MATSCIENCE REPORT 9

LECTURES ON

GRAVITATION

BY

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LECTURES ON GRAVITATION +

(A course of four lectures)

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by

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P R E F A C E.

The contents of this report are the lecture notes of 4 lectures delivered by Professor L. I. Schiff of Stanford University, California during his stay in Madras as a visiting Professor of the Institute of Mathematical sciences, Madras in the spring of 1963. The first two lectures deal with an introduction to the formal aspects of the theory of gravitation and general relativity while the rest are concerned with a noval verification of the theory using a spinning object as a test particle in a gravitational field. This preliminary version of the lecture notes was prepared by Dr. R. Vasudevan.

These lecture notes were not looked into by Professor L.I. Schiff. Since these notes are not complete in themselves this is intended for private circulation only.

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ON GRAVITATION AND GENERAL RELATIVITY

LECTURE I.

INTRODUCTION:

Einstein's theory of gravitation, (the theory of Relativity,) which has been accepted as the most satisfactory description of gravitational phenomenon for more than forty years, is a theory of great conceptual and structural elegance, and it is also so designed that it automatically agrees in appropriate limits, with Newton's mechanics of gravitating bodies, and with Einstein's theory of special relativity. We shall in these lectures be concerned not so much with the formal aspects of the theory, as with the experimental basis of the theory. We shall in general be interested in the calculation of the geodesic orbits for mass points and spinning objects, and calculate the change in the spin axis of the gyroscope in a gravitational field. Also we shall investigate the various cosmological models in a brief manner and relate the astronomica evidences obtained for these theories.

The Difference between Special and General theory of relativity:

The dimensionless parameter characteristic of the special theory is $\frac{v^2}{c^2}$ which is very small compared to unity in the Newtionian theory. The general theory is characterised by the constant parameter $\frac{GM}{c^2 r}$ where G is the newtonian, gravitational constant, M the gravitating mass, r the distance from the centre of the gravitating mass and C of course is the velocity of light. Newtonian theory is the limit when these two constant parameters become very small compared to 1. To get a feeling for the magnitude of these constants the value of the constant $\frac{GM}{c^2 r}$

at the surface of the sun and that of the earth is 10^{-6} and 10⁻ respectively. Hence any terrestrial or solar experiments canno reveal any special consequences of the general theory.

THE PRINCIPLE OF EQUIVALENCE:-

This celebrated principle states the identity of the gravitational and inertial mass of a body. The inertial mass o body, by definition is that which appears in the equation, $F = m$ in a dynamical experiment. While the gravitational mass enters the definition of the force on a spring $W = m_G g$; The equality of gravitational and inertial masses was originally formulated i terms of equal accelations for all freely falling test particles regardless of mass or chemical composition. The pendulum experi ment again is a free fall experiment in which the force is deter mined by the gravitational mass, and the acceleration by the inertial mass. Much more precise experiments, were performed ha a century ago, by Eotvios[1] and his collaborators which are being repeated with improved techniques by Dicke[2] at M.I.T. and the accuracy in such experiments is of the order of one in 10^8; The mass term that appears in the mass energy relation, and that indi cated in the mass spectrograph experiments are all inertial masse While from the days of Newton, the equality of these two has been a puzzle, the equivalence principle, as stated below arose, in th formulation of general relativity of Einstien about half a centur ago namely: all observations made locally on a system, in a stat uniform gravitational field in the absence of local background matter, agree with the corresponding operations made on the same system, when it is subject to an equivalent acceleration in the absence of the field. Implicit in this statement is a lot of physics and leads to the concept that the space time orbit of a particle.is independent of the structure of internal forces in the particle. Gravitation, is exhibited as a distortion of space time by the presence of mass and the test particles move in accordance with the geodesic equations for a Reimannian metric, intimately related to the gravitational field.

EXPERIMENTAL VERIFICATIONS:

Unlike the principle of covarience, the principle of equivalence cannot be regarded as a necessarily inescapable axiom of physics, since it makes perfectly definite statements as to the interrelated character of the coordinate system and gravitational fields which might or might not be true. Hence the final justification for the introduction of the principle must depend on the comparison of these predicted consequences with the results of observation.

Of course the simplest of the consiquences is that the gravitational acceleration of all bodies is the same in the same field. Again the general theory leads to Newtonian theory of gravitation as a first approximation furnishing support for the confirmed laws of celestial motion. But our belief in this theory receives compelling sanction by the three crucial tests which distinguish between the predictions of the Newtonian theory and those of the precise Einstien's theory. The three well known crucial tests of the tyeory are:-

(1) The Gravitational Red shift.

(2) Precession of the Planetary Orbits.

(3) The deflection of light in a gravitational field.

(1) The Red shift:- This is a direct consequence of

the equivalence principle and occurs when we compare the spectrum of light emitted from the same atom on the surface of the sun and that on the surface of the earth. Earlier measurements of St. John and Adams, have given evidence to the fact that the light gets degraded in its energy by passing through the gravitational field. Later experiments at Harvard, and Harwell (3) (using Mansbauer effect techniques) have proved this with an accuracy of about 3 per cent. But this can be explained by the mere assumption that the measurements in a uniform gravitational field are identical with those done with a frame work which is subject to the same acceleration g as in the gravitational field. Take two identical clocks placed in a frame with constant acceleration g, one ahead of the other by a distance h in the direction of the field. Let the clocks have natural period T_0 and let light signals be sent at the end of each period from one clock to the other to permit a comparison of the observed rates. The time necessary for the signal to pass between the two clocks is approximately $t = \frac{h}{c}$, c being the velocity of light (1). The forward clock will acquire added velocity in the direction of motion, in this interval of time given by $v = gt = g\frac{h}{c^2}$ (2) Hence by ordinary Doppler effect (first order), the period of the rear clock when measured in terms of the forward clock with the help of the arriving light signals is

$$T = T_0\left(1 + \frac{v}{c}\right) = T_0\left(1 + g\frac{h}{c}\right) \quad \ldots\ldots \quad (1)$$

By the principle of equivalence, this result can be interpreted as also applying to the analogous situation of two stationary clocks, separated by a distance in the direction of a uniform field of intensity g so that we may write $T_2 = T_1\left(1 + \frac{\Delta\psi}{c^2}\right)$

with a difference in gravitational potential $\Delta\Psi = gh$, the clock at the lower end having a longer observed period. Two atoms of the same substance, can replace the identically constructed clocks and the period or correspondingly the wavelength of a given spectral line of the atom, should show a shift given by

$$\delta\lambda = \lambda \frac{\Delta\Psi}{c^2}$$

when it passes through a gravitational potential $\Delta\Psi$ or equivalently for the frequency $\frac{\delta\nu}{\nu} = \frac{\Delta\Psi}{c^2} = -\frac{gh}{c^2}$

For sodium light orginating from the surface of the sun

$$\frac{\delta\lambda}{\lambda} = 2\cdot 12 \times 10^{-6}$$

Before taking up the detailed derivation of the other two effects it is interesting to point out, how the orbit precession experiment is a very conclusive test of the theory more than anything else. On the simplest interesting case, i.e., that of the gravitational field about the stationary spherically symmetric mass M, the metric that represents the solution of the field equations (to be given later), can be written in the Schwarzchlid standard form.

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta . d\phi^2\right) \quad (2$$

where units have been chosen so that $M = \frac{G.M}{c^2}$; $G = c = 1$

A reformulation of the above form in which the radial as well as the polar coordinates appear symmetrically is obtained by a redefinition of the scale of length and is given by the <u>isotropic form as</u>

$$ds^2 = \frac{\left(1 - \frac{M}{2r}\right)^2}{\left(1 + \frac{M}{2r}\right)^2} dt^2 - \left(1 + \frac{M}{2r}\right)^4 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right) \quad \ldots\ldots (3)$$

Hence we have to look into the other two effects to get a decisive verification of the General relativity.

The special relativity is valid whenever $\frac{M}{r} \ll 1$ and the metric has to be supplemented by the geodesic for the motion of the test particle and the nullgeodesic for the light ray.

Suppose we wish to verify the structure of the equation by Comparison with observations. We may write the metric in the form

$$ds^2 = \left(1 + \alpha \cdot \frac{M}{r} + \beta \frac{M^2}{r^2} + \ldots\right)dt^2 - \left(1 + \gamma \frac{M}{r} + \delta \frac{M^2}{r^2} + \cdots\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta . d\phi^2\right) \quad \text{---- (4)}$$

Where α, β, γ , and δ are of order unity and for planetary cicular orbits of the Newtonian type $\frac{GMm}{r^2} = \frac{mv^2}{r}$ or $\frac{GM}{c^2 r} = \frac{v^2}{c^2}$ i.e., the parameters in general and special relativity are the same. Writing the above equation as

$$\frac{ds^2}{dt^2} = \left(1 + \alpha \cdot \frac{GM}{c^2 r} + \beta \cdot \left(\frac{GM}{c^2 r}\right)^2 + \ldots\right) - \left(\quad\right)\frac{1}{c^2}\frac{dr^2}{dt^2} + \left(\quad\right)$$

$$\frac{dr^2}{dt^2} \sim \frac{v^2}{c^2} \quad \text{---- (5)}$$

Hence each term in the first bracket is greater than the corresponding term in 2nd bracket by one order. Hence the term $\alpha \frac{GM}{c^2 r}$ in the first bracket corresponds to the term unity in the second bracket, and for reduction to Newtonian orbits, α should be equal to -2. The theory of the gravitational red shift also can be accounted for in the same way. The theory of the gravitational deflection of light resluits from the null geodesic for a light ray and together with the above value of α and the choice γ = +2. Finally the theory of the precession of the perehelion of mercury results form the geodesi equation of motion for a test particle, with the forgoing values of α & γ and the choice of β = o. Higher terms in (5) have not been subjected to experimental test It is argued elsewhere(*) that the values α & γ and the null

(*) - See American J. of Physics - 28,340 (1960) B=). Eddington Page 105.

geodesic equation for the light ray can be obtained correctly from the equivalence principle and special theory of relativity. On the other hand the value of β which depends on the nonlinearity of the equations of motion and the geodesic equation for a test particle cannot be obtained in this fashion. Hence the plenatary precession is the most crucial experimental test of the theory.

FORMALS ASPECTS OF THEORY:-

According to the principle of covariance the line element which should hold in all cases can be expressed, in the usual way as $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$. The $g_{\mu\nu}$ are called fundamental tensors and we can take them to be symmetric; $g_{\mu\nu}$, g^μ_ν & $g^{\mu\nu}$ are the covariant, and mixed and contravaniant tensors respectively, and also

$$g^\mu_\nu = g_{\nu\sigma} g^{\mu\sigma} \begin{array}{l} = 0 \text{ if } \mu \neq \nu \\ = 1 \text{ if } \mu = \nu \end{array} \quad \cdots \quad (6)$$

$g_{\mu\nu}$ are ten independent quantities and since in V_4 space there are only four transformations the $g_{\mu\nu}$ which represent in a way the gravitational potentials cannot be all made unity, i.e., A permanent gravitational field cannot be transformed away by choosing any suitable coordinate system.

Christofel 3 index symbols:-

$$[\mu\nu, \sigma] = \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)$$

$$\{\mu\nu, \sigma\} = g^{\sigma\lambda}[\mu\nu, \lambda] = \Gamma^\sigma_{\mu\nu} \quad \cdots (7)$$

There are 40 different 3 index symbols and these correspond to a generalised force in gravitational theory since these include the derivatives of $g_{\mu\nu}$

The Reiman-Christofel Tensors:-

We have the definitions

$$R^{\tau}_{\mu\nu\sigma} = \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\tau}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\tau}_{\alpha\sigma} + \frac{\partial}{\partial x^{\nu}}\Gamma^{\tau}_{\mu\sigma} - \frac{\partial}{\partial x^{\sigma}}\Gamma^{\tau}_{\mu\nu}$$

$$R_{\mu\nu\sigma\rho} = g_{\rho\tau} R^{\tau}_{\mu\nu\sigma}$$

and we can see that

$$R_{\mu\nu\sigma\rho} + R_{\mu\sigma\rho\nu} + R_{\mu\rho\nu\sigma} = 0 \quad \ldots (8)$$

$R^{\tau}_{\mu\nu\sigma}$'s are general tensors of the 4th rank involving second derivatives of $g_{\mu\nu}$'s and the $g_{\mu\nu}$'s in a product form. Though in general they can have 256 components because of the symmetries, etc. that are pertinent to these tensors we get only 20 independent quantities.

