STUDY OF SOME ELEMENTARY PARTICLE INTERACTIONS WITH SPECIAL REFERENCE TO THE USE OF STOCHASTIC METHODS

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PREFACE

This thesis embodies the research work done by the author during the years 1959-1962 under the guidance of Professor Alladi Ramakrishnan, formerly Professor of Physics at the Univefsity of Madras and now the Director of the Institute of Mathematical Sciences, Madras, in the field of elementary particle physics.

The problems dealt with in the thesis are classified into three well-defined groups. The first deals with a re-examination of the established results to facilitate a better understanding and includes refinements in the methods of electrodynamicx calculations. The second consists of an application of the recent technique of dispersion theory to problems relating to strange particles and the third an illustration of the phenomenological approach to particle physics.

Fifteen papers dealing with part of the subject matter of this thesis have been published by the author in scientific journals periodically, the available reprints being enclosed in the form of a booklet. The range and complexity of problems dealt with in these papers has necessitated collaboration either with my guide or with my colleagues and due acknowledgment of this collaboration has been made in each chapter.

My grateful thanks are due to Professor Alladi Ramakrishnan for his constant guidance and encouragement throughout the course of this work. I am indebted to the University of Madras and the Institute of Mathematical Sciences for providing me with excellent facilities for research work and to the Atomic Energy Commission, Government of India, for the award of a Research Fellowship during the period of study.

The Institute of Mathematical Sciences, Madras,
August, 1962.

R. Thunga

(R. Thunga)

CONTENTS

PART I.

STUDY OF SOME ELEMENTARY PARTICLE INTERACTIONS WITH SPECIAL REFERENCE TO THE USE OF STOCHASTIC METHODS.

INTRODUCTION

It is now well-recognised that after quantum electrodynamics had reached a stage of "conceptual consistency and physical completeness" new difficulties arose in quantum field theory with the inadequacy of perturbation methods for strong interactions on the one hand and the ignorance of the nature of the interactions involving strange particles on the other. Excluding attempts at a fundamental departure from the present concepts, any further study should naturally deal with the following three aspects.

1. As far as quantum electrodynamics is concerned, refinements in the methods of calculation and inclusion of higher order processes are the main possibilities. Apart from this, it is felt by the author that a reexamination of the motivation for quantum field theory will lead to a better understanding of its effectiveness and limitations in the study of physical processes.

2. Recently new techniques, for example, dispersion methods have been introduced to tackle the problem of strong interactions in a non-perturbative manner. Such methods can be extended to various problems involving strange particles. This approach has become particularly fruitful in view of the existence of resonant states of elementary particles.

3. Explanation of experimental data on strange particles, albeit through phenomenological methods.

This thesis comprising three parts concerns itself with the study of processes which illustrate the nature and scope of these attempts with particular emphasis on a critical examination of the physical basis of quantum field theory. This is not only important but necessary at a time when extreme views for and against the concept of fields are being put forward and attempts at an "abstract approach to field theory" are being received with extreme optimism on the one hand and scepticism on the other. It seems paradoxical that though the application of perturbation methods to strong interactions may be inadequate and considered "clumsy" , it is the field concept that has led to the discovery of important principles such as the analytic continuation of the S-matrix and symmetry laws.[1)]

At this point we wish to make a comment upon what is meant by the inadequacy of perturbation theory. It may be that perturbation approximations are not valid for strong interactions but in studying the process from a mathematical point of view and taking all the terms in a perturbation expansion into account we may still arrive at results which are as good and valid as those obtained by intrinsically non-perturbative methods as for example the dispersion theoretic approach. This amounts to taking the integrand

1) G.F.Chew , "The S-matrix theory of strong interactions", UCRL - 9701 (1961)

of the S-matrix 'seriously' and physically interpreting it as an amplitude for a "series of events in space-time". The question of the realizability of these events need not concern us as long as we do not establish any correspondence with physical phenomena before integrating over space-time points and intermediate dynamical variables. This principle has been emphasized only recently[2] and we wish to carry it to its logical conclusions. We wish to assert in addition that the study of interactions within the framework of perturbation expansions is not only conceptually satisfying but bears an elegant correspondence with the description of evolutionary stochastic processes. This also leads quite naturally to the 'derivation' of the field operators - a derivation which is an essential departure from the conventional mode since the concept of interaction in this viewpoint precedes the definition of annihilation and creation operators. Thus we build a "physical basis of quantum field theory"[3], on such an interpretation of the integrand of the S-matrix. We are encouraged to believe that this attempt is not a mere reformulation of known axioms since it leads to a new proof of the equivalence between the Feynman and field theoretic formalisms.

The above re-examination has a larger objective - to analyse the role played by causality and the Pauli

2) A. Ramakrishnan and N.R.Ranganathan, "Stochastic Methods in quantum mechanics", Journal of Mathematical Analysis and Applications (1961) (in press).

3) A. Ramakrishnan, T.K.Radha and R.Thunga, "The Physical basis of quantum field theory", Journal of Mathematical Analysis and Applications, 1962 (in press).

exclusion principle, both of which are there to stay irrespective of the views on field theory.

Part I consisting of six chapters deals with these three aspects. In Chapter 1 , a re-examination of the physical basis of quantum theory leads naturally to the concept of field operators in essential contrast with the conventional viewpoint in which their introduction is ad hoc. In Chapter 2 , the Feynman kernel is derived by extending the concept of equal time density functions in quantum mechanics to different times in close analogy with what are called "product densities" in stochastic theory. The expressions are generalized to give the density correlations at m points due to n particles.

In Chapter 3 , we wish to revive interest in what is considered to be a closed subject viz., the distinction between positive and negative energies even in virtual states[4]. While it is generally felt that the main advantage of the Feynman formalism is its inherent covariance, it should be emphasized that a deeper insight into the structure of the Feynman kernel is essential particularly from the point of view of causality. Bearing this in mind, we decompose the Feynman propagator into ~~free~~ positive and negative energy parts. The relative contributions from the two parts to the matrix element for electrodynamic processes like Compton scattering and Bremmstrahlung are calculated.

