MATSCIENCE REPORT 2

LECTURES ON

ANGULAR MOMENTUM

G. RAMACHANDRAN

The Institute of Mathematical Sciences, Madras-4. (India)

THE INSTITUTE OF MATHEMATICAL SCIENCES

MADRAS - 4 (India)

ON ANGULAR MOMENTUM

Lectures given by

G.Ramachandran*.

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* Senior Research Fellow of The Department of the Atomic Energy, Government of India.

ON ANGULAR MOMENTUM

by

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Lecture 1.

According to the quantum theory the angular momentum of a system is quantized and an eigen state of angular momentum has two eigen values j , m associated with it, $j(j+1)$ denoting† the square of the angular momentum and m the projection along an arbitrary direction usually chosen as the z-axis; the allowed values of j being all non-negative integers and half-integers and for a given j the allowed values of m being $j, j-1, \cdots, -j+1, -j$. This can be shown to be a consequence of the commutation relations

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y \qquad (1)$$
$$[J^2, J_{x,y,z}] = 0$$

satisfied by the components J_x, J_y, J_z of the angular momentum operator $\vec{J}$.

The further development of the quantum theory of angular momentum rests basically on two concepts (1) the addition of two angular momenta and (2) the transformations produced under rotations of the coordinate system.

If $\vec{J}_1$ and $\vec{J}_2$ are the angular momentum operators (each obeying the commutation relations (1)) associated with two systems one easily verifies that the total angular momentum $\vec{J}$ defined by

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† In units of $\hbar = 1, c = 1$

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \tag{2}$$

also satisfies similar commutation relationships and that J_1^2, J_2^2, J^2 and J_z form a set of commuting operators while $J_1^2, J_2^2, J_{1z}, J_{2z}$ also form obviously a commuting set of operators. It therefore follows that we can either have states $|j_1 j_2 j m\rangle$ which are eigen states of the first set of operators or equally well the eigen states $|j_1 j_2 m_1 m_2\rangle \equiv |j_1 m_1\rangle |j_2 m_2\rangle$ of the second set of operators. Since both the sets of states $|j_1 j_2 j m\rangle$ and $|j_1 m_1\rangle |j_2 m_2\rangle$ describe the same system it should be possible to express any state of one set as a linear superposition of the states of the second set. Thus

$$|j_1 j_2 j m\rangle = \sum_{m_1} C(j_1 j_2 j; m_1 m_2)\, |j_1 m_1\rangle |j_2 m_2\rangle \tag{3}$$

and

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{j} C(j_1 j_2 j; m_1 m_2)\, |j_1 j_2 j m\rangle \tag{4}$$

The transformation coefficients are referred to as the Clebsch-Gordon coefficients and we shall for the present state that these coefficients

(1) are real (hence the appearance of the same coefficients in the inverse expansion also),

(2) vanish if $m_1 + m_2 \neq m$ (hence the summation in (3) over only one of the variables m_1, m_2 and we choose also not to write m within the C symbol as $C(j_1 j_2 j; m_1 m_2 m)$ whenever no confusion arises).

(3) vanish if J lies outside $(J_1+J_2), (J_1+J_2-1), \cdots, |J_1-J_2|$.
(or if J_1 J_2 J do not form a triangle) and also if
$J < |m_1 + m_2|$.

and (4) satisfy the following symmetry properties:

$$(a)\ C(J_1 J_2 J;\ m_1 m_2) = (-1)^{J_1+J_2-J}\, C(J_2 J_1 J;\ m_2 m_1)$$

$$(b)\ C(J_1 J_2 J;\ m_1 m_2 m) = (-1)^{J_1+J_2-J}\, C(J_1 J_2 J;\ -m_1 -m_2 -m) \qquad (5)$$

and $$(c)\ C(J_1 J_2 J;\ m_1 m_2 m) = (-1)^{J_2+m_2}\left(\frac{2J+1}{2J_1+1}\right)^{1/2} C(J J_2 J_1;\ -m\ m_2\ -m_1)$$

which are sufficient to derive any further symmetry property, e.g.

$$C(J_1 J_2 J;\ m_1 m_2 m) = (-1)^{J_1-m_1}\left(\frac{2J+1}{2J_2+1}\right)^{1/2} C(J_1 J J_2, m_1 -m) \qquad (5\ d)$$

obtained using (a) and (c). Again,

$$C(J J 0;\ m, -m) = (-1)^{J-m}\frac{1}{(2J+1)^{1/2}} \qquad (5\ e)$$

which is obtained using (c) and noting that $C(J 0 J;\ m 0) = 1$.

From (5 b) we have

$$C(J_1 J_2 J;\ 0 0) = (-1)^{J_1+J_2-J}\, C(J_1 J_2 J;\ 0 0)$$

and since J is an integer (half integers cannot obviously have zero projection)

$$C(J_1 J_2 J;\ 0 0) = (-1)^{J_1+J_2+J}\, C(J_1 J_2 J;\ 0 0)$$

and therefore

$$C(J_1 J_2 J;\ 0 0) = 0 \text{ unless } J_1 + J_2 + J \text{ is even.} \qquad (5\ f)$$

We note that the Clebsch-Gordon transformation describes, <u>given</u> J_1, J_2 the change of basis from $m_1 m_2$ to $J m$ or vice versa. The states $|J_1 J_2 J m\rangle$ form a complete orthogonal set and arenormalised if $|J_1 m_1\rangle$ and $|J_2 m_2\rangle$ are normalised. The unitarity of the transformation is explicitly expressed through

$$\sum_{m_1} C(j_1 j_2 J;\, m_1 m_2 m)\, C(j_1 j_2 J';\, m_1 m_2 m) = \delta_{JJ'} \qquad (6)$$

and

$$\sum_{J} C(j_1 j_2 J;\, m_1 m_2 m)\, C(j_1 j_2 J;\, m_1' m_2' m) = \delta_{m_1 m_1'} \qquad (7)$$

$m = m_1 + m_2 = m'$ assures $m = m''$ in (6) while there is no summation over m in (7) since $m = m_1 + m_2$ given m_1, m_2. Also since $m = m_1' + m_2'$, $\delta_{m_1 m_1'}$ implies $\delta_{m_2 m_2'}$.

The Clebsch-Gordon coefficients are the basic quantities which occur even when we are coupling many angular momenta, the various quantities known as the Racah coefficients, 9-j and 12-j symbols etc. being certain aggregates comprising of a number of Clebsch-Gordon coefficients.

We shall therefore take up a detailed study of the coupling of two angular momenta the conventions employed in the definition of the Clebsch-Gordon coefficients and their evaluation etc. later.

Let us now consider the behaviour of the angular momentum states $|J\, m\rangle$ under rotations of the coordinate system. The operator R representing rotations has the general form

$$R = e^{i(\vec{n}\cdot\vec{J})\theta} \qquad (8)$$

representing a rotation through an angle θ about a given direction $\vec{n}$; and $\vec{J}$ is the angular momentum operator. It is clear from (8) that R commutes with J^2 and hence j is a good quantum number under rotations. Therefore

$$R\,|jm\rangle = \sum_{m'} \langle jm'|R|jm\rangle\,|jm'\rangle$$

$$= \sum_{m'} D^{j}_{m'm}\,|jm'\rangle \qquad (9)$$

where the states $|jm'\rangle$ refer to the original coordinate system and we denote $\langle jm'|R|jm\rangle$ by the symbol $D^{j}_{m'm}$ referred to as the rotation matrices.

In other words, we can consider the various states $|jm\rangle$ for a given j to be a set of $(2j+1)$ quantities transforming under rotations according to the $D^{j}_{m'm}$ rule (9). In fact we define a set of $(2k+1)$ quantities (whether states or operators) which transform under rotations according to (9) as the components of spherical tensor T^{k}_{q} of rank k. The motivation for this notion arises inview of the considerable simplification resulting in the evaluation of the matrix elements $\langle j'm'|O|jm\rangle$ of an operator O between angular momentum states, if O is a spherical tensor O^{k}_{q}. In such a case we have a powerful result which asserts that the $(2j'+1)(2k+1)(2j+1)$ elements are related to each other through Clebsch-Gordon coefficients and consequently it is enough to know any one of them to determine the rest. This is the Wigner-Eckart theorem which states that

$$\langle j'm' | O^k_q | jm \rangle = C(jkj'; mqm') \langle j' \| O^k \| j \rangle \quad (10)$$

where $\langle j' \| O^k \| j \rangle$ is a quantity independent of the projection quantum numbers, referred to as the reduced matrix element, which is usually evaluated by calculating L.H.S. for a chosen m, q, m' and dividing by the corresponding Clebsch-Gordon coefficient. The theorem also tells us that the matrix elements vanish when $m + q \neq m'$ or if j, k, j' do not form a triangle.

Examples of commonly occurring spherical tensor operators are:

(1) the angular momentum operators J_{+1}, J_0, J_{-1} defined by

$$J_{\pm} = \mp \frac{J_x \pm i J_y}{\sqrt{2}}$$

$$J_0 = J_z \quad (11)$$

which form a spherical tensor of rank 1 . Since

$$\langle j'm' | J_0 | jm \rangle = m \, \delta_{mm'} \, \delta_{jj'}$$

$$= C(j1j; m0m) \langle j \| J \| j \rangle$$

$$\langle j' \| J \| j \rangle = \delta_{jj'} \sqrt{j(j+1)} \quad (12)$$

Substituting for the Clebsch-Gordon coefficient. The linear combinations of the spinmatrices $\sigma_{\pm 1}$, σ_0 defined at above provide a particular example and

$$\langle \tfrac{1}{2} \| \sigma \| \tfrac{1}{2} \rangle = 2\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)} = \sqrt{3} \quad (13)$$

since $\frac{1}{2}\vec{\sigma}$ are the matrices of the spin operator.

(2) The spherical harmonics Y_{lm} are obviously the components of a spherical tensor of rank l .

(3) The components $r_{\pm 1}$, r_0 of a vector $\vec{r}$ defined as in (11)

$$r_{\pm 1} = \mp \frac{x \pm i y}{\sqrt{2}} = r\, Y_{1,\pm 1}\sqrt{\frac{4\pi}{3}} \, .$$

$$r_0 = z = \sqrt{\frac{4\pi}{3}}\, r\, Y_{1,0} \qquad (14)$$

form a spherical tensor of rank 1.

(4) The components $\nabla_{\pm 1}$, ∇_0 of the gradient operator defined similarly form also a spherical tensor of rank 1.

Apart from these, various techniques are used in calculations involving angular momentum states, to express a given operator in terms of spherical tensor operators, for e.g.

$$\exp i\vec{k}\cdot\vec{r} = \sum_{l=0}^{\infty} i^{l}(2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

$$= 4\pi \sum_{l=0}^{\infty} i^{l}\, j_l(kr) \sum_{m=-l}^{l}(-1)^m\, Y_{l,-m}(\theta_k,\phi_k)\, Y_{lm}(\theta_r,\phi_r) \qquad (15)$$

where $j_l(kr)$ are the spherical Bessel functions and θ is the angle between $\vec{k}(k\theta_k\phi_k)$ and $\vec{r}(r\theta_r\phi_r)$.

We shall consider examples of the use of these concepts in the next lecture.

Lecture 2 :

The magnetic dipole moment operator $\vec{\mu}$ of a particle possessing orbital and spin angular momenta is

$$\vec{\mu} = g_L \vec{L} + g_s \vec{S} \tag{16}$$

where $\vec{L}$ and $\vec{S}$ are the orbital and spin angular momentum operators. If $T^k_\nu(1)$ and $T^k_\nu(2)$ --- are spherical tensors of rank k, it is easily seen that a linear combination

$$T^k_\nu = \sum_i a_i \, T^k_\nu(i) \tag{17}$$

is also a spherical tensor of rank k ; thus $\vec{\mu}$ is a spherical tensor of rank 1 . To evaluate the expectation value of this operator for a state of orbital angular momentum ℓ spin s total angular momentum J and projection m we make use of a result known as the Projection theorem for first rank tensors which states that

$$\langle J m' | T^1_M | J m \rangle = \frac{\langle J m' | J_M | J m \rangle \langle J \| \vec{T} \cdot \vec{J} \| J \rangle}{J(J+1)} \tag{18}$$

Therefore

$$\langle \ell s J m | \vec{\mu} | \ell s J m \rangle = \frac{\langle J m | \vec{J} | J m \rangle \langle J \| g_L L^2 + g_s S^2 + (g_L + g_s) \vec{L} \cdot \vec{S} \| J \rangle}{J(J+1)}$$

since

$$2 \vec{L} \cdot \vec{S} = J^2 - L^2 - S^2$$

$$J^2 | \ell s J m \rangle = J(J+1) | \ell s J m \rangle , \quad S^2 | \ell s J m \rangle = s(s+1) | \ell s J m \rangle \text{ etc}$$

we have

$$\langle \ell s J m | \mu_M | \ell s J m \rangle = m \, \delta_{M0} \left[\tfrac{1}{2}(g_L + g_s) + \tfrac{1}{2}(g_L - g_s) \times \frac{\ell(\ell+1) - s(s+1)}{J(J+1)} \right] \tag{19}$$

The electric quadrupole moment operator Q of a distribution of charge is (w.r.t a given direction)

$$Q = r^2 \, Y_{2,0}(\cos\theta) \tag{20}$$

θ being the polar angle. Obviously Q is a spherical tensor of rank 2 and operates only on the orbital angular state whereas the system may also possess spin. In an actual case the situation is further complicated in that the distribution is due to more than a single particle when the operator Q for the system will be defined as

$$Q = \sum_k Q(k) \tag{21}$$

and is in virtue of (17) ~~is~~ a spherical tensor of rank 2 . However the Wigner-Eckart theorem is a result derived from transformation properties under rotation and therefore we can formally write

$$\langle J m \mid Q \mid J m \rangle = C(J2J;m0)\cdot\langle J \| Q \| J \rangle \tag{22}$$

where $|Jm\rangle$ denote eigen states of total angular momentum of the system. Substituting the value of the Clebsch-Gordon coefficient

$$\langle J m \mid Q \mid J m \rangle = \frac{3m^2 - J(J+1)}{[J(J+1)(2J-1)(2J+1)]^{1/2}} \langle J \| Q \| J \rangle \tag{23}$$

The values $\langle JJ \mid \mu \mid JJ \rangle$ and $\langle JJ \mid Q \mid JJ \rangle$ are usually given as the magnetic dipole moment and electric quadrupole moment.

We shall now consider the detailed evaluation of the matrix element of an operator O between states $|J_1 J_2 J m\rangle$ when the operator is a spherical tensor of rank k and projection q and operates only on one of the sub-systems; say 1 .

$$\langle j_1' j_2' j' m' | O_q^k(1) | j_1 j_2 j m \rangle = C(j\, k\, j';\, m\, q\, m') \langle j_1' j_2' j' \| O^k(1) \| j_1 j_2 j \rangle \quad (24)$$

by Wigner-Eckart theorem. We canwrite the L.H.S. explicitly as

$$\sum_{m_1} \sum_{m_1'} C(j_1' j_2' j';\, m_1' m_2' m') \langle j_1 m_1' j_2 m_2' | O_q^k(1) | j_1 m_1 j_2 m_2 \rangle \times C(j_1 j_2 j;\, m_1 m_2 m)$$

$$= \sum_{m_1} \sum_{m_1'} C(j_1' j_2' j';\, m_1' m_2' m')\, C(j_1 j_2 j;\, m_1 m_2 m) \times C(j_1 k j_1';\, m_1 q m_1') \langle j_1' \| O^k \| j_1 \rangle \delta_{j_2 j_2'} \delta_{m_2 m_2'}$$

applying the Wigner-Eckart theorem once again. Multiplying now L.H.S. and R.H.S. of (24) by $C(j\, k\, j';\, m\, q\, m')$ and summing over q and using (6) we obtain

$$\langle j_1' j_2' j' \| O^k(1) \| j_1 j_2 j \rangle = \sum_{m_1} \sum_{q} C(j_1 j_2 j;\, m_1 m_2 m) \times C(j\, k\, j';\, m\, q\, m')\, C(j_1 k j_1';\, m_1 q m_1') \times C(j_1' j_2 j';\, m_1' m_2 m') \langle j_1' \| O^k \| j_1 \rangle \delta_{j_2 j_2'}$$

We use the symmetry property (5 a) on the first and the fourth Clebsch-Gordon coefficients in the above equation and defining a quantity

$$U(j_1 j_2 j\, j_3;\, j_{12} j_{23}) \equiv \sum_{m_1} \sum_{m_2} C(j_1 j_2 j_{12};\, m_1 m_2) \times C(j_{12} j_3 j;\, m_1 + m_2\, m_3)\, C(j_2 j_3 j_{23};\, m_2 m_3) \times C(j_1 j_{23} j;\, m_1\, m_2 + m_3) \quad (25)$$

we have

$$\langle j_1' j_2' j' \| O^k(1) \| j_1 j_2 j \rangle = (-1)^{j_1+j_2-j} (-1)^{j_1'+j_2-j'}$$

$$\times\, U(j_2 j_1 j' k ; j j_1') \langle j_1' \| O^k \| j_1 \rangle \delta_{j_2 j_2'} \qquad (26\text{ a})^{*}$$

which expresses the reduced matrix element of $O^k(1)$ between states of total angular momentum in terms of the reduced matrix element of $O^k(1)$ between states of the sub-system 1 . By a similar procedure we can show

$$\langle j_1' j_2' j' \| O^k(2) \| j_1 j_2 j \rangle = U(j_1 j_2 j' k ; j j_2') \delta_{j_1 j_1'} \langle j_2 \| O^k \| j_2 \rangle \qquad (26\text{ b})$$

The quantity $U(j_1 j_2 j j_3 ; j_{12} j_{23})$ referred to as the recoupling coefficients which by definition(25) is a product of four Clebsch-Gordon coefficients summed over two independent projection quantum numbers and which would be read as j_1 j_2 coupling to form j_{12} coupled with j_3 forms j ; and j_2 j_3 coupling to form j_{23} and j_1 j_{23} couple to form j -- which corresponds to two ways of coupling j_1 j_2 j_3 pair by pair to form j . In fact if we denote by $|(j_1 j_2) j_{12} j_3 j m\rangle$ and $|j_1 (j_2 j_3) j_{23} j m\rangle$ the total angular momentum states formed according to the two schemes outlined above, the recoupling coefficients can be defined as the transformation coefficients connecting the two sets of states

$$|(j_1 j_2) j_{12} j_3 j m\rangle = \sum_{j_{23}} U(j_1 j_2 j j_3 ; j_{12} j_{23}) \, |j_1 (j_2 j_3) j_{23} j m\rangle \qquad (27)$$

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* Since j_1+j_2-j is an integer we can write the phase factor as $(-1)^{-j_1-j_2+j}(-1)^{j_1'+j_2-j'} = (-1)^{j-j'+j_1'-j_1}$

The order of coupling should also be borne since any change would contribute a phase factor according to (5 a). It is clear from (25) that these coefficients are real and since

$$U(j_1 j_2 j\, j_3 ; j_{12}\, j_{23}) \equiv \langle j_1 (j_2 j_3)\, j_{23}\, j\, m \mid (j_1 j_2)\, j_{12}\, j_3\, j\, m \rangle \tag{27 a}$$

could be considered to be the matrix element of the identity operator (which is spherical tensor of rank 0) between states $|jm\rangle$ its dependence on the projection quantum numbers is $C(j0j; m0) = 1$ and hence the absence of m within the symbol.+ From (25) it is clear that if any one of $j_1\, j_2\, j_{12}$; $j_{12}\, j_3\, j$; $j_2\, j_3\, j_{23}$ or $j_1\, j_{23}\, j$ do not form a triangle, the recoupling coefficient vanishes. The coefficients satisfy the orthogonality conditions

$$\sum_{j_{12}} U(j_1 j_2 j\, j_3 ; j_{12}\, j_{23})\, U(j_1 j_2 j\, j_3 ; j_{12}\, j_{23}') = \delta_{j_{23}\, j_{23}'} \tag{28 a}$$

$$\sum_{j_{23}} U(j_1 j_2 j\, j_3 ; j_{12}\, j_{23})\, U(j_1 j_2 j\, j_3 , j_{12}'\, j_{23}) = \delta_{j_{12}\, j_{12}'} \tag{28 b}$$

The Racah coefficients $W(j_1\, j_2\, j\, j_3 ; j_{12}\, j_{23})$ defined by

$$\left[(2j_{12}+1)(2j_{23}+1)\right]^{1/2} W(j_1 j_2 j\, j_3 ; j_{12}\, j_{23}) = U(j_1 j_2 j\, j_3 ; j_{12}\, j_{23}) \tag{29}$$

exhibit the symmetry propertiesm more elegantly

$$W(a\, b\, c\, d ; e\, f) = W(b\, a\, d\, c ; e\, f) \tag{30 a}$$

$$= W(c\, d\, a\, b ; e\, f) \tag{30 b}$$

$$= W(a\, c\, b\, d ; f\, e) \tag{30 c}$$

$$= (-1)^{e+f-b-c}\, W(a\, e\, f\, d ; b\, c) \tag{30 d}$$

+and the freedom in (25) to sum over any two of the three independent projection quantum numbers.

$$= (-1)^{e+f-a-d} W(e\,b\,c\,f\,;\,a\,d) \qquad (30\ e)$$

which would be deduced from (25), (29) and (5). Clearly if j_1 or j_2 or j_3 is zero, the recoupling coefficients reduce to unity; and together with (30) we have

$$W(a\,b\,c\,d\,;\,o\,f) = (-1)^{f-b-d} \frac{\delta_{ab}\,\delta_{cd}}{[(2b+1)(2d+1)]^{1/2}} \qquad (30\ f)$$

As an example of the use of the relations (26) and (27) derived above, let us consider the evaluation of the matrix element for the photo production of π -mesons from a bound nucleonwhich has the form

$$\langle L' \tfrac{1}{2} J' M' \,|\, (\vec{\sigma}\cdot\vec{K} + \lambda) \exp i\, \vec{k}\cdot\vec{r} \,|\, L \tfrac{1}{2} J M \rangle \qquad (32)$$

where $\vec{K}$ and λ do not operate on the nucleon states.

We easily verify that $\vec{\sigma}\cdot\vec{K}$ can be written as $\sum_{\mu=1,0,-1}(-1)^{\mu}\,\sigma_{\mu} K_{-\mu}$ where σ_{μ} and K_{μ} are defined as in (11) and (14). We make use of the expansion (15) for $e^{i\vec{k}\cdot\vec{r}}$ and observe that $Y_{lm}(\theta_k\,\phi_k)$ do not obviously operate on the angular momentum states of the nucleon. To bring (32) into a form so as to enable application of (26) and (27) we introduce a complete set of states $|L'' \tfrac{1}{2} J'' M''\rangle$ between the two operators $(\vec{\sigma}\cdot\vec{K} + \lambda)$ and $\exp(i\vec{k}\cdot\vec{r})$ thus expressing (32) as

$$\sum_{L''}\sum_{J''}\sum_{M''} \langle L' \tfrac{1}{2} J' M' \,|\, \sum_{\mu}(-1)^{\mu}\sigma_{\mu}K_{-\mu} + \lambda \,|\, L'' \tfrac{1}{2} J'' M'' \rangle$$

$$\times \langle L'' \tfrac{1}{2} J'' M'' \,|\, 4\pi \sum_{l=0}^{\infty} i^{l} j_l(kr) \sum_{m=-l}^{l} (-1)^{m} Y_{l\,-m}(\theta_k\,\phi_k)\, Y_{lm}(\theta_r\,\phi_r) \,|\, L \tfrac{1}{2} J M \rangle$$

(33)

Substituting now for the matrix elements of σ_μ and $Y_{lm}(\theta_r \phi_r)$ using (27) and (26) respectively we have (32) as

$$4\pi \sum_{l=0} i^l \langle j_l(kr)\rangle \langle L' \| Y_l \| L\rangle \sum_{m=-l}^{l} (-1)^m Y_{l-m}(\theta_k, \phi_k)$$
$$\times \sum_{J''=L'-\frac{1}{2}}^{L'+\frac{1}{2}} C(J l J''; M m)(-1)^{L-L'-J+J''} \, U(\tfrac{1}{2} L J'' l; J L')$$
$$\times \Big[C(J'' 1 J'; M+m, \mu, M')\, \delta_{\mu, M'-M-m} K_{-\mu}$$
$$\times U(L' \tfrac{1}{2} J' 1; J'' \tfrac{1}{2}) \sqrt{3} + \delta_{J'' J'}\, \lambda \Big] \qquad (34)$$

The summation over L'' drops since the matrix elements of $(\vec{\sigma}\cdot\vec{K} + \lambda)$ contributes a $\delta_{L''L'}$ and M'' is fixed by $M'' = M + m$. The ~~summation-on~~ summation over μ is also unnecessary and the summation over m is restricted since $m + \mu$ is fixed by $m + \mu = M' - M$. The summation over l is also limited to $|L - L'|$ to $L + L'$. $\langle j_l(kr)\rangle$ denotes the matrix element of the spherical Bessel functions between initial and final radial states. We shall consider in the next lecture the evaluation of the reduced matrix element $\langle L' \| Y_l \| L\rangle$ and we have used (13) for $\langle \tfrac{1}{2} \| \sigma \| \tfrac{1}{2}\rangle$.

It is clear also that the reduced matrix element $\langle l s j \| Q \| l s j\rangle$ of the quadrupole moment operator of a single bound particle with spin is

$$\langle l s j \| Q \| l s j\rangle = (-1)^{2l+2s-2j}\, U(\tfrac{1}{2}\, l\, j\, 2; j\, l) \times \langle l \| Y_2 \| l\rangle \qquad (35)$$

using (26); and the phase factor is 1 since $l + s - j$ is an integer.

Lecture 3.

