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MATSCIENCE REPORT 14

LECTURES ON
THE STUECKELBERG FORMALISM OF VECTOR MESON FIELDS

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THE STUECKELBERG FORMALISM OF VECTOR MESON FIELDS

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1. INTRODUCTION

Until very recently the vector field has not been popular among physicists. The main reason is the following: Experimental physicists said that except for photon there seemed to be no elementary particle which had spin one. Also, for higher spins there exists the unpleasant feature that the theory involves singularities stronger than those of spin or spin 1/2 fields, so that one cannot derive reliable answers from the theory.

However, the situation has changed recently; experimental physicists have found some resonance states which have spin one e.g. particles such as the ω and ρ mesons. On the other hand, some theoreticians have proposed the so-called gauge theory of strong and weak interactions, that is, they formulate the theory of these interactions on the basis of the gauge principle by introducing vector fields.

Usually, to describe particles with spin one and with non-zero mass one uses a vector field u_μ. This field must satisfy the subsidiary condition $\partial_\mu u_\mu = 0$ in order to reduce the number of degrees of freedom from 4 to 3, which correspond to the different orientations of spin. However there exists another formalism originally due to Stueckelberg, in which one uses five variables A_μ and B satisfying some subsidiary conditions.

It is well-known that the Stueckelberg formalism of vector meson fields is sometimes more convenient than the ordinary formalism. The most characteristic features of the former are the following:

(i) The vector field A_μ behaves in a way very similar to the photon field with which we are more familiar, and so it is suitable for discussing the connection between the meson and photon fields;

(ii) The gauge covariance property manifests itself more explicitly here than in the ordinary formalism and is very similar to that of quantum electrodynamics;

(iii) The derivative terms in the commutation relation of the vector field, which is the cause of non-renormalizable divergences in most vector field interactions, arises, in this formalism, solely from an auxiliary scalar field B. Thus, the question of the renormalizability of interactions can be reduced to another question of whether or not the field B can be eliminated from Hamiltonian by certain transformations. For these reasons the Stueckelberg formalism has been applied to various problems by many authors. It seems to us, however, that a detailed account of the formalism itself has not been given in the literature. Thus, the aim of the present lectures is to give such an account by discussing the most general properties of this formalism.

Our main results are summarized as follows: To formulate the theory in a rigorous manner, we need to introduce an indefinite metric. We can show, however, that there does not arise any difficulty in connection with negative probabilities. A proof of the equivalence between the Stueckelberg formalism and the ordinary formalism is given.

In this series of lectures I would like to discuss a detailed account of the formalism thereby showing the equivalence between this formalism and the ordinary one. I will also discuss some applications to the problems of gauge transformation and of the renormalizability of vector field interactions.

2. General Formalism

In the ordinary formalism, we have a field operator u_μ which satisfies the (free) equations of motion

$$\partial_\nu u_{\mu\nu} + x^2 u_\mu = 0 ,$$
$$u_{\mu\nu} = \partial_\mu u_\nu - \partial_\nu u_\mu \tag{2.1}$$

These also lead to the condition

$$\partial_\mu u_\mu = 0 \qquad \text{SUBSIDIARY CONDITION} \tag{2.2}$$

Condition (ii) states that of the 4 components of u_μ, $\mu = 1,2,3,4$, only three are independent, corresponding to the three independent spin orientations.

The field equations (i) may be obtained from the free-field Lagrangian

$$\mathcal{L} = -\frac{1}{4} u_{\mu\nu} u_{\mu\nu} - \frac{1}{2} x^2 u_\mu u_\mu \tag{2.3}$$

The field operators u_μ obey the commutation relation

$$\left[u_\mu(x), u_\nu(x') \right] = i \left[\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\kappa^2} \right] \Delta(x-x') \tag{2.4}$$

In contrast to the ordinary formalism, we introduce in the Stueckelberg formalism* 5 variables, a vector field A_μ and a scalar field B, and impose a subsidiary condition on the state vectors to reduce the number of degrees of freedom to three.

* Ref. E. C. G. Stueckelberg Helv. Phys. Acta 11 (1938) 299

We introduce the field operator U_μ

$$U_\mu = A_\mu + \frac{1}{\varkappa} \partial_\mu B \tag{2.5}$$

The free field Lagrangian is

BB·

$$\begin{aligned} \mathcal{L} = &- \frac{1}{2} \left[(\partial_\mu A_\nu)(\partial_\mu A_\nu) + \varkappa^2 A_\nu A_\nu \right] \\ &- \frac{1}{2} \left[(\partial_\mu B)(\partial_\mu B) + \varkappa^2 B_\mu B_\mu \right] \end{aligned} \tag{2.6}$$

and the commutation relations in the absence of interaction are

$$\begin{aligned} \left[A_\mu(x), A_\nu(x') \right] &= i \delta_{\mu\nu} \Delta(x - x'), \\ \left[B(x), B(x') \right] &= i \Delta(x - x') \end{aligned} \tag{2.7}$$

The commutation relation for the operator $U_\mu(x)$ is the same as that for $u_\mu(x)$ in the ordinary formalism, eqn.(2.4) is the derivative term coming from the B-field.

<u>The Interaction of a Vector Field with a Spinor Field:</u>

The total Lagrangian for a spinor field ψ interacting with a vector field U_μ, with a conserved current, may be written as follows:

$$\mathcal{L} = - \overline{\psi} \left[\gamma_\mu (\partial_\mu - ie U_\mu) + \varkappa \right] \psi +$$

free field – Lagrangian for the vector field (2.8)

Putting $U_\mu = A_\mu + \frac{1}{x} \partial_\mu B$

this becomes

$$\mathcal{L} = -\overline{\psi}\left[\gamma_\mu\left(\partial_\mu - ie\, U_\mu\right) + x\right]\psi - \frac{1}{2}\left[(\partial_\mu A_\nu)(\partial_\mu A_\nu) + x^2 A_\nu A_\nu\right] - \frac{1}{2}\left[(\partial_\mu B)(\partial_\mu B) + x^2 B^2\right] \tag{2.9}$$

From the Lagrangian (2.9) follow the equations of motion:

$$(\gamma_\mu \partial_\mu + x)\psi = ie(\gamma_\mu \bar{U}_\mu \psi), \tag{2.10}$$

$$(\Box - x^2) A_\mu = -ie\,\overline{\psi}\gamma_\mu\psi = -J_\mu, \tag{2.11}$$

$$(\Box - x^2) B = \frac{1}{x}\partial_\mu J_\mu = 0 \tag{2.12}$$

As seen above, U_μ plays a role very similar to u_μ

One can easily check the conservation of the current

$$\partial_\mu J_\mu = 0 \tag{2.13}$$

In the Stueckelberg formalism we have to impose the following subsidiary condition on the state vector

$$(\partial_\mu A_\mu + x B)\Psi = \Omega\Psi = 0 \tag{2.14}$$

It is easy to see that the above condition (2.14) is consistent with eqns.(2.1[illegible]) (2.12), Thus, if $\Omega\Psi = \dot{\Omega}\Psi = 0$ at $t = t_0$, then (2.11) is satisfied for all t. Notice that by virtue of (2.12), the condition (2.14) leads to

$$(\partial_\mu v_\mu)\Psi = 0 \tag{2.15}$$

The Lagrangian (2.9) and the subsidiary condition (2.14) are invariant under the 'gauge' transformation

$$\begin{aligned} A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x), \\ B(x) &\to B'(x) = B(x) - \varkappa \Lambda(x), \\ \psi(x) &\to \psi'(x) = \psi(x) \end{aligned} \tag{2.16}$$

use this to elim the

provided $\Lambda(x)$ satisfies

$$(\Box - \varkappa^2)\Lambda(x) = 0 \tag{2.17}$$

We also note that (2.9) is invariant under

either (I)

$$\begin{aligned} \psi(x) &\to \psi'(x) = e^{ie\Lambda(x)}\psi(x), \\ A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x), \\ B(x) &\to B'(x) = B(x) \end{aligned} \tag{2.18a}$$

or (II)

$$\begin{aligned} \psi(x) &\to \psi'(x) = e^{ie\Lambda(x)}\psi(x), \\ A_\mu(x) &\to A'_\mu(x) = A_\mu(x) \\ B(x) &\to B'(x) = B(x) - \varkappa \Lambda(x). \end{aligned} \tag{2.18b}$$