The contracted Reimann-Christofel tensors are formed by setting $\tau = \sigma$, $\quad R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma}$.

Because of duplication of indices,

$$\Gamma^{\sigma}_{\mu\sigma} = \frac{\partial}{\partial x^{\mu}} \log\sqrt{-g}.$$

$$R_{\mu\nu} = -\frac{\partial}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} + \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\log\sqrt{-g} - \Gamma^{\alpha}_{\mu\nu}\frac{\partial}{\partial x^{\alpha}}\log\sqrt{-g} \quad \ldots (9)$$

The law that $R_{\mu\nu} = 0$ in empty space was chosen by Einstein as the law of gravitation. $R^{\alpha}_{\mu} = g^{\alpha\beta} R_{\mu\beta}$ The spur $R = R^{\alpha}_{\alpha} = g^{\alpha\mu} R_{\alpha\mu}$ is called the Gaussian curvature or simply curvature of space time. $R_{\mu\nu}$ is a symmetrical tensor; and the above statement means that there are 10 differential equations, to determine $g_{\mu\nu}$. Since it can be seen that there are 4 identical relations between these and so the number of equations is effectively The condition for 'flat' space time can be expressed by setting the Reimann Christofel tensor equal to zero giving us the covariant equation $R^{\tau}_{\mu\nu\sigma} = 0$. $R_{\mu\nu} = 0$ is satisfied not only when condition for flat space time is satisfied but also under less stringent conditions. We must now turn to the fundamental problem of obtaining a covariant relation between the gravitation potentials $g_{\mu\nu}$ and the components of the energy momentum tensor $T_{\mu\nu}$, This relation will be the appropriate relativistic analogus of the poisson's equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi = 4\pi K\rho \quad \ldots . (10)$$

which in Newtonian theory connects the single gravitational potential ψ with the density of the matter. ρ and the gravitational constant K. Poisson's equation does not involve higher derivatives of the Newtonian potential than the second. Moreover with the principle of covariance we know that the convariant divergence of the energy momentum tensor $T^{\mu\nu}$ for any coordinate system, should equal zero; This is a fundamental equation of the mechanics of a continuous medium

$$T^{\mu\nu}_{,\nu} = \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\alpha\nu} T^{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu} T^{\mu\alpha} = 0 \quad \ldots (11)$$

which becomes $\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0$ for the Cartesian coordinate system. It is to be noted that $\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu}\right)$ is a most general tensor of second rank, constructed solely from $g_{\mu\nu}$ and their first and second derivatives, whose contracted covariant derivative would be identically zero. All these and other consideration led Einstien to formulate the appropriate field equations for relativistic theory of gravitation, which is the following.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -K T_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

Of course $T_{\mu\nu}$ is the energy momentum tensor of all the things in the field excluding the gravitation field itself. To obtain the poisson equation in the Newtonian limit K is chosen to be equal to $8\pi G/c^4$.

The geodesic paths in general are defined by the condition $\delta\int ds = 0$, i.e., the total interval along the trajectory shall be an extremum for small variations which vanish at the two limits of the integration. A covariant expression of the geodesic is given by

$$\frac{d^2x^\sigma}{ds^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \quad \ldots (12)$$

and Einstein assumed that this is the trajectory of a particle or a light ray as a postulate in a gravitational field. Furthermore the additional limitation that $ds = 0$ for a light ray is also taken as an appropriate generalisation in the presence of a gravitational field. The effects of the Λ term would increase with the size of the region considered and in a region within the size of a solar system, Λ is small enough to be neglected and so we shall be justified in taking the field equations as

$$-KT_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \quad \ldots (13)$$

together with $KT = R$

as a result of contraction Λ is retained for cosmological consideration and is called the cosmological constant.

The constants which enter the theory can be listed as

$C = 2.9979 \times 10^{10}\ cm/sec$. $G = 6.664 \times 10^{-8}\ cm^3 gm^{-1} sec^{-2}$

and hence

$$\frac{8\pi G}{c^4} = 2.073 \times 10^{-48}\ cm^{-1} gm^{-1} sec^2 \quad \ldots (14)$$

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LECTURE-II.

Let us think about the motion of a test particle and a light ray in a gravitational field, taking a general metric, with $g_{\mu\nu}$ coefficients being given by power series expansion in terms of the quantity $\frac{M}{r}$. This will enable us to fix the coefficients in the power series, to a certain extent; let us therefore write the standard line element as

$$ds^2 = \left(1 + \alpha\frac{M}{r} + \beta\frac{M^2}{r^2} + \cdots\right)dt^2 - \left(1 + \gamma\frac{M}{r} + \frac{\delta M^2}{r^2} + \cdots\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \quad (1)$$

The geodesic equations of motion are:-

$$\frac{d^2x^\sigma}{ds^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{dx}\frac{dx^\nu}{ds} = 0 \quad (2)$$

where r, θ, ϕ and t are the four components of x^σ. It can be seen that with $g_{11} = -\left(1 + \gamma\frac{M}{r} + \delta\frac{M^2}{r^2} + \cdots\right)$,

$g_{22} = r^2$, $g_{33} = r^2\sin^2\theta$, $g_{44} = \left(1 + \alpha\frac{M}{r} + \beta\frac{M^2}{r^2} + \cdots\right)$ with $c = G = 1$

the r equation reduces to the line element itself and we get

$$1 = \left(1 + \alpha\frac{M}{r} + \beta\frac{M^2}{r^2} + \cdots\right)\left(\frac{dt}{ds}\right)^2 - \left(1 + \gamma\frac{M}{r} + \delta\frac{M^2}{r^2} + \cdots\right)\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\theta}{ds}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 \quad (3)$$

For the θ equation, we get

$$\frac{d^2\theta}{ds^2} - \cos\theta\sin\theta\left(\frac{d\phi}{ds}\right)^2 + \frac{2}{r}\frac{dr}{ds}\cdot\frac{d\theta}{ds} = 0 \quad \cdots(4)$$

If initially $\theta = \frac{\pi}{2}$ & $\frac{d\theta}{ds} = 0$ so that $\frac{d^2\theta}{ds^2} = 0$ then the particle continues to move in this plane itself. The equation for ϕ becomes

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\cdot\frac{d\phi}{ds} = 0 \quad \cdots\cdots(5)$$

which integrates to $\frac{d\phi}{ds} = \frac{h}{r^2}$ where h is a constant. We get $\frac{1}{\psi}\frac{d\psi}{ds} = -\frac{2}{r}\frac{dr}{ds}$ ....(6), where $\psi = \frac{d\phi}{ds}$

In the non-relativistic limit h is the angular momentum since

$$\psi = \frac{c'}{r^2}, \quad \frac{d\phi}{ds} = \frac{h}{r^2} = \frac{d\phi}{dt}$$

If we substitute e^{ν} for $g_{44} = \left(1 + \frac{\alpha M}{r} + \beta\frac{M^2}{r^2} + \cdots\right)$

We get for the t equation

$$\frac{d^2t}{ds^2} + \frac{d\nu}{ds}\frac{dt}{ds} = 0 \qquad \text{----} (7)$$

which integrates out to $\frac{dt}{ds} = k e^{-\nu}$ where k is a constant and in the non-relativistic limit it is the energy-integral. Taking the r equation given above and putting $r = \frac{1}{u}$ we obtain

$$1 - \frac{k^2}{1 + \alpha M u + \beta M u^2 + \cdots} + \left(1 + \gamma M u + \delta M^2 u^2 + \cdots\right) u^2 \left(\frac{du}{d\theta}\right)^2 + h^2 u^2 =$$

Since

$$\frac{dr}{ds} = -\frac{1}{u^2}\frac{du}{ds} = -\frac{1}{u^2}\frac{du}{d\phi}\cdot\frac{d\phi}{ds} = -h\frac{du}{d\phi} \quad .... (8)$$

To eliminate k related to energy in the non-relativistic limit put

$$\left(1 + \alpha M u + \cdots\right)\left(1 + \gamma M u + \cdots\right) h^2 \left(\frac{du}{d\phi}\right)^2 + \left(1 + h^2 u^2\right)\left(1 + \alpha M u + \cdots\right) = k^2 \quad ...(9)$$

h is of order $r^2\omega \sim rv$ and so $h^2u^2 \sim v^2$,

and $Mu \sim \frac{GM}{c^2 r} \sim \frac{v^2}{c^2} \sim v^2$ for purposes of order of magnitude calculations and $v^2 \ll 1$ for a particle and also $\frac{du}{d\phi}$ is of order u for a general elliptic orbit. Hence to go to one order beyond the lowest we keep the terms as follows:

$$\left(1 + \alpha M u + \cdots\right)\left(1 + \gamma M u + \cdots\right) h^2 \left(\frac{du}{d\phi}\right)^2 \approx \left[1 + (\alpha + \gamma) M u + \ldots\right] \times v^2$$

$$\left(1 + h^2 u^2 + \ldots\right)\left(1 + \alpha M u + \ldots\right) \simeq 1 + \left(h^2 u^2 + \alpha M u\right) + \beta M^2 u^2 + \alpha M h^2 u^3 \quad .... (10)$$

(i.e.) if we keep the terms of order up to $\frac{v^4}{c^4}$ only; the equation becomes after differentiating with respect to ϕ

$$\frac{d^2u}{d\phi^2}\left\{1+(\alpha+\gamma)Mu\right\}+\frac{1}{2}(\alpha+\gamma)M\left(\frac{du}{d\phi}\right)^2+\left(u+\frac{\alpha M}{2h^2}\right)+\left(\beta\frac{M^2u}{h^2}+\frac{3\alpha Mu^2}{2}\right)=0$$

Rewriting this putting the lowest order terms on the left we get

$$\frac{d^2u}{d\phi^2}+u+\frac{\alpha M}{2h^2}=-(\alpha+\gamma)M\left[u\frac{d^2u}{d\phi^2}+\frac{1}{2}\left(\frac{du}{d\phi}\right)^2\right]-\left(\beta\frac{M^2u^2}{h^2}+\frac{3\alpha}{2}Mu^2\right) \quad \text{---- (12}$$

Compare this with $\frac{d^2u}{d\phi^2}+u=\frac{m}{h^2}+3mu^2$

If M = 0, we get $\frac{d^2u}{d\phi^2}+u=0$ which is the equation for the straight line in the absence of gravitation fields. Let us solve this equation by perturbation.

Neglecting the right side of the equation we get the first order solution as

$$u_0=-\frac{\alpha M}{2u^2}\left[1+e\cos(\phi-\omega)\right] \quad \text{---- (13)}$$

which is an ellipse with e and ω as arbitrary constants. This is the Newtonian case and for (e = 0) i.e. circular orbit

$$u_0=-\frac{\alpha M}{2h^2}=\frac{1}{R}, \quad \alpha=-\frac{2h^2}{MR}$$

Taking $h=R^2\frac{d\phi}{dt}=R^2\omega=Rv$ and $\frac{M}{R^2}=\frac{v^2}{R}$

We have to take $\alpha=-\frac{2Rv^2}{M}=-2$

Putting this solution on the right side of the equation we get

$$\frac{d^2u}{d\theta^2}+u+\frac{\alpha M}{2R^2}=-(\alpha+\gamma)M\left(\frac{\alpha M}{2h}\right)^2\left\{-e\cos(\phi-\omega)\left[1+e\cos(\phi-\omega)\right]+\frac{1}{2}e^2\sin^2(\phi-\omega)\right\}$$
$$+\frac{\beta M^2}{h^2}\cdot\frac{\alpha M}{h^2}\left[1+e\cos(\phi-\omega)\right]$$
$$-\frac{3\alpha M}{2}\cdot\left(\frac{\alpha M}{2h^2}\right)^2\left\{1+2e\cos(\phi-\omega)+e^2\cos^2(\phi-\omega)\right\} \quad \text{---(1}$$

The significant terms on the right for secular precession are there proportional to $\cos(\phi-\omega)$ which is

$$A\cos(\phi-\omega)=\frac{eM^3}{4h^4}\left[(\alpha+\gamma)\alpha^2+2\alpha\beta-3\alpha^3\right]\cos(\phi-\omega) \quad \text{---- (15)}$$