Any rotation R of a coordinate system is usually specified by giving the Euler angles α, β, γ of rotation. If the transformation is from $(x_I\ y_I\ z_I)$ to $(x_{II}\ y_{II}\ z_{II})$ fixed in space and if $(x_r\ y_r\ z_r)$ denotes the rotating coordinate system and $(x'\ y'\ z')$ and $(x''\ y''\ z'')$ some fixed frames in space, we have in this order:

(1) a rotation about the z_r axis through an angle α

$$(x_r\ y_r\ z_r) = (x_I\ y_I\ z_I) \xrightarrow{\alpha} (x_r\ y_r\ z_r) = (x'\ y'\ z' = z_I)$$

(2) a rotation about the y_r axis through an angle β,

$$(x_r\ y_r\ z_r) = (x'\ y'\ z') \xrightarrow{\beta} (x_r\ y_r\ z_r) = (x''\ y'' = y'\ z'')$$

and (3) a rotation about the z_r axis again through an angle γ

$$(x_r\ y_r\ z_r) = (x''\ y''\ z'') \xrightarrow{\gamma} (x_r\ y_r\ z_r) = (x_{II}\ y_{II}\ z_{II} = z'')$$

Writing for convenience $(x_I\ y_I\ z_I) = (x\ y\ z)$, we have the operator R in the form

$$e^{-i\gamma J_{z_r}}\, e^{-i\beta J_{y_r}}\, e^{-i\alpha J_{z_r}} = R(\alpha,\beta,\gamma)$$

$$= e^{-i\gamma J_{z'}}\, e^{-i\beta J_{y'}}\, e^{-i\alpha J_z} \qquad (36)$$

And the states $|\psi\rangle$ and operators O in the two frames are related through

$$|\psi\rangle_{II} = R\ |\psi\rangle_I \qquad (37)$$

and

$$O_{II} = R\, O_I\, R^{-1} \qquad (38)$$

Using (38) now we can write

$$e^{-i\gamma J_{z''}} = e^{-i\beta J_{y'}}\, e^{-i\gamma J_{z'}}\, e^{+i\beta J_{y'}}$$

and again

$$e^{-i\beta J_y}\, e^{-i\gamma J_z} = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\gamma J_z}\, e^{+i\alpha J_z}$$

thus obtaining from (36) an alternative form for $R(\alpha\beta\gamma)$ as

$$R(\alpha\beta\gamma) = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\gamma J_z} \tag{39}$$

which we shall use later. For the present we observe that

$$|jm\rangle_{II} = R(\alpha\beta\gamma)\,|jm\rangle_{I} = \sum_{m'} D^{j}_{m'm}(\alpha\beta\gamma)\,|jm'\rangle_{I} \tag{40}$$

which in particular gives

$$Y_{lm}(\theta,\phi) = R(\phi,\theta,\phi)\; Y_{lm}(0,0)$$

$$= \sum_{m'} D^{l}_{m'm}(\phi,\theta,\phi)\,\delta_{m'0}\left(\frac{2l+1}{4\pi}\right)^{1/2}$$

substituting for $Y_{lm'}(0,0)$; and hence

$$D^{l}_{0m}(\phi\,\theta\,\phi) = \left(\frac{4\pi}{2l+1}\right)^{1/2} Y_{lm}(\theta,\phi) \tag{41}$$

Let us consider the effect of operating by $R(\alpha\beta\gamma)$ on both sides of (4). Since

$$R\,|j_1 m_1\rangle_{I}\,|j_2 m_2\rangle_{I} = |j_1 m_1\rangle_{II}\,|j_2 m_2\rangle_{II}$$

we have

$$\sum_{m_1'}\sum_{m_2'} D^{j_1}_{m_1'm_1}(\alpha\beta\gamma)\,D^{j_2}_{m_2'm_2}(\alpha\beta\gamma)\,|j_1 m_1'\rangle\,|j_2 m_2'\rangle$$

$$= \sum_{m'}\sum_{j'} C(j_1 j_2 j'; m_1 m_2 m)\, D^{j'}_{m'm}(\alpha\beta\gamma)\,|j_1 j_2 j' m'\rangle$$

Expressing $|j_1 j_2 j' m'\rangle$ on R.H.S. using (3) and equating coefficients, we obtain

$$D^{j_1}_{m_1' m_1}(\alpha\beta\gamma)\, D^{j_2}_{m_2' m_2}(\alpha\beta\gamma) = \sum_{j'} C(j_1 j_2 j'; m_1 m_2 m) \times C(j_1 j_2 j'; m_1' m_2' m')\, D^{j'}_{m' m}(\alpha\beta\gamma) \quad (42)$$

which is known as the Clebsch-Gordon series.

In a similar way operating by R on (3) and expressing $|j_1 j_2 j m\rangle$ on L.H.S. in terms of $|j_1 m_1'\rangle |j_2 m_2'\rangle$ using (3) and equating coefficients we have

$$C(j_1 j_2 j; m_1' m_2' m')\, D^{j}_{m' m}(\alpha\beta\gamma) = \sum_{m_1} C(j_1 j_2 j; m_1 m_2 m) \times D^{j_1}_{m_1' m_1}(\alpha\beta\gamma)\, D^{j_2}_{m_2' m_2}(\alpha\beta\gamma)$$

and multiplying both sides by sides by $C(j_1 j_2 j; m_1 m_2' m')$ and summing over m_1' and using (6) we obtain

$$D^{j}_{m' m}(\alpha\beta\gamma) = \sum_{m_1} \sum_{m_1'} C(j_1 j_2 j; m_1 m_2 m)\, C(j_1 j_2 j; m_1' m_2' m') \times D^{j_1}_{m_1' m_1}(\alpha\beta\gamma)\, D^{j_2}_{m_2' m_2}(\alpha\beta\gamma) \quad (42\ a)$$

which is referred to as the inverse Clebsch-Gordon series, which can also be obtained from (42) transferring the Clebsch-Gordon coefficients to the L.H.S. using (6) twice. Successive application of (42 a) enables one, in principle, to evaluate the rotation matrices $D^{j}_{m' m}(\alpha\beta\gamma)$ for all j if it is known for one (non-zero) j'.

Returning now to (42) and specialising to the case $m_1' = m_2' = 0 = m'$ we obtain, using (41) a coupling rule for spherical harmonics

$$Y_{l_1 m_1}(\theta,\phi)\, Y_{l_2 m_2}(\theta,\phi) = \sum_{l'} \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2l'+1)}\right]^{1/2} C(l_1 l_2 l'; m_1 m_2 m)\, C(l_1 l_2 l'; 00)\, Y_{l'm}(\theta,\phi) \quad (43)$$

The distinction between (43) and (4) should be clear: The states $|l_1 l_2 l m\rangle$ are not $Y_{lm}(\theta,\phi)$ even if the arguments $(\theta_1\phi_1),(\theta_2\phi_2)$ in $Y_{l_1m_1}$ and $Y_{l_2m_2}$ tend to the same value (θ,ϕ), the dependence of $|l_1 l_2 l m\rangle$ on the arguments $(\theta_1\phi_1)(\theta_2\phi_2)$ is defined only through (3). (42 a) can also be specialised using (41) to give

$$\left(\frac{4\pi}{2l+1}\right)^{1/2} Y_{l0}(\theta,\phi) = \sum_{m_1 m_2}\sum C(l_1 l_2 l, m_1 m_2)\, C(l_1 l_2 l; m_1' m_2' 0)$$

$$\times D^{l_1}_{m_1 - m_1}(\phi\theta\phi)\, D^{l_2}_{m_1' - m_1'}(\phi\theta\phi) = P_l(\theta) \qquad (43\text{ a})$$

Considering now the matrix element

$$\langle L'M'|Y_{lm}|LM\rangle = \int Y^*_{L'M'}(\theta,\phi)\, Y_{lm}(\theta,\phi)\, Y_{LM}(\theta,\phi) \sin\theta\, d\theta\, d\phi$$

$$= \sum_{l'}\left[\frac{(2L+1)(2l+1)}{4\pi\,(2l'+1)}\right]^{1/2} C(Ll l'; Mm)\, C(Ll l'; 00)$$

$$\times \int Y^*_{L'M'}(\theta,\phi)\, Y_{l',M+m}(\theta,\phi)\, d\Omega$$

$$= \left[\frac{(2L+1)(2l+1)}{4\pi\,(2L'+1)}\right]^{1/2} C(LlL'; MmM')\, C(LlL'; 00) \qquad (44)$$

on account of the orthonormality of the spherical harmonics. Or

$$\langle L'\| Y_l \| L\rangle = \left[\frac{(2L+1)(2l+1)}{4\pi\,(2L'+1)}\right]^{1/2} C(LlL'; 00) \qquad (44\text{ a})$$

We define composition of two spherical tensors T^{k_1}, T^{k_2}, by the rule

$$\left(T^{k_1}\times T^{k_2}\right)^k_q = \sum_{q_1} C(k_1 k_2 k; q_1\, q - q_1)\, T^{k_1}_{q_1}\, T^{k_2}_{q-q_1} \qquad (45)$$

which is easily seen to be a spherical tensor of rank k.

Conversely

$$T^{k_1}_{q_1} T^{k_2}_{q_2} = \sum_k C(k_1 k_2 k; q_1 q_2 q)\left(T^{k_1} \times T^{k_2}\right)^k_q \tag{45 a}$$

It is of interest in some cases to calculate matrix elements of the type $\langle J_1' J_2' J' m' | \left(T^{k_1}_{(1)} \times T^{k_2}_{(2)}\right)^k_q | J_1 J_2 J m\rangle$. Clearly the reduced matrix element

$$\langle J_1' J_2' J' \| \left(T^{k_1}(1) \times T^{k_2}(2)\right)^k \| J_1 J_2 J\rangle$$

$$= \sum_m C(J k J'; m q m') \langle J_1' J_2' J' m' | T^k_q(1,2) | J_1 J_2 J m\rangle$$

$$= \sum_m \sum_{m_1} \sum_{q_1} \sum_{m_1'} C(J_1 J_2 J; m_1 m_2 m) C(k_1 k_2 k; q_1 q_2 q)$$

$$\times C(J_1' J_2' J'; m_1' m_2' m') C(J k J'; m q m')$$

$$\times \langle J_1' m_1' | T^{k_1}_{q_1} | J_1 m_1\rangle$$

$$\times \langle J_2' m_2' | T^{k_2}_{q_2} | J_2 m_2\rangle \tag{46}$$

which can be written using Wigner-Eckart theorem as

$$\begin{bmatrix} J_1 & J_2 & J \\ k_1 & k_2 & k \\ J_1' & J_2' & J' \end{bmatrix} \langle J_1' \| T^{k_1}_{(1)} \| J_1\rangle \langle J_2' \| T^{k_2}_{(2)} \| J_2\rangle \tag{47}$$

where

$$\begin{bmatrix} J_1 & J_2 & J \\ k_1 & k_2 & k \\ J_1' & J_2' & J' \end{bmatrix} = \sum_{m_1} \sum_{m_2} \sum_{q_1} C(J_1 J_2 J; m_1 m_2 m)$$

$$\times C(k_1 k_2 k; q_1 q_2 q) C(J_1' J_2' J'; m_1' m_2' m')$$

$$\times C(J_1 k_1 J_1'; m_1 q_1 m_1') C(J_2 k_2 J_2'; m_2 q_2 m_2')$$

$$\times C(J k J'; m q m') \tag{48}$$

~~(A2)~~

is referred to as the 9-J coefficient* which can also be

* to distinguish from Wigner's 9-J symbol

$$\begin{Bmatrix} J_1 & J_2 & J \\ k_1 & k_2 & k \\ J_1' & J_2' & J' \end{Bmatrix} = \left[(2J+1)(2k+1)(2J_1'+1)(2J_2'+1)\right]^{-1/2} \begin{bmatrix} J_1 & J_2 & J \\ k_1 & k_2 & k \\ J_1' & J_2' & J' \end{bmatrix}$$

defined, as in the case of recoupling coefficients, to be the transformation coefficient between states $|(j_1 j_2) j_{12} (j_3 j_4) j_{34} j m\rangle$ and $|(j_1 j_3) j_{13} (j_2 j_4) j_{24} j m\rangle$ which are eigen states of total angular momentum of a system consisting of four angular momenta constructed in two different ways of coupling them two by two

$$|(j_1 j_2) j_{12} (j_3 j_4) j_{34} j m\rangle = \sum_{j_{13} j_{24}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} |(j_1 j_3) j_{13} (j_2 j_4) j_{24} j m\rangle \tag{49}$$

In going from (46) to (47) the summation over say m_1' drops since it is determined by $m_1 + \nu_1 = m_1'$ and we have replaced the summation over m by the equivalent summation over m_2 to make comparison with the definition (49) transparent. We also see that in R.H.S. of (48) we have four independent projection quantum numbers while the summation is only on three which corresponds to the appearance of m in the definition (49) viz.,

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \langle (j_1 j_3) j_{13} (j_2 j_4) j_{24} j m | (j_1 j_2) j_{12} (j_3 j_4) j_{34} j m\rangle \tag{49 a}$$

but the co-efficient however does not depend on m as can be seen by considering the r.h.s. of (49 a) (as we did in the case of recoupling coefficients) to be the matrix element of the identity operator between states $|j m\rangle$. It is clear from (48) that these coefficients are real which consequently imposes the symmetry property

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \begin{bmatrix} j_1 & j_3 & j_{13} \\ j_2 & j_4 & j_{24} \\ j_{12} & j_{34} & j \end{bmatrix} \tag{50}$$

and further symmetry properties could be deduced using (48) and (5) for e.g.

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = (-1)^{j_1+j_3+j_{13}} \left[\frac{(2j_{12}+1)(2j_{34}+1)(2j+1)}{(2j_2+1)(2j_4+1)(2j_{24}+1)}\right]^{1/2} \begin{bmatrix} j_1 & j_{12} & j_2 \\ j_3 & j_{34} & j_4 \\ j_{13} & j & j_{24} \end{bmatrix} \qquad (51)$$

If we consider two particles with spin and j_1 j_2 denote their orbital angular momenta and j_3 j_4 their spins the wave functions on L.H.S. and R.H.S. of (49) represent respectively the $j-j$ coupling and L.S. coupling wave functions of the system and the $9-j$ ~~symbols~~ coefficients are referred to as the $LS-JJ$ transformation coefficients.

It is clear from the definition (45) that if $k_1 = k_2$ one of the possible values of k is zero when it defines a scalar

$$(T^k \times T^k)^0_0 = \sum_q C(k\,k\,0; q\,-q)\, T^k_q\, T^k_{-q} = \sum_q (-1)^{k-q} \frac{1}{(2k+1)^{1/2}}\, T^k_q\, T^k_{-q}$$

using (5 e). Conventionally the quantity

$$(-1)^{-k}\sqrt{2k+1}\,(T^k \times T^k)^0_0 = \sum_q (-1)^q\, T^k_q\, T^k_{-q} = (T^k \cdot T^k) \qquad (52)$$

is referred to as the scalar product of two spherical tensors (of equal rank). In particular one can easily verify that the scalar product of two vectors $\vec{r}_1$ and $\vec{r}_2$ as defined above in the spherical basis is identical with the usual $\vec{r}_1 \cdot \vec{r}_2$; we shall use the brackets (.) to denote the scalar product in spherical basis.

spherical basis.

To specialise (47) now to the case $k_1 = k_2 = K$ and $k = 0$ we shall write

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} \equiv \langle (j_1 j_2) j_{12} (j_3 j_4) j_{34} j m | (j_1 j_3) j_{13} (j_2 j_4) j_{24} j m \rangle$$

$$= \sum_{j_{234}} \sum_{j_{324}} \langle (j_1 j_2) j_{12} (j_3 j_4) j_{34} j m | (j_1 j_2) (j_3 j_4) j_{34} j_{234} j m \rangle$$

$$\times \langle j_1 j_2 (j_3 j_4) j_{34} j_{234} j m | j_1 j_3 (j_2 j_4) j_{24} j_{324} j m \rangle$$

$$\times \langle j_1 j_3 (j_2 j_4) j_{24} j_{324} j m | (j_1 j_3) j_{13} (j_2 j_4) j_{24} j m \rangle \qquad (53)$$

and change the order of coupling in the second element of j_2, j_{34} and j_2, j_4 thereby acquring phase factors $(-1)^{j_2+j_{34}-j_{234}}$ and $(-1)^{j_2+j_4-j_{24}}$ respectively. Since the angular momentum states with different j of a composite system are orthogonal to each other irrespective of the manner in which the constitutent angular momenta are coupled, the second element in (53) contains a $\delta_{j_{234}\, j_{324}}$. It is clear now from inspection that

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{bmatrix} = \sum_{j_{234}} (-1)^{j_4 - j_{24} - j_{34} + j_{234}} \, \mho(j_1 j_2 j\, j_{34} ; j_{12} j_{234})$$

$$\times \mho(j_3 j_4 j_{234} j_2 ; j_{34} j_{24}) \qquad (54)$$

$$\times \mho(j_1 j_3 j\, j_{24} ; j_{13} j_{234})$$

which is a useful expansion of the $9j$-coefficients in terms of the recoupling coefficients. Using the expansion (54) and (30 f) in (47) we have

$$\langle j_1' j_2' j' \| (T^{k_1}(1) \times T^{k_2}(2))^0 \| j_1 j_2 j \rangle = \delta_{jj'} (-1)^{K - j_2' + j_2} (-1)^{j_2' - K - j_2} [(2j_2+1)(2K+1)]^{-\frac{1}{2}}$$
$$\times (2j_2'+1)^{\frac{1}{2}} \, U(j_1 K j' j_2' ; j_1' j_2) \langle j_1' \| T^{k_1} \| j_1 \rangle \langle j_2' \| T^{k_2} \| j_2 \rangle$$
$$= \sqrt{\frac{2j_2'+1}{(2j_2+1)(2k+1)}} \; U(j_1 K j' j_2' ; j_1' j_2) \, \delta_{jj'} \langle j_1' \| T^{k_1}(1) \| j_1 \rangle$$
$$\times \langle j_2' \| T^{k_2}(2) \| j_2 \rangle \qquad (55\text{ a})$$
$$k_1 = k_2 = K$$

or using the definition (52)

$$\delta_{mm'} \langle j_1' j_2' j' \| (T^{K_1}(1) \cdot T^{K_2}(2)) \| j_1 j_2 j \rangle = \langle j_1' j_2' j' m' | (T^{K_1}(1) \cdot T^{K_2}(2)) | j_1 j_2 j m \rangle$$
$$= \delta_{jj'} \delta_{mm'} (-1)^{-K} \sqrt{\frac{2j_2'+1}{2j_2+1}} \; U(j_1 K j' j_2' ; j_1' j_2)$$
$$\times \langle j_1' \| T^K(1) \| j_1 \rangle \langle j_2' \| T^K(2) \| j_2 \rangle \qquad (55\text{ B})$$

which find many applications, as we shall see later.

LECTURE IV

We have already defined a set of $2k+1$ quantities T^k_q which transform under rotation as

$$T^k_q(II) = R\left(T^k_q(I)\right) = \sum_{q'} D^k_{q'q}\, T^k_{q'}(I) \tag{9a}$$

to form a spherical tensor of rank k, $R(T^k_q)$ being

$$R(T^k_q) = R\,T^k_q \quad \text{if } T^k_q = |kq\rangle \tag{37}$$

and $$R(T^k_q) = R\,T^k_q\,R^+ \quad \text{if } T^k_q = O^k_q \tag{38}$$

To see how the $2k+1$ quantities $\langle k,q|$ transform under rotations, we observ e from (8), (36) or (39) that

$$R^+ = R^{-1} \tag{56)*}$$

and since the scalar product $\langle k,q\,|\,k,q\rangle = 1$ is invariant under rotations of coordinate system, $R^+ = R^{-1}$ is the appropriate operator to effect transformation of " *bra* " states under rotations.

$$\underset{II}{\langle} k,q| = \underset{I}{\langle} k,q|\,R^{-1} \tag{57}$$

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* and consequently

$$D^J_{m'm}(\alpha,\beta,\gamma)^* = D^J_{mm'}(-\alpha,-\beta,-\gamma) \tag{56a}$$

since $\quad R^{-1}(\alpha,\beta,\gamma) = R(-\alpha,-\beta,-\gamma)$ from (36) or (39)

$$= \sum_{q'} \langle k\, q | R^{-1} | k\, q' \rangle \langle k\, q' |_{I}$$

$$= \sum_{q'} D^{k\,*}_{q'q} \langle k\, q' |_{I}$$

(58)

Making use of the representation (39) for $R(\alpha\beta\gamma)$ we can write

$$D^{J}_{m'm}(\alpha\beta\gamma) = e^{-i\alpha m'} \langle J\, m' | e^{-i\beta J_y} | J\, m \rangle e^{-i\gamma m}$$

$$= e^{-i\alpha m'}\, d^{J}_{m'm}(\beta)\, e^{-i\gamma m} \qquad (59)$$

The matrices $d^{J}(\beta)$ can be evaluated for $J = 1/2$, for example, by expanding $exp(-i\beta J_y)$ in power series, introducing complete set of states $\sum_m |\tfrac{1}{2}\, m\rangle\langle \tfrac{1}{2}\, m|$ in between operators J_y , replacing $\langle \tfrac{1}{2} | J_y | \tfrac{1}{2} \rangle$ by the Panli spin matrices σ_y and using the property $\sigma_y^{\,2} = 1$ we obtain

$$d^{1/2}(\beta) = \begin{array}{cc} -1/2 & +1/2 \\ \end{array} \; {}^{m'}\!/_{m'}$$

$$d^{1/2}(\beta) = \begin{bmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{bmatrix} \begin{array}{c} -1/2 \\ +1/2 \end{array}$$

(60)

where the rows and columns are numbered $(-1/2 \;\; , +1/2)$ and we easily verify that the rotation matrices $D^{J}_{m'm}(\alpha\beta\gamma)$ have the symmetry property

$$D^{J}_{m'm}(\alpha\beta\gamma)^{*} = (-1)^{m'-m} D^{J}_{-m',-m}(\alpha\beta\gamma) \tag{61}$$

at least for $J = 1/2$. We can now show that since (61) holds for $J = 1/2$ it holds for all J ~~for~~, we can construct any $D^{J}_{m'm}$ from $D^{1/2}$'s by repeated applications of the inverse Clebsch-Gordon series (42a).

Let (61) hold for $J = J_1, J_2$. Taking the complex conjugate of (42a) and using (61) on R.H.S. We have since the C's are real

$$D^{J}_{m'm}(\alpha\beta\gamma)^{*} = \sum_{m_1'}\sum_{m_2'} C(J_1J_2J; m_1 m_2 m)\, C(J_1J_2J; m_1' m_2' m')$$

$$\times (-1)^{m_1'-m_1}(-1)^{m_2'-m_2} D^{J_1}_{-m_1',-m_1}(\alpha\beta\gamma)\, D^{J_2}_{-m_2',-m_2}(\alpha\beta\gamma)$$

$$= (-1)^{m'-m}\sum_{m_1'}\sum_{m_2'}(-1)^{J_1+J_2-J} C(J_1J_2J; -m_1, -m_2, -m)$$

$$\times(-1)^{J_1+J_2-J} C(J_1J_2J; -m_1', -m_2', -m')\, D^{J_1}_{-m_1',-m_1}(\alpha\beta\gamma)\, D^{J_2}_{-m_2',-m_2}(\alpha\beta\gamma)$$

$$= (-1)^{m'-m} D^{J}_{-m',-m}(\alpha\beta\gamma)$$

since $m_1' + m^2 = m'$ and $m_1 + m_2 = m$ and ~~xxx~~ using (5b). We can write one of $(-1)^{J_1+J_2-J}$ as $(-1)^{J-J_1-J_2}$ since $J_1 + J_2 - J$ is an integer so that (61) follows on using (42a) to replace the double summation on R.H.S. by $D^{j}_{-m'-m}(\alpha\beta\gamma)$.

Making use of the above result we write (58) as

$$(-1)^{q} \underset{II}{<} kq| = \sum_{q'} (-1)^{q'} \langle kq'| \, D^{k}_{-q'-q}(\alpha\beta\gamma) \tag{62}$$

and we observe that the set of quantities $(-1)^{-q} \langle k\,{-q}|$ transform under totations as a spherical tensor of rank k , which we shall denote by $T^{+\,k}_{q}$ so that

$$(-1)^{q} \langle kq| = T^{+\,k}_{-q} = (-1)^{q} \left(T^{k}_{q}\right)^{+} \tag{63}$$

if $T^{k}_{q} = |kq\rangle$. We can generalise and define the hermitian conjugate spherical tensor $T^{+\,k}_{q}$ of a given spherical tensor T^{k}_{q} through (63) and $T^{+\,k}_{q}$ transforms accordings to

$$T^{+\,k}_{q}(\mathrm{II}) = \sum_{q'} D^{k}_{+q'+q} \, T^{+\,k}_{q}(\mathrm{I}) \tag{64}$$

or using (63)

$$\left(T^{k}_{q}(\mathrm{II})\right)^{+} = \sum_{q'} (-1)^{q'-q} D^{k}_{-q'-q} \left(T^{k}_{q}(\mathrm{I})\right)^{+} \tag{64}$$

which is simply the hermitian conjugate of (9a).

Let us now consider a matrix element $\langle J'm'| O^{k}_{q} |Jm\rangle$ of a spherical tensor operator O^{k}_{q} between angular momentum states. If O^{k}_{q} is an operator operating on the states and does not depend on any external orientations, the matrix element is simply a number and invariant under rotations. For example, the operators J_μ or ∇_μ or Y_{lm}. operating on $|Jm\rangle$ are such operators whereas $\vec{J}.\vec{H}$ where $\vec{H}$ is an external field, is not since in thatcase the matrix element is $\langle J'm'|\vec{J}|Jm\rangle$. $\vec{H}$ and hence transforms like a vector, the matrix elements $\langle J'm'|\vec{J}|Jm\rangle$ being numbers. ~~Again~~ But, $\langle l's'J'm|\vec{L}\cdot\vec{S}|lsJm\rangle$ in a scalar since both $\vec{L}$ and $\vec{S}$ operate on the states and $\vec{L}.\vec{S}$ does not depend on any external (to the system represented by the states) orientations.

Considering now the matrix elements $\langle J'm'|O^k_q|Jm\rangle$ which are scalars we observe that each is also a prooduct of three spherical tensors of rank J k and J' respectively. By virtue of (45a) we see that $O^k_q|Jm\rangle$ is a linear combination of spherical tensors of rank $K = J+k, J+k-1, \ldots, |J-k|$ viz.,

$$\sum_K C(kJK;qmM)\left(O^k \times T^J\right)^K_M$$

where $T^J_m \equiv |Jm\rangle$ and the matrix element itself is a linear combination of spherical tensors of rank $(J'+K), \ldots, |J'-K|$. But we already know that this can only be of rank zero being a scalar and therefore K can only take the value J' so that the matrix element is the zero rank spherical tensor

$$(-1)^{-m'}C(J'J'0;-m'm'0)\,C(kJJ';qmm')\left\langle\left(T^{+J'}\times\left(O^k\times T^J\right)^{J'}\right)^0_0\right\rangle$$

and using (5e) the above can be written as

$$\frac{(-1)^{J'}}{\sqrt{2J'+1}}\,C(kJJ';qmm')\left\langle\left(T^{+J'}\times\left(O^k\times T^J\right)^{J'}\right)^0_0\right\rangle \qquad (65)$$

from which it is clear that the matrix elements vanish if $m + q \neq m'$ or if $j\ k\ j'$ do not form a triangle and that the dependence of the matrix element on the projection quantum numbers could be factored out in the form of a clebsch-Gordon coefficient; which is simply the Wigner-Eckart theorem.*

We can also rewrite (65) using (5a) so that we can identify the reduced matrix element in (10) as

$$\langle j' \| O^k \| j \rangle = \frac{(-1)^{j+k}}{\sqrt{2j'+1}} \left\langle \left(T^{+j} \times \left(O^k \times T^j \right)^{j'} \right)^0 \right\rangle \tag{66}$$

$$= \frac{(-1)^{j+k-j'}}{2j'+1} \left\langle \left(T^{+j'} \cdot \left(O^k \times T^j \right)^{j'} \right) \right\rangle \tag{66a}$$

using (52)

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* In arriving at this result above we have used the symmetry (5e) of the Clebsh-Gordon coefficients and (61) which was obtained using the reality of the Clebsch-Gordon coefficients, both of which we have not as yet justified. However (5e) and (61) could be obtained from basic considerations (see Lectures 11, 12) so that the above is quite an adequate proof of the theorem.

LECTURE V

As a good exercise in the composition of spherical tensors, we shall consider now the multipole expansion of a vertex field, in particular the electro magnetic field.