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4) A. Ramakrishnan, T.K.Radha and R. Thunga, Proc. Ind. Acad. Sc., LII , 228 (1960)

In Chapter 4 , we demonstrate the equivalence between energy denominators occurring in the field theoretic formalism and the method of the decomposed Feynman propagator respectively.[7] This is done by identifying the energy denominators in the two formalisms upto fourth order perturbation expansions. A generalized proof for nth order is also given.

Chapter 5 consists of a new proof of the equivalence in the operator formalism with a redefined interaction term. The justification for this redefinition is sought by the application of stochastic methods to quantum mechanics. The equivalence is also established in the usual operator formalism in a simple manner by working with the integrand of the S-matrix. Despite their simplicity, we consider these methods to be as complete and rigorous as those of Dyson[5] and Wick.

In Chapter 6 , we study higher order processes like double Compton scattering. The cross-sections for this process with circularly polarized photons are given[6] .

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5) See for example, "Mesons and Fields", Vol. I , H.A.Bethe, S. Schweber and F. de Hoffmann, Row Peterson, Evanston, Illinois (1955).

6) T. K. Radha and R. Thunga, Zeit, Fur Physik, 161, 20 (1961)

7) A. Ramakrishnan, T.K. Radha and R. Thunga, Jour. of Math. Anal. (1962) (in press)

Part II of this thesis consists of an application of the recent developments in dispersion theoretic techniques to various problems involving strange particles. The smallness of the fine structure constant for electromagnetic interactions was the only stimulus for perturbative expansions in powers of this constant and the validity of perturbative approximations was amply borne out by the fantastic accuracy of the results obtained in quantum electrodynamics. It was evident that such a treatment in the case of strong interactions will not lend itself to any suitable approximation. Of the different techniques that were developed to tackle the problem of strong interactions, the most successful has been the dispersion theoretic approach.

The fundamental postulates of this theory lie in the requirement of the analyticity properties of the scattering amplitude and the imposition of symmetry and conservation principles such as Lorentz invariance, unitarity and crossing symmtry. The problem then reduces to an investigation of the consequences of these restrictions. The analytical properties of the S-matrix as a function of the dynamical variables are studied, ~~the location and strength of the dynamical variables are studied~~, the location and strength of the singularities being determined generally from field theoretic concepts[1]. In the complex

1) G.F.Chew, "S-matrix theory of Strong Interactions", Reprint volume, W.A.Benjamin Inc. (1961).

energy plane the sum of the masses of the strongly interacting particles determines the location of the singularities, the ones near the origin being due to one and two particle configurations.

This would have been quite formal but for the experimental discovery of resonances. In order to relate a problem of theoretical interest to one of experimental feasibility one is forced to make assumptions and approximations which in turn are to be justified by the accuracy of the results obtained thereby. The assumption that is now made is that the physical aspects of any problem are governed by the 'nearby' singularities alone, a complete understanding of the effect of multiparticle configurations being unnecessary. The existence of the resonances introduces considerable simplification in two important respects: (1) Two and multiparticle intermediate states can be assumed to be dominated by the corresponding resonances and can be approximated by these 'single' particle states and (2) the experimental knowledge of the mass of the resonance amounts to a reduction in the number of parameters involved. As an illustration of this we have made use of the Y^* resonances.

The major advantage of the dispersion theoretic approach is the absence of an explicit interaction Hamiltonian, a feature that makes it the only candidate for

problems involving strange particles. The S matrix approach therefore permits "an orderly and systematic series of approximations whose validity is subject to a realistic appraisal without any assumption as to the magnitudes of the coupling constants".[1]

In Chapter I of this part, we have applied the method of single variable dispersion relations to the problem of pion-hyperon scattering.[2] In the application of fixed momentum transfer dispersion relations to pion-nucleon scattering by Chew and others[3], it was found that when the 'absorptive' part of the scattering amplitude occurring in the dispersion integral is expressed in terms of the scattering amplitude by means of the unitarity condition, the results were identical with those of the static theory.[4] The effective range formulae that were obtained were based essentially on the assumption that barring the single nucleon pole, the effect of multiparticle configurations is dominated by the P_{33} resonance and can therefore be approximated by a constant fitted by experiment. In the case of pion-hyperon scattering the recent discovery of the Y* resonance enables us to adopt a somewhat similar procedure.

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1) G.F.Chew, "S-matrix Theory of Strong Interactions", Reprint volume, W.A.Benjamin Inc. (1961)
2) Alladi Ramakrishnan, T.K.Radha, R.Thunga and A.P. Balachandran, Nuovo Cimento, (in press) (1962)
3) G.F.Chew, M.L.Goldberger, F.E.Low and Y.Nambu, Phys. Rev., 106, 1337 (1957).
4) G.F.Chew and F.E.Low, Phys. Rev. 101, 1570 (1956)

While it is true that the one-dimensional (fixed momentum transfer) dispersion relations are extremely useful in the correlation of experimental data, a knowledge of the behaviour of the scattering amplitude as a function of momentum transfer is naturally desirable. This has been achieved in the Mandelstam[1] representation which is the relativistic analog of the Chew-Low method with both energy and momentum transfer being treated as complex variables. Though a 'complete' solution in the Mandelstam representation seems too ambitious at present, results being possible only in the limit of one-dimensional dispersion relations or at most in the 'strip approximation', there are several aspects of the representation that provide a better insight into a problem. For instance, the explicit dependence on $\cos\theta$ is very useful in projecting out partial wave amplitudes and in examining their analytical properties. This is particularly fruitful because the unitarity condition for these amplitudes takes a simple form in the low energy region before inelastic processes set in. Frazer and Fulco[2] in employing this method for pion-nucleon scattering have been able to obtain the Chew-Low effective range formulae in an elegant manner by neglecting all but the closest singularities to the physical region, i.e. the physical cut and the contribution from the crossed pole (the direct pole

1) S.Mandelstam, Phys. Rev. 112 , 1344 (1958).