However, the subsidiary condition is not invariant under either (2.18a) or (2.18b)

The above gauge transformation can be generated by the unitary transformation $\exp\left(i G[\sigma]\right)$ defined on a space-like surface σ :

$$G[\sigma] = \int_\sigma d\sigma_\mu \left[\Lambda \partial_\mu (\partial_\lambda A_\lambda + \varkappa B) - \partial_\mu \Lambda \, (\partial_\lambda A_\lambda + \varkappa B)\right] \tag{2.19}$$

'Λ' may depend explicitly on the time. $G[\sigma]$ is hermitian; thus $\exp\left(i G(\sigma]\right)$ is a unitary operator. Under the unitary transformation effected by this operator, $A_\mu(x)$, $B(x)$ and $\psi(x)$ transform as follows (for x on the space-like surface σ):

$$\begin{aligned}
A_\mu(x) \to A'_\mu(x) &= e^{iG[\sigma]} A_\mu(x) e^{-iG[\sigma]} \\
&= A_\mu(x) + \partial_\mu \Lambda(x) \\
B(x) \to B'(x) &= e^{iG[\sigma]} B(x) e^{-iG[\sigma]} \\
&= B(x) - \varkappa \Lambda(x), \\
\psi(x) \to \psi'(x) &= e^{iG[\sigma]} \psi(x) e^{-iG[\sigma]} \\
&= \exp\left(ie\Lambda(x)\right) \psi(x)
\end{aligned} \tag{2.20}$$

Note: (2.9) is not invariant under (2.20)

In the case of free field, i.e. e = 0, ψ is unchanged by G. Then (2.9) is invariant. The subsidiary condition (2.14) also is invariant in this case.

By using the equations of motion one can easily show the surface independence of G that is, that

$$\frac{\delta}{\delta\sigma(x)} G[\sigma] = \partial_\mu\left[\Lambda \partial_\mu \Omega_\Lambda - \partial_\mu \Lambda \, \Omega_\Lambda\right] = 0, \tag{2.21}$$

$$\Omega_\Lambda = \partial_\lambda A_\lambda + \varkappa B .$$

provided $\Lambda(x)$ satisfies eq.(2.17)

This property of G, however, does not mean that G is a constant of motion, because G does not commute with P_μ the energy momentum 4-vector.

$$i\left[G, P_\mu\right] = \delta_\nu P_\mu = -i\left[P_\mu, G\right]$$
$$= \int_\sigma d\sigma_\nu \left[\Lambda \partial_\mu \partial_\nu \Omega_\lambda - \partial_\nu \Lambda \partial_\mu \Omega_\lambda\right] \tag{2.22}$$
$\neq 0$ in general.

This is due to the explicit dependence of Λ on x i.e. the fact that Λ is not a dynamical variable and can depend explicitly on x .

3. The free field:

In order to prepare for our later discussion in the interaction representation, let us study in this section some general properties of the free fields A_μ and B , combined by the subsidiary condition (2.14).

The field operators now satisfy the following commutation relations:

$$\left[A_\mu(x), A_\mu(x')\right] = i\,\delta_{\mu\nu}\,\Delta(x-x'), \quad \mu,\nu=1,2,3,4$$
$$\left[B(x), B(x')\right] = i\,\Delta_\chi(x-x') \qquad (3.1)$$

The equations of motion are

$$\left[\Box-\chi^2\right]A_\mu(x) = 0,$$
$$\left[\Box-\chi^2\right]B(x) = 0 \qquad (3.1a)$$

First, we may regard, as usual, all the operators A_i, $A_0 = -iA_4$ and B as Hermitian operators and expand them in terms of creation and annihilation operators $a_i(k)$, $a_0(k)$, $b(k)$ and $a_i^+(k)$, $a_0^+(k)$, & $b^+(k)$

Then, we have, for the free Hamiltonian, the following expression

$$H_0 = \sum_k E_k\left\{\left(\sum_i a_i^+(k)a_i(k)\right) - a_0^+(k)a_0(k) + b^+(k)b(k)\right\} \qquad (3.2)^*$$

Here, let us introduce the following terminology: In a special reference frame in which $k=0$ particles described by

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* We use the following notations: $+$ Hermitian conjugate;
* Complex conjugate, $(\pm)$ positive (negative) frequency part.

$a_i(0)$, $a_i^{\dagger}(0)$, $i = 1, 2, 3$ are called 'a_i-particles' or the 'ordinary particles', since these are the particles which appear in the ordinary formalism and the three states $i = 1, 2, 3$ correspond to three spin oriéntations. Particles described by $a_0(0)$, $a_0^{\dagger}(0)$ and $b(0)$, $b^{\dagger}(0)$ are called 'a_0-particles' and 'b-particles' respectively. These particles have no spin, since spin is defined by referring to space rotation. In (3.2) we see that the a_0-particles have negative energies, and, moreover, for this component the roles of creation and annihilation are interchanged, i.e. (3.1); a_0 is creation operator and $a_0^{\dagger}$ the annihilation operator (This can be seen from (3.1). We have $[A_0(x), A_0^{\dagger}(x')] = -i\Delta(x-x')$ so that $[a_0(k), a_0^{\dagger}(k)] = -1$, $[a_0^{\dagger}(k), a_0(k)] = +1$

In connection with this there arise the same kind of difficulties as those we encounter in quantum electrodynamics.

First, we look at the normalization of state vectors which satisfy the subsidiary condition (2.14). In terms of a's and b's. (2.14) takes the form

$$\left.\begin{aligned} &[i\vec{k}\, a(k) - i k_0 a_0(k) + \varkappa\, b(k)]\Psi = 0 \\ &[-i\vec{k}\, a(k)^{\dagger} + i k_0 a_0(k)^{\dagger} + \varkappa\, b(k)^{\dagger}]\Psi = 0 \end{aligned}\right\} \text{for all } k$$

(3.3)

In a special coordinate system in which $\vec{k} = 0$ (3.3) becomes

$$[-i\, a_0(0) + b(0)]\,\Psi = 0$$
$$[i\, a_0(0)^\dagger + b(0)^\dagger]\,\Psi = 0 \qquad (3.3')$$

Let us denote, by $\Psi(n, m)$ an eigenstate of H_0 with eigen-values n and m for the number operators of a_0 and b particles. By substituting the expansion

$$\Psi = \sum_{n,m} d(n,m)\,\Psi(n,m) \qquad (3.4)$$

into (3.3') we obtain the coefficients:

$$d(n,m) = c(-i)^n\,\delta_{nm}\,;$$

OR

$$i\, d(n,m)\sqrt{n+1} - c(m+1, n+1)\sqrt{m+1} = 0\,, \qquad (3.5)$$
$$i\, d(n,m)\sqrt{n} + d(n-1, m-1)\sqrt{m} = 0$$

where C is a constant. Thus, in the physical states satisfying (3.3'), the negative energies due to a_0-particles are exactly cancelled out by the contributions from b-particles. However, such states are no longer normalizable. As in the case of quantum electrodynamics, this difficulty is also manifested in the commutation relations. Since for the state $\Psi(0,0)$ there hold the relations $A_i^{(+)}\Psi(0,0) = B^\dagger\Psi(0,0) = 0$ and $A_i^{(-)}\Psi(0,0) = 0$, We have

$$0 = \langle \{ \Omega_\mu , A_\nu' \} \rangle_0 = -i \langle [\hat{\partial}_\mu \hat{A}_\mu^{(1)}, A_\nu'] \rangle_0$$

$$= \hat{\partial}_\nu \Delta^{(1)}(x - x') \neq 0 ,$$

where

$$\Delta^{(1)} = -i \ (\Delta^{+} - \Delta^{-}) ;$$

$$\hat{A}_\mu^{(1)} = \begin{Bmatrix} A_\nu^{(1)} \\ -A_4^{(1)} \end{Bmatrix} ; \quad \hat{\partial}_\nu = \begin{bmatrix} \partial_i \\ -\partial_4 \end{bmatrix}$$

In order to remove these difficulties we have to invoke the technique of an indefinite metric in the Hilbert space.

