MATSCIENCE REPORT

LECTURES ON

QUANTUM ELECTRODYNAMICS

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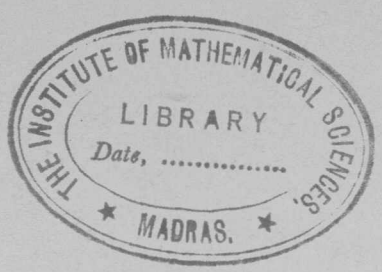

THE INSTITUTE OF MATHEMATICAL SCIENCES, MADRAS-4, INDIA.

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LECTURES ON
QUANTUM ELECTRODYNAMICS

I. SOME TOPICS IN QUANTUM ELECTRODYNAMICS

By
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II. QUANTISATION OF THE ELECTROMAGNETIC FIELD

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SOME TOPICS IN QUANTUM ELECTRODYNAMICS

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LECTURE I.

SCATTERING THEORY AND QUANTUM ELECTRODYNAMICS

The aim of this series of lectures is to give an outline of quantum electrodynamic calculations with special emphasis on some work done by this group which might lead to some fresh problems. A brief introduction on the theory of scattering will also be given for completeness.

It is well-known that the Schrodinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi$$

represents the temporal evolution of the wave function ψ. The wave function is considered stationary if the only time dependence is of the exponential type i.e.

$$\psi(x,t) = e^{-iEt} \psi(x)$$

where $\psi(x)$ satisfies the time independent Schrodinger equation. In the presence of interaction with an external field, which can be included in H, stationary solutions may exist in which case the particle is said to be bound to the external field and the corresponding wave function ψ_n represents the bound state. For example we have the hydrogen atom, the coulomb field $V(r)$ being included in H. Now, the equation of motion need not have stationary solutions and when the interaction is such that we do not get bound state solutions, it leads to collision theory. In this case, the Hamiltonian H is split into two parts H_0 and H' and the evolution of the system from $t = -\infty$ to $t = +\infty$ is studied assuming H_0 to be time independent and that H' vanished at $t = +\infty$ and $t = -\infty$. This leads to the study of transitions from eigen states of H_0, the free particle Hamiltonian. The inclusion of relativity into the wave

mechanics of single particles led to the negative energy states which in turn were physically interpreted by the postulate of the unobservable 'sea' with obervable 'holes' in it. Thus even the study of 'single' particles became an inherently multiparticle phenomenon--(due to the presence of the 'sea) and necessitated the formulation of the quantum theory of fields dealing with the temporal evolution of systems of a changing numbers of particles. The essentially single particle formatism of Feynman however, was able to achieve the results of quantum field theory, at least in the field of quantum electrodynamics.

In this lecture, we shall confine ourselves to the treatmen of the single particle scattering by the old-fashioned perturbatio theory. Consider a particle in the stationary state corresponding to the time-independent Hamiltonian H_0, the eigenfunction of H_0 being $\psi_n(\vec{x})\, e^{-iE_n t}$ where E_n is the eigenvalue of H_0. Time dependence other than the exponential occurs when obeys

$$i\frac{\partial \psi(t)}{\partial t} = H(t)\,\psi(t)$$

For any physical process, H (t) is such that $H \to H_0$ as $t \to \pm \infty$. Equivalently

$$H = H_0 + H'$$

with $H' \to 0$ as $t \to \pm \infty$ or more generally

$$H' = 0 \quad \text{for} \quad t < T_0$$

$$H' = H'(t) \quad \text{for} \quad T_0 < t < T.$$

$$H' = 0 \quad \text{for} \quad t > T.$$

and we let $T_0 \to -\infty$ and $T \to +\infty$. And even if H' does not vary with time in the interval $T_0 < t < T$, it is still time dependent since it is zero outside this interval.

Since $\varphi_n - \delta$ form a complete orthonormal set, we have

$$\psi(x) = \sum_n b_n(t)\,\varphi_n(\vec{x})\,e^{-iE_n t} = \sum_n b_n(t)\,\psi_n(x) = \sum_n C_n(t)\,\varphi_n(\vec{x})$$

we note that $C_n(t)$ and $b_n(t)$ satisfy

$$i\frac{dC_n(t)}{dt} = \sum_m H'_{nm}\,C_m(t) + E_n\,C_n(t) \tag{1}$$

$$i\frac{db_n(t)}{dt} = \sum_m H'_{nm}(t)\,b_m(t) \tag{2}$$

(3)

where

$$H'_{nm} = \int \varphi_n^*(\vec{x})\,H'\,\varphi_m(\vec{x})\,d^3x \tag{4}$$

(5)

$$H'_{nm}(t) = \int \psi_n^*(\vec{x})\,H'\,\psi_m(x)\,d^3x \tag{6}$$

since

$$\psi_n(x) = e^{-iH_0 t}\,\varphi_n(\vec{x}) = e^{-iE_m t}\,\varphi_m(\vec{x}) \tag{7}$$

(2) can be put in the vector matrix form

$$i\frac{d\vec{C}(t)}{dt} = [H']\,\vec{C}(t) + [H_0]\,\vec{C}(t) \tag{8}$$

where $\vec{C}(t)$ is a column matrix with typical elements $C_n(t)$ and $[H']$ and $[H_0]$ are matrices with elements H'_{nm} and $E_n\delta_{nm}$ respectively. And for (3), we have

$$i\frac{d\vec{b}(t)}{dt} = [H'(t)]\,\vec{b}(t) \tag{9}$$

where $[H'(t)]$ has typical elements $H'_{nm}(t)$. The calculation of the coefficients $b_n(t)$ is the main problem in a quantum mechanical collision process, $b_n(t)$ representing the amplitude for the particl to be found in the state n. By a solution of (9) is meant that given $\vec{b}(t_0)$ we have to find $\vec{b}(t)$ for $t > t_0$. Iterating we get

$$\vec{b}(t) = [U(t,t_0)]\,\vec{b}(t_0) \tag{10}$$

where

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} H'(\tau_1)\,d\tau_1 + (-i)^2\int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1\, H'(\tau_2)H'(\tau_1) + \cdots + (-i)^k \int_{t_0}^{t} d\tau_k \int_{t_0}^{\tau_k} d\tau_{k-1} \cdots \int_{t_0}^{\tau_2} H'(\tau_k)\cdots H'(\tau_1) + \cdots \tag{11}$$

where the time parameters are ordered such that

$$t > \tau_k > \cdots > \tau_1 > t_0 \tag{12}$$

Since we are interested in computing the amplitude of transition from a definite initial to a definite final state, we assume all elements of $\vec{b}\,(t_0)$ are zero except a specific $b_i\,(t_0) \equiv 1$. Denoting a typical element of $\vec{b}\,(t)$ as $b_{fi}\,(t)$, we have

$$b_{fi}(t) = \langle f|S|i\rangle = \sum_n \langle f|S^{(n)}|i\rangle \tag{12}$$

with $S = \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} U(t, t_0)$

and

$$S^{(n)} = (-i)^n \int_{t_0}^{t} d\tau_n \int_{t_0}^{\tau_n} d\tau_{n-1} \cdots \int_{t_0}^{\tau_2} d\tau_1\, H'(\tau_n)\cdots H'(\tau_1) \tag{13}$$

By definition each element of $H'(\tau)$ is an integral over space by (5). Defining

$$R_{nm}(\vec{x},t) = \psi_n^*(x)\, H'(x)\, \psi_m(x) \tag{14}$$

so that

$$H'_{nmn'm'}(t) = \int R_{nm}(x)\, d^3x \tag{15}$$

and

$$\langle f|S|i\rangle = \vec{b}(t)\left[1 + (-i)\int_{-\infty}^{+\infty} d^3x_1 \int_{t_0}^{t} [R(x_1)]\,d\tau_1 + \cdots + (-i)^k \int_{-\infty}^{+\infty} d^3x_k \int_{-\infty}^{+\infty} d^3x_{k-1} \cdots \int_{-\infty}^{+\infty} d^3x_1 \int_{t_0}^{t} d\tau_k \cdots \int_{t_0}^{\tau_2} d\tau_1\, [R(x_k)]\cdots [R(x_1)] + \cdots \right] \tag{16}$$

with

$$b_i(t_0) = 1 \quad \text{if } k \neq i$$

$$b_k(t) = 1 \quad \text{if } k \neq f$$

We shall now calculate the probability per unit time that a particle makes a transition from i to f under H'. Taking the first order term in the expansion

$$\langle f|S|i\rangle = \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} (-i)\int_{t_0}^{t} dt_1 \int_{-\infty}^{+\infty} d^3x_1\, \psi_f^*(x_1)\, H'(x_1)\, \psi_i(x_1) \quad (17)$$

with

$$\psi_f = \varphi_f e^{-iE_{p_f}t}\,; \quad \psi_i = \varphi_i e^{-iE_{p_i}t} \quad (18)$$

$\vec{p}_i$ and $\vec{p}_f$ are the initial and final momenta of the particle. Assuming H' (x) can be written as $H'(\vec{x})_0\, e^{\pm iq_4 t}$, $+iq_4$ when H' imparts energy q_4 and $-iq_4 t$ when it takes away an energy q_4 we have

$$b_{fi} = H'_{fi}(-i)\int_{t_0}^{t} e^{-iE_{p_f}t} e^{\pm iq_4 t} e^{+iE_{p_i}t}\, dt \quad (19)$$

where

$$H'_{fi} = \int \varphi_f^*(\vec{x})\, H'(\vec{x})\, \varphi_i(\vec{x})\, d^3x \quad (20)$$

i.e. the space integration is performed first and then the time integration. Ignoring normalization factors etc. and assuming

$$\varphi_i = e^{+i\vec{p}_i\cdot\vec{x}}\,; \quad \varphi_f = e^{+i\vec{p}_f\cdot\vec{x}}$$

$$H'(\vec{x}) = e^{\pm i\vec{q}\cdot\vec{x}}\,; \quad H'(x) = e^{\mp iq\cdot x}$$

(which implies an absorption of four momentum q with q_4 as the +ve energy component). We are considering the first order electrodynamic interaction of a photon and an electron. For an emission we would have $H'(x) = e^{+iq\cdot x}$. In the case of absorption, the space integration in H'_{fi} gives $(2\pi)^3\delta(\vec{p}_f - \vec{p}_i - \vec{q})$ and the time integration $(2\pi)\delta(E_{p_f} - E_{p_i} - E_q)$. This implies that the matrix element will vanish unless

$$\vec{p}_f = \vec{p}_i + \vec{q} \quad \text{and} \quad E_{p_f} = E_{p_i} + E_q \quad (21)$$

But evidently both these conditions cannot be simultaneously satisfied. Thus a first order process cannot occur.

For the second order interaction we have

$$\langle f|S^{(2)}|i\rangle = (-i)^2 \sum_m \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \int_{-\infty}^{+\infty} d^3x_2 \int_{-\infty}^{+\infty} d^3x_1 \times$$

$$\cdot \psi_f^*(x_2)\, H'(x_2)\, \psi_m(x_2)\, \psi_m^*(x_1)\, H'(x_1)\, \psi_i(x_1)$$

$$= (-i)^2 \sum_m \langle f|H'|m\rangle\langle m|H'|i\rangle \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \times$$

$$\cdot e^{-i(E_m - E_f)\tau_2 - i(E_i - E_m)\tau_1} \qquad (22)$$

$t_0 \to -\infty, t \to +\infty$

the sum over m implying a summation or (integration) over all possible intermediate states. In this case, we find that we have first to integrate over the intermediate variable τ_1 between t_0 and τ_2 and let only $t_0 \to -\infty$. To evaluate the integral, we multiply the integrand by $e^{+\epsilon\tau_1}$ where ϵ is a small infinitesimal positive quantity and can be made to go to zero. Keeping ϵ as non-zero and integrating we get

$$\lim_{\epsilon \to 0} \frac{e^{i(E_m - E_i)\tau_2}}{-i(E_i - E_m + i\epsilon)} \qquad (23)$$

and the second integration over τ_2 gives the δ function $\delta(E_f - E_i)$. While f should be on the energy shell, m need not. In fact m should include states both off and on the energy shell. As before, setting

$$\psi_f = e^{-i p_f \cdot x_2} \text{ etc. and } H'(x_2) = e^{+i q_2 \cdot x_2}$$

and $H'(x_1) = e^{-i q_1 \cdot x_1}$ corresponding to emission and absorption of a photon,

$$\langle f|S^{(2)}|i\rangle = (-i)^2 \sum_m \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \int_{-\infty}^{+\infty} d^3x_1 \int_{-\infty}^{+\infty} d^3x_2\, e^{i p_f \cdot x_2}$$

$$\cdot e^{i q_2 \cdot x_2} e^{-i p_m \cdot x_2} e^{i p_m \cdot x_1} e^{-i q_1 \cdot x_1} e^{-i p_i \cdot x_1}$$

as $t \to +\infty$; $t_0 \to -\infty$ (24)

where p and q satisfy the correct energy-momentum relation.

Space integration leads $\vec{p}_m = \vec{p}_i + \vec{q}_1$, $\vec{p}_f + \vec{q}_2 = \vec{p}_m$ (25)

Thus the initial system consists of an electron of 4-momentum p_i and photon with 4-momentum q_1, the energy $E_i = E_{p_i} + E_{q_1}$ The intermediate state is that of a single electron with momentum $\vec{p}_i + \vec{q}_1$ and energy $E_m = E_{p_i} + E_{q_1} \neq E_i$ or E_f . Thus energy is not conserved in the intermediate state even though momentum is. In any physical problem, considering the absorption of a real photon q_1, p_i and q_1 are given so that for a required p_f, q_2 is fixed. Thus $\sum_m$ does not imply any sum over q_1 or q_2. But the relative sequence of absorption of q_1 and emission of q_2 is not specified so that emission of q_2 can occur at x_1 and absorption at x_2 and the corresponding intermediate state will have 3 particle electron of momentum $\vec{p}_i - \vec{q}_2$ with energy $E_{\vec{p}_i - \vec{q}_2} = E_m$, and photons q_1 and q_2 ., i.e. $\langle f | S^{(2)} | i \rangle$ will have a contribution from

$$\lim_{\substack{t \to +\infty \\ t_0 \to -\infty}} (-i)^2 \sum_m H'_{fm} H'_{mi} \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \, e^{-i(E_f - E_i)\tau_2} \times e^{+i(E_i - E_m)(\tau_2 - \tau_1)} \quad (26)$$

with

$$E_f = E_{p_f} + E_{q_2} \; ; \quad E_i = E_{p_i} + E_{q_1} \, , \; E_m = E_{p_i - q_2} + E_{q_1} + E_{q_2}$$

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Lecture II

The four-dimensional Kernel function formalism

The only difference between this and the old fashioned perturbation theory is that in this case the space and time integrations are performed together so that all expressions are inherently four-dimensional.

In $\langle f|S|i\rangle$, replacing R_{nm} by $\psi_n^* H' \psi_m$ we get for a typical term

$$\lim_{\substack{t\to+\infty \\ t_0\to-\infty}} (-i)^n \int_{-\infty}^{+\infty} d^3x_n \cdots \int_{-\infty}^{+\infty} d^3x_1 \int_{t_0}^{t} d\tau_n \cdots \int_{t_0}^{\tau_2} d\tau_1 \times$$

$$\times \sum \psi_f^*(\vec{x}_n, \tau_n) H'(\tau_n) \psi_{m_n}(\vec{x}_n \tau_n) \cdots$$

$$\cdots \psi_{m_1}^*(\vec{x}_1 \tau_1) H'(\tau_1) \psi_i(\vec{x}_1 \tau_1) \qquad (27)$$

And edefining

$$K(2,1) = \sum_m \psi_m(2) \psi_m^*(1) \qquad (28)$$

we get

$$\lim_{\substack{t\to+\infty \\ t_0\to-\infty}} (-i)^n \int_{t_0}^{t} d\tau_n \cdots \int_{t_0}^{\tau_2} d\tau_1 \int_{-\infty}^{+\infty} d^3x_n \cdots \int_{-\infty}^{+\infty} d^3x_1 \times$$

$$\psi_f^*(n) H'(n) K(n, n-1) \cdots H'(2) K(2,1)$$

$$H'(1) \psi_i(1) \qquad (29)$$

where 1,2 etc refer to space-time (29) points $(\vec{x}_1, t_1)$ etc. Now $K(2,1)$ in (23) satisfies the homogeneous equation

$$i\frac{\partial}{\partial t_2} K(2,1) - H(2)\, K(2,1) = 0 \qquad (30)$$

But in (23) as time is ordered $K(2,1) = 0$ is only for $t_2 > t_1$. If we define $K_0(2,1)$ such that

$$K_0(2,1) \equiv K(2,1) \text{ for } t_2 > t_1 \text{ and}$$
$$= 0 \quad \text{for } t_2 < t_1$$

then $K_0(2,1)$ satisfies the inhomogeneous equation

$$i\frac{\partial}{\partial t_2} K_0(2,1) - H(2)\, K_0(2,1) = \delta(2,1) \qquad (31)$$

Physically we can visualize the process by imagining the perturbation to act at space-time points 1,2 etc. and the particle between being represented by a propagator. These 4-dimensional 'vertex' points are integrated out. We shall deal with the aspect in greater detail when considering the Feynman formalism for the Dirac electron.

<u>Feynman's Kernel function formalism for the Dirac electron.</u>

The Dirac equation for a free particle perturbed by an electromagnetic field (represented by $\vec{A}$, φ the vector and scalar potentials) is given by

$$(\not{p} - e\not{A} - m)\psi = 0 \qquad (32.)$$

with the usual notation $\not{p} = \gamma_\mu p_\mu$ and $\not{A} = \gamma_4 A_4 - \vec{\gamma}\cdot\vec{A} = \gamma_\mu A_\mu$. The extension of the kernel function formalism to the Dirac equation is made by replacing the single component wave function by the four component and have the kernel is defined by

$$K_0(2,1) = \sum_m \psi_m(2)\,\psi_m^{+}(1)\,\beta$$
$$= \sum_m \psi_m(2)\,\bar{\psi}_m(1) \qquad \text{for } t_2 > t_1$$
$$= 0 \qquad \text{for } t_2 < t_1 \tag{33}$$

The matrix $\beta\,(\equiv \gamma_4)$ is attached to the kernel because, the interaction term H' is defined as $e\beta\not{A}$ and as we shall see presently this definition of K facilitates the removal of the β using the result $\beta^2 = 1$. The sum over in (33) is over both positive and negative energies. Evidently $K_0(2,1)$ is a 4x4 matrix. However, this definition of the kernel leads to the following difficulty. Now for every positive energy solution of the Dirac equation, there is also a negative energy solution. A free particle solution $\psi(2)$ at t_2 can be obtained from the wave function $\psi(1)$ at t_1 from

$$\psi(2) = \int K_0(2,1)\,\psi(1)\,d^3x_1 \tag{34}$$

i.e. a positive energy wave function at t_1 is propagated to with positive energy. The same would apply to an initial negative energy solution since K_0 is a sum over both over positive and negative energies. This does not lead to any complication

as long as there is no perturbation present since in this case initial positive energy states will continue to exist as such. In the presence of interaction however, a transition to a negative energy state may occur and the use of will give rise to a definite probability of realizing a negative energy state. at later time. This is solved by the modified definition of the kernel by Feynman

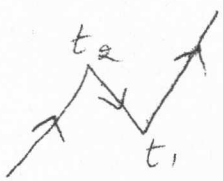
$$K_F(2,1) = \sum_{m,E>0} \psi_m^{(2)} \psi_m^{\dagger(1)} \quad \text{for } t_2 > t_1$$

$$= -\sum_{m,E<0} \psi_m^{(2)} \psi_m^{\dagger(1)} \quad \text{for } t_2 < t_1 \qquad (35)$$

so that we have imposed the requirements

(1) only positive energy states are propagated forward in time and

(2) K_F satisfies the inhomogeneous equation. According to the Dirac position formalism, in the presence of a perturbation, one of the negative energy electrons $(-E, \vec{p})$ from the 'sea' can jump into a positive energy state $(+E', \vec{p}')$ and the 'hole' thus created is identified with the positron of positive energy +E. and momentum $-\vec{p}$. Thus we deal only with positive energy states. Thus considering a second order process, initially we have an electron and at t_1 a pair is created and at t_2 $(t_2 > t_1)$ the incident electron and thepositron are annihilated and we have the final electron. According to the Feynman picture, the incident electron of positive energy



makes a transition to a negative energy state at t_2 and is propagated 'backward' in time to t_1 $(t_1 < t_2)$ where it again jumps to a positive energy state due to some perturbation and we have the final positive energy electron. Thus we follow the path of the single electron forward and backward in time. The negative sign in K_F for the negative energy part is connected to the Pauli-principle and we shall deal with this when proving the equivolence between the Feynman and field theoretic formalisms.

The momentum transform of the Feynman Kernel.

K_F in configuration representation is not of much use in calculations but the momentum transform has a very elegant form

$$K_F(2,1) = \sum_{+E} u_n(\vec{p})\,\bar{u}_n(\vec{p})\,e^{-i\vec{p}\cdot(x_2-x_1)} \qquad t_2>t_1$$

$$= -\sum_{-E} u_n(\vec{p})\,\bar{u}_n(\vec{p})\,e^{-ip\cdot(x_2-x_1)} \qquad t_2<t_1$$

The summation becomes an integration over p since E, $\vec{p}$ can take continuous values

$$K_F(2,1) = \sum_{spin} \int_{E_p>0} \frac{1}{2E_p}\, u(\vec{p})\,\bar{u}(\vec{p})\, e^{-ip\cdot x}\, d^3p \qquad \text{for } t_2>t_1 \tag{37}$$

$$= -\sum_{spin} \int_{E_p<0} \frac{1}{2E_p}\, u(\vec{p})\,\bar{u}(\vec{p})\, e^{-ip\cdot x}\, d^3p \qquad \text{for } t_2<t_1$$

where $x = x_2 - x_1$ and $\bar{u}\,u = 2E_p$. It can be seen

$$\sum_{\text{spin}} u(\vec{p})\,\bar{u}(\vec{p}) = \not{p} + m \tag{38}$$

Thus

$$K_F(2,1) = \int (\not{p} + m)\, e^{-i(p\cdot x}\, \frac{1}{2E_p}\, \frac{d^3p}{(2\pi)^3} \tag{39}$$

$$= (\not{\partial} + m) \int \frac{1}{2E_p}\, e^{-i(p\cdot x}\, \frac{d^3p}{(2\pi)^3}$$

$$\text{for } t > 0 \qquad E_p > 0 \tag{40}$$

Since

$$\frac{1}{E_p}\, e^{-iE_p t} = \lim_{\epsilon \to 0} \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-ip_4 t}}{p_4^2 - E_p^2 + i\epsilon}\, dp_4 \tag{41}$$

we have

$$K_F(2,1) = i \int e^{-i(p\cdot x}\, \frac{\not{p} + m}{p^2 - m^2}\, d^4p \tag{42}$$

or

$$K_F(p) = i\,\frac{(\not{p} + m)}{p^2 - m^2} = \frac{i}{\not{p} - m}$$

Calculation of matrix elements

In the presence of a perturbation due to an electromagnetic field, the perturbed kernel $K_F^A(2,1)$ satisfies the equation

$$(i\not\nabla_2 - e\not A - m) K_F^A(2,1) = i\,\delta(2,1) \tag{43}$$

and K_F^A can be expanded in terms of K_F, the unperturbed kernal as

$$K_F^A(2,1) = K_F(2,1) - ie\int K_F(2,3)\not A(3) K_F(3,1)\,d(3)$$
$$+ i^2e^2\int K_F(2,4)\not A(4) K_F(4,3)\not A(3) K_F(3,1)\,d(3)\,d(4) \tag{44}$$

And as before, the matrix element

$$b_{fi} = \int \bar\psi_f(2)\, K_F^A(2,1)\, \psi_i(1)\, d(2)\, d(1) \tag{45}$$

where $\psi_i(1) = u(p_i)\, e^{-ip_i\cdot x_1}$

$\bar\psi_f(2) = \bar u(p_f)\, e^{ip_f\cdot x_2}$

Using this in (44), the zeroth order term vanishes since ψ-s are orthogonal. For the 1st order we have

$$(-i)\int \bar\psi_f(2)\, e\not A(2)\, \psi_i(1)\, d(1) \tag{46}$$

Defining the momentum transform of $\not A(x)$ as

$$\not a(q) = \int \not A(x)\, e^{iq\cdot x} d^4x \tag{47}$$

the first order matrix element becomes

$$\bar{u}(p_f)\,\not{A}(q)\,u(p_i) \tag{48}$$

with $p_f - p_i = q$

The first order transition from p_i to p_f is possible only when q has no time component is $\not{A}(q)$ must be the transform of a function of space only as in the case of the coulomb potentia. The second order matrix element is

$$(-i)^2 \int\int \bar{\psi}_f(2)\, e\not{A}(2)\, K_F(2,1)\, e\not{A}(1)\, \psi_i(1)\, d(1)\wedge d(2) \tag{49}$$

and substituting

$$\not{A}(1) = \frac{1}{(2\pi)^4}\int \not{A}(q)\, e^{-iqx}\, d^4q \tag{50}$$

and the kernel as

$$\int \frac{1}{\not{p}-m}\, e^{2p\cdot(x_2-x_1)}\, \frac{d^4p}{(2\pi)^4} \tag{51}$$

and integrating over space-time points (1) and (2)

$$(-i)^2 \int \bar{u}_2\, e\not{A}(q_2)\, \frac{1}{\not{p}_1+\not{q}_1-m}\, e\not{A}(q_1)\, u_1\, \frac{d^4q_1}{(2\pi)^4} \tag{52}$$

when the perturbation is due to photons,

$$\not{A}(q_1) = \not{\varepsilon}\sqrt{4\pi}$$

and the matrix element is

$$M = \bar{u}_2\, \not{\varepsilon}_2\, \frac{1}{\not{p}_1+\not{q}_1-m}\, \not{\varepsilon}_1\, u_1$$

Since all expressions are four-dimensional the entire formalism is relativstically invariant.