The solution of $\frac{d^2u}{d\phi^2} + u + \frac{\alpha M}{2h^2} = A\cos(\phi - \omega)$ .... (16)

of period 2π is given by

$$u = -\frac{\alpha M}{2h^2}\left[1 + e\cos(\phi-\omega)\right] + \frac{1}{2}A\phi\sin(\phi-\omega)$$

$$\simeq -\frac{\alpha M}{2h^2}\left[1 + e\cos\left(\phi - \omega + \frac{Ah^2}{\alpha M e}\phi\right)\right)$$

for small angles of ϕ. This the angle ω which is the longitude of the perhilion changes by an amount

$$-\frac{2\pi A h^2}{\alpha M e} = \frac{\pi M^2}{h^2}\left(\alpha - \frac{\alpha\gamma}{2} - \beta\right) \quad \text{.... (17)}$$

Knowing that $\alpha = -2$ this becomes $(4 - \beta + \gamma)\frac{\pi M^2}{h^2}$

The value for mercury within an observational accuracy of 1 per cent is given by $6\pi M^2/h^2$ so that $\gamma - \beta = 2$

Motion of a light ray:

The equations are the same as above except that now $ds = 0$ i.e. the motion is a null geodesic. This means that we take $h \to \infty$ and $k \to \infty$ while $\frac{h}{k}$ remains finite.

$$(1 + \alpha M u + \ldots)(1 + \gamma M u + \ldots)\left(\frac{du}{d\phi}\right)^2 + u^2(1 + \alpha M u + \cdots) = \frac{k^2}{h^2} = \text{finite} \quad \ldots (1$$

Keeping lowest and next lowest order terms

$$\left[1 + (\alpha+\gamma)Mu\right]\left(\frac{du}{d\phi}\right)^2 + u^2(1 + \alpha M u) = \frac{k^2}{h^2}$$

and differentiating

$$2\frac{du}{d\phi}\cdot\frac{d^2u}{d\phi^2}\left[1 + (\alpha+\gamma)uM\right] + (\alpha+\gamma)M\left(\frac{du}{d\phi}\right)^3 + \left[2u + 3\alpha M u^2\right]\frac{du}{d\phi} = 0$$

$$\text{i.e. } \frac{d^2u}{d\phi^2} + u = -(\alpha+\gamma)M\left[u\frac{d^2u}{d\phi^2} + \frac{1}{2}\left(\frac{du}{d\phi}\right)^2\right] - \frac{3}{2}\alpha M u^2 \quad \text{---- (20)}$$

The unperturbed orbit is a straight line, $u_0 = \frac{1}{R}\cos\theta$ where R is the impact parameter. Again proceeding by perturbation

$$\frac{d^2u}{d\phi^2} + u = -\frac{(\alpha+\gamma)M}{R^2}\left(1 - \cos^2\phi + \frac{1}{2}\sin^2\phi\right) - \frac{3}{2}\frac{\alpha M}{R^2}\cos^2\phi$$

$$= -\frac{(\alpha+\gamma)}{2R^2}M + \frac{3M\gamma}{2R^2}\cos^2\phi \quad \text{---- (21)}$$

The approximate solution of this is

$$u = \frac{\cos\phi}{R} + \frac{(\gamma-\alpha)}{2R^2}M - \frac{M\gamma}{2R^2}\cos^2\phi \quad \text{---- (22)}$$

Changing this into cartesian Coordinates.

$$X = \frac{\cos\phi}{u}\,, \quad x^2 + y^2 = \frac{1}{u^2}$$

$$1 = \frac{x}{R} + \frac{(\gamma - \alpha)M}{2R^2}\sqrt{x^2+y^2} - \frac{\gamma M}{2R^2}\frac{x^2}{\sqrt{x^2+y^2}} \quad \cdots (23)$$

or $$x = R + \frac{\gamma M}{2R}\frac{x^2}{\sqrt{x^2+y^2}} - \frac{(\gamma - \alpha)}{2R}M\sqrt{x^2+y^2}$$

Comparing this with the unperturbed solution $x = R$ We see that as $y \to \pm \alpha$ this becomes

$$x \to R - \frac{(\gamma - \alpha)}{2R}M.|y|. \quad \ldots (24)$$

The angle between the symptotes are $\theta = \frac{(\gamma - \alpha)M}{R}$ and the ray is concave towards the sum if $\gamma - \alpha > 0$. The observed value is $\frac{4M}{R}$ so that $\gamma - \alpha - 4$ since $\alpha = -2$, $\gamma = 2$,

Therefore the value of $\beta = .0$

To summarise

1) Motion of orbit requires $\alpha = -2$. This accounts for the red shift.

2) Light deflection require $\gamma = 4 + \alpha = 2$.

3) Mercuty precession therefore requires $\beta = -6 + \alpha^2 - \frac{\alpha\gamma}{2} = 0. \ldots (25)$

In the case of satellite motion the precession due to the bulge of the earth is many times bigger than the general relativity effect and hence could not be tested.

<u>Cosmological Problems:</u>

We start with the field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \quad \ldots (26)$$

where Λ is the cosmological constant. We assume that the Universe is homogeneous, isotropic and static, and regard the stars and galaxies as local fluctuations. Then the ρ_0 the density and P_0 the pressure are ~~perturbed~~ constants and the line element is that due to Schwartzchild:--

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad \text{.... (27)}$$

λ and ν are functions of r only and $\lambda(0) = \nu(0) = 0$

since at the position of the origin the space is locally galiliean

$\therefore T_1^1 = T_2^2 = T_3^3 = P_0$ & $T_4^4 = \rho_0$

These lead to two independent equations:-

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda = 8\pi P_0$$

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda = 8\pi \rho_0 \quad \text{.... (28)}$$

$\dfrac{dP_0}{dr} = -\dfrac{(P_0 + \rho_0)}{2}\nu' = 0$ since P_0 is constant. Three cases are possible--

a) $\nu' = 0$ (b) $\rho_0 + P_0 = 0$ (c) both $\nu' = 0$ & $P_0 + \rho_0 = 0$ ...(29)

These three cases correspond to the Einstien, De sitter and the special relativity line elements. Let us take

a) <u>Einstien line element:</u>

$\nu = 0$ since $\nu' = 0$ and $\nu(0) = 0$ Hence $e^{\nu} = 1$ & $e^{\lambda} \neq 1$

The first equation becomes

$$e^{-\lambda} = 1 - (\Lambda - 8\pi P_0)r^2$$

If you put $R^2 = \dfrac{1}{\Lambda - 8\pi P_0}$ the line element becomes

$$ds^2 = dt^2 - \frac{dr^2}{(1 - r^2/R^2)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Also the second equation plus the first give

$$8\pi(\rho_0 + P_0) = \frac{\lambda' e^{-\lambda}}{r} = -2(8\pi P_0 - \Lambda)$$

$$\Lambda = 4\pi(\rho_0 + 3P_0), \quad \frac{1}{R^2} = 4\pi(\rho_0 + P_0) \quad \text{.... (31)}$$

Thus Λ and R are positive if ρ_0 and P_0 are. The parameter R plays the role of the raius of the Universe, if $R^2 > 0$ it is said to be an open universe, sinc after r = R the line element blows up. If $R^2 < 0$ then it is called a closed universe.

The observed value of ρ_0 is $\rho_0 \simeq 3\times 10^{-29}$ gms/cm^3
This is matter density and radiation density is $\simeq 3\times10^{-34}$ gm/cm^3
and P_0 is negligible. Therefore $\Lambda \simeq \frac{1}{R^2} = 4\pi\rho_0$

or $R = \frac{1}{\sqrt{4\pi\rho_0}} = \sqrt{\frac{c^2}{4\pi G\rho_0}} \simeq 6\times10^{27}$ cm $\simeq 6\times10^{9}$ light years.

And observations of galaxies usually go out to 0.6×10^{9} light years. The effect of the Λ term on the solar system can be found by comparing the $\frac{r^2}{R^2}$ term with $\frac{2m}{r}$ at the orbit of the neptune where $r \simeq 5\times10^{14}$ cms. The ratio is

$$\left(\frac{r^2}{R^2}\right)\Big/\left(\frac{2m}{r}\right) \simeq \left(\frac{r}{R}\right)^2 \times \frac{rc^2}{2GM} \simeq 10^{-17} \quad \text{----} (33)$$

Hence the cosmological term is completely negligible inside the solar system since for a particle at rest,

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{44}\left(\frac{dt}{ds}\right)^2 = 0 \quad \text{----} (34)$$

and $\Gamma^1_{44} = \frac{1}{2}e^{(\nu-\lambda)}\nu'$ and zero for all other components.

Hence $\frac{d^2x^\mu}{ds^2} = 0$ and the particle remains at rest.

Hence there is no intrinsic doppler shift in the model.
Morever $g_{44} = 1$ and hence there is no gravitational shift.

This disagrees with Hubble's observations of a red shift that can be interpreted as Doppler shift with $v = \frac{r}{T}$ where $T = 4\times10^{17}$ sec $= 1.3\times10^{10}$ years. Note however that red shift corresponding to the distance R would be $\frac{\delta\lambda}{\lambda} \simeq \frac{1}{2}$ which agrees with the most distant observed extragalactic nebula. The Einstein universe may be actually regarded to be in equilibrium between Λ repulsion and gravitational attraction due to ρ_0

<u>De Sitter's line element:</u>

We take $\rho_0 + P_0 = 0$ so that ρ_0 and P_0 are separately zero. This universe is empty $4\ \lambda' + \nu' = 0$ or $\lambda = -\nu$ since $\lambda(0) = \nu(0) = 0$. The equation

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda = 8\pi\rho_0 \quad \text{------} (35)$$

can be integrated after multiplication by r^2

$$-re^{-\lambda}+r-\frac{\Lambda r^3}{3}=8\pi\rho_0\frac{r^3}{3}+A \quad \ldots\ldots (36)$$

$$e^{-\lambda}=1-\frac{\Lambda+8\pi\rho_0}{3}r^2-\frac{A}{r}$$

where A = 0 since $\lambda(0)=0$

$$R^2=\frac{3}{\Lambda+8\pi\rho_0}=\frac{3}{\Lambda} \quad \text{since } \rho_0=0$$

and the line element becomes

$$ds^2=\left(1-\frac{r^2}{R^2}\right)dt^2-\frac{dr^2}{\left(1-\frac{r^2}{R^2}\right)}-r^2(d\theta^2+\sin^2\theta\, d\phi^2) \quad \ldots (37)$$

R^2 can be positive or negative according to whether the universe is open or closed. We now must insert test particles (Nebulae) into this empty universe. Since $g_{44}\neq 1$. A particle initially at rest does not stay that way. Thus there is motion without matter in universe instead of matter without motion in Eistien universe. The observable effects of this are obtained by transforming to coordinates such that $g_{44}=1$. This can be done by the transformations of Robertston and Lemartie (Ref. Tolman 1930s).

$$r'=\frac{re^{-t/R}}{\sqrt{1-\frac{r^2}{R^2}}},\quad t'=t+\tfrac{1}{2}R\log\left(1-\frac{r^2}{R^2}\right),\ \theta'=\theta,\ \phi'=\phi \quad \ldots (38)$$

Hence

$$dr'=\frac{\partial r'}{\partial r}.dr+\frac{\partial r'}{\partial r}.dt=\frac{e^{-t/R}}{\left(1-\frac{r^2}{R^2}\right)^{3/2}}dr-\frac{(r/R)\,e^{-t/e}}{\sqrt{1-r^2/R^2}}dt$$

$$dt'=-\frac{r/R}{1-r^2/R^2}dr+dt$$

$$e^{-2t'/R}dr'^2+dt'^2=-\frac{dr^2}{(1-r^2/R^2)}+dt^2\left(1-\frac{r^2}{R^2}\right)$$

$$r^2(d\theta^2+\sin^2\theta\, d\phi^2)=r'^2e^{2t'/R}\left[d\theta'^2+\sin^2\theta' d\phi'^2\right]$$

$$ds^2=dt'^2-e^{2t'/R}\left[dr'^2+r'^2(d\theta'^2+\sin^2\theta' d\phi'^2)\right] \quad \ldots (39)$$

In these coordinates a particle initially at rest remains at rest. The metric is isotropic and light ray move along straight paths.

We assume that $\Lambda > 0$ and $R^2 > 0$ so R is real. Dropping the primes the length scale changes like $e^{t/R}$ and the metric is isotropic t is universal time and so all the clocks have the same proper time. The proper distance from the origin to r is given by $r e^{t/R}$, which changes steadily with t.