To do this we follow exactly the same procedure as was applied by Gupta and Bleuler in quantum electrodynamics. First, we regard all operators A_i , A_4 and B as Hermitian operators, and expand them as follows:

$$A_\mu(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2E_k}} \left[a_\mu(k) e^{ik\cdot x} + a_\mu^+(k) e^{-ik\cdot x} \right]$$

$$B(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2E_k}} \left[b(k) e^{ik\cdot x} + b^+(k) e^{-ik\cdot x} \right] \tag{3.6}$$

From (3.1), we get, for a_μ, a_μ^+, b and b^+

$$[a_\mu(k), a_\nu(l)] = \delta_{\mu\nu} \delta_{kl} ,$$

$$[b(k), b(l)] = \delta_{kl} \tag{3.7}$$

all other commutators = 0

The free Hamiltonian becomes

$$H_0 = \sum_k E_k \left\{ \left(\sum_{\mu=1}^{4} a_\mu^+(k) \, a_\mu \right) + b^+(k) b(k) \right\} \tag{3.8}$$

In contrast with (3.2) all 4 components of a_μ are treated here on an equal footing, and so all a_μ's are now annihilation operators and all a_μ^+s creation operators. The a_4 — particles have positive energies.

Secondly, we replace the subsidiary condition (2.14) by a weaker one

$$\Omega^{(+)} \Psi = 0 \, ; \qquad (3.9)$$
$$\Omega_\mu^{(+)} = (\partial_\mu A_\mu + x B)^{(+)}$$

This is done in order to remove the difficulty in compatibility of the commutation relations and the subsidiary condition. In momentum space, (3.9) reads

$$\left[i\vec{k}\, a(\vec{k}) - E_k\, a_4(k) + x\, b(k) \right] \Psi = 0, \quad \text{for all } \vec{k} \qquad (3.9')$$

and in a special system in which $\vec{k} = 0$

$$\left[a_4(0) - b(0) \right] \Psi = 0 \qquad (3.9'')$$

Thirdly, we introduce an indefinite metric, characterized by

$$\eta = (-1)^{N_4} = \eta^{-1} = \eta^\dagger \qquad (3.10)$$

where $N_4 = a_4^\dagger a_k$ and define the scalar product of a vector Ψ and the expectation value of an operator A by

$$\langle \Psi, \Psi \rangle = \Psi^* \eta \Psi \, , \qquad (3.11)$$

and

$$\langle A \rangle = (\Psi^* \eta A \Psi) \qquad (3.12)$$

respectively. The reality condition of an operator A is thus given by

$$A = \Psi^{*} \eta A \Psi \tag{3.13}$$

From (3.10) we have

$$A_i \eta = \eta A_i \qquad \text{(Consider only one mode } \vec{k}\text{)}$$
$$A_4 \eta = - \eta A_4 \tag{3.14}$$

We first assume these relations and (3.10) only for the part of A_μ with wave vector $\vec{k} = 0$ Then one can show, by means of a Lorentz transformation they also hold for all $\vec{k}$.

Hence, from (3.13) and (3.14),

$$\langle A_i \rangle = \langle A_i \rangle^{*},$$
$$\langle A_4 \rangle = - \langle A_4 \rangle^{*}. \tag{3.15}$$

as required. H_0 is real in this sense.

We are now in a position to study the property of state vectors which satisfy the condition (3.9''). Let us denote by $\Phi(n, m)$ an eigenstate of H_0 given by (3.8) in which occupation numbers n of a_4, b and a_i particles are n, $m+$ respectively. Substituting again the expansion

$$\Psi = \sum_{n,m} c(n, m) \, \Phi(n, m) \tag{3.16}$$

into (3.9'), we find, by use of the relation $a_4 \Phi(n,m) = \sqrt{n}\, \Phi(n-1, m)$, etc,

$$\frac{d\,(n+1,\, m)}{d\,(n,\, m+1)} = \sqrt{\frac{(m+1)}{(n+1)}} \,. \tag{3.17}$$

whence one gets the following set of orthonormal vectors, each of which satisfies the condition (3.9'')

$$\begin{aligned}
\Psi_0 &= \Phi(0,0) \\
\Psi_1 &= \Phi(1,0) + \Phi(0,1) \\
\Psi_2 &= \Phi(2,0) + \sqrt{2}\,\Phi(1,1) + \Phi(0,2), \\
\Psi_3 &= \Phi(3,0) + \sqrt{3}\,\big(\Phi(2,1) + \Phi(1,2)\big) \\
&\quad + \Phi(0,3), \\
&\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots \\
\Psi_n &= \sqrt{{}_nC_0}\,\Phi(n,0) + \sqrt{{}_nC_1}\,\Phi(n-1,1) \\
&\quad + \cdots + \sqrt{{}_nC_m}\,\Phi(n-m,m) + \\
&\quad \cdots + \sqrt{{}_nC_{n-1}}\,\Phi(1,n-1) + \sqrt{{}_nC_n}\,\Phi(0,n)
\end{aligned} \tag{3.18}$$

From (3.10) and (3.11) we see that the NORMS of all these vectors but Ψ_0 are zero.

$$\begin{aligned}
\langle \Psi_0, \Psi_0 \rangle &= 1 \,, \\
\langle \Psi_n, \Psi_n \rangle &= 0 \quad (n = 1, 2 \cdots n)
\end{aligned} \tag{3.19}$$

Or generally,

$$\langle \Psi_n , \Psi_{n'} \rangle = \delta_{n0} \delta_{n'0} \tag{3.19'}$$

It should be noted also that Ψ_n is the eigenstate of H_0 with the eigenvalue $n E_k$

$$H_0 \Psi_n = n_k E_k \Psi_n \tag{3.20}$$

but the expectation value of H_0 is nil:

$$\langle H_0 \rangle_n = \langle \Psi_n^* \eta H_0 \Psi_n \rangle = 0 \tag{3.21}$$

Any physical state which satisfies the subsidiary condition (3.9') can thus be expressed as a linear combination of Ψ_n's

$$\Psi = \sum_n C_n \Psi_n \tag{3.22}$$

which has certainly a finite Norm $|C_0|^2$ and the vanishing expectation value of H_0; $\langle H_0 \rangle = 0$, for $N_i = 0$, $i = 1, 2, 3$ Therefore, (3.22) can be regarded in general as a vacuum state. Sometimes we shall call Ψ_0 the true vacuum. The difficulty related to the commutation relation also disappears. Any state in which n_i, a_i particles exist can be given as a direct product of $\Phi(n_1, n_2, n_3)$ an eigenstate of $E_k a_i(k)^\dagger a_i(k)$ and the vector (3.22). When considering all the modes $\vec{k}$, we have to take an infinite product of the above kind of vectors.

Finally, let us consider effects of the gauge transformation defined by (2.16), (2.19). Since $\left[\Omega^{(+)}(x), \Omega^{-}(x')\right] = 0$ for any point x and x', the gauge transformation operator (for the free field) (2.19), commutes with $\Omega^{(+)}$. Therefore, any state vector which satisfies the subsidiary condition (3.9) is is transformed, tunder an arbitrary gauge transformation, into another vector which also satisfies the subsidiary condition, that is, if $\Omega^{(+)} \Psi = 0$ then

$$\Omega^{(+)} \left(e^{iG} \Psi \right) = 0 \tag{3.23}$$

In other words, the space spanned by the physical states $\Psi_0, \Psi_1, \Psi_2, \ldots\ldots$ is invariant under gauge transformations. We can see this more explicitly in momentum space as follows: When we write

$$\Lambda(x) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2E_k}} \left[\lambda(k) e^{ikx} + \lambda^*(k) e^{-ik\cdot x} \right], \tag{3.24}$$

then the gauge transformation (2.16) taken, in momentum space, the following form.

$$\vec{a}(k) \rightarrow \vec{a}'(k') = \vec{a}(k) + i\vec{k}\Lambda(k)$$

$$a_4(k) \rightarrow a_4(k') = a_4(k) - k_0 \Lambda(k),$$

$$b(k) \rightarrow b(k') = b(k) - x\Lambda(k)$$

(3.25)

$$e^{iG[\sigma]} = \exp\Big[\sum_k \Lambda(k)\left(i\vec{k}\,\vec{a}(k)^{+} + k_0\, a_4(k)^{+} + \chi\, b(k)^{+}\right) - \Lambda(k)^{*}\left(-i\vec{k}\,\vec{a}(k) + k_0\, a_4(k) + \chi\, b(k)\right)\Big] \tag{3.26}$$

If we take an infinitesimal gauge transformation such that

$$\Lambda(k) \neq 0 \quad \text{only for} \quad \vec{k} = 0,$$

then the effect of such a gauge transformation on the true vacuum state Ψ_0 for example, is

$$\begin{aligned} e^{iG[0]}\,\Psi_0 &= \left[1 + \Lambda\chi\left(a_4^{+}(0) + b^{+}(0)\right)\right]\Phi(0,0) \\ &= \Psi_0 + \Lambda\chi\,\Psi_1 \end{aligned} \tag{3.27}$$

Thus the true vacuum state Ψ_0 with the eigenvalue $E = 0$ of H_0 is transformed, by the gauge transformation (3.26), into a non-eigenstate. This implies that the true vacuum state Ψ_0 is not gauge-invariant, This we can also see from the fact that the Hamiltonian (3.8) is not invariant under the gauge transformation (3.25). (cf Eq (2.22) However, it is easy to show that the expectation value of H_0, $\langle H_0\rangle_0 = (\Psi_0^{*}\eta H_0 \Psi_0)$ remains zero and so it is gauge invariant*.

