b(E')$ bounded $\tau_s(E')$ closed sets and their subsets	all absolute convex $\tau_s(E')$ bounded sets and their subsets
3			The topology $\tau_s(E'',E')$ lies somewhere between columns 1 and 5, but as it is not comparable with $\tau_k(E,E')$, it does not fit in this table			$\tau_k(E'',E')$	$\tau_{b^*}(E'',E')$	$\tau_b(E'',E')$
2		$\tau_s(E,E')$	$\tau_k(E,E')$	$\tau_{b^*}(E,E')$	$\tau_b(E,E')$			
1	E	all finite dimensional $\tau_s(E')$ bounded sets and their subsets	all absolute convex $\tau_s(E')$ compact sets and their subsets	all absolute convex $\tau_b(E')$ bounded $\tau_s(E')$ closed sets their subsets	all absolute convex $\tau_s(E')$ bounded sets and their subsets			

1 2 3 4 5 6 7

<———E is semireflexive———> (columns 2 to 4)

<———E is distinguished———> (columns 4 to 7)

<——E semidistinguished——> (columns 4 to 5)

Note: For normed linear spaces

Columns 4 to 7 coalesce

C H A P T E R IV

C O N J U G A T E L O C A L L Y C O N V E X S P A C E S

4.1 In this section we attempt to answer the question: 'When is a l.c. space the strong dual of another l.c. space?' In particular, when is a Banach space a conjugate Banace space? Not all Banach spaces are conjugate Banach spaces; for example, the space C_0 of all convergent sequences, is not a conjugate space -- a fact which is an easy consequence of Krein Milman Theorem. The above question for Banach spaces was settled by Dixmier (Duke Math. J. 15 (1948) 1057-1071).

Suppose there exists a Banach space E, which is isometrically, isomorphic to F', where F is a Banach space, then $E' \cong F'' \supset F$. So F as a subspace of F'' is a weak*-dense, norm-closed subspace of F'' (i.e., E').

Dixmier's results:

THEOREM A: A Banach space E is linearly homeomorphic to the norm dual of another Banach space if and only if there exists in the dual E' of E, a subspace V, with the following properties:

a) V is $\tau_s(E)$ dense in E'

b) V is $\tau_b(E)$-closed in E'

c) no proper subspace of V has both these properties. Such a subspace V of E' is said to be minimal in E'.

THEOREM B: <u>A Banach space E is isometrically isomorphic to the norm dual of another Banach space F if and only if there exists in the dual E' of E, a minimal subspace V, which has further the property: $V \cap S_1$ is $\tau_s(E)$-dense in S_1, where S_1 is the unit ball of E'.</u>

A subspace V with the last property is said to be a subspace of characteristic 1.

In order to prove these theorems, Dixmier introduced the following numbers.

$$r = \max\{\rho \geq 0 : V \cap S_1 \text{ is } \tau_s(E)\text{-dense in } S_\rho\},\ S_\rho$$

being the ball of radius ρ in E'

$$\delta = \inf_{\substack{x \in E \\ x \neq 0}} \left\{ \sup_{F \in V \cap S_1} \frac{|F(x)|}{\|x\|} \right\}$$

$$R = \frac{1}{\sup_{x \in \Sigma^*} \|x\|},\ \Sigma^* \text{ is } \tau_s(V,E)\text{-closure of } \Sigma, \text{ the}$$

unit ball in E.

$$t = \inf_{\substack{x \in E \\ x \neq 0 \\ z \in V^\perp}} \left\{ \frac{\|x + z\|}{\|x\|} \right\}$$ where $V^\perp$ is the orthogonal complement in E''.

Dixmier's critical theorem states that

$$r = s = \frac{1}{R} = t$$

In what follows, we attempt a partial answer to the problem proposed in the beginning without giving into the generalizations of the above numbers of Dixmier.

4.2 DEFINITION: A linear subspace V of E' is said to be minimal if a) V is weakly dense

b) V is strongly closed

c) no proper subspace of V has both these properties.

DEFINITION: A linear subspace V of E' is duxial in E'. If every convex, weakly compact subset of E' is contained in the weak closure of some bounded subset of $V[\tau_{bi}(E, E')]$ where τ_{bi} denotes the induced topology on V.