If instead of performing the space-time integration simultaneously we perform the space integration first and then the time integration in two parts.

We get some interesting results. In obtaining the 4-dimensional transform of the kernel in this method, we find that the space-integration amounts to picking out terms corresponding to momentum $\vec{p}+\vec{q}$ and energy $E(\vec{p}+\vec{q})$ which can be positive or negative
/If we now split the time integration from $-\infty$ to $+\infty$ into two parts $t = 0$ to $+\infty$ with $+E(\vec{p}+\vec{q})$ and $t = -\infty$ to 0 with $-E(\vec{p}+\vec{q})$, we get the kernel as

$$\frac{1}{E_{p+q}}\left[\frac{\overset{+}{\not{P}}+m}{E(\vec{p})+E(\vec{q})-E(\vec{p}+\vec{q})} - \frac{\overset{-}{\not{P}}+m}{E(\vec{p})+E(\vec{q})+E(\vec{p}+\vec{q})}\right]$$

where $P = p+q$ and $\overset{+}{P}$ implies that we have used $+E(\vec{p}+\vec{q})$ as coefficient of γ_t. Thus the first term corresponds to the positive energy part of the intermediate state while the second represents the negative energy part. With the use of thi decomposed propagator we may calculate the relative contributions to any process from positive and negative energy intermediate states respectively. Besides this, this was used in demonstrating the equivalence between Feynman and field theoretic formalism in an elegant manner.

Calculation of matrix element for Compton effect

Consider the scattering of a photon $q_1(\vec{q}_1, \omega)$ and polarisation $\vec{e}_1$ incident on a free electron at rest, the final state consisting of a photon $q_2(\vec{q}_2, \omega_2)$ and polarization $\vec{e}_2$ and electron of 4-momentum $p_2(\vec{p}_2, E_2)$. In the

Feynman picture, the electron $p_1(0,m)$ absorbs q_1 and propagates as $\frac{i}{\not p_1 + \not q_1 - m}$ and then emits q_2 and becomes an electron of momentum p_2. Conservation requires.

$$\not p_1 + \not q_1 = \not p_2 + \not q_2$$

We have

$$A_{1\mu} = (4\pi e^2)^{1/2} e_{1\mu} e^{-i q_1 \cdot x}$$
$$A_{2\mu} = (4\pi e^2)^{1/2} e_{2\mu} e^{+i q_2 \cdot x}$$

Since the photon polarization is perpendicular to direction of motion.

$$\vec{e}_1 \cdot \vec{q}_1 = 0 \quad ; \quad \vec{e}_2 \cdot \vec{q}_2 = 0$$

Also $q_1^2 = 0 \quad ; \quad q_2^2 = 0$

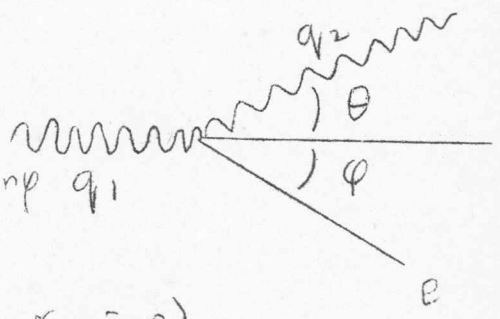
and $\psi_i = u_1 e^{i p_1 \cdot x} \qquad \psi_2 = u_2 e^{-i p_2 \cdot x}$

$$\not p_1 u_1 = m u_1 \quad ; \quad \not p_2 u_2 = m u_2$$

$$p_1 \cdot p_1 = m^2 \quad ; \quad p_2 \cdot p_2 = m^2$$

Thus taking the initial photon to be incident along the x axis. we have

$$\not p_1 = m \gamma_t$$
$$\not p_2 = E_2 \gamma_t - \gamma_x p_2 \cos\varphi + \gamma_y p_2 \sin\varphi$$
$$\not q_1 = \omega_1 (\gamma_t - \gamma_x)$$
$$\not q_2 = \omega_2 (\gamma_t - \gamma_x \cos\theta - \gamma_y \sin\theta)$$



The initial photon can be polarised along z (A) mutually or along y (B): (A) $\not{e}_1 = \not{\gamma}_z$ (B) $\not{e}_1 = \not{\gamma}_y$

(A') $\not{e}_2 = \not{\gamma}_z$ (B') $\not{e}_2 = \not{\gamma}_y \cos\theta - \not{\gamma}_x \sin\theta$

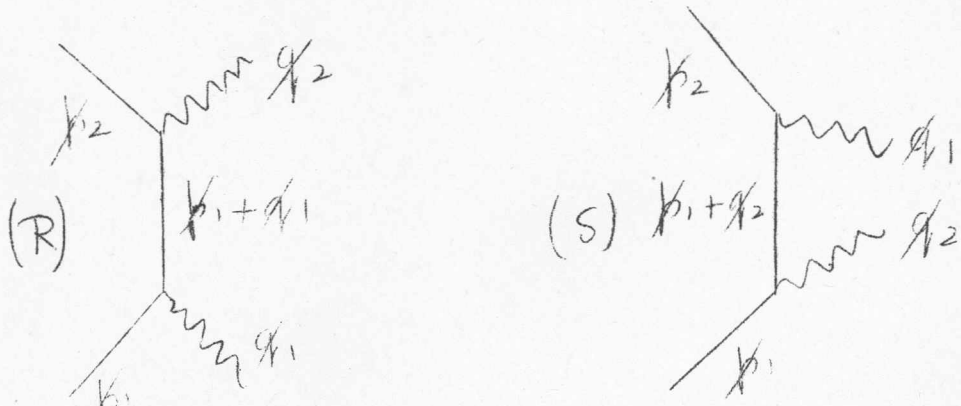
From the conservation relations

$$\not{p}_1 + \not{q}_1 = \not{p}_2 + \not{q}_2$$

We obtain the frequency shift as

$$\frac{m}{\omega_2} - \frac{m}{\omega_1} = 1 - \cos\theta$$

The matrix element can be directly written using the Feynmal rules



For the (R) & (S) diagrams the have

$$-i(4\pi e^2)\left(\bar{u}_2 \not{e}_2 \frac{1}{\not{p}_1 + \not{q}_1 - m} \not{e}_1 u_1\right) = -i 4\pi e^2 \bar{u}_2 R u_1$$

and

$$-i(4\pi e^2)\left(\bar{u}_2 \not{e}_1 \frac{1}{\not{p}_1 - \not{q}_2 - m} \not{e}_2 u_1\right) = -i(4\pi e^2)\bar{u}_2 S u_1$$

In order to calculate the differential cross-section we have to compute $|\bar{u}_2 (R+S) u_1|^2$

(i.e. the square of the sum of the amplitudes)

To reduce this we make use of the relation

$$\frac{1}{\not{p} - m} = \frac{\not{p} + m}{p^2 - m^2}$$

and $\not{a}\not{b} = -\not{b}\not{a} + 2a\cdot b$

Thus

$$\bar{u}_2 \not{\varepsilon}_2 \frac{1}{\not{p}_1+\not{q}_1-m} \not{\varepsilon}_1 u_1 = \bar{u}_2 \not{\varepsilon}_2 \frac{(\not{p}_1+\not{q}_1+m)}{(p_1+q_1)^2-m^2} \not{\varepsilon}_1 u_1$$

$$\left[\not{p}_1 \not{\varepsilon}_1 u_1 = -\not{\varepsilon}_1 \not{p}_1 u_1 + 2\varepsilon_1\cdot p_1 u_1 \right.$$
$$\left. = -\not{\varepsilon}_1 \not{p}_1 u_1 = -\not{\varepsilon}_1 m u_1\right]$$

$$= \frac{\bar{u}_2 \not{\varepsilon}_2 (\not{q}_1) \not{\varepsilon}_1 u_1}{(p_1+q_1)^2 - m^2}$$

$$\not{p}_1^2 + \not{q}_1^2 + 2p_1\cdot q_1 - m^2 = 2p\cdot q_1 = 2m\omega_1$$

Hence

$$R = \not{\varepsilon}_2 \not{q}_1 \not{\varepsilon}_1 / (2m\omega_1)$$

Illy $S = \not{\varepsilon}_1 \not{q}_2 \not{\varepsilon}_2 / (2m\omega_2)$

Now since there are four possible sets of polarization combinations for the photons, i.e. AA' AB' A'B and BB' we have to sum over these. We substitute the appropriate values for $\not{\varepsilon}_1$ & $\not{\varepsilon}_2$ in $\bar{u}_2(R+S)u_1$ and find the square in each case. We have

Polarisation	Matrix element
AA'	$\frac{(\omega_1-\omega_2)^2}{\omega_1\omega_2} + 4$
AB'	$\frac{(\omega_1-\omega_2)^2}{\omega_1\omega_2}$
BA'	$\frac{(\omega_1-\omega_2)^2}{\omega_1\omega_2}$
BB'	$\frac{(\omega_1-\omega_2)^2}{\omega_1\omega_2} + 4\omega^2\theta$

These four can be more generally written as

$$|M|^2 = \frac{(\omega_1 - \omega_2)^2}{\omega_1 \omega_2} + 4(e_1 \cdot e_2)^2$$

The differential cross-section may now be found, i.e.

$$d\sigma = \frac{2\pi}{(\text{Normalisation term})} |M|^2 \times \left[\text{density of final states}\right]$$

and in this case we get the Klein Nishnia formula

$$d\sigma = \frac{2\pi}{2E_1 2E_2 2\omega_1 2\omega_2} \times (4\pi e^2)^2 \, |u_2 (R+S) u_1|^2 \times (2\pi)^{-3} \frac{E_2 \omega_2^3 \, d\Omega_{\omega_2}}{m\omega_1}$$

$\downarrow$ [density of final states for a two particle system]

$$+ \frac{e^4}{4m^2} \frac{\omega_2^2}{\omega_1^2} d\Omega_{\omega_2} \left[\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} - 2 + 4(e_1 \cdot e_2)^2 \right]$$

<u>To derive the density of states.</u>

Physically we are interested in the transition per unit time into a small energy internal ΔE in the neighbourhood of the final state E_f. If the density of final states is $\rho(E)\,dE$ then the transition probability for the infinite time internal is

$$\omega = K \langle f | S | i \rangle \rho(E)\, dE$$

The number of states available for a particle with momentum between $\vec{k}$ and $k + dk$ is $\frac{d^3\vec{k}}{h^3}$ or $\frac{d^3k}{(2\pi)^3}$ (when $\hbar = 1$)

when $|\vec{k}|$ is between $\vec{p}$ and $\vec{p}+d\vec{p}$ the number is $p^2dp/(2\pi)^3$ Thus the number of states per unit energy interval is $f(E_p) = \frac{p^2}{(2\pi)^3}\frac{dp}{dE} = \frac{E_p}{(2\pi)^3}$ as $\frac{dp}{dE} = E/p$

For a two particle system, with momenta p_1 and p_2 and energy E_1 and E_2 with the total as

$$\vec{P} = \vec{p_1} + \vec{p_2} \qquad \text{and} \qquad E = E_1 + E_2$$

the number available in the internal dE is just the number for a particle of momentum between $\vec{p_1}$ and $\vec{p_1}+d\vec{p_1}$ (since $\vec{p_2}$ is fixed) i.e.

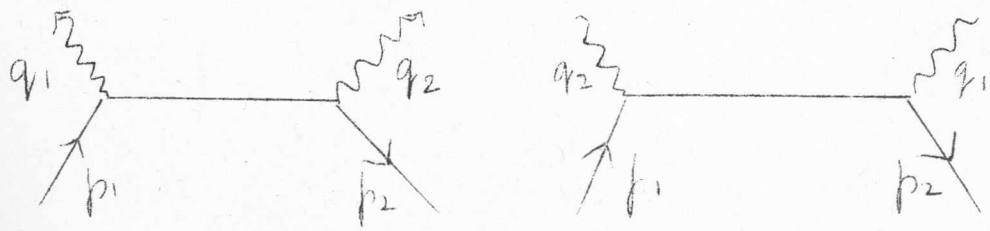
$$\frac{1}{(2\pi)^3} p_1^2\, dp_1 = \frac{1}{(2\pi)^3} p_1^2 \left(\frac{dp_1}{dE}\right) dE$$

and can be computed

1) Other q.e.d. processes

Two photon pair annihilation

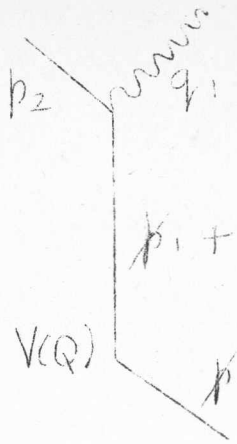
The diagram will be

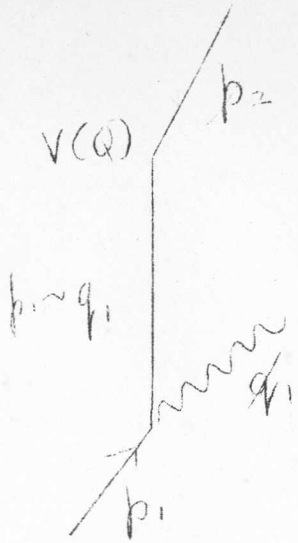


a free electron is annihilated in a collision with a free positron conservation of energy requires two photons. We see that the diagrams differ from Compton effect only in the time ordering.

(2) Bremstrahlung

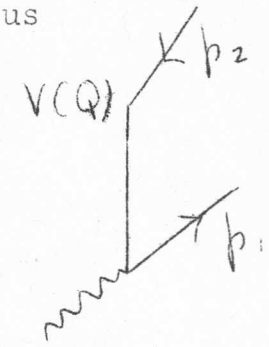
The e^- in the field of a nucleus emits a photon making a radiative transition.

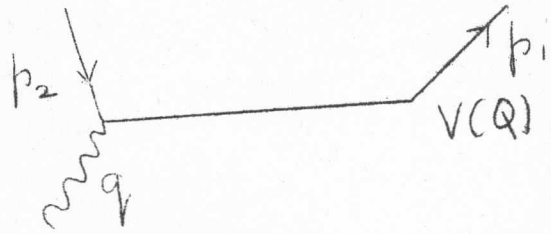




3) Pari production

A Photon of energy $>$ 2m can create a pair in the field of a nucleus





Here again we see that the difference from Bremsstrahlung is only in the time ordering.

LECTURE III

THE FORMALISM OF QUANTUM FIELD THEORY

This as has been already mentioned was an attempt to deal with the multiplicity of particles. Even though the field concept is generally introduced in an axiomatic manner; we shall try to develop the idea from the single particle in wave function formalism.

Essentially, the stall of a system of particles is described by a state vector involving the occupation numbers of particles in different state characterized by the values of dynamical variables. Evidently, in the presence of interaction the occupation numbers will change. This change is described with the help of field operators which operating on the state vector yield the occupation numbers and their mode of variation. Thus while in a single particle formalism we deal with the wave function Ψ, the field theory we have to deal with two types of quantities, the field operator Ψ and the state vector Ψ. Ψ bears a correspondence with the single particle wave function Ψ in that both are state vectors while the field operator Ψ also is similar to the wave function in that both obey equations of motion. It is felt that the deduction of in analogy with the single particle wave function and then obtaining the operators is a more natural and satisfactory method.

Considering the wave function of a free particle in the absence of interaction we know

$$\Psi(\vec{x},t) = (2\pi)^{-\frac{3}{2}} \int \phi(\vec{p},t)\, e^{i\vec{p}\cdot\vec{x}}\, d^3p \qquad (1)$$

and $\phi(\vec{p},t) = (2\pi)^{-3/2} \int \psi(\vec{x},t)\, e^{-i\vec{p}\cdot\vec{x}}\, d^3x$ (1)

and the energy can take two values

$$E_p = \pm\sqrt{p^2+m^2} \tag{2}$$

ψ can be expressed as a linear combination of eigenfunctions ψ_p as

$$\psi(\vec{x},t) = \int C_I(p)\, \psi_p(\vec{x},t)\, d^3p \tag{3}$$

where $C_I(p)$ is a complex number. Since $\psi_p = u_p e^{-ip\cdot x}$ where u_p is either a vector or a spinor (depending on the transformation properties) which we shall say is the structural part of the wave function, we find

$$\phi(\vec{p},t) = (2\pi)^{3/2} C_I u_p e^{-iE_p t} = C_S u_p \tag{4}$$

Thus we may represent a state by C_I or C_S itself since ψ and φ are connected to it. If eigenvalues of p are discrete then the state can be represented as a row vector $\vec{C}$ with as many components $C(p)$ as we have p values.

Considering a many particle system in the absence of interaction, Ψ can be expressed as

$$\Psi = \sum_{\{n_i\}} C_n(\{n_i\};t)\, \Phi^n_{\{n_i\}} \tag{5}$$

where Φ is the configuration representation of a state corresponding to an aggregate of particles $\{n_i\}$ with n_i

particles in the i^{th} state ($i=1, 2, 3 \ldots$) the total number of particles being n. C represents the amplitude for the members of the aggregate $\{n_i\}$ being their respective states and is just the analogue of $C_I(p)$ of the single particle case. Φ obeys the Schrodinger equation in the absence of interaction when n is fixed but Ψ by (5) need not be confined to fixed n since n can vary in Φ_n. If we now want Ψ to obey the Schrodinger equation H cannot be described in terms of a fixed number of variables but in terms of field operators.

Thus comparing with the single particle case we find from (1) while e^{-ipx} is a trivial form of the Φ function $C_S(p)$ representing the amplitude that the particle has momentum p is a trivial form of the C-function with $n_i = 1$; $p_1 = p$ and all $n_r = 0$. Thus the amplitude in occupation number space is just the momentum representation of Ψ.

If Ψ should obey the Schrodinger equation, then H will have to consist of operators which connect the $\Phi^{n'}_{\{n_i\}}$ - s in other words, it should be defined in terms of creation and annihilation operators. These operators a_i^+ and a_i are defined usually by

$$a_i \Phi^n_{n_1 \ldots n_i}(x_1 \ldots x_n) = \sqrt{n_i}\, \phi^{n-1}_{n_1 \ldots n_{i-1}}(x_1 \ldots x_{n-1}$$

$$a_i^+ \Phi^n_{n_1 \ldots n_i}(x_1 \ldots x_n) = \sqrt{n_i+1}\, \phi^{n+1}_{n_1 \ldots n_i+1}(x_1 \ldots x_n$$

(c)

and also by

$$a_i^{+} C_n(n_1, n_2 \cdots n_i \cdots) \rightarrow \sqrt{n_i+1}\, C_{n+1}(n_1 n_2 \cdots n_i+1)$$

$$a_i\, C_n(n_1 n_2 \cdots n_i \cdots) \rightarrow \sqrt{n_i}\, C_{n-1}(n_1 \cdots n_i - 1 \cdots)$$

(7)

We shall now try to define it through Ψ . We may write

$$a_k^{+} \sum_n C_n \Phi^n = \sum_n C_n' \Phi^n \qquad (8)$$

where $C'_{n+1}(k_1 \cdots k_{n+1}) = \sqrt{n_{k+1}}\, C_n(k_1 \cdots k_n)\delta_{k,k_{n+1}}$

where $k_1 \cdots$ etc. indicates the individual momento of particles i.e a_k^{+} operates on both C_n and Φ^n and changes C_n to C_{n+1} and Φ^n to Φ^{n+1} . Taking the particular case when

$$\Psi = \Phi^n \{n_i\}$$

with all coefficients zero except $C^n\{n_i\} = 1$, we have

$$a_k^{+}\Psi = a_k^{+}\Phi^n = C'_{n+1}(k_1 \cdots k_n, k)\Phi^{n+1}\{n_i\} = \sqrt{n_{k+1}}\;\Phi^{n+1}$$

(9)

From the definition of a - s in (7) we get the following commutation rules, i.e.

$$[a_\alpha, a_\beta] = [a_\alpha^{+} a_\beta^{+}] = 0$$

$$[a_\alpha\, a_\beta^{+}] = \delta_{\alpha\beta} \qquad (10)$$

which are the rules for bosons. Thus we have tried to introduce the concept of creation and annihilation operators through the concept of their interaction and not introduced in an ad hoc manner. We shall further show that the form of the Hamiltonian itself can be deduced by this method.

TEMPORAL EVOLUTION OF THE STATE VECTOR IN A COLLISION PROCESS

It is found more convenient to write the Schrodinger equation of the state vector in momentum representation, i.e.

$$i \frac{\partial C_s(t)}{\partial t} = H(t) C_s(t) \tag{11}$$

and in the usual manner we split

$$H(t) = H_0 + H'(t) \tag{12}$$

and writing

$$C_s(t_2) = U_s(t_2, t_1) C_s(t_1) \tag{13}$$

we have

$$U_s(t_2, t_1) = e^{-iH_0(t_2-t_1)} + (-i)\int_{t_1}^{t_2} e^{-iH_0(t_2-\tau_1)} H(\tau) e^{-iH_0(\tau-t_1)} d\tau$$

$$+(-i)^2 \int_{t_1}^{t_2} d\tau_2 \int_{t_1}^{\tau_2} e^{-iH_0(t_2-\tau_2)} H'(\tau_2) e^{-iH_0(\tau_2-\tau_1)} H'(\tau_1) e^{-iH_0\tau_1} d\tau_1$$

$$+ \text{- - - - - - - -} \tag{14}$$

We can also write

$$U_S(t_2,t_1) = e^{-iH_0t_2}\, U_I(t_2,t_1)\, e^{iH_0t_1} \tag{15}$$

where

$$U_I(t_2,t_1) = 1 + (-i)\int_{t_1}^{t_2} H_I(\tau)\,d\tau + (-i)^2\int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_2} d\tau_1\, H_I'(\tau_2) H_I'(\tau_1) + \cdots \tag{16}$$

i.e. since $C_S(t) = e^{-iH_0t}\, C_I(t)$

or $C_I(t_2) = U_I(t_2,t_1)\, C_I(t_1)$ (17)

In any collision process we are interested in the matrix element of transition $C_{fi}(t_2,t_1)$ from the state i at t_1 to f at t_2 as $t_1 \to -\infty$ and $t_2 \to +\infty$

The general final state $C(t_2)$ is

$$C(t_2) = \sum_f C_{fi}(t_2-t_1)\, C_{Ii}(t_1) \tag{18}$$

Taking $C_I(t_1)$ such that $C_{Ij}(t_1) = 0$ for all $j \neq i$ and $C_{Ii}(t_1) = 1$, we have

$$C_S(t_1) = C_{Ii}(t_1)\, e^{-iE_0t_1} = e^{-iE_0t_1} \tag{19}$$

where E_0 is the energy of the initial system.