A scalar field $\phi(r,\theta,\phi)$ satisfying

$$\nabla^2 \phi(r\theta\phi) + k^2 \phi(r\theta\phi) = 0 \tag{67}$$

has solutions ofthe form

$$\phi_{l,m}(r\theta\phi) = \mathfrak{Z}_l(kr)\, Y_{lm}(\theta,\phi) \tag{68}$$

where $\mathfrak{Z}_l(kr)$ are spherical Bessel or Neumann functions of order l or a linear combination of both. $\phi_{lm}(r\theta\phi)$ are clearly eigen states of orbital angular momentum $\vec{L} = -i\vec{r}\times\vec{\nabla}$, i.e.,

$$L^2 \phi_{lm} = l(l+1)\phi_{lm}$$

$$L_z \phi_{lm} = m\,\phi_{lm} \tag{69}$$

and have parity $(-1)^l$

$$\phi_{lm}(r,\pi-\theta,\phi+\pi) = (-1)^l \phi_{lm}(r\theta\phi) \tag{70}$$

A plane wave solution $e^{i\vec{k}\cdot\vec{r}}$ of (67) could be expanded using the Rayliegh expansion of a plena wave as

$$e^{i\vec{k}\cdot\vec{r}} = (4\pi)^{1/2}\sum_{l=0}^{\infty} i^l (2l+1)^{1/2} j_l(kr)\, P_l(\cos\theta) \tag{71}$$

Where Θ is the angle between $\vec{k}$ and $\vec{r}$ and we can choose the direction of $\vec{k}$ as the Z- axis without loss of generality so that (71)represent an expansion of the plane wave solution in terms of angular momentum and parity states. Our object now is to set up (1) the angular momentum and parity eigen states of a vector field $\vec{A}(r\,\theta\,\phi)$ satisfying

$$\nabla^2 \vec{A}(r\,\theta\,\phi) + k^2 \vec{A}(r\,\theta\,\phi) = 0 \tag{72}$$

and (ii) to expand a plane wave solution of (72) in terms of these solutions. For this purpose we observe that

$$\vec{A}(r\,\theta\,\phi) = \vec{i}\,A_x(r\theta\phi) + \vec{j}\,A_y(r\theta\phi) + \vec{k}\,A_z(r\theta\phi) \tag{73}$$

can also be written as

$$= \sum_{\mu=-1,0,1} (-1)^{\mu}\, {}^1\chi_{\mu}\, A_{-\mu}(r\,\theta\,\phi)$$

where $A_{\mu}(r\,\theta\,\phi)$ are the components of $\vec{A}(r\,\theta\,\phi)$ in spherical basis (11) and ${}^1\chi_{\mu}$ are the unit vectors in spherical basis:

$$\begin{aligned} {}^1\chi_{\pm 1} &= \mp \frac{\vec{i} \pm \sqrt{-1}\,\vec{j}}{\sqrt{2}} \\ {}^1\chi_0 &= \vec{k} \end{aligned} \tag{73a}$$

We also observe that under a transformation of the coordinate system $\vec{A}(r\,\theta\,\phi)$ under goes changes due to two reasons (1) since the components A_x, A_y, A_z form a vector in space and (2) due to the change in the variables. This could conveniently be described if we allow the unit vectors undergo the necessary changes to account for (1) treating $A_x(r,\theta,\phi)$, $A_y(r\theta\phi)$ and $A_z(r\theta\phi)$ as three functions of $r\,\theta\,\phi$ which transform suitably to account for (2). For, the effect of transformation on a vector

$$\vec{r} = \vec{i}\,x + \vec{j}\,y + \vec{k}\,z = \sum_{\mu=-1}^{1} (-1)^{\mu}\, {}^{1}\chi_{\mu}\, r_{-\mu} \tag{74}$$

can be described either by sperifying the new components $x'\, y'\, z'$ of the vector on the basis vectors $\vec{i}, \vec{j}, \vec{k}$ or by finding a new basis $\vec{i}\,', \vec{j}\,', \vec{k}\,'$ in terms of which the components of $\vec{r}$ remain the same x, y, z. Thus we describe the transformation of a vector field as

$$\vec{i} A_x(r\theta\phi) + \vec{j} A_y(r\theta\phi) + \vec{k} A_z(r\theta\phi)$$
$$\rightarrow \vec{i}\,' A_x(r'\theta'\phi') + \vec{j}\,' A_y(r'\theta'\phi') + \vec{k}\,' A_z(r'\theta'\phi')$$
$$= \vec{i}\,' A_x'(r\theta\phi) + \vec{j}\,' A_y'(r\theta\phi) + \vec{k}\,' A_z'(r\theta\phi)$$

or

$$\sum_{\mu} (-1)^{\mu}\, {}^{1}\chi_{\mu}\, A_{-\mu}(r\theta\phi) \rightarrow \tag{75}$$
$$\sum_{\mu} (-1)^{\mu}\, {}^{1}\chi_{\mu}'\, A_{-\mu}(r'\theta'\phi')$$
$$= \sum_{\mu} (-1)^{\mu}\, {}^{1}\chi_{\mu}'\, A_{-\mu}'(r,\theta\,\phi)$$

in opherical basis.

In particular under rotations, one can by explicit construction show that the basis vectors ${}^{1}\chi_{\mu}$ transform for infinitisimal rotations like a spherical tensorof rank 1. Since, the matrices of the angular momentum operators J_x, J_y and J_z are just $\sqrt{-1}$ times the corresponding matrices (i.e. in the same representation) for respective infinitisimal rotations, we can consider $\sqrt{-1}$ times the infinitisimal rotation matrices for ${}^1\chi_\mu$ as the 3 x 3 matrices of an 'angular momentum' operator $\vec{S}$ whose eigen states ${}^1\chi_\mu$ are. Viz.,

$$S^2\,{}^1\chi_\mu = 2\,{}^1\chi_\mu$$

$$S_z\,{}^1\chi_\mu = \mu\,{}^1\chi_\mu \qquad (76)$$

and we call ~~a~~ vector field a spin 1 field. Also (72) implies that each component of $\vec{A}(r\theta\phi)$ satisfies an equation of the type (67) and consequently have solutions of the form (68) which are eigenstates of L^2 and L_z. And since $\vec{A}(r\theta\phi)$ consists of products of the type $A_{lm}(r\theta\phi)\,{}^1\chi_\mu$ we can in a nology with the coupling of two angular momenta define a total angular momentum operator $\vec{J}$ of the vector field as

$$\vec{J} = \vec{L} + \vec{S} \qquad (77)^*$$

The linear combination (73) of terms $A_{lm}(r\theta\phi)\,{}^1\chi_\mu$ which is a vector in space is not (always) an eigen state of J^2 and J_z and clearly the spherical tensor

$$\vec{T}_{jlM}(\theta\phi) = \sum_m C(l1J;m\mu M)\,Y_{lm}(\theta,\phi)\,{}^1\chi_\mu \qquad (78)$$

---

* The commutation relations (1) themselves represent a generalisation of the classical concept of angular momentum as the moment of momentum, the precise nature of the generalisation physically being to identify the angular momentum operaor of a system with $\sqrt{-1}$ times the infinitisimal rotation operator for the system.

is the proper quantity we are looking for; ~~but~~ it should immediately be seen that (78) is not a vector in space though it can be written the form $\vec{i}\,x + \vec{j}\,y + \vec{k}\,z$, $x\,y\,z$ being, in general, complex*. The arrow on top of $T_{J\ell M}$ merely indicates that we are coupling a spherical tensor of rank 1 i.e. a spin 1 field and it is in this sense that (78) is a solution of the vector field. In what follows we drop the arrow since we are considering only a spin 1 field, to avoid confussion with a physical vector.

It is obvious from (78) that we can have $J = \ell + 1$, ℓ or $\ell - 1$ or with a given J we can have three spherical tensors $T_{J\ell M}$ with $\ell = J+1, J, J-1$. The spherical tensors $T_{J,J+1,M}$ and $T_{J,J-1,M}$ have clearly the same parity while $T_{J,J,M}$ has parity opposite to this so that we have two eigen states of total angular momentum and parity with same J but opposite parity Π. Viz.,

$$A_{J\Pi M}(r\theta\phi) = C\, \mathcal{J}_J(kr)\, T_{JJM}(\theta,\phi)$$

and
$$A_{J\Pi' M}(r\theta\phi) = C_1\, \mathcal{J}_{J+1}(kr)\, T_{J,J+1,M}(\theta,\phi) + C_{-1}\, \mathcal{J}_{J-1}(kr)\, T_{J,J-1,M}(\theta,\phi) \quad (79)$$

---

* An interesting particular example is the case

$$\vec{T}_{010}(\theta,\phi) = \sum_m C(110, m-m0)\, Y_{1m}(\theta,\phi)\, {}^1\chi_{-m} \quad (78a)$$

$$= -\frac{1}{\sqrt{4\pi}}\,\frac{\vec{r}}{r} \quad (78b)$$

the spherical symmetry of which follows from the fact it is invariant under rotations (J being zero). A physical situation corresponding to the above is the case of the could~~mb~~ force field $\vec{F}(r\theta\phi)$ due to a charge placed at the origin:

$$\vec{F}(r\theta\phi) \sim \frac{\vec{r}}{r^2} = -\frac{\sqrt{4\pi}}{r}\,\vec{T}_{010}(\theta,\phi) \quad (78c)$$

~~and~~

which are solutions of (72), the coefficients C, C_1 and C_{-1} to be determined appropriate to the physical situation under consideration.

Let us now apply the above considerations to the electromagnetic field defined by Maxwell's equations for the electric and magnetic field strengths $\vec{E}$ and $\vec{H}$.

$$\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{H} + \vec{\nabla} \times \vec{m}$$

$$\frac{\partial \vec{H}}{\partial t} = \vec{\nabla} \times \vec{E} + 2\pi \vec{J}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \tag{80}$$

$$\vec{\nabla} \cdot \vec{H} = 4\pi \vec{\nabla} \cdot \vec{m}$$

Where ρ and $\vec{J}$ are the electric charge and current densities and $\vec{m}$ the distribution of magnetisation at the source. The vector and scalar potentials $\vec{A}$ and ϕ are defined through

$$\vec{\nabla} \times \vec{A} = \vec{H} + 4\pi \vec{m}$$

$$\vec{\nabla} \phi = -\vec{E} - \frac{\partial \vec{A}}{\partial t}$$

'1)

which however do not determine the field strengths completely since we can add a vector field $\vec{\nabla} \chi$ generated out of a scalar field χ to $\vec{A}$ and still obtain the same $\vec{E}$ and $\vec{H}$. Replacing $\vec{A}$ by $\vec{A} + \vec{\nabla}\chi$, χ **satisfie~~d~~s**

$$\nabla^2 \chi - \partial^2 \chi \quad - \vec{\nabla} \cdot \vec{A} \quad \partial \phi$$

(82)

and if we restrict the arbitraryness in $\vec{A}$ to the family of χ's which satisfies (82) $= 0$ the field is said to be in the Lorentz gauge when $\vec{A}$ and ϕ satisfy

$$\nabla^2 A - \frac{\partial^2 \vec{A}}{\partial t^2} = 4\pi(\vec{J} + \vec{\nabla}\times\vec{m})$$

$$\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = 4\pi\rho \tag{83}$$

together with

$$\vec{\nabla}\cdot\vec{A} + \frac{\partial\phi}{\partial t} = 0$$

which is referred to as the Lorentz condition.

For a source free monochromatic field (i.e

$$\vec{A}(r\theta\phi, t) = \vec{A}(r\theta\phi)\, e^{-i\omega t}$$

being the frequency) (83) reduce to

$$\nabla^2\vec{A}(r\theta\phi) + k^2\vec{A}(r\theta\phi) = 0 \tag{84a}$$

$$\nabla^2\phi(r\theta\phi) + k^2\phi(r\theta\phi) = 0 \tag{84b}$$

and

$$\vec{\nabla}\cdot\vec{A}(r\theta\phi) + ik\,\phi(r\theta\phi) = 0 \tag{84c}$$

where $\omega^2 = k^2$, k being the wave number.

We are now concerned with the solutions of (84) which are eigen states of angular momentum and parity. The solutions of (84a) are clearly of the form (79) which are referred to as the multipole solutions of order J

(and usually the letter L is used in literature instead of J) The first one of the solutions (79) is referred to as the magnetic multipole solutions and the second one as the electric multipole solution

$$A^{(m)}_{LM}(r\theta\phi) = C\, \mathcal{J}_L(kr)\, T_{LLM}(\theta\phi) \tag{85a}$$

$$A^{(e)}_{LM}(r\theta\phi) = C_1\, \mathcal{J}_{L+1}(kr)\, T_{L,L+1,M}(\theta,\phi) + C_{-1}\, \mathcal{J}_{L-1}(kr)\, T_{L,L-1,M}(\theta,\phi) \tag{85b}$$

It should be noticed that corresponding to both the solutions, there will be electric as well as magnetic field strengths $\vec{E}$ and $\vec{H}$ determined throgh

$$\vec{\nabla} \times \vec{A} = \vec{H}$$

and

$$\vec{\nabla}\phi = -\vec{E} + ik\vec{A} \tag{81a}$$

the solutions of ϕ corresponding to the solutions (86) of $\vec{A}$ being determined through (84c) which we shall calculate in the next lecture to be

$$\phi^{(m)}_{LM}(r\theta\phi) = 0 \tag{86a}$$

$$\phi^{(e)}_{LM}(r\theta\phi) = i\left[C_1\left(\frac{L+1}{2L+1}\right)^{1/2} + C_{-1}\left(\frac{L}{2L+1}\right)^{1/2}\right]\mathcal{J}_L(kr)\, Y_{LM}(\theta\phi) \tag{86b}$$

which clearly do not involve the unit vectors, ϕ being scalar.

We see from (80) that the eletric and magnetic fields $\vec{E}$ and $\vec{H}$ if they are eigen states of parity, must always have opposite parity so that the names 'eldctric' and 'magnetic' to the two solutions in (85) are appropriate to the extent they are solutions of opposite parity. Since it is found that for small values of kr, i.e $kr << 1$, the electric field $E_{LM}^{(m)}$ derived from $A_{LM}^{(m)}$ is much smaller compared to the $H_{LM}^{(m)}$ derived from the same and similarly $H_{LM}^{(e)} << E_{LM}^{(e)}$ for $kr << 1$, the first of the solutions in (85) is referred to as the 'magnetic' multipole solutions and the second as the 'electric' multipole solution.

LECTURE VI

We have already pointed out that $A_{J\ell M}(r\theta\phi)$ could be written in the form $\vec{i}\,x + \vec{j}\,y + \vec{k}\,z$ so that we can now define quantities like $\vec{V}\cdot A_{J\ell M}$ or $\vec{V}\times A_{J\ell M}$ in the usual way if $\vec{V}$ is a vector or again, in general, a quantity which can be written in the form $\vec{i}\,V_x + \vec{j}\,V_y + \vec{k}\,V_z$.

Writing,

$$A_{J\ell M}(r\theta\phi) = \mathcal{J}_\ell(kr)\, T_{J\ell M} = \sum_\mu C((\ell 1);\, m-\mu\, M)\, \mathcal{J}_\ell(kr)\, Y_{\ell m}\, {}^1\chi_\mu \tag{87}$$

in the form

$$= \sum_\mu (-1)^\mu (A_{J\ell m})_\mu\, {}^1\chi_{-\mu} \tag{87a}$$

where

$$(A_{J\ell m})_\mu = (-1)^\mu C((\ell 1);\, m-\mu\, M)\, \mathcal{J}_\ell(kr)\, Y_{\ell\, m+\mu}(\theta\phi) \tag{88}$$

we have by (52)

$$\vec{V}\cdot A_{J\ell M} = V_x (A_{J\ell m})_x + V_y (A_{J\ell m})_y + V_z (A_{J\ell m})_z$$
$$= \sum_\mu (-1)^\mu V_\mu (A_{J\ell m})_{-\mu}$$

$$= \sum_{\mu} C(\ell 1 J; m \mu M)\, V_{\mu}\, J_{\ell}(kr)\, Y_{\ell\, M-\mu}(\theta, \phi) \tag{89}$$

using (88). In particular, therefore,

$$\vec{\nabla} \cdot A_{J\ell M} = \sum_{\mu} C(\ell 1 J; m \mu M)\, \nabla_{\mu}\, J_{\ell}(kr)\, Y_{\ell\, M-\mu}(\theta, \phi) \tag{89a}$$

where the components of the gradient operator in spherical basis is defined as usual.

$$\nabla_{\pm 1} = \mp \frac{\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y}}{\sqrt{2}}, \quad \nabla_0 = \frac{\partial}{\partial z}$$

To evaluate (89a), it is convenient to write the operator $\vec{\nabla}$, using the identify

$$\vec{r} \times (\vec{r} \times \vec{\nabla}) = \vec{r}\,(\vec{r} \cdot \vec{\nabla}) - (\vec{r} \cdot \vec{r})\, \vec{\nabla} \tag{90}$$

as

$$\vec{\nabla} = \frac{1}{r^2} \left\{ \vec{r}\,(\vec{r} \cdot \vec{\nabla}) - \vec{r} \times (\vec{r} \times \vec{\nabla}) \right\}$$

$$= \frac{1}{r^2} \left\{ \vec{r}\, r \frac{\partial}{\partial r} - i\, \vec{r} \times \vec{L} \right\} \tag{91}$$

in terms of the operators $\frac{\partial}{\partial r}$ and $\vec{L}$ which operate respectively on the radial and angular parts only. Also from (74) and (14) we can write the vector $\vec{r}$ as

$$\vec{r} = \sqrt{\frac{4\pi}{3}}\, r \sum_{\mu} (-1)^{\mu}\, Y_{1\mu}(\theta \phi)\, {}^{1}\chi_{-\mu} \tag{74a}$$

and if $\vec{C} = \vec{A} \times \vec{B}$ where $\vec{A}$ and $\vec{B}$ are two vectors, the components of $\vec{C}$, (where $C_x = A_y B_z - A_z B_y$ ~~By~~ etc.) could be expressed in the spherical basis as

$$C_\mu = i\sqrt{2} \sum_{\mu_1} C(111;\mu_1,\mu_2,\mu) A_{\mu_1} B_{\mu_2} \tag{92}$$

which is readily verified. Thus from (91) we have

$$\nabla_\mu = \sqrt{\frac{4\pi}{3}} \Big[Y_{1\mu}(\theta\phi) \frac{\partial}{\partial r} + \frac{\sqrt{2}}{r} \times \sum_{\mu_1} C(111;\mu_1,\mu_2\,\mu) Y_{1\mu_1}(\theta\phi) L_{\mu_2} \Big] \tag{93}$$

using (74a) and (92).

To evaluate (89a) now we have to evaluate

$$\nabla_\mu \mathfrak{Z}_\ell(kr) Y_{\ell m}(\theta\phi) = \sqrt{\frac{4\pi}{3}} \Big[\frac{d}{dr} \mathfrak{Z}_\ell(kr) Y_{1\mu}(\theta\phi) Y_{\ell m}(\theta,\phi) + \frac{\sqrt{2}}{r} \mathfrak{Z}_\ell(kr) \sum_{\mu_1} C(111;\mu_1\mu_2\mu) Y_{1\mu_1}(\theta\phi) L_{\mu_2} Y_{\ell m}(\theta\phi) \Big] \tag{94}$$

we have

$$L_\mu Y_{\ell m}(\theta\phi) = C(\ell 1 \ell; m\mu\, m+\mu) \sqrt{\ell(\ell+1)}\, Y_{\ell, m+\mu}(\theta\phi) \tag{95}$$

Using the Wigner-Eckart theorem and (12) since L_μ is the component of orbital angular momentum operator $\vec{L}$. Therefore (94) can be written as

$$= \sqrt{\frac{4\pi}{3}}\Big[\frac{d}{dr}\mathcal{J}_\ell(kr)\,Y_{1\mu}(\theta\phi)\,Y_{\ell m}(\theta\phi)$$

$$+\frac{\sqrt{2}}{r}\mathcal{J}_\ell(kr)\sum_{\mu_1} C(111;\mu_1\mu_2\mu)\sqrt{\ell(\ell+1)}$$

$$\times C(\ell 1\ell; m\mu_2 m+\mu_2)\,Y_{1\mu_1}(\theta\phi)\,Y_{\ell\, m+\mu_2}(\theta\phi)\Big] \quad (96)$$

and using the coupling rule (43) we can write (96) as

$$\sqrt{\frac{4\pi}{3}}\sum_{\ell'}\sqrt{\frac{3(2\ell+1)}{4\pi(2\ell'+1)}}\;C(\ell 1\ell';00)$$

$$\times\Big[\frac{d}{dr}\mathcal{J}_\ell(kr)\,C(\ell 1\ell'; m\mu M)$$

$$+\frac{\sqrt{2}}{r}\mathcal{J}_\ell(kr)\sqrt{\ell(\ell+1)}\sum_{\mu_1} C(111;\mu_1\mu_2\mu)$$

$$\times C(\ell 1\ell; m\mu_2)\,C(\ell 1\ell'; m+\mu_2\,\mu_1 M)\Big]Y_{\ell' M}(\theta\phi) \quad (97)$$

Using the symmetry property (5a) we can write

$C(111;\mu_1\mu_2\mu) = (-1)\,C(111;\mu_2\mu_1\mu)$ and we can now express

$$\sum_{\mu_1} C(111;\mu_2\mu_1\mu)\,C(\ell 1\ell; m\mu_2)\,C(\ell 1\ell'; m+\mu_2\,\mu_1 M)$$

$$= U(\ell 1\ell' 1;\ell 1)\,C(\ell 1\ell'; m\mu M) \quad (98)^*$$

---

* Since we have from (27)

$$C(J_1J_2J_{12};m_1m_2)\,C(J_{12}J_3J;m_1+m_2\,m_3)$$

$$= \sum_{J_{23}} U(J_1J_2JJ_3;J_{12}J_{23})\,C(J_2J_3J_{23};m_2m_3)$$

$$\times C(J_1J_{23}J;m_1\,m_2+m_3\,m) \quad (98a)$$

(Continued on next Page.)

We have thus

$$\nabla_{\mu}\, \mathfrak{z}_{l}(kr)\, Y_{lm}(\theta\phi) = \sum_{l'=l-1}^{l+1} C(l1l';\, m\mu M)\, C(l1l';\, 00)$$

$$\times \sqrt{\frac{2l+1}{2l'+1}} \left[\frac{d}{dr} + \frac{\sqrt{2}}{r}\sqrt{l(l+1)}\; W(l1l'1;\, l1) \right]$$

$$\times\, \mathfrak{z}_{l}(kr)\, Y_{lM}(\theta\phi) \tag{99}$$ *

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equating coefficients of $|j_1 m_1\rangle \; |j_2 m_2\rangle \; |j_3 m_3\rangle$ on both sides of (27). We can now multiply (98a) by $C(j_2 j_3 j'_{23};\, m_2 m_3)$ and sum over m_3 to obtain.

$$\sum_{m_3} C(j_1 j_2 j_{12};\, m_1 m_2)\, C(j_{12} j_3 j;\, m_1+m_2\, m_3\, m)$$

$$\times\, C(j_2 j_3 j'_{23};\, m_2 m_3) = W(j_1 j_2 j\, j_3;\, j_{12} j'_{23})$$

$$\times\, C(j_1 j'_{23} j;\, m_1\, m_2+m_3) \tag{98b}$$

using the unitarity condition (6) on right hand side.

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* (99) holds clearly with any arbitrary $\phi(r)$ instead of $\mathfrak{z}_l(kr)$ since we have not made use of the special form of the radial function anywhere so far. Also the summation over, l' is only $l' = l \pm 1$ since $C(l1l;\, 00) = 0$ by $(5f)$. Substituting for

$$C(l1l+1;\, 00) = \left(\frac{l+1}{2l+1}\right)^{1/2};\quad C(l1l-1;\, 00) = -\left(\frac{l}{2l+1}\right)^{1/2} \tag{101}$$

nt :

and

$$\mathcal{V}\,(\ell\,|\,\ell+1\,1\,;\ell\,1) = -\left(\frac{\ell}{2(\ell+1)}\right)^{1/2}$$

$$\mathcal{V}(\ell\,1\,\ell-1\,1\,;\ell\,1) = \left(\frac{\ell+1}{2\ell+1}\right)^{1/2} \tag{102}$$

We can write (99) as

$$\nabla_\mu \phi(r)\, Y_{\ell m}(\theta\phi) = \left(\frac{\ell+1}{2\ell+3}\right)^{1/2} C(\ell\,1\,\ell+1;\, m\,\mu\,M) \times \left(\frac{d}{dr} - \frac{\ell}{r}\right)\phi(r)\, Y_{\ell+1,M}(\theta\phi)$$
$$-\left(\frac{\ell}{2\ell-1}\right)^{1/2} C(\ell\,1\,\ell-1;\, m\,\mu\,M)\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\phi(r)\, Y_{\ell-1,M}(\theta\phi) \tag{103}$$

so that

$$\vec{\nabla}\phi(r)\, Y_{\ell m}(\theta\phi) = \sum_{\mu=-1}^{1} (-1)^\mu\, \nabla_\mu\, {}^1\chi_{-\mu}\, \phi(r)\, Y_{\ell m}(\theta\phi)$$
$$= -\left(\frac{\ell+1}{2\ell+3}\right)^{1/2}\left(\frac{d}{dr} - \frac{\ell}{r}\right)\phi(r) \sum_\mu \left(\frac{2\ell+3}{2\ell+1}\right)^{1/2} \times C(\ell+1,1,\ell;\, M,-\mu,m)\, Y_{\ell+1,M}(\theta\phi)\, {}^1\chi_{-\mu}$$
$$+\left(\frac{\ell}{2\ell-1}\right)^{1/2}\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\phi(r) \sum_\mu \left(\frac{2\ell-1}{2\ell+1}\right)^{1/2} C(\ell-1,1,\ell;\, M,-\mu,m) \times Y_{\ell-1,M}(\theta\phi)\, {}^1\chi_{-\mu} \tag{104}$$

using (103) and the symmetry properties (5c) and (5d) in that order and since $(-1)^{\ell\pm1+1-\ell} = 1$; $(-1)^{2\mu} = 1$, μ being integer.