REMARK: Note that for Banach spaces this means and is equivalent to saying, V is of characteristic 1.

We characterize below:

1) Minimal subspaces of E'

2) Minimal and duxial subspaces of E'

3) those l.c. spaces for which E is minimal and duxial in E''

4) l.c. spaces E which are strong duals of l.c. spaces which belong to class (3).

4.3 LUXEMBURG'S THEOREM: (Proc. International Symposium on linear spaces, Jerusalen, 1961, 307-318).

If V_1 and V_2 are two $\tau_s(E)$-dense subspaces of E', then the $\tau_s(E)$ - closures of V_1 and V_2 are identical if and only if the topologies $\tau_s(V_1, E)$ and $\tau_s(V_2, E)$ coincide on each $M \in \mathcal{M}$ where $\mathcal{M}$ is the class of all convex bounded closed sets of $E(\tau)$.

4.4 THEOREM: Let V be a $\tau_s(E)$-dense, $\tau_b(E)$-closed linear subspace of E', Then the following are equivalent:-

a) V is minimal in E'

b) $E'' = E \oplus V^{\perp}$ (Algebraically)

c) Each $M \in \mathcal{M}$ is relatively $\tau_s(V, E)$-compact in E.

PROOF: a) ==> b) First note that because V is $\tau_s(E)$-dense in E', $V^{\perp} \cap E = \{0\}$. Let $z \in E''$ and $z \notin V^{\perp}$. Let $W = \{y \in V \mid \langle z, y \rangle = 0\} = Z_{\perp} \cap V$ where Z is the one-dimensional subspace spanned by z. V and $Z_\perp$ are both strongly closed and so W is $\mathcal{T}_b(E)$-closed. Also W is not all of V. Hence by hypothesis of minimality of V, W cannot be weakly dense in E'. So there exists $x \neq 0$, $x \in E$ such that $\langle x, \omega \rangle = 0$ for all $\omega \in W$. Thus z and x are two linear functionals both vanishing on the same hyperplane W of V. Consider $z - \lambda x$, where $\lambda = \dfrac{\langle z, y_0 \rangle}{\langle x, y_0 \rangle}$ for some $y_0 \in V \setminus W$. This vanishes on W and y_0. So $z - \lambda x$ vanishes on all of V. i.e. $z - \lambda x \in V^\perp$. Let $z'' = z - \lambda x$ so $z = z'' + \lambda x \in V^\perp \oplus E$. This proves (a) $\Longrightarrow$ (b).

b) $\Longrightarrow$ c) From (b) $E''/V^\perp \cong E$ (algebraically). Also it is the dual of $V[\mathcal{T}_{bi}(E,E')]$. $\langle \frac{E''}{V^\perp}, V \rangle$ is therefore a dual system, isomorphic to $\langle E, V \rangle$. So the spaces $E''/V^\perp[\mathcal{T}_s(V)]$ and $E[\mathcal{T}_s(V)]$ are, linearly homeomorphic. For every $M \in \mathcal{M}$, $M^\circ \cap V$ is a $\mathcal{T}_{bi}(E,E')$-neighborhood of 0 in V. Hence by Alaoglu Bourbaki theorem, $(M^\circ \cap V)^\circ$ is a $\mathcal{T}_s(V)$-compact subset of $E''/V^\perp$. But $(M^\circ \cap V)^\circ = (M^\circ \cap V)_\circ$ when we identify E and $E''/V^\perp$. So $(M^\circ \cap V)_\circ$ is $\mathcal{T}_s(V)$-compact in E. But $(M^\circ \cap V)_\circ$ is the bipolar of M in the dual system $\langle E, V \rangle$ and is therefore $\mathcal{T}_s(V,E)$-closure of M in E. Thus M is relatively $\mathcal{T}_s(V)$ compact in E. This proves(c).

c) $\Longrightarrow$ a)

Every $M \in \mathcal{M}$ is relatively $\mathcal{T}_s(V)$-compact in E. Let $W \subsetneqq V$ be $\mathcal{T}_b(E)$-closed in E. If W were also $\mathcal{T}_s(E)$-dense then $\mathcal{T}_s(W)$ would be a Hausdorff topology on E and therefore on Every $\overline{M}$, the $\mathcal{T}_s(V)$-closure of M in E. But $\overline{M}$ is $\mathcal{T}_s(V)$-compact by hypothesis, and the Hausdorff topology $\mathcal{T}_s(W)$ is a coarser topol gy on $\overline{M}$. Hence $\mathcal{T}_s(W,E)$ and $\mathcal{T}_s(V,E)$ coincide on every $\overline{M}$ and therefore on every $M \in \mathcal{M}$. By Luxemburg's theorem, W and V have the same $\mathcal{T}_b(E)$-closure in E', which is a contradiction. Thus W cannot be $\mathcal{T}_s(E)$-dense in E' and so V is minimal.