Now consider a light ray that starts from $x_0 > 0$ at t_0 and moves to the left. The trajectory is given by $ds = 0$,

$$dy = dz = 0 \text{ or } R > 0 \;, \quad \frac{dx}{dt} = -e^{-t/R} \qquad \ldots (40)$$

Integrating this we have

$$x = x_0 + R\left(e^{-t/R} - e^{-t_0/R}\right)$$

Thus this ray may never get to X = 0 unless $x_0 < R e^{-t_0/R}$

Hence the farthest proper distance of the source an observer can see at the emission time 't_0' is $x_0 e^{t_0/R} = R$. The light emitted from x_0 at t_0 arrives at $x = 0$ at time given by

$$0 = x_0 + R\left(e^{-t_1/R} - e^{-t_0/R}\right)$$

$$0 = -\Delta t_1 e^{-t_1/R} + \Delta t_0 e^{-t_0/R}$$

$$\Delta t_1 = \Delta t_0 e^{(t_1 - t_0)/R} \qquad \ldots (41)$$

since t is the proper time of the clock.

$$\frac{\lambda + \delta\lambda}{\lambda} = \frac{\Delta t_1}{\Delta t_0} = e^{(t_1 - t_0)/R}$$

Now the proper distance of the source at the time of the observation (Not emission) i.e.

$$d_1 = x e^{t_1/R} = R\left(e^{(t_1 - t_0)/R} - 1\right)$$

$$\frac{\delta\lambda}{\lambda} = \frac{d_1}{R} \qquad \ldots (42)$$

This is in agreement with Hubble's observation. Also in terms of time of emission,

$$d_0 = x_0 e^{t_0/R} = R\left(1 - e^{(t_0 - t_1)/R}\right)$$

$$\frac{\nu + \delta\nu}{\nu_0} = \frac{\Delta t_0}{\Delta t_1} = e^{t_0 - t_1} = 1 - \frac{d_0}{R};$$

$$\frac{\delta\nu}{\nu} = \frac{d_0}{R} \qquad \ldots\ldots\ldots \quad (43)$$

WEYL'S POSTULATE

In the foregoing we solved the field equations under certain assumptions, Weyl prepared a general model, which makes certain physical assumptions, and leads directly to a choice of metric. Then the field equations can be used to see what $T_{\mu\nu}$ correspond to these metrics.

Weyl assumed that the 'particles' of the universe the nubulae, lie in space time on a bundle of geodesics that diverge from a point in the finitely or infinitely distant past. Thus the geodesics behave like streamlines, and do not intersect except for a singular point in the distant past and possibly in the distant future. At each point in space time there is a unique velocity. This model agrees well with observation since random velocities of nebulae are of order 100 km/sec. while, the expansion goes up to velocities upto .4 C.

We now introduce the cosmological principle according to which the universe presents the same statistical aspect to all observers, regardless of position. It may be shown that Weyl's postulate implies that coordinates may be chosen such that t = constant hyper surfaces are orthogonal to the geodesies which correspond to (x^1, x^2, x^3) = constant, It follows then that

$$ds^2 = dt^2 - h_{ij}\, dx^i\, dx^j \quad (i, j = 1, 2, 3 \ldots) \quad \ldots\ldots (1)$$

Here t is a universal time, since $g_{44} = 1$.

The cosmological principle now tells that h_{ij} can depend on t only through a scale factor $h_{ij} = f(t) . l_{ij}$

Finally since $l_{ij}\, dx^i\, dx^j$ describe a homogeneous isotropic, time dependent 3 space it can be shown to have constant curvature. Robertson and Walker[(4)] (1939) showed that the most gener metric is $ds^2 = dt^2 - \frac{f^2(t)}{(1+kr^2)^2}(dx^2+dy^2+dz^2)$,

k a constant and $r^2 = x^2+y^2+z^2$ (2)

The sign of the 3 space curvature is the same as that of k.

We now see how this line element fits in with the field equations. $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu}$ (3)

We use rectangular coordinates and write down only the typical non vanishing coponents. Remembering that there is isotropy between x, y, z $\Gamma^1_{11} = \Gamma^2_{12} = -\frac{2kx}{1+kr^2}$, $\Gamma^1_{22} = \frac{2kx}{1+kr^2}$, $\Gamma^1_{14} = \frac{\dot{f}(t)}{f(t)}$

and $\Gamma^4_{11} = f\dot{f}/(1+kr^2)^2$

$$R_{11} = R_{22} = R_{33} = -\frac{8k + f\ddot{f} + 2\dot{f}^2}{(1+kr^2)^2}, \quad R_{44} = \frac{3\ddot{f}}{f}$$

$$R^1_1 = g^{11}R_{11} + g^{12}R_{12} + g^{13}R_{13} + g^{14}R_{14} = \frac{8k + f\ddot{f} + 2\dot{f}^2}{f^2}$$

and this equals R^2_2 and R^3_3. Also $R^4_4 = 3\ddot{f}/f$

$$R = R^1_1 + R^2_2 + R^3_3 + R^4_4 = \frac{6(4k + f\ddot{f} + \dot{f}^2)}{f^2} \qquad (4)$$

Note that the 3 space curvature is obtained by calculating R_{ij} as though f were constant, it is $\frac{24k}{f^2}$ and hence the sign of k is as expected.

The field equations in mixed form gives

$$4k + 2f\ddot{f} + \dot{f}^2/f^2 + \Lambda = 8\pi p_0$$
$$-3(4k + \dot{f}^2)/f^2 + \Lambda = -8\pi \rho_0 \qquad (5)$$

Thus in general p_0 and ρ_0 are functions of t only. The De Sitter metric is a special case with $p_0 = \rho_0 = 0$

Then $f^2\Lambda = 4k + 2f\ddot{f} + \dot{f}^2$

$f^2\Lambda = 12k + 3\dot{f}^2$

Eliminating k $f^2\Lambda = 3f\ddot{f}$; $\ddot{f} - \frac{\Lambda}{3}f = 0$

(or) $f(t) =$

$$Ae^{\sqrt{\frac{\Lambda}{3}}t} + Be^{-\sqrt{\frac{\Lambda}{3}}t} \text{ where } AB = \frac{3k}{\Lambda} \qquad (6)$$

The expanding De Sitter case corresponds to A = 1. B = 0, where $k = 0$ and $\frac{1}{R} = \sqrt{\frac{\Lambda}{3}}$ as previously. Various non-static soluti can be obtained from the above equations. When they expand they tend asymptotically to the de sitter case since ρ_0 and P_0 becom arbitrarily small.

One general thermodynamic result follows, from the equa tion, consider a volume element $dv = dx\,dy\,dz$. The proper vo element is $\frac{f^3 dv}{(1+kr^2)^3} = dv_p$. We now show that the relation

$$\delta(dE) + P_0\,\delta(dv_p) = 0$$

is satisfied where $dE = \rho_0 dv_p$ is the energy contained in dv and δ represents a small change in time

$$\delta(dE) = \delta\left[\frac{\rho_0 f^3 dv}{(1+kr^2)^3}\right] = dv_p\,\delta\rho_0 + \rho_0\,\delta(dv_p) \qquad (8$$

$$= dv_p \cdot \delta\rho_0 + \frac{3\dot{f}}{f}\delta t \cdot \rho_0 \cdot dv_p$$

$$\delta\rho_0 = \delta t \cdot \frac{d\rho_0}{dt} = \frac{3}{8\pi}\delta t\left[\frac{2\dot{f}\ddot{f}}{f^2} - \frac{2\dot{f}(4k+\dot{f}^2)}{f^3}\right]$$

$$\delta(dE) + \rho_0\,\delta(dv_p) = \delta t \cdot dv_p\left\{\frac{3}{8\pi}\left[\frac{2\dot{f}\ddot{f}}{f^2} - \frac{2\dot{f}(4k+\dot{f}^2)}{f^3}\right] + 3\dot{f}/f\,(\rho_0 + P_0)\right\}$$

However $\rho_0 + P_0 = \frac{1}{8\pi}\,\frac{8k - 2f\ddot{f} + 2\dot{f}^2}{f^2}$

so that this expression is actually zero. Thus the energy loss pe unit volume that accompanies the thinning out of the material duri the expansion goes into work done by the pressure on the outside universe.

If we neglect the pressure compared to ρ_0 we have that $\delta(dE) = 0$ or $\rho_0 f^3 = \text{Constant} = C$. We then get the simple differential equation for f,

$$\dot{f}^2 = \frac{8\pi C}{3f} - 4k + \Lambda\frac{f^2}{3}$$

Bondi shows that this equation arises also in Newtonian cosmology, and discusses numerical solution in several cases. Thus for low pressure systems there is little formal difference between the Newtonian and general relativistic case although the interpretation of some of the parameters is different. Further there is little difference in dynamics in the two theories if the gravitational potential energy of a particle is small compared to its rest energy.

Bondi[(5)] discusses the relation between various non-static models and observation.

The steady state theory:

Bondi[(6)] and Gold in 1948 said that in physics we can distinguish between inherent and accidental phenomena by varying the former and repeating observation. But in cosmology there is only one universe. So we could not reasonably say what things would be like if they were radically changed.

Their starting point is not general relativity, but the 'perfect' cosmological principle. This asserts that the universe presents the same statistical aspects, not only at all positions but also at all times. They also assume isotropy. Observations not only show that the universe explodes but also an enormous lack of thermodynamic equilibrium. For example there is much less radiation than needed for equilibrium, with the matter that is present. This means that the universe is running down. But if it is in a steady state something of low entropy must continually be added.

The rate of addition of material can be found by considering moderate sized sphere

$$\frac{4\pi}{3} r^2 \frac{d\rho}{dt} = 4\pi r^2 v \rho_0 \quad \text{——} \quad \text{--(matter going out)} \qquad (4)$$

$$\frac{d\rho}{dt} = \frac{3\rho_0 v}{r} = \frac{3\rho_0}{T} = \frac{3 \times 3 \times 10^{-29}}{1.3 \times 10^{10}} = 7 \times 10^{-39}\ gm/cm^3/sec$$

which corresponds to one hydrogen atom being created per cubic meter, on the average in 2 x 10^8 years. This is far too small to observe directly in the laboratory. So there is no contradiction with physics.

All of this in agreement with Weyl's postulate. However Robertson and Walker metric leads to a 3 space curvature $3k/f^2$ which changes with time, since the curvature is observable in principle, the perfect cosmological principle requires $k = 0$. We thus arrive at the line element

$$ds^2 = dt^2 - f^2(t)\,(dx^2 + dy^2 + dz^2) \qquad (5)$$

Light propagation is that governed by $\frac{dx}{dt} = \pm \frac{1}{f(t)}$ (6)

For a ray starting at x_0 at t_0 and running into $x = 0$

$$x = x_0 - \int_{t_0}^{t} \frac{dt'}{f(t')} \quad \text{- - - . . . . .} \qquad (7)$$

So that there is a horizon if $\int_0^\infty \frac{dt}{f(t)}$ converges

If the ray arrives at $x = 0$ at time t,

$$x_0 = \int_{t_0}^{t_1} \frac{dt}{f(t)} \simeq \frac{t_1 - t_0}{f}$$ if the distance is not too great. The proper distance is then $d = f.\,x_0 = t_1 - t_0$. (8)

To calculate the red shift $x = \int_{t_0}^{t_1} \frac{dt}{f(t)}$

$$0 = \frac{\Delta t_1}{f(t_1)} - \frac{\Delta t_0}{f(t_0)}$$

$$\frac{\Delta t_1}{\Delta t_0} = 1 + \frac{\delta}{\lambda} = \frac{f(t_1)}{f(t_0)} \simeq \frac{f(t_0) + (t_1 - t_0)\dot{f}(t_0)}{f(t_0)}$$

$$\boxed{\frac{\delta\lambda}{\lambda} \simeq (t_1 - t_0)\frac{\dot{f}}{f} = d \times \frac{\dot{f}}{f}} \qquad (9)$$

Thus T or R is equal to $f/\dot{f}$

The perfect cosmological principle says that this is a constant so that $f = e^{t/T}$. We thus get the De sitter line element. However it now need not correspond to an empty universe since general relativity is not accepted on the cosmological scale; for example $T_{\mu\nu}$ is not conserved, since matter is continuously created.

Thus the steady state theory predicts everything, except the value of T. Nuclear counts should give the same proper density everywhere, and thus far this seems to be the case. Also nebulae of all ages are present in this case.

It is assumed that matter is created everywhere regardless of what matter is present. It is assumed to be neutral since otherwise electrostatic forces would dominate. If separate protons, electrons and noucleons are created there would be almost more radiation from and heating of intergalactic matter than observed. Thus hydrogen atoms are favoured. Other atoms seem unlikely, since hydrogen seems to be the main constituent of the universe. The observed element distributions can be explained without recourse to an initial calastrophic formation (Bursbidge, Bursbrigle Fouler, Hoyle, R.M.P. 1957, Hoyle Annals of Physics 1960).