2) W. R. Frazer and J. Fulco, Phys. Rev. 117 1609 (1960)

cannot occur for a P_{33} amplitude). We have followed a similar procedure for pion-hyperon scattering. An important feature of this extension is the role of the relative $\Sigma\Lambda$ parity which determines the sign and magnitude of the residue at the pole. Since the spin and parity assignments of the Y* have not yet been experimentally established, partial wave dispersion relations are set up of $\Lambda\pi$ and $\Sigma\pi$ scattering amplitudes in the resonance channels for various spin and parity assignments. An effective range analysis is then shown to give definite indications regarding the most likely spin-parity assignment for the Y* if the $\Sigma\Lambda$ relative parity were known or vice versa.[1] An order of magnitude estimate of the $\Sigma\Lambda\pi$ coupling constant can also be found from the observed width of the resonance. This application is yet another example of a "calculation being reinforced at its weak links by experimental information", a feature characteristic of all dispersion theoretic methods.

In Chapter II , we tackle the problem of photoproduction of pions from Λ hyperons.[2] Making use of the unitarity condition which relates this inelastic process to the elastic $\Lambda\pi$ scattering, solutions are obtained in terms of the $\Lambda\pi$ scattering phase shifts

1) Alladi Ramakrishnan, T.K.Radha, R.Thunga and A.P.Balachandran, Nuovo Cimento (in press)
2) Alladi Ramakrishnan, T.K.Radha, R.Thunga and A.P.Balachandran, Letter to Nuovo Cimento(in press)

which were evaluated in the previous chapter. We further may extend this to electro-production of pions as well. Though a direct experimental observation of the process may not be possible it is felt that these considerations will have a direct bearing on the study of processes like $Y + N \rightarrow \Lambda + K + \pi$ The results are also of some importance in the study of the Λ form factor.

In Chapter III, we give the partial wave dispersion analysis for the production process $\bar{K} + N \rightarrow \Xi + K$.[1] Experimental data on this particular process is just coming out and it is hoped that such an analysis will throw light on some of the aspects of the problem. The existence of crossing symmetry for this process is found to be very useful in the analysis of such a production process.

In Chapter IV , we present results for production processes in hyperon-nucleon collisions in which our previous results find a direct application. The technique employed is the extrapolation method of Chew and Low and finer details such as the final state interactions[2] have been incorporated. In the case of pion production in Λp collisions, the total cross-sections in the physical region can be evaluated making use of our previous results for $\Lambda\pi$ scattering.

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1) Alladi Ramakrishnan, G.Bhamathi, S.Indumathi, T.K.Radha and R.Thunga, Nuclear Physics (in press)
2) G.Bhamathi, S.Indumathi, T.K.Radha and R.Thunga, Nuovo Cimento, (in press)

Problems in Part III of this thesis illustrate the phenomenological approach to strange particle physics. In general, a systematic classification of known experimental facts leads to certain well-defined rules regarding the behaviour of elementary particles. The application of these phenomenological rules to other problems leads to results which when experimentally verified strengthen their validity. The theoretical justification for these rules is of course a different question.

The best example of this is perhaps the postulate of charge independence for strongly interacting systems which requires that all physical processes be invariant under rotations in isospin space. In Chapter I of this Part, we have analysed the consequences of this hypothesis in cascade-nucleon collisions [1]. With the recent production of the Λ cascade particle in the laboratory, any theoretical investigation of its interactions is very relevant especially since even its intrinsic attributes like spin and parity have not yet been established. Our analysis of ΞN collisions resulting in three particle final states on the assumption of charge independence leads to interesting equalities and inequalities among the cross-sections for the different channels. We further take into account the possibility of resonances

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1) Alladi Ramakrishnan, T.K.Radha and R.Thunga, Nuclear Physics (in press)(1962)

in the final state. It is interesting to note that this reaction can throw light not only on the now well-established Y^* resonance but also on possible $\Xi\pi$, YY and YN resonances. It is also shown that an analysis of the angular distribution of the meson produced in the reactions can be used to determine the spin and parity of these resonances. The distributions are given for both even and odd ΞN parity.

In Chapter II, we have investigated the consequences of attributing spin 3/2 for the cascade particle[1]. If the cascade were to be an 'elementary' particle it should of course have spin 1/2 but in the absence of any clear cut and experimental evidence in support of this the possibility of higher spins cannot be excluded. We here present some spin dependent features in strong and weak interactions involving the cascade particle. The experimentally most feasible reaction $\Xi^- + p \rightarrow \Lambda + \Lambda$ is analysed retaining only S and P waves in the final state and it is seen that for even ΞN parity this reaction cannot occur if Ξ has spin 3/2. Similarly the charge exchange reaction if it were to have a threshold would give rise to a crisp in the $\Lambda\Lambda$ production cross-section and our study of this leads to some definite information regarding the cascade-nucleon relative parity besides the Ξ spin. We have

1) Alladi Ramakrishnan, G.Bhamathi, T.K.Radha and R.Thunga, Nuovo Cimento, (1961).

further calculated the binding energy of double hyper-fragments assuming the cascade to have spin 3/2 and the correlations between the two charged pions arising from the decay of the two $\Lambda \to$ are calculated using the technique of spin 3/2 in the angular distribution of the reaction $\bar{\Xi}^{+} + p \to K^{+} + K^{+}$. The Ξ decay distribution and that of the subsequent Λ are also given under the assumption of spin 3/2 for the Ξ . This in addition to reflecting the dependence on the spin of the initial cascade is also expected to reveal the nature of the coupling in weak interactions.

In Chapter III we investigate some of the strong and electromagnetic decay modes of the recently observed K* resonances. Besides the strong decay mode by which the resonances are usually identified there may exist other modes with small but finite branching ratios. A phenomenological study of such modes viz., the energy spectra and angular correlation of the decay products is expected to provide a better understanding of these resonances. We here present calculations assuming some probable mechanisms for the two and three particle decay modes of the K* resonances.[1)]

In the final chapter we have analysed the possible effects of baryon-baryon interaction which can be studied

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1) To be submitted to the Nuovo Cimento.

in cascade-nucleon collisions resulting in two-hyperon final states.[1] We have envisaged the possibility of $\Sigma\Sigma$ bound systems and the decay distributions are also presented.