4.5 Specializing the above theorem for the case $V = E'$ and using the definition of semireflexivity, we obtain:

COROLLARY: <u>For any l.c. space, following are equivalent</u>:

a) E' <u>is minimal</u>

b) $E = E''$ i.e. E <u>is semireflexive</u>

c) <u>Every</u> $M \in \mathcal{M}$ <u>is</u> $\mathcal{T}_s(E',E)$-compact.

4.6 Since a l.c. space is reflexive if and only if it is quasibarrelled and semireflexive, we have the following:

COROLLARY: A quasibarrelled space E is reflexive if and only if its strong dual does not contain any proper closed weakly dense subspace.

4.7 DEFINITION: An l.c. space E is *quasi-M-barrelled* if in E', every absolutely convex, $\tau_b(E)$-bounded set is $\tau_b(E)$-compact; or equivalently, if the strong bidual induces the Mackey topology on E.

4.8 THEOREM: *If E is an l.c. space, the following are equivalent:*

a) *E is minimal in E''*

b) *E is quasi-M-barrelled and is $\tau_b(E')$-closed in E'.*

An elegant use of the tables yields the proof.

4.9 THEOREM: *If E is a complete quasi M-barrelled space, then E is minimal in E''. As a special case, Frechet (Banach) spaces are minimal in their biduals.*

4.10 THEOREM: *In order that a $\tau_b(E)$-closed, $\tau_s(E)$-dense subspace V of E' is minimal and duxial in E', the following are necessary and sufficient:*

a) *Every $M \in \mathcal{M}$ is relatively $\tau_s(V,E)$-compact.*

b) *$\tau_k(E',E)$ is coarser than $\tau_b(V,E)$ on E.*

(See VK's paper for proof)

4.11 DEFINITION: An l.c. space E is <u>semidistinguished</u> if every absolute convex $\tau_s(E')$-compact subset of E'' is contained in the $\tau_s(E')$-closure in E'' of some M in E or equivalently, if the strong dual $E'[\tau_b(E)]$ has its Mackey topology.

4.12 Applying (b) of 4.10 to $E(\subset E'')$ we get $\tau_b(E,E') \supset \tau_k(E,E')$. Reverse is already true and hence E is semidistinguished. Thus.

THEOREM: E <u>is minimal and duxial in E'' if and only if</u>

a) E <u>is</u> $\tau_b(E')$-<u>closed in</u> E''

b) E <u>is quasi M-barelled and</u>

c) E is <u>semidistinguished.</u>

4.13 DEFINITION: <u>E is said to be V-reflexive</u> if

a) $V[\tau_{bi}(E\ )]' = E$

b) $E[\tau_b(V)] = E(\tau)$

4.14 THEOREM: <u>If E is any l.c. space, the following are equivalent.</u>

a) $E'(\tau_b(E))$ <u>is E-reflexive</u>

b) E is <u>quasi-M-barrelled</u>.

4.15 THEOREM: <u>Let</u> $E(\tau)$ <u>be an l.c. space with its Mackey topology. Then the following are equivalent</u>.

a) E is linearly homeomorphic to the strong dual of an l.c. space F, which has the following 3 properties.

1) F is $\tau_b(F')$-closed in F''

2) F is quasi-M-Barrelled

3) F is semi-distinguished

b) There exists a minimal and duxial subspace V of E' with the additional property that

$$\tau_b(V, E) \subseteq \tau_k(E', E) \text{ on } E$$

c) $E(\tau)$ is V-reflexive, where V is $\tau_b(E)$-closed and $\tau_s(E)$-dense subspace of E'

(For proof see VK's paper)

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