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* In quantum electrodynamics we have also the same situation. The true vacuum state in which no longitudinal and scalar photons exist is changed, under a gauge transformation, into a state which is not as eigenstate of the free Hamiltonian. Thus, the true vacuum state is not gauge invariant.

4. Relativistic Invariance of the Theory.

In the above we have considered state vectors only for the mode $\vec{k} = 0$. However, a general state vector is a linear combination of vectors of the form

$$\Psi = \prod_{\vec{k}} \Psi(\vec{k}) \tag{4.1}$$

where the $\Psi(\vec{k})$'s, is the state vectors for the modes $\vec{k}$, are all independent since the corresponding creation and annihilation operators commute with each other. We shall show in the following that in our discussion it is convenient to obtain $\Psi(\vec{k})$ with $\vec{k} \neq 0$ from $\Psi(\vec{k}=0)$ by means of a Lorentz transformation.

Let us consider a Lorentz transformation

$$x' = LX, \quad \text{or} \quad X'_\mu = a_{\mu\nu} X_\nu \tag{4.2}$$

For simplicity we can take, without loss of generality of our argument,

$$L = \| a_{\mu\nu} \|$$

$$= \frac{1}{\sqrt{1-v^2}} \begin{bmatrix} 1 & 0 & 0 & -iv \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ iv & 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

We assume that the transformation of A_μ and B

$$A_\mu(x) \longrightarrow A'_\mu(x') = a_{\mu\nu} A_\nu(x)$$

$$B(x) \longrightarrow B'(x') = B(x) \tag{4.4}$$

căn be generated by an operator $U(L)$ such that

$$U^{-1}(L)\, A_\mu(x)\, U(L) = a_{\mu\nu} A_\nu(L^{-1}x) = A'_\mu(x)$$

$$U^{-1}(L)\, B(x) U(L) = B(L^{-1}x) = B'(x) \tag{4.5}$$

We can then prove the relation

$$(\Psi^* \eta\, A'_\mu(x)\, \Psi) = (\Phi^* \eta\, A_\mu(x)\, \Phi)$$

$$(\Psi^* \eta\, B'(x)\, \Psi) = (\Phi^* \eta\, B(x)\, \Phi) \tag{4.6}$$

where

$$\Phi = U\Psi \tag{4.6'}$$

provided the operator U satisfies

$$U^\dagger \eta U = \eta \tag{4.7}$$

which is a generalized unitarity condition. Incidentally (4.6), (4.6'), and (4.7) enable us, as in the usual theories, to interpret the transformation in two ways i.e. as 'passive' and active' transformations (cf Wigner) For example, (6') implies that under the Lorentz transformation the state Ψ is transformed into another state state Φ - an active transformation.

Next, let us consider the transformation properties of $a_\mu(k)$, $a_\mu^+(k)$ defined by (3.6) and (3.7). If we interpret $a_\mu(k)$, $a_\mu^+(k)$ as the Belinfante operators which are defined as the averaged values over a small Lorentz invariant domain ϕ in the 4-dimensional momentum space ($\vec{k}$ being a projection of this region or to the space plane) of the operators $\tilde{a}_\mu(k)$, $\tilde{a}_\mu^+(k)$ introduced by

$$A_\mu(x) = \frac{1}{4}\frac{1}{\sqrt{\pi^3}}\int\frac{d^3k}{E_k}\left(\tilde{a}_\mu(k)e^{ik\cdot x}+\tilde{a}_\mu^+(k)e^{-ik\cdot x}\right) \tag{4.7a}$$

The operators $a_\mu(k)$, $a_\mu^+(k)$ are transformed in a covariant way under Lorentz transformations and so are $\tilde{a}_\mu(k)$ and $\tilde{a}_\mu^+(k)$ (*)

Substituting (3.6) into (4.5) we obtain

$$\begin{aligned}
U^{-1}a_\mu(k)U &= a_{\mu\nu}\,a_\nu(L^{-1}k)\\
U^{-1}a_\mu^+(k)U &= a_{\mu\nu}\,a_\nu^+(L^{-1}k)\\
U^{-1}b(k)U &= b(L^{-1}k)\\
U^{-1}b^+(k)U &= b^+(L^{-1}k)
\end{aligned} \tag{4.8}$$

(*) It should be remarked that the operators $a_\mu(k)$, $a_\mu^+(k)$ defined by $A_\mu(x) = \frac{1}{\sqrt{V}}\sum_k\frac{1}{\sqrt{2E_k}}\left(a_\mu(k)e^{ikx}+a_\mu^+(k)e^{-ik\cdot x}\right)$ do not transform in a covariant way, although the last expression looks as if it tends to the expression (4.7a) as $V\to\infty$

whence

$$a_\nu(k') = \sum_\mu a_{\mu\nu}\, U^{-1} a_\mu(k) U\,,$$

$$b(k') = U^{-1} b(k)\, U\,, \text{ etc} \tag{4.8'}$$

where $k' = L^{-1} k$

For the L given by (4.3) we have

$$a_1(k) = \frac{1}{\sqrt{1-v^2}}\, U^{-1}\left(a_1(0) + iv\, a_4(0)\right) U\,,$$

$$a_2(k) = U^{-1}\, a_2(0)\, U\,,$$

$$a_3(k) = U^{-1}\, a_3(0)\, U\,,$$

$$a_4(k) = \frac{1}{\sqrt{1-v^2}}\, U^{-1}\left(a_4(0) - iv\, a_1(0)\right) U\,,$$

$$b(k) = U^{-1} b(0)\, U\,; \tag{4.9}$$

and

$$a_1^+(k) = \frac{1}{\sqrt{1-v^2}}\, U^{-1}(a_1^+(0) + iv\, a_4^+(0) U\,,$$

$$a_2^+(k) = U^{-1}\, a_2^+(0)\, U\,,$$

$$a_3^+(k) = U^{-1}\, a_3^+(0)\, U\,,$$

$$a_4^+(k) = \frac{1}{\sqrt{1-v^2}}\, U^{-1}\left(a_4^+(0) - iv\, a_1^+(0)\right) U\,,$$

$$b^+(k) = U^{-1} b(0)\, U \tag{4.10}$$

where we have put $\vec{k} = 0$ and dropped the prime from k'

and so $k_\mu \left(= k_\mu'\right) = \left(\frac{-vx}{\sqrt{1-v^2}}, 0, 0, \frac{ik}{\sqrt{1-v^2}} \right)$

By taking the Hermitian conjugate of (4.10) and comparing it with (4.9), we obtain an alternative expression for $a(k)$ and $a^+(k)$

$$a_1(k) = \frac{1}{\sqrt{1-v^2}} \; u^+ \left(a_1(0) - iv\, a_4(0) \right) (u^+)^{-1}$$

$$a_2(k) = u^+ \; a_2(0) \, (u^+)^{-1}$$

$$a_3(k) = u^+ \; a_3(0) \, (u^+)^{-1},$$

$$a_4(k) = \frac{1}{\sqrt{1-v^2}} \; u^+ \left(a_4(0) + iv\, a_1(0) \right) (u^+)^{-1},$$

$$a_1^+(k) = \frac{1}{\sqrt{1-v^2}} \; u^+ \left[a_1^+(0) - iv\, a_4^+(0) \right] (u^+)^{-1},$$

$$a_2^+(k) = u^+ \, a_2(0) \, (u^+)^{-1},$$

$$a_3^+(k) = u^+ \, a_3(0) \, (u^+)^{-1},$$

$$a_4^+(k) = \frac{1}{\sqrt{1-v^2}} \; u^+ \left(a_4^+(0) + iv\, a_1^+(0) \right) (u^+)^{-1} \tag{4.11}$$

We may regard (4.9) and (4.10) as the defining equations for $a_\mu(k)$, $a_\mu^+(k)$ in terms of $a_\mu(0)$, $a_\mu^+(0)$

From (3.14) follows the relations given below for $a_\mu(k)$, $k \neq 0$

$$a_i(k)\eta = \eta\, a_i(k),$$
$$a_i^{+}(k)\eta = \eta\, a_i^{+}(k),$$
$$a_4(k)\eta = -\eta\, a_4(k)$$
$$a_4^{+}(k)\eta = -\eta\, a_4^{+}(k)$$

(3.14')

We assume (3.14) only for $\vec{k} = 0$ and obtain (3.14') by a Lorentz transformation from the corresponding relations for $a_\mu(0)$, $a_\mu^{+}(0)$ This can be proved by using (4.9). (4.10) (4.11) and (4.7) This indicates that our formalism of the indefinite metric is self-consistent and also that although the properties of η look non-covariant, the whole formalism is essentially covariant.