It is also assumed that matter is created at rest in the coordinate system given above. This system can be found by observation since only to such an observer does the universe look isotropic. There could be some random motion but not much or the intergalactic temperature will be too high.

In a sense energy is conserved in this model since matter is continually produced and matter and radiation (strongly Doppler shifted) are continually being pushed over the horizon. Bondi and

Gold say that it is hydrodynamic continuity that is violated. Also Baryon and Lepton numbers are not conserved since there is a preference for normal matter over antimatter. This is a consequence of the perfect cosmological principle since otherwise there would be large scale annihilation and the universe will run down.

Hoyle's theory:

Hyle[7] has attempted to put the steady state theory, on the basis of field equations, rather than the perfect cosmological principle. In effect he replaces the term by an equally small non-conservative term.

Hoyle introduces a time like vector of constant length α at each point. It points towards future time along the geodesic through that point. $C_\mu = (0, 0, 0, \alpha)$

From this a symmetrical Tensor field is derived

$$C_{\mu\nu} = \frac{\partial C_\mu}{\partial x^\nu} - \Gamma_{\mu\nu}^{\alpha} C_\alpha$$

by covaraint differentiation. We adopt the metric of the steady state theory.

$$ds^2 = dt^2 - f^2(t)\,(dx^2 + dy^2 + dz^2) \quad \text{.....} \ (11)$$

Now $\dfrac{\partial C_\mu}{\partial x^\nu} = 0$ and Γ' s are given with K = 0. The non-vanishing ones are

$$\Gamma_{14}^{1} = \Gamma_{24}^{2} = \Gamma_{34}^{3} = \frac{\dot f}{f}, \ \Gamma_{11}^{4} = \Gamma_{22}^{4} = \Gamma_{33}^{4} = f\dot f \quad \text{....} \ (12)$$

The only non-vanishing components of $C_{\mu\nu}$ are

$$C_{11} = C_{22} = C_{33} = -\alpha f \dot f \quad \text{........} \ (13)$$

We generalise the field equations:-

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + C_{\mu\nu} = -8\pi T_{\mu\nu} \quad \text{.......} \ (14)$$

where $R_{11} = R_{22} = R_{33} = -(f\ddot{f} + 2\dot{f}^2),\ R_{44} = \frac{3\ddot{f}}{f}$ .... (15)

In mixed form we have

$$R_1^1 = R_2^2 = R_3^3 = \frac{f\ddot{f} + 2\dot{f}^2}{f^2},\ R_4^4 = \frac{3\ddot{f}}{f},\ R = \frac{6(f\ddot{f} + \dot{f}^2)}{f^2}$$

$C_1^1 = C_2^2 = C_3^3 = \alpha \dot{f}/f,\ C_4^4 = 0$ .... (16) The equations are

$$-\frac{2f\ddot{f} + \dot{f}^2}{f^2} + \frac{\alpha\dot{f}}{f} = 8\pi P_0$$

$$-3\dot{f}^2/f^2 = -8\pi\rho_0$$

We solve the first equation neglecting pressure

$$2f\ddot{f} + \dot{f}^2 = \alpha f\dot{f}$$

$$\frac{1}{\dot{f}}\frac{d}{dt}(f\dot{f}^2) = \alpha f\dot{f}$$

$$\frac{d}{dt}(f\dot{f}^2) = \alpha f\dot{f}^2,\ f\dot{f}^2 = A^2 e^{\alpha t}.$$

$$f^{1/2}\frac{df}{dt} = Ae^{\frac{1}{2}\alpha t},\ \frac{2}{3}f^{3/2} = \frac{2A}{\alpha}e^{\frac{1}{2}\alpha t} + B \quad .... (17)$$

The second equation then gives

$$\rho_0 = \frac{3}{8\pi}\frac{A^2 e^{\alpha t}}{\left(3A/\alpha\, e^{1/2\alpha t} + 3/2\, B\right)^2} \quad .... (18)$$

For $\alpha > 0$ this approaches a positive constant as $t \to \infty$. Thus for long times we might as well take B = 0 and make

$$f(t) = \left(\frac{3A}{\alpha}\right)^{2/3} e^{\frac{1}{3}\alpha t};\ \rho_0 = \frac{\alpha^2}{24\pi} \quad ...... (19)$$

Now let us determine duly the zero of time, so choising $\frac{3A}{\alpha} = 1$ we get the de sitter metric again. Where now

$$\alpha = \frac{3}{T} \text{ and } \rho_0 = \frac{3}{8\pi T^2} \quad .... (20)$$

Thus the present theory relates ρ_0 and T which no other theory does.

with $T = 1{\cdot}3 \times 10^{10}$ years $\rho_0 = \frac{3}{8\pi T^2 G} = 1{\cdot}1 \times 10^{-29}$ gm/cm^3 .... (21)

which is remarkably close to the observed value. This relation between ρ_0 and T is very much like the relation ρ_0 and R in the Einstein case. However that case does not admit of red shift. Finally we see that $T_{\mu\nu}$ is not conserved. From the field equation

$$C^{\mu\nu}_{,\nu} = -8\pi T^{\mu\nu}_{,\nu} \qquad \ldots\ldots(22)$$

That this is not zero, is seen directly

$$C^{\mu\nu}_{,\nu} = \frac{\partial C^{\mu\nu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\alpha\nu} C^{\alpha\nu} + \Gamma^{\nu}_{\alpha\nu} C^{\mu\alpha}$$

$$C^{1\nu}_{,\nu} = C^{2\nu}_{,\nu} = C^{3\nu}_{,\nu} = 0, \quad C^{4\nu}_{,\nu} = -\frac{3\alpha\dot{f}^{2}}{f^{2}} \qquad \ldots(23)$$

From our solution, the last is

$$C^{4\nu}_{,\nu} = -9/T^{3}$$

Thus

$$T^{4\nu}_{,\nu} = \frac{9}{8\pi T^{3}} = \frac{3\rho_0}{T} \qquad \ldots\ldots\ldots(24)$$

The sign corresponds of course to the creation of matter. Finally the $C_{\mu\nu}$ term in the field equations is of the same order of magnitude as the former $\Lambda g_{\mu\nu}$ term and hence has no observable effect on solar system dynamics. (Merea PRS 206, 562, 1957).

C) Field inside a rotating shell:-

Mach's principle. Thirring calculated[(8)] ... iety. 19,

the field inside of a massive rotating shell. This was a crude model of Mach's principle, which showed that rotating shell, (Universe) produces effects like the centrifugal and corioli forces that would occur if the universe were at rest and the internal system were rotating.

For the rotating mass

$$T^{\mu\nu} = \rho_0 u^{\nu} u^{\mu}, \quad u^{\mu} = \frac{dx^{\mu}}{ds} \qquad \ldots(25)$$

where pressure is neglected, we need ~~the~~ metric only to zero order

$$ds^2 = \delta_{\mu\nu}\, dx^\mu\, dx^\nu$$

$$\left(\frac{ds}{dt}\right)^2 = (1 - v^2), \quad v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

$$u^1 = \frac{v_x}{\sqrt{1-v^2}} \; \ldots \quad u^4 = \frac{1}{\sqrt{1-v^2}}$$

For rotating about the Z axis

$$v = \bar{\omega} \times \bar{r} \text{ or } v_x = -\omega y, \quad v_y = \omega x, \quad v_z = 0$$
$$v^2 = \omega^2(x^2+y^2) \qquad (26)$$

For the lowest order $T_{\mu\nu} = \pm T^{\mu\nu}$ and $T_{11} = \frac{\rho_0 \omega^2 y^2}{1-v^2} \simeq \rho_0 \omega^2 y^2$ through the terms of order which we must keep to get a centrifugal force.

Also $T_{22} = \rho_0 \omega^2 x^2$; $T_{12} \simeq -\rho_0 \omega^2 xy$, $T_{44} = \frac{\rho_0}{1-v^2} = \rho_0(1+v^2)$

$$T_{14} = -T^{14} = \frac{\rho_0 \omega y}{1-v^2} \simeq \rho_0 \omega y, \quad T_{24} \simeq -\rho_0 \omega x \qquad (27)$$

others zero.

$$T = T_1^1 + T_2^2 + T_3^3 + T_4^4 \simeq \rho_0(1-v^2) - \rho_0\omega^2 y^2 - \rho_0 \omega^2 x^2$$
$$= \rho_0 \ldots (28)$$

For stationary $T_{\mu\nu}$, ρ_0 must be axially symmetric. Then with $\frac{\partial h_{\mu\nu}}{\partial t} = 0$ we get

$$\nabla^2 h_{\mu\nu} = 16\pi \left(T_{\mu\nu} - \tfrac{1}{2} T \delta_{\mu\nu}\right) \text{ with}$$

$$T_{11} - \tfrac{1}{2} T\delta_{11} = \tfrac{1}{2}\rho_0(1 + 2\omega^2 y^2)$$

$$T_{22} - \tfrac{1}{2} T\delta_{22} = \tfrac{1}{2}\rho_0(1 + 2\omega^2 x^2)$$

$$T_{33} - \tfrac{1}{2} T\delta_{33} = \tfrac{1}{2}\rho_0$$

$$T_{44} - \tfrac{1}{2} T\delta_{44} = \tfrac{1}{2}\rho_0\left[1 + 2\omega^2(x^2+y^2)\right]$$

$$T_{12} - \tfrac{1}{2} T\delta_{12} = \rho_0 \omega^2 xy$$

$$T_{14} - \tfrac{1}{2} T\delta_{14} = \rho_0 \omega y$$

$$T_{24} - \tfrac{1}{2} T\delta_{24} = -\rho_0 \omega x \qquad \text{others zero} \ldots (29)$$

For simplicity assume that ρ_0 is a function of r only. Then each component of $16\pi\left(T_{\mu\nu} - \frac{1}{2}T\delta_{\mu\nu}\right)$ can be written in the form

$$b_0(r) + \sum_{m=-1}^{1} b_{1m}(r)\, y_{1m}(\theta,\phi) + \sum_{m=-2}^{2} b_{2m}(r)\, y_{2m}(\theta,\phi) \tag{30}$$

so that the corresponding components of $h_{\mu\nu}$ can be written as

$$a_0(r) + \sum_1 a_{1m}(r)\, y_{1m}(\phi,\theta) + \sum_1 b_{2m}(r)\, y_{2m}(\theta,\phi) \tag{31}$$

Now $X = r\sin\theta\cos\phi$ and $Y = r\sin\theta\sin\phi$ are equal to r times linear combination of Y_{1m}. Also $xy = r^2\sin^2\theta\cos\phi\sin\phi$ is r^2 times a linear combination of y_{2m}

Finally $\frac{x^2}{y^2} = \sin^2\theta\cos^2\phi = \frac{1}{2}\sin^2\theta\cos 2\phi - \frac{1}{6}(3\cos^2\theta - 1) + \frac{1}{3}$

$$\frac{y^2}{r^2} = \sin^2\theta\sin^2\phi = -\frac{1}{2}\sin^2\theta\cos 2\phi - \frac{1}{6}(3\cos^2\theta - 1) + \frac{1}{3} \tag{32}$$

In each case the first two terms are y_{2m}. Thus we get for each component a set of equation like $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}a_0\right) = b_0$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{da_{1m}}{dr}\right) - \frac{2}{r^2}a_{1m} = b_{1m}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{da_{2m}}{dr}\right) - \frac{6}{r^2}a_{2m} = b_{2m} \tag{33}$$

We [illegible]cialize to a thin shell of mass M and radius R

$\rho_0(r) = \frac{M}{4\pi R^2}\,\delta(r-R)$ Analytical equation is then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}a_\ell\right) - \frac{\ell(\ell+1)}{r^2}a_\ell \doteq b_\ell\,\delta(r-R) \tag{34}$$

for $r < R$, $a_\ell = A_\ell r^\ell$

$r > R$, $a_\ell = B_\ell / r^{\ell+1}$

integrating $\quad r = R$

$$\lim_{\epsilon \to 0} \int_{R-\epsilon}^{R+\epsilon} \frac{d}{dr}\left(r^2 \frac{da_l}{dr}\right) dr = R^2 b_l$$

$$= R^2 \left[\left(\frac{da_l}{dr}\right)_{R+\epsilon} - \left(\frac{da_l}{dr}\right)_{R-\epsilon}\right]$$

$$= R^2 \left[- \frac{(l+1)}{R^{l+2}} B_l - l A_l R^{l-1}\right] \qquad (35)$$

Continuity at $r = R$ requires $A_l R^l = B^l / R^{l+1}$

$$B_l = A_l R^{2l+1}$$

and

$$A_l = - b_l / (2l+1) R^{l-1}$$

$$a_l = - \frac{b_l R}{(2l+1)} \left(\frac{r}{R}\right)^l \text{ for } r > R \qquad (36)$$

We can now write down $h_{\mu\nu}$

$$h_{11} = - \frac{2M}{R} - 4M\omega^2 R \left[\frac{1}{5}\left(\frac{x}{R}\right)^2 - \frac{1}{15}\left(\frac{r}{R}\right)^2 + \frac{1}{3}\right]$$

$$h_{22} = - \frac{2M}{R} - 4M\omega^2 R \left[\frac{1}{5}\left(\frac{x}{R}\right)^2 - \frac{1}{15}\left(\frac{r}{R}\right)^2 + \frac{1}{3}\right]$$

$$h_{33} = - \frac{2M}{R}$$

$$h_{44} - \frac{2M}{R} - 4M\omega^2 R \left[\frac{1}{5} \frac{x^2+y^2}{R^2} - \frac{2}{15}\left(\frac{r}{R}\right)^2 + \frac{2}{3}\right]$$

$$h_{12} = \frac{4M\omega^2 R}{5} \times \frac{xy}{R} \qquad (37)$$

$$h_{24} = \frac{4M\omega}{3} \times \frac{x}{R}$$

Others zero.