Using (78) in (104) we have

(continued next page)

and consequently (89λ) is,

$$\vec{\nabla}\cdot A_{J\ell M} = \sqrt{\frac{2\ell+1}{2J+1}}\, C(\ell 1 J;00)\left(\frac{d}{dr} + \frac{\sqrt{2\ell(\ell+1)}}{r}\, v(\ell 1 J 1;\ell 1)\right) \times \mathfrak{Z}_\ell(kr)\, Y_{JM}(\theta,\phi) \tag{100}$$

using again the unitarity property (6). It is clear from (100) that

$$\vec{\nabla}\cdot A_{LLM} = 0 \tag{106a}$$

since $C(L1L;00) = 0$, L being integer. Substituting in (100) for the Clebsch-Gordon and recoupling coefficient using (101) (102)

$$\vec{\nabla}\cdot A_{L\lambda M} = \left(\frac{\lambda+1}{2\lambda+3}\right)^{1/2}\left(\frac{d}{dr} - \frac{\lambda}{r}\right)\mathfrak{Z}_\lambda(kr)\, Y_{LM} \quad \text{for } \lambda = L-1 \tag{106b}$$

$$\vec{\nabla}\cdot A_{L\lambda M} = -\left(\frac{\lambda}{2\lambda-1}\right)^{1/2}\left(\frac{d}{dr} + \frac{\lambda+1}{r}\right)\mathfrak{Z}_\lambda(kr)\, Y_{LM} \quad \text{for } \lambda = L+1 \tag{106c}$$

We now make use of the property,

(Continued previous Page)

$$\vec{\nabla}\phi(r)\, Y_{\ell m}(\theta\,\phi) = -\left(\frac{\ell+1}{2\ell+1}\right)^{1/2}\left(\frac{d}{dr} - \frac{\ell}{r}\right)\phi(r)\, T_{\ell,\ell+1,m} + \left(\frac{\ell}{2\ell+1}\right)^{1/2}\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\phi(r)\, T_{\ell,\ell-1,m} \tag{107}$$

as

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(105)

Equations (013) and (105) are alternate forms of what is referred to as the gradient formula.

$$\frac{d}{dx} \mathfrak{z}_\lambda(x) = \frac{\lambda}{x} \mathfrak{z}_\lambda(x) - \mathfrak{z}_{\lambda+1}(x)$$

$$= \frac{\lambda+1}{x} \mathfrak{z}_\lambda(x) + \mathfrak{z}_{\lambda-1}(x) \qquad (107)$$

satisfied by $\mathfrak{z}_\lambda(kr)$ so that we can rewrite (106) as

$$\vec{\nabla} \cdot A_{LLM} = 0 \qquad (108a)$$

$$\vec{\nabla} \cdot A_{L,L-1,M}(r\theta\phi) = -\left(\frac{L}{2L+1}\right)^{1/2} k \mathfrak{z}_L(kr) Y_{LM}(\theta\phi) \qquad (108b)$$

and

$$\vec{\nabla} \cdot A_{L,L+1,M}(r\theta\phi) = -\left(\frac{L+1}{2L+1}\right)^{1/2} k \mathfrak{z}_L(kr) Y_{LM}(\theta\phi) \qquad (108c)$$

From (84c) we have $\phi_{L\lambda M} = \frac{1}{ik} \vec{\nabla} \cdot A_{L\lambda M}$ so that (86) follows from (108) and (85). The expressions (85) and (86) are required solutions (Viz eigen states of angular momentum and parity) of the electromagnetic field in the Lorentz gauge, the constants C, C_+, C_- to be determined from the physical situation the field is to represent. For example, if the field is to represent photons* we can immediately impose that the field should be transverse. The transversality condition could be written as

$$\vec{k} \cdot \vec{A}(k, \theta_k, \phi_k) = 0 \qquad (109)$$

where $\vec{A}(k\theta_k\phi_k)$ is the Fourier transform of $\vec{A}(r\theta\phi)$ in to momentum space. The condition corresponding to (109) in configuration space is

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* It should however be pointed out that $\vec{A}(r\theta\phi)$ cannot be considered the wave function of a photon in configuration space since photons can never be localized and consequently no probability interpretation can be given to the wave function; but the Fourier transform of $\vec{A}(r\theta\phi)$ in momentum space is a good wave function of a photon.

$$\vec{\nabla} \cdot \vec{A}(r\,\theta\,\phi) = 0 \tag{110}$$

and clearly (110) restricts the (gauge) arbitrariness in $\vec{A}$ to a sub class of the Lorentz gauge which is termed the solenoidal guage. (84c), with (110) implies that $\phi = 0$ in the solenoidal guage. Imposing this restriction on (86b) we have

$$C_1 = -\left(\frac{L}{L+1}\right)^{1/2} C_{-1} \tag{111A}$$

The functions $A^{(\lambda)}_{LM}(r\,\theta\,\phi)$ could for all calculational purposes be considered as wave functions for the photon and with such usage in mind we impose that the solutions be normalised to unity. From (85) therefore we have

$$|C|^2 = 1 \quad ; \quad |C_1|^2 = -|C_{-1}|^2 + 1 \tag{111 B}$$

since $T_{L\lambda M}$ are normalised as $Y_{\ell m}$ denote normalised spherical harmonics and ${}^{1}\chi_{\mu}$ are unit vectors in spherical basis; and also $\mathcal{J}_\ell(kr)$ denote normalised radial functions. From (111A) and (111B) we have

$$\begin{aligned} C &= 1 \\ C_1 &= \mp\left(\frac{L}{2L+1}\right)^{1/2} \\ C_{-1} &= \pm\left(\frac{L+1}{2L+1}\right)^{1/2} \end{aligned} \tag{112}$$

choosing the constants to be real. Choosing also $C_1 < 0$ we have

$$A^{(m)}_{LM}(r\,\theta\,\phi) = \mathcal{J}_L(kr)\; T_{LLM}(\theta\,\phi) \tag{113}$$

$$A^{(e)}_{LM}(r\theta\phi) = -\left(\frac{L}{2L+1}\right)^{1/2} \mathcal{J}_{L+1}(kr)\, T_{L,L+1,M}(\theta\phi)$$
$$+\left(\frac{L+1}{2L+1}\right)^{1/2} \mathcal{J}_{L-1}(kr)\, T_{L,L-1,M}(\theta\phi) \tag{114}$$

$$\phi^{(m)}_{LM}(r\theta\phi) = 0 \tag{115}$$

$$\phi^{(e)}_{LM}(r\theta\phi) = 0 \tag{116}$$

Using the definitions (81) we have now

$$\vec{H} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = ik\vec{A}$$

and written explicitly

$$H^{(m)}_{LM}(r\theta\phi) = ik\left[-\left(\frac{L}{2L+1}\right)^{1/2} \mathcal{J}_{L+1}(kr)\, T_{L,L+1,M}(\theta\phi) + \left(\frac{L+1}{2L+1}\right)^{1/2} \mathcal{J}_{L-1}(kr)\, T_{L,L-1,M}(\theta\phi)\right]$$
$$= E^{(e)}_{LM}(r\theta\phi) \tag{117}$$

$$H^{(e)}_{LM}(r\theta\phi) = -ik\,\mathcal{J}_L(kr)\, T_{LLM}(\theta\phi)$$
$$= -E^{(e)}_{LM}(r\theta\phi)$$

on calculation using (92) and the gradient formula. It is clearly a property of the source free Maxwell's equations that if $\vec{E}$ and $\vec{H}$ are solutions then $\vec{E}_d = \mp \vec{H}$; $\vec{H}_d = \pm \vec{E}$ are also solutions. $\vec{E}_d$, $\vec{H}_d$ are said to be 'dual' fields to $\vec{E}$ and $\vec{H}$. We therefore see form (117) that 'electric' and 'magnetic' multipole solutions are dual to each other.

The set of solutions (113)..(116) form a complete set of angular momentum and parity solutions for the transverse electro-magnetic field only, and they represent only two independent linear combinations of the three spherical tensors T_{LLM}, $T_{LL\pm 1M}$ with a same L which are orthogonal to each other.

We can now form a third independent (i.e. orthogonal both to (113) & (114)) linear combination of the $T_{L\lambda M}$'s, which clearly will not satisfy the transversality condition, and referred to as the longitudinal multipole solution:

$$A^{(l)}_{LM}(r\theta\phi) = \left(\frac{L+1}{2L+1}\right)^{1/2} j_{L+1}(kr)\, T_{L\,L+1\,M}(\theta\phi)$$

$$+ \left(\frac{L}{2L+1}\right)^{1/2} j_{L-1}(kr)\, T_{L\,L-1\,M}(\theta\phi) \tag{118}$$

The scalar potential $\phi^{(l)}_{LM}(r\theta\phi)$ corresponding to the above solution is form (84c) and (108)

$$\phi^{(l)}_{LM}(r\theta\phi) = i\, j_L(kr)\, Y_{LM}(\theta\phi) \tag{119}$$

Now the set of solutions (113),(114),(115),(116),(118) and (119) form a complete ortho~~gonal~~normal set of solutions of the electromagnetic field equations (83) in Lorentz guage and any general solution $\vec{A}(r\theta\phi)$ and $\phi(r\theta\phi)$ could be expanded in terms of these states.

$$\vec{A}(r\theta\phi) = \sum_{L,M}\left[a_{LM} A^{(m)}_{LM} + b_{LM} A^{(e)}_{LM} + c_{LM} A^{(l)}_{LM}\right]$$

$$\phi(r\theta\phi) = \sum_{LM} d_{LM}\, \phi^{(l)}_{LM} \tag{120}$$

where a_{LM}, b_{LM}, c_{LM} and d_{LM} are appropriate coefficients. In particular if the field is transverse $c_{LM} = 0 = d_{LM}$ for all L,M.

LECTURE VII

We shall now take up problem (ii) viz., that of the expansion of a plane wave solutions of (83) in terms of the multipole solutions (112) (113) (114) (117) and (118). The plane wave solution of (83) is of the form ${}^1\chi_\mu \, e^{i\vec{k}\cdot\vec{r}}$ and if we choose a coordinate system such that the z axis lies along the direction of propagation $\vec{k}$, then clearly ${}^1\chi_0$ is the unit vector representing longitudinal polarisation and ${}^1\chi_\pm$ correspond to left and right circular polarisations.*

Using (71)

$$
\begin{aligned}
{}^1\chi_\mu \, e^{i\vec{k}\cdot\vec{r}} &= (4\pi)^{1/2} \sum_{l=0}^{\infty} i^l (2l+1)^{1/2} j_l(kr) Y_{l,0}(\theta')\, \chi_\mu \\
&= (4\pi)^{1/2} \sum_{l=0}^{\infty} i^l (2l+1)^{1/2} j_l(kr) \sum_{\lambda=l-1}^{l+1} C(l1\lambda;0\mu)\, T_{\lambda l \mu}(\theta') \quad + \qquad (121)
\end{aligned}
$$

using the inverse of (78). Substituting for the values of the Clebsch-Gordon coefficients,

$$
\begin{aligned}
C(l\,1\,l;0\mu) &= \frac{-\mu}{\sqrt{2}}, \quad l > 0 \\
C(l\,1\,l+1;0\mu) &= \left[\frac{l+2}{2(2l+1)}\right]^{1/2} \quad \text{for } \mu = \pm 1 \\
C(l\,1\,l+1;0\mu) &= \left(\frac{l+1}{2l+1}\right)^{1/2} \quad \text{for } \mu = 0
\end{aligned} \qquad (122)
$$

---

* Usually $\frac{1}{\sqrt{2}}(\vec{i} \pm \sqrt{-1}\,\vec{j})$ are used to denote the unit vectors corresponding to left and right circular polarisations, in which case $-{}^1\chi_1$ and ${}^1\chi_{-1}$ are the desired unit vectors which could collectively be represented as $-\mu\, {}^1\chi_\mu$ for transverse fields.

+ θ' is the angle between $\vec{k}$ and $\vec{r}$; in the coordinate system $\vec{k}$ along z, the polar angles of $\vec{r}$ are obviously (θ', ϕ). (121) is however, independent of the angle ϕ as is also (125) and (128).

$$c(l\,1\,l-1;\,0\mu) = \left(\frac{l-1}{2(2l+1)}\right)^{1/2} \quad \text{for} \quad \mu = \pm 1$$
$$l > 1$$
$$c(l\,1\,l-1;\,0\mu) = -\left(\frac{l}{2l+1}\right)^{1/2} \quad \text{for} \quad \mu = 0$$

We have for tranverse waves

$$^{1}\chi_{\mu}\, e^{i\vec{k}\cdot\vec{r}} = (2\pi)^{1/2} \sum_{l=0}^{\infty} i^{l} j_{l}(kr) \Big[(-\mu)(2l+1)^{1/2} T_{l l \mu} + (l+2)^{1/2} T_{l+1,l,\mu} + (l-1)^{1/2} T_{l-1,l,\mu} \Big] \tag{123}$$

The summation over l extends from 1 to ∞ in the first term (since $l > 0$) from 0 to ∞ in the 2nd term and essentially from 2 to ∞ in the third term in (123), so that

$$\begin{aligned} ^{1}\chi_{\mu}\, e^{i\vec{k}\cdot\vec{r}} = (2\pi)^{1/2} \Big[& \sum_{L=1}^{\infty} i^{L} (2L+1)^{1/2} j_{L}(kr)(-\mu) T_{LL\mu} \\ & + \sum_{L=1}^{\infty} i^{L-1} (L+1)^{1/2} j_{L-1}(kr)\, T_{L,L-1,\mu} \\ & + \sum_{L=1}^{\infty} i^{L+1} L^{1/2} j_{L+1}(kr)\, T_{L,L+1,\mu} \Big] \end{aligned} \tag{124}$$

changing the summation variable l to L, $L = l$ in the first term $L = l+1$ in the second and $L = l-1$ in the third terms.

Which can be rewritten as

$$= (2\pi)^{1/2} \sum_{L=1}^{\infty} i^{L} (2L+1)^{1/2} \Big[(-\mu)\, j_{L}(kr)\, T_{LL\mu} + \frac{1}{i}\left(\frac{L+1}{2L+1}\right)^{1/2} j_{L-1}(kr)\, T_{L,L-1,\mu} + i\left(\frac{L}{2L+1}\right)^{1/2} j(kr)\, T \ldots \Big]$$

$$= (2\pi)^{1/2} \sum_{L=1}^{\infty} i^{L} (2L+1)^{1/2} \times \left[-\mu A^{(m)}_{L\mu}(r\theta'\phi') - i A^{(e)}_{L\mu}(r\theta'\phi') \right] \tag{125}$$

Using (113) and (114). For longitudinal polarisation

$${}^{1}\chi_0 e^{i\vec{k}.\vec{r}} = (4\pi)^{1/2} \sum_{l=0}^{\infty} i^{l} j_l(kr)\left[(l+1)^{1/2} T_{l+1,l,0} - l^{1/2} T_{l-1,l,0} \right] \tag{126}$$

from (121) and (122). We can rewrite (126) as

$${}^{1}\chi_0 e^{i\vec{k}.\vec{r}} = (4\pi)^{1/2} \left[\sum_{L=1}^{\infty} i^{L-1} j_{L-1}(kr) L^{1/2} T_{L,L-1,0} - \sum_{L=0}^{\infty} i^{L+1} j_{L+1}(kr)(L+1)^{1/2} T_{L,L+1,0} \right] \tag{127}$$

and we can trivially extend the summation over L in the first term to include $L = 0$, so that

$${}^{1}\chi_0 e^{i\vec{k}.\vec{r}} = (4\pi)^{1/2} \sum_{L=0}^{\infty} i^{L-1} (2L+1)^{1/2} A^{(l)}_{L0}(r\theta'\phi') \tag{128}$$

using (116).

Expressions (125) and (128) represent respectively the expansions of transverse and longitudinal plane wave solutions of the vector potential in terms of multipole solutions, and (71) itself represents the expansion of a plane wave solution of the scalar potential in terms of its multipole solutions (118). These expressions however refer to a co-ordinate system in which the direction of propagation is along the Z-axis. If $\vec{k}$ makes polar angles (θ_k, ϕ_k) in some other frame of reference the spherical tensors T_{Jlm} in the two frames are connected by

$$T_{Jlm}(\theta, \phi)_{II} = \sum_{m'} D^J_{m'm}(\phi_k, \theta_k, \phi_k)\, T_{Jlm'}(\theta, \phi)_{I} \tag{40a}$$ *

so that we have for right and left circular polarised waves

$$\vec{A}_{\vec{k}}(r\theta\phi)_{\substack{l.c\\ r.c}} = (+1)\, {}^{1}\chi_{\mu}\, e^{i\vec{k}\cdot\vec{r}} \; ; \qquad \mu = \begin{matrix} +1 & \text{for } l.c \\ -1 & \text{for } r.c \end{matrix} \text{ in a r.h system}$$

$$= (2\pi)^{1/2} \sum_{L=1}^{\infty} \sum_{M=-L}^{L} i^{L} (2L+1)^{1/2}\, D^{L}_{M\mu}(\phi_k, \theta_k, \phi_k) \Big[A^{(m)}_{LM}(r\theta\phi) + i\mu\, A^{(e)}_{LM}(r\theta\phi) \Big] \tag{129}$$

For the longitudinal component,

$$\vec{A}_{\vec{k}}(r\theta\phi)_{long} = (4\pi)^{1/2} \sum_{L=0}^{\infty} \sum_{M=-L}^{L} i^{L+1} (2L+1)^{1/2}\, D^{L}_{M0}(\phi_k, \theta_k, \phi_k) \times A^{(l)}_{LM}(r\theta\phi) \tag{130}$$

And,

$$\phi_{\vec{k}}(r\theta\phi) = (4\pi)^{1/2} \sum_{L=0}^{\infty} \sum_{M=-L}^{L} i^{L} (2L+1)^{1/2}\, D^{L}_{M0}(\phi_k, \theta_k, \phi_k) \times \phi^{(l)}_{LM}(r\theta\phi) \tag{131}$$

gives the expansion for the scalar potential.

The above expansions are useful in considering for example, the electromagnetic transitions between nuclear states whose matrix elements

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* The Eulerian angle

are of the form $\langle J'm' \mid J_N \cdot A \mid J\, m \rangle$ where J_N denotes the nucleon current-charge density $\vec{J}_N$, ρ_N and A the four vector $\vec{A}$, ϕ, and $J_N \cdot A = \vec{J}_N \cdot \vec{A} - \rho_N \phi$. From (129) it follows that emission or absorption of radiation is not possible from a nuclear state with angular momentum $J = 0$ transition to another state also with $J = 0$ since the appropriate multipole expansion starts with $L = 1$ and $\vec{J} \cdot A_{LM}$ is clearly a spherical tensor of rank L from (89). However $0 \to 0$ transitions can occur through longitudinal and scalar components whose multipole expansions (130),(131) start with $L = 0$ when no radiation is emitted or absorbed; one such example being internal conversion.

We shall now take up for consideration the formulae (47) and (55) derived earlier and before we take up examples of their use let us consider the scalar product $\vec{A} \cdot \vec{B}$ of two first rank tensor operators

$$\vec{A} \cdot \vec{B} = \sum_\mu (-1)^\mu A_\mu B_{-\mu}$$

In particular $\vec{A}$, $\vec{B}$ may both be the angular momentum operator $\vec{J}$ itself ~~so that~~ when the matrix element

$$\langle J'm' \mid \vec{J} \cdot \vec{J} \mid J\, m \rangle = \langle J'm' \mid J^2 \mid J m \rangle = \delta_{JJ'}\, \delta_{mm'}\, J(J+1) \tag{132}$$

can also be written as

$$\sum_{m'} \sum_\mu (-1)^\mu \langle J m \mid J_\mu \mid J m' \rangle \langle J m' \mid J_{-\mu} \mid J m \rangle \tag{133}$$

introducing a oomplete set of states $|J\, m'\rangle$ and since

$J_\mu^\dagger = (-1)^\mu J_{-\mu}$ ... (134), $\vec{J}$ being hermitian. (133) can be written as

$$=\sum_{\mu}\sum_{m'}\langle J\, m \,|\, J_{\mu} \,|\, J\, m'\rangle \langle J\, m \,|\, J_{\mu} \,|\, J\, m'\rangle^{*} \tag{135}$$

$$=\sum_{m'} C(J1J;\, m'\mu m)\, C^{*}(J1J;\, m'\mu m)\, |\langle J \| J \| J\rangle|^{2} \tag{136}$$

using the Wigner-Echart theorem. On account of the unitarity of the Clebsch-Gordon transformation, the above is simply $|\langle J \| J \| J\rangle|^{2}$ and is equal to $J(J+1)$ by (132). Therefore,

$$\langle J \| J \| J\rangle = \sqrt{J(J+1)} \tag{137}$$

apart from any phase factor.

If we now consider the case when one of the operators, say, $\vec{B}$ is $\vec{J}$ and $\vec{A}$ is an arbitrary first rank tensor operator $\vec{T}$, the diagonal matrix element $\langle J\, m \,|\, \vec{T}\cdot\vec{J} \,|\, J\, m\rangle$ can be written as

$$\sum_{m'}\sum_{\mu}(-1)^{\mu}\langle J\, m \,|\, T_{\mu} \,|\, J\, m'\rangle\langle J\, m' \,|\, J_{-\mu} \,|\, J\, m\rangle \tag{138}$$

introducing a complete set of states and since J_{μ} connects only states of same J. Using (134) and the Wigner-Eckart theorem (138) is

$$\sum_{m'}\sum_{\mu}(-1)^{\mu} C(J1J;\, m'\mu m)\, \langle J \| T \| J\rangle \times (-1)^{\mu} C(J1J;\, m'\mu m)^{*}\langle J \| J \| J\rangle^{*} \tag{139}$$

which can be written using the unitarity ~~(5c) and (5b) in that order~~ as

$$= \langle J \| T \| J\rangle\langle J \| J \| J\rangle^{*} \tag{140}$$

$$= \langle J \| T \| J\rangle\, J(J+1)/\langle J \| J \| J\rangle \tag{141}$$

It should be noticed that the transition from (139) to (140) and (141) is possible only for matrix elements of ($\vec{J} \cdot \vec{T}$) diagonal in J .

Using Wigner-Eckart theorem

$$\langle Jm | \vec{T} \cdot \vec{J} | Jm \rangle = \langle J \| \vec{T} \cdot \vec{J} \| J \rangle \tag{142}$$

Therefore using (141) for L.H.S., we have

$$\frac{J(J+1) \langle J \| T \| J \rangle}{\langle J \| J \| J \rangle} = \langle J \| \vec{T} \cdot \vec{J} \| J \rangle$$

or $$\langle J \| T \| J \rangle = \frac{\langle J \| J \| J \rangle \langle J \| \vec{T} \cdot \vec{J} \| J \rangle}{J(J+1)}$$

and multiplying both sides by $C(J1J; m \mu m')$ we have

$$\langle J m' | T_\mu | J m \rangle = \frac{\langle J m' | J_\mu | J m \rangle \langle J \| \vec{T} \cdot \vec{J} \| J \rangle}{J(J+1)} \tag{143}$$

which is ~~ix~~ the projection theorem for first rank tensors and holds only for diagonal (in J) elements of any arbitrary first rank tensor operator T^1_μ . In general the T^1_μ may connect states J with states J and $J \pm 1$. It is clear that in the above proof we have made use of only the Wigner-Eckart theorem and the unitarity of the Clebsch-Gordon transformation.

LECTURE VIII

As a simple example of the use of (55) we can evaluate the matrix element $\langle l s J \| \vec{L} \cdot \vec{S} \| l s J \rangle$ encountered in the evaluation of the magnetic moment

$$\langle l s J \| \vec{L} \cdot \vec{S} \| l s J \rangle = - U(l 1 J s; l s) \langle l \| L \| l \rangle \langle s \| S \| s \rangle \tag{144}$$

Or, in general

$$\langle J_1' J_2' J \| \vec{J}_1 \cdot \vec{J}_2 \| J_1 J_2 J \rangle = -U(J_1 1 J J_2 ; J_1 J_2)\, \delta_{J_1 J_1'} \delta_{J_2 J_2'} \times \sqrt{J_1(J_1+1)}\, \sqrt{J_2(J_2+1)} \tag{145}$$

which can also be evaluated (as in lecture 2) writing

$$\vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right)$$

so that

$$\langle J_1 J_2 J \| \vec{J}_1 \cdot \vec{J}_2 \| J_1 J_2 J \rangle = \frac{1}{2}\left[J(J+1) - J_1(J_1+1) - J_2(J_2+1)\right] \tag{146}$$

and we can indirectly evaluated from (145) = (146) the recoupling coefficient

$$U(J_1 1 J J_2 ; J_1 J_2) = \frac{J_1(J_1+1) + J_2(J_2+1) - J(J+1)}{2\sqrt{J_1(J_1+1)\, J_2(J_2+1)}} \tag{147}$$

Or

$$W(J_1 J_1 J_2 J_2 ; 1 J) = (-1)^{J_1+J_2-J} \frac{J(J+1) - J_1(J_1+1) - J_2(J_2+1)}{2\sqrt{J_1(J_1+1)(2J_1+1)\, J_2(J_2+1)(2J_2+1)}} \tag{148}$$

Using (29) and (30d)

Let us consider the scalar product

$$\left(Y_\ell(\theta_1 \phi_1) \cdot Y_\ell(\theta_2 \phi_2)\right) = \sum_{m=-\ell}^{\ell} (-1)^m\, Y_{\ell m}(\theta_1 \phi_1)\, Y_{\ell -m}(\theta_2 \phi_2) \tag{149}$$

which is obivously a spherical tensor of rankzero and thus invariant under rotations. Let us consider a new orientation of the coordinate system such that the new Z axis lies along the direction (Θ_1 , ϕ_1) in the old system be now ($\Theta\ \phi$) so that the invariant quantity (149) now is

$$\sum_m (-1)^m Y_{l,m}(0\,0)\, Y_{l,-m}(\theta\,\phi) = \sum_m (-1)^m \delta_{m0} \sqrt{\frac{2l+1}{4\pi}}\, Y_{l,0}(\theta,\phi)$$

$$= \frac{2l+1}{4\pi} P_l(\cos\theta)$$

or

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_m (-1)^m \, Y_{l,m}(\theta_1 \phi_1)\, Y_{l,-m}(\theta_2 \phi_2) \tag{150}$$

where θ, the angle between the two directions $(\theta_1\ \phi_1)$ and $(\theta_2\ \phi_2)$ is clearly an invariant under rotations. $(\theta_1\ \phi_1)$ $(\theta_2\ \phi_2)$ can clearly be <u>any</u> two directions making an angle θ between them and (150) is referred to as the spherical harmonic addition theorem.

We shall now take up an important application of the formulae (55) as also (47), to evaluate the matrix element, of interaction between two particles; which is referred to as the energy matrix. If the interaction $V(1,2)$ between particles 1 and 2 is central i.e.

$$V(1,2) = V(r) \; ; \; r = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta} \tag{151}$$

θ being the angle between the position coordinates $\vec{r}_1$ and $\vec{r}_2$ of the particles and in particular, if $V(r) = \frac{1}{r}$, we have

$$V(r) = \frac{1}{r_1\left[1 + \left(\frac{r_2}{r_1}\right)^2 - 2\frac{r_2}{r_1}\cos\theta\right]^{1/2}} = \frac{1}{r_1} \sum_{l=0}^{\infty} \left(\frac{r_2}{r_1}\right)^l P_l(\cos\theta) \; ;$$

$$r_1 > r_2 \tag{152}$$

We now assert that any well-behaved function of the form (151) could be expanded in terms of the Legendre polynominals $P_l(\cos\theta)$ so that

$$V(r) = \sum_{l=0}^{\infty} J_l(r_1, r_2)\, P_l(\cos\theta) \tag{153}$$

where $J_l(r_1, r_2)$ are to be determined from a knowledge of $V(r)$. To evaluate matrix elements $\langle J_1' J_2' J \| V(r) \| J_1 J_2 J \rangle$ for central interactions we use (150) to express $P_l(\cos\theta)$ in (153) in terms of the arguments (θ_1, ϕ_1), (θ_2, ϕ_2) of particles 1 and 2 respectively so that

$$V(r) = \sum_{l=0}^{\infty} J_l(r_1, r_2) \left(C^l(\theta_1 \phi_1) \cdot C^l(\theta_2 \phi_2) \right) \tag{154}$$

where

$$C^l_m(\theta, \phi) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\theta\,\phi) \tag{155}$$

and using (55b)

$$\langle J_1' J_2' J \| V(r) \| J_1 J_2 J \rangle = \sum_{l=0}^{\infty} \langle J_l(r_1, r_2) \rangle$$

$$\times (-1)^l \sqrt{\frac{2J_2'+1}{2J_2+1}}\; U(J_1\, l\, J\, J_2' ; J_1' J_2)$$

$$\times \langle J_1' \| C^l(1) \| J_1 \rangle \langle J_2' \| C^l(2) \| J_2 \rangle \tag{156}$$

$\langle J' \| C^l \| J \rangle$ are easily evaluated using (44) if J and J' refer to the orbital states or using (26a) if J, J' refer to the total angular momentum, (i.e orbital + spin) states of the individual particles.

$\langle J_l(r_1, r_2) \rangle$ denotes the matrix element of $J_l(r_1, r_2)$ between the radial wave functions $\phi_1(r_1)\, \phi_2(r_2)$ and $\phi_1'(r_1)\, \phi_2'(r_2)$ of the 2 particle system i.e.

$$\langle J_\ell(r_1, r_2) \rangle = F_\ell$$

$$= \int_0^\infty r_1^2 dr_1 \int_0^\infty r_2^2 dr_2 \, \phi_1'(r_1)^* \phi_2'(r_2)^* J_\ell(r_1, r_2)$$

$$\times \phi_1(r_1) \, \phi_2(r_2) \qquad (157)$$

and is known as the generalised Slater integral, the method being due to Slater who applied it to atomic problems where $V(r)$ is $\propto \frac{1}{r}$.