Until recently, the experimental feasibility of these reactions appeared quite remote but with the recent identification of even the $\bar{\Xi}$ in the laboratory it is to be expected that data on cascade-nucleon collisions would be quite extensive.

The Appendix contains a note on the theory of geomagnetic effects.

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1) G. Bhamathi, S.Indumathi, T.K.Radha and R.Thunga, Letter to Progress of Theoretical Physics, 25, 870 (1961)

PART I.

CHAPTER I.

THE PHYSICAL BASIS OF QUANTUM FIELD THEORY [1]

1. Introductory Remarks.

The development of any physical theory is usually marked by alternate stages of mathematical formulation and physical interpretation. A theory starts in an attempt to explain a series of connected physical phenomena or atleast to systematise the factual knowledge relating to such phenomena. A mathematical formulation is made which embraces all these facts within its scope; once such a formulation is available it is developed to its logical conclusions some of which may not have been originally anticipated. We are then compelled to interpret physically the newly derived conclusions.

Perhaps the best example is the quantum theory of scattering which passed through the following stages of development :

1. Wave mechanics of single particles - a mathematical scheme, the creation of which was demanded by the physical facts relating to the dual nature of light and matter.

2. The theory of 'antiparticles'. One of the mathematical consequences of a relativistic theory was the negative energy states which could only be explained in the physical world through the postulate of antiparticles and in particular, positrons in electrodynamics.

[1] A. Ramakrishnan, T. K. Radha and R. Thinga, Jour. of Math. Anal. (1962) (in press)

3. The quantum theory of fields ~~x~~ - a mathematical formulation introducing field operators in an attempt to tackle the multiplicities of particles like the electron and the positron.

4. The analytic continuation of the S-matrix with complex dynamical variables, and

5. The re-interpretation of the singularities in terms of particles, resonances and decaying states.

The essential difference between classical and wave mechanics consists in the principle of complementarity. A pair of canonically conjugate dynamical variables no longer obey the commutative law of multiplication. The replacement of the dynamical variables by operators led to the quantum theory which was found to be of general applicability to systems formed by an arbitrary number of particles. This generalization gave birth to the quantised theory of fields where the wave function is replaced by a field 'operator' containing the 'creation' and 'annihilation' operators for the particles. We think it worthwhile to re-examine critically this development since we feel too much is being taken for granted about the inadequacy of perturbation methods on the one hand and the extraordinary efficacy of the analytic approach on the other. It is gratifying that such a re-examination leads to the establishment of a correspondence between probabilistic and quantum mechanical concepts - a thing which is of interest since the first interpretation of the modulus of the wave function was as a probability density.

We now maintain that the passage from the wave function to the field operator is not as profound as the replacement of the dynamical variables by operators. The introduction of field operators "is not a quantization but just a definition of convenient operators related to the occupation numbers in a way compatible with the exclusion principle".[1] In support of this contention we need only point out that the single particle formalism of Feynman is able to arrive at all the results of field theory within the framework of perturbation. This in fact is not surprising since the axioms and postulates of the quantum theory of fields were in fact suggested by physical considerations.

In this chapter we emphasize that the development of quantum field theory is merely an extension of wave mechanics. This, as we have already stated is a departure from the customary tendency to study quantum theory from an axiomatic point of view since the concept of interaction precedes that of the definition of annihilation and creation operators. The close analogy with the unfolding of a stochastic process is established. Besides this, we have a larger objective - to discuss the ways in which the Pauli principle has been imbedded in the field theoretic and Feynman formalisms. It is well recognised that the Feynman and field theoretic formalisms are equivalent. Then why should the Pauli principle be completely expressible by the commutation relations in one case while it is to be invoked ad hoc on the other ?

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1) L.Rosenfeld, "Introductory considerations on Elementary Constituents and their Interactions", NORDITA Publications No. 44.

2. State-Vector in quantum field theory

In wave mechanics a free particle in the absence of interaction is represented by a wave function $\psi(\vec{x}, t)$ in configuration space or $\phi(\vec{p}, t)$ in momentum space, the two being connected by the relation

$$\psi(\vec{x}, t) = (2\pi)^{-3/2} \int \phi(\vec{p}, t)\, e^{i\vec{p}\cdot\vec{x}}\, d^3p \qquad 1.$$

$$\phi(\vec{p}, t) = (2\pi)^{-3/2} \int \psi(\vec{x}, t)\, e^{-i\vec{p}\cdot\vec{x}}\, d^3x \qquad 2.$$

When a particle is free and has a momentum $\vec{p}$ its energy is given by the relativistic formula

$$E_p^2 = p^2 + m^2 \qquad 3.$$

In such a case it would be possible to choose a wave function ψ_p which is an eigen function of momentum corresponding to the eigen value $\vec{p}$. Due to the quadratic relation, E_p can take two values corresponding to a given momentum, $E_p = \pm\sqrt{p^2 + m^2}$ and the eigen functions corresponding to these are called the positive and negative energy eigen functions. Any free particles wave function can then be expressed as a linear combination of ψ_p - s (i.e.)

$$\psi(\vec{x}, t) = \int C_I^{(p)}\, \psi_p(\vec{x}, t)\, d^3p \qquad 4.$$

where $C_I^{(p)}$ is a complex number.

We know that the wave function ψ_p for every elementary particle is postulated, the postulate sometimes being derived (the Dirac equation) or suggested (the Klein-Gordon equation) from first principles. Quite generally ψ_p can

be written as $u_p\, e^{-ip\cdot x}$ where u_p is an entity with more than one component -- either a vector or a spinor (depending on its transformation properties). We shall for convenience call u_p the structural part of the wave function. We recognise from (2) and (4)

$$\phi(\vec{p}, t) = (2\pi)^{3/2} C_I\, u_p\, e^{-iE_p t} = C_S\, u_p \qquad 5$$

and if u_p is congruent with unity then $\phi(\vec{p}, t)$ degenerates to C_S itself. In any case we may represent a state by C_I or C_S itself since we know how ψ and ϕ are related to it. If the eigen values of p are discrete, the state can be represented as row vector $\vec{C}$ with as many components $C(p)$ as we have p values of $C(p)$.