We now define the vacuum state $\rangle_0$ as the sequence of amplitudes $C(i)$ with $C(i) = 0$ for all except the null aggregate $\{0\}$ and $C(0) = 1$ And also define the hermitian conjugate $\langle_0$ as $\langle_0 \rangle_0 = 1$. Now a state representing an aggregate $\{i\}$ of particles of momentum $p_1 \cdots p_n$, can be represented by

$$|i\rangle = a^+_{p_1} \cdots a^+_{p_n} \rangle_0 \tag{19}$$

which is the analogue of the column vector in the single particle case consider a system of fermions. Now $a^+_p |i\rangle$ will give a $\{i\}$ plus one particle of momentum p. If $\{i\}$ itself has a particle of momentum p then $a^+_p |i\rangle = 0$ according to the Pauli principle. In such a case we can define the annihilation operator a_p as

$$a^+_p a_p |i\rangle = |i\rangle \tag{20}$$

Since $a^+_p |i\rangle = 0$ we also have

$$a_p a^+_p |i\rangle = 0 \tag{21}$$

On the other hand if $|i\rangle$ does not contain a particle of momentum p, then $a_p |i\rangle = 0$ so that

$$a_p a^+_p |i\rangle = |i\rangle \tag{21}$$

and

$$a^+_p a_p |i\rangle = 0 \tag{22}$$

Thus if we impose the condition $a\,a_p^\dagger + a_p^\dagger a = 1$ the Pauli principle is ensured in the representation of states $a(a_p a_p^\dagger + a_p^\dagger a_p)|i\rangle = |i\rangle$ irrespective of whether $|i\rangle$ has a particle of momentum p or not. Also since $\langle\ \rangle_0 = 1$ we would also like $\langle i|i\rangle = 1$ i.e. $\langle i| = {}_0\langle a\{i\}$.

Since $|i\rangle = a^\dagger\{i\}\rangle_0$

Now the matrix element C_{fi} is

$$C_{fi} = \langle f|e^{iH_0t_2}U_S(t_2-t_1)e^{-iH_0t_1}|i\rangle \tag{23}$$

$$= \langle f|U_I(t_2,t_1)|i\rangle$$

$$= \langle f|S|i\rangle$$

(24)

where $$S = \underset{\substack{t_2\to\infty \\ t_1\to-\infty}}{Lt}\ U_I(t_2t_1)$$

is just the S-matrix.

We shall now identify H' in terms of the creation and annihilation operators.

Now when $\{i\}$ say changes to $\{k\}$, only a few particles in $\{i\}$ say $\{\alpha\}$ are destroyed and few say $\{\beta\}$ are created so that $\{\beta\}$ along with $\{i\}-\{\alpha\}$ comprise $\{k\}$. Thus the interaction Hamiltonian will involve only $\alpha+\beta$ particles and $E_i - E_k = E_\alpha - E_\beta$. For instance in electrodynamics $\alpha+\beta=3$ consists of two fermions and a photon. We shall hence confine ourselves

to the subaggregate $\{\alpha\}$ and $\{\beta\}$ Since we know

$$\int \exp\left[i(\lambda_1+\lambda_2).x\right] d^3x \supset \delta(\lambda_1+\lambda_2) \quad (25)$$

and $$\int \exp\left[i\left(\sum_i \lambda_i + \sum_i \mu_i\right)x\right] d^3x \supset \delta\left(\sum_i \lambda_i + \sum_i \mu_i\right)$$

and since free particle wave functions have $e^{-ip.x}$

it is reasonable to assume that

$$R(\beta \leftarrow \alpha) = \int H(\beta \leftarrow \alpha) d^3x$$

where $$H(\beta \leftarrow \alpha) \supset [\text{Wave function of } \{\beta\}]^+ \times [\quad ,, \quad ,, \quad \{\alpha\}] \quad (26)$$

where d^3x implies integration over space point X and all wave functions have the same spatial argument X . We note that the total momentum of $\{\alpha\}$ need not be equal to that of $\{\beta\}$ in $H\{\beta \leftarrow \alpha\}$ since space integration ensures momentum conservation. Also in the wave function we ~~xx~~ omit the time dependence. If we include it we have

$$H_I(\beta \leftarrow \alpha) = H(\beta \leftarrow \alpha) e^{-i(E_\alpha - E_\beta)t}$$

$$= e^{iH_0 t} H(\beta \leftarrow \alpha) e^{-iH_0 t} \quad (27)$$

We would now like to introduce the concept of interaction only through R or H so that it may be independent of the particular states $\{i\}$ and $\{f\}$. For this we should have the following two conditions satisfied viz. (1) $\{\alpha\}$ should be a subaggregate of $\{i\}$ and (2) $\{\beta\}$ can be added to $\{i\} - \{\alpha\}$ only if the Pauli principle is not violated i.e. fermions in $\{\beta\}$ should not be contained in $\{i\} - \{\alpha\}$ For the first condition, assuming discrete momentum states, if a particle is known to exist in p_α then $|\alpha\rangle$ is represented by a column vector with its α^{th} element equal to unity and the rest zero. Then if $\langle\beta|$ is a row vector, $R_{\beta\alpha} = \langle\beta|R|\alpha\rangle$ where R is a matrix with components $R_{\beta\alpha}$ so that transitions can occur only if $|\alpha\rangle$ exists at t. We now see that if we attach the annihilation operator to the wave function and the creation operator to the complex conjugate, both conditions are satisfied. Thus defining $\psi = \sum_k a_k \psi_k$ as the field operator so that

$$\bar{\psi} = \sum_k a_k^+ \psi_k$$

$$\sum_{\beta\alpha} H_I(\beta \leftarrow \alpha) \supset \bar{\psi}\psi \qquad (28)$$

We know that for bosons, the probability of absorption is proportional to n while emission is proportional to n + 1 or amplitudes are proportional to $\sqrt{n}$ and $\sqrt{n+1}$ respectively.

Thus if $|n\rangle$ represents a state of n bosons

$$a_p^+ |n\rangle = \sqrt{n+1}\, |n+1\rangle \tag{29}$$

and $$a_p |n\rangle = \sqrt{n}\, |n-1\rangle$$

This implies the commutation relation

$$(aa^+ - a^+a) = 1$$

If we now allow for negative energy states and postulate that the annihilation of a negative energy particle is equivalent to the creation of a positive energy antiparticle, we have

$a_{-k} = b_k^+$, $u_{-k} = V_k$ so that

$$\psi = \sum_k a_k u_k e^{-ik\cdot x} + \sum_k b_k^+ V_k e^{+ik\cdot x} \tag{30}$$

which is also valid for bosons except that u and V do not occur.

Lecture IV

EVALUATION OF THE MATRIX ELEMENT

Having arrived at the form of the interaction Hamiltonian in terms of the creation and annihilation operators we shall now evaluate the matrix element in field theory and identify it with the Feynman matrix element. Our method will differ from the conventional in that we shall deal with the integrand of the S-matrix instead of the integral when we reorder the operators. The justification for this is sought from Stochastic arguments

Consider a collision process due to an interaction operator in the time interval To-T. We are interested in the amplitude $C_S(f \leftarrow i\ ;\ T, T_0)$ as $T_0 \to -\infty$ and $T \to +\infty$

We can think of the interaction as follows. At To we have all aggregate $\{i\}$ the amplitude for its existence being $e^{-iE_i T_0}$ Till some time point τ_1 it remains the same with amplitude $e^{-iE_i(\tau_1 - T_0)}$ and in the infinitesimal interval between τ_1 and $\tau_1 + \Delta$ if changes to $\{m_1\}$ (momentum being conserved) with amplitude $R_S(m_1 \leftarrow i)\Delta$. Since energy need not be conserved $\{m_1\}$ is not physically realizable. $\{m_1\}$ survives until τ_2 with an amplitude $e^{-iE_{m_1}(\tau_2 - \tau_1)}$ and changes to $\{m_2\}$ between τ_2 and $\tau_2 + \Delta$ and so on.

Thus for realising $\{k\}$ at T the amplitude is

$$V_S(\tau_n, \cdots \tau_1\ ;\ T) = e^{-iE_k(T-\tau_n)} R_S(k \leftarrow m_n)\, e^{-iE_{m_{n-1}}(\tau_n - \tau_{n-1})} \cdots e^{-iE_{m_1}(\tau_2 - \tau_1)} R_S(m \leftarrow i)\, e^{-iE_i \tau_1} d\tau_n \cdots d\tau_n \qquad (1)$$

which we shall call the amplitude in the Schrodinger representation for a typical temporal realization of 'complexion' to occur, a 'complexion' being characterized by the aggregate attributed to the various time points. A sum over all complexions - i . performing the time ordered integration and summing over all agg egate. We obtain the amplitude for transition to $\{k\}$ as

$$C_s(k;T) = \sum_{n=1}^{\infty} \sum_{\{m_r\}} \frac{e^{iE_i T} R_s(k \leftarrow m_{n-1}) \cdots R_s(m_1 \leftarrow i)}{(E_i - E_k)(E_i - E_{m_{n-1}}) \cdots (E_i - E_{m_1})} \qquad (2)$$

Of course to obtain the matrix element for transition to $\{f\}$ it is not just enough to let $T \to +\infty$ we have to replace E_k by E_f and omit $(E_k - E_i)$ in the denominator and introduce a $\delta(E_f - E_i)$ in the numerator

In evaluation the matrix element when H_I is defined in terms of field operators, we refrain from performing the space-time integration and reorder the vertices in the Feynman square, . The space time integration is subsequently performed This rearrangement can be done in two ways.

We know that the interaction term contains annihilation and creation operators with the corresponding wave functions attached to them. It is obvious that once the expression for a 'typical realization' is written it is a mathematically valid operation to rearrange the operators without reference to the wave functions

The result of such a rearrangement of the annihilation and creation operators is ultimately to yield the numerical factor +1 or -1 in view of the commutation relations. The wave functions are suitably grouped so that we have a product of scalar quantities; and because of their scalar nature they can then be rearranged so that we get what is known as a Feynman sequence of wa functions. This, with the appropriate sign factor ± 1 obtained from the rearrangement of the operators, yields the exact Feynman matrix element. We shall describe this process of rearragnement in detail for an nth order scattering of a single electron by considering typical realization, the amplitude of which makes a nonvanishing contribution to the matrix element.

The nth order term of the matrix element is an integral over n space-time points, the time points alone being ordered. The integrand is therefore

$$\langle f | H_I(n) \cdots H_I(k) \cdots | i \rangle \qquad (3)$$

If the initial state consists of an electron of momentum $\vec{p}_1$, i.e. $|i\rangle = a_{p_1}^{\dagger} \rangle_0$ and the final state, an electron of momentum $\vec{p}_2$ then $|f\rangle = a_{p_2}^{\dagger} \rangle_0$ and every $H_I(i) \supset \bar{\psi}(i) \psi(i) \phi(i)$. . As mentioned before the symbol $\supset$ means that the factors on the right-hand side are contained in $H_I(i)$ and is used to indicate that we ignore constants and matrices which are not relevant to the present discussion. We shall also, for the present, ignore the photon or meson operator $\varphi(i)$ and concern ourselves with the bilinea fermion operators $\bar{\psi}(i) \psi(i)$ only. Thus

$$[n] = H(n) \supset \bar{\psi}(n)\,\psi(n)$$

$$= \left[\hat{a}_p^{+}\,\hat{a}_{p_1}(n) + \hat{a}_p^{+}\,\hat{b}_{p_1}^{+}(n) + \hat{b}_p\,\hat{b}_{p_1}^{+}(n) + \hat{b}_p\,\hat{a}_{p_1}(n)\right] \tag{4}$$

where the symbol $\hat{}$ denotes that the operators are accompanied by their corresponding wave functions. We shall now separate the operators from their wave functions and, first, order the operators as follows:

Consider a typical 'realization' or complexion'

$$\langle f|\,[n]\cdots[j]\cdots[k]\cdots[i]\,|i\rangle \tag{5}$$

Since the initial electron of momentum $\vec{p}_1$ has to be annihilated there should be an a_{p_1} in some bracket $[k]$. Let this bracket be the 'nearest' with an a_{p_1} to occur to the left of $a_{p_1}^{+}\rangle_0$. This bracket would contain in addition an a_p^{+} or b_p to the left of a_{p_1}. We now move a_{p_1} to the left of $a_{p_1}^{+}$ through all the intervening brackets $[k-1]\cdots[1]$ and this results in no change of sign since each bracket contains two fermion operators none of which can be an a_{p_1} or an $a_{p_1}^{+}$ (the former by assumption and the latter since $a_{p_1}^{+}\,a_{p_1}\rangle_0 = 0$). Thus we have

$$[n]\cdots[k+1]\,a_{p_1}^{+}(k)\,[k-1]\cdots[1]\,a_{p_1}\,a_{p_1}^{+}\rangle_0 \tag{6}$$

or

$$[n]\cdots[k+1]\,b_{p_1}(k)\,[k-1]\cdots[1]\,a_{p_1}\,a_{p_1}^{+}\rangle_0 \tag{7}$$

We recognize $a_{p_1}\, a_{p_1}^{\dagger} >_0 = >_0$ if the realization is as in ($^{(6)}$) we next move $a_{p'}^{\dagger}(k)$ to the right of a_{p_1}. There should now be another bracket, say $[j]$ with $t_j > t_k$ containing an $a_{p'}$ which is shifted through all the intervening brackets to the left, which again leads to a numerical factor +1 and we have

$$[n] \cdots [j+1]\, a_{p''}^{\dagger}\, [j-1] \cdots [k+1]\,[k-1] \cdots a_{p'}(j)\, a_{p_1}^{\dagger}(k) >_0 \qquad (8)$$

or

$$[n] \cdots [j+1]\, b_{p''}(j)\, [j-1] \cdots [k+1]\,[k-1] \cdots a_{p'}(j)\, a_{p_1}^{\dagger}(k) >_0 \qquad (9)$$

Thus we have given a complete prescription for the arrangement of the operators a and a^{+}. Turning to the wave functions corresponding to the above typical realization, we can arrange t them as

$$[n] \cdots [\bar{u}_{p''}(j)]\,[\bar{u}_{p'}(k)\; u_{p_1}(k)] \qquad (10)$$

and $u_p(j)\, \bar{u}_{p'}(k)$ can be identified to be a single element of the positive energy part of what is called the Feynman kernel. The sum of all such terms for all possible realization will give the positive part of the Feynman kernel $K_+(j,k)$ with $t_j > t_k$.

If on the other hand $[k]$ had a $b_{p'}$ to the left of a_{p_1} then there should be a $b_{p'}^{\dagger}$ in a bracket $[m]$ with $t_m < t_k$ since the positron should be created before it can be annihilated. Thus, if we have

$$[n] \cdots [k+1]\,[b_{p'}(k)\, a_{p_1}(k)]\,[k-1] \cdots [a_{p''}^{\dagger}(m)\, b_{p'}^{\dagger}(m)] \cdots a_{p_1}^{\dagger} >_0 \qquad (11)$$

we move a_{p_1} to the left of $a_{p_1}^{\dagger}$ so that

$$a_{p_1}\, a_{p_1}^{\dagger} \rangle_0 = \rangle_0 \tag{12}$$

with a numerical factor (+1). We now move $b_{p_1'}(k)$ to the left of $b_{p_1'}^{\dagger}(m)$ i.e.

$$[n] \cdots [k+1][k-1] \cdots b_{p_1'}(k)\left[a_{p_1''}^{\dagger}(m)\, b_{p_1'}^{\dagger}(m)\right] \cdots a_{p_1}(k)\, a_{p_1}^{\dagger} \rangle_0 \tag{13}$$

or

$$[n] \cdots [k+1][k-1] \cdots b_{p_1'}(k)\left[b_{p_1''}(m)\, b_{p_1''}^{\dagger}(m)\right] \cdots a_{p_1}\, a_{p}^{\dagger} \rangle_0 \tag{14}$$

If we now shift $b_{p_1'}(k)$ to the left of $b_{p_1'}^{\dagger}(m)$ we obtain a negative sign i.e.

$$-\, a_{p_1''}^{\dagger}(m)\, b_{p_1'}(k)\, b_{p_1'}^{\dagger}(m) \cdots a_{p_1}\, a_{p_1}^{\dagger} \rangle_0 \tag{15}$$

or

$$-\, b_{p_1''}(m)\, b_{p_1'}(k)\, b_{p_1'}^{\dagger}(m) \cdots a_{p_1}\, a_{p_1}^{\dagger} \rangle_0 \tag{16}$$

We now shift the pair $(-)\, b_{p_1'}(k)\, b_{p_1'}^{\dagger}(m)$ to the extreme right without any change in sign and identify $b_{p_1'}\, b_{p_1'}^{\dagger} \rangle_0 \rightarrow$ This completes the prescription for all the operators.

As regards the wave functions, we have

$$[n] \cdots \left[\bar{v}_{p_1'}(k)\, u_{p_1}(k)\right]\left[\bar{u}_{p_1''}(m)\, v_{p_1'}(m)\right] \cdots [1] \tag{17}$$

which may be rewritten as

$$[n] \cdots \left[\bar{u}_{p_1''}(m)\, v_{p_1'}(m)\right]\left[\bar{v}_{p_1'}(k)\, u_{p_1}(k)\right] \cdots [1] \tag{18}$$

It is to be noted that the positron operators occur in the order $b_{p_1'}(k)\, b_{p_1'}^{\dagger}(m)$ so that if they carried their wave functio with them we would have had $\bar{v}_{p_1'}(k)\, v_{p_1'}(m)\, e^{ip_1'(x_m - x_k)}$

which cannot be identified with an element of the Feynman kernel for negative energies. However by shifting the product of wave functions in such a manner as to arrange them on the Feynman sequence, and taking over the negative sign obtained from the shifting of the operators, we can recognize

$$-v_{p'}\bar{v}_{p'}e^{ip'\cdot(x_m-x_k)} = -u_{-p'}\bar{u}_{-p'}e^{ip'\cdot(x_m-x_k)} \quad \text{with } t_m < t_k \tag{19}$$

to be an element of the Feynman kernel for negative energies, the sum of all such terms for all possible realizations giving the negative energy part of K_+. In this case we have detached the operators from the wave functions though the derivation of the field operators was based on the concept that the wave function was attached to the operator.

The essential feature of this method is that we move the wave functions along with the operators and redefine the interaction Hamiltonian as

$$H(n) = \sum_{p,p'}\left[\hat{a}_{p'}^{+}\hat{a}_p + \hat{b}_{p'}^{+}\hat{a}_p^{+} + \hat{b}_{p'}^{+}\hat{b}_p + \hat{b}_{p'}\hat{a}_p\right] \tag{20}$$

where the creation operator b^+ always occurs to the left of either a^+ or b, unlike the ordering obtained from $\bar{\psi}\psi$. At any vertex one of the four fundamental processes can occur and the operators should occur in pairs but not necessarily in the order prescribed by $\bar{\psi}\psi$. A unique prescription can be given for the choice of the correct order based on the following arguments

The process of pair annihilation at t represents the transition of a positive energy electron at t to a negative energy state at $t + \Delta$, the perturbation acting in the interval Δ and hence the electron destruction operator should be placed to the right of the positron destruction operator. In the case of pair creation in the interval between $t - \Delta$ and t, we view the process as though we trace the negative energy state of the electron at $t + \Delta$ back to a positive energy state at t so that in this case $b_{p'}^{\dagger}$ should be placed to the left of $a_{p}^{\dagger}$. For electron and positron scattering the creation operators will be to the left of the annihilation operators. Thus the interaction Hamiltonian reads as (52).

Having postulated the interaction Hamiltonian in the above form we now proceed to derive the matrix element for the nth order scattering of a single electron. The integrand as before will be given by

$$\langle f | [n][n-1] \cdots [2] \cdots [1] | i \rangle \qquad (21)$$

where $[n]$ is now given by (20).

If we use the new interaction term at each vertex, we can ignore the commutation relations of operators corresponding to the different time points. We know that this commutation relation was used to obtain the negative sign when we were moving the positron operators in the previous method. Ignoring the commutation relations corresponding to different times amounts to viewing the process, ab initio, in the Feynman sequence. Thus,

for a typical realization, we move the entire bracket $[k]$ containing an a_{p_i} to the left of $a^{\dagger}_{p_i}>_0$ i.e.

$$\langle f|[n]\cdots[k+1][k-1]\cdots[1][k]\,|\,a^{\dagger}_{p_i}>_0 \tag{22}$$

The rearrangement is identical to that in Method I in the case when the operator attached to a_{p_i} in $[k]$ is an $a^{\dagger}_{p_i}$. If on the otherhand it is a $b_{p'}$ we place the bracket $[m]$ with $t_m < t_k$ containing $b^{\dagger}_{p'}$ to the left of $[k]$

$$[n]\cdots[k+1][k-1]\cdots[m+1][m-1]\cdots[1][m][k] \tag{23}$$

so that we have either

$$\left[\hat{b}^{\dagger}_{p'}(m)\,b_{p''}(m)\right]\left[\hat{b}_{p'}(k)\,\hat{a}_{p_i}(k)\right]a^{\dagger}_{p_i}>_0 \tag{24}$$

or

$$\left[\hat{b}^{\dagger}_{p'}(m)\,\hat{a}^{\dagger}_{p''}(m)\right]\left[\hat{b}^{\dagger}_{p'}(k)\,\hat{a}_{p_i}(k)\right]a^{\dagger}_{p_i}>_0 \tag{25}$$

Thus in shifting the operator $b^{\dagger}_{p'}(m)$ to the left of $b_{p'}(k)$ we acquire a negative sign i.e.

$$-\hat{b}_{p''}(m)\,\hat{b}^{\dagger}_{p'}(m)\,\hat{b}_{p'}(k)>_0 \tag{26}$$

or

$$\begin{aligned}-\hat{b}^{\dagger}_{p'}(m)\,\hat{b}_{p'}(k) &= b^{\dagger}_{p'}b_{p'}\,v_{p'}(m)\,\bar{v}_{p'}(k)\\ &= -u_{-p'}\bar{u}_{-p'}\,e^{ip'\cdot(x_m-x_k)}\end{aligned} \tag{27}$$

and we can straightaway identify

$$-u_{-p'}\,\bar{u}_{-p'}\,e^{ip'\cdot(x_m-x_k)} \tag{28}$$

with the negative energy part of the Feynman kernel.

It is to be noted that the negative sign in Method I comes from the switching of adjacent operators in two successive bracket while in the present method we obtain the negative sign by commuting the operators a single bracket.

Lecture V

It is well-known that the concepts of virtual states in nonrelativistic wave mechanics, Feynman formulation, and field theory are in principle the same though the detailed structure of the 'state' is different in the three descriptions.

1. In nonrelativistic wave mechanics, we exclude negative energy states--in other words we do not envisage the creation and annihilation of pairs. In the temporal evolution of the system, the number of fermions and antifermions are assumed to be separately conserved. In calculating the matrix elements we perform spatial integration first and the temporal evolution of the states in momentum representation is studied. It is therefore possible to speak of the state of a system at a particular time t.

2. In the Feynman formalism we include particles in the negative energy states, but the sequence of events in a perturbation expansion is not temporally ordered. We usually take the four-dimensional transform and, in this case, we can speak of the initial and final states being connected by a Feynman sequence of intermediate states.