From these we compute the acceleration of a particle

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \cdot \frac{dx^\beta}{ds} = 0 \qquad (38)$$

To the lowest order this becomes to first order in velocities

$$\frac{d^2 x^i}{dt^2} \simeq \Gamma^i_{44} - 2\Gamma^i_{4j} \frac{dx^j}{dt}$$

$$\Gamma^i_{44} = \frac{1}{2} g^{i\alpha}\left(2\frac{\partial g_{\alpha 4}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^\alpha}\right) = -\frac{1}{2} g^{i\alpha} \frac{\partial g_{44}}{\partial x^\alpha}$$

$$\ldots (39)$$

Since the g's do not depend on t also:-

$$\Gamma^{i}_{4j} = \frac{1}{2} g^{i\alpha}\left(\frac{\partial g_{\alpha 4}}{\partial x^{j}} + \frac{\partial g_{\alpha j}}{\partial x^{4}} - \frac{\partial g_{4j}}{\partial x^{\alpha}}\right)$$

$$\Gamma^{1}_{44} = -\frac{1}{2} \cdot 4M\omega^{2}R \cdot \frac{2x}{15R^{2}} = -\frac{4M\omega^{2}x}{15R}$$

$$\Gamma^{2}_{44} = -\frac{4M\omega^{2}y}{15R} \;;\quad \Gamma^{3}_{44} = \frac{8M\omega^{2}z}{15R}$$

$$\Gamma^{i}_{4j} = \frac{1}{2}\left(\frac{\partial g_{4j}}{\partial x^{i}} - \frac{\partial g_{4i}}{\partial x^{j}}\right)$$

$$\Gamma^{1}_{24} = \frac{1}{2} \times \frac{4M\omega}{3R} \times 2 = \frac{4M\omega}{3R} \qquad (40)$$

$$\Gamma^{2}_{14} = -\frac{4M\omega}{3R}$$

We get

$$\frac{d^{2}x}{dt^{2}} = \frac{4M\omega^{2}}{15R} - \frac{8M\omega}{3R}\frac{dy}{dt}$$

$$\frac{d^{2}y}{dt^{2}} = \frac{4M\omega^{2}y}{15R} + \frac{8M\omega}{3R}\frac{dx}{dt}$$

$$\frac{d^{2}z}{dt^{2}} = -\frac{8M\omega^{2}R}{15R} \qquad (41)$$

For comparison a particle in a system rotating in the opposite direction with angular velocity would experience the following acceleration

$$\frac{d^{2}x}{dt^{2}} = \omega^{2}x - 2\omega\frac{dy}{dt}$$

$$\frac{d^{2}y}{dt^{2}} = \omega^{2}y + 2\omega\frac{dx}{dt}$$

$$\frac{d^{2}z}{dt^{2}} = 0$$

Thus the shell is less effective than the universe by a factor of order $\left(\frac{M}{R}\right)$. For actual universe

$$\frac{M}{R} \rightarrow \int_0^T \frac{4\pi\rho_0 r^2 dr}{r} = 2\pi\rho_0 T^2;$$

We use Hoyle's expression

$$2\pi\rho_0 T^2 = 3/4.$$

This gives a kind of intutive justification for Mach's principle.

We shall now think of a spinning article in a gravitational field and try to calculate the change in the spin axis during its free fall in a gravitational field. This can prove to be a new test of the general theory of relativity.

EQUATION OF MOTION OF MASS POINT

Papapetru (9) starts from $T^{\alpha\beta}_{\ \ ,\beta}=0$ for material test object using the earth centred non-rotating coordinate system with Schuratzchild metric. He chooses a world line X^{α} to specify the motion of C.M. of the object (choice not unique) X^{i} are functions of $x^4 = t$ or of s along the world line. With $\delta x^{\alpha} = x^{\alpha} - x^{\alpha}$, $T^{\mu\nu}$ is concentrated in a narrow tube around each path so that we can expand in terms of integrals like $\int T^{\mu\nu}\, dv$, $\int \delta x^{\alpha}\, T^{\mu\nu}\, dV$, $\int \delta x^{\alpha}\, \delta x^{\beta}\, T^{\mu\nu}\, dv$ etc. the integration being over 3 space for t = constant.

Pole particle:- Only $\int T^{\mu\nu}\, dv \neq 0$, all other moments are zero.

Pole-dipole particle (spinning) only this $\int \delta x^{\alpha}\, T^{\mu\nu}\, dv \neq 0$.

Consider a pole particle but include non-gravitational constraint F^{α} acting on the centre of mass.

$$T^{\alpha\beta}_{\ \ ,\beta} = \frac{F^{\alpha}}{u^4}\, \delta\left[x^1 - X^1(t)\right] \delta\left[x^2 - X^2(t)\right] \delta\left[x^3 - X^3(t)\right]$$

the force acting at the point X here $u^{\alpha} = \frac{dx^{\alpha}}{ds}$ where s is along the world line of Centre of Mass not necessarily a geodesic.

Assume $u^{\alpha} F_{\alpha} = 0$

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\mu\nu}\, T^{\mu\nu} = \frac{F^{\alpha}}{u^4}\, \delta_1 \delta_2 \delta_3, \qquad (1)$$

consider:-

$$\frac{\partial}{\partial x^{\gamma}}\left(x^{\alpha} T^{\beta\gamma}\right) = x^{\alpha}\frac{\partial T^{\beta\gamma}}{\partial x^{\gamma}} + T^{\beta\alpha} \qquad (2)$$

with (1) this becomes

$$\frac{\partial}{\partial x^{\sigma}}\left(x^{\alpha} T^{\beta\sigma}\right) = T^{\beta\alpha} - x^{\alpha}\Gamma^{\beta}_{\mu\nu}\, T^{\mu\nu} + x^{\alpha}\frac{F^{\beta}}{u^4}\, \delta_1 \delta_2 \delta_3. \qquad (3)$$

Integrate (1) over v for fixed t. Since $\int \frac{\delta T''}{\delta x^{\prime}_{\alpha}}$ etc; vanish

$$\frac{d}{dt}\int T^{\alpha 4}dv = -\int \Gamma^{\alpha}_{\mu\nu} T^{\mu\nu} dv + \frac{F}{u^4}$$

where F^{α} is evaluated at X also . integrating (3)

$$\frac{d}{dt}\int X^{\alpha} T^{\beta 4} dv = \int T^{\alpha\beta} dv - \int X^{\alpha}\Gamma^{\beta}_{\mu\nu} T^{\mu\nu} dv + \frac{X^{\alpha}F^{\beta}}{u^4}$$

For a pole particle X^{α} and $\Gamma^{\alpha}_{\mu\nu}$ can be removed, from integrals and evaluated on world line:- This can be done by expanding X^{α} & $\Gamma^{\prime\alpha}_{\mu\nu}$ about C.M.

$$\frac{d}{dt}\int T^{\alpha 4} dv + \Gamma^{\alpha}_{\mu\nu}\int T^{\mu\nu} dv = \frac{F^{\alpha}}{u^4} \tag{5}$$

and (4):

$$\frac{dX^{\alpha}}{dt}\int T^{\beta 4} dv + X^{\alpha}\frac{d}{dt}\int T^{\beta 4} dv = \int T^{\alpha\beta} dv - X^{\alpha}\Gamma^{\beta}_{\mu\nu}\int T^{\mu\nu} dv + \frac{X^{\alpha}F^{\beta}}{u^4} \tag{6}$$

Take (6) minus X^{α} times. (5) with $\alpha \to \beta$

$$\int T^{\alpha\beta} dv = \frac{dx^{\alpha}}{dt}\int T^{\beta 4} dv \tag{7}$$

Define $M^{\alpha\beta} = u^4 \int T^{\alpha\beta} dv$

Then 5 becomes

$$\frac{d}{ds}\left(\frac{M^{\alpha 4}}{u^4}\right) + \Gamma^{\alpha}_{\mu\nu} M^{\mu\nu} = F^{\alpha} \tag{8}$$

and (7) becomes

$$M^{\alpha\beta} = \frac{u^{\alpha} M^{\beta 4}}{u^4} \tag{9}$$

Put $\beta = 4$ in (9)

$$M^{\alpha 4} = \frac{u^{\alpha} M^{44}}{u^4}$$

and putting this back in (9)

$$M^{\alpha\beta} = m u^{\alpha} u^{\beta}, \quad m = \frac{M^{44}}{(u^4)^2} = \frac{1}{u^4}\int T^{44} dv$$

Then form 8

$$\frac{d}{ds}(m u^{\alpha}) + m\,\Gamma^{\alpha}_{\mu\nu} u^{\mu} u^{\nu} = F^{\alpha} \tag{10}$$

We now obtain an identity starting from

$$u_\alpha u^\alpha = g_{\alpha\beta} u^\alpha u^\beta = \frac{ds^2}{ds^2} = 1$$

$$0 = u_\alpha \frac{d}{ds} u^\alpha + u^\alpha \frac{d}{ds} u_\alpha = u_\alpha \frac{d}{ds} u^\alpha + u^\alpha \frac{d}{ds} (g_{\alpha\beta} u^\beta)$$

$$= u_\alpha \frac{d}{ds} u^\alpha g_{\alpha\beta} \frac{du}{ds} + 2 u^\alpha u^\beta u^\gamma g_{\alpha\mu} \Gamma^\mu_{\beta\nu}$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\epsilon} \left(\frac{\partial g_{\epsilon\mu}}{\partial x^\nu} + \frac{\partial g_{\epsilon\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\epsilon} \right)$$

$$g_{\alpha\mu} \Gamma^\mu_{\beta\nu} = \frac{1}{2} \left(\frac{\partial g_{\beta\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\beta\nu}}{\partial x^\alpha} \right)$$

$$\therefore 2 g_{\alpha\mu} \Gamma^\mu_{\beta\nu} u^\alpha u^\beta u^\nu = u^\alpha u^\beta u^\nu \frac{\partial g_{\beta\alpha}}{\partial x^\nu}$$

$$= u^\alpha u^\beta \frac{\partial g_{\beta\alpha}}{\partial s}$$

$$u_\alpha \frac{du^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} u_\alpha u^\mu u^\nu = 0 \quad \ldots (11)$$

Equation (10) becomes

$$u^\alpha \frac{dm}{ds} + m \frac{du^\alpha}{ds} + m \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = F^\alpha$$

or multiplying by u_α and using (11)

$$\frac{dm}{ds} = u_\alpha F^\alpha = u^\alpha F_\alpha$$

We assume right side in zero, whence m - constant

$$m \left[\frac{du^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu \right] = F^\alpha \qquad (12)$$

when $F^\alpha = 0$ this is geodesic equation.