Now, a let us consider the subsidiary condition (3.9') for $\Psi(k)$,

$$\left[c\, k_1 a_1(k) - E_k\, a_4(k) + x\, b(k)\right] \Psi(k) = 0$$

(3.9')

By using (4.8') the expression in the bracket can be written as

$$-x\, U^{-1}\left(a_4(0) - b(0)\right) U$$

So (3.9') becomes

$$\left(a_4(0) - b(0)\right) U\, \Psi(k) = 0$$

(3.9'')

The solution of eqn. (3.''') is

$$U \Psi(k) = \sum \Psi_i (n_1, n_2, n_3) \Psi_i \qquad (4.12)$$

where $\Psi_i(n_1, n_2, n_3)$ is the wave function for the a_1, a_2, a_3 modes and Ψ_i is the wave function for the a_4, b_4 mode

or

$$\Psi(k) = U_k^{-1} \left(\sum_i \Psi_i (n_1, n_2, n_3) \Psi_i \right. \qquad (4.12a)$$

where we have attached a suffix k to U as U is different for different k. We can easily show that the state vector (4.12) is an eigenvector of

$$H_k = E_k \left(\sum_\mu a_\mu^+(k) a_\mu(k) + b^+(k) b(k) \right) \qquad (4.12b)$$

$$= E_k U^{-1} \left(\sum_\mu a_\mu^+(k) a_\mu(k) + b^+(k) b(k) \right) U .$$

H_k is the part of the Hamiltonian for the mode k. Thus, to specify state vedtors $\Psi(k)$ we can use eigenvalues of the number of operators in the corresponding rest system viz. $a_i^+(0)\, a_i(0)$, $a_4^+(0)\, a_4(0)$ Since the operator U satisfies (4.7) the norm of $\Psi(k)$ is

$$\left(\Psi(k')^* \eta\, \Psi(k) \right) = \left(\Psi_0 (n_1, n_2, n_3), \Psi_0 (n_1, n_2, n_3) \right)$$

$$\equiv c_0^2 \qquad (4.13)$$

and the expectation value of H_k is

$$\langle H_k \rangle = (n_1 + n_2 + n_3) E_k |c|^2 \qquad (4.14)$$

To specify the states $\Psi(k)$ we can also use the eigenvalues of $a_i^+(k)\,a_i(k)$, $a_4^+(k)\,a_4(k)$ and $b^+(k)\,b(k)$;

$$\Psi_{\substack{n_1\\n_2\\n_3\\n_4\\b}}(k) = \frac{1}{\sqrt{n_1!\cdots n_b!}} \prod_i \left(a_i^+(k)\right)^{n_i}\left(a_4^+(k)\right)^{n_4} \times \left(b^+(k)\right)^{n_b} |0\rangle \tag{4.14a}$$

$$= \frac{1}{\sqrt{n_1!\ldots n_b!}} \left[\frac{1}{\sqrt{1-v^2}}\right]^{n_4+n_1} \left\{\left(a_1^+(0) + iv\, a_4^+(0)\right)^{n_1}\right\} \left\{\left(a_4^+(0) - iv\, a_1^+(0)\right)^{n_4}\right\}\left(a_2^+(0)\right)^{n_2}\left(a_3^+(0)\right)^{n_3} \times \left[b^+(0)\right]^{n_b} |0\rangle \tag{4.14b}$$

where we have chosen the X axis in the direction of the momentum $\vec{k}$

We can then show that from the relation

$$\left(\Psi_{n_i', n_4', n_b'}(0), \Psi_{n_i, n_4, n_b}(0)\right) = (-1)^{n_4(0)} \delta_{n_i' n_i}\, \delta_{n_4' n_4} \times \delta_{n_b' n_b} \tag{4.14c}$$

follows the relation

$$\langle \Psi_{n_i' n_4' n_b'}(k), \Psi_{n_i, n_4, n_b}(k)\rangle = (-1)^{n_4(k)} \delta_{n_i n_i'}\, \delta_{n_4 n_4'}\, \delta_{n_b n_b'} \tag{4.14d}$$

This, together with the argument given after eqn. (3.14') completes the proof of the covariance of the metric. However, the use of such vectors as given by eqn. (4.14a) is inconvenient because of the complicated subsidiary condition (3.9')

So far we have assumed the existence of the operator U which satisfies the conditions (7), (9) and (10) (or (5)). We shall now prove this by explicitly constructing the operator.

It is sufficient to discuss the case of an infinitesimal Lorentz transformation $L(v)$ of the form (4.3) v being small enough so that v^2 may be neglected. The corresponding transformation U may be constructed as follows:

Write U as a product of two transformations U_a and U_b, such that U_a and U_b commute and satisfy the following relations:

$$U = U_a U_b = U_b U_a ,$$

$$U_a^{-1} a_1(0) U_a = a_1(k) + iv\, a_4(k) ,$$

$$U_a^{-1} a_2(0) U_a = a_2(k) ; \quad U_a^{-1} a_3(0) U_a = a_3(k)$$

$$U_a^{-1} a_4(0) U_a = a_4(k) + iv\, a_1(k)$$

$$U_b^{-1} b(0) U_b = b(k) , \tag{4.15}$$

and

$$U_a^{-1} a_1^{\dagger}(0) U_a = a_1^{\dagger}(k) - iv\, a_4^{\dagger}(k)$$

$$U_a^{-1} a_2^{\dagger}(0) U_a = a_2^{\dagger}(k) ; \quad U_a^{-1} a_3^{\dagger}(0) U_a = a_3^{\dagger}(k)$$

$$U_a^{-1} a_4^{\dagger}(0) U_a = a_4^{\dagger}(k) + iv\, a_1^{\dagger}(k) ,$$

$$U_a^{-1} b^{\dagger}(0) U_a = b^{\dagger}(k) \tag{4.15a}$$

It is easily verified that $U = U_a U_b$ satisfies the conditions (7) (9), and (10).

U_a may be constructed as a product of a non-infinitesimal transformation U_{a_1} and an infinitesimal transformation U_{a_2}

$$U_a = U_{a_1} U_{a_2} , \tag{4.16}$$

The field operators get transformed as

$$U_{a_1}^{-1} a_1(0) U_{a_1} = a_1(k) ; \quad U_{a_1}^{-1} a_2(0) U_{a_1} = a_2(k)$$

$$U_{a_1}^{-1} a_4(0) U_{a_1} = a_4(k) , \quad U_{a_1}^{-1} a_3(0) U_{a_1} = a_3(k)$$

$$U_{a_1}^{-1} a_1^{+}(0) U_{a_1} = a_1^{+}(k) , \quad U_{a_1}^{-1} a_2^{+}(0) U_{a_1} = a_2^{+}(k)$$

$$U_{a_1}^{-1} a_4^{+}(0) U_{a_1} = a_4^{+}(k) , \quad U_{a_1}^{-1} a_3^{+}(0) U_{a_1} = a_3^{+}(k)$$