3. In field theory, we envisage the creation and annihilation of particles and antiparticles. If, in the calculation of matrix elements the spatial integration is performed first, the situation is the same as in (1) except that we can have a multiplicity of particles. If however, we wish to integrate space and time together, we have first to rearrange the operators suitably and

this leads to the Feynman matrix element. It was considered that the essential merit of the four-dimensional integration was the inherent covariance of the theory at every stage, while separate integration with respect to time led to energy denominators. This seems to have been accepted for so long that no effort was made to find if it was possible to decompose the Feynman matrix element into relativistically invariant components. We now find that this is indeed possible in such a manner that some of the revealing features of the temporal ordering are preserved while at the same time the elegance of relativistic invariance is not lost. In fact if this is done the concept of virtual states becomes identical in all three formalisms.

The nth order matrix element for the scattering of an electron from momentum p_1' to momentum p_2' is given by

$$M = \int d^4x_n \cdots \int d^4x_1 \; \bar{\psi}_{p_2'}(x_n) K(x_n, x_{n-1}) \cdots \psi_{p_1'}(x_1) \tag{1}$$

In the Feynman formalism, the matrix element in momentum representation obtained by performing the four-dimensional integration is given by

$$M = \bar{u}(p_2') \not{e}_n \frac{1}{\not{p}_n - m} \not{e}_{n-1} \cdots \not{e}_2 \frac{1}{\not{p}_1 - m} \not{e}_1 u(p_1') \tag{2}$$

where the $\not{p}_i$ refer to intermediate virtual states with energy

$$p_4 \neq E_{p_i} = \sqrt{\vec{p}_i^{\,2} + m^2}$$

For a given order in the sequence of perturbations $\mathcal{L}_n \cdots \mathcal{L}_1$ (i.e. for a single Feynman diagram), the above can be decomposed into 2^{n-1} terms which are individually covariant as follows: The space and time integrations of (1) are separately performed the former leading to conservation of three momentum at every vertex, and the time integration which is subsequently performed is split into two parts corresponding to the ranges $t = -\infty$ to 0 with energy -E and $t = 0$ to $+\infty$ with energy +E respectively. It was shown in an earlier paper that this leads to the decomposition of the Feynman propagator

$$\frac{1}{\not p + \not q - m} = \frac{1}{2(E_{p+q})} \left\{ \frac{\overset{+}{\not P} + m}{E_p + E_q - E_{p+q}} - \frac{\overset{-}{\not P} + m}{E_p + E_q + E_{p+q}} \right\} \quad (3)$$

where $(E_{p+q})^2 = (\vec{p} + \vec{q})^2 + m^2$ and $\overset{+}{P}$ is the Feynman four vector with energy $+E_{p+q}$ and $\overset{-}{P}$ has the fourth component $-E_{p+q}$.

If we use this propagator it is more convenient to think of the energy of the 'virtual' particle to be $+E_{p+q}$ with momentum $\vec{p} + \vec{q}$. It is virtual in the sense that its energy does not correspond to $E_p + E_q$ the energy of the system before its creation. In a similar way, $-E_{p+q}$ corresponds to a negative energy 'virtual' particle. These two parts are taken together in the Feynman formalism where we attribute an energy p_4 to the **virtual** particle.

The main advantage of this decomposed propagator is that it lends itself to a method of comparison with field theory due to the presence of the energy denominators so that the equivalence between Feynman and field theoretic formalism can be established even in the old fashioned manner, that is, after space integration.

In a field theoretic picture for a given order in the sequence of perturbations, the nth order term has $n!$ diagrams, each of which will give different energy denominators. This is because the position of every vertex relative to all other vertices is important since the time integration is performed in a temporarily ordered way. Thus every new complexion gives a different energy denominator and a sum over intermediate states implies a sum over all such diagrams.

If, on the other hand, we employ the method of the decomposed Feynman propagator, the position of every vertex on the time axis with respect to the previous (in the Feynman sense) one is relevant since the integration over interval $t_n - t_{n-1}$ is divided into two ranges, positive and negative, or whether the nth vertex lies above or 'below' the $n-1$th vertex. Thus, since for an nth order process there are $n-1$ propagators each of which can be split into two parts, it follows that we can have 2^{n-1} diagrams. It now remains to be shown that the $n!$ diagrams of field theory are equivalent to the 2^{n-1} such diagrams. $(n! \geq 2^{n-1}$ for all $n \geq 2)$

Calculations

We here demonstrate explicitly the equivalence upto the fourth order. We have for definiteness considered the Compton scattering of an electron.

In second order, since $2^{n-1} = n!$ for $n = 2$ the correspondence is immediate. For the third order, let us consider the sequence in which an initial photon of four momentum $q(\omega, \vec{q})$ is absorbed by an electron at rest and two photons of four momenta $q_1(\omega_1, \vec{q}_1)$ and $q_2(\omega_2, \vec{q}_2)$ are emitted in this order along the Feynman path, the final electron having momentum $p_2(E_2, \vec{p}_2)$ The calculation of the entire matrix element would of course involve all permutations of the above sequence. For our present purpose, it is sufficient to consider this particular sequence only.

A. Field theoretic formalism

The matrix element in field theory for this third order process is

$$M_3 = \sum \frac{H_{f\,II}\; H_{II\,I}\; H_{I\,i}}{(E_0 - E_I)(E_0 - E_{II})} \qquad (4)$$

where f and i refer to final and initial states respectivel and the summation is over all intermediate states II and I.

$$H_{f\,II} = (\Phi_f, H_{int}\, \Phi_{II}) \text{ , etc.} \qquad (5)$$

and

$$H_{int} = e \int \bar{\psi} \gamma_\mu \psi \phi \, d^3x \qquad (6)$$

where ψ and ϕ are the electron and photon field operators respectively.

We now evaluate M_3 for each of the $\frac{n!}{2} = 3$ diagrams. The other half is just an exact counterpart with all positive energy intermediate states replaced by negative energy states and vice versa. Thus corresponding to diagram (i) we have

$$M_{3i} = \frac{\sum H_{f\,II} \; H_{II\,I} \; H_{I\,i}}{(m+\omega - E_{p_1+q_1})(m+\omega-\omega_1-E_{p_2+q_2})}$$

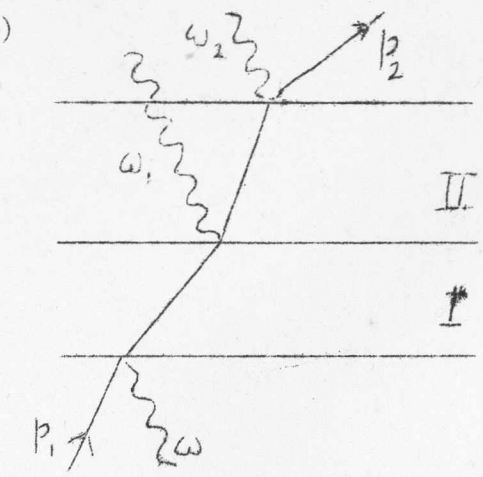
$$= \frac{\langle \Phi_f | H_{int}(x_3) | \Phi_{II} \rangle \langle \Phi_{II} | H_{int}(x_2) | \Phi_I \rangle \langle \Phi_I | H_{int}(x_1) | b_{p_1}^{+} \rangle_0}{(m+\omega - E_{p+q})(m+\omega-\omega_1-E_{p_2+q_2})} \qquad (7)$$

where $b_{p_1}^{+}$ is the creation operator of the electron with momentum $p_1 (0, m)$. The photon field operators are all omitted in what follows since they always commute and are hence not relevant for our arguments. We shall also omit numerical factors for convenience. Conservation of energy implies

$$m + \omega = E_2 + \omega_1 + \omega_2 \qquad (8)$$

and the energy denominators in (7) correspond to diagram (i)

(i)



Expanding H_{int} - s inserting the photon operators and integrating over the space variables $X_1 \cdots X_3$ leads ultimately to an overall δ-function which implies momentum conservation. Thus the electron operators in the numerator of (7) reduce to

$$\sum_{\substack{\text{all mom,}\\ \text{indices}}} \langle \Phi_f | (b^+ b)(b^+ b)(b^+ b) b^+ p_1 \rangle_0 \, \bar{u}_{p_2} [u \bar{u} u \bar{u}] u_{p_1}$$

where the b-s are the annihilation operators for electrons. Making use of

$$b_{p'}, b_p^+ \rangle_0 = \delta(p - p') \qquad \text{(a)}$$

and the overall δ-function resulting from the space integration of (7) becomes)

$$M_{3,i} = \frac{\bar{u}_{p_2} [u \bar{u}_{p_2 + q_2}][u \bar{u}_{p_1 + q}] u_{p_1}}{(m + \omega - E_{p_1 + q})(m + \omega - \omega_1 - E_{p_2 + q_2})} \qquad (9)$$

(ii) Similarly for the diagram 3 ii we have

$$M_{3ii} = \frac{\langle \Phi_f | H_{int}(x_3) | \Phi_{II} \rangle \langle \Phi_{II} | H_{int}(x_2) | \Phi_I \rangle \langle \Phi_I | H_{int}(x_1) b^+_{p_1}}{(m + \omega - E_{p+q})(\omega_1 - E_{p_1+q} - E_{p_2+q_2})} \qquad (10)$$

and the numerator

$$= \sum \langle \Phi_f | (d b)(b^+ d^+)(b^+ b) \; | b^+ p_1 \rangle_0 \; \bar{u}_{p_2} [v \bar{v} \, u \bar{u}] u_{p_1}$$

where d and d^+ are the annihilation and creation operators of the positron; this is now rearranged as

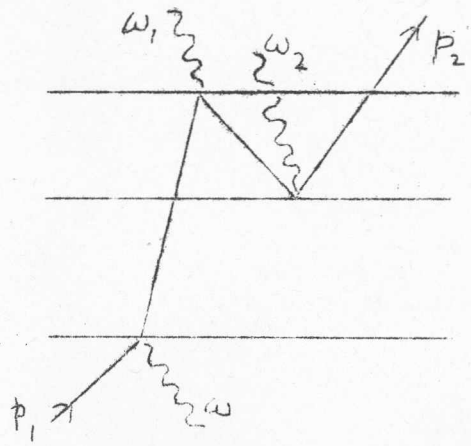
$$- \sum \langle \Phi_f | \, b^+ (d d^+)(b b^+) \, b \, | b^+ p_1 \rangle_0 \; \bar{u}_{p_2} [v \bar{v} \, u \bar{u}] u_{p_1}$$

so that we can now apply and we have

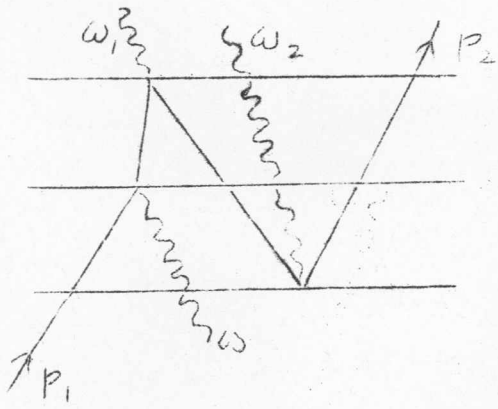
$$M_{3.ii} = \frac{-\bar{u}_{p_2} [v \bar{v}_{p_2 + q_2}][u \bar{u}_{p_1 + q}] u_{p_1}}{(m + \omega - E_{p_1 + q})(\omega_1 - E_{p_1 + q} - E_{p_2 + q_2})} \qquad (11)$$

(iii) In the case of diagram iii which differs from ii in that the vertex at which ω_2 is emitted is 'below' the vertex at which ω is absorbed, the energy denominators are obviously different.

(ii)



(iii)



We have

$$M_{3iii} = \frac{\sum \langle \Phi_f | (dd)(b^+b)(b^+d^+) | b^+ p_1 \rangle_0 \, \bar{u}_{p_2} [v\bar{v} u\bar{u}] u_{p_1}}{(-E_2 - \omega_2 - E_{p_2+q_2})(\omega_1 - E_{p_1+q} - E_{p_2+q_2})} \quad (12)$$

the numerator when arranged is $-\langle \Phi | b^+_{p_2} (dd^+)(bb^+) b | b^+ p_1 \rangle_0$

and again

$$M_{3iii} = \frac{-\bar{u}_{p_2} [v\bar{v}_{p_2+q_2}] [u\bar{u}_{p_1+q}] u_{p_1}}{(-E_2 - \omega_2 - E_{p_2+q_2})(\omega_1 - E_{p_1+q} - E_{p_2+q_2})} \quad (13)$$

It is interesting to note that the numerators in (11) and (13) are identical though the denominators are different so that they are equivalent to a single Feynman diagram.

3. Feynman formalism

The Feynman matrix elements for a given type of diagram can be immediately written down with the use of the decomposed propagator (2). Thus we have for the Feynman diagram equivalent to the (i)

$$M^F_{3i} = \frac{\bar{u}_{p_2} [(p_2 \!\!\!/ + q_2 \!\!\!/) + m][(p_1 \!\!\!/ + q \!\!\!/) + m] u_{p_1}}{(m + \omega - \omega_1 - E_{p_2+q_2})(m + \omega - E_{p_1+q})} \quad (14)$$

and corresponding to the other Feynman diagram equivalent to (ii) and (iii) we have

$$M^{F}_{3,2} = \frac{-\bar{u}_{p_2}\left[(\not{p}_2 + \not{q}_2) + m\right]\left[(\not{p}_1 + \not{q}) + m\right] u_{p_1}}{(E_2 + \omega_2 + E_{p_2+q_2})(m + \omega - E_{p_1+q})}$$

3C. Equivalence

We now have to show that (11) and (13) together reduce to (15).

1) It is seen that the expression (2) is the same as (14)

i.e. $M_{3,i} = M^{F}_{3,1}$ since $\sum_{spins} u\bar{u}_{p_2+q_2} = \left[(\not{p}_2 + \not{q}_2) + m\right]$ (16)

2) To show

$$M_{3,ii} + M_{3,iii} = M^{F}_{3,2} \qquad (17)$$

From (11) and (13)

$$M_{3,ii} + M_{3,iii} = N/(\omega_1 - E_{p_1+q} - E_{p_2+q_2}) \times \left[\frac{1}{m + \omega - E_{p_1+q}} + \frac{1}{E_2 + \omega_2 + E_{p_2+q_2}}\right]$$

$$= -N/(m+\omega-E_{p_1+q})(E_2+\omega_2+E_{p_2+q_2}) \qquad (18)$$

and since

$$-\sum_{\text{spins}}[v\bar{v}_{p_2+q_2}] = -\sum_{\text{spins}} u_{-\epsilon}\,\bar{u}_{-\epsilon}(p_2+q_2)$$

$$= [(\bar{p}_2 \not{-} q_2) + m] \qquad (19)$$

$$M_{3,ii} + M_{3,iii} = \frac{\bar{u}_{p_2}[(\bar{p}_2 \not{-} q_2)+m][(p_1 \overset{+}{\not{-}} q)+m]u_{p_1}}{(m+\omega-E_{p_1+q})(E_2+\omega_2+E_{p_2+q_2})}$$

$$\equiv M^{F}_{3,2} \qquad (20)$$

Lecture VI.

DENSITY CORRELATIONS IN QUANTUM MECHANICS

This work was stimulated by an old paper of Weisskopf on the self-energy of the electron. In this paper it was shown by Weisskopf that when the Dirac sea of negatice energy electrons is taken into account the electron is no longer a point charge but has a finite charge spread and consequently the selfenergy is not infinite but logarithmatically divergent. In order to arrive at this result, Weisskopf has defined a joint probability function which gives the probability of finding charge simultaneously at two points due to the presence of an electron or in other words gives the charge distribution of a single electron. Now in the theory of stochastic processes the distribution of a discrete number of variables in a continuous space is defined through product density functions for a particular value of the parameter t with respect to which the process evolves. This was adapted to the study of density correlations corresponding to n-points on the t-axis through the concept of sequent product density functions. (Now it was found that an application of the concept of sequent product densitites of quantum mechanical systems leads to rather intergsting results. When the product densities of Weisskopf are extended to include different time points, it is found to lead immediately to the modified Feynman propagator- a result which was historically obtained by Feynman some ten years later. Besides, when applied to many particle systems the role of the Pauli principle and its relation with the Feynman kernel is also clarified the

We shall now give a brief outline of the relevant positions of Weisskopf's paper before going into the concept of density correlations.

In order to determine the charge distribution in the neighbourhood of an electron he defines the joint probability function

$$G(\xi) = \int \rho(\vec{r} - \vec{\xi}/2)\, \rho(\vec{r} + \vec{\xi}/2)\, d\vec{r} \qquad (1)$$

where $\rho(\vec{r})$ is the charge density at $\vec{r}$ and $G(\xi)$ is the probability of finding / charge simultaneously at two points separated by distance $\vec{\xi}$.

When a single electron alone is present, it will give information of the charge spread of the electron. Now

$$\rho(\vec{r}) = e\left\{\psi^*(\vec{r})\,\psi(\vec{r})\right\} - \sigma \qquad (2)$$

where $\psi(\vec{r})$ is the wave function is and

$$\left\{\psi^*\,\psi\right\} = \sum_{\mu=1}^{4} \psi_\mu^*\,\psi_\mu \qquad (3)$$

where ψ is the spinor with four components ψ_μ. σ is the charge density of the unperturbed electrons in the 'sea' which is unobservable and is to hence to be subtracted. In the one electron theory σ is zero. ψ — the wave function is expanded in terms of ϕ_q — the stationary states q of the free electron

$$\psi(\vec{r}) = \sum_q a_q\, \phi_q(\vec{r}) \qquad (4)$$

and

$$\left\{ \phi_q^*(\vec{r}) , \phi_q(\vec{r}) \right\} = 1/V \tag{5}$$

where V is the total volume of the system. The ψ can be regarded as an operator and we have the usual $a_q^* a_q = N_q$ the number operator and $a_q a_q^* = 1 - N_q$

Thus in $G(\xi) = \int \rho(r_2)\, \rho(r_1)\, d\xi$

$$\rho(r_2)\rho(r_1) =$$

$$\sum_q \sum_{q'} \sum_{q''} \sum_{q'''} a_q^* \phi_q^*(r_2) a_{q'} \phi_{q'}(r_2) a_{q''}^* \phi_{q''}^*(r_1) a_{q'''} \phi_{q'''}(r_1)$$

$$- \sigma \left(\sum_q \sum_{q'} a_q^* \phi_q^*(r_2)\ a_{q'} \phi_{q'}(r_2) \right)$$

$$+ \sum_{q''} \sum_{q'''} \left(a_{q''}^* \phi_{q''}^*(r_1)\ a_{q'''} \phi_{q'''}(r_1) \right)$$

$$+ \sigma^2 \tag{6}$$

The only combination that can contribute is

$$a_q^*\ a_q\ a_{q'}^*\ a_{q'} = N_{q'} \tag{7}$$

or

$$a_q^*\ a_{q'}\ a_{q'}^*\ a_q = q(1 - N_{q'})$$

Hence

$$\rho(r_2)\,\rho(r_1) = e^2 \sum_q \sum_{q'} N_q N_{q'} + e^2 \sum_q \sum_{q'} N_q (1 - N_{q'})$$

$$\times \left\{ \phi_q^*(r_2) \phi_{q'}(r_2) \right\} \left\{ \phi_{q'}^*(r_1) \phi_q(r_1) \right\}$$

$$- 2\sigma e \sum_q N_q + \sigma^2 \tag{8}$$

From this expression Weisskopf obtains for a single electron, $\sigma = 0$ and $N_{q_0} = 1$ and all $N_q = 0$ for $q \neq q_0$

$$G(\xi) = e^2 \sum_q \int \phi^*_{q_0}(r_2)\, \phi_q(r_2)\, \phi^*_q(r_1)\, \phi_{q_0}(r_1)\, dr$$

and when evaluated by inserting the electron wave functions

$$G(\xi) = e^2\, \delta(\xi)$$

i.e. in the one electron case $G_1(\xi)$ is a δ-function.

Now (3) for vacuum is $\sigma = \sum_{-q} N_{-q}$ and $N_{+q} = 0$, $N_{-q} = 1$

$$G_1^{vac}(\xi) = e^2 \sum_{+q} \sum_{-q'} \int \phi^*_{-q'}(r_2)\, \phi_{+q}(r_2)\, \phi^*_{+q}(r_1)\, \phi_{-q}(r_1)\, dr \qquad (9)$$

which he attributes to charge fluctuation, i.e. electron pairs produced in the presence of the vacuum. And for an electron in the presence of the vacuum,

$$N_{q_0} = 1 \; ; \; N_{+q} = 0 \; ; \; N_{-q} = 1$$

$$\bar{G}(\xi) = G_{vac+1}(\xi) - G_{vac}(\xi)$$

$$e^2 (\quad) \int \{\phi_{q_0}^*(r_2)\, \phi_q(r_2)\}\{\phi_q^*(r_1)\, \phi_{q_0}(r_1)\}\, dr$$

(10)

and gives rise to the Haukel function.

We now extend this definition by replacing all space coordinations by space time coordinations. and apply it to quantum mechanical many particle systems. This can be done in two ways in the case of fermions-

(a) In the formalism of the electron-field with the sea of negative energy particles and

(b) The electron-positron field with only positive energy particles. We shall assume, that there is no interaction between the particles.

The Dirac Picture

Extending the instant product density functions of Weiss-kopf different times,

$$\rho(k) = \psi^*(k)\,\psi(k) - \sigma \qquad (11)$$

and the field variable $\psi(k)$ at x_k (space-time coord)

$$\psi(k) = \sum_q a_q\, \psi_q(k) \qquad (12)$$

We now define the sequent correlation operator of $\mathcal{F}_n^m$ for m particle and n spacetime points

a) $\mathcal{F}_n^m$ for $n = 2$, $m = 1$

$$\mathcal{F}_2 = \rho(2)\,\rho(1) = \sum_q \sum_{q'} \sum_{q''} \sum_{q'''} \left(a_q^* \phi_q^*(2)\, a_{q'} \phi_{q'}(2)\, a_{q''}^* \phi_{q''}^*(1)\, a_{q'''} \phi_{q'''}(1) \right)$$

$$- \sigma \left(\sum_q \sum_{q'} a_q^* \phi_q^*(2)\, a_{q'} \phi_{q'}(2) + \sum_{q''} \sum_{q'''} a_{q''}^* \phi_{q''}^*(1)\, a_{q'''} \phi_{q'''}(1) \right) \qquad (13)$$

$$+ \sigma^2$$

Since a^* and a satisfy the usual anticommutation relations for creation and annihilation operators only combinations of the form

$$a_q^* a_q \, a_{q'}^* a_{q'} = N_q N_{q'}$$

and

$$a_q^* a_{q'} \, a_{q'}^* a_q = N_q (1 - N_{q'})$$

which are diagonal give contribution to the expectation value of $\mathcal{F}_2$. Thus

$$\rho(2)\rho(1) = \sum_q \sum_{q'} N_q N_{q'} + \sum_q \sum_{q'} N_q (1-N_q) \phi_q^*(2) \phi_{q'}(2) \phi_{q'}^*(1) \phi_q(1) - 2\sigma \sum_q N_q + \sigma^2 \tag{14}$$

In the one electron theory we have $\sigma = 0$ and assuming the single particle to be in the state q_0, we have $N_{q_0} = 1$ and $N_q = 0$ for $q \neq q_0$, then for $t_2 > t_1$,

$$\mathcal{F}_2^{(1)} = \rho(2)\rho(1) = \sum_q \phi_{q_0}^*(2) \phi_q(2) \phi_q^*(1) \phi_{q_0}(1) \tag{15}$$

$\sum_q \phi_q(2) \phi_q^*(1)$ can be easily identified as the unmodified Feynman kernel $K_0(2,1)$. Thus $\rho(2)\rho(1) = \mathcal{F}_2^{(1)}$ is given by

$$\mathcal{F}_2^{(1)} = \phi_{q_0}^*(2) K_0(2,1) \phi_{q_0}(1) \tag{16}$$

In the Dirac hole theory $\sigma = \sum_{-q} N_{-q}$ and for the vacuum all $N_{-q} = 1$ and all $N_{+q} = 0$. So the product density function reduces to

$$\mathcal{F}_2^{vac} = \sum_{+q} \sum_{-q} \phi_{-q}^*(2)\, \phi_{+q}(2)\, \phi_{+q}^*(1)\, \phi_{-q}(1) \tag{17}$$

This can be interpreted as the 'vacuum fluctuation' effect due to the negative energy electrons in the 'sea' when there are no positive energy electrons present. Now for vacuum + one positive energy electron we have all $N_{-q} = 1$, $N_{q_0} = 1$ and $N_{+q} = 0$ for $q \neq q_0$. Thus

$$\mathcal{F}_2^{vac+1} = \sum_{+q} \phi_{q_0}^*(2)\, \phi_{+q}(2)\, \phi_{+q}^*(1)\, \phi_{q_0}(1) + \sum_{-q} \phi_{-q}^*(2) \left[\sum_{+q} \phi_{+q}(2)\, \phi_{+q}^*(1) - \phi_{q_0}(2)\, \phi_{q_0}^*(1) \right] \phi_{-q}(1) \tag{18}$$

and

$$\mathcal{F}_2^{vac+1} - \mathcal{F}_2^{vac} = \sum_{+q} \phi_{q_0}^*(2) \left[\phi_{+q}(2)\, \phi_{+q}^*(1) \right] \phi_{q_0}(1) - \sum_{-q} \phi_{q_0}^*(1) \left[\phi_{-q}(1)\, \phi_{-q}^*(2) \right] \phi_{q_0}(2) \tag{19}$$

Thus we see that the contribution to the density consists of two parts:

(i) starting with ϕ_{q_0} at (1) and going to space-time point (2) and

(ii) starting with ϕ_{q_0} at (2) and going to (1).