From wave mechanical considerations, it is known that particles fall into two classes - fermions, the total wave function for such a system being antisymmetric in the individual wave functions and bosons with a total wave function which is symmetric. Such anti-symmetrised or symmetrized functions can be expressed in both configuration and momentum representations. In either case the analogue of C_p for the many particle system is obviously $C(n_1, n_2, \cdots n_r)$ representing n_1 particles of momentum p_1 ; n_2 of momentum p_2 etc. and n_r of momentum p_r or more generally $C(\{k\})$ where $\{k\}$ is some typical aggregate characterised by the occupation numbers of the various types of particles in different momentum states. If the state vector in configuration space for a typical aggregate $\{k\}$ is $\psi(\{k\}; t)$ then any state can be represented by $\sum_{\{k\}} C(\{k\})\, \psi(\{k\}; t)$ and as in the case of single

particle

particle theory the momentum transform of $\psi(\{k\})$ would involve $C(\{k\})$ and the structural part of the wave function of the individual particles. The description of the state by the amplitudes C is adequate though they do not involve the structural part. For such a free particle system the energy is given by

$$E\{k\} = n_1 E_{p_1} + n_2 E_{p_2} + \cdots n_\lambda E_{p_\lambda} \qquad 6.$$

where p_i are the momenta of the components of $\{k\}$ and the E_{p_i}'s are given by the relativistic formula. E can also be treated as an eigen value of an operator H_0 corresponding to $\{k\}$, the representation of H_0 depending on whether we are working with ψ, C or ϕ.

We can now write $\psi(\{k\};t)$ as $\phi(\{k\})\, e^{-iE(\{k\})t}$ and $\psi(\vec{x},t) = \psi(x) e^{-iE(\{k\})t}$ in the absence of interaction. Or equally well, we can describe the state by $C_I(k)$ which we shall call the interaction representation of the occupation number state or by $C_S(k) = C_I(\{k\}) e^{-iE(\{k\})t}$ which may be called the Schrodinger representation of the occupation number state.

3. Temporal evolution of the state vector in a collision process

The concept of interaction can be best introduced in a negative manner by statingx that if we have an aggregate $\{i\}$ of free particles it continues to exist as such for all time. The amplitude that it continues so, for a duration t is $e^{-iE_i t}$ the modulus of which of course is unity. We now assume an interaction to be operative in the time interval T_0 to T, the result of the interaction being that if

interactions occurring at n time points $\tau_n \cdots \tau_1$ with $\tau_n > \tau_{n-1} \cdots > \tau_1$ is given by

$$V_S(\tau_n, \cdots \tau_1; T) = \exp\left[-i E_R (T - \tau_n)\right] R_S(k \leftarrow m_n) \cdot \exp\left[-i E_{m_{n-1}} (\tau_n - \tau_{n-1})\right] \cdots \exp\left[-i E_{m_1} (\tau_2 - \tau_1)\right] R_S(m_1 \leftarrow i)\, e^{-i E_i \tau_1} \times d\tau_n \cdots d\tau_1 \quad 7.$$

We can call this the amplitude in Schrodinger representation for a typical temporal realization or for a typical temporal "complexion" to occur, a "complexion" being characterized by the aggregates attributed to the various time points. On summing over all "complexions", i.e. performing the ordered integration over the variables, $\tau_1 \cdots, \tau_n$ from T_0 to T and summing over all possible intermediate aggregates, we obtain the amplitude for $\{k\}$ in the Schrodinger representation as

$$C_S(k; T) = \sum_{n=1}^{\infty} \sum_{\{m_j\}} \frac{e^{-i E_i T} R_S(k \leftarrow m_{n-1}) \cdots R_S(m_1 \leftarrow i)}{(E_i - E_k) \cdots (E_i - E_{m_1})} \quad 8$$

where $\sum_{\{m_j\}}$ implies summation over all possible intermediate aggregates $\{m_1\} \cdots \{m_n\}$ and $\sum_{n=1}^{\infty}$ denotes summation over the number of temporary vertices. In writing this expression it is assumed that on performing the integration over the range T_0 to T the term corresponding to the

lower limit in the indefinite integral vanishes if $T_0 \to -\infty$

This is ~~the~~ equivalent to multiplying the interaction amplitude by a factor $e^{+\epsilon T}$ with ϵ chosen as small as we please so that $e^{\epsilon \tau} = 1$ for finite τ and zero for $\tau \to -\infty$ i.e. the interaction is "switched on" only at T_0 . In a strict sense therefore we should write any term in the energy denominator as .. ..

$E_i - E_m + i\epsilon$. In the above expression for C , none of the energies E_j , $E_{m_n} \ldots E_{m_1}$ need be equal to E_i . The study of the contribution to the integral due to any of them becoming equal to E_i i.e. due to real intermediate states raises difficult mathematical problems due to the singularities. For the purposes of our discussion we may assume that these energies are different from E_i i.e. they are "off the energy shell".

Therefore it is clear that just by letting $T \to +\infty$ in the expression for C we will not obtain the matrix element for a transition to $\{f\}$ or what is the same we cannot replace $\{j\}$ by $\{f\}$ since to perform the integral we have assumed $E_j \neq E_i$. Since we know that $\frac{1}{2\pi i}\int e^{-i(E_f - E_i)T} dT = \delta(E_f - E_i)$, we note that if we replace E_j by E_f we must omit the factor $E_f - E_i$ in the denominator and introduce the δ – function in the numerator. This is also equivalent to setting $T = 0$ and writing

$$C_I(\{f\} \leftarrow \{i\}) = \sum_{\{j\}} R(\{f\} \leftarrow \{j\})\, C(\{j\}) \qquad 9.$$

In writing this it is also assumed that to $R(\{f\} \leftarrow \{m_n\})$ we attach the factor $e^{-\epsilon T}$ (i.e.) the interaction is "switched off" at T as $T \to \infty$. Thus we see that $C(\{f\} \leftarrow \{i\})$ may be computed if $R(\{m_n\} \leftarrow \{m_{n-1}\})$ the fundamental transition amplitudes in the Schrodinger representation are known.