(4.17)

and

$$U_{a_2}^{-1} a_1(k) U_{a_2} = a_1(k) - i v a_4(k) ,$$

$$U_{a_2}^{-1} a_4(k) U_{a_2} = a_4(k) + i v a_1(k) ,$$

$$U_{a_2}^{-1} a_1^{+}(k) U_{a_2} = a_1^{+}(k) - i v a_4^{+}(k)$$

$$U_{a_2}^{-1} a_4^{+}(k) U_{a_2} = a_4^{+}(k) + i v a_1^{+}(k)$$

(4.18)

(and U_{a_2} leaves $a_2(k)$, $a_3(k)$, $a_2^{\dagger}(k)$, & $a_3^{\dagger}(k)$ unchanged)

An operator U_{a_1} with the properties (4.17) is given by

$$U_{a_1} = \exp\left\{ -\frac{\pi}{2} \sum_{\mu=1}^{4} \left[a_\mu^{\dagger}(k)\, a_\mu(0) - a_\mu^{\dagger}(0)\, a_\mu(k) \right] \right\} \tag{4.19}$$

Similarly, U_{a_2} and U_b are given by

$$U_{a_2} = 1 + iv \left[a_4^{\dagger}(k)\, a_1(k) - a_1^{\dagger}(k)\, a_4(k) \right], \tag{4.20}$$

$$U_b = \exp\left\{ -\frac{\pi}{2} \left[b^{\dagger}(k)\, b(0) - b^{\dagger}(0)\, b(k) \right] \right. \tag{4.21}$$

We have thus given an explicit construction of the operator U obeying the general unitarity condition (4.7)

Till now we have dealt only with the free Stueckelberg field. We shall now construct the interacting Stueckelberg field.

5. The field in interaction

Let us go back again to the case of the interacting field, the Lagrangian of which is given by (2.9). The total Hamiltonian is

$$H = H_0 - ie\,\overline{\Psi}\gamma_\mu\Psi\,\overline{U}_\mu - \frac{ie^2}{2\varkappa^2}\left(\overline{\Psi}\gamma_\mu\Psi\right)^2 \tag{5.1}$$

when H_0 is given by (3.8) and the different sign of the third term is due to the use of a non-canonical variable in the second term. Since A_4 is now taken to be Hermitian the interaction term $-ie(\overline{\Psi}\gamma_4\Psi)A_4$ is not Hermitian in the usual sense, but satisfies the reality condition (3.13)

$$H = \eta^{-1}H^{+}\eta \tag{5.2}$$

which in fact guarantiees the conservation of the norm of the wave functions, $\langle\Psi,\Psi\rangle = (\Psi^{*}\eta\Psi)$ That is, if $\Psi(t) = \sum_\alpha c_\alpha(t)\,\Phi_\alpha$ and $\langle\Psi(0),\Psi(0)\rangle = 1$ then for any t we have

$$1 = \sum_\alpha |c_\alpha(t)|^2\eta_\alpha$$

The 'Unitarity' of the S-matrix with such a Hamiltonian is now expressed in the form

$$S^{+}\eta S = \eta \tag{4.3}$$

The completeness relation is written in the form

$$\sum_\alpha \Phi_\alpha \Phi_\alpha^* = 1 \ ; \left(\overset{\text{with}}{\eta \Phi_\alpha = \eta_\alpha \Phi_0} \right) \tag{4.4}$$ *

From (4.3) and (4.4) follows the following relations

$$\eta_\alpha = \sum_\beta \eta_\beta \left| (\Phi_\beta^* \eta S \Phi_\alpha) \right|^2 = \sum_\beta \eta_\beta \left| \langle \beta | S | \alpha \rangle \right|^2 \tag{4.5}$$

Thus, for an initial state Φ_α with $\eta_\alpha = 1$, we have the conservation law of probability in the form

$$1 = \sum_\beta \eta_\beta \left| \langle \beta | S | \alpha \rangle \right|^2$$

where $\left| \langle \beta | S | \alpha \rangle \right|^2$ gives the transition probability for $\alpha \to \beta$

Now, we should notice that even in the case of interacting field, the operator $\Omega_\mu = \partial_\mu A_\mu + \varkappa B$ in the Heisenberg representation still satisfies the free Klein-Gordon equation, as can be seen from (2.11), (2.12). Therefore in this representation we can define the positive-frequency part in an invariant way and impose a subsidiary condition

$$\Omega_\mu^{(+)} \Psi_H = 0 \tag{4.6}$$ +

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* Notice that our notation and definition are somewhat different from those employed in some literatures.

\+ The suffices H, I and S refer to the Heisenberg interaction and Schrodinger representations, respectively.

Thus, if we assume $\Omega^{(+)}_H \Psi_H = \dot{\Omega}^{(+)}_H \Psi_H = 0$ at a certain time $t = t_0$ then (4.6) will hold at any time t.

In terms of canonical variables the operator Ω can be written in the form

$$\Omega_H = \partial_i A_{iH} - i \Pi A_H + \varkappa B_H \tag{4.6'}$$

In order to obtain the corresponding subsidiary conditions in the interaction and Schrodinger representation one has only to apply canonical transformations to (4.6'). Since canonical variables go over into the corresponding ones in other representations and the field A_H has no coupling of derivative types, the subsidiary conditions in other representations are of the same form as (4.6') but with canonical variables in the respective representations. Thus, in the interaction representation we have

$$\left(\partial_\mu A_\mu(x)_I + \varkappa B(x)_I \right)^+ \Psi_I(t) = 0 \tag{4.7}$$

In the Schodinger representation it is convenient to use the expression in momentum space,

$$\left[i\vec{k}\, \vec{a}_S(k) - k_0\, a_{4S}(k) + \varkappa\, b_S(k) \right] \Psi_S(t) \overset{?}{=} 0 \tag{J.8}$$

We shall now assume in the Schrodinger representation that all initial states are confined to vectors of the form

$$\Psi_{iS}(t = -\infty) = \Psi_\alpha \prod_k \Psi_0(k) \tag{4.9}$$

+Where Ψ_α is a wave function for a_i particles and $\Psi_0(k)$ the true vacuum state for a_4 and b particles in the mode $\vec{k}$. (4.9) clearly satisfies the subsidiary condition (4.8). Since the final state $\Psi_{f_S}(t=\infty)$ also satisfies (4.8) it should have the form

$$\Psi_{f_S}(t=+\infty) = \sum_\beta \Psi_\beta \prod_k \left(\sum_n C_{\beta n}(k)\, \Psi_n(k) \right)$$

with Ψ_n's defined by (3.18)

Here we shall make the following remarks (i) If an initial state is an eigenstate of the total Hamiltonian, then so is the corresponding state at any time t, and the eigenvalue E is, so to speak conserved, (ii) since $\langle H \rangle$ satisfies (4.2) is also conserved, We can also assume the asymptotic condition $H \longrightarrow H_0$ as $t \longrightarrow \pm\infty$. The energy eigenvalue of the initial state $E_i = E_\alpha$ and its expectation value $\langle H \rangle_i = E_\alpha$ when E_α is the eigenvalue of Ψ_α . The energy eigenvalue of the final state $E_f = E_\beta + \sum E_n = E_{\beta'} + \sum_{n'} E_{\beta'} = \ldots\ldots$ where the second terms come from the states $\Psi_n(k)$, $\Psi_{n'}(k)$ From (i), we have $E_i = E_f$ or

$$E_\alpha = E_\beta + \sum E_n, \quad = E_{\beta'} + \sum E_{n'} \qquad (4.11)$$

and from (ii) we have $\langle H \rangle_i = \langle H \rangle_f$ or

$$E_\alpha = E_\beta \qquad (4.12)$$

Therefore, from (4.11) and (4.12),

$$\sum_n E_n \equiv 0 \tag{4.13}$$

Since the E_i's cannot be negative, (4.13) implies that every $E_n = 0$ This means that (4.10) must have the same form as (4.9) i.e. $\Psi_f = \Psi \prod_k \Psi_0(k)$. Thus, we can conclude that actual transitions take place only between states in which only the ordinary vector particles exist. The condition (4.5) now takes the form

$$1 = \sum_\beta | \langle\beta|S|\alpha\rangle |^2 = \sum_\beta |(\Psi_\beta^* S \Psi_\alpha)|^2 \tag{4.5}$$

Hence, in the sub-space of our Hilbert space the S-matrix is unitary in the conventional sense.