Since $t_2 > t_1$, (i) represents the electron going forward in time and the kernel to be used turns out to be

$$\sum_{+q} \phi_{+q}(2)\ \phi_{+q}^{*}(1)$$

with summation over positive energies only

while (ii) represents the electron 'going' backward in time and the kernel for this

$$-\sum_{-q} \phi_{-q}(1)\ \phi_{-q}^{*}(2)$$

with the summation only over negative energies. Thus if we define

$$K_{+}(x';x) = \sum_{+q} \phi_{+q}(x')\,\phi_{+q}^{*}(x) \quad \text{for } t' > t$$

$$= -\sum_{-q} \phi_{-q}(x')\,\phi_{-q}^{*}(x) \quad \text{for } t' < t$$

(20)

we recognize $K_+(x,'x)$ to the the 'Feynman' kernel and the product density can then be rewritten as

$$\mathcal{F}_2^{(1)} = \phi_{q_0}^*(1)\, K_+(1,2)\, \phi_{q_0}(2) + \phi_{q_0}^*(2)\, K_+(2,1)\, \phi_{q_0}(1) \quad (21)$$

This form is also valid for $t_2 < t_1$

(b) $\mathcal{F}_n^m$ for $n = 3$ and $m = 1$

$$\mathcal{F}_3^{(1)} = \rho(3)\,\rho(2)\,\rho(1) \qquad \text{for } t_3 > t_2 > t_1$$

$$= \phi_{q_0}^*(3)\, K_+(3,2)\, K_+(2,1)\, \phi_{q_0}(1) + \phi_{q_0}^*(2)\, K_+(2,1)\, K_+(1,3)\, \phi_{q_0}($$

$$+ \phi_{q_0}^*(1)\, K_+(1,3)\, K_+(3,2)\, \phi_{q_0}(2) + \phi_{q_0}^*(1)\, K_+(1,2)\, K_+(2,3)\, \phi_{q_0}(3$$

$$+ \phi_{q_0}^*(2)\, K_+(2,3)\, K_+(3,1)\, \phi_{q_0}(1) + \phi_{q_0}^*(3)\, K_+(3,1)\, K_+(1,2)\, \phi_{q_0}($$

$$+ \sum_{-q} \phi_{-q}^*(3)\, K_+(3,2)\, \phi_{-q}(2)\, \phi_{q_0}^*(1)\, \phi_{q_0}(1)$$

$$+ \sum_{-q} \phi_{-q}^*(2)\, K_+(2,1)\, \phi_{-q}(1)\, \phi_{q_0}^*(3)\, \phi_{q_0}(3)$$

$$+ \sum_{-q} \phi_{-q}^*(3)\, K_+(3,1)\, \phi_{-q}(1)\, \phi_{q_0}^*(2)\, \phi_{q_0}(2)$$

(22)

The last three terms correspond to the contributions to the density from the negative energy electrons in the sea or the physical vacuum and are generally known as 'vacuum fluctuation terms'. These can be eliminated as should be done in any physically valid theory, by adopting the subtraction procedure for each pair of points when the particle q_0 is at the third. This procedure can be generalized when there are more than three points.

(c) $\mathcal{F}_3^{1,2} = \rho(3)\rho(2)\rho(1)$ for $t_3 > t_2 > t_1$ is given by

$$\mathcal{F}_3^2 = \phi^*_{q_0}(3) K_+(3,2) K_+(2,1) \phi_{q_0}(1) + \phi^*_{q_0}(2) K_+(2,1) K_+(1,3) \phi_{q_0}(3)$$

$$+ \phi^*_{q_0}(1) K_+(1,3) K_+(3,2) \phi_{q_0}(2) + \phi^*_{q_0}(1) K_+(1,2) K_+(2,3) \phi_{q_0}(3)$$

$$+ \phi^*_{q_0}(2) K_+(2,3) K_+(3,1) \phi_{q_0}(1) + \phi^*_{q_0}(3) K_+(3,1) K_+(1,2) \phi_{q_0}(2)$$

$$+ \phi^*_{q_1}(3) K_+(3,2) K_+(2,1) \phi_{q_1}(1) + \phi^*_{q_1}(2) K_+(2,1) K_+(1,3) \phi_{q_1}(3)$$

$$+ \phi^*_{q_1}(1) K_+(1,3) K_+(3,2) \phi_{q_1}(1) + \phi^*_{q_1}(1) K_+(1,2) K_+(2,3) \phi_{q_1}(3)$$

$$+ \phi^*_{q_1}(2) K_+(2,3) K_+(3,1) \phi_{q_1}(1) + \phi^*_{q_1}(3) K_+(3,1) K_+(1,2) \phi_{q_1}(2)$$

$$+ \phi^*_{q_0}(3) K_+(3,2) \phi_{q_0}(2) \phi^*_{q_1}(1) \phi_{q_1}(1)$$

$$+ \phi^*_{q_1}(3) K_+(3,2) \phi_{q_1}(2) \phi^*_{q_1}(1) \phi_{q_0}(1)$$

$$+ \phi^*_{q_0}(2) K_+(2,1) \phi_{q_0}(1) \phi^*_{q_1}(3) \phi_{q_1}(3)$$

$$+ \phi^*_{q_1}(2) K_+(2,1) \phi_{q_1}(1) \phi^*_{q_0}(3) \phi_{q_1}(3)$$

$$+\phi_{q_0}^{*}(3)\,K_+(3,1)\,\phi_{q_0}(1)\,\phi_{q_1}^{*}(2)\,\phi_{q_1}(2)+\phi_{q_1}^{*}(3)\,K_+(3,1)\,\phi_{q_1}(1)\,\phi_{q_0}^{*}(2)\,\phi_{q_0}(2)$$

$$+\phi_{q_0}^{*}(2)\,K_+(2,3)\,\phi_{q_0}(3)\,\phi_{q_1}^{*}(1)\,\phi_{q_1}(1)+\phi_{q_1}^{*}(2)\,K_+(2,3)\,\phi_{q_1}(3)\,\phi_{q_0}^{*}(1)\,\phi_{q_0}(1)$$

$$+\phi_{q_0}^{*}(1)\,K_+(1,2)\,\phi_{q_0}(2)\,\phi_{q_1}^{*}(3)\,\phi_{q_1}(3)+\phi_{q_1}^{*}(1)\,K_+(1,2)\,\phi_{q_1}(2)\,\phi_{q_0}^{*}(3)\,\phi_{q_0}(3)$$

$$+\phi_{q_0}^{*}(1)\,K_+(1,3)\,\phi_{q_0}(3)\,\phi_{q_1}^{*}(2)\,\phi_{q_1}(2)+\phi_{q_1}^{*}(1)\,K_+(1,3)\,\phi_{q_1}(3)\,\phi_{q_0}^{*}(2)\,\phi_{q_0}(2)$$

$$-\phi_{q_0}^{*}(3)\,K(3,1)\,\phi_{q_0}(1)\,\phi_{q_0}^{*}(2)\,\phi_{q_0}(2)-\phi_{q_1}^{*}(3)\,K_+(3,1)\,\phi_{q_1}(1)\,\phi_{q_1}^{*}(2)\,\phi_{q_1}(2)$$

$$+\ \text{vacuum fluctuation terms.}$$

(23)

Thus we have the six diagrams corresponding to those in (b) for the two electrons of energy q_0 and q_1 and in addition we also have the terms corresponding to the existence of the particle q_0 (q_1) at one of the points while the other two points are connected by the kernel which takes q_1 (q_0) from one to the other.

(31) $\mathcal{F}_n^m$ for $m=2$, $n=4$.

This is merely an extension of the previous results, and no new principle is involved. We shall however group those terms which lead to the antisymmetric Feynman kernel for a two particle state, i.e.

$$\phi_{q_0}^{*}(4)\,\phi_{q_1}^{*}(3)\,K_+(3,2)\,K_+(4,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)$$

$$+\,\phi_{q_0}^{*}(3)\,\phi_{q_1}^{*}(4)\,K_+(4,2)\,K_+(3,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)$$

$$=\phi_{q_0}^{*}(4)\,\phi_{q_1}^{*}(3)\,K_+(3,2)\,K_+(4,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)$$

$$-\left[-\phi_{q_0}^{*}(3)\,\phi_{q_1}^{*}(4)\,K_+(4,2)\,K_+(3,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)\right]$$

(24)

If we require the final state to be antisymmetric, we find that the kernel $K_+(4, 3; 2, 1)$ is also antisymmetric (i.e.)

$$K_+(4,3;2,1) = K_+(4,1)\,K_+(3,2) - K_+(4,2)\,K_+(3,1) \tag{25}$$

The extension to more than two particles follows immediately and the n particle kernel can be expressed in the form of a Slater determinant of the single particle kernels.

It is relevant to draw attention to some observations made by Feynman in this connection. In the fundamental paper in which he introduced the kernel formalism he made a special reference to the connection between the minus sign occurring before the sum over the negative energy states in the definition of $K_+(2, 1)$ (when $t_2 < t_1$) and the Pauli principle.

Later in introducing the kernel corresponding to two particles, the antisymmetrical form was used by calling into aid the Pauli principle. Therefore it was clear that the existence of the minus sign in the negative energy part of K_+ while consistent with the Pauli principle, is not equal with it. In spite of this Feynman seems to have attempted to deduce the Pauli principle from the single particle kernel itself by studying the probability for an electron to go from (1) to (2) with any number of virtual pairs occurring and proving that an inconsistency occurs unless the Pauli principle is invoked. Our considerations however lead to the conclusion that the kernel of single particle, two particles and for can be deduced by once assuming the Pauli principle and therefore it is not possible to speak of the single particle kernel itself as implying the principle in its entirely. Or in other words, the Pauli principle implies at once that the kernel for the single particle is K_+ and for two symmetric.

Electron-positron field

The field variable at the space-time point x_k is

$$\psi(x_k) = \sum_q \left[a_q u_q e^{-iq\cdot x_k} + b_q^{+} v_q e^{+iq\cdot x_k} \right] \qquad (2)$$

where a_q and b_q^{+} are the annihilation operator of an electron and the creation operator of a positron of four-momentum q respectively attached to u and v the

corresponding spinors. The bilinear term $\bar{\psi}\psi$ at a space-time point x_k would read

$$\sum_{q_i q_i'} \left[\hat{a}_{q_1}^{+} \hat{a}_{q_1'}(k) + \hat{b}_{q_2} \hat{a}_{q_2'}(k) + \hat{a}_{q_3}^{+} \hat{b}_{q_3'}^{+}(k) + b_{q_4} b_{q_4'}(k) \right] \tag{26}$$

where the symbol $\hat{}$ indicates that the operators are accompanied by their corresponding wave functions.

It has been pointed out by Ramakrishnan that in bilinear fermion operators corresponding to the same space-time point it is not necessary that the ordering should be prescribed by $\bar{\psi}\psi$ In fact, on the basis of physical arguments, it has been shown by him that the ordering of the individual operators in the above four terms should be

$$\sum_{q_i q_i'} \left[\hat{a}_{q_1}^{+} \hat{a}_{q_1'}(k) + \hat{b}_{q_2} \hat{a}_{q_2'}(k) + \hat{b}_{q_3}^{+} \hat{a}_{q_3'}(k) + \hat{b}_{q_4}^{+} \hat{b}_{q_4'}(k) \right]$$

$$= \sum_{\langle q \rangle} \sum_{\alpha} [k] \tag{27}$$

where the creation operator of the positron b_q^{+} always occurs to the left of either a_q or b_q . The bracket $[k]$ contains any one of the four terms and sum over α indicates a sum over all the terms. It is this ordering that has been shown to lead directly to the reduction of an S-matrix expansion to the Feynman formalism.

We shall now apply the above prescription to determine the density correlation function $\mathcal{F}_n^m$ for the following cases:-

a) One electron and two space-time points $m = 1, n = 2$

b) One electron and three space-time points $m = 1, n = 3$

c) Two electrons and three space-time points $m = 2, n = 3$

d) Two electrons and four space-time points $m = 2, n = 4$

(a)

$$\mathcal{F}_2' = \rho(2)\,\rho(1) = \sum_{\langle q \rangle} \sum_{\alpha} [2]\,[1] \qquad \text{where } t_2 > t_1$$

The initial system consists of an electron of momentum q_0 and hence one of the brackets has to contain the annihilation operator $\hat{a}_{q_0}$. Associated with this $\hat{a}_{q_0}$ we can have either an $\hat{a}^{+}_{q_0'}$ or a $\hat{b}_{q_1}$. If it is an $\hat{a}^{+}_{q_0'}$, this will be annihilated at a later time (i.e. at t_2) by an $\hat{a}_{q_1}$, which has to contain an $\hat{a}^{+}_{q_0}(2)$ since the final system is an electron of momentum q_0. If on the other hand $\hat{a}_{q_0}$ has a $\hat{b}_{q_1}$ associated with it, a positron should have existed before it could be annihilated so that $\hat{b}^{+}_{q_1}$ should be associated with space-time point (1). Thus the only possible combinations are

$$\sum_{q'} \hat{a}^{+}_{q_0}(2)\,\hat{a}_{q'}(2)\,\hat{a}^{+}_{q'}(1)\,\hat{a}_{q_0}(1)$$

and

$$\sum_{q'} \hat{b}_{q'}(2)\ \hat{a}_{q_0}(2)\ \hat{b}^{\dagger}_{q'}(1)\ \hat{a}^{\dagger}_{q_0}(1)$$

The second term can be rewritten as

$$-\hat{a}^{\dagger}_{q_0}(1)\ \hat{b}^{\dagger}_{q'}(1)\ \hat{b}_{q'}(2)\ \hat{a}_{q_0}(2)$$

the negative sign resulting due to the anti-commutation of the operators in the rearrangement.

Thus we see that the contribution to the density consists of two parts similar to that in the previous section, and if we define

$$K_+(x', x) = \sum_{+q'} \hat{a}_{q'}(x)\ \hat{a}^{\dagger}_{q'}(x) \quad \text{for } t' > t$$

$$= -\sum_{-q'} \hat{a}_{-q'}(x')\ a^{\dagger}_{-q'}(x) \quad \text{for } t' < t$$

we recognise $K_+(x', x)$ to be the Feynman kernel and the product density can then be rewritten as before for $t_2 > t_1$ as

$$\mathcal{F}_2' = \hat{a}_{q_0}(2)\, K_+(2,1)\, \hat{a}_{q_0}(1) + \hat{a}_{q_0}(1)\, K_+(1,2)\, \hat{a}_{q_0}(2)$$

This form is also valid for the case $t_2 < t_1$.

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QUANTISATION OF THE ELECTROMAGNETIC FIELD

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QUANTISATION OF THE ELECTROMAGNETIC FIELD

Lecture I

Though the oldest known elementary particle field, the quantisation of the electromagnetic field presents difficulties. The set of Maxwell's equations without external sources can be written in terms of the antisymmetric electromagnetic tensor, in the relativistic form

$$\frac{\partial F_{\mu\nu}(x)}{\partial x_\mu} = 0 \qquad (1)$$

where the components of $F_{\mu\nu}$ are given in terms of the electric (E) and magnetic (M) field strengths by

$$F_{4k} = -F_{k4} = i\vec{E}_k \; ; \quad k = 1,2,3$$

$$F_{12} = -F_{21} = H_3 \; , \qquad \text{(cyclically)}$$

Equation (1) can also be derived from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F_{\mu\nu} = \frac{1}{2}(E^2 - H^2) \qquad (2)$$

if we treat the potentials defined by

$$F_{\mu\nu}(x) = \frac{\partial A_\nu(x)}{\partial x_\mu} - \frac{\partial A_\mu(x)}{\partial x_\nu}$$

as the dynamical variables (and not the field strengths) and vary with respect to them. The resulting equation

$$\Box A_\mu(x) - \frac{\partial^2 A_\nu(x)}{\partial x_\mu \partial x_\nu} = 0 \qquad (3)$$

is obviously invariant under the gauge transformations

$$A_\mu(x) \longrightarrow A_\mu(x) + \frac{\partial \Lambda(x)}{\partial x_\mu} \qquad (4)$$

where $\Lambda(x)$ is a scalar function. A restriction is put on this transformation to get the equation of motion in the form

$$\Box A_\mu(x) = 0 \qquad (5)$$

This is done by imposing the Lorentz condition

$$\frac{\partial A_\mu(x)}{\partial x_\mu} = 0$$

and choosing the gauge function $\Lambda(x)$ such that $\Box \Lambda(x) = 0$

If we start from the Lagrangian (2) and compute the canomical momentum function conjugate to $A_\mu(x)$ we get

$$\pi_\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu(x)} = i F_{4\mu}(x)$$

which means that the momentum conjugate to $A_4(x)$ vanishes identically. ($F_{44} = 0$). Thus we cannot have the usual commutation relation for the fourth component of the field and its momentum conjugate. Fermi, therefore, modified the Lagrangian to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} \frac{\partial A_\mu(x)}{\partial x_\mu} \frac{\partial A_\nu(x)}{\partial x_\nu} \qquad (2a)$$

for which $\pi_k(x) = i F_{4k}(x)$, $\pi_4(x) = i \partial A_\nu(x)/\partial x_\nu$

Therefore we can write out the canomical commutation relations which hold only for equal times ($x_0 = x_0'$) for all the 4 components of the field (For the present we shall neglect the Lorentz condition for a reason to be mentioned later). These are

$$[A_\mu(x), A_\nu(x')] = [A_k(x), \pi_4(x')]$$
$$= [\pi_k(x), \pi_\mu(x')] = 0$$

which can also be written as

$$[A_\mu(x), A_\nu(x')] = \left[\frac{\partial A_\mu(x)}{\partial x_0}, \frac{\partial A_\nu(x')}{\partial x_0'}\right] = 0$$

Also

$$[A_\mu(x), \pi_k(x')] = i \delta_{\mu k} \delta(\vec{x} - \vec{x}')$$
$$[A_4(x), \pi_4(x')] = i \delta(\vec{x} - \vec{x}')$$

which can be combined into

$$\left[\frac{\partial A_\mu(x)}{\partial x_0}, A_\nu(x')\right] = -i \delta_{\mu\nu} \delta(\vec{x} - \vec{x}')$$

We now make a Fourier decomposition of the electromagnetic field:

$$A_\mu(x) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} A_\mu(k)$$

where k is the momentum which gives, from the equation of motion (5) that $k^2 = \vec{k}^2 - k_0^2 = 0$ which has two solutions $k_0 = \pm \omega$; $\omega = +\sqrt{\vec{k}^2}$. Thus we may write

$$A_\mu(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \left[e^{ikx} A_\mu(\vec{k}) + e^{-ikx} A_\mu^*(\vec{k}) \right]$$

Since $A_k(x)$ is a hermitian and $A_4(x) (= iA_0(x))$ is an antihermitian operator, it follows that $A_k(\vec{k})$ and $iA_4(\vec{k})$ are hermitian conjugates of $A_k^*(\vec{k})$ and $iA_4^*(\vec{k})$. Also since $A_\mu(x)$ is a vector, there are for every $\vec{k}$ four mutually independent polarisation directions possible described by the components of the vector $e_\mu^{(\lambda)}$, $\lambda = 1 \cdots 4$. To get the most general solution of the equation of motion (5) we must sum over the four possible polarisation directions.

$$A_\mu(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \sum_{\lambda=1}^{4} \frac{e_\mu^{(\lambda)}}{\sqrt{2\omega}} \left[e^{ikx} a^{(\lambda)}(\vec{k}) + e^{-ikx} a^{*(\lambda)}(\vec{k}) \right]$$

with $e_\mu^{(\lambda)} e_\mu^{(\lambda')} = \delta_{\lambda\lambda'}$

e_μ^λ can also be thought of as the tensor effecting the transition from $a^\lambda(\vec{k})$ and $a^{*(\lambda)}(\vec{k})$ to $A_\mu(x)$ _7.

We can choose the polarisation vectors s ch that $e^{(3)}$ is along the longitudinal direction so that

$$e_\ell^{(3)} = \frac{k_\ell}{\omega} \quad \text{and} \quad e_\ell^{(4)} = e_4^{(1)} = e_4^{(2)} = e_4^{(3)} = e_\ell^{(1)} \cdot k_\ell = e_\ell^{(2)} \cdot k_\ell = 0 \,, \; e_4^{(4)} = 1$$

Then $\lambda = 1, 2$ will correspond to be transverse photons and $\lambda = 4$ to the scalar (time-like) photon.