It is also customary to write the interaction representation of the matrix element for the transition $\{i\} \to \{f\}$ as

$$C_I[\{f\}|\{i\}; T] = \int R_I(\{f\} \leftarrow \{m_n\}) \ldots R_I(\{m_1\} \leftarrow \{i\})\, d\tau_{n+1} \ldots d\tau_1$$

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and may be computed if the fundamental transition amplitudes

$$R_I\left(\{m_n\} \leftarrow \{m_{n-1}\}\right) = e^{i(E_m - E_{m_{n-1}})t}\, R_S\left(\{m_n\} \leftarrow \{m_{n-1}\}\right)$$

in the interaction representation are known. It is not only cumbersome but also an impossible task to enumerate $R(\{j\} \leftarrow \{i\})$ for all $\{j\}$ and $\{i\}$. It is obviously a reasonable assumption that when an aggregate $\{i\}$ changes to the aggregate $\{j\}$ only a few particles are involved i.e. only a few say $\{\alpha\}$ in $\{i\}$ cease to exist and a few others say $\{\beta\}$ are created which together with those remaining i.e. $\{i\} - \{\alpha\}$ form the aggregate $\{j\}$. Thus the R will involve only $\alpha + \beta$ particles and $E_i - E_j = E_\alpha - E_\beta$ where E_j is the sum of energy of all particles of aggregate $\{j\}$. This is precisely the assumption that is made in quantum theory. In electrodynamics $\alpha + \beta = 3$ consists of two fermions and a boson (photon) in the Yukawa interaction two fermions and a massive boson and in some 'catastrophic' interactions more than two fermions and bosons. Confining ourselves to the subaggregate $\{\gamma\}$ in $\{i\}$ and $\{\beta\}$ in $\{j\}$ it is enough if the momentum of the subaggregate $\{\beta\}$ is equal to the momentum of $\{\alpha\}$. Since we know from elementary mathematical analysis that

$$\int e^{i(\lambda_1 + \lambda_2)x}\, d^3x \supset \delta(\lambda_1 + \lambda_2) \qquad 18.$$

and

$$\int e^{i\left(\sum_i \lambda_i + \sum_i \mu_i\right)x} d^3x = \delta\left(\sum_i (\lambda_i + \mu_i)\right) \quad 13$$

and since free particle wave functions have coefficient $e^{-ip.x}$ it is a reasonable postulate in quantum mechanics to say that

$$R\left(\{\beta\} \leftarrow \{\alpha\}\right) = \int H\left(\{\beta\} \leftarrow \{\alpha\}\right) d^3x \quad 14.$$

$$H\left(\{\beta\} \leftarrow \{\alpha\}\right) \supset [\text{wave function of } \{\beta\}]^{+} [\text{that of } \{\alpha\}] \quad 15.$$

where d^3x implies integration of space point x and all the wave functions have the same spatial argument x and $\supset$ denotes that H contains all the terms to the right of $\supset$. The total momentum of $\{\alpha\}$ need not be equal to that of $\{\beta\}$ in the definition of $H\left(\{\beta\} \leftarrow \{\alpha\}\right)$ since the space integration ensures momentum conservation.

We can use $H_I\left(\{\beta\} \leftarrow \{\alpha\}\right)$ instead of $R\left(\{\beta\} \leftarrow \{\alpha\}\right)$ in the expression for the matrix element provided we make spatial integration over all intermediate spatial vertices.

From the point of view of elegance it would be desirable that the concept of interaction should be introduced in such a manner as to be independent of the particular states i and f under consideration i.e. only through $R_{\beta\alpha}$ or $H_{\beta\alpha}$ provided we know that we can enumerate its value for all α and β and take care that $R_{\beta\alpha}$ when operating on the amplitude for an aggregate $\{i\}$ to exist yields a non-zero value only if it satisfies two fundamental requirements :

1) That $\{\alpha\}$ is a subaggregate of $\{i\}$.

2) $\{\beta\}$ dould be added to $\{i\} - \{\alpha\}$ without contradicting the Pauli principle (i.e.) the fermions in aggregate $\{\beta\}$ should not be contained in $\{i\} - \{\alpha\}$.

This is achieved in elementary wave mechanics when we are dealing with only single particle systems in a very simple way since $\{\alpha\}$ and $\{\beta\}$ contain only one particle and $\{i\} \equiv \{\alpha\}$. Assuming that only discrete momentum values p_1 , p_2 ... p_n are possible if a particle is known to exist with momentum p_α , the state $|\alpha\rangle$ can be represented by a column vector with its α-th element equal to unity and the rest zero. Then if $\langle\beta|$ is a row vector

$$R_{\beta\alpha} = \langle\beta| R |\alpha\rangle \qquad 16.$$

provided R is a matrix with components $R_{\beta\alpha}$. Thus the representation of states by such column vectors and interactions by matrices R ensures that the transition occurs from state $|\alpha\rangle$ to $|\beta\rangle$ at t only if α exists at time t .

In this ~~xxixxx~~ single particle picture we should of course ask the question how can a state α change to β since rearrangement with conservation is possible only if we have more than one type of particle. In such a case we are constrained to treat the perturbation as "external" and define $R_{\beta\alpha}$ as

$$R_{\beta\alpha} = \int \psi^*_\beta(x)\, H\, \psi_\alpha(x)\, d^3x \qquad 17.$$

where H is an external perturbation which is a function of space and time. If it has an exponential dependence of

the form $e^{-iq\cdot x}$ then an absorption of momentum q occurs and

$$\vec{p}_\beta = \vec{p}_\alpha + \vec{q}$$

The perturbation may be due to the particle like a photon but is not included in the description of the state which envisages only electrons (i.e.) single type of particles) though the nomenclature that $E_{p_i} + E_{p_i}$ is the energy of the photon-electron system is used in wave mechanics. This is anomalous but has come to stay due to historical reasons. This is dispensed with in the more recent single particle formalism of Feynman to be discussed presently.