6. Comparison of various formulations:

In this section we shall discuss the connection between different formulations or representations of the vector field. For this purpose the interaction representation is the most convenient. The basic relations in the 'Stueckelberg representation' are as follows:

The Tomonaga equation is

$$i \frac{\delta \Psi[\sigma]_S}{\delta \sigma[x]} = H_{int}(x)\, \Psi[\sigma]_S \qquad (6.1)$$

with

$$H_{int}(x) = -J_\mu U_\mu + \frac{1}{2\chi^2}(J_\mu \chi_\mu)^2 \qquad (5.1')$$

$$\overline{U}_\mu = A_\mu + \frac{1}{\chi}\partial_\mu B \qquad (5.1'')$$

$$J_\mu = ie\,\overline{\Psi}\gamma_\mu \Psi \qquad (5.1''')$$

The operators A_μ and B satisfy the commutation relations given by (3.1). The subsidiary condition (4.7) in now generalized to

$$\left[\partial_\mu A_\mu(x) + \chi B(x)\right]^{(+)} \Psi[\sigma]_{(S)} = 0 \qquad (5.2)$$

where the point x need not necessarily be on the space like surface σ *. It is easy to check the following relations:

(i) the integrability condition $\left[i\frac{\delta}{\delta\sigma(x)} - H_{int}(x),\ i\frac{\delta}{\delta\sigma(x')} - H_{in}\right. = 0$

for a space-like $(x - x')$

* This generalized subsidiary condition is simpler in form than the one in quantum electrodynamics which contains also a term depending on J_μ

(ii) $\left[\Omega^{+}(x), \Omega^{+}(x')\right] = 0$ for any points x, x'

(iii) $\left[i\frac{\delta}{\delta\sigma(x)} - H_{int}(x), \Omega^{(+)}(x)\right] = 0$ for any points. x, x'

. The last two guarantee the consistency of (5.2) with (5.1). The commutation relations for $\overline{U}_{\mu}$ are from (3.1),

$$\left[U_{\mu}(x), U_{\nu}(x')\right] = i\left[\delta_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\chi^{2}}\right]\Delta_{\chi}(x-x') \quad (3.3)$$

Since (5.1') and (5.3) (and consequently the propagation function for the U_{μ}) have the same form as the corresponding quantities in the ordinary formalism of vector field, it is evident therefore that as far as the part of S-matrix elements which contain only virtual meson lines is concerned, the Stueckelberg formalism gives the same result is the ordinary formalism. As for the part which involves external meson lines we can make the following observation: As was proved in the previous section, if there are no a_4 or b particles present in initial states, such particles will not appear in final states either. Thus, under this circumstance, the incoming and outgoing external mesons are restricted only to the ordinary vector particles. The part of matrix elements which are responsible for absorption and emission of these particles are exactly the same in both the formalisms, since the interaction Hamiltonian is the same, Therefore, we can conclude that both formalisms are quite equivalent with each other as far as the S-matrix is concerned. We shall now make the unitary transformation

$$\Psi[\sigma]_S \rightarrow \Psi[\sigma]_{(MG)} = exp\left[-\frac{i}{\chi}\int_\sigma d\sigma_\mu J_\mu B\right] \times \Psi[\sigma]_S \qquad (5.4)$$

The Tomonaga equation is changed to

$$\frac{i\,\delta\Psi[\sigma]_{MG}}{\delta\sigma[x]} = -(J_\mu A_\mu)\,\Psi[\sigma]_{MG} \qquad (5.5)$$

and the subsidiary condition to

$$\left[\left(\partial_\mu A_\mu(x) + \chi B(x)\right)^+ - \int_\sigma \Delta^{(+)}(x-x')\,J'_\mu\,d\sigma'_\mu\right] \times \Psi[\sigma]_{MG} = 0 \qquad (5.6)$$

This we shall call the 'Matthews-Glauber representation' which is the most convenient one for discus the connection between a vector field and the photon field. The limit $\chi \longrightarrow 0$ can be made without encountering singularities. The complete disappearance of the field B from the interaction Hamiltonian is due to the conservation of the current J_μ. Thus, we can interpret (5.6) as a defining equation of B and regard all A_μ's as independent variables. The whole formalism is invariant under the gauge transformation (similar to (2.9).

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda \quad \text{(with } \Lambda \text{ arbitrary)} \qquad (5.7)$$

if this is followed by a unitary transformation on the state vector

$$\Psi[\sigma]_{MG} \rightarrow \Psi[\sigma']_{MG} = exp\left[-i\int_\sigma d\sigma_\mu J_\mu \Lambda\right] \times \Psi[\sigma]_{MG} \qquad (5.7')$$

The subsidiary condition remains unchanged if Λ satisfies $\Box\Lambda = 0$. Such a restriction, however, may not be necessary unless we need the variable B. At any rate, because of this invariance the effective number of independent variables reudces to 3. This representation is also very suitable for discussing the renormalizability problem of vector field interactions. Since non-renormalizable divergences come only from the B-field, whether an interaction is renormalizable or not depends on whether the B-field can be eliminated from the Interaction Hamiltonian by some unitary transformation. Notice here that the eqns. in the represent are of similar form to those of Quantum Electro Dynamics. Moreover in this representation one can make the limit $\varkappa \to 0$ without encountering singularities such as those which appear in the ordinary formalism. The connection ~~course~~ between the two theories and $\varkappa \neq 0$ and $\varkappa = 0$ (photon) are very clear here.

We shall make a further unitary transformation

$$\Psi_{MG}[\sigma] = \Psi[\sigma]_{op} = \exp\left[\frac{-i}{\varkappa^2}\int_\sigma d\sigma_\mu\, J_\mu \frac{\partial A_\lambda}{\partial x_\lambda}\right]\Psi[\sigma] \tag{5.8}$$

to get the Tomonaga equation

$$i\,\frac{\delta\Psi[\sigma]_{OP}}{\delta\sigma[x]} = \left\{ -J_\mu\left[A_\mu - \frac{1}{\varkappa^2}\partial_\mu\left(\frac{\partial A_\lambda}{\partial x_\lambda}\right)\right] + \frac{1}{2\varkappa^2}\left(J_\mu n_\mu\right)^2\right\}\Psi[\sigma]_{OP} \tag{5.9}$$

and the subsidiary condition

$$\Omega^{(+)} \Psi[\sigma]_{oP} = 0 \tag{5.10}$$

$$\Omega^{(+)} = \partial_\mu A_\mu(x) + \varkappa B(x)$$

This we shall call the Ogievetski-Polubarinov representation. If we put

$$u_\mu(x) = A_\mu - \frac{1}{\varkappa^2}\partial_\mu\left(\frac{\partial A_\lambda}{\partial x_\lambda}\right) \tag{5.11}$$

then u_μ satisfies

$$[u_\mu(x), u_\nu(x')] = i\left[\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\varkappa^2}\right]\Delta_\varkappa(x - x') \tag{5.12}$$

and $\partial_\mu u_\mu = 0$ (as an operator Eq) (5.13)

$$\left[\, u_\mu(k)^+ \sim a_\mu(k) - \frac{1}{\varkappa^2}(ik_\mu)(ik_\lambda)\, a_\lambda(k)\right.$$

$$\text{For } \vec{k} = 0, \quad u_\mu = 0 \sim a_\mu(0) - \delta_{\mu 4}\, a_4(0)$$

Here, we can clearly see that only the field described by u_μ has actually the interaction with the ψ- field. The relations (5.13), (5.14) show that this field is nothing but the ordinary vector field. Its Fourier components $u_\mu(k)$ in the rest system $\vec{k} = 0$ are $u_i(k) \neq 0$, $u_4(k) = 0$ which imply that u_μ describes particles with three independent states. The equivalence between the Stueckelberg formalism and the ordinary formalism is most obvious in this representation. The subsidiary condition (5.10) plays the role only of restricting Hilbert space into its sub-space of ordinary vector particles*. Finally, we should remark the following point: A question such as whether a_4 or b -particles exist in virtual state has no definite physical meaning; as is clear from the above arguments, the existence of these particles depends on which representation is being employed. For example, no b-particles appears at any time t in the Matthews-Glauber representation if it does not exist at $t = -\infty$ but this is not the case with the Stuckelberg representation.

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* Ogievetski and Polubarinov claim in their paper, that such a subsidiary condition is unnecessary in their formalism: However, when they restrict themselves into a particular sub-space, a subsidiary condition is essentially being introduced.