We can now derive the commutation relation and interpret the operators $a^{(\lambda)}$ and $a^{*(\lambda)}$. We have

$$[A_\mu(x), A_\nu(x')]_{x_0 = x_0'} = 0$$

$$= \frac{1}{V} \sum_{\vec{k}} \sum_{\vec{k}'} \frac{e_\mu^{(\lambda)} e_\nu^{(\lambda')}}{2\sqrt{\omega\omega'}} \{ e^{i(\vec{k}\cdot\vec{x} + \vec{k}'\cdot\vec{x}') - i(\omega+\omega')x_0}$$

$$\times [a^{\lambda}(\vec{k}), a^{(\lambda')}(\vec{k}')] + e^{-i(\vec{k}\cdot\vec{x} + \vec{k}'\cdot\vec{x}') + i(\omega+\omega')x_0}$$

$$\times [a^{*(\lambda)}(\vec{k}), a^{*(\lambda')}(\vec{k}')] + e^{i(\vec{k}\cdot\vec{x} - \vec{k}'\cdot\vec{x}') - i(\omega-\omega')x_0}$$

$$\times [a^{(\lambda)}(\vec{k}), a^{*(\lambda')}(\vec{k}')] + e^{-i(\vec{k}\cdot\vec{x} - \vec{k}'\cdot\vec{x}') + i(\omega-\omega')x_0}$$

$$\times [a^{*(\lambda)}(\vec{k}), a^{(\lambda')}(\vec{k}')] \} \qquad (9)$$

Since ω and ω' are positive, there is no way of clubbing the first two terms so that for a solution we must have

$$[a^{(\lambda)}(\vec{k}), a^{(\lambda')}(\vec{k}')] = [a^{*(\lambda)}(\vec{k}), a^{*(\lambda')}(\vec{k}')] = 0 \qquad (10a)$$

But the third and fourth terms can be combined if we set $\vec{k} = \vec{k}'$ so that we can write

$$[a^{(\lambda)}(\vec{k}), a^{*(\lambda')}(\vec{k}')] = c^{(\lambda\lambda')}(\vec{k})\, \delta_{\vec{k}\vec{k}'}$$

with $c^{(\lambda\lambda')}(\vec{k}) = c^{(\lambda'\lambda)}(-\vec{k})$

The value of $c^{(\lambda'\lambda)}$ is obtained by substituting (10b) in (6b) which gives

$$-i\,\delta_{\mu\nu}\,\delta(\vec{x} - \vec{x}') = \frac{1}{V} \sum_{\vec{k}} \sum_{\lambda,\lambda'} \frac{e_\mu^{(\lambda)} e_\nu^{(\lambda')}}{2\omega}$$

$$\times (-i\omega)\, c^{(\lambda\lambda')}(\vec{k})\, 2\, e^{i\vec{k}\cdot(\vec{x} - \vec{x}')}$$

or $\qquad c^{(\lambda\lambda')}(\vec{k}) = \delta_{\lambda\lambda'}$

Therefore $\left[a^{(\lambda)}(\vec{k}), a^{*(\lambda')}(\vec{k}')\right] = \delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}$ (10c)

To obtain an interpretation for a and a* we have first to compute the Hamiltonian of the field which, using the Lagrangian (2a) is

$$H = \frac{1}{2}\int d^3x \left[\frac{\partial A_\mu(x)}{\partial x_0}\frac{\partial A_\mu(x)}{\partial x_0} + \frac{\partial A_\mu(x)}{\partial x_k}\frac{\partial A_\mu(x)}{\partial x_k}\right]$$

$$= -\frac{1}{4V}\sum_{\vec{k}}\sum_{\vec{k}'}\sum_{\lambda,\lambda'}\int\frac{d^3x}{\sqrt{\omega\omega'}}\left[e^{ikx(\lambda)}a^{(\lambda)}(\vec{k}) - e^{-ikx(\lambda)}a^{*(\lambda)}(\vec{k})\right]$$

$$\times\left[e^{ik'x}a^{(\lambda')}(\vec{k}') - e^{-ik'x}a^{*(\lambda')}(\vec{k}')\right]$$

$$\times\,\delta_{\lambda\lambda'}(\omega\omega' + \vec{k}\cdot\vec{k}') \qquad (11)$$

which reduces to the expression

$$\frac{1}{2}\sum_{\vec{k},\lambda}\omega\left\{a^{(\lambda)}(\vec{k}), a^{*(\lambda)}(\vec{k})\right\}$$

on using the commutation relations)10). The curly braces denote the anticommutator. If we now introduce the hermitian operators,

$$q^{(\lambda)} = \frac{1}{\sqrt{2\omega}}\left(a^{(\lambda)} + a^{*(\lambda)}\right), \quad \lambda = 1,2,3$$

$$q^{(4)} = \frac{1}{\sqrt{2\omega}}\left(a^{(4)} - a^{*(4)}\right)$$

$$p^{(\lambda)} = -i\sqrt{\omega/2}\left(a^{(\lambda)} - a^{*(\lambda)}\right), \quad \lambda = 1,2,3$$

$$p^{(4)} = i\sqrt{\omega/2}\left(a^{(4)} + a^{*(4)}\right) \qquad (12)$$

then using (10) we see that p and q satisfy the usual canonical commutation relations

$$\left[p^{(\lambda)}, q^{(\lambda')}\right] = -i\delta_{\lambda\lambda'} \text{ for all } \lambda,\lambda'$$

Also $$H = \frac{1}{2}\sum_{\vec{k}}\left\{\sum_{\lambda=1}^{3}\left(p^{(\lambda)2} + \omega^2 q^{(\lambda)2}\right) - \left(p^{(4)2} + \omega^2 q^{(4)2}\right)\right\} \qquad (13)$$

i.e. the electromagnetic field can be considered as composed of four independent harmonic oscillators the energy of the system being given by

$$E = \sum_{\vec{k}}\left\{\sum_{\lambda=1}^{3} n^{(\lambda)}(\vec{k}) - n^{(4)}(\vec{k})\right\}\omega \qquad (14)$$

(omitting the zero point energy). n represents the number of

oscillators of each type.

We see that depending on the number of scalar photons, present in the system of energy can be even negative which is not admissible in field theory.

If we designate by $|n\rangle$ the state containing n particles of given k (momentum and polarisation), we have

$$H a^* |n\rangle = [H, a^*] |n\rangle = a^* H |n\rangle$$
$$= \begin{cases} (1+n)\omega a^* |n\rangle & \text{for } \lambda \neq 4 \\ (1-n)\omega a^* |n\rangle & \text{for } \lambda = 4 \end{cases}$$

Thus $a^* |n\rangle$ is an eigenvector of H with eigenvalue $(n+1)$ for $\lambda = 1, 2, 3$ and $-(n-1)$ for $\lambda = 4$. So we can generate a state containing an arbitrary number of these particles by operating a^* (for $\lambda = 1, 2, 3$) and $a^{(4)}$ on the vacuum. More specifically

$$\prod_k \prod_{\lambda=1}^{4} |n^{(\lambda)}(k)\rangle = \prod_k \prod_{\lambda=1}^{3} \frac{[a^{*(\lambda)}(k)]^{n^{(\lambda)}(k)}}{\sqrt{n^{(\lambda)}(k)}} \times \frac{[a^{(4)}(k)]^{n^{(4)}(k)}}{\sqrt{n^{(4)}(k)}} |0\rangle \qquad (15)$$

The factors in the denominator arise from normalizing the state vector. From (15) we see that

$$\langle n | a^{(\lambda)} | n+1 \rangle = \langle n+1 | a^{*(\lambda)} | n \rangle = \sqrt{n+1} \quad \text{for } \lambda \neq 4 \qquad (16a)$$

$$\langle n+1 | a^{(4)} | n \rangle = -\langle n | a^{*(4)} | n+1 \rangle = \sqrt{n+1} \qquad (16b)$$

Thus we can interpret $a^{(\lambda)}$ and $a^{*(4)}$ as annihilation operators and $a^{*(\lambda)}$ and $a^{(4)}$ as creation operators.

We shall now mention the difficulty in quantising the electromatnetic field, viz., the difficulty of including the Lorentz condition as an operator equation. For this would contradict the

commutation relation

$$[A_4(x), \pi_4(x')]_{x_0=x_0'} = i\left[A_4(x), \frac{\partial A_\nu(x')}{\partial x_\nu'}\right]_{x_0=x_0'}$$

$$= i\,\delta(\vec{x}-\vec{x}')$$

instead of being zero, as required by the Lorentz condition. That is why we neglected (it) at the beginning. In the subsequent lectures we shall see how this difficulty has been overcome and how the quantisation of the Lorentz condition also removes the trouble mentioned earlier about the energy being not positive definite.

For completeness, we shall also give the commutation relations for unequal times which can be derived from the commutation relations (10).

$$[A_\mu(x), A_\nu(x')] = \frac{1}{V}\sum_{\vec{k},\vec{k}'}\sum_{\lambda,\lambda'}\frac{e_\mu^{(\lambda)} e_\nu^{(\lambda')}}{2\sqrt{\omega\omega'}}$$

$$\times\left[\left(e^{ikx}a^{(\lambda)}(\vec{k}) + e^{-ikx}a^{*(\lambda)}(\vec{k})\right)\right.$$

$$\left.\left(e^{ik'x'}a^{(\lambda')}(\vec{k}') + e^{-ik'x'}a^{*(\lambda')}(\vec{k}')\right)\right]$$

$$= \frac{1}{V}\sum_{\vec{k},\lambda}\frac{e_\mu^{(\lambda)} e_\nu^{(\lambda)}}{2\omega}\left[e^{ik(x-x')} - e^{-ik(x-x')}\right]$$

$$= \frac{\delta_{\mu\nu}}{(2\pi)^3}\int\frac{d^3k}{2\omega}\left[e^{ik(x-x')} - e^{-ik(x-x')}\right] \qquad (17)$$

This can be written in the form

$$[A_\mu(x), A_\nu(x')] = -i\,\delta_{\mu\nu}\,D(x'-x)$$

$$= -\frac{\delta_{\mu\nu}}{(2\pi)^3}\int dk\, e^{ik(x-x')}\,\delta(k^2)\,\epsilon(k) \qquad (18)$$

with

$$\epsilon(k) = k_0/|k_0|$$

LECTURE II

To overcome the difficulty regarding inclusion of the Lorentz condition as an operator equation, Fermi suggested that it is not necessary that all the classical Maxwell equations should correspond to operator-identities in the quantized theory. It is enough if the expectation values of the field operators satisfy these equations for every physically realisable state. Fermi therefore suggested instead of the Lorentz condition, the subsidiary condition

$$\frac{\partial A_\nu(x)}{\partial x_\nu} |\psi\rangle = 0 \qquad (1)$$

which in momentum language will correspond to the two equations

$$[a^{(3)}(\vec{k}) + i\, a^{(4)}(\vec{k})]\,|\psi\rangle = 0 \qquad (2a)$$

$$[a^{*(3)}(\vec{k}) + i\, a^{*(4)}(\vec{k})]\,|\psi\rangle = 0 \qquad (2b)$$

as can be seen by using the expansion for $A_\nu(x)$ and choosing the polarisation vectors as in the previous lecture.

Thus the Fermi subsidiary condition is a prescription for only the longitudinal and scalar photons, leaving the transverse degrees of freedom unaffected. Now any physical state vector containing the photons can be written

$$|\psi\rangle = |\psi_T\rangle \prod_{\vec{k}} |\Phi_{\vec{k}}\rangle \qquad (3)$$

where $|\psi_T\rangle$ contains only the transversal photons and $|\Phi_k\rangle$ the longitudinal and scalar photons. We can write the latter as

$$|\Phi_k\rangle = \sum_{n^{(3)}, n^{(4)}} \alpha_{n^{(3)} n^{(4)}} |n^{(3)}, n^{(4)}\rangle$$

$$= \sum_{n^{(3)}, n^{(4)}} \alpha_{n^{(3)} n^{(4)}} [a^{*(3)}]^{n^{(3)}} [a^{(4)}]^{n^{(4)}} \qquad (4)$$

remembering that $a^{(4)}$ is the creation operator for the scalar photon. The constant $\alpha_{n^{(3)} n^{(4)}}$ is to be determined by

substituting (3) in (2) which gives the two equations

$$\sum \alpha_{n^{(3)} n^{(4)}} \left[\sqrt{n^{(3)}} \, | n^{(3)}-1, n^{(4)} \rangle + c \sqrt{n^{(4)}+1} \, | n^{(3)}, n^{(4)}+1 \rangle \right] = 0$$

$$\sum \alpha_{n^{(3)} n^{(4)}} \left[\sqrt{n^{(3)}+1} \, | n^{(3)}+1, n^{(4)} \rangle - c \sqrt{n^{(4)}} \, | n^{(3)}, n^{(4)}-1 \rangle \right] = 0 \qquad (5)$$

It is seen that (5) can be solved with $\alpha_{n^{(3)} n^{(4)}} = c \left(\delta_{n^{(3)}, n^{(4)}-1} \right) (-1)^{n^{(4)}}$ where c is a constant to be determined by the normalisation condition $\langle \Phi_h | \Phi_h \rangle = 1$

We thus see that the Fermi subsidiary condition (1) or (2) eliminates the longitudinal and scalar degrees of freedom and leaves only the transverse photons in a physically realisable state. We also see that the condition (3) which gives $n^{(3)} = n^{(4)}$ makes the energy

$$E = \sum_{\bar{k}} \left\{ \sum_{\lambda=1}^{3} n^{(\lambda)}(\bar{k}) - n^{(4)}(\bar{k}) \right\} \omega$$

positive definite since the energies of the longitudinal and scalar photons cancel. Thus the quantisation of the Lorentz condition has removed the positive nondefiniteness of the electromagnetic field without the Lorentz condition.

Though the Fermi method of quantisation seems to achieve the purpose, there are inherent difficulties in it. Thus,

$$\langle \psi | \left[\frac{\partial A_\mu(x)}{\partial x_\mu}, A_\nu(x') \right] | \psi \rangle = -i \frac{\partial}{\partial x_\nu} D(x'-x) \neq 0 \qquad (7)$$

instead of being zero as we should expect from (1). This is because the state vector $|\Phi_h\rangle$ (equation 4) cannot be normalized and hence (1) cannot be satisfied with a finite c. Thus from (7), we obtain instead of (7) an indefinite expression of the form $0 \times \infty$ which has therefore to be defined by a limiting procedure. Coester

gave one such procedure. The photon is given a small mass, μ so that it obeys the equation of motion

$$(\Box - \mu^2)\ A_\mu(x) = 0 \qquad (8)$$

Introduce a scalar field B(x) and a new vector field $U_\mu(x)$ defined by

$$B(x) = -\frac{1}{\mu}\frac{\partial A_\mu(x)}{\partial x_\mu}\ ;\ U_\mu(x) = A_\mu(x) + \frac{1}{\mu}\frac{\partial B(x)}{\partial x_\mu} \qquad (9)$$

which satisfy (8). $U_\mu(x)$ also satisfies the operator equation

$$\frac{\partial U_\mu(x)}{\partial x_\mu} = -\mu B(x) + \frac{1}{\mu}\Box B(x) = 0 \qquad (10)$$

so that only three of its four components are independent. Hence we can use the first three components $U_k(x)$ and $B(x)$ as the independent components. Since B (x) represents a scalar field with mass μ

$$[B(x), B(x')] = -i\Delta(x'-x) = -\frac{1}{\mu^2}\frac{\partial^2}{\partial x_\mu \partial x_\nu}[A_\mu(x), A_\nu(x')] \qquad (11)$$

where $\Delta(x'-x)$ is the singular function for a massive boson just as D (x' - x) is for the photon. From (11) and (9) we have

$$[U_\mu(x), B(x')] = -\frac{1}{\mu}\left[A_\mu(x), \frac{\partial A_\nu(x')}{\partial x_\nu'}\right] + \frac{i}{\mu}\frac{\partial}{\partial x_\mu}\Delta(x'-x) = 0 \qquad (12)$$

from which it follows that

$$[U_\mu(x), U_\nu(x')] = [U_\mu(x), A_\nu(x')] = -i\left(\delta_{\mu\nu} - \frac{1}{\mu^2}\frac{\partial^2}{\partial x_\mu \partial x_\nu}\right)\Delta(x'-x) \qquad (13)$$

From (11), (12), and (13) and using the Fourier decompositions

$$U_k(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega}}\Big\{e^{ikx}\Big[\sum_{\lambda=1}^{2} e_k^{(\lambda)} u^{(\lambda)}(\vec{k}) + \frac{\omega}{\mu} e_k^{(3)} u^{(3)}(\vec{k})\Big] + e^{-ikx}\Big[\sum_{\lambda=1}^{2} e_k^{(\lambda)} u^{*(\lambda)}(\vec{k}) + \frac{\omega}{\mu} e_k^{(3)} u^{*(3)}(\vec{k})\Big]\Big\}$$

$$B(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega}}\left[e^{ikx} b(\vec{k}) + e^{-ikx} b^*(\vec{k})\right]$$

We have

$$[u^{\lambda}(\vec{k}),\ u^{*(\lambda)}(\vec{k}')] = \delta\lambda\lambda'\delta\vec{k}\vec{k}'$$
$$[b(\vec{k}),\ b^{*}(\vec{k}')] = -\delta\vec{k}\vec{k}' \qquad (15)$$

all other commutators vanishing. The roles of the annihilation and creation operators for the B-field are seen to be interchanged, just as was the case for $A_4(x)$

Now we make the postulate that for a physically realisable state vector does not contain a B-particle, so that

$$-\mu \langle \psi | B(x) | \psi \rangle = \langle \psi | \frac{\partial A_{\nu}(x)}{\partial x_{\nu}} | \psi \rangle = 0 \qquad (16)$$

Also

$$\lim_{\mu \to 0} \langle \psi | B(x) B(x') | \psi \rangle \mu^2$$
$$= \lim_{\mu \to 0} \langle \psi | \frac{\partial A_{\mu}(x)}{\partial x_{\mu}} \frac{\partial A_{\nu}(x')}{\partial x_{\nu'}} | \psi \rangle$$
$$= i \lim_{\mu \to 0} \mu^2 \Delta^{(+)}(x' - x) = 0 \qquad (17)$$

etc. for any product of B operators. In other words,

$b^{*}(k)|\psi\rangle = 0$: Thse are statements of the modified Lorentz condition. Let us also require that the physical state does not contain longitudinal particles (*) i.e. $U^{(3)}(\vec{k})|\psi\rangle = 0$

From (9) and (14), we have, on using the Fourier decomposition of $A_{\mu}(x)$,

$$b(\vec{k}) = -\frac{i}{\mu}\left(|\vec{k}|\, a^{(3)}(\vec{k}) + (\omega\, a^{(4)}(\vec{k}))\right)$$
$$u^{(\lambda)}(\vec{k}) = a^{(\lambda)}(\vec{k}); \quad \lambda = 1, 2$$
$$\frac{\omega}{\mu} u^{(3)}(\vec{k}) = a^{(3)}(\vec{k}) + \frac{i}{\mu}|\vec{k}|\, b(\vec{k})$$
$$= \frac{\omega}{\mu^2}\left[\omega\, a^{(3)}(\vec{k}) + i|\vec{k}|\, a^{(4)}(\vec{k})\right]$$

(10)

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* There is no prima facie reason for this requirement but investigations (by Coester and Belinfante) have shown that even if they exist, they cannot interact with the Dirac field, in the limit $\mu \to 0$, and hence can be dropped from the beginning.

Therefore the non-existence of B and $U^{(3)}$ fields can be represented by

$$\left[\omega\, a^{(3)} + i\,|\vec{k}|\, a^{(4)}\right]|\Phi_{\vec{k}}\rangle = 0$$
$$\left[|\vec{k}|\, a^{*(3)} + i\,\omega\, a^{*(4)}\right]|\Phi_{\vec{k}}\rangle = 0 \qquad (19)$$

which in the limit $\mu \to 0$ (when $|\vec{k}| = \omega$) we recognize to be the Fermi subsidiary conditions (1). But the difference is that when the mass is different from zero Φ_k can be normalized with a finite norm, and can be written in the form

$$|\Phi_{\vec{k}}\rangle = \frac{\mu}{\omega} \sum_{n^{(3)}, n^{(4)}} \left(-i\,\frac{|\vec{k}|}{\omega}\right)^{n^{(3)}} \delta_{n^{(3)}, n^{(4)}}\, |n^{(3)}, n^{(4)}\rangle \qquad (20)$$

Therefore, the expectation value of the commutator $\left[\frac{\partial A_\mu(x)}{\partial x_\nu}, A_\nu(x')\right]$ taken between such a state vector vanishes identically in the limit $\mu \to 0$ so that the contradiction mentioned earlier is removed.

LECTURE III.

We saw that the essential difficulty about the Fermi method of quantizing the electromagnetic field was that we had to do with unnormalisable state vectors which could be traced to the fact that we had a couple of equations (Equations 2a and 2b of last lecture) for the operators $a^{(3)}$ and $a^{(4)}$ which led to a system of equations for the state vectors the solution of which necessarily contained an infinite number of particles so that the state vectors could not be explicitly normalized. Also though the Fermi subsidiary condition is sufficient to ensure that

$$\langle \psi | \frac{\partial A_\nu(x)}{\partial x_\nu} | \psi \rangle = 0 \qquad (1)$$

it is not necessary as we shall see presently. Now the Fermi condition implies that not only a longitudinal or scalar photon does not exist but it cannot also be created (as the presence of the creation operator $a^{*(4)}$ shows). Now we cannot modify the subsidiary condition to remove this objection just by dropping one of the two equations (2a) or (2b) of last lecture; for, though (1) would still be satisfied in such a case each equation contains both a creation and annihilation operator. What we require is that the fourth component of the field behaves like the other three so that $a^{(4)}$ and $a^{*(4)}$ are annihilation and creation operators, respectively, of the scalar photon, i.e.

$$\langle n+1 | a^{*(4)} | n \rangle = \langle n | a^{(4)} | n+1 \rangle = \sqrt{n+1} \qquad (2)$$

instead of equation (16b) of Lecture I. But this would make $A_4^{(\alpha)}$ also hermitian which is in conflict with the classical reality condition where the fourth component is pure imaginary. Gupta resolved this difficulty by introducing the indefinite metric operator, η (used earlier by Dirac) which is defined by

$$\text{Norm} \, |\psi\rangle = \langle \psi | \eta | \psi \rangle$$

$$\bar{F} = \langle \psi | \eta F | \psi \rangle \qquad (3)$$

for the norm of a state vector and the expectation value of an operator. If the norm is to be always real, η must be hermitian. But the norm need not necessarily be positive; we can have states with zero and negative norms, for which, of course, there is no probability interpretation. Any phisically realisable state vector should have only a positive norm. Though the introduction of the indefinite metric in the Gupta-Bleuler method seems to violate this

requirement, we shall see presently that the form of the subsidiary condition they chose makes the theory to be in agreement with this requirement.

With the modified definition of norms and expectation values it is very well possible for a hermitian operator to have an imaginary eigenvalue which is what we want. Thus we can write

$$\langle \psi | \eta A_k(x) | \psi \rangle = \langle \psi | A_k^*(x) \eta^* | \psi \rangle = \langle \psi | A_k(x) \eta | \psi \rangle \qquad (4a)$$

or $$[A_k(x), \eta] = 0$$

for the first three components

and $$\langle \psi | \eta A_4(x) | \psi \rangle = - \langle \psi | A_4(x) \eta | \psi \rangle$$

or $$\{ A_4(x), \eta \} = 0 \qquad (4b)$$

for the scalar photon.

With the redefined operators for the scalar photon, we have, retaining only the first equation in the Fermi subsidiary condition (equation 2a of last lecture)

$$[a^{(3)}(\vec{k}) + i\, a^{(4)}(\vec{k})] | \psi \rangle = 0 \qquad (5)$$

with both the operators being annihilation operators. (By this, the stringent condition regarding non-creatability of the undesirable photons is removed.) In configuration space (5) reads

$$\frac{\partial A_\nu^{(+)}(x)}{\partial x_\nu} | \psi \rangle = 0 \qquad (6)$$

where (+) denotes the positive frequency part of the field, the x dependence of which is given by e^{ikx}. By taking the Hermitian conjugate of (6) and using (4) we have

$$\langle \psi | \left(\frac{\partial A_k^{(-)}(x)}{\partial x_k} - \frac{\partial A_4^{(-)}(x)}{\partial x_4} \right) \eta = \langle \psi | \eta \frac{\partial A_\nu^{(-)}(x)}{\partial x_\nu} = 0 \qquad (7)$$

showing that (c) is sufficient to ensure (1) in the modified metric

Since we have only one equation (5), as the subsidiary condition there is no restriction on the state vector as we had before. Thus, e.g., we can choose

$$|\Phi^{(0)}\rangle = |0,0\rangle$$

$$|\Phi^{(1)}\rangle = |1,0\rangle + i\,|0,1\rangle$$

$$|\Phi^{(n)}\rangle = \sum_{r=0}^{n} (i)^r \sqrt{n c_r}\, |n-r, r\rangle \qquad (c)$$

which obviously form an orthogonal set in the sence that

$$\langle \Phi^{(n)} | \eta | \Phi^{(n')} \rangle = 0 \quad \text{for } n \neq n' \qquad (9)$$

Also we have the normalisation

$$\langle \Phi^{(n)} | \eta | \Phi^{(n)} \rangle = \sum_{r=0}^{n} (-1)^r n c_r$$

$$= \delta_{n,0} \qquad (10)$$

The last step follows by taking the binomical expansion for $(a+b)^n$ and setting $a = 1$, $b = -1$.