Turning now to the multiparticle systems we emphasise that a column vector and matrix representations of α and R respectively are not suitable and we also have the additional difficulty in choosing $\{\beta\}$ in $R_{\beta\alpha}$ consistent with the Pauli principle. This is done by the following formalism.

Let us define a vacuum state $|\rangle_0$ as follows: the sequence of amplitudes $C\{i\}$ with $C\{i\} = 0$ for all except the null aggregate $\{0\}$ and $C\{0\} = 1$. Though we will not represent any general state by a column vector we can define $|\rangle_0$ by the above prescription and $\langle|_0$ as its hermitian conjugate such that

$$\langle|\rangle_0 = 1$$

19.

We then define a state representing an aggregate $\{i\}$ with particles of momentum $p_1 \cdots p_n$ as

$$|i\rangle = \{p_1 \cdots p_{iN}\} = a^\dagger_{p_1} \cdots a^\dagger_{p_{iN}}\rangle_0 \qquad 20.$$

which is the analogue of the column vector in the single particle case. $a^\dagger_p$ is called for obvious reasons the creation operator. We shall for the moment concern ourselves with a system of fermions. An $a^\dagger_p$ operating on a state $|i\rangle$ will represent a aggregate $\{i\}$ plus one particle of momentum p. If however $|i\rangle$ contains a particle of momentum p then $a^\dagger_p|i\rangle = 0$. In such a case we can define an annihilation operator a_p such that $a_p|i\rangle$ represents the system without the particle of momentum p and then

$$a^\dagger_p[(a_p)|i\rangle] = |i\rangle \qquad 21.$$

since $a^\dagger_p|i\rangle = 0$ we can also write

$$a_p a^\dagger_p|i\rangle = 0 \qquad 22.$$

On the other hand if $|i\rangle$ does not contain a particle of momentum p then $a_p|i\rangle = 0$ and $a^\dagger_p|i\rangle$ represents the system $|i\rangle$ plus a particle ofmomentum p and $a_p a^\dagger_p|i\rangle$ is identical with state $|i\rangle$. Thus

$$a_p a^\dagger_p|i\rangle = |i\rangle \qquad 23.$$

and

$$a^\dagger_p a_p|i\rangle = 0 \qquad 24.$$

Therefore if we impose the condition that $a^\dagger a + aa^\dagger = 1$ the Pauli principle is ensured in the representation of states (i.e.) $(aa^\dagger + a^\dagger a)|i\rangle = |i\rangle$ irrespective of whether $|i\rangle$ contains a particle of momentum p or not. It is therefore clear that if by $a^\dagger|i\rangle$ we imply a state containing an aggregate $\{i\}$ ~~sequence of operators~~ then ~~$a^\dagger\{i\}$~~ ~~where~~ $a^\dagger\{i\}\rangle_0$ indicates a series of

creation operators, i.e. then $a a^{\dagger} \{i\}\rangle_0 = |i\rangle$ (but we know that $\langle\ \rangle_c = 1$) ; we would also like the hermitian conjugate of $|i\rangle$ to be $\langle i|$ and therefore we note that $\langle i|i\rangle = 1$ and $\langle i|$ can be represented as $\langle a\{i\}$.

For bosons we can represent states and their conjugate by the operations of the creation and annihilation operators on the vacuum but we will define their commutation relations only after invoking the concept of interaction.

We are now in a position to find a suitable representation for $\sum_{\alpha\beta} R_{\alpha\beta}$ so that the two fundamental requirements are satisfied. Extending these postulates of ordinary wave mechanics we require that $H(\beta \leftarrow \alpha) \supset$ (product of the wave function of α) x (product of conjugates of wave functions of β).

If we attach to each wave function the annihilation operator and its complex conjugate the creation operator we will satisfy both the fundamental conditions. Thus defining $\psi = \sum_k a_k \psi_k$ as a field operator corresponding to the particle represented by the wave function ψ and $\bar{\psi}$ its complex conjugate as

$$\bar{\psi} = \sum a_k^{\dagger} \bar{\psi}_k$$

$\sum_{\beta\alpha} R_{\beta\alpha}$ is a product of the field variables and their conjugates,

$$\sum_{\alpha\beta} H_I(\beta \leftarrow \alpha) \supset \bar{\psi}\psi \qquad 25.$$

In the case of bosons we are guided by one important rule that the probability of the absorption of a boson when there are bosons is proportional to n while the creation

is proportional to $n+1$. The amplitudes should therefore be proportional to $\sqrt{n}$ and $\sqrt{n+1}$ respectively. If $|n\rangle$ represent a state with n bosons, we would require that

and $$a_p^+ |n\rangle = \sqrt{n+1}\, |n+1\rangle \quad (26.)$$
$$a_p |n\rangle = \sqrt{n}\, |n-1\rangle \quad (27.)$$

This implies the commutation relation that $aa^+ - a^+a = 1$

Till now we have envisaged only positive energy states and the field operator obtained is $\sum_k a_k \psi_k$. If we now allow for negative energy states as well and postulate that the annihilation of a negative energy particle is equivalent to the creation of positive energy antiparticle we can write

$$a_{-k} = b_k^+ \qquad \text{and} \qquad u(-k) = v(k)$$

Then the field operator ~~and~~ is

$$\psi = \sum_k a_k u_k e^{-ikx} + b_k^+ v_k e^{ikx} \quad (28.)$$

This expansion is also valid for bosons except that the u-s and v-s do not occur. In this case we have

$$\phi = \sum_k \left(a_k e^{-ikx} + b_k^+ e^{ikx} \right) \quad (29)$$

It is interesting at this stage to discuss whether the interpretation that the annihilation and creation of a negative energy particle is equivalent to creation and annihilation of positive energy antiparticle also implies the existence of an unobservable sea of negative energy particles. Postulating the sea it is clear that the destruction of a negative energy particle in the sea would imply the creation of a hole - that is the creation of a positive energy antiparticle