7. The case of non-conserved current:

In this section we shall briefly discuss the case in which the current J_μ is not conserved. The Stueckelberg formalism is still possible in this case. This is because the operator satisfies the free Klein-Gordon equation, regardless of whether J_μ is conserved or not. The subsidiary condition (2.14) is thus compatible with the field equations. It should be noticed that in case of non-conserved current the corresponding condition in the ordinary formalism is not satisfied,

$$\partial_\mu u_\mu = \frac{1}{\lambda^2}\partial_\mu J_\mu \neq 0$$

. The proof of equivalence between the two formalisms can be given, however, along lines similar to the one in section 4.

The only difference between two cases is in the behaviour under gauge transformations: the non-conserved current J_μ is intimately connected with non-invariance under gauge transformation. As discussed elsewhere in great details, the B-field cannot be completely eliminated in the Matthews-Glauber representation.[1] It remains, after unitary transformation, in gauge non-invariant terms. Accordingly we do not have any simple M.G or Ogievetski-Polubarinov representations.[2] Such a theory is non-renormalizable in general.

(1) P. T. Matthews Phys. Rev. 76 (1949) 1254
R. J. Glauber Prog. Theor. Phys 9 (1953) 295

(2) V. I. Ogievelsky and I.V. Polubarinov: Proc. International conf. on High-Energy Physics, CERN (1962), p. 666

8. Interaction of a Single Hermitian Stueckelberg Field with a Charged Field.

We have discussed, in the previous lectures, a fairly simple system of a real Stueckelberg field interacting with a spinor field via vector coupling: such a system has the nice feature that the theory is gauge invariant. As a consequenc of this, we saw that the MG or OP representations take very simple forms. Today we shall consider a more general system such as a Stueckelberg field which is not hermitian or the system with the more general forms of interactions, with a special emphasis on the renormaliability problem.

To begin with, we shall discuss in a general way the system of a single Hermitian Stueckelberg field interacting with a complex or charged spin 0, 1/2 or 1 fields.

In QED, we introduce the electromagnetic interaction by replacing, in the free field Lagrangian, $\partial_\mu \to \partial_\mu - ieA_\mu$ As is well-known, the theory thus obtained in gauge invariant. For a single Stueckelberg field let us do the same thing; We introduce the interaction by the replacement

$\partial_\mu \to \partial_\mu - if\, U_\mu$ where

$$U_\mu = A_\mu + \frac{1}{\varkappa}\partial_\mu B \qquad (7.1)$$

Since the free Lagrangian in general contains second order terms in ∂_μ, the interaction terms contain not only terms like $J_\mu U_\mu$ but also terms such as $J_{\mu\nu} A_\mu A_\nu$, where $J_\mu, J_{\mu\nu}$ consist only of the source field, and so the interaction Hamiltonian in the interaction representations takes the following form.

$$H_{int}(x) = - f J_\mu U_\mu - \frac{1}{2} f^2 J_{\mu\nu}(x) U_\mu U_\nu + \frac{1}{2\chi^2} f^2 (J_\mu U_\mu)^2 \tag{7.2}$$

The Tomonaga-Schwinger equation is given by

$$i \frac{\delta \Psi[\sigma]}{\delta\sigma} = H_{int} \Psi[\sigma] \tag{7.3}$$

The second term in (7.2) is absent for the interaction with a spinor field, as $J_{\mu\nu}$ must be symmetric in μ, ν. Noting this, (7.2) describes the interactions with source fields having spin 0, 1/2 or 1.

For spin 1/2, $J_{\mu\nu} = 0$ so the Lagrangian becomes linear in U_μ. Note also that $J_\mu, J_{\mu\nu}$ are the same as the corresponding quantities of Quantum Electrodynamics.

In terms of the Duffin-Kemmer operators β_μ's the interaction Hamiltonian may be written

$$H_{int} = -i f (\bar{\Psi} \beta_\mu \Psi) U_\mu + f^2/\chi \; \bar{\Psi} \beta_\mu [1 + (\beta_\nu U_\nu)^2] \beta_\rho \Psi U_\mu U_\rho \tag{7.4}$$

Now, as we introduced the interaction is a gauge-invariant manner, we have the conservation of the current J_μ in the Heisenberg representation,

$$\partial_\mu J_\mu = 0 \tag{7.5}$$

This can be expressed in the interaction representation by

$$n_\mu(x)\left[J_\mu(x), J_\nu(x')\right] = i\, n_\mu(x)\, \partial_\rho\, J_{\rho\nu}(x,n) \times \partial_\mu\, \Delta(x-x')$$

$$n_\mu n_\nu \left[J_\mu(x), J_\nu(x')\right] = 0 \tag{7.6}$$

$$n_\mu \left[J_\mu(x), J_{\mu\nu}(x', n')\right] = 0$$

(Remarks: $(x - x')$ is space-like; hence ∂_μ does not operate on n in $J_{\mu\nu}$).

The above relations were first obtained by Umezawa.*

The Tomonaga-Schwinger equation (7.3) is invariant under the gauge transformation

$$U_\mu \to U_\mu' = U_\mu + \partial_\mu \Lambda, \quad A_\mu \to A_\mu' = A_\mu + \partial_\mu \Lambda, \quad B' \to B$$

OR

$$B \to B' = B - x\Lambda, \quad A_\mu' = A_\mu \tag{7.7}$$

provided one makes the following transformation for $\Psi[\sigma]$

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* Ref. H. Umezawa. and S. Kamefuchi Nucl Phys 23 (1961) 399
See also C.N. Yang and R.L Mills Phys Rev 96 (1959) 191

$$\Psi[\sigma] \to \Psi[\sigma']$$
$$= \exp\left[-i\int d\sigma_\mu J_\mu(x)\,\Lambda(x)\right]\Psi[\sigma] \tag{7.8}$$

In the course of the proof of this statement the use of made the relations (7.6)

We notice here that the unitary transformation (7.8) with the gauge function $\Lambda(x)$ applied to U_μ eliminates from a term with the form $\partial_\mu \Lambda$. Thus, it is expected that when applying a q-number gauge transformation which is obtained from (4) by replacing $\Lambda(x)$ by $\frac{1}{\varkappa} B(x)$, the term $\partial_\mu B$ in U_μ may disappear. In fact in spite of the non-commutability of $\partial_4 B$ with $\partial_l B$, one can completely eliminate the B-field from the interaction Hamiltonian.

That is to say, when we apply a transformation

$$\Psi \to \Psi' = \exp\left[-i\frac{1}{\varkappa}\int d\sigma_\mu J_\mu(x) B(x)\right], \tag{7.9}$$

the Tomonaga equation for the new state vector becomes

$$i\frac{\delta\Psi[\sigma]}{\delta\sigma[x]} = -\left(-f J_\mu A_\mu - \frac{1}{2}f^2 J_{\mu\nu}A_\mu A_\nu\right) \times \Psi'[\sigma] \tag{7.10}$$

So, in general, the q-number gauge transformation (7.9) leads to the MG-representation, The subsidiary condition is changed into the form, we saw in the last lecture.

The above general discussion is orginally due to Umezawa and we call (7.9), (7.10) the Glauber-Umezawa theorem.

9. Conclusion

Now, this is all that we know about the Stueckelberg formalism and all that, I believe, is known by physicists about the Stueckelberg formalism. Sakurai, Salam and Ward * and others have proposed a theory with which to explain various conservation laws on the basis of gauge particles, vector bosons, and in fact, the vector resonances ω
ω, ρ and K^* with isospin 0,1 and 1/2. have been found. These resonances may or may not be the gauge particles of the above-mentioned authors. When we want to make such an identification, there arises a serious difficulty. That is the mass of these particles. Theoretically these particles are introduced so as to make the theory 'gauge invariant'. However, as we have seen, expect for a case of a singlet, the gauge invariance, requires the vanishing mass of the gauge particles. For the range-Mills-triplet particles, the mass terms violates the invariance under the gauge trasnformation with a general gauge function. So, we have, to add in an ad hoc manner non-zero mass terms to the theory, which contradict the very principle of gauge invariance on which we base our theory.

There have been many attempts to derive a non-zero mass from such theories. e.g. Schwinger and Goldstone, Salam and Weinberg * applied Goldstone's method. But, Schwinger's attempt is nothing more than a conjecture, and the latter people have actually failed, because of the Goldstone theorem. Our program is to apply the method of inequivalent representations to this problem which will be discussed latter by Umezawa. For this purpose, the Stueckelberg formalism is very convenient and this is where Professor Umezawa's lectures and mine meet.

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