(10) gives the important result that none of the norms is negative (in spite of introducing the indefinite metric) and in fact only the state vector $|\Phi^{(0)}\rangle$ has a norm different from 0. Thus a physical state vector satisfying (5) can be written as

$$|\Phi(\vec{k})\rangle = |\Phi^{(0)}_{(\vec{k})}\rangle + \sum_{n \neq 0} c^{(n)}(\vec{k})\, |\Phi^{(n)}_{(\vec{k})}\rangle \qquad (11)$$

with arbitrary $c^{(n)}(\vec{k})$

The equivalence of the states $|\Phi_{\vec{k}}\rangle$ and $|\Phi^{(0)}_{\vec{k}}\rangle$ corresponds to the lack of uniqueness in the choise of the gauge function, $\Lambda(x)$ which can be seen by computing the expectation value of the potentials, $A_\mu(x)$. If for simplicity we assume

that there are no transverse photons present, then

$$\langle \Psi | \eta A_\mu(x) | \Psi \rangle = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega}} \times \{ e^{ikx} [e_\mu^{(3)} \langle \Phi_{\vec{k}} | \eta a^{(3)}(\vec{k}) | \Phi_{\vec{k}} \rangle + e_\mu^{(4)} \langle \Phi_{\vec{k}} | \eta a^{(4)}(\vec{k}) | \Phi_{\vec{k}} \rangle] + e^{-ikx} [e_\mu^{(3)} \langle \Phi_{\vec{k}} | \eta a^{*(3)}(\vec{k}) | \Phi_{\vec{k}} \rangle + e_\mu^{(4)} \langle \Phi_{\vec{k}} | \eta a^{*(4)}(\vec{k}) | \Phi_{\vec{k}} \rangle \quad (12)$$

Since

$$a^{(3)} | \Phi^{(n)} \rangle = \sqrt{n} \, | \Phi^{(n-1)} \rangle$$
$$a^{(4)} | \Phi^{(n)} \rangle = i \sqrt{n} \, | \Phi^{(n-1)} \rangle$$

it follows that

$$\langle \Phi_{\vec{k}} | \eta a^{(3)}(\vec{k}) | \Phi_{\vec{k}} \rangle = [\sum_{n \neq 0} \langle \Phi^{(0)} | \eta | \Phi^{(n-1)} \rangle + \sum_{n \neq 0} c^{*(n')} \langle \Phi^{(n')} | \eta | \Phi^{(n-1)} \rangle] \times \sqrt{n} \, c^{(n)} = c^{(1)}(\vec{k}) \text{ by (9) \& (10)} \quad (13)$$

Similarly

$$\langle \Phi_{\vec{k}} | \eta a^{(4)}(\vec{k}) | \Phi_{\vec{k}} \rangle = i \, c^{(1)}(\vec{k}) \quad (14)$$

Substituting in (12) we have

$$\langle \Psi | \eta A_\mu(x) | \Psi \rangle = \frac{\partial \Lambda(x)}{\partial x_\mu}$$

with

$$\Lambda(x) = \frac{i}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega^3}}$$

Thus the arbitrariness in the choice of the gauge function $\Lambda(x)$ which can satisfy the equation

$$\Box \Lambda(x) = 0$$

by a suitable choice of the coefficients is associated with the

arbitrariness in $c^{(1)}$ and hence with the nonuniqueness of the state vectors $|\Phi_{\vec{k}}\rangle$

Finally let us calculate the expectation value of the Hamiltonian to find out whether the energy is positive definite

$$\langle \psi | \eta H | \psi \rangle = \sum_{\vec{k}} \left(\Phi_{\vec{k}}^{(0)} + \sum_{n \neq 0} c^{*(n)}(\vec{k}) | \Phi_{\vec{k}}^{(n)} \right)$$

$$* \; \eta \sum_{n' \neq 0} n' \omega \, c^{(n')}(\vec{k}) | \Phi_{\vec{k}}^{(n')} \rangle$$

$$+ \sum_{\vec{k}} \omega \left(n_{\vec{k}}^{(1)} + n_{\vec{k}}^{(2)} \right) = \sum_{\vec{k}} \omega \left(n_{\vec{k}}^{(1)} + n_{\vec{k}}^{(2)} \right) \quad (15)$$

the last following by again using the orthogonality and normalisation conditions, (9) and (10).

Thus we see that the energy is positive definite and is the energy of the transverse photons alone and independent of the mixing of the longitudinal and scalar photons.

The Gupta-Blenler method thus gives an eminently satisfactory way of quantising the electromagnetic field.

LECTURE IV.

We saw that the difficulty in quantising the electromagnetic field arose only when we wanted to include the Lorentz condition as an operator identity. The Lorentz condition itself is an offshoot of using a potential description of the electromagnetic field so that the question arises whether we can consider the electromagnetic field and its interaction with charged matter without introducing potentials. Mandelstam has given such a description of quantum electrodynamics. For simplicity consider a scalar particle interacting with the photon. The commutation relations for its field, φ , are

$$[\dot{\varphi}^*(x), \varphi(y)] = [\varphi(x), \varphi^*(y)] = -i\delta(x-y) \quad (17)$$

with all other commutators vanishing. φ and φ^* obey the equations of motion

$$\left\{\left(\frac{\partial}{\partial x_\mu} - ieA_\mu\right)^2 - m^2\right\}\varphi = 0$$
$$\left\{\left(\frac{\partial}{\partial x_\mu} + ieA_\mu\right)^2 - m^2\right\}\varphi^* = 0 \qquad (2)$$

which are obviously invariant under the combined gauge-transformations

$$\varphi \to \varphi e^{ie\Lambda(x)}; \quad \varphi^* \to \varphi^* e^{-ie\Lambda(x)} \qquad (3a)$$

$$A_\mu \longrightarrow A_\mu + \frac{\partial \Lambda(x)}{\partial x_\mu} \qquad (3b)$$

(3a) is an extension of the usual phase gauge transformation for any complex field to a co-ordinate dependent gauge function Now the question arises whether we can set up field operators which are themselves gauge-invariant quantities. In the case of the electromagnetic field this is well-known and is in fact the electromagnetic tensor $F_{\mu\nu}$. For the matter field we can construct such an operator, by writing

$$\underline{\Phi}(x,P) = \varphi(x)\exp\left\{-ie\int_\infty^x d\xi_\mu A_\mu(\xi)\right\} \qquad (4a)$$

$$\underline{\Phi}^*(x,P) = \varphi^*(x)\exp\left\{ie\int_\infty^x d\xi_\mu A_\mu(\xi)\right\} \qquad (4b)$$

where the integral is over the space-like path P ending at the point x. We see that unlike the conventional field variable $\varphi(x)$, $\underline{\Phi}(x,P)$ depends in addition to the field point, x, also on the path P. $\underline{\Phi}$ does not depend on the gauge selected for the electromagnetic potential A_μ but only on P.

Now if the path is moved through an infinitesimal area $\sigma_{\mu\nu}$ at a point Z, to a new path P', while the end point remains fixed, it follows from the relativistic generalisation

of Stoke's theorem that

$$\delta \int_{-\infty}^{x} d\xi_\mu A(\xi) = \{\partial_\mu A_\nu(z) - \partial_\nu A_\mu(z)\}\sigma_{\mu\nu} = F_{\mu\nu}(z)\sigma_{\mu\nu} \qquad (5)$$

so that the change in the operators $\underline{\Phi}(x,P)$ and $\underline{\Phi}^*(x,P)$ due to the change in the path is given by

$$\delta_z \underline{\Phi}(x,P) = -ie\,\underline{\Phi}(x,P) F_{\mu\nu}(z)\sigma_{\mu\nu}$$
$$\delta_z \underline{\Phi}^*(x,P) = ie\,\underline{\Phi}^*(x,P) F_{\mu\nu}(z)\sigma_{\mu\nu} \qquad (6)$$

Since the path dependence of $\underline{\Phi}$ and $\underline{\Phi}^*$ is known there is only one independent variable for each field point x. If two points P_1 and P_2 differ by a finite area, then to ensure a unique result, we must have the following two conditions. First, if we choose a surface bounded by P_1 and P_2 and gradually move the path from P_1 to P_2, the result must not depend on the order in which two elements σ_1 and σ_2 of this surface are crossed. Secondly, the result must not depend on which surface, bounded by P_1 and P_2 we take in the four-dimensional space. If we have two such surfaces the difference in the integral $\int F_{\mu\nu} d\sigma_{\mu\nu}$ over them will be, using Gauss' theorem

$$\Delta \int F_{\mu\nu} d\sigma_{\mu\nu} = \int \epsilon_{\kappa\lambda\mu\nu} \frac{\partial F_{\mu\nu}}{\partial x_\lambda} d\tau$$

and consistency requires that

$$\epsilon_{\kappa\lambda\mu\nu} \frac{\partial F_{\mu\nu}}{\partial x_\lambda} = 0 \qquad (7)$$

This condition is automatically satisfied if $F_{\mu\nu}$ is defined in terms of the potentials (and is in fact the Maxwell equations) while here it has to be introduced as a consistency condition.

Next we define the derivative of the gauge-invariant field

variable $\Phi(x,P)$ by

$$\partial_\mu \Phi(x,P) = \lim_{dx_\lambda \to 0} \frac{\Phi(x+dx_\lambda, P') - \Phi(x,P)}{dx_\lambda} \qquad (8)$$

where the path P' is obtained from P by giving it an extension of magnitude dx_μ in the μ-direction, i.e. the new path P' passes through the same end point x. Thus the derivative has a double role of acting on the coordinate, x, and also of changing the path. The meaning of this differentiation can be seen by writing in terms of the conventional variables

$$\begin{aligned}\partial_\mu \Phi(x,P) &= \partial_\mu \{\varphi(x) \exp[-ie\int_{-\infty}^{x} A_\nu(\xi) d\xi_\nu]\} \\ &= \frac{\partial \varphi(x)}{\partial x_\mu} \exp[-ie\int_{-\infty}^{x} A_\nu(\xi) d\xi_\nu] \\ &\quad - ie A_\mu(x) \varphi(x) \exp[-ie\int_{-\infty}^{x} A_\nu(\xi) d\xi_\nu] \\ &= \{(\frac{\partial}{\partial x_\mu} - ie A_\mu(x)) \varphi(x)\} \exp[-ie\int_{-\infty}^{x} A_\nu(\xi) d\xi_\nu] \qquad (9)\end{aligned}$$

Thus the differentiation of Φ corresponds to the usual gauge-invariant derivative of the conventional field variable

Another point to be noted regarding the differentiation of Φ is that the operation is not commutative when an electro-magnetic field is present. For, if we denote the points (x_μ, x_ν), $(x_\mu + dx_\mu, x_\nu)$, $(x_\mu, x_\nu + dx_\nu)$ and $(x_\mu + dx_\mu, x_\nu + dx_\nu)$ by A, B, C and D, then

$$\begin{aligned}\partial_\nu \partial_\mu \Phi(x,P) &= \frac{\partial_\nu \Phi(x+dx_\mu, P+AB) - \partial_\nu \Phi(x,P)}{dx_\mu} \\ &= \frac{1}{dx_\mu dx_\nu}\Big[\{\Phi(x+dx_\mu+dx_\nu, P+ABD) \\ &\quad - \Phi(x+dx_\mu, P+AB)\} - \{\Phi(x+dx_\nu, P+AC) \\ &\quad - \Phi(x,P)\}\Big] \qquad (10)\end{aligned}$$

where P + ABD indicates that the new path goes to A as before

ther along the lines AB, BD,. For $\partial_\mu \partial_\nu \Phi(x,P)$ the first term in the numerator in the final expression is $\Phi(x+dx_\mu+dx_\nu, P+ACD)$ the other terms being the same as in (10). Hence

$$(\partial_\nu \partial_\mu - \partial_\mu \partial_\nu)\Phi(x,P) = \frac{1}{dx_\mu dx_\nu} \times \{\Phi(x+dx_\mu+dx_\nu, P+ABD) - \Phi(x+dx_\mu+dx_\nu, P+ACD)\}$$
$$= -ie\Phi(x,P)F_{\mu\nu}$$

so that the differential operation is commutative only when

$$F_{\mu\nu} = 0$$

The physical meaning of the above formulation of path dependent, gauge-independent field variabables can be understood as follows. The path dependance is intimately connected with the arbitrariness in the choice of the phase factors in the operators for charged fields, we can choose the phase factors arbitrarily at one point which we have taken to be at infinity. Once this has been done the phase factors at a neighbouring point will be fixed. If we choose different phase factors at two neighbouring points we have to add extra terms to the equations of motion. Such terms are unphysical and can be removed by a gauge transformation which re-establishes the correct choice of phase at the second point. The correct choice at a field point distant from the reference point cannot be determined directly. We can, however, specify a path between the two points and pass continuously from the reference to the field point, keeping the phase factors of all the operators the same along the path. We thus obtain the path-dependent operators. If the result obtained is independent of the

path chosen, the particles in the theory would have no interaction with the electromagnetic field, as shown earlier. In general, the operator may depend on the path and this is in fact so when there is an electromagnetic field present. If the electromagnetic field is not quantized, two operators at the same space-time, but defined by different paths differ just by a phase factor. For a quantized electromagnetic field, the two operators will differ by a q-number phase factor.

Mandelstam draws an abanogy between this path dependent formulation of electrodynamics and general relativity. In the latter theory the choice of the co-ordinate system is the arbitrary factor instead of the phase. Once the system has been specified at one point, we must obtain it at a neighbouring point by parallel displacement if we do not wish to add extra terms to the equations of the motion. There will be no gravitational field so long as the coordinate systems obtained at a point by parallel displacement from the reference point do not depend on the path. A path difference however, will introduce a difference between the field variables for two neighbouring paths with the same end-point, which can be expressed in terms of the Riemann tensor representing a gravitational field. For such a situation the space is said to be curved. So by analogy we can talk of the "phase" or "gauge" curvature when there is a dependence of phase on the path. At present only the charge gauge seems to have such a curvature (There is an arbitrariness in phase associated with other conservation laws like those of baryon, isospin or hypercharge). An immediate consequence of this analogy between phase and co-ordinate systems is that once we know how to quantize the path-dependent formulation of electro-

dynamics, we can try a similar method for the much-vexed problem of quantizing the gravitational field.

But before proceeding to quantise electrodynamics, we may mention, an experiment, proposed by Aharonov and Bohm, which can measure the path dependence of operators. The proposal is to separate and recombine a beam of electrons and to examine the interference pattern produced. The path of neither beam passes through an electromagnetic field, but the region between them contains a field. Each of the beams would then obey the equations of motion in the absence of a field; still, their phases would be displaced relative to one another and the interference fringes would be altered. No experiment performed at any point in the electron beams would show up a difference between this situation and the field-free situation where the interference is unaffected.

LECTURE V.

We now wish to quantize the path-dependent fields operators, $\underline{\Phi}$, $\underline{\Phi}^*$, and the electromagnetic field variable which we have chosen to be $F_{\mu\nu}$. Now since we have an interacting system, we are dealing with Heisenberg operators and the commutation relations for arbitrary times of such operators cannot be obtained from the usual canonical commutation rules. This would require the knowledge of the solutions of the equations of motion for $\underline{\Phi}$ and $F_{\mu\nu}$ for all times which, for the interacting systems, is, in fact, the problem to be solved. We have to use more modern methods for obtaining commutation rules and one such is due to Peierls.

In Peierl's method, co-variant commutation relations are derived from a knowledge of the Heisenberg equations of motion

together with the field equations. (The consistency of the two set of equations implies a particular type of commutation relations as can be checked in the simple case of the canonical commutation relations.) Let the desired commutation relationships be between two functions A, B of the field variables at two different space-time points, x_1, x_2 respectively. Consider a modified Lagrangian

$$\mathcal{L}'(x) = \mathcal{L}(x) + \lambda A(x_1)\,\delta(x - x_1) \qquad (1)$$

where δ is a four dimensional δ-function i.e.,

$$\mathcal{L}'(t) = \mathcal{L}(t) + \lambda A(x_1)\,\delta(t - t_1)$$

By varying (1) with respect to the field variable, we get the modified field equations. This change in the Lagrangian will induce a corresponding change in the field variable or any function of the field variable, e.g.

$$B'(x) = B(x) + \lambda \delta_1 B(x) \qquad (2a)$$

$$= S B(t) S^{-1}, \qquad (2b)$$

S being the transformation matrix, B'(t) will be determined completely by the equation of motion together with a boundary condition which we shall take to be δ, $B \to 0$ as $t \to -\infty$. If A contains no time derivatives of field quantities the modified Hamiltonian of the system will be

$$H'(t) = H - \lambda A(x)\,\delta(t - t_1) = H - \delta H \qquad (3)$$

In order that B'(t) be a solution of the modified Heisenberg equation of motion for it, viz.,

$$\frac{\partial B'(t)}{\partial t} = i\,[H'(t), B'(t)] \qquad (4)$$

S must satisfy the equation

$$i \frac{\partial S}{\partial t} = \delta H S \qquad (5)$$

To the firstorder in λ the solution for S can be written as

$$S(t) = 1 - i\lambda \theta(t - t_1) A(x_1) \qquad (6)$$

where θ is the step function. Using this expression in (2b) and comparing with (2a) we have

$$\delta_1 B(x_2) = \frac{1}{i} [A(x_1), B(x_2)] \theta(x_2 - x_1) \qquad (7)$$

Similarly if we had chosen $t_2 < t_1$ and the boundary condition $\delta_2 B(x_2) \to 0$ as $t \to +\infty$, we would have

$$\delta_2 B(x_2) = -\frac{1}{i} [A(x_1), B(x_2)] \theta(x_1 - x_2) \qquad (8)$$

Combining (7) and (8) we get the symmetrical form

$$[A(x_1), B(x_2)] = i \{\delta_1 B(x_2) - \delta_2 A(x_1)\} \qquad (9)$$

Let us apply this method to obtain the commutation relations for Φ, Φ^* and $F_{\mu\nu}$. The Lagrangian is

$$L = -\{\partial_\mu \Phi^*(x)\}\{\partial_\mu \Phi(x)\} - m^2 \Phi^*(x)\Phi(x) - \frac{1}{4}\{F_{\mu\nu}(x)\}^2 \qquad (10a)$$

which in terms of the conventional path-independent, gauge-dependent operators will be

$$L = -\left\{\frac{\partial \varphi^*}{\partial x_\mu} + ie A_\mu \varphi^*\right\}\left\{\frac{\partial \varphi}{\partial x_\mu} - ie A_\mu \varphi\right\} - m^2 \varphi^* \varphi - \frac{1}{4}\{\partial_\mu A_\nu - \partial_\nu A_\mu\}^2 \qquad (10b)$$

(10a) has no path dependence since Φ and Φ^* appear always multiplied in it. We note that there is no separate interaction term in it; the interaction arises from the dependence of Φ on $F_{\mu\nu}$. The equations of motion generated by varying with respect to the field variables are

$$(\partial_\mu^2 - m^2)\Phi = 0, \quad (\partial_\mu^2 - m^2)\Phi^* = 0$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} + J_\nu = 0 \quad \text{where} \quad J_\nu = -ie(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) \qquad (11)$$

(11)

To derive the commutation relations we add to (10a) a term $\epsilon\,\delta(x-x_1)\,\Phi(x_1)$ with ϵ, a vanishingly small quantity. i.e. (in terms of the conventional quantities) we add

$$\epsilon\,\delta(x-x_1)\,\varphi(x_1)\exp\left\{-ie\int_{-\infty}^{x_1} d\xi_\mu A_\mu(\xi)\right\}$$

The modified equations of motion will be

$$\left\{\left(\frac{\partial}{\partial x_\mu}-ieA_\mu\right)^2-m^2\right\}\varphi = 0 \qquad (12a)$$

$$\left\{\left(\frac{\partial}{\partial x_\mu}+ieA_\mu\right)^2-m^2\right\}\varphi^* + \epsilon\,\delta(x-x_1)\times\exp\left\{-ie\int_{-\infty}^{x_1} d\xi_\mu A_\mu(\xi)\right\} = 0 \qquad (12b)$$

$$\frac{\partial F_{\mu\nu}(x)}{\partial x_\nu} + j_\nu(x) - \epsilon\,ie\left\{\int_{-\infty}^{x_1} d\xi_\nu\,\delta(x-\xi)\right\}\times\exp\left\{-ie\int_{-\infty}^{x_1} d\xi_\mu A_\mu(\xi)\right\} = 0 \qquad (12c)$$

(Since the added term contains only φ and A_μ it affects only the equations for φ^* and $F_{\mu\nu}$). The last term in (12c) is obtained by taking the functional derivative of the exponential with respect to $A_\mu(x)$. We now assume that all the variables are unchanged before the time corresponding to x_1. Immediately afterwards, the changes $D\Phi$ in the variables due to the extra term in the Lagrangian will be

$$D\Phi\left\{\left(\frac{\partial}{\partial t}-ieA_\mu\right)\varphi^*(x)\right\} = \epsilon\,\delta(\bar{x}-\bar{x}_1)\exp\left\{-ie\int_{-\infty}^{x_1} d\xi_\mu A_\mu(\xi)\right\} \qquad (13a)$$

$$D\Phi\left\{F_{0i}(x)\right\} = \epsilon\,ie\int_{-\infty}^{x_1} d\xi_i\,\delta(\bar{x}-\bar{\xi})\,\varphi(x)\times\exp\left\{-ie\int_{-\infty}^{x} d\xi_\mu A_\mu(\xi)\right\} \qquad (13b)$$

where i takes the values 1, 2, 3. The changes in the other

variables are obviously zero. Multiplying (13a) on both sides by $\exp\{-ie\int_{-\infty}^{x} d\xi_\mu A_\mu(\xi)\}$ and using Peierls' rule that $e^{[\Phi, X]} = D_\Phi$ where X is Φ, Φ^* or $F_{\mu\nu}$, we get the following commutation relations

$$\begin{aligned}
[\dot{\Phi}^*(x,P), \Phi(y,P)] &= [\dot{\Phi}(x,P), \Phi^*(y,P)] \\
&= -i\delta(\vec{x}-\vec{y}) \\
[\dot{\Phi}(x,P), F_{0i}(y)] &= -e\int_{-\infty}^{x} d\xi_i\, \delta(\vec{y}-\vec{\xi})\, \dot{\Phi}(x,P) \\
[\dot{\Phi}^*(x,P), F_{0i}(y)] &= e\int_{-\infty}^{x} d\xi_i\, \delta(y-\xi)\, \dot{\Phi}(x,P) \qquad (14)
\end{aligned}$$

all other commutators being zero. The dot indicates gauge-invariant differentiations defined earlier.

The equations of motion, the commutation relations, the equations giving the change in the field variable (equation 6 of last lecture) and the consistency condition (equation 7 of last lecture) together give us a theory of interacting electromagnetic and charged scalar fields which involve only gauge-independent quantities and with out potentials (these last appearing only as calculational aids, without any further role).

Finally, we may mention how the introduction of these path-dependent operators will affect the quantities usually computed like the S-matrix elements for various processes. Now using the procedure of Low in the conventional Lagrangian field theory using the canonical commutation relations or that of Lehmann, Symanzik and Zimmermann where no specific interaction is assumed, but the asymptotic condition which connects the limit of a Heisenberg field operator at time t as $t \to \pm\infty$ with an ''out'' and ''in'' field operators obeying free field equations, is used the matrix

element for any process can be reduced to Fourier transforms of the expectation values of time-ordered products of field operators (with differential operators operating on them), e.g., the matrix element for two particles going over to two other particles have the form

$$\int dx\,dy\,dz\,dw\; e^{-iwr - isz + ipx + iqy} \times K\, T\langle \varphi(x)\varphi(y)\varphi(z)\varphi(w)\rangle$$

where p and q are the momenta of the incoming particles, r and s those of the outgoing particles and K is a product of differential operators. We could have written the time-ordered product in terms of the path-dependent variables but the result should be independent of the path and dependent only on the quantum numbers of the particles and the Heisenberg equations of motion they obey. But if we are interested in the matrix element of any arbitrary operator between specified states, the path corresponding to this particular operator will be path-dependent. Thus the arbitrariness in specifying the paths on which the operators depend does not propagate itself into the specification of the states (unless the states or operators are themselves unspecified).

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L E C T U R E VI.

In this lecture, we shall study a simple but interesting application of Mandelstam's theory regarding quantisation of the electromagnetic field, viz., derivation of Dirac's expression connecting the electric charge and the magnetic monopole (if it exists). Dirac was led to conclude that the quantisation of the electric charge would immediately imply the existence of the magnetic monopole the strength of which is also quantised. We shall describe the considerations which led Dirac to the existence of the magnetic pole and then give the derivation by Caribbo and Ferrari of the Dirac expression.