For the evaluation of the energy matrix in nuclear problems we have not only to consider central interactions (known as Wigner interactions) but also exchange and spin-dependent interactions as well as non-central interactions. We have three types of exchange interactions (1) space exchange (or Majorana interaction V_M) (2) spin exchange (Bartlett interaction V_B) and (3) space and spin exchange (or Heisenberg interaction V_H). If $P_{12}^{(r)}$ $P_{12}^{(s)}$ denote the space and spin exchange operators respectively

$$V_M(1,2) = V_M(r) \, P_{12}^{(r)}$$

$$V_B(1,2) = V_B(r) \, P_{12}^{(s)} \qquad (158)$$

and

$$V_H(1,2) = V_H(r) \, P_{12}^{(r)} \cdot P_{12}^{(s)}$$

where

$$P_{12}^{(r)} \, \psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$$

$$P_{12}^{(s)} \, \chi(1,2) = \chi(2,1)$$

The two particle wave function Ψ (1,2) is a product of a space wave function $\psi(\vec{r}_1, \vec{r}_2)$ spin wave function χ (1,2) and an isotopic spin wave function τ (1,2). By Pauli principle, $\Psi(1,2)$ must be antisymmetric. i.e.

$$P_{12}^{(r)} P_{12}^{(s)} P_{12}^{(\tau)} \Psi(1,2) = + \Psi(2,1) = - \Psi(1,2) \tag{159}$$

where $P_{12}^{(\tau)} \tau(1,2) = \tau(2,1)$

is the isotopin spin exchange operator. Since the operation of any of the exchange operators twice leads to identify we have using (159)

$$P_{12}^{(r)} P_{12}^{(s)} = - P_{12}^{(\tau)} \tag{159a}$$

the operation on Ψ being understood, so that we can write

$$V_H(1,2) = - V_H(r) P_{12}^{(\tau)} \tag{160}$$

If we consider the two particle system in $L-S$ coupling the spin-orbit wave function is of the form

$$|LSJM\rangle = \sum_{M_L} |l_1 l_2 L M_L\rangle \, |\tfrac{1}{2}\tfrac{1}{2} S M_S\rangle \, C(LSJ; M_L M_S M)$$

where l_1 l_2 are the individual orbital angular momenta of the particles and $P_{12}^{(r)}$, $P_{12}^{(s)}$ operate only on $|l_1 l_2 L M_L\rangle$ and $|\tfrac{1}{2}\tfrac{1}{2} S M_S\rangle$ respectively.

The isotopic spin wave function $\tau(1,2)$ is again of the form

$$|\tfrac{1}{2}\tfrac{1}{2} T M_\tau\rangle$$

isotopic spin being in all its operations similar to angular momentum and $P_{12}^{(\tau)}$ operates only on this state.

We shall show later (lecture 11) that the interchange of arguments in a state $|j_1 j_2 J m\rangle$ results in $(-1)^{j_1+j_2-J} |j_1 j_2 J m\rangle$ which is also ~~related to~~ evident from the symmetry property (5a). ~~Thus~~ Also

$$P_{12}^{(r)} |l_1 l_2 L M_L\rangle = (-1)^{l_1+l_2-L} P_{r_1,r_2} |l_1 l_2 L M_L\rangle$$

where $P_{r_1,r_2} \phi_1(r_1)\phi_2(r_2) = \phi_1(r_2)\phi_2(r_1)$

Since the arguments in the radial wavefunctions are also

$$\langle (l_1' l_2') L' (\tfrac{1}{2}\tfrac{1}{2}) S' J M ; (\tfrac{1}{2}\tfrac{1}{2}) T M_\tau | V_M(1,2) + V_B(1,2) + V_H(1,2) | \\ |(l_1 l_2) L (\tfrac{1}{2}\tfrac{1}{2}) S J M ; (\tfrac{1}{2}\tfrac{1}{2}) T M_\tau \rangle$$

$$= \delta_{LL'} \delta_{SS'} \langle l_1' l_2' L \| (-1)^{l_1+l_2-L} V_M(r) [-(-1)^S V_B(r) + (-1)^T V_H(r) \| l_1 l_2 L \rangle \quad (161)$$

Which being the matrix element of a sum of central interactions can be evaluated as described earlier.* We also see that for a symmetric (under exchange 1,2) orbital state the contributions of $V_B(r)$ and $V_H(r)$ add up while for an antisymmetric orbital state $V_B(r)$ and $V_H(r)$ oppose each other. The energy matrix of exchange interactions in $j-j$ coupling could also be obtained from (161) applying the $L-S$, $J-J$ transformation. i.e. by expressing the states

$$|(l_1 \tfrac{1}{2}) J_1 (l_2 \tfrac{1}{2}) J_2 J M \rangle = \sum_{L,S} \begin{bmatrix} l_1 & l_2 & L \\ \tfrac{1}{2} & \tfrac{1}{2} & S \\ J_1 & J_2 & J \end{bmatrix} |(l_1 l_2) L (\tfrac{1}{2}\tfrac{1}{2}) S J M \rangle \quad (162)$$

An explicit representation for the spin (or isotopic spin) exchange operator $P_{12}^{(s)}$ would be given in terms of the Pauli spin matrices $\vec{\sigma}_1$ operating on particle 1 and $\vec{\sigma}_2$, operating on 2

$$P_{12}^{(s)} = \frac{1}{2} \left[1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right] \quad (163)$$

To verify that (163) operating on any state ${}^{1/2}\chi_m (1)\ {}^{1/2}\chi_{m'} (2)$ leads to ${}^{1/2}\chi_m (2)\ {}^{1/2}\chi_{m'} (1)$, we can write

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \left[2 (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) + \sigma_{1z} \sigma_{2z} \right]$$

where

$$\sigma^{\pm} = \frac{\sigma_x \pm i \sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and consequently

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* The radial integrals in the first term alone being the

$$\sigma^{+}\,{}^{1/2}\chi_{1/2} = 0 \quad ; \quad \sigma^{+}\,{}^{1/2}\chi_{-1/2} = {}^{1/2}\chi_{-1/2}$$

$$\sigma^{-}\,{}^{1/2}\chi_{1/2} = {}^{1/2}\chi_{-1/2} \quad ; \quad \sigma^{-}\,{}^{1/2}\chi_{-1/2} = 0$$

so that if we consider

$$\frac{1}{2}\left[1 + 2(\sigma_1^{+}\sigma_2^{-} + \sigma_1^{-}\sigma_2^{+}) + \sigma_{1z}\,\sigma_{2z}\right]{}^{1/2}\chi_{m}(1)\;{}^{1/2}\chi_{m'}(2)$$

we see by direct calculation that for all the possible 4 states with $m = \pm \frac{1}{2}$, $m' = \pm \frac{1}{2}$ the above leads to the desired result.

A central spin-dependent interaction is of the general form.

$$V(1,2) = V_W(r) + V_S(r)\,(\vec{\sigma}_1 \cdot \vec{\sigma}_2) \tag{164}$$

the evaluation of the matrix elements of which can for example be made rewriting (164) as

$$\{V_W(r) - V_S(r)\} + V_S(r)\,P_{12}^{(\delta)}$$

Uisng (163), or more directly noting that,

$$\langle \tfrac{1}{2}\,\tfrac{1}{2}\,S' \,\|(\vec{\sigma}_1 \cdot \vec{\sigma}_2)\|\, \tfrac{1}{2}\,\tfrac{1}{2}\,S\rangle = \left[\frac{1}{4}\,\delta_{S1} - \frac{3}{4}\,\delta_{S0}\right] 4\,\delta_{SS'} \tag{165}$$

The spin dependent term adds up to the Wigner interaction for triplet states and opposes the same for singlet states.

Under charge independence, we have uniquely two types of non-central interactions (1) the vector interaction

$$V(1,2) = V(r)\,\vec{L}_r \cdot \vec{S} \tag{166}$$

where

$$\vec{L}_r = (\vec{r}_1 - \vec{r}_2) \times (\vec{p}_1 - \vec{p}_2) \tag{166a}$$

is the orbital angular momentum operator referring to the relative motion between the two particles and

$$\vec{S} = \vec{s}_1 + \vec{s}_2 \tag{166b}$$

is the total spin operator.

and (2) the tensor interaction

$$V(1,2) = V(r)\, S_{12} \tag{167}$$

where

$$S_{12} = \left[\frac{(\vec{s}_1 \cdot \vec{r})(\vec{s}_2 \cdot \vec{r})}{r^2} - \frac{1}{3}(\vec{s}_1 \cdot \vec{s}_2) \right] \tag{167a}$$

$\vec{r}$, being the relative coordinate

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{167b}$$

To evaluate the matrix element for the vector interaction, we observe that $V(r)\,\vec{L}_r$ is a first rank spherical tensor operator operating on the spatial part of the wave function and $\vec{S}$ obviously is a first rank tensor operator over spin space. Therefore,

$$\langle L'S'JM | V(r)\,\vec{L}_r \cdot \vec{S} | LSJM \rangle$$

$$= -\sqrt{\frac{2S'+1}{2S+1}}\; U(L1JS';L'S)\, \langle L' \| V(r)\vec{L}_r \| L \rangle \langle S' \| \vec{S} \| S \rangle \tag{168}$$

Using (55b). But

$$\langle s' \| s \| s \rangle = \delta_{s,s'} \sqrt{2}\, \delta_{s,1} \tag{169}$$

or the vector interaction is zero in singlet states and we have for $L' = L$ which is usually of interest,

$$\langle L 1 J M | V(r) \vec{L}_r \cdot \vec{s} \| L 1 J M \rangle = -\sqrt{2}\, U(L1J1; L1) \langle L \| V(r) \vec{L}_r \| L \rangle \tag{170}$$

$$= \frac{J(J+1) - L(L+1) - 2}{2\sqrt{L(L+1)}} \langle L \| V(r) \vec{L}_r \| L \rangle \tag{171}$$

using (147).

LECTURE IX.

To evaluate $\langle l_1' l_2' L' \| V(r) \vec{L}_r \| l_1 l_2 L \rangle$ in (168) we observe that

$$\begin{aligned} \vec{L}_r &= \vec{L}_1 + \vec{L}_2 - \vec{r}_1 \times \vec{p}_2 - \vec{r}_2 \times \vec{p}_1 \\ &= \vec{L} - \vec{r}_1 \times \vec{p}_2 - \vec{r}_2 \times \vec{p}_1 \end{aligned} \tag{172}$$

so that

$$\begin{aligned} \langle l_1' l_2' L' \| V(r) \vec{L}_r \| l_1 l_2 L \rangle &= \langle l_1' l_2' L' \| V(r) \vec{L} \| l_1 l_2 L \rangle \\ &\quad - \langle l_1' l_2' L' \| V(r) \{ \vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1 \} \| l_1 l_2 L \rangle \\ &= \langle l_1' l_2' L' \| V(r) \| l_1 l_2 L \rangle \langle l_1 l_2 L \| \vec{L} \| l_1 l_2 L \rangle \\ &\quad - \langle l_1' l_2' L' \| V(r) \{ \vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1 \} \| l_1 l_2 L \rangle \end{aligned} \tag{173}$$

$$= \delta_{LL'} \langle l_1' l_2' L \| V(r) \| l_1 l_2 L \rangle \sqrt{L(L+1)}$$

$$- \langle l_1' l_2' L' \| V(r) \{ \vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1 \} \| l_1 l_2 L \rangle$$

(174)

since $\vec{L}$ connects states of same $l_1\, l_2$ and L only and $V(r)$ being scalar connects states of same L only. The reduced matrix element in the first term can be evaluated using (156) and therefore we have only to evaluate the second term in (174). For this we replace $\vec{p}$ by $-i\,\vec{\nabla}$ and recalling the expression (92) for the vector product of two vectors in spherical basis, we have.

$$\{\vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1\}_\mu = -i\sqrt{2}\, i \sum_{\mu_1} C(111;\mu_1\mu_2\mu)$$

$$\times \{ r_1 C^1_{\mu_1}(1)\, \nabla_{\mu_2}(2) + r_2 C^1_{\mu_1}(2)\, \nabla_{\mu_2}(1) \}$$

(175)

$$= \sqrt{2}\left[r_1 \left(C^1(1) \times \nabla(2) \right)^1_\mu + r_2 \left(C^1(2) \times \nabla(1) \right)^1_\mu \right]$$

(176)

where we have written the components $\sqrt{\frac{4\pi}{3}}\, r\, Y_{1\mu}$ of a vector $\vec{r}$ in spherical basis as $r\, C^1_\mu$ using the definition (155) and using the notation of (45) to express the composition of spherical tensors. Changing* the order of coupling in the second term of (176), thereby acquiring a negative sign and expanding $V(r)$ suing (154) we have,

$$V(r)\{\vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1\}_\mu = \sqrt{2} \sum_{l=0}^{\infty} J_l(r_1, r_2) \left(C^l(1) \cdot C^l(2) \right)$$

$$\times \left[r_1 \left(C^1(1) \times \nabla(2) \right)^1_\mu - r_2 \left(\nabla(1) \times C^1(2) \right)^1_\mu \right]$$

(177)

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* Since $\nabla(1)$ does [illegible]

Expressing ($C^{\ell}(1)$ $C^{\ell}(2)$) in terms of $(C^{\ell}(1) \times C^{\ell}(2))^{0}_{0}$ using the definition (58) and writing the product $T^{0}\,T^{1}_{\mu}$ of zero rank and first rank tensors as $(T^{0} \times T^{1})^{1}_{\mu}$ we can rewrite (177) as

$$V(r)\left\{\vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1\right\}_{\mu} = \sqrt{2}\sum_{\ell=0}^{\infty} J_{\ell}(r_1, r_2)(-1)^{\ell}\sqrt{2\ell+1}$$

$$\times\Big[r_1 \left(\left(C^{\ell}(1)\times C^{\ell}(2)\right)^{0} \times \left(C^{1}(1)\times \nabla(2)\right)^{1}\right)^{1}_{\mu}$$

$$- r_2 \left(\left(C^{\ell}(1)\times C^{\ell}(2)\right)^{0} \times \left(\nabla(1)\times C^{1}(2)\right)^{1}\right)^{1}_{\mu}\Big] \qquad (178)$$

We observe that each of the terms in (178) is a first rank spherical tensor obtained after coupling four spherical tensors, and since the coupling law (45) for spherical tensors is identical to the coupling law (3) for angular momenta, we can use (49) to reexpress (178) as

$$= \sqrt{2}\sum_{\ell=0}^{\infty} J_{\ell}(r_1, r_2)\sum_{s=\ell-1}^{\ell+1}\sum_{s'=\ell-1}^{\ell+1}\begin{bmatrix} \ell & \ell & 0 \\ 1 & 1 & 1 \\ s & s' & 1\end{bmatrix}(-1)^{\ell}\sqrt{2\ell+1}$$

$$\times\Big[r_1 \left(\left(C^{\ell}(1)\times C^{1}(1)\right)^{s} \times \left(C^{\ell}(2)\times \nabla(2)\right)^{s'}\right)^{1}_{\mu}$$

$$- r_2 \left(\left(C^{\ell}(1)\times \nabla^{1}(1)\right)^{s} \times \left(C^{\ell}(2)\times C^{1}(2)\right)^{s'}\right)^{1}_{\mu}\Big]$$

(179)

and now each term of (179) has the form of a spherical tensor of rank 1 composed of two spherical tensors of rank s and s', one operating on particle 1 only and the other operating on particle 2 only so that we can apply (47) to evaluate the matrix element

$$\langle (l_1' l_2') L' \| V(r) \{ \vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1 \} \| (l_1 l_2) L \rangle$$

$$= \sqrt{2} \sum_{l=0}^{\infty} \sum_{s} \sum_{s'} \begin{bmatrix} l & l & 0 \\ 1 & 1 & 1 \\ s & s' & 1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & L \\ s & s' & 1 \\ l_1' & l_2' & L' \end{bmatrix} (-1)^{l} \sqrt{2l+1}$$

$$\times \int_0^\infty r_1^2 dr_1 \int_0^\infty r_2^2 dr_2 \, \phi_1'(r_1)^* \phi_2'(r_2)^* J_l(r_1, r_2)$$

$$\times \Big[r_1 \langle l_1' \| (C^l(1) \times C^1(1))^s \| l_1 \rangle \langle l_2' \| (C^l(2) \times \nabla(2))^{s'} \| l_2 \rangle$$

$$- r_2 \langle l_1' \| (C^l(1) \times \nabla(1))^s \| l_1 \rangle \langle l_2' \| (C^l(2) \times C^1(2))^{s'} \| l_2 \rangle \Big]$$

$$\times \phi_1(r_1) \phi_2(r_2) \qquad (180)$$

remembering that the operation of $\vec{\nabla}$ over the radial wave functions also should be taken into account. The quantity

$$(C^l \times C^1)^\lambda_m = \frac{4\pi}{\sqrt{(2l+1)3}} \sum_{m_1} C(l 1 \lambda; m_1 m_2 m) Y_{l m_1} Y_{1 m_2} \qquad (181)$$

expressing the $C's$ back into the $Y's$ using (155). Expressing the product of the two spherical harmonics using (43) and performing the sum over m_1 using (6)

$$(C^l \times C^1)^\lambda_m = \frac{4\pi}{\sqrt{3(2l+1)}} \sum_{l'} \sum_{m_1} C(l 1 \lambda; m_1 m_2 m) \sqrt{\frac{(2l+1)3}{4\pi(2l'+1)}}$$
$$\times C(l 1 l'; 00) C(l 1 l'; m_1 m_2 m) Y_{l' m} \qquad (182)$$

$$= C(\ell 1 \lambda; 00)\; C^{\lambda}_{m} \quad * \tag{183}$$

[illegible] the reduced matrix element of (183) between states ℓ', ℓ'' is

$$\langle \ell'' \| (C^{\ell} \times C^{1})^{\lambda} \| \ell' \rangle = C(\ell' \lambda \ell''; 00)\, C(\ell 1 \lambda; 00) \sqrt{\frac{2\ell'+1}{2\ell''+1}} \tag{184}$$

[illegible]ing (44a)

We can also write

$$(C^{\ell} \times \nabla)^{\lambda}_{M} = \sum_{m} C(\ell 1 \lambda; m \mu M)\, C^{\ell}_{m} \nabla_{\mu}$$

[illegible]d using (102)

$$(C^{\ell} \times \nabla)^{\lambda}_{M}\, Y_{\ell_1 m_1} \phi_1(r) = \sum_{m} C(\ell 1 \lambda; m \mu M)\, C^{\ell}_{m}$$

$$\times \Big[\Big(\frac{\ell_1+1}{2\ell_1+3}\Big)^{1/2} C(\ell_1 1 \ell_1+1; m_1 \mu)\, Y_{\ell_1+1, m_1+\mu} \Big(\frac{d}{dr} - \frac{\ell_1}{r}\Big) \phi_1(r)$$

$$- \Big(\frac{\ell_1}{2\ell_1-1}\Big)^{1/2} C(\ell_1 1 \ell_1-1; m_1 \mu)\, Y_{\ell_1-1, m_1+\mu} \Big(\frac{d}{dr} + \frac{\ell_1+1}{r}\Big) \phi_1(r) \Big]$$

6

(185)

[illegible]nd expressing C^{ℓ}_{m} as Y^{ℓ}_{m} using (155) and coupling the two spherical [illegible]armonics using (43) we have the above as

$$= \sum_{m} C(\ell 1 \lambda; m \mu M)$$

$$\times \Big[(\ell_1+1)^{1/2}\, C(\ell_1 1 \ell_1+1; m_1 \mu) \sum_{\ell'} (2\ell'+1)^{-1/2}$$

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or $(Y_{\ell} \times Y_{\ell_1}$ [illegible] $C($ [illegible]

$$\times\, C(\ell,\ell_1+1,\ell';00)\cdot C(\ell,\ell_1+1,\ell';m,m_1+\mu,M+m_1)\,Y_{\ell',M+m_1}$$

$$\times\left(\frac{d}{dr}-\frac{\ell_1}{r}\right)\phi_1(r)$$

$$-(\ell_1)^{1/2}C(\ell_1\,1\,\ell_1-1;m_1\,\mu)\sum_{\ell'}(2\ell'+1)^{-1/2}C(\ell,\ell_1-1,\ell';00)$$

$$\times\, C(\ell,\ell_1-1,\ell';m,m_1+\mu,M+m_1)\,Y_{\ell',M+m_1}$$

$$\left.\times\left(\frac{d}{dr}+\frac{\ell_1+1}{r}\right)\phi_1(r)\right] \qquad (186)$$

156) could again be written using (5a) as

$$=\left[\sum_{\ell'}\sqrt{\frac{\ell_1+1}{2\ell'+1}}\,C(\ell,\ell_1+1,\ell';00)(-1)^{-(\ell+1-\lambda)}(-1)^{\ell_1+1+\ell-\ell'}\right.$$

$$\times\sum_{m}C(\ell_1\,1\,\ell_1+1;m_1\,\mu)\,C(\ell_1+1,\ell,\ell';m_1+\mu,m,M+m_1)$$

$$\times\, C(1\,\ell\,\lambda\,;^{\mu}_{\lambda}\,m\,\cancel{\mu}\,M)\left(\frac{d}{dr}-\frac{\ell_1}{r}\right)$$

$$-\sum_{\ell'}\sqrt{\frac{\ell_1}{2\ell'+1}}\,C(\ell,\ell_1-1,\ell';00)(-1)^{-(\ell+1-\lambda)}(-1)^{\ell_1-1+\ell-\ell'}$$

$$\times\sum_{m}C(\ell_1\,1\,\ell_1-1;m_1\,\mu)\,C(\ell_1-1,\ell,\ell';m_1+\mu,m,M+m_1)$$

$$\left.\times\, C(1\,\ell\,\lambda;\mu\,m\,M)\left(\frac{d}{dr}+\frac{\ell_1+1}{r}\right)\right]$$

$$\times\,\phi_1(r)\,Y_{\ell',M+m_1} \qquad (187)$$

nd performing now the summation over m using (98b) we have

$$(c^{\ell} \times \nabla)^{\lambda}_{M} \, \phi_1(r) \, Y_{\ell_1 m_1}(\theta\phi) = \sum_{\ell'} C(\ell_1 \lambda \ell'; m_1 M \, m_1+M)$$

$$\times (-1)^{\ell_1+\lambda-\ell'} \Big[U(\ell_1 1 \ell' \ell; \ell_1+1, \lambda) \sqrt{\frac{\ell_1+1}{2\ell'+1}} \, C(\ell, \ell_1+1, \ell'; 00) \times \left(\frac{d}{dr} - \frac{\ell_1}{r}\right)$$

$$- U(\ell_1 1 \ell' \ell; \ell_1-1, \lambda) \sqrt{\frac{\ell_1}{2\ell'+1}} \, C(\ell, \ell_1-1, \ell'; 00) \times \left(\frac{d}{dr} + \frac{\ell_1+1}{r}\right) \Big] \phi_1(r) \, Y_{\ell'; M+m_1}(\theta\phi)$$

(188)

so that

$$\langle \ell_1' \| (c^{\ell} \times \nabla)^{\lambda} \| \ell_1 \rangle = (-1)^{\ell_1+\lambda-\ell_1'} (2\ell_1'+1)^{-1/2}$$

$$\times \Big[U(\ell_1 1 \ell_1' \ell; \ell_1+1, \lambda) \, C(\ell, \ell_1+1, \ell_1'; 00) \sqrt{\ell_1+1} \left(\frac{d}{dr} - \frac{\ell_1}{r}\right)$$

$$- U(\ell_1 1 \ell_1' \ell; \ell_1-1, \lambda) \, C(\ell, \ell_1-1, \ell_1'; 00) \sqrt{\ell_1} \left(\frac{d}{dr} + \frac{\ell_1+1}{r}\right) \Big]$$

(189)

Using (184) and (189) in (180) and writing

$$\begin{bmatrix} \ell & \ell & 0 \\ 1 & 1 & 1 \\ s & s' & 1 \end{bmatrix} = (-1)^{s-s'} \begin{bmatrix} 1 & 1 & 1 \\ \ell & \ell & 0 \\ s & s' & 1 \end{bmatrix}$$

(190)

$$= (-1)^{\cancel{\ell}+\cancel{s}-s'} \sqrt{\frac{2s'+1}{3(2\lambda+1)}} \; U(1 \ell 1 s'; s 1)$$

(191)

sing (54) and (30f), we have

$$\langle l_1' l_2' L' \| V(r) \{ \vec{r}_1 \times \vec{p}_2 + \vec{r}_2 \times \vec{p}_1 \} \| l_1 l_2 L \rangle$$

$$= \sqrt{\frac{2}{3}} \sum_{l=0}^{\infty} (-1)^l \sum_{s=l-1}^{l+1} \sum_{s'=l-1}^{l+1} (-1)^{s-s'} \begin{bmatrix} l_1 & l_2 & L \\ s & s' & 1 \\ l_1' & l_2' & L' \end{bmatrix} \sqrt{2s'+1}$$

$$\times U(1 l 1 s'; s 1) \Big[C(l 1 s; 00)\, C(l_1 s l_1'; 00) \sqrt{\frac{2l_1+1}{(2l_1'+1)(2l_2'+1)}}$$

$$\times (-1)^{l_2+s'-l_2'} \Big\{ \sqrt{l_2+1}\, U(l_2 1 l_2' l; l_2+1, s')\, C(l, l_2+1, l_2'; 00)$$

$$\times \Big\langle J_l(r_1, r_2)\, r_1 \Big(\frac{d}{dr_2} - \frac{l_2}{r_2} \Big) \Big\rangle$$

$$- \sqrt{l_2}\, U(l_2 1 l_2' l; l_2-1, s')\, C(l, l_2-1, l_2'; 00)$$

$$\times \Big\langle J_l(r_1, r_2)\, r_1 \Big(\frac{d}{dr_2} + \frac{l_2+1}{r_2} \Big) \Big\rangle \Big\}$$

$$- C(l 1 s'; 00)\, C(l_2 s' l_2'; 00) \sqrt{\frac{2l_2+1}{(2l_2'+1)(2l_1'+1)}}$$

$$\times (-1)^{l_1+s-l_1'} \Big\{ \sqrt{l_1+1}\, U(l_1 1 l_1' l; l_1+1, s)\, C(l, l_1+1, l_1'; 00)$$

$$\times \Big\langle J_l(r_1, r_2)\, r_2 \Big(\frac{d}{dr_1} - \frac{l_1}{r_1} \Big) \Big\rangle$$

$$- \sqrt{l_1}\, U(l_1 1 l_1' l; l_1-1, s)\, C(l, l_1-1, l_1'; 00)$$

$$\times \Big\langle J_l(r_1, r_2)\, r_2 \Big(\frac{d}{dr_1} + \frac{l_1+1}{r_1} \Big) \Big\rangle \Big\} \Big] \qquad (192)$$

where the quantities $\langle F(r_1, r_2) \rangle$ are defined similar to (157)

The first term in (174) on using (156) is explicitly,

$$\delta_{LL'} \sqrt{L(L+1)} \sum_{l=0}^{\infty} (-1)^l \sqrt{\frac{2l_2'+1}{2l_2+1}}\, U(l_1 l L l_2'; l_1' l_2)$$

$$\times \sqrt{\frac{(2l_1+1)(2l+1)}{4\pi (2l_1'+1)}} \sqrt{\frac{(2l_2+1)(2l+1)}{4\pi (2l_2'+1)}} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l+1}}\, C(l_1 l l_1'; 00)$$

$$\times C(l_2 l l_2'; 00) \langle J_l \rangle$$

$$= \delta_{LL'} \sqrt{L(L+1)} \sum_{l=0}^{\infty} (-1)^l \sqrt{\frac{2l_1+1}{2l_1'+1}}\, U(l_1 l L l_2'; l_1' l_2)$$

$$\times C(l_1 l l_1'; 00)\, C(l_2 l l_2'; 00)\, F_l \qquad (193)$$

so that the desired matrix element,

$$\langle l_1' l_2' L' \| V(r) \vec{L}_r \| l_1 l_2 L \rangle = \qquad (193) - (192) \ldots (194)$$

The case $l_1' = l_1$; $l_2' = l_2$ and $L' = L$ is usually of interest in spectroscopy when only even l terms contribute to (193) In (192) the $s = s' = l$ term contributes zero in general, x since $C(l 1 l; 00) = 0$.

LECTURE X.