Equation of Motion of Spinning Test Particle

We also know

$$\frac{\partial}{\partial x^{\alpha}}\left(x^{\alpha} x^{\beta} T^{\gamma\delta}\right) = x^{\alpha} x^{\beta} \frac{\partial T^{\gamma\delta}}{\partial x^{\alpha}} + x^{\beta} T^{\gamma\alpha} + x^{\alpha} T^{\gamma\beta}$$

$$= x^{\beta} T^{\gamma\alpha} + x^{\alpha} T^{\gamma\beta} - x^{\alpha} x^{\beta} \Gamma_{\mu\nu}^{\gamma} T^{\mu\nu}$$

$$+ \frac{x^{\alpha} x^{\beta} F^{\gamma}}{u^{4}} \quad \delta_1 \quad \delta_2 \quad \delta_3$$

or using (1). Then integrate and expand replacing X^{α} under integral by $X^{\alpha} + \delta X^{\alpha}$ and $\Gamma_{\mu\nu}^{\alpha}$ by

$$\left(\Gamma_{\mu\nu}^{\alpha}\right)_0 + \left(\frac{\partial \Gamma_{\mu\nu}^{\alpha}}{\partial x^{\sigma}}\right)_0 \delta x^{\sigma} \equiv \Gamma_{\mu\nu}^{\alpha} + \Gamma_{\mu\nu,\sigma}^{\alpha}\, \delta x^{\sigma}$$

We can then obtain

$$\frac{d}{dt}\int T^{\alpha 4} dv + \Gamma_{\mu\nu}^{\alpha} \int T^{\mu\nu} dv + \Gamma_{\mu\nu,\sigma}^{\alpha} \int \delta x^{\sigma} T^{\mu\nu} dv = \frac{F^{\alpha}}{u^{4}}$$

$$\frac{dx^{\alpha}}{dt} \int \delta x^{\beta} T^{\gamma 4} dv + \frac{dx^{\beta}}{dt} \int \delta x^{\alpha} T^{\gamma 4} dv - \int \delta x^{\alpha} T^{\gamma\beta} dv$$

$$+ \int \delta x^{\beta} T^{\gamma\alpha} dv$$

It may be shown that these lead to

$$\frac{d}{ds}\left(\frac{M^{\alpha 4}}{u^4}\right) + \Gamma^{\alpha}_{\mu\nu} M^{\mu\nu} + \Gamma^{\alpha}_{\mu\nu,\sigma} u^4 \int \delta x^{\sigma} . T^{\mu\nu} dv = F^{\alpha}$$

This is the same as (8) if last term on the left is neglected.

Also with $S^{\alpha\beta} = \int \delta x^{\alpha} T^{\beta 4} dv - \int \delta x^{\beta} T^{\alpha 4} dv$ $S^{\alpha\beta}$ being the spin tensor.

$$\frac{DS^{\alpha\beta}}{Ds} = \frac{dS^{\alpha\beta}}{ds} + \Gamma^{\alpha}_{\mu\nu} S^{\mu\beta} u^{\nu} + \Gamma^{\beta}_{\mu\nu} S^{\alpha\mu} u^{\nu}$$

We can obtain the spin equation

$$\frac{DS^{\alpha\beta}}{Ds} + u^{\alpha} u_{\rho} \frac{DS^{\beta\rho}}{Ds} - u^{\beta} u_{\rho} \frac{DS^{\alpha\rho}}{Ds} = 0 \qquad (13)$$

which does not involve F^{α}.

Papapetrou shows that $S^{\alpha\beta}$ is a tensor and that the mass

$$m = \frac{1}{u^4}\left(M^{\alpha 4} + \Gamma^{\alpha}_{\mu\nu} S^{\mu 4} u^{\nu}\right) u_{\alpha}$$

which reduces to $\frac{M^{44}}{(u^4)^2}$ when $S^{\alpha} = 0$ is a scalar.

We neglect the effect of spin on motion and hence use (12). Actually only newtonian approximation to (12) is used. Use of Equation (13):

$$S^{12} = \int \delta x^1 T^{24} dv - \int \delta x^2 T^{14} dv$$

$$= \int (x P_y - y P_x) dv = S_2 \text{ etc}$$

$$S^{14} - \int \delta x^1 T^{44} dv - \int \delta x^4 T^{14} dv$$

second term is zero, since space integral is carried out for t = constant. First term is proportional to X coordinate of Centre of Mass and hence is zero if X^1 is the Centre of Mass. This is different in different coordinate systems.

It is convenient to write (13) in a particular non-covaniant form

$$\frac{DS^{\alpha\beta}}{Ds} + \frac{u^{\alpha}}{u^4} \frac{DS^{\beta 4}}{Ds} - \frac{u^{\beta}}{u^4} \frac{DS^{\alpha 4}}{Ds} = 0 \qquad (14)$$

(14) may be obtained from (13) by putting $\beta = 4$ and multiplying by $\frac{u^\beta}{u^4}$

$$\frac{u^\beta}{u^4}\frac{Ds^{\alpha 4}}{Ds} + \frac{v^\alpha u^\beta v_\rho}{u^4}\frac{Ds^{4\rho}}{Ds} - u^\beta v_\rho \frac{Ds^{\alpha\rho}}{Ds} = 0$$

Then put $\alpha = 4$ and multiply by u^α / u^4

$$\frac{u^\alpha}{u^4}\frac{Ds^{4\beta}}{Ds} + u^\alpha u_\rho \frac{Ds^{\beta\rho}}{Ds} - \frac{u^\alpha u^\beta v_\rho}{u^4}\frac{Ds^{4\rho}}{Ds} = 0$$

Then add and substitute into (13)

We now see that 3 of the 6 equations are trivial, put $\alpha = i$, $\beta = 4$

$$\frac{Ds^{i4}}{Ds} + \frac{u^i}{u^4}\frac{Ds^{44}}{Ds} - \frac{u^4}{u^4}\frac{Ds^{i4}}{Ds} = 0$$

Thus we need a supplementary condition corinaldesi and papapetrou. Chose $S^{i4} = 0$ in earth system i.e. Centre of Mass is that measured by the earth centred observer. Pirani choses $S^{\alpha\beta} u_\beta = 0$ i.e. Centre of Mass is that measured by co-moving observer since then $u^0_i = 0$ and therefore $S^{i4} = 0$.

6. <u>The C.P. Equations:</u>

Then (14) with $S^{i4} = 0$.

$$\frac{Ds^{ik}}{Ds} = -\frac{u^i}{u^4}\left[\frac{ds^{k4}}{ds} + \Gamma^k_{\ell\mu} s^{\ell 4} u^\mu + \Gamma^4_{\ell\mu} s^{k\ell} u^\mu\right] + \frac{u^k}{u^4}\left[\frac{ds^{i4}}{ds} + \Gamma^i_{\ell\mu} s^{\ell 4} u^\mu + \Gamma^4_{\ell\mu} s^{i\ell} u^\mu\right]$$

$$\frac{Ds^{ik}}{Ds} = \Gamma^4_{\ell\mu}\frac{u^\mu}{u^4}\left(s^{i\ell}u^k - s^{k\ell}u^i\right) \quad \ldots\ldots (15)$$

This can be written in rectangular coordinates to first order using the standard form

$$\frac{d\vec{s}}{dt} = \frac{m}{r^3}\left[2\vec{s}(\vec{r}\,\vec{v}) + 2\vec{v}(\vec{v},\vec{s}) - \vec{r}(\vec{v}.\vec{s}) - \frac{3r(r.v)(r.s)}{r^2}\right] \quad (16)$$

using the isotropic form

$$\frac{d\vec{s}}{dt} = \frac{m}{r^3}\left[3\vec{s}(\vec{r}\,\vec{v}) + \vec{v}(\vec{r},\vec{s}) - 2r(\vec{v}.\vec{s})\right] \quad (17)$$

<u>The Pirani Equation:-</u>

We start from the covariant from (13) and put on the supplementary condition $S^{\alpha\beta}u_\beta = 0$.

Differentiating this

$$u_\beta \frac{dS^{\alpha\beta}}{ds} + S^{\alpha\beta}\frac{du_\beta}{ds} = 0$$

Then (13) becomes

$$\frac{DS^{\alpha\beta}}{Ds} = -u^\alpha u_\rho\left[\frac{dS^{\beta\rho}}{ds} + \Gamma^\beta_{\mu\nu}S^{\mu\rho}u^\nu + \Gamma^\rho_{\mu\nu}S^{\beta\mu}u^\nu\right]$$
$$+ u^\beta u_\rho\left[\frac{dS^{\alpha\rho}}{ds} + \Gamma^\alpha_{\mu\nu}S^{\mu\rho}u^\nu + \Gamma^\rho_{\mu\nu}S^{\alpha\mu}u^\nu\right]$$
$$= -u^\alpha\left[-S^{\beta\rho}\frac{du_\rho}{ds} + u_\rho u^\nu\Gamma^\rho_{\mu\nu}S^{\beta\mu}\right]$$
$$+ u^\beta\left[-S^{\alpha\rho}\frac{du_\rho}{ds} + u_\rho u^\nu\Gamma^\rho_{\mu\nu}S^{\alpha\mu}\right]$$

$$\frac{du_\rho}{ds} = \frac{d}{ds}(g_{\rho\sigma}u^\sigma) = u^\sigma u^\gamma\frac{\partial g_{\rho\sigma}}{\partial x^\gamma} + g_{\rho\sigma}\frac{du^\sigma}{ds}$$

$$\frac{du^\sigma}{ds} = -\Gamma^\sigma_{\mu\nu}u^\mu u^\nu + f^\sigma + O(S^{\alpha\beta}) \text{ where } f^\sigma = \left(\frac{f^\sigma}{m}\right)$$

(from eq. 12). After some reduction (to the lowest order in $S^{\alpha\beta}$)

$$\frac{DS^{\alpha\beta}}{Ds} = \frac{1}{2}u^\nu u^\sigma\left(\frac{\partial g_{\rho\nu}}{\partial x^\sigma} - \frac{\partial g_{\sigma\beta}}{\partial x^\nu}\right)\left(u^\alpha S^{\beta\rho} - u^\beta S^{\alpha\rho}\right)$$
$$+ f_e\left(u^\alpha S^{\beta e} - u^\beta S^{\alpha e}\right)$$

It can be shown that the first line involves 3 powers of u^i and hence can be neglected. Then taking $\alpha = i$, $\beta = k$ and relating S^{i4} in $\frac{DS^{ik}}{Ds}$ on right to S^{ik} through supplementary condition, and f_4 to f_i then $f_\alpha u^\alpha = 0$.

We get with the standard form

$$\frac{d\bar{S}}{dt} = \frac{3m}{r^3}\left[\vec{v}(\vec{r}\cdot\vec{S}) - \frac{\vec{r}\,(r\cdot v)(r\cdot S)}{r^2}\right. \tag{18}$$

with the isotropic form $\quad + \vec{S}(\vec{v}\cdot\vec{f}) - \vec{f}(\vec{v}\cdot\vec{S})$.

$$\frac{d\vec{S}}{dt} = \frac{m}{r^3}\left[\vec{S}(\vec{r}\cdot\vec{v}) + 2\vec{v}(\vec{r}\cdot\vec{S}) - \vec{r}(\vec{v}\cdot\vec{S})\right]$$
$$+ \vec{S}(\vec{v}\cdot\vec{f}) - \vec{f}(\vec{v}\cdot\vec{S}).$$

8. Reconciliation:

We now reconcile (16) (19) with each other by transforming to the comoving system. First consider the metric. A length δx^c has a proper length $ds = \delta x^i\sqrt{-g_{ii}}$ and it is this that is measured by a comoving observer. Thus in the isotropic case all coordinate lengths are to be multiplied by $\left(1 + \frac{m}{r}\right)$ and S^{ik} is to be multiplied by $\left(1 + \frac{2m}{r}\right)$ all to the first order. In the standard case radial lengths are multiplied by $\left(1 + \frac{m}{r}\right)$ and tangential lengths left unchanged. Therefore tangential components of $\vec{S}$ are multiplied by $\left(1 + \frac{m}{r}\right)$ and radial components left unchanged.