The phase of a wave function is arbitrary at any given point and only the phase difference at two points has a meaning. If φ_m and ψ_n are two wave functions, $\int \varphi_m \psi_n \, d\tau$ is a number the square of the modulus of which has a physical meaning, namely, the probability of agreement of the two states. If the integral is to have a definite modulus, the integral though it need not have a definite phase at each point, must have a definite phase difference between any two points. Thus the change in phase in $\varphi_m \psi_n$ round a closed curve must vanish. This requires that the change in phase in ψ_n round a closed curve is equal and opposite to that in φ_m and hence the same as in ψ_m. Thus the change in phase of a wave function round any closed curve must be the same for all the wave functions which means that the change in phase must be determined by the dynamical system itself. The nonintegrability of the phase must be connected with the field of force in which the particle moves.

There can however be a change in phase round a closed curve by arbitrary multiples of 2π, for different wave functions. Conside a closed curve. The wave equation (in the electromagnetic case, for instance) requires the wave function to be continuous and hence the change in phase round a small closed curve must be small. Thus this change cannot now be different by multiples of 2π for different wave functions. It must have one definite value and may therefore be interpreted without ambiguity in terms of the flux of the 6-vector $\vec{E}$, $\vec{H}$ through the small closed curve, the flux being small.

If the wave function itself vanishes which implies (since the wave function is complex) two conditions so that in general the points at which it vanishes will be along a line which will be called the nodal line, then the phase of the wave function does not have a meaning. (In 4-dimensions, the nodal line becomes a two-dimensional nodal surface). If we now take a wave function having a nodal line passing through our small closed curve, consideration of continuity no longer enable us to infer that the change in phase round the small closed curve must be small. All we shall be able to say is that the phase change will be close to $2\pi n$ where n is some integer, positive or negative, which will be characteristic of an nodal line, the sign being associated with a direction encircling the nodal line, which in turn may be associated with a direction along the nodal line. The difference between the phase change round the small closed curve and the nearest $2\pi n$ must now be the same as the change in phase round the closed curve for a wave function with no nodal line through it. Hence

this difference must be interpreted in terms of the flux of the 6-vector ($\vec{E}$, $\vec{H}$) through the closed curve. In 3-dimensional space only the magnetic flux will come into play and hence we get for the phase change round the small closed curve $2\pi n + c \int (H \cdot d\vec{S})$. A large closed curve can be treated by division into a network of small closed curves. The total change in phase will equal the sum of all the changes round the small closed curves and hence is equal to $2\pi \Sigma n + c \int \vec{H} \cdot d\vec{S}$, the integration being taken over the surface and the summation over all nodal lines that pass through it, the proper sign being attached. $c \int \vec{H} \cdot d\vec{S}$ must be the same for all wave functions and $2\pi\Sigma n$ may be different for different wave functions. For a closed surface the two terms together must vanish. Hence the sum over all nodal lines must be the same for all wave functions and must equal $-\frac{e}{2\pi c}$ times the total magnetic flux crossing the surface. If Σn does not vanish, some nodal lines must have end points inside the closed area, since a nodal line without such end points must cross the surface twice (atleast) and will contribute equal and opposite amounts to Σn at the two points of crossing. The value of Σn for the closed surface will thus equal the sum of the values of n for all nodal lines having end points inside the surface. This sum must be the same for all wave functions. Since this applies to any closed surface, it follows that the end points of the nodal lines must be the same for all wave functions. These end points are singular points in the electromagnetic field. The total flux of the magnetic crossing a small closed surface surrounding one of these points is $g = 4\pi G = \frac{2\pi n}{e}$ where n is the characteristic of the nodal line that ends there, or the sum of the characteristics of all

nodal lines ending there when there is more than one. Thus at the end point there will be a magnetic monopole of strength

$$g = 2\pi n/e \qquad (1)$$

Following Caribbo and Ferrari, we shall derive expression (1). For this purpose, let us introduce a vector potential, B_μ in addition to the usual one when there are magnetic sources present in addition to electric ones. The introduction of the second potential is compensated by extending the group of gauge transformations. We have also two sets of Maxwell's equations,

$$\partial_\nu F_{\mu\nu}(x) = j_\mu(x) \qquad (2)$$

$$\partial_\nu \tilde{F}_{\mu\nu}(x) = g_\mu(x) \text{ or } 0 \qquad (3)$$

depending on whether there is a magnetic current (and density of magnetic charge) g_μ present or not. The two equations (2) and (3) can be solved by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon^{\mu\nu\rho\sigma}\partial_\rho B_\sigma \qquad (4)$$

with

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda\,, \quad B_\mu \longrightarrow B_\mu + \partial_\mu \Gamma \qquad (5)$$

as well as the meaning transformations.

$$A_\mu \longrightarrow A_\mu + A_\mu'\,; \quad B_\mu \longrightarrow B_\mu + B_\mu' \qquad (6)$$

A' and B' should satisfy the zero field conditions

$$\partial_\mu A_\nu' - \partial_\nu A_\mu' + \varepsilon^{\mu\nu\rho\sigma}\partial_\rho B_\sigma' = 0 \qquad (7)$$

We can use (6) to impose Lorentz conditions on A and B so that they satisfy the set of equations

$$\partial_\mu A_\mu = \partial_\mu B_\mu = 0\,, \quad \Box A_\mu = j_\mu\,; \quad \Box B_\mu = g_\mu \qquad (8)$$

Without magnetic sources, we introduce as in Lecture IV,

the gauge invariant operator for the matter field

$$\underline{\Phi}(x,P) = \varphi(x) \exp\left[-ie \int_{(P)}^{\infty} A_\mu(\xi)\, d\xi_\mu\right] \qquad (9)$$

which depends only on P and does not depend on the gauge selected for A_μ. We have the relation

$$\underline{\Phi}(x,P') = \underline{\Phi}(x,P) \exp\left[-i\frac{e}{2}\int_S F_{\mu\nu}\, d\sigma_{\mu\nu}\right] \qquad (10)$$

which can be used in the general case in which magnetic sources are present. But this requires that the change in $\underline{\Phi}$ does not depend on the particular choice of the two-dimensional surface S. If S_1 and S_2 are two such surfaces, then

$$\underline{\Phi}(x,P) \exp\left[-i\frac{e}{2}\int F_{\mu\nu}\, d\sigma_{\mu\nu}\right] = \underline{\Phi}(x,P) \exp\left[-i\frac{e}{2}\int F_{\mu\nu}\, d\sigma_{\mu\nu}\right] \qquad (11)$$

so that for the closed two-dimensional surface $S = S_1 - S_2$, we have

$$\exp\left[-i\frac{e}{2}\int_S F_{\mu\nu}\, d\sigma_{\mu\nu}\right] = 1 = \exp\left[-ie\int_V g_\mu\, dV_\mu\right] \qquad (12)$$

by changing over to a volume integral over V enclosed by S. $g_\mu \neq 0$ corresponds to the existence of Dirac monopoles. If g_μ is a classical (C number) source than (12) requires that

$$q_V = \int_V g_\mu\, dV_\mu = 2\pi n/e \qquad (13)$$

Since V is completely arbitrary this can be so only if g_μ is due to one or more point sources each with a magnetic charge which is a multiple of $g = 2\pi/e$. If g_μ is an operator, then (12) is satisfied if all the eigenvalues of q_V are multiples of g. This is true if g_μ represents the current of one or more quantized fields, each of them carrying a magnetic charge which is a multiple of g.

These fields would then be associated with monopoles.

Some consequences of the existence of the Dirac monopole are the following. Parity is not conserved. For if $\tilde{F}_{\mu\nu}$ is a tensor, $\partial_\mu \tilde{F}_{\mu\nu}$ is an axial vector while j_μ is a vector. Similarly, in addition to the operation of charge conjugation, C, we have the magnetic pole conjugation M and only C' = CM and P' = PM are conserved (for strong and electromagnetic interactions). The CPT theorem has to be replaced by a CPTM theorem. Now if particles had an intrinsic electric moment, μ_e (just as they can have a magnetic moment μ_m) the interaction Lagrangian for the electric moment interaction in the non-relativistic can be written as $\mu_e (\vec{\sigma} \cdot \vec{E})$. We have

$$P \vec{\sigma} P^{-1} = \vec{\sigma} \; ; \quad P \vec{E} P^{-1} = -\vec{E}$$

so that if P is good such an interaction cannot exist. But since parity can be violated (in weak interactions), let us study the above interaction under the Wigner time reversal T.

$$T \vec{\sigma} T^{-1} = -\vec{\sigma} \; ; \quad T \vec{A} T^{-1} = -\vec{A}$$

and hence $T \vec{E} T^{-1} = \vec{E}$

so that there can be no intrinsic electric moments for particles if T which is good, is generally found to be true. But with the existence of a magnetic monopole, CP (or equivalently the old T operation) means only the invariance under TM and not each alone. The electric dipole moment could be proportional to the product of a magnetic pole and a spin angular momentum in which case each would change sign under TM but their product and the resulting electric field would not.

The large value of g ($g^2 = 137$ for n = 1) would cause the magnetic poles to have a very high energy loss by ionization in matter. At high velocity the ionization of a pole of strength g is expected to be thousands of times the minimum ionisation produced by a particle of charge e.

Other properties of the magnetic monopole enumerated by Amaldi et al are; They could be bound to a paramagnetic atom or molecule (like oxygen) with energies comparable to the chemical bond. They would interact with matter in bulk being repelled by diamagnetic and attracted by paramagnetic substances. They can be bound to a nucleus which can give rise to two effects. The first is due to the interaction of the magnetic pole with the magnetic moment of the nucleus. The second effect would take place only if the Dirac poles were fermions, since they would then necessarily show an electric dipole moment (as discussed earlier) which would interact with the electric charge of the nucleus.

The search for the magnetic monopole has been going on but till now with negative results. The attempt is to produce them in proton-nucleon collisions or using secondary γ-rays, the irradiated target being placed in a pulsed magnetic field with sufficient strength to extract the tightly bound magnetic poles from it and accelerate them through nuclear emulsions. Upper bounds for the production cross-sections have been estimated. The cross-section for the production of pairs of magnetic poles with masses below or around three proton masses at about 28 Gev has a limit varying from 10^{-38} to 10^{-40} cm^2.

LECTURE VII

We mentioned earlier that only the charge conservation law appears to give rise to a path dependence and that it might not be possible to attach gauge curvature with other conservation laws like those of isotopic spin, baryon and hypercharge as there is complete arbitrariness in the phases associated with these laws. (We may also think of this as due to the non-realizability of the ''fields'' associated with them). We shall consider these latter conservation laws which are strictly conserved in strong interactions. In electrodynamics we have the notion of a conserved current and universality of coupling to the vector particle, the proton. Similarly in weak interactions the vector part is found to be associated with a conserved current. So Sakurai and others have suggested vector or gauge theories of strong interactions to bring these latter in unison with the electromagnetic and weak interactions. Wigner had noted the similarity between the electric charge and baryonic charge, viz., that they are both conserved to a very high degree of accuracy. But the two are also dissimilar since in the first case ''charge'' means both a conserved additive number and a (measurable) coupling constant whereas in the other case, measurement is not possible. The dissimilarity can also be looked at as follows. In the electromagnetic case charge conservation is an immediate consequence of Maxwell's equations, since the continuity equation

$$\frac{\partial}{\partial x_\mu} j_\mu = 0 \qquad (1)$$

follows from combining the equations

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \; ; \; \vec{\nabla} \vec{E} = \rho \qquad (2)$$

(This can also be seen very easily in terms of the electromagnetic tensor, $F_{\mu\nu}$ The Maxwell equation in terms of it, viz,

$$\partial_\nu F^{\mu\nu} = j^\mu$$

gives because of the antisymmetry of $F_{\mu\nu}$ the continuity equation (1), on differentiating both sides).

The analogy between the two charges cannot be pushed, further by taking the pion to be the particle which is the "carrier" of the baryonic field since the "current" to which it is coupled is a pseudoscalar and not a vector particle as in the case of the photon. Therefore a vector meson has to be introduced which is universally coupled to the baryon current. Lee and Yang have shown that if the vector meson had mass zero there would be a long-ranged anti-gravity effect (like Coulomb repulsion) between two macroscopic objects. The interaction would be much weaker than the gravitational interaction. So for strong interactions the vector meson coupled to the baryonic current is assumed to be massive.

Similarly vector mesons can be associated with the other conserved quantities--isotopic spin and hypercharge. The isospin current is isovector in nature and even under G-conjugation whereas the baryon and hypercharge currents are isoscalar and odd under G. So we can expect one T=1, even G vector meson and two T = 0 odd G, vector mesons. (If other quantum numbers can be found which are conserved there will be more such particles. In the unitary symmetry model, there is a strangeness changing current with isospin 1/2 which is approximately conserved so that we may conjecture a T = 1/2 vector meson).

The ρ-meson may be the T = 1 particle, the ω meson, one of the T = 0, G = -1 vector mesons and K* (of mass 880 MeV) the T = 1/2, S = $\pm$ 1 particles.

The vector meson theory has been tried to be justified on the basis of a gauge principle. The existence of a coordinate dependent gauge transformation

$$\varphi \longrightarrow \varphi\, e^{ie\Lambda(x)} \tag{3}$$

does not leave the Lagrangian of the matter field invariant, e.g. for a Dirac particle, the transformed Lagrangian would be

$$\bar{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi + ie\,\bar{\psi}\,\gamma_\mu\,\psi\,\frac{\partial \Lambda(x)}{\partial x_\mu} \tag{4}$$

which contains an extra term. But if we start from the Lagrangian,

$$\bar{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi - ie\,\bar{\psi}\,\gamma_\mu\,\psi A_\mu - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{5}$$

it is evidently invariant under the gauge transformations for (3) and for A_μ which is

$$A_\mu \rightarrow A_\mu + \frac{\partial \Lambda(x)}{\partial x_\mu} \tag{6}$$

Thus the existence of the co-ordinate dependent gauge transformation is supposed to ensure the existence of a vector particle the free field for which is given by the last term of (5). But this need not be the case. Gauge invariance will be assured if we take insteadof the above Lagrangian,

$$L = \bar{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right)\psi - ie\,\bar{\psi}\,\gamma_\mu\,\psi\,\frac{\partial B}{\partial x_\mu} \tag{7}$$

such that

$$B'(x) = B(x) + \Lambda(x) \tag{8}$$

where B(x) is a scalar. (7) does not have a free field part for B (x) and hence does not lead to the equation of motion of this

field. The same situation arises for (5). Write

$$A_\mu = \left(A_\mu - \Box^{-1}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right) + \Box^{-1}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}$$
$$= A_\mu^{(1)} + A_\mu^{(0)} \qquad (9)$$

where $A_\mu^{(1)}$ does not change by the gauge-transformation on A_μ and only $A_\mu^{(0)}$ does

$$A_\mu^{(0)\prime} = A_\mu^{(0)} + \frac{\partial \Lambda}{\partial x_\mu}\,, \quad A_\mu^{(1)\prime} = A_\mu^{(1)} \qquad (10)$$

$F_{\mu\nu}$ being gauge-invariant depends only on $A_\mu^{(1)}$ and does not depend on $A_\mu^{(0)}$ Thus (5) does not lead to the equation of motion for A_μ but only for the gauge-independent part $A_\mu^{(1)}$ only. $A_\mu^{(0)}$ enters the Lagrangian (5) which however does not lead to an equation for it. The components of $A_\mu^{(1)}$ for which (5) gives equations of motion have nothing to do with gauge-invariance as they fail to undergo a gauge-transformation. The gauge-principle only causes the appearance of some scalar field without any dynamical manifestation which can be removed by a canonical point transformation of the field variable,

$$\psi'(x) = \exp\left[(e\beta(x)\right]\psi(x) \qquad (11)$$

The requirement that the gauge transformation associated with a conservation law be local is supposed to imply that the vector meson coupled to the corresponding current has zero-mass. Consider the equation

$$\vec{\nabla}\cdot\vec{E} = \rho$$

An integration over a large volume gives on using Gauss' theorem

$$\int (d\tau)\,\rho = Q = \int d\vec{S}\cdot\vec{E} \qquad (12)$$

where Q is the total charge of the system and a constant of motion. At large distances, therefore, $\vec{E} \sim (Q/4\pi)\frac{\vec{r}}{r^2}$.

Thus we could argue that if one finds a static field which has long range there must be a zero-mass particle associated with it or else the field must be of finite range. But for this to be true the total charge Q must be different from zero. When a charge is introduced into the vacuum accompanying it the electric field, polarizes the vacuum, which produces a partial compensation of the charge. This is the origin of charge renormalisation. But it is possible that the compensation of the charge is not partial but complete. In this case, the constant total charge that will be observable in any arbitrarily large volume will be zero.

Schwinger shows this is actually the case by taking specific examples. Consider the Lagrangian

$$L \equiv -\frac{1}{2} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{e^2}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\gamma_\mu \partial_\mu + m_0)\psi + \bar{\psi}\gamma_\mu \psi A_\mu \qquad (13)$$

which is the usual Lagrangian if $A_\mu \longrightarrow eA_\mu$. The action operator $W = \int (dx) \mathcal{L}$ is dimensionless if $\frac{1}{\hbar} = 1$. The dimension of the Lagrangian density is $\frac{1}{L^{n+1}}$ and that of the coupling constant e^2 is L^{3-n}. In the one-dimensional case which we shall be considering e^2 is of the dimensions of L^2 i.e. it carries a length or mass. There are only two γ-matrices, γ_1 and γ_0 which are represented by 2 x 2 matrices. Let us calculate the polarization of the vacuum of the Fermi field ψ by an external electromagnetic field A_μ. The simplified Lagrangian is

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - iA_\mu)\psi \qquad (14)$$

i.e. the mass of the fermion is zero. The current induced in the vacuum by the external field is

$$\langle j_\mu(x) \rangle = \langle \bar{\psi}(x) \gamma_\mu \bar{\psi}(x) \rangle$$

xixmxx which is a singular expression since

$$\langle j_\mu(x)\rangle_0 = e \lim_{x\to x'} \sum_{\alpha\beta} (\Phi_0, \bar\psi_\alpha(x)\psi_\beta(x')\Phi_0)$$

$$= e \lim_{x\to x'} \sum_{\alpha\beta} S^{(-)}_{\beta\alpha}(x-x')(\gamma_\mu)_{\alpha\beta}$$

$$= e\, Tr\,(\gamma_\mu S^{(-)}(0)) \qquad (15)$$

which is infinite. So we should consider $j_\mu(x)$ to be always defined by a suitable limiting process on the Green's function.

$$G(x,x') = \langle T(\psi(x)\bar\psi(x'))\rangle \qquad (16)$$

which obeys the inhomogeneous differential equation.

$$\gamma_\mu(\partial_\mu - iA_\mu(x))G(x,x') = \delta(x-x') \qquad (17)$$

Under the gauge-transformation (u), the Green's function transforms according to

$$G(x,x') = e^{i\lambda(x)} G(x,x') e^{-i\lambda(x')} \qquad (18)$$

We have, therefore,

$$\langle j_\mu(x)\rangle = Tr\,(\gamma_\mu G(x,x))$$

where

$$G(x,x) = \lim_{x'\to x} G(x,x') e^{-i\int_{x'}^{x} d\xi^\mu A_\mu(\xi)} \qquad (19)$$

The exponential factor is required in order to maintain gauge invariance for $x' \neq x$. The solution of (17) can be written as

$$G(x,x') = G_0(x,x')\, e^{i(\varphi(x)-\varphi(x'))} \qquad (20)$$

where

$$\alpha^\mu \partial_\mu \varphi(x) = \alpha^\mu A_\mu(x) \qquad (21)$$

and G_0 is the Green's function for the free field, i.e. when $A_\mu = 0$.

$$G_0(x,x') = \int_0^\infty \frac{dp}{2\pi} e^{-ip\gamma_\mu(x_\mu - x'_\mu)}; \; x_0 > x'^0$$
$$= -\int_{-\infty}^0 \frac{dp}{2\pi} e^{-ip\alpha^\mu(x_\mu - x'_\mu)}, \; x_0 < x'_0 \tag{22}$$

For equal times,

$$G_0 = \frac{i}{2\pi} \frac{\gamma_1}{x_1 - x_1'} \tag{23}$$

Expanding the exponential in a Taylor series for $x_1 \to x_1'$ we obtain

$$G \simeq \frac{i}{2\pi} \frac{\gamma_1}{x_1 - x_1'} \left[1 + i(x_1 - x_1')(\partial_1 \varphi - A_1)\right] \tag{24}$$

$$G(x,x) = -\frac{1}{2\pi} \gamma (\partial_1 \varphi(x) - A_1(x))$$
$$= \frac{1}{2\pi} \gamma (\partial_0 \varphi(x) - A_0(x)) \tag{25}$$

on taking a symmetric limit, i.e. taking an average of the values of the limits attained from the left and from the right.

We therefore have

$$\langle j_\mu(x) \rangle = \frac{1}{2\pi} \left[A_\mu(x) - \partial_\mu \operatorname{Tr} \tfrac{1}{4} \varphi(x)\right] \tag{26}$$

Writing equation (21) explicitly,

$$(\gamma^0 \partial_0 + \gamma^1 \partial_1) \varphi(x) = \gamma^0 A_0(x) + \gamma^1 A_1(x) \tag{27}$$

and multiplying it from the left by $(\gamma^0 \partial_0 - \gamma^1 \partial_1)$ we obtain the second order differential equation

$$-\partial^2 \varphi(x) = -\partial_\mu A^\mu(x) + \gamma^1 (\partial_0 A_1(x) - \partial_1 A_0(x)) \tag{28}$$

Taking the trace reduces this to

$$-\partial^2 \tfrac{1}{4} \operatorname{Tr} \varphi(x) = -\partial_\mu A^\mu(x) \tag{29}$$

which can be solved for $\operatorname{Tr} \varphi$ by the corresponding Green's function $D(x, x')$. Symbolically

$$\frac{1}{4} \operatorname{Tr} \varphi = -D \partial_\nu A^\nu \tag{30}$$

Hence

$$\langle j_\mu \rangle = -\frac{1}{2\pi}\left[A_\mu + \partial_\mu D \partial_\nu A^\nu\right] \tag{31}$$

Now let us think of the idea of self-consistency of the field and the current induced by it/considering A_μ not as an external field but as being somehow brought into existence propagating in accordance with Maxwell's equations. Then this field induces a current which in turn changes the nature of the field. From our Lagrangian (13), we have

$$\partial_\nu F^{\mu\nu} = j_\mu \;,\; e^2 F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{32}$$

from which

$$e^2 j_\mu = -\partial^2 A_\mu + \partial^\mu \partial_\lambda A^\lambda \tag{33}$$

Using the Lorentz gauge

$$\partial_\mu A^\mu = 0 \tag{34}$$

and employing (32) we get the equation for A_μ

$$\left(-\partial^2 + \mu^2\right) A_\mu = 0 \tag{35}$$

where $\mu^2 = e^2/2\pi$. This is an equation describing a non-interacting vector particle of finite mass $\mu = \sqrt{e^2/2\pi}$. Thus we see that gauge invariance of a vector field does not necessarily require zero mass particles.

We can see this point also simply as follows. In the Lorentz gauge

$$\langle j \rangle = -\frac{1}{\pi} \cdot A^0 \tag{36}$$

The potential A^0 has its source in the total charge density $J_0 + \langle j_0^0 \rangle$ where J_0 is the external charge density

$$-\partial^2 A^0 = e^2\left(J^0 + \langle j_0^0 \rangle\right)$$

or

$$\left(\frac{d^2}{dx^2} - \mu^2\right) A^0 = -e^2 J^0 \tag{37}$$

The solution of this is

$$A^0 = \frac{e^2}{2\mu} \int (dx')^2 e^{-\mu(x'-x'')} J^0(x') \qquad (38)$$

The total charge induced in the vacuum is therefore

$$\int \langle j^0(x') dx' = -\frac{e^2}{2\mu\pi} \int dx' dx'' e^{-\mu|x'-x''|} \cdot J^0(x') \qquad (39)$$
$$= -q_0$$

which exactly cancels the inserted charge q_0. Thus there is no long-range field and hence there is no argument for a zero-mass particle.

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