The tensor interaction between two nucleons is of the form

$$V_T = S_{12} V(r) = V(r) \left\{ \frac{(\vec{s}_1 \cdot \vec{r})(\vec{s}_2 \cdot \vec{r})}{r^2} - \frac{1}{3}(\vec{s}_1 \cdot \vec{s}_2) \right\}$$

where S_{12} is clearly the scalar product

$$S_{12} = \sum_{\alpha, \beta = x, y, z} T_{\alpha\beta} S_{\alpha\beta} \qquad (195)$$

of two cartesian tensors of rank 2

$$T_{\alpha\beta} = r_\alpha r_\beta - \frac{1}{3}\delta_{\alpha\beta} \tag{196}$$

and

$$S_{\alpha\beta} = s_{1\alpha} s_{2\beta} \tag{197}$$

constructed respectively out of the cartesian components of the relative position vector $\vec{r}$ and the spin operators $\vec{s}_1$, $\vec{s}_2$ of the two particles.

Allowing ourselves a little diggression it may be pointed out that a correspondence could be established between the components of the 2nd rank tensor $A_\alpha B_\beta$ and the components of the rank 2 spherical tensor $(A^1 \times B^1)^2_\mu$ as we have previosuly done in the case of zero and first rank tensors Viz., the scalar and vector products respectively ((52) and (92))

It is easily seen that

$$(A^1 \times B^1)^2_{\pm 2} = \frac{(A_x B_x - A_y B_y) \pm i (A_x B_y + A_y B_x)}{2}$$

$$(A^1 \times B^1)^2_{\pm 1} = \frac{1}{\sqrt{2}}\left[(A_z B_x \mp A_x B_z) \pm i (A_z B_y \mp A_y B_z)\right]$$

$$\text{and } (A^1 \times B^1)^2_0 = \sqrt{\tfrac{1}{6}}\,(A_x B_x + A_y B_y) + \sqrt{\tfrac{2}{3}}\, A_z B_z \tag{198}$$

Returning to our problem, we shall first express S_{12} as a scalar product of spherical tensor operators, writing

$$(\vec{s}.\vec{r}) = (-1)\sqrt{3}\,(s^1 \times r^1)^0_0$$

using (52) and

$$(\vec{s}_1.\vec{r})(\vec{s}_2.\vec{r}) = 3\left((s_1^1 \times r^1)^0 \times (s_2^1 \times r^1)^0\right)^0_0$$

$$= 3 \sum_{k=0}^{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ k & k & 0 \end{bmatrix} \left((s_1^1 \times s_2^1)^k \times (r^1 \times r^1)^k\right)^0_0 \tag{199}$$

s in (179) and using (54) for the 9-J coefficient

$$(\vec{s}_1\cdot\vec{r})(\vec{s}_2\cdot\vec{r}) = \sum_{k=0}^{2}\sqrt{2k+1}\,U(110k;k1)\left((s_1'\times s_2')^k\times(r'\times r')^k\right)_0 \quad (201)$$

$$= \sum_k \sqrt{2k+1}\,C(11k;00)\,r^2\left((s_1'\times s_2')^k\times C^k(\vec{r})\right)_0 \quad (202)$$

since $U(110k;k1) = U(11k0;k1) = 1$ in virtue of (29) and (30a) and using (183) for $(r'\times r')^k$. Clearly the summation over k is now 0 and 2 only.

Thus,

$$(\vec{s}_1\cdot\vec{r})(\vec{s}_2\cdot\vec{r}) = r^2\left[\frac{1}{3}(\vec{s}_1\cdot\vec{s}_2) + \sqrt{5}\,C(112;00)\left(S^{(2)}\times C^2(\vec{r})\right)_0\right] \quad (203)$$

or $$S_{12} = \sqrt{5}\,C(112;00)\left(S^{(2)}\times C^2(\vec{r})\right)_0^0 \quad (204)$$

where $$S^{(2)}_\mu = (s_1'\times s_2')^2_\mu \quad (205)$$

is clearly a spherical tensor operator of rank 2 over spin space. We can also write

$$C(112;00)\,C^2_\mu(\vec{r}) = \frac{1}{r^2}(r'\times r')^2_\mu$$

$$= \frac{1}{r^2}\left[(r_1'\times r_1')^2_\mu + (r_2'\times r_2')^2_\mu - (r_1'\times r_2')^2_\mu - (r_2'\times r_1')^2_\mu\right]$$

$$= \frac{1}{r^2}\left[C(112;00)\left\{r_1^2\,C^2_\mu(1) + r_2^2\,C^2_\mu(2)\right\} - 2r_1r_2\left(C^1(1)\times C^1(2)\right)^2_\mu\right] \quad (206)$$

so that we can express

$$V_T = \left(\mathcal{L}^{(2)} \cdot S^{(2)}\right) \tag{207}$$

where

$$\mathcal{L}^{(2)}_{\mu} = \frac{V(r)}{r^2}\Big[C(112;00)\left\{r_1^2 C^2_{\mu}(1) + r_2^2 C^2_{\mu}(2)\right\} - 2 r_1 r_2 \left(C^1(1) \times C^1(2)\right)^2_{\mu}\Big] \tag{208}$$

is clearly a spherical tensor of rank 2 over orbital space. A matrix element of V_T in $L-S$ coupling can now be written using (55b) as

$$\langle (l_1' l_2') L' S' J \| V_T \| (l_1 l_2) L S J \rangle = \delta_{S,S'} \delta_{S,1} U(L 2 J 1; L' 1) \langle L' \| \mathcal{L}^{(2)} \| L \rangle \times \langle 1 \| S^{(2)} \| 1 \rangle \tag{209}$$

since S^2 being a spherical tensor of rank 2 can connect only $S = 1$, $S' = 1$. Thus the tensor interaction is effective only in triplet states and also it does not contribute to singlet triplet transitions, where as the vector force would allow such a transition, for example, in collisions. The quantity

$$\langle \tfrac{1}{2}\, \tfrac{1}{2} 1 \| S^{(2)} \| \tfrac{1}{2}\, \tfrac{1}{2}\, 1 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \langle \tfrac{1}{2} \| s_1 \| \tfrac{1}{2} \rangle \langle \tfrac{1}{2} \| s_2 \| \tfrac{1}{2} \rangle \tag{210}$$

using (47) and is

$$= \frac{3}{4} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \tag{211}^*$$

* If instead of s_1, s_2 we have σ_1, σ_2 the value will be four times as much.

To calculate the reduced matrix element $\langle l_1' l_2' L' \| \mathcal{L}^{(2)} \| l_1 l_2 L \rangle$ ... can expand $\dfrac{V(r)}{r^2}$ in terms of $P_l(\cos\theta)$ as in (153), so that

$$\mathcal{L}^{(2)}_{\mu} = \sum_{k=0}^{\infty} J_k(r_1, r_2) \left(C^k(1) \cdot C^k(2) \right)$$
$$\times \Big[\left\{ r_1^2\, C^2_{\mu}(1) + r_2^2\, C^2_{\mu}(2) \right\} C(112;00)$$
$$- 2 r_1 r_2 \left(C^1(1) \times C^1(2) \right)^2_{\mu} \Big] \qquad (212)$$

...hich can be rewritten as

$$= \sum_{k=0}^{\infty} (-1)^k \sqrt{2k+1} \Big[-2 r_1 r_2 J_k(r_1, r_2) \sum_{s=k-1}^{k+1} \sum_{s'=k-1}^{k+1}$$
$$\times \begin{bmatrix} k & k & 0 \\ 1 & 1 & 2 \\ s & s' & 2 \end{bmatrix} \left(\left(C^k(1) \times C^1(1) \right)^s \times \left(C^k(2) \times C^1(2) \right)^{s'} \right.$$
$$+ C(112;00) \sum_{s=k-2}^{k+2} \begin{bmatrix} k & k & 0 \\ 2 & 0 & 2 \\ s & k & 2 \end{bmatrix} \Big\{ r_1^2\, J_k(r_1, r_2)$$
$$\left(\left(C^k(1) \times C^2(1) \right)^s \times C^k(2) \right)^2_{\mu} + r_2^2 J_k(r_1, r_2) \left(\left(C^k(2) \times C^2(2) \right)^s \times C^k(1) \right)^2_{\mu} \Big\} \Big]$$

$$(213)$$

We can write

$$\begin{bmatrix} k & k & 0 \\ 1 & 1 & 2 \\ s & s' & 2 \end{bmatrix} = (-1)^{s-s'} \begin{bmatrix} 1 & 1 & 2 \\ k & k & 0 \\ s & s' & 2 \end{bmatrix} = (-1)^{s-s'} \sqrt{\frac{2s'+1}{3(2k+1)}}\, U(1k2s; s1)$$

sing the symmetry and (54) of the $9-j$ coefficients. Similarly

$$\begin{bmatrix} k & k & 0 \\ 2 & 0 & 2 \\ s & k & 2 \end{bmatrix} = (-1)^{s-k} U(2k2k;s0)$$
$$= (-1)^{s-k-(s-k)}\sqrt{2s+1}\, W(sk20;2k) = \sqrt{\frac{2s+1}{5(2k+1)}}$$

using (54) and (30). Thus,

$$\mathcal{L}^{(2)}_{\mu} = \sum_{k=0}^{\infty}\Big[C(112;00) \sum_{s=k-2}^{k+2} (-1)^{k} \sqrt{\frac{2s+1}{5}}\, C(k2s;00)$$
$$\times \Big\{ r_1^2 J_k(r_1,r_2) \left(C^{s}(1)\times C^{k}(2)\right)^2_{\mu}$$
$$+ (-1)^{-(k+s)} r_2^2 J_k(r_1,r_2) \left(C^{k}(1)\times C^{s}(2)\right)^2_{\mu} \Big\}$$
$$- 2 r_1 r_2 J_k(r_1,r_2) \sum_{s}\sum_{s'} (-1)^{s+k-s'} \sqrt{\frac{2s'+1}{3}}$$
$$\times U(1k2s';s1)\, C(k1s;00)\, C(k1s';00)$$
$$\times \left(C^{s}(1)\times C^{s'}(2)\right)^2_{\mu} \Big] \tag{214}$$

using (185) and changing the order of coupling in the last term. From (214) the matrix element

$$\langle l_1' l_2' L' \| \mathcal{L}^{(2)} \| l_1 l_2 L \rangle$$
$$=$$

$$= \sum_{k=0}^{\infty} \Big[C(112;00) \sum_{s=k-2}^{k+2} \sqrt{\frac{2s+1}{5}}\, C(k2s;00)$$

$$\times \Big\{ \langle r_1^2 J_k(r_1,r_2)\rangle \begin{bmatrix} l_1 & l_2 & L \\ s & k & 2 \\ l_1' & l_2' & L' \end{bmatrix} (-1)^{s} C(l_1' s l_1;00) \times C(l_2' k l_2;00)$$

$$+ \langle r_2^2 J_k(r_1,r_2)\rangle \begin{bmatrix} l_1 & l_2 & L \\ k & s & 2 \\ l_1' & l_2' & L' \end{bmatrix} (-1)^{k} C(l_1' k l_1;00) \times C(l_2' s l_2;00) \Big\}$$

$$- 2\langle r_1 r_2 J_k(r_1,r_2)\rangle \sum_{s,s'=k-1}^{k+1} (-1)^k \sqrt{\frac{2s'+1}{3}}$$

$$\times U(1k2s';s1)\, C(k1s;00)\, C(k1s';00)(-1)^{s+s'}$$

$$\times \begin{bmatrix} l_1 & l_2 & L \\ s & s' & 2 \\ l_1' & l_2' & L' \end{bmatrix} C(l_1' s l_1;00)\, C(l_2' s' l_2;00) \Big] \quad (215)$$

using (47) and since

$$\langle l' \| C^k \| l \rangle = \sqrt{\frac{4\pi}{2k+1}}\, \langle l' \| Y_k \| l \rangle = \sqrt{\frac{2l+1}{2l'+1}}\, C(l k l';00) \quad (216a)$$

$$= (-1)^k C(l' k l;00) \quad (216b)$$

using (44a) and (5c).

The summation over in (215) is restricted clearly to a few terms by the various Clebsch-Gordon coefficients.

LECTURE XI.

We shall now briefly take up the basic question of coupling of two angular momenta. For completeness we recall that the commutation relationship (1) would be equivalently be written in the form

$$[J_z , J_\pm] = \pm J_\pm \quad ; \quad [J_+, J_-] = 2 J_z$$

and

$$[J^2, J_{\pm, z}] = 0 \tag{217}$$

where $J_\pm$ are the non-hermitian operators

$$J_\pm = J_x \pm i J_y \tag{218*}$$

Setting up matrix representations of, (217) in a basis in which J^2, J_z are diagonal $[J_z, J_\pm] = J_\pm$ shows that $J_\pm$ increase or decrease respectively the eigen value of J_z by unity; and $[J_+, J_-] = 2J_z$ gives a difference equation from which we deduce that

$$|\langle j\, m\pm 1 | J_\pm | j m\rangle|^2 = (j \mp m)(j \pm m + 1) \tag{219}$$

where j is the maximum value amongst m, the eigen values of J_z. (219) also enables identification of $j(j+1)$ as the eigen value of $J^2 = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+)$. It is clear that (219) does not give the phase of the matrix elements of $J_\pm$ and since this cannot be ascertained from any other considerations either, it is conventionally choosen to be real and positive (convention 1).

For a composite system consisting of two components we define the angular momentum $\vec{J}$ by (2) and from the fundamental principles of quantum mechanics we have the expansions

$$|j_1 j_2 j m\rangle = \sum_{m_1} \sum_{m_2} C(j_1 j_2 j; m_1 m_2 m) |j_1 m_1\rangle |j_2 m_2\rangle \tag{3}$$

* to be distinguished from

$$J_\pm = J_x \pm i J_y$$

and

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_j \sum_m C^*(j_1 j_2 j;\, m_1 m_2 m) |j_1 j_2 j m\rangle \tag{4}$$

The property (2) of the coefficients is easily obtained by operating both sides of the expansion by $J_z = J_{1z} + J_{2z}$. The maximum value of j cannot be $>(j_1 + j_2)$ since otherwise we should also have $m > j_1 + j_2$; but m can at the most be $j_1 + j_2$ corresponding to $m_1 = j_1$, $m_2 = j_2$. With $m = j_1 + j_2$ we have only one state $|j_1 j_1\rangle |j_2 j_2\rangle$ in the product space and we verify that this is also an eigenstates of J^2 with $j = j_1 + j_2$.

Since the states $|j_1 m_1\rangle |j_2 m_2\rangle$ are normalised and we want the states $|j_1 j_2 j m\rangle$ also to be normalised the coefficient $C(j_1 j_2\, j_1+j_2;\, j_1\, j_2\, j_1+j_2)$ must have modulus unity and again by convention the phase is chosen to be zero, since this phase could not be determined from any other consideration either (convention II). Thus,

$$|j_1 j_2\, j_1+j_2\, j_1+j_2\rangle = |j_1 j_1\rangle |j_2 j_2\rangle \tag{220}$$

With $m = j_1 + j_2 - 1$ we can have two states $|j_1\, j_1-1\rangle |j_2 j_2\rangle$ and $|j_1 j_1\rangle |j_2\, j_2-1\rangle$ from which we can have two independent linear combinations one corresponding to $j = j_1 + j_2$ and another with $j = j_1 + j_2 - 1$ and so on. Since the total number of states in the product space is $(2j_1+1)(2j_2+1)$ and for each j we must have $(2j+1)$ states

$$(2j_1+1)(2j_2+1) = \sum_{j=j_{min}}^{j_1+j_2} (2j+1) \tag{221}$$

rom which one can argue that J_{min} has to be $|J_1 - J_2|$. .
ence the property 3.

Operating on equation (3) on both sides by $J_{\pm} = J_{1\pm} + J_{2\pm}$
e have

$$\sqrt{(J \mp m)(J \pm m+1)}\,|J\; m\pm 1\rangle = \sum_{m_1} C(J_1 J_2 J; m_1\, m_2\, m)$$

$$\times \Big\{ \sqrt{(J_1 \mp m_1)(J_1 \pm m_1+1)}\,|J_1\, m_1 \pm 1\rangle\,|J_2\, m_2\rangle$$

$$+ \sqrt{(J_2 \mp m_2)(J_2 \pm m_2+1)}\,|J_2\, m_2 \pm 1\rangle\,|J_1\, m_1\rangle \Big\} \quad (222)$$

We can now express $|J\; m\pm 1\rangle$ on L.H.S. Using (4) and equating coefficients of $|J_1\, m_1\rangle\,|J_2\, m_2\rangle$ we obtain

$$\sqrt{(J-m)(J+m+1)}\; C^*(J_1 J_2 J; m_1\, m_2\, m+1)$$

$$= \sqrt{(J_1+m_1)(J_1-m_1+1)}\; C(J_1 J_2 J; m_1-1, m_2, m)$$

$$+ \sqrt{(J_2+m_2)(J_2-m_2+1)}\; C(J_1 J_2 J; m_1, m_2-1, m) \quad (223)$$

and

$$\sqrt{(J+m)(J-m+1)}\; C^*(J_1 J_2 J; m_1\, m_2\, m-1)$$

$$= \sqrt{(J_1-m_1)(J_1+m_1+1)}\; C(J_1 J_2 J; m_1+1, m_2, m)$$

$$+ \sqrt{(J_2-m_2)(J_2+m_2+1)}\; C(J_1 J_2 J; m_1, m_2+1, m) \quad (224)$$

Which together show that if the coefficients with a particular value of m are real (for given $j_1\, j_2\, j$) then all the coefficients have to be real. For $j = j_1 + j_2$ at least we therefore assert that all the coefficients must be real on account of (220).

Operating (3) by J_{1z}, for example, we have

$$\langle j_1 m_1\, j_2 m_2 \,|\, J_{1z} \,|\, j_1 j_2 j m\rangle = m_1\, C((j_1 j_2 j; m_1 m_2 m)$$

,

$$\sum_{j'} C^*(j_1 j_2 j'; m_1 m_2 m)\,\langle j_1 j_2 j' m \,|\, J_{1z} \,|\, j_1 j_2 j m\rangle$$

$$= m_1\, C(j_1 j_2 j; m_1 m_2 m) \qquad (225)$$

We have at our disposal the $\binom{12}{2} = 66$ ~~(66)~~ commutation relationships between the quantities $J_1^2, J_2^2, J^2, J_{1z}, J_{2z}, J_z$, $J_{1\pm}, J_{2\pm}, J_\pm$ taking matrix representations of which yield however only the information that

(i) all these 12 operators connect states of same $j_1\, j_2$ only

(ii) J_{1z}, J_{2z} connect states $|j_1 j_2 j m\rangle$ with same m only while $J_{1\pm}, J_{2\pm}$ connect states $|j_1 j_2 j m\rangle$ respectively to $|j_1 j_2 j' m \pm 1\rangle$ only.

(iii) the matrix elements $\langle j_1 j_2 j m \,|\, J_{1z} \,|\, j_1 j_2 j m\rangle$ (i.e. between same j) are real, which can also be seen on writing

$$J_1^2 = J_{1z}(J_{1z} - 1) + J_{1+} J_{1-} \qquad (226)$$

and taking matrix elements

$$j_1(j_1+1) = \sum_{j'} |\langle j_1 j_2 j m \,|\, J_{1z} \,|\, j_1 j_2 j' m\rangle|^2$$

$$- \langle j_1 j_2 j m \,|\, J_{1z} \,|\, j_1 j_2 j m\rangle + |\langle j_1 j_2 j m \,|\, J_{1+} \,|\, j_1 j_2 j' m-1\rangle|^2$$

---

* ... †

and so on, and does not allow determiniation of $\langle j_1 j_2 j' m | J_{1z} | j_1 j_2 j m \rangle$ which is of interest.

We can, however, elicit some further information from the recursion relations (223) or (224). Putting $j_1 = j_2$; $j=0$; $m_1 = -m_2+1$ in (223) we have

$$C(j_1, j_1, 0 \;;\; m_1 - 1, -m_1 + 1, 0) = -C(j_1, j_1, 0 \;;\; m_1, -m_1, 0) \tag{228}$$

or

$$C(j\,j\,0\,;\, m, -m) = (-1)^{j-m}\, C(j\,j\,0\,;\, j, -j) \tag{229}$$

The cenitarity of the Clebsch-Gordon transformation (3) imposes

$$\sum_{m=-j}^{j} |C(j\,j\,0\,;\, m, -m)|^2 = 1$$

i.e

$$(2j+1)\,|C(j\,j\,0\,;\, j, -j)|^2 = 1$$

using (229). Or, we have

$$C(j\,j\,0\,;\, j, -j\, 0) = \frac{1}{\sqrt{2j+1}}\, e^{i\delta(j)} \tag{230}$$

Therefore, from (229)

$$C(j\,j\,0\,;\, m, -m) = \frac{(-1)^{j-m}}{\sqrt{2j+1}}\, e^{i\delta(j)} \tag{231}$$

This together with the property (61) of the rotation matrices enables us to prove the result (65). (61) is a basic property which can be obtained (refer next lecture) independent of any considerations of the Clebsch-Gordon coefficients. We can therefore write

$$\langle J_1 J_2 J' m | J_{1z} | J_1 J_2 J m\rangle = C(1 J J'; 0 m m)\langle J' \| J_1 \| J\rangle' \tag{232a}$$

$$\langle J_1 J_2 J' m \pm | J_{1\pm} | J_1 J_2 J m\rangle = C(1 J J'; \pm 1, m, m \pm 1)\langle J' \| J_1 \| J\rangle' \tag{232b,c,}$$

and clearly $\vec{J}_1$ connect states J to $J+1$, J and $J-1$.

To calculate the coefficients occuring in (232) we observe that for $J' = J+1$

$$C(1 J\ J+1; 1\ J\ J+1) = 1$$

from (220) and we can calculate all other coefficients with $J' = J+1$ using the recursion relations.

Specialising (223) we obtain (for $J > 0$)

$$0 = \sqrt{2}\ C(1 J J; 0 J J) + \sqrt{2J}\ C(1 J J; 1 J-1 J) \tag{233}$$

which together with the cenitarity

$$|C(1 J J; 0 J J)|^2 + |C(1 J J; 1, J-1, J)|^2 = 1 \tag{234}$$

determines

$$C(1 J J; 1, J-1, J) = \frac{1}{\sqrt{J+1}}\ e^{i\delta'(j)} \tag{235}$$

$$C(1 J J; 0 J J) = -\sqrt{\frac{J}{J+1}}\ e^{i\delta'(j)} \tag{236}$$

apart from an unspecified phase factor $e^{i\delta'(J)}$ common to all the coefficients with $J' = J$; all the coefficients with $J' = J$

being determined in terms of (235) and (236) through the recursion relations.

In a similar way we can again specialise (223) to give for $J' = J-1$; (J

$$0 = \sqrt{2}\, C(1, J, J-1; -1, J, J-1) + \sqrt{2J}\, C(1, J, J-1; 0, J-1, J-1) \tag{237}$$

$$0 = \sqrt{2}\, C(1, J, J-1; 0, J-1, J-1) + \sqrt{2(2J-1)}\, C(1, J, J-1; 1, J-2, J-1) \tag{238}$$

and unitarity requires

$$|C(1, J, J-1; -1, J, J-1)|^2 + |C(1, J, J-1; 0, J-1, J-1)|^2 + |C(1, J, J-1; 1, J-2, J-1)|^2 = 1 \tag{239}$$

From (237), (238) and (239) we have

$$C(1, J, J-1; -1, J, J-1) = -\sqrt{\frac{2J-1}{2J+1}}\, e^{i\delta(J-1)} \tag{240}$$

$$C(1, J, J-1; 0, J-1, J-1) = \sqrt{\frac{2J-1}{J(2J+1)}}\, e^{i\delta'(J-1)} \tag{241}$$

$$C(1, J, J-1; 1, J-2, J-1) = -\sqrt{\frac{1}{J(2J+1)}}\, e^{i\delta'(J-1)} \tag{242}$$

and all other coefficients with $J' = J-1$ can now be determined using the recursion relations.

In fact the procedure out-lined above is quite general and one can obtain all the $C(J_1 J_2 J; m_1 m_2 m)$'s in the above fashion except for the ambiguity in phase.

If, for example, $\langle j_1, j_2, j\pm 1, m \,|\, J_{1z} \,|\, j_1 j_2 j\, m\rangle$ are real then it is clear from (225) and (220) that all the Clebsch-Gordon coefficients have to be real. We have already seen that the commutation relations do not determine $\langle j_1 j_2 j' m \,|\, J_{1z} \,|\, j_1 j_2 j\, m\rangle$. Making use of (232) we can only determine $|\langle j\pm 1 \| J_1 \| j\rangle'|^2$ from say,

$$j_1(j_1+1) = \sum_{j'} \Big[|C(1j'j;\, 0\, m\, m)|^2 + \tfrac{1}{2}\times 2\, |C(1j'j;\, -1, m+1)|^2 + \tfrac{1}{2}\times 2\, |C(1j'j;\, 1, m-1)|^2 \Big] |\langle j_1 j_2 j' \| J_1 \| j_1 j_2 j\rangle|^2$$

$$= \sum_{j'} |\langle j_1 j_2 j' \| J_1 \| j_1 j_2 j\rangle|^2 \qquad (243)^{*}$$

and

$$\sum_{j'} \Big[|C(1\, j' j';\, +1, m-1, m)|^2 - |C(1\, j\, j';\, -1, m+1, m)|^2 \Big]$$

$$= 2\, \langle j_1 j_2 j\, m \,|\, J_{1z} \,|\, j_1 j_2 j\, m\rangle \qquad (244)$$

obtained on taking matrix representation of

$$J_1^2 = J_{1z}^2 + \tfrac{1}{2}\left[J_{1-} J_{1+} + J_{1+} J_{1-} \right] \qquad (243a)$$

and

$$\left[J_{1+}, J_{1-} \right] = 2 J_{1z} \qquad (244a)$$

Therefore, again by convention, all matrix elements $\langle j_1 j_2 j\pm 1\, m \,|\, J_{1z} \,|\, j_1 j_2 j\, m\rangle$ are chosen to be real and positive (convention III). Consequently,

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* using unitarity

(i) all Clebsch-Gordon coefficients are real

(ii) all matrix elements $\langle J_1, J_2, J\pm 1, m \mid J_{2z} \mid J_1 J_2 J m\rangle$ are real and negative, since all off-diagonal matrix elements of J_z are zero. Reminding ourselves that when we write a state $|J_1 J_2 J m\rangle$ the first number J_1, denotes the eigen value $J_1(J_1+1)$ of J_1^2 operating on space 1, the second J_2 that of J_2^2 etc, so that if we now call space 1 as space 2 and vice versa J_2 will now represent the eigen-value of J_1^2 and J_1 the eigen value of J_2^2 and therefore the total state will be written as $|J_2 J_1 J m\rangle$. And as a result of the above convention we find

$$\langle J_1 J_2 J\pm 1\, m \mid J_{1z} \mid J_1 J_2 J m\rangle \neq \langle J_2 J_1 J\pm 1\, m \mid J_{2z} \mid J_2 J_1 J m\rangle$$

(since L.H.S. is always positive and R.H.S. negative) But,

$$|\langle J_1 J_2 J\pm 1\, m \mid J_{1z} \mid J_1 J_2 J m\rangle|^2 = |\langle J_2 J_1 J\pm 1\, m \mid J_{2z} \mid J_2 J_1 J m\rangle|^2 \tag{245}$$

since the Clebsch-Gordon transformation is unitary. Therefore,

$$\langle J_1 J_2 J\pm 1 m \mid J_{1z} \mid J_1 J_2 J m\rangle = -\langle J_2 J_1 J\pm 1\, m \mid J_{2z} \mid J_2 J_1 J m\rangle \tag{246}$$

We have, from (220) that for $J = J_1 + J_2$

$$|J_1 J_2\ J_1+J_2\ J_1+J_2\rangle = |J_1 J_1\rangle |J_2 J_2\rangle = |J_2 J_1\ J_1+J_2\ J_1+J_2\rangle$$

and using (223) we find that all states with $J = J_1 + J_2$ satisfy

$$|J_2, J_1, J_1+J_2, m\rangle = |J_1, J_2, J_1+J_2, m\rangle \tag{247}$$

Therefore, to satisfy (246) with $J = J_1 + J_2 - 1$ we must have

$$|j_2, j_1, j_1+j_2-1, m\rangle = -|j_1, j_2, j_1+j_2-1, m\rangle \qquad (248)$$

and extending the argument upto any j ; we have

$$|j_2\ j_1\ j\ m\rangle = (-1)^{j_1+j_2-j}\ |j_1\ j_2\ j\ m\rangle \qquad (249)$$

or equivalently,

$$C(j_2 j_1 j;\ m_2 m_1 m) = (-1)^{j_1+j_2-j}\ C(j_1 j_2 j;\ m_1 m_2 m) \qquad (250)$$

which is just the symmetry property (5a). The L.H.S. in (248) is clearly $P_{12}\ |j_1\ j_2\ j\ m\rangle$ and (249) is just the result utilised in Lecture 8. Using (250) in (65) we obtain now the familiar (10).