Next consider the Lorentz transformation to the comoving system For $\vec{v}$ along x.

$$\alpha_{ik} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

In general

$$S^{120} = \gamma S^{12} + v\gamma S^{24}, \quad S^{230} = S^{23} \quad S^{310} = \gamma S^{31} - v\gamma S^{34}$$

$$S^{140} = S^{14}, \quad S^{240} = \gamma S^{24} + v\gamma S^{12}, \quad S^{340} = \gamma S^{34} - v\gamma S^{31}$$

In the C-P case

$$S^{14} = S^{24} = S^{34} = 0$$

So

$$S^{120} = \gamma S^{12}, \quad S^{230} = S^{23} \quad S^{310} = \gamma S^{31}$$

$$S^{140} = 0, \quad S^{240} = v\gamma S^{12}, \quad S^{340} = -\gamma S^{31}$$

Thus the component of $\vec{S}$ along $\vec{v}$ is unchanged and the $\perp r$ components are increased by $\left(1 + \frac{v^2}{2}\right)$

In the pirani case $S^{14} = 0$, $S^{24} = -vS^{12}$, $S^{34} = vS^{31}$,

$S^{120} = \frac{1}{r} S^{12}$, $S^{230} = S^{23}$, $S^{310} = \frac{1}{r} S^{31}$

$S^{140} = S^{240} = S^{340} = 0$

as expected.

There again the components of $\vec{S}$ along $\vec{v}$ is unchanged, but now the $\perp$ components are multiplied by $\left(1 - \frac{v^2}{2}\right)$

We thus get the 4 combinations

$$C-P-S,\quad \overset{0}{\overline{S}} = \vec{S} + \frac{m}{r}\left[\vec{S} - \vec{r}(\vec{r}\cdot\vec{S})\right] + \frac{v^2}{2}\left[\vec{S} - \vec{v}(v\cdot s)\right]$$

$$C.P.I,\quad \overline{S}^0 = \vec{S} + \frac{2m}{r}\vec{S} + \frac{v^2}{2}\left[\vec{S} - v(v\cdot s)\right]$$

$$P.S,\quad \vec{S}^0 = \vec{S} + \frac{m}{r}\left[\hat{S} - \hat{r}(\hat{r}\cdot\hat{S})\right] - \frac{v^2}{2}\left[\vec{S} - v(v\cdot s)\right]$$

$$P.I,\quad \overline{S}^0 = \overline{S} + \frac{2m}{r}\vec{S} - \frac{v^2}{2}\left[\vec{S} - v(v\cdot s)\right]$$

<u>In the C.P.S. Case:</u>

$$\frac{d\vec{S_0}}{dt} = \frac{d\vec{S}}{dt} + \frac{d}{dt}\left[\frac{m\vec{S}}{r} - \frac{m\vec{r}(\vec{r}\cdot\vec{S})}{r^3}\right] + \frac{1}{2}\frac{d}{dt}\left[v^2\vec{S} - \vec{v}(\vec{v}\cdot\vec{S})\right]$$

$$= \frac{d\vec{S}}{dt} - \frac{m}{r^2}\dot{r}\vec{S} + \frac{3m}{r^4}\dot{r}\vec{r}(\vec{r}\cdot\vec{S}) - \frac{m}{r^3}\vec{v}(\vec{r}\cdot\vec{S})$$
$$- \frac{m}{r^3}\vec{r}(\vec{v}\cdot\vec{S})$$
$$+ v\cdot\dot{v}\vec{S} - \frac{1}{2}\vec{v}(\vec{v}\cdot\vec{S}) - \frac{1}{2}\vec{v}(\overset{o}{\vec{v}}\cdot\vec{S}).$$

Now $\frac{d\vec{S}}{dt}$ is gotten from (16) and

$$r^o = \frac{\vec{r}\,\vec{v}}{r},\quad v^o = \frac{\vec{v}\cdot\overset{o}{\vec{v}}}{v}$$

Further from (12) $\overset{o}{\vec{v}} = -\frac{m}{r^3}\vec{r} + \vec{f}$

$$\frac{d\vec{S^o}}{dt} = \frac{m}{r^3}\Big[2\vec{S}(\vec{r}\cdot\vec{v}) + 2\vec{v}(\vec{r}\cdot\vec{S}) - \vec{r}(\vec{v}\cdot\vec{S})$$
$$- \frac{3\vec{r}(\vec{r}\cdot\vec{v})(r\cdot\vec{S})}{r^2} - \vec{S}(\vec{r}\,\vec{v}) + \frac{3\vec{r}(\vec{r}\,\vec{v})(\vec{r}\cdot\vec{S})}{r^2}$$
$$- \vec{v}(\vec{r}\,\vec{S}) - \vec{r}(\vec{v}\cdot\vec{S}) - \vec{S}(\vec{r}\cdot\vec{v}) + \frac{1}{2}\vec{r}(\vec{v}\cdot\vec{S})$$
$$+ \frac{1}{2}\vec{v}(\vec{r}\cdot\vec{S})\Big] + \vec{S}(\vec{v}\cdot\vec{f}) - \frac{1}{2}\vec{f}(\vec{v}\cdot\vec{S}) - \frac{1}{2}\vec{v}(\vec{f}\cdot\vec{S})$$

$$= \frac{3m}{2r^3}\left[\vec{v}(\vec{r}\cdot\vec{S}) - \vec{r}(\vec{v}\cdot\vec{S})\right] + \vec{S}(\vec{v}\cdot\vec{f}) - \frac{1}{2}\overline{f}(\vec{v}\cdot\vec{S})$$
$$- \frac{1}{2}\vec{v}(\vec{f}\cdot\vec{S}) \quad ..(20)$$

The C.P.I. case gives the same result.

In the P.I. case

$$\frac{d\vec{s}^{\circ}}{dt} = \frac{d\vec{s}}{dt} - \frac{2m}{r^2}\dot{r}\vec{s} - v.\dot{v}\vec{s} + \frac{1}{2}\dot{v}(\vec{v}.\vec{s}) + \frac{1}{2}\vec{v}(\vec{v}.\vec{s})$$

[From equation (19)]

$$= \frac{m}{r^3}\Big[\vec{s}(\vec{r}.\vec{v}) + 2\vec{v}(\vec{r}.\vec{s}) - \vec{r}(\vec{v}.\vec{s}) - 2s(\vec{r}.\vec{v}) + \vec{s}(\vec{r}.\vec{v}) - \frac{1}{2}\vec{r}(\vec{v}.\vec{s}) - \frac{1}{2}\vec{v}(\vec{r}.\vec{s})\Big] + \vec{s}(\vec{v}.\vec{f}) - \vec{f}(\vec{v}.\vec{s}) - \vec{s}(\vec{v}.\vec{f}) + \frac{1}{2}\vec{f}(\vec{v}.\vec{s}) + \frac{1}{2}\vec{v}(\vec{f}.\vec{s})$$

$$= \frac{3m}{2r^3}\Big[\vec{v}(\vec{r}.\vec{s}) - r(\vec{v}.\vec{s})\Big] + \frac{1}{2}\vec{v}(\vec{f}.\vec{s}) - \frac{1}{2}\vec{f}(\vec{v}.\vec{s}) \quad \ldots(21)$$

The P.S. case gives the same result. To reconcile the two C-P and the two P cases with each other we note that the C-P supplementary condition means, that there is a torque in the comoving system. This is spinning if the gyro is supported at its Centre of Mass in this system

$$s^{i4} = \int \delta x^i T^{44} dv = m u^4 \delta x^i$$

In the comoving system $\vec{v} = 0, \quad u^4 = 1$

$$\overline{\delta x^{\circ}} = \frac{1}{m} s^{14} = 0$$

$$\overline{\delta y^{\circ}} = \frac{1}{m} s^{24} = 0 = \frac{v}{m}\gamma s^{12} = \frac{v\gamma s}{m}$$

$$\overline{\delta z^{\circ}} = \frac{1}{m} s^{34} = -\frac{v}{m}\gamma s^{31} = -\frac{v\gamma s_y}{m}$$

where v is along Y. The above gives the coordinate of the comoving Centre of Mass, with respect to the earth centred Centre of Mass. Since $\overline{F}$ is applied at the later point the torque of

$\vec{T}$ of $\vec{F}$ is $T_x = -(\overline{\delta y^\circ} F_z - \overline{\delta z^\circ} F_y) = \frac{v\gamma}{m}(F_y \delta_y + F_z \delta_z)$

$$T_y = -(\overline{\delta_z^\circ} F_x - \overline{\delta x_\circ} F_z) = \frac{v\gamma}{m}(F_x S_y)$$

$$T_z = -(\overline{\delta x^\circ} F_y - \overline{\delta y_\circ} F_x) = \frac{v\gamma}{m} F_x S_z$$

In rotation convariant form this is

$$\overline{T} = \frac{\gamma}{M}\left[\vec{S}(\vec{v}\cdot\vec{F}) - \vec{v}(\vec{F}\cdot\vec{S})\right] \simeq \vec{S}(\vec{v}\cdot\vec{f}) - \vec{v}(\vec{f}\cdot\vec{S})$$

Thus substracting $\vec{T}$ from the right side of (20) we get (21).

Thus finally in all 4 cases

$$\frac{d\vec{S^\circ}}{dt} = \frac{3m}{2r^2}\left[(\vec{r}\times\vec{v})\times\vec{S^\circ}\right] + \frac{1}{2}\left[(\vec{f}\times\vec{v})\times\vec{S^\circ}\right] \tag{22}$$

9. <u>Earth rotation effect:</u>

For $\vec{\omega}$ along $+z$ Lense and Thirring (10) got from the isotropic metric to first order

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2m}{r} & 0 & 0 & -\frac{2I\omega Y}{r^3} \\ 0 & -1-\frac{2m}{r} & 0 & \frac{2I\omega X}{r^3} \\ 0 & 0 & -1-\frac{2m}{r} & 0 \\ -\frac{2I\omega Y}{r^3} & \frac{2I\omega X}{r^3} & 0 & 1-\frac{2m}{r} \end{pmatrix} \tag{23}$$

where $I = \frac{8\pi}{15}\rho R^5 = \frac{2}{5}MR^2$, ρ = constant

Including this and putting in proper units (2?) becomes

$$\frac{d\vec{S^\circ}}{} = \vec{\Omega}\times\vec{S_0}$$

$$\vec{\Omega} = \frac{\vec{F}\times\vec{v}}{2mc^2} + \frac{3GM}{2c^2r^3}(\vec{r}\times\vec{v}) + \frac{GI}{c^2r^3}\left[3\frac{(\vec{\omega}\cdot\vec{r})\vec{r}}{r^2} - \vec{\omega}\right] \tag{24}$$

To understand (23) suppose the metric were normally flat with respect to the rotating earth so that $ds^2 = dt'^2 - dx'^{}- dY'^2 - dz'^2$

Then in the non-earth system, which rotates in direction $-\vec{\omega}$ with respect to the earth

$$x' = x\cos\omega t + Y\sin\omega t\,, \quad Y' = -x\sin\omega t + Y\cos\omega t$$

$$z' = z\,, \quad t' = t\,.$$

$$ds^2 = dt^2 - \left(dx^2 + dy^2 + r^2\omega^2 dt^2 + \omega y\,dx\,dt - \omega x\,dY\,dt + dz^2\right)$$

$$g_{\mu\nu} \begin{pmatrix} -1 & 0 & 0 & -\frac{\omega Y}{2} \\ 0 & -1 & 0 & \frac{\omega x}{2} \\ 0 & 0 & -1 & 0 \\ -\frac{\omega Y}{2} & \frac{\omega x}{2} & 0 & 1-\omega^2 r^2 \end{pmatrix}$$

Thus the metric dragging effect is of fractional order

$$\frac{4I}{r^3} = \frac{4GI}{r^3c^2} = \frac{4G\cdot{}^2/_5 MR^2}{r^3c^2}$$

at the surface of the earth this is $\frac{8}{5}\frac{GM}{Rc^2}$, note that $g = \frac{GM}{R^2}$

so that $\frac{GM}{Rc^2} = \frac{gR}{c^2} = 6\times 10^{-10}$

<u>Earth bound laboratory:</u>

$$\vec{v} = \vec{\omega}\times\vec{r}\,, \qquad \dot{\vec{v}} = \vec{\omega}\times\vec{v}$$

$$f = \frac{F}{m} = \overset{o}{\vec{v}} + \frac{GM}{r^3}\vec{r}$$

Thus (24) becomes, with λ being the lattitude.

$$r = \left[\frac{4gR}{5c^2}\left(1+\cos^2\lambda\right) - \frac{\omega^2R^2}{2c^2}\cos^2\lambda\right] + \frac{4g\sin\lambda}{5\omega c^2}\left(\vec{\omega}\times\vec{v}\right)$$

6×10^{-9} rotations/day $\simeq$ 45" / century

Earth axis precession = .8"/ century.

Attempts are being made to verify these results using the satellites. The gyroscopes inside these are spinning particles falling freely and change of rotation axis can be checked, at different intervals of time.

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