LECTURE XII

We recall now that $\vec{J} = -i\vec{r}\times\vec{\nabla}$, in particular, satisfies the commutation relations (1) and that correspondingly we have eigenstates of J^2 and J_z for integer values of j only which also have the explicit spartial representation.

$$Y_{l,m}(\theta,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} \tag{251}$$

where we use l instaed of j in conformity with normal usage and P_l^m 's are the associated Legendre polynomials; while for half integral values of j, the operators as also the eigen states do not admit of such a spatial representation. If however we denote by $\chi_\pm$ the two eigenstates $|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$ and $|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$ with $J = \tfrac{1}{2}$ and

$$\partial_\pm \equiv \frac{\partial}{\partial\chi_\pm}$$ then

$$\begin{aligned} J_x &\backsim \frac{1}{2}\left[\chi_-\partial_+ + \chi_+\partial_-\right] \\ J_y &\backsim \frac{i}{2}\left[\chi_-\partial_+ - \chi_+\partial_-\right] \\ \text{and}\quad J_z &\backsim \tfrac{1}{2}\left[\chi_+\partial_+ - \chi_-\partial_-\right] \end{aligned} \tag{252}$$

clearly provide a convenient representation of the operator $\vec{J}$ over spin $\frac{1}{2}$ space, and the operator

$$J^2 = J_z(J_z - 1) + J_+J_- = k(k+1) \tag{253}$$

where

$$k = \frac{1}{2}\left[\chi_+\partial_+ + \chi_-\partial_-\right]$$

Let us now consider monomials of the form $x_+^a \, x_-^b$. Operating by J^2 and J_z of (253) and (252) we see that $x_+^a \, x_-^b$ is an eigenstate of J^2, J_z with eigen values $\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}-1\right)$ and $\frac{1}{2}(a-b)$ respectively. Or, clearly $x_+^{J+m} \, x_-^{J-m}$ are eigen states of J^2, J_z with eigen values J, m. To satisfy also (219) we consider quantities

$$\frac{x_+^{J+m} \; x_-^{J-m}}{\left[(J+m)!\,(J-m)!\right]^{1/2}} \tag{254}$$

which for all purposes behave like states $|J, m\rangle$ with respect to the angular momentum operators (252). (254) is referred to as the spinorial representation of angular momentum states. Which we now use to give matrix representation to the rotation operators $R(\alpha\beta\gamma)$. If we denote $R(\alpha\beta\gamma)\,|Jm\rangle = |Jm\rangle'$ we have

$$\frac{(x_+')^{J+m}\,(x_-')^{J-m}}{\left[(J+m)!\,(J-m)!\right]^{1/2}} = \sum_{m'} D^J_{m'm}(\alpha\beta\gamma)\frac{(x_+)^{J+m'}\,(x_-)^{J-m'}}{\left[(J+m')\,(J-m')\right]^{1/2}} \tag{255}$$

and in particular for $\alpha = \gamma = 0$, we can write using (60)

$$x_+' = x_+ \cos\beta/2 - x_- \sin\beta/2$$

$$x_-' = x_+ \sin\beta/2 + x_- \cos\beta/2$$

nd

$$\frac{\left(x_+\cos\beta/_2 - x_-\sin\beta/_2\right)^{J+m}\left(x_+\sin\beta/_2 + x_-\cos\beta/_2\right)^{J-m}}{\left[(J+m)!(J-m)!\right]^{1/2}} = \sum_{m'} d^J_{m'm}(\beta)\,\frac{(x_+)^{J+m'}(x_-)^{J-m'}}{\left[(J+m')!(J-m')!\right]^{1/2}} \qquad (256)$$

from which one obtains,

$$d^J_{m'm}(\beta) = \left[\frac{(J+m')!\,(J-m')!}{(J+m)!\,(J-m)!}\right]^{1/2} \sum_{\sigma} \binom{J+m}{J-m'+\sigma}\binom{J-m}{\sigma}$$
$$\times(-1)^{J-m'-\sigma}\left(\cos\beta/_2\right)^{2\sigma+m'+m}\left(\sin\beta/_2\right)^{2J-2\sigma-m'-m} \qquad (257)$$

Clearly from (257) all matrix elements $d^J_{m'm}(\beta)$ are real i.e.,

$$d^J_{m'm}(\beta)^* = d^J_{mm'}(-\beta) = d^J_{m'm}(\beta) \qquad (258)$$

Also, from (257)

$$d^J_{m'm}(\pi) = (-1)^{J-m'}\,\delta_{m',-m}$$

and $d^J_{m'm}(-\pi) = (-1)^{J+m'}\,\delta_{m',-m}$ (259)

(260)

Since if we [illegible] the axis of quantization
$|J\,m\rangle$ [illegible] $m\rangle$ apart from any [illegible]

so that representing a rotation through 2π about the Y-axis as two successive rotations through π, we have

$$d^J_{m'm}(2\pi) = \sum_{m''} d^J_{m'm''}(\pi)\, d^J_{m''m}(\pi) = (-1)^{2J}\delta_{mm'}$$

(261)

and since

$$D^J_{m'm}(\alpha\beta\gamma) = e^{-im'\alpha}\, d^J_{m'm}(\beta)\, e^{-im\gamma}$$

We see that while a rotation through 2π leaves an integral spin state invariant, a rotation through 4π is necessary to restore a half integral state back to the same phase.

Let us now write

$$\begin{aligned} D^J_{m'm}(\alpha\beta\gamma)^* &= D^J_{mm'}(-\gamma,-\beta,-\alpha) \\ &= e^{i\gamma m}\, d^J_{mm'}(-\beta)\, e^{i\alpha m'} \\ &= e^{i\alpha m'}\, d^J_{m'm}(\beta)\, e^{i\gamma m} \end{aligned}$$

(262)

using (258).

We can write now using (259) and (260)

$$\begin{aligned} d^J_{m'm}(\beta) &= \sum_k d^J_{m'k}(\pi)\, d^J_{km}(\beta-\pi) = (-1)^{J-m'} \sum_k d^J_{-m'k}(\beta)\, d^J_{km}(-\pi) \\ &= (-1)^{-(J-m')}(-1)^{J-m}\, d^J_{-m'-m}(\beta) = (-1)^{m'-m}\, d^J_{-m',-m}(\beta) \end{aligned}$$

(263)

so that we can now write (262) as

$$D^{J}_{m'm}(\alpha\beta\gamma)^{*} = (-1)^{m'-m} D^{J}_{-m',-m}(\alpha\beta\gamma) \tag{264}$$

which is just the result (61) used earlier. Thw Wigner-Eckart Theorem and the unitarity of the Clebsch-Gordon transformation were sufficint (refer Lecture 7) to prove the Projection theorem, which we now make use of to evaluate $\langle J_1 J_2 J m | J_{1z} | J_1 J_2 J m \rangle$ occuring in (243)

$$\langle J_1 J_2 J m | J_{1z} | J_1 J_2 J m \rangle = \frac{m}{2J(J+1)} \left[J(J+1) + J_1(J_1+1) - J_2(J_2+1) \right] \tag{265}$$

and consequently,

$$\left| \langle J_1 J_2 J \| J_1 \| J_1 J_2 J \rangle \right| = \frac{\left[J(J+1) + J_1(J_1+1) - J_2(J_2+1) \right]}{2\sqrt{J(J+1)}} \tag{266}$$

Using (265) and (266) in (243) and (244) we have

$$\left| \langle J_1 J_2 J+1 \| J_1 \| J_1 J_2 J \rangle \right|^2 + \left| \langle J_1 J_2 J-1 \| J_1 \| J_1 J_2 J \rangle \right|^2$$

$$= J_1(J_1+1) - \frac{\left[J(J+1) + J_1(J_1+1) - J_2(J_2+1) \right]^2}{4J(J+1)} \tag{243A}$$

and

$$\frac{1}{J+1}\left|\langle J_1 J_2 J+1 \| J_1 \| J_1 J_2 J\rangle\right|^2 + \frac{1}{J}\left|\langle J_1 J_2 J-1 \| J_1 \| J_1 J_2 J\rangle\right|^2$$

$$= \frac{J^2(J+1)^2 + \{J_1(J_1+1) - J_2(J_2+1)\}^2}{4J^2(J+1)^2} \;{}^{*} \qquad (244A)$$

from which one can solve for the two unknowns

$$\left|\langle J_1 J_2 J-1 \| J_1 \| J_1 J_2 J\rangle\right| = \sqrt{\frac{(J - J_1 + J_2)(J + J_1 - J_2)}{4J(2J+1)}}$$

$$\times \sqrt{(J_1 + J_2 + 1 - J)(J_1 + J_2 + 1 + J)} \qquad (267)$$

$$\left|\langle J_1 J_2 J+1 \| J_1 \| J_1 J_2 J\rangle\right| = \sqrt{\frac{(J+1-J_1+J_2)(J+1+J_1-J_2)}{4(J+1)(2J+1)}}$$

$$\times \sqrt{(J_1 + J_2 + J + 2)(J_1 + J_2 - J)} \qquad (268)$$

Multiplying (267) and (268) by $C(J\,1\,J+1;\, m\, 0)$ and $C(J\,1\,J-1;\, m\, 0)$ respectively (obtained from (220)(240-242) etc.) and choosing the resulting quantities to be positive (by convention III) and substituting in (225) one obtains the recursion relation

$$C(J_1 J_2 J+1;\, m_1 m_2 m)\sqrt{\frac{(J+m+1)(J-m+1)}{(2J+1)(J+1)}}$$

$$\times \langle J_1 J_2 J+1 \| J_1 \| J_1 J_2 J\rangle$$

$$+ C(J_1 J_2 J-1;\, m_1 m_2 m)\sqrt{\frac{(J-m)(J+m)}{J(2J+1)}}\langle J_1 J_2 J-1 \| J_1 \| J_1 J_2 J\rangle$$

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* Using (250) [illegible]

$$= C(J_1 J_2 J; m_1 m_2 m)$$

$$\times \{ m_1 - m \langle J_1 J_2 J \| J_1 \| J_1 J_2 J \rangle \} \tag{269}$$

where the reduced matrix elements are given by (266), (267) and (268) with phase factor + 1.

The basic recursion relations (223) (224) and (269) are sufficient to derive (Racah) a general expression for the Clebsch-Gordon coefficients. From the form of the relation ( ) one can write

$$C(J_1 J_2 J; m_1 m_2 m) = (-1)^{J_1 - m_1} f(m_1 m_2; J m)$$

$$\times \left[\frac{(J_1 + m_1)!\,(J_2 + m_2)!\,(J + m)!}{(J_1 - m_1)!\,(J_2 - m_2)!\,(J - m)!} \right]^{1/2} \tag{270}$$

so that

$$f(m_1 m_2; J m-1) = (J_2 + m_2 + 1)(J_2 - m_2) f(m_1, m_2 + 1; J m)$$

$$- (J_1 + m_1 + 1)(J_1 - m_1) f(m_1 + 1, m_2; J m) \tag{271}$$

$$(J - m)(J + m + 1) f(m_1 m_2; J m+1)$$

$$= f(m_1, m_2 - 1; J m) - f(m_1 - 1, m_2; J m) \tag{272}$$

are the recursion relations satisfied by the f's . Putting $m = J$ in (272) we see that $f(m_1 m_2 ; J \underset{=J}{m})$ is independent of m's so that

$$f(m_1 m_2 ; J J) = A_J \tag{273}$$

From (271) we now have

$$f(m_1 m_2 ; J\, J-1) = \left[(j_2+m_2+1)(j_2-m_2) - (j_1+m_1+1)(j_1-m_1)\right] \times A_J \tag{274}$$

and again using the above and (271)

$$\begin{aligned} f(m_1 m_2 ; J\, J-2) = \big[&(j_2+m_2+1)(j_2+m_2+2)(j_2-m_2)(j_2-m_2-1) \\ &- 2(j_2+m_2+1)(j_2-m_2)(j_1+m_1+1)(j_1-m_1) \\ &+ (j_1+m_1+1)(j_1+m_1+2)(j_1-m_1)(j_1-m_1-1)\big] A_J . \end{aligned} \tag{275}$$

and so on

$$\begin{aligned} f(m_1 m_2 ; J\, J-u) = A_J \sum_t (-1)^t \binom{u}{t} \frac{(j_1-m_1)!}{(j_1+m_1)!} \\ \times \frac{(j_1+m_1+t)!\,(j_2+m_2+u-t)!\,(j_2-m_2)!}{(j_1-m_1-t)!\,(j_2-m_2-u-t)!\,(j_2+m_2)!} \end{aligned} \tag{276}$$

Using (270), (276) and (273) in (269) with $m = J$ we have since $C(J1\, J-1; m_1 m_2 J) = 0$ a relationship between A_J and A_{J+1}

$$A_j = \left[(j_1+j_2+j+2)(j_1+j_2-j)(j_1+j-j_2+1)(j+j_2-j_1+1)\right]^{1/2}$$

$$\times \left(\frac{2j+1}{2j+3}\right)^{1/2} A_{j+1}$$

(277)

which together with (220) gives

$$A_j = \left[\frac{(2j+1)(j_1+j_2-j)!}{(j_1+j_2+j+1)!\,(j_1+j-j_2)!\,(j+j_2-j_1)!}\right]^{1/2}$$

(278)

Thus, one has finally.

$$C(j_1 j_2 j;\, m_1 m_2 m) = \left[\frac{(2j+1)(j_1+j_2-j)!}{(j_1+j_2+j+1)!\,(j_1+j-j_2)!\,(j+j_2-j_1)!}\right]^{1/2}$$

$$\times \left[\frac{(j_1-m_1)!\,(j_2-m_2)!\,(j-m)!\,(j+m)!}{(j_1+m_1)!\,(j_2+m_2)!}\right]^{1/2}$$

$$\times \sum_t (-1)^{j_1-m+t}\,\frac{(j_1+m_1+t)!\,(j_2+j-m_1-t)!}{t!\,(j-m-t)!\,(j_1-m_1-t)!\,(j_2-j+m+t)!} \quad (279)$$

where as also in (276) the summation is over all integer values consistent with the factional notation the factorial of a negative number being meaningless.

LECTURE XIII

In our considerations of the angular momentum states $|J\, m\rangle$ so far we have been choosing a direction in space called the Z- axis the projection along which together with the magnitude of the angular momentum vector was quantised. The question naturally arises whether in fact there exists any preferential direction in space to be called the Z- axis in delaing with physical systems. If for example a system is in state $|J, J\rangle$ with respect to a Z- axis, the angular momentum of the system is 'oriented'* along this direction in which case the system is said to be polarised. If, on the other hand, the system has no preferred directions in space, then there is equal probability for finding the system in any of the $(2J+1)$ states $|J\, m\rangle$ (referred to an arbitrary Z axis); when the system is said to be unpolarized. We have also a third (or more general) alternative in which, referred to a conveniently fixed Z axix, each state $|J\, m\rangle$ has a probability p_m assigned to it, $\sum_m p_m =$ 1; the first two cases being now the particular cases when $p_m = \delta_{mJ}$ and $p_m = \frac{1}{2J+1}$ respectively.

We also know from basic principles that if a system, is in a stationary state $|\alpha\rangle$, say, then it can be wirtten as a superposition of a complete set of basic states $|\beta_i\rangle$ of the system i.e.,

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* In quantum theory, we know the direction of angular momentum is unobservable; even in the case of maximum projection $m = J$ the angular momentum vector can only be considered to precess around the Z axis, making an angle Θ with it

$$\Theta = \cos^{-1} \frac{J}{\sqrt{J(J+1)}} \approx 0 \text{ for large } J .$$

$$|\alpha\rangle = \sum_i |\beta_i\rangle\langle\beta_i|\alpha\rangle \tag{280}$$

so that $|\langle\beta_i|\alpha\rangle|^2$ is the probability for the system being found in state $|\beta_i\rangle$. The distinction between the two must however be clear; what we considered formerly is a case of statistical distribution while (280) is a quantum mechanical (or coherent) superposition of states. If we have now probabilities p_j assigned to the states $|\alpha_j\rangle$ forming also a complete set, p_j can be considered to form a diagonal matrix $\langle\alpha_j|\rho|\alpha_i\rangle = p_j\,\delta_{ij}$ in the $|\alpha\rangle$ representation, and is referred to as the density matrix ρ. Transforming the matrix ρ into the $|\beta\rangle$ representation

$$\langle\beta_m|\rho|\beta_n\rangle = \sum_i\sum_j \langle\beta_m|\alpha_j\rangle\langle\alpha_j|\rho|\alpha_i\rangle\langle\alpha_i|\beta_n\rangle$$

it clear that ρ is not necessarily diagonal in the $|\beta\rangle$ representation. Thus, in particular, an assignment of probabilities P_m to states $|Jm\rangle$ becomes for example under rotation a matrix with off diagonal elements; and the concept of the density matrix can be used to represent a statistical distimibution in any reference frame, through it should be diagonalised to allow interpretation of the elements as statistical probabilities (or statistical weights)

If we now consider an operator O of the system, $\langle\alpha_j|O|\alpha_j\rangle$ represents the expectation value of the operator between states α_j; and since the system has probability p_j to be in state $|\alpha_j\rangle$

$$\sum_j \langle\alpha_j|O|\alpha_j\rangle p_j$$

represents the 'average expectation value', denoted by $\langle O \rangle$, of the operator O. We also see that

$$\langle O \rangle = \sum_j \langle \alpha_j | O | \alpha_j \rangle p_j = \sum_j \sum_m \sum_n p_j \langle \alpha_j | \beta_m \rangle \langle \beta_m | O | \beta_n \rangle \langle \beta_n | \alpha_j \rangle$$

$$= \sum_m \sum_n \langle \beta_m | O | \beta_n \rangle \langle \beta_n | \rho | \beta_m \rangle$$

$$= \mathrm{Tr} \, (\rho O) \tag{281}$$

and is independent of the representation used, as one might expect.

Let us consider now scattering from states $|\alpha_j\rangle$ (the $2J+1$ two particle states, for example corresponding to the $2J+1$ projection states of total angular momentum J) which are weighed with probabilities p_j. The cross-section $d\sigma$ summed over all initial and final states is

$$d\sigma = C \sum_i \sum_j |\langle \alpha_j | T | \alpha_i \rangle|^2 p_i = C \sum_i \sum_j \langle \alpha_j | T | \alpha_i \rangle p_i \langle \alpha_i | T^+ | \alpha_j \rangle$$

$$= C \, \mathrm{Tr} \, (T \rho T^+) = C \langle T^+ T \rangle \tag{282}$$

where T represents the scattering matrix and C is the constant of proportionality. On the other hand summing only over the initial states, the probability for transition into a final state $|\alpha_j\rangle$ is $|\langle \alpha_j | T | \alpha_i \rangle|^2 p_i$ and if $\rho(f)$ denotes the density matrix after scattering

$$\langle \alpha_j | \rho(f) | \alpha_j \rangle = \sum_i |\langle \alpha_j | T | \alpha_i \rangle|^2 p_i = \langle \alpha_j | T \rho T^+ | \alpha_j \rangle$$

or in general, we write (283)

$$\rho(f) = T \rho(i) T^+$$

[illegible]

where we have now written the density matrix ρ before collision as $\rho(i)$, to avoid confussion. The statistical distributions before and after the transition are not, in general, the same. We also observe that $Tr\,\rho(f)$ is not necessarily unity even though $Tr\,\rho(i) = \sum_j p_j(i) = 1$. It is clear, however that the unnormalised density matrix $\rho(f)$ can be normalised by deviding it by its own trace, $Tr\,\rho(f)$. Thusm to accomadate the possibility of dealing with unnormalised density matrices we rewrite the equations (281), (283) and (282) as

$$\langle O \rangle = \frac{Tr(O\rho)}{Tr\,\rho} \tag{284}$$

$$\rho(f) = T\,\rho(i)\,T^{+} \tag{285}$$

$$d\sigma = c\,\frac{Tr(T\,\rho(i)\,T^{+})}{Tr\,\rho(i)} = c\,\frac{Tr\,\rho(f)}{Tr\,\rho(i)} \tag{286}$$

The density matrix ρ is obviosuly a hermitian matrix.* It is of order n, where n is the number of the basic states and consequently could be expressed as a linear combination of n^2 independent matrices S^{μ} (of order n) which may be chosen suitably

$$\rho = \sum_{\mu} C_{\mu} S^{\mu} \tag{287}$$

The linear independence of the base matrices is expressed by the 'orthogonality' relation

$$Tr(S^{\mu} S^{\nu}) = n\,\delta_{\mu\nu} \tag{288}$$

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* $\rho_{ij} = \sum_k \langle \beta_i | \alpha \rangle\, p_k \langle \alpha | \beta_j \rangle$ and if $\rho(i)$ is hermitian [illegible] also hermitian

We see [illegible]

suitably normalising the base matrices.

Since the average expectation value of S^{μ} is

$$\langle S^{\mu} \rangle = \frac{Tr\, \rho\, S^{\mu}}{Tr\, \rho} = \frac{n\, C_{\mu}}{Tr\, \rho}$$

$$C_{\mu} = \frac{Tr\, \rho}{n} \langle S^{\mu} \rangle \qquad (289)$$

or

$$\rho = \frac{Tr\, \rho}{n} \sum_{\mu=1}^{n^2} \langle S^{\mu} \rangle S^{\mu}$$

One of the S^{μ} can always be chosen to be the unit matrix I, so that

$$\rho = \frac{Tr\, \rho}{n} \left[I + \sum_{\mu=1}^{n^2-1} \langle S^{\mu} \rangle S^{\mu} \right] \qquad (290)$$

which provides a representation for the density matrix in terms of a set of n^2-1 matrices (which may be chosen to correspond to the matrices of n^2-1 linearly independent operators on the system) and (n^2-1) parameters, $\langle S^{\mu} \rangle$.

In particular, for example for spin ½ particles in the linear momentum representation the density matrix is 2 x 2 matrix which can be written following (290) as

$$\rho = \frac{Tr\, \rho}{2} \left[I + \vec{P} \cdot \vec{\sigma} \right] \qquad (291)$$

where $\vec{P} \equiv \langle \vec{\sigma} \rangle$ (292)

chosing the S^{μ} to be the 3 Panli spin matrices $\vec{\sigma}\,(\sigma_x, \sigma_y, \sigma_z)$.

Thus given the parameters $P_x \; P_y \; P_z$ the state of polarisation of the spin $\frac{1}{2}$ system is completely specified and since the spin operators transform like components of an axial vector, $\vec{P}$ is also an axial vector and is referred to as the 'polarisation vector'. Since σ_x, σ_y are non-diagonal, to diagonalise the density matrix one has obvisouly to find the coordinate system in which $\vec{P}$ is along the z-axis, when we have the probabilities associated with the states $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$ respectively as $\frac{1+P}{2}$ and $\frac{1-P}{2}$ so that $P = |\vec{P}|$ is clearly

$$P = \frac{(\text{number of particles with spin up}) - (\text{number of particles with spin down})}{\text{Total number of particles}} \tag{293}$$

in the ensemble. The system is said to be polarised along the direction $\vec{P}$; P being the degree of polarisation. If $p_{1/2} = 1$, $P = 1$ and the system is completely polarised; if $p_{1/2} = 1/2 = p_{-1/2}$, then $P = 0$, and the system is unpolarised.

Since $\sigma_x, \sigma_y, \sigma_z$ are hermitian it is clear from (281), (292) that P_x, P_y, P_z are real. In fact, to determine a $n \times n$ density matrix we need clearly $\frac{n^2 - n}{2}$ parameters which may be complex plus $n - 1$ real parameters, so that $n^2 - 1$ is the total number of real parameters to be specified. The $n^2 - 1$ parameters $\langle S^\mu \rangle$ may in general be complex when, however, not all of them will be independent though the S^μ are <u>linearly</u> independent.

LECTURE 14.

Let us now consider scattering of a beam of spin $\frac{1}{2}$ particles (polarised along a unit vector $\vec{n}$ inspace with a degree of polarisation P) by a target spin zero system. We have from (285) and (293) the polarisation $\vec{P}_{①}$ after scattering as

$$\vec{P}_{①} = \frac{Tr(\vec{\sigma}\rho_f)}{Tr(\rho_f)} = \frac{Tr\,(T\rho_i T^{+}\vec{\sigma})}{Tr\,(T\rho_i T^{+})}$$

$$= \frac{Tr(TT^{+}\vec{\sigma}) + Tr\,(T\vec{\sigma}.\vec{P}\,T^{+}\vec{\sigma})}{Tr(TT^{+}) + Tr(T\vec{\sigma}.\vec{P}\,T^{+})} \tag{294}$$

and the differential scattering cross-section

$$d\sigma = \frac{c\;Tr\,\rho_f}{Tr\;\rho_i} = \frac{c}{2}\left[Tr(TT^{+}) + Tr(T\vec{\sigma}.\vec{P}\,T^{+})\right] \tag{295}$$

If $\vec{k}, \vec{k}_1$ are the momenta of the particles before and after scattering, one can argue (from invariance considerations) that T must have the form

$$T = F(\theta_1) + G(\theta_1)\,\vec{\sigma}\cdot\frac{\vec{k}\times\vec{k}_1}{|\vec{k}\times\vec{k}_1|} = F(\theta_1) + G(\theta_1)\,\vec{\sigma}\cdot\vec{n}_1 \tag{296}$$

where F and G are functions of the scattering angle θ_1 , and also energy. Using (296) we find from (295) and (294) respectively that for an unpolarised incident beam, the differential cross-section $d\,\sigma^{(0)}(\theta_1)$ and polarisation $\vec{P}_1^{(0)}(\theta_1)$ are

$$d\sigma^{(0)}(\theta_1) = \frac{1}{2}\, c \left[|F(\theta_1)|^2 + |G(\theta_1)|^2 \right]$$

(297)

and

$$\vec{P}_1^{(0)}(\theta_1) = \frac{F^* G + G^* F}{|F|^2 + |G|^2} \times \frac{\vec{k} \times \vec{k}_1}{|\vec{k} \times \vec{k}_1|}$$

$$= \frac{2\, Re\left(G^*(\theta_1)\, F(\theta_1)\right)}{|F(\theta_1)|^2 + |G(\theta_1)|^2} \cdot \vec{n}_1$$

(298)

using the relations

$$Tr\, \vec{\sigma} = 0 = Tr\, (\vec{\sigma} \cdot \vec{A}) \qquad (299)$$

$$Tr\, (\vec{\sigma} \cdot \vec{A})\, \vec{\sigma} = 2\, \vec{A} \qquad (300)$$

and

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = (\vec{A} \cdot \vec{B}) + i\, \vec{\sigma} \cdot (\vec{A} \times \vec{B}) \qquad (301)^{*}$$

---

* NOTE: $\vec{\sigma} \cdot \vec{A}$ is the matrix $\begin{pmatrix} A_z & A_- \\ A_+ & -A_z \end{pmatrix}$ and consequently $(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) \neq (\vec{\sigma} \cdot \vec{B})(\vec{\sigma} \cdot \vec{A})$, in general. Note however that $Tr\, (A\, B \cdots C) = Tr\, (C\, A\, B \cdots)$ and so on.

The polarisation arises, clearly due to the interference between the spin-dependent and spin-independent terms in T and is perpendicular to the plane of scattering. Using (298) to define a quantity $\vec{P}_1^{(0)}(\theta)$ we can write the cross-section (295) as [illegible] from time reversal invariance argument, the cross-section (295) as

$$d\sigma(\theta_1) = \frac{C}{2}\left[|F|^2 + |G|^2\right]\left(1 + \vec{P}\cdot\vec{P}_1^{(0)}(\theta_1)\right)$$

$$= \frac{C}{2}\left(|F(\theta_1)|^2 + |G(\theta_1)|^2\right)\left(1 + P\,P_1^{(0)}(\theta_1)\cos\phi\right) \qquad (302)$$

where ϕ is the angle between $\vec{n}$ and $\vec{n}_1$. Thus, for scattering taking place in a plane containing the (incident) polarisation vector, we have the same differential scattering cross-section as we would have with an unpolarised beam; the cross-section reaching the extrem values $d\sigma^{(0)}(\theta_1)\left\{1 \pm P\,P_1^{(0)}(\theta_1)\right\}$, corresponding to $\phi = 0^\circ, 180^\circ$ in a plane perpendicular to the incident polarisation vector. As a corollary we see that with particles polarised along the direction of motion the cross-section is the same as with unpolarised particles, the polarisation $\vec{P}_1^{(\ell)}(\theta_1)$ after scattering is however different.

The polarisation $\vec{P}_1^{(0)}(\theta_1)$ produced as a result of scattering an unpolarised beam gives rise, after a second scattering (through θ_2) under identical conditions, to the cross-section

$$d\sigma^{(0)}_{\vec{k}_1,\vec{k}_2}(\theta_1,\theta_2,\phi) = \frac{C}{2}\left[|F(\theta_2)|^2 + |G(\theta_2)|^2\right] \times \left(1 + \vec{P}_1^{(0)}(\theta_1)\cdot\vec{P}_1^{(0)}(\theta_2)\right)$$

$$= \frac{C}{2}\left(|F(\theta_2)|^2 + |G(\theta_2)|^2\right)\left(1 + P_1^{(0)}(\theta_1)\,P_1^{(0)}(\theta_2)\cos\phi\right) \qquad (303)$$

Where ϕ is the angle between $\vec{n}_1$ and $\vec{n}_2 = \dfrac{\vec{k}_1 \times \vec{k}_2}{|\vec{k}_1 \times \vec{k}_2|}$, $\vec{k}_2$ being the momentum after the second scattering. In particular if the two scatterings take place in the same plane (303) shows clearly a right-left asymmetry

$$d\sigma^{(0)}(\theta_1, \theta_2)_{\text{left}}^{\text{right}} = \frac{c}{2}\left(|F(\theta_2)|^2 + |G(\theta_2)|^2\right)\left(1 \pm P_1^{(0)}(\theta_1)\, P_1^{(0)}(\theta_2)\right) \tag{304}$$

if the first scattering is to the right

$$\text{and } d\sigma^{(0)}(\theta_1\, \theta_2)_{\text{left}}^{\text{right}} = \quad '' \quad \left(1 \mp P_1^{(0)}(\theta_1)\, P_1^{(0)}(\theta_2)\right) \tag{305}$$

if the first scattering is to the left. One experimentally determines from these expressions, (303 or (304), (305) the degree of polarisation $P_1^{(0)}(\theta)$ produced by the scattering, for example, by choosing $\theta_1 = \theta_2 = \theta$ and thus finding $[P_1^{(0)}(\theta)]^2$

The polarisation $\vec{P}_2$ after two scatterings is clearly

$$\vec{P}_2 = \frac{Tr\{T(\theta_2)\, T^+(\theta_2)\, \vec{\sigma}\} + Tr\{T(\theta_2)\, \vec{\sigma}\cdot\vec{P}_1(\theta_1)\, T^+(\theta_2)\, \vec{\sigma}\}}{Tr\{T(\theta_2)\, T^+(\theta_2)\} + Tr\{T(\theta_2)\, \vec{\sigma}\cdot\vec{P}_1(\theta_1)\, T^+(\theta_2)\}} \tag{306}$$

and in particular for incident unpolarised particles

$$\vec{P}_2^{(0)} = \frac{1}{\{1 + \vec{P}_1^{0}(\theta_1)\cdot\vec{P}_1^{(0)}(\theta_2)\}}\left[\vec{P}_1^{0}(\theta_2) + \frac{F^*(\theta_2)F(\theta_2) - G^*(\theta_2)G(\theta_2)}{|F(\theta_2)|^2 + |G(\theta_2)|^2}\,\vec{P}_1^{(0)}(\theta_1) + \vec{n}_2\cdot\vec{P}_1^{(0)}(\theta_1)\,\frac{2|G(\theta_2)|^2\,\vec{n}_2}{|F(\theta_2)|^2 + |G(\theta_2)|^2} - \frac{2\,\mathrm{Im}\left(F(\theta_2)\,G^*(\theta_2)\right)}{|F(\theta_2)|^2 + |G(\theta_2)|^2}\left(\vec{P}_1^{(0)}(\theta_1) \times \vec{n}_2\right)\right] \tag{307}$$

The form (291) of the density matrix can also be used to describe the polarisation of photons (in linear momentum representation) though the spin of the photon is 1 since for a (real) photon there are only two allowed (transverse) states and consequently one needs only a 2 x 2 density matrix. Labelling the rows and columns by two orthogonal linear polarised states ψ_1 ψ_2 any pure state Ψ of the photon can be written as

$$\Psi = a_1 \psi_1 + a_2 \psi_2 \tag{308}$$

and is represented by the density matrix

$$\rho_\Psi = \begin{bmatrix} a_1 a_1^* & a_1 a_2^* \\ a_2 a_1^* & a_2 a_2^* \end{bmatrix} \tag{309}$$

in the ψ_1, ψ_2 representation and the $\langle S^\mu \rangle$ are

$$\langle I \rangle = a_1 a_1^* + a_2^* a_2 \equiv I \tag{310}$$

$$\langle \sigma_z \rangle = a_1 a_1^* - a_2^* a_2 \equiv P_1 \tag{311}$$

$$\langle \sigma_x \rangle = a_1 a_2^* + a_2^* a_1^* \equiv P_2 \tag{312}$$

and

$$\langle \sigma_y \rangle = i (a_1 a_2^* - a_2 a_1^*) \equiv P_3 \tag{313}$$

Though we are working in 2 by 2 representation, it should be remembered that we are dealing with a spin 1 field, and ψ_1 ψ_2 transform like (two of the) components of a vector and cosequently the transformation matrix, for example, for a rotation through θ about the direction of propagation is

$$\begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \tag{314}$$

and consequently

$$\rho' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \rho \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Writing ρ' again in the form (291) we see that (314a)
the parameters I, P_1, P_2 and P_3 transform under the above change of frame according to

$$\begin{bmatrix} I' \\ P_1' \\ P_2' \\ P_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{315}$$

Obviously I and P_3 are invariant under the particular transformation chosen and for a rotation through $\pi/4$, $P_1' = P_2$. We also observe that the parameters P_1, P_2, P_3 for photons do <u>not</u> transform like components of a vector. If we now choose I to be proportional to the intensity of the beam (when the density matrix will however be unnormalised) the four parameters I, P_1, P_2, P_3 admit respectively of the physical interpretation as (1) the total intensity (2) the difference intensity along Ψ_1, Ψ_2 directions (3) the difference in intensities along directions inclined at an angle $\pi/4$ to Ψ_1 Ψ_2 and (4) the difference between the intensities of the left circular and right circular components. These parameters are referred to as the Stokes parameters. The same applies w.r.t. any general density matrix

$$\rho = p\,\rho_{\Psi} + p'\,\rho_{\Psi'} \tag{316}$$

where p, p' are the statistical weights attached to the pure states

Ψ and Ψ', Ψ' being orthogonal to Ψ .

To represent density matrices of systems of higher angular momentum, say, j , we need clearly $(2J+1)^2$ linearly independent operators S^μ (including unit operator, if chosen to be one of the S^μ). We observe that a spherical tensor operator T^k_q of rank k, $k = 0, 1, \ldots, 2j$. can connect states $|jm\rangle$ of the system and (as we have seen in considering addition of angular momenta) since with each k we can have $(2k+1)$ operators we can consider the $(2J+1)(2J+1)$ such operators T^k_q with k ranging form 0 to $2j$ to form the basis operators. And we can build this set of basis operators in terms of the angular momentum operator $\vec{J}$ for the system, by defining for example

$$T^k_q = \sqrt{\frac{4\pi}{2k+1}} \frac{1}{\left(\sqrt{J(J+1)}\right)^k} \mathcal{Y}_{k,q}(\vec{J}) \tag{318}$$

where $\mathcal{Y}_{k,q}$ are the solid harmonics

$$\mathcal{Y}_{k,q}(\vec{r}) = (r)^k \; Y_{k,q}(\theta,\phi) \tag{319}$$

$Y_{k\,q}(\theta\phi)$ denoting, as usual, the normalised spherical harmonics.

In particular

$$T^0_0 = 1 \tag{318.1}$$

$$T^1_\mu = \frac{J_\mu}{\sqrt{J(J+1)}} \tag{318.2}$$

$$T^2_0 = \frac{3J_z^2 - J(J+1)}{2\,J(J+1)} \tag{318.3}$$

and so on, and the choice of the factors in (318) agrees with the choice of unit operator as T_0^0. For $J = 1/2$ however, T_μ^1 of (318) is not first σ_μ but

$$T_\mu^1 = \frac{1}{\sqrt{3}}\,\sigma_\mu \qquad (320)$$

To describe polarisation of deuterons, for example the density matrix ρ could be represented in terms of T_0^0 T_μ^1 and T_μ^2 and thus one has, in this case, to specify 9 parameters.

In general, we can write (in operator form)

$$\rho = \frac{1}{2J+1} \sum_{k=0}^{2J} \sum_{q=-k}^{k} \langle T_q^k \rangle \, T_q^k \qquad (321)$$

and a complete description of the ensemble is given either by giving ρ or by specifying the $(2J+1)^2$ parameters, $\langle T_q^k \rangle$ in any particular frame. $\langle T_q^k \rangle$ are some times referred to as 'tensor moments'. A specific advantage in choosing the basis operators as spherical tensor operators lies clearly in the fact that under rotations of coordinate system the transformation of the density matrix is simply desired by

$$\rho' = R(\alpha\beta\gamma)\rho = \frac{1}{2J+1} \sum_{k=0}^{2J} \sum_{q'=-k}^{k} \sum_{q=-k}^{k} D_{q'q}^{k}(\alpha\beta\gamma) \times \langle T_q^k \rangle \, T_{q'}^k \quad \cdot\cdot (322)$$

or writing ρ' in the form (321) in the new system the parameters are easily seen to transform (322) as

$$\langle T_q^k \rangle = \sum_{q'=-k}^{k} D_{q q'}^{k}(\alpha\beta\gamma) \langle T_{q'}^k \rangle_I \qquad (322a)$$

If ρ is diagonalisable, the diagonal elements, as mentioned earlier, could be interpreted as the statistical weights associated with the various substates referred to the particular Z axis; in which case the system is said to be oriented with respect to that axis. If $\langle T_0^1 \rangle$ is non-vanishing the system is said to be polarised about the axis and the polarisation P_I is defined in terms of $\langle T_0^1 \rangle$ as

$$P_I = \sqrt{\frac{J+1}{J}} \langle T_0^1 \rangle \tag{323}$$

and if $\vec{n}$ denotes a unit vector along this axis,

$$\vec{P}_I = P_I \vec{n} . \tag{324}$$

is referred to as the polarisation vector. It is readily seen that (323) and (324) agree with the earlier definitions in the case of spin $\frac{1}{2}$ particles. Also

$$P_I = \frac{1}{J} \sum_m m \, p_m \tag{323a}$$

clearly represents the ratio of the average spin along the axis of polarisation to the actual spin, lies between -1 to $+1$, and is zero for random orientation, i.e. for $p_m = \frac{1}{\sqrt{2J+1}}$. If, however, p_m is quadratically dependent on m, P_I vanishes clearly but

$$\sum_m m^2 p_m \neq 0$$

The system is then said to be aligned which is conveniently described by defining

$$P_{II} = \frac{2(J+1)}{2J-1} \langle T_0^2 \rangle \tag{325}$$

(325a)

which again lies between and and is zero for random orientation.

Lecture 15

Considering an 'oriented' system with spin J we observe that to specify the state of polarisation of the system, we need specify only $2J$ parameters $\langle T^k_0 \rangle$, $k = 0, 1, \dots, 2J$ in the frame of reference in which the density matrix is diagonal. This is equivalent to specifying the $2J$ statistical weights p_m for the $2J+1$ states, the $(2J+1)$th weight being determined by normalisation. From (281) and (318) it is clear that these two sets of parameters are connected through

$$\langle T^k_0 \rangle = \sum_m p_m P_k \left(\frac{m}{\sqrt{J(J+1)}} \right)$$

(326)

where $P_k(x)$ is the Legendre polynomial of order k. $P_k\left(\frac{m}{\sqrt{J(J+1)}}\right)$ is identical apart from factors with $C(J k J; m0m)$. Therefore the related quantities

$$B_k(j) = \sum_m (2k+1)^{1/2} C(J k J; m0) p_m$$

(327)

are sometimes used to specity the state of polarisation of an oriented system and are referred to 'orientation parameters'. The quantities $(2J+1)^{-1/2} B_k(j)$ are Fano's 'statistical tensors'

$$G_k(j) = \sum_m (-1)^{J-m} p_m \, C(J J k; m, -m) \qquad (328)$$

which again provide a convenient description of the state of polarisation of/an 'oriented' system.

Let us consider, as an example, the problem of radiation emitted by 'oriented' nuclei of spin J in decaying from states $|\alpha J m \Pi\rangle$ to states $|\alpha' J' m' \Pi'\rangle$ where Π, Π' are parities of the initial and final states and α denote other quantum numbers characterising the states. The matrix element for the process as pointed out earlier (lecture 7), is

$$\langle \alpha' J' m' \pi' | \vec{J}_N \cdot \vec{A} | \alpha J m \Pi \rangle \qquad (329)$$

If E, E' are the energies of the initial and final states respectively, $E - E' = k$ is the energy of the photon, which may be emitted along a direction $\vec{k}(\vec{k}, \theta_k \phi_k)$. To calculate the angular distribution

$$w(\theta_k \phi_k) = c \sum_{m'} \sum_m \left| \langle \alpha' J' m' \pi' | \vec{J}_N \cdot \vec{A} | \alpha J m \Pi \rangle \right|^2 p_m \qquad (330)$$

where c is a constant of proportionality $\vec{A}$ can conveniently be expressed in terms of the multipole solutions using the complex conjugate of (129) (for $(\pm 1)^{\prime}\chi_{\mp}\, e^{-i\vec{k}\cdot\vec{r}}$) so that angular momentum and parity conservations reduce (329) to a few terms of the type

$$M_L^{\mu} = (2\pi)^{1/2}(-1)^L(2L+1)^{1/2} D^L_{M\mu}(\phi_k, \theta_k, \phi_k)^* \times \langle\alpha' J' m' \pi' | \vec{J}_N \cdot [A^{(m)*}_{LM} - i\mu A^{(e)*}_{LM}] | \alpha J m \pi\rangle \tag{331}$$

which add up coherently so that (330) is

$$\omega^{\mp 1}(\theta_k \phi_k) = \sum_L \omega_L^{\mp 1}(\theta_k \phi_k) + \sum_{\substack{L, L' \\ L' > L}} \omega^{\mp 1}_{LL'}(\theta_k \phi_k) \tag{332*}$$

---

*NOTE: The angular momentum conservation limit the contributing multipoles to the orders $L = |J - J'|, \dots, J + J'$ since $J_N \cdot A_{LM}$ is a spherical tensor of rank L. Out of these, the parity conservation selects only

(i) those electric multipoles satisfying

$$\pi(-1)^L = \pi'$$

and (ii) those magnetic multipoles which satisfy

$$\pi(-1)^{L+1} = \pi'$$

since using (89) for $\vec{J}_N, \vec{A}$ we observe that $J_{N\mu}$ have parity (-1) and

$$A^{(m)}_{LM}(r, \pi-\theta, \pi+\phi) = (-1)^L A^{(m)}_{LM}(r\theta\phi)$$

$$A^{(e)}_{LM}(r, \pi-\theta, \pi+\phi) = (-1)^{L+1} A^{(e)}_{LM}(r\theta\phi) \tag{333}$$

(334)

---

for right circular or left circular polarised radiation where $\omega_1^{\mu}(\theta_k, \phi_k)$ are the pure multipole terms and $\omega_{LL'}^{\mu}(\theta_k,\phi_k)$ are the interferance terms; the summation over L and L' being over the allowed multipole transitions.

Considering a pure 2^L - pole term

$$\omega_L^{\mu}(\theta_k,\phi_k) = C(2\pi)(2L+1)\sum_{m'=-J'}^{J'}\sum_{m=-J}^{J} p_m D_{M,\mu}^{L}(\phi_k\theta_k\phi_k)^*$$

$$\times D_{M\mu}^{L}(\phi_k\theta_k\phi_k)\{C(J L J'; m,-M,m')\}^2$$

$$\times\Big[|\langle\alpha'J'\pi'\|J_N\cdot A_L^{(m)*}\|\alpha J\pi\rangle|^2\,\delta_{\pi',\pi(-1)^{L+1}}$$

$$+|\langle\alpha'J'\pi'\|J_N\cdot A_L^{(e)*}\|\alpha J\pi\rangle|^2\,\delta_{\pi',\pi(-1)^{L}}\Big] \quad (335)$$

using (63) and Wigner-Eckart theorem and since with a given L we can have only either an electric multipole transition or a magnetic multipole transition. The term within brackets is clearly a number and is independent of (θ_k ϕ_k) so that we shall denote by a $a_L^{e\text{ or }m}$ the quantity $C(2\pi)(2L+1)|\langle\alpha'J'\pi'\|J_N\cdot A_L^{e\text{ or }m}$ $\|\alpha J\pi\rangle|^2$ and since the shape of the distribution is the same for either electric or magnetic multipole transitions, we shall simply write

$$w_L^{\mu}(\theta_k,\phi_k) = a_L \sum_{m'} \sum_{m} (-1)^{M-\mu} p_m$$

$$\times \sum_{\nu=0}^{2L} C(LL\nu;\, M,-M)\, C(LL\nu;\, \mu,-\mu)\, D^{\nu}_{00}(\phi_k\theta_k\phi_k)$$

$$\times \left\{ C(JLJ';\, m,-M,m') \right\}^2$$

(336)

using (264) and the Clebsch-Gordon series (42). Also from (42a) and (43a)

$$D^{\nu}_{00}(\phi_k\theta_k\phi_k) = P_{\nu}(\cos\theta_k) \tag{337}$$

we can also express

$$C(LL\nu;\, M,-M)\, C(JLJ';\, m,-M,m')$$

$$= (-1)^{J-m}\sqrt{\frac{2J'+1}{2L+1}}\; C(JJ'L;\, m,-m',M)\, C(LL\nu;\, M,-M)$$

$$= (-1)^{J-m}\sqrt{\frac{2J'+1}{2L+1}} \sum_{s} U(JJ'\nu L;\, Ls)$$

$$\times C(J'Ls;\, -m',-M,-m)\, C(Js\nu;\, m,-m)$$

(338)

using (5d) and (98a). Using (5c) and (338) in (336) we have

$$w_L^{\mu}(\theta_k\phi_k) = a_L \sum_{m=-J}^{J} p_m \sum_{\nu=0}^{2L} P_{\nu}(\cos\theta_k) \times$$

$$\times \sum_{s} U(JJ'\nu L; Ls)\, C(LL\nu; \mu, -\mu)$$

$$\times\, C(Js\nu; m, -m)\, (-1)^{M-\mu} (-1)^{J-m} \sqrt{\frac{2J'+1}{2L+1}}\, (-1)^{L-M} \sqrt{\frac{2J'+1}{2J+1}}$$

$$\times \sum_{m'} C(J'LJ; -m', -M, -m)\, C(J'Ls; -m' - M, -m)$$

The summation over m' is clearly δ_{sJ} by unitarity, (6) so that using (29) and (30)

$$w_L^{\mu}(\theta_k \phi_k) = a_L (-1)^{L-\mu} (2J'+1)(-1)^{J-J'} \sum_{\nu=0}^{2L} P_\nu(\cos$$

$$\times\, W(LLJJ; \nu J')\, C(LL\nu; \mu, -\mu)$$

$$\times \sum_{m} (-1)^{J-m} p_m C(JJ\nu; m, -m) \tag{339}$$

which is on using (328)

$$w_L^{\mu}(\theta_k \phi_k) = b_L \sum_{\nu=0}^{2L} G_\nu(j)\, P_\nu(\cos\theta_k)$$

$$\times\, W(LLJJ; \nu J')\, C(LL\nu; \mu, -\mu) \tag{340}$$

where b_L is a constant $b_L^{\text{ear } m} = -a_L^{\text{ear } m} (-1)^{L+J-J'} (2J'+1)$ and is independent of μ. (340) is independent of ϕ_k and we also observe that the statistical weighing of the initial nuclear states affect the angular distribution and polarisation of the emitted radiation not directly in terms of p_m but through certain moments $G_\nu(j)$ of p_m

(i) suppose the initial nucleus is randomly oriented or $p_m = \frac{1}{2J+1}$. Then,

$$G_\nu(j) = \frac{1}{2J+1} \sum_m (-1)^{J-m} C(JJ\nu; m,-m)$$

which can be written using (5e) as

$$= \frac{1}{\sqrt{2J+1}} \sum_m C(JJ\nu; m,-m)\, C(JJ0; m,-m)$$

$$= \delta_{\nu 0} \frac{1}{\sqrt{2J+1}} \tag{341}$$

using (6). That is, only the zeroeth moment is nonvanishing and consequently (since $P_0(\cos\theta_k) = 1$) the angular distribution $W_L^\mu(\theta_k)$ is spherically symmetric; also since (by (5a))

$$C(LL0; \mu,-\mu) = C(LL0; -\mu,\mu)$$

the intensities of the left circular and right circular components are equal.

(ii) suppose the initial nucleus is oriented but the polarisation of the radiation is not observed. Summing over μ in (340) we have using (5a)

$$w_L(\theta_k) = \sum_{\mu=-1,1} w_L^{\mu}(\theta_k) = k_L \sum_{\nu=0}^{2L} G_\nu(j)$$

$$\times W(LLjj;\nu j')\,C(LL\nu;1,-1)\,P_\nu(\cos\theta_k)$$

$$\times \{1+(-1)^\nu\} \qquad (342)$$

i.e. Only the even statistical tensors contribute to the angular distribution (342)

(iii) Rewriting, for convenience, (340) as

$$w_L^{\pm 1}(\theta_k) = \sum_{\nu=0}^{2L} (\pm 1)^\nu A_\nu P_\nu(\cos\theta_k) \qquad (343)$$

where $A_\nu = k_L\, G_\nu(j)\, W(LLjj;\nu j')\, C(LL\nu;1,-1)$ (344) we see that for even ν terms $w_L^{+1}(\theta_k) = w_L^{-1}(\theta_k)$ or that the even ν terms in (343) are polarisation insensitive. Expressing,

$$w_L^{\pm 1}(\theta_k) = \sum_{\nu\ \mathrm{even}} A_\nu P_\nu(\cos\theta_k) \pm \sum_{\nu\ \mathrm{odd}} A_\nu P_\nu(\cos\theta_k)$$

$$= w_L^{\mathrm{even}}(\theta_k) \pm w_L^{\mathrm{odd}}(\theta_k)$$

(345)

we see that $w_L^{\mathrm{even}}(\theta_k)$ has a forward-backward symmetry (or fore and aft symmetry) about $\theta_k = 90^\circ$ (since $P_{\nu\,\mathrm{even}}(\theta_k)$ contain only even powers of $\cos\theta_k$). When polarisation of the emitted radiation is not observed (case ii) $\omega_L(\theta_k) = \omega_L^{even}(\theta_k)$ and consequently the distribution shows a fore and aft symmetry. We see also that the degree of circular polarisation of the emitted radiation is

$$= \frac{\omega_L^{+1}(\theta_k) - \omega_L^{-1}(\theta_k)}{\omega_L^{+1}(\theta_k) + \omega_L^{-1}(\theta_k)}$$

$$= \omega_L^{odd}(\theta_k) \Big/ \omega_L^{even}(\theta_k) \tag{345}$$

$$= \pm \frac{\omega_L^{\pm 1}(\theta_k) - \omega_L(\theta_k)}{\omega_L(\theta_k)} \tag{346}$$

Considering the interferance terms

$$\omega_{LL'}^{\mu}(\theta_k \phi_k) = 2 (\mathrm{Re}\Big[\sum_{m'} \sum_{m} (M_L^{*} M_L)\, p_m \Big] \tag{347}$$

it is clear that they are again of the form (336)

$$\omega_{LL'}^{\mu}(\theta_k \phi_k) = a'_{LL} \sum_{m'} \sum_{m} (-1)^{M-\mu} p_m \sum_{\nu=|L-L'|}^{L+L'}$$
$$\times C(LL'\nu; M,-M)\, C(LL'\nu; \mu,-\mu)\, D_{00}^{\nu}(\phi_k \theta_k \phi_k)$$
$$\times C(JLJ'; m,-M,m')\, C(JL'J'; m,-M,m') \tag{348}$$

Where

$$a_{LL'} = (4\pi)(2L+1)^{1/2}(2L'+1)^{1/2}\, c\, \mathrm{Re}\Big[i^{L'}(-i)^{L}$$

$$\times \langle \alpha' J' \pi' \| J_N \cdot \{ A_{L'}^{(m)*} - i\mu A_{L'}^{(e)*} \} \| \alpha J \pi \rangle^{*} \langle \alpha' J' \pi' \| J_N \cdot \{ A_{L}^{(m)*} - i\mu A_{L}^{(e)*} \} \| \alpha J \pi \rangle \Big] \quad (349)$$

and in terms of $G_\nu(j)$ as

$$w_{LL'}^{\mu}(\theta_k) = b_{LL'} \sum_{\nu=|L-L'|}^{L+L'} G_\nu(j) P_\nu(\cos\theta_k)$$

$$\times W(LL'JJ;\nu J')\, c(LL'\nu;\mu,-\mu) \quad (350)$$

The matrix elementsin (349) can either (i) be both magnetic or electric transitions in which case (349) is independent of the photon polarisation or (ii) may be of opposite kind in which case (349) will be $\pm$, depending on the circular polarisation; in any case, since $a_{LL'}$ occurs outside the ν summation the discussion above with respect to the pure multipole terms holds also in the case of the interferance terms and consequently for the whole radiation emitted. It is, however, found sufficient to consider the lowest $L = |J - J'|$ mostly, (there may be other selection rules forbidding the transition) or the two lowest multipole terms in (332) to obtain the distribution, the transition probability decreasing rapidly with increasing multipole order. To obtain the angular distribution of radiation polarised lineraly a suitable linear combination of the two circular polarisations should be taken in the matrix element stage (331). Observation of linear polarisation provides information on the electric or magnetic character of the radiation.

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Lecture IV:

This proof of Wigner-Ekart theorem is in effect similar to that given in Ref. 1) but in more familiar language.

Lecture V

7) V.F.Weisskopf: Relativistic Quantum mechanics (CERN)

Lecture VI.
see also

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Lecture VII: This simple proof of the Projection Theorem follows as a continuation of the arguments given in Lecture IV.

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