But is not this interpretation valid even when there is no sea, as in the case of bosons ? The answer is found in recognising that the postulate of the sea implies something more - that when a particle of energy $-E$ is destroyed in the sea of a negative energy, it implies that it has made a transition either to a positive energy state or another negative energy state if there is a vacancy. Thus the total number of particles in the world if we include the sea remains the same. This is equivalent to stating that the total number of particles minus the number of holes remains invariant (i.e.) the conservation of the number of particles minus antiparticles. That is why in nature this conservation applies to baryons and leptons which are fermions though the concept of particles and anti-particles needs only the interpretation that the annihilation of one implies the creation of other and vice versa. The additional postulate of conservation is equivalent to that of the sea.

4. Evaluation of the matrix element in field theory

We now give a brief discussion on the various ways of evaluating the matrix element defined as the integral over space and time in (10) .

a) The old fashioned approach.

This consists in performing the space integration first at each vertex so that we obtain the formula given by (7). The evaluation is complete only if we are able to specify all the intermediate states and this is feasible only if we consider approximations which involve only the first few terms in a perturbation expansion. In such cases, the intermediate states can be enumerated from elementary considerations and the corresponding energies ascribed to them. If we used the field theoretic representation for H_{int} we have the annihilation and creation operators corresponding to each vertex and their rearrangement results in a numerical factor $+1$ or -1 . This factor can also be obtained from general arguments of Pauli principle when we omit the annihilation and creation operators in H_{int} .

The perturbation theory in this form is felt to be "awkward" and theoretical physicists "have been at great lengths to set up a relativistically invariant theory". We shall however show in Chapter V that the 'old-fashioned' approach could also be used to establish the equivalence between the Feynman and field theoretic formalisms in an elegant manner.

4.6 The covariant approach.

In this method we refrain from performing the space-time integration and reorder the vertices in the Feynman sequence. The space-time integration is subsequently performed. The re-arrangement can be done as follows:

We know that the interaction term containing annihilation and creation operators ~~with~~ have the corresponding wave functions attached to them. It is obvious that once the expression for a "typical realisation" is written it is a mathematically valid operation to rearrange the operators without reference to the wave functions. The result of such a re-arrangement of the annihilation and creation operators is ultimately to yield the numerical factor $+1$ or -1 in view of the commutation relations. The wave functions are suitably grouped so that we have a product of scalar quantities; and because of their scalar nature they can then be rearranged so that we get a Feynman sequence of wave functions. This, with the appropriate sign factor ± 1 obtained from the rearrangement of the operators yields the exact Feynman matrix element. We shall describe this process of rearrangement in detail for an n th order scattering of a single electron by taking a typical realisation of the amplitude which makes a non-vanishing contribution to the matrix element.

The n -th order term of the matrix element is an integral over ' n ' space-time points, the time points alone being ordered. The integrand is therefore

$$\langle f | H(n) \cdots H(j) \cdots H(1) | i \rangle \qquad 30.$$

If the initial state consists of an electron of momentum p_1 i.e. $|i\rangle = a^\dagger_{p_1}|_0\rangle$ and the final state, an electron of momentum p_2 then $|f\rangle = a^\dagger_{p_2}|_0\rangle$ and every $H_I(j) \supset \bar{\psi}(j)\psi(j)\phi(j)$ The symbol $\supset$ means that the factors on the right hand side are contained in $H(j)$ and is used to indicate that we ignore constants and matrices which are not relevant for the present discussion. We shall also for the present ignore the photon or meson operator $\phi(j)$ and concern ourselves with the bilinear fermion operators $\bar{\psi}(j)\psi(j)$ only. Thus

$$[n] = H(n) \supset \bar{\psi}(n)\psi(n) = [\hat{a}^\dagger_p \hat{a}_{p'}(n) + \hat{a}^\dagger_p \hat{b}^\dagger_{p'}(n) + \hat{b}_p \hat{b}^\dagger_{p'}(n) + \hat{b}_p \hat{a}_{p'}(n)] \qquad 31.$$

where the symbol $\wedge$ denotes that the operators are accompanied by their corresponding wave functions. We shall now separate the operators from their wave functions and first order the operators as follows:

Consider a typical realisation

$$\langle f | [n] \cdots [j] \cdots [k] \cdots [1] | i \rangle \qquad 32.$$

since the initial electron of momentum $\vec{p}_1$ has to be annihilated there should be an a_{p_1} in some bracket $[k]$. Let this bracket be the first with an a_{p_1} to occur to the left of $|a^\dagger_{p_1}\rangle_0$. This bracket would contain in addition an $a^\dagger_{p'}$ or $b_{p'}$ to the left of a_{p_1}. We now move a_{p_1} to the left of $a^\dagger_{p_1}$ through all the intervening brackets $[k-1] \cdots [1]$

and this results in no change of sign since each bracket contains two fermion operators none of which can be an a_{p_1} or an $a^{\dagger}_{p_1}$ (the former by assumption and the latter since $a^{\dagger}_{p_1} a^{\dagger}_{p_1} \rangle_0 = 0$). Thus we have

$$[n] \cdots [k+1]\, a^{\dagger}_{p'}(k)\, [k-1] \cdots [1]\, a_{p_1} a^{\dagger}_{p_1} \rangle_0 \qquad 33.$$

or

$$[n] \cdots [k+1]\, b_{p'}(k)\, [k-1] \cdots [1]\, a_{p_1} a^{\dagger}_{p_1} \rangle_0 \qquad 34$$

We recognise $a_{p_1} a^{\dagger}_{p_1} \rangle_0 = \rangle_0$. If the realisation is as in (33) we next move $a^{\dagger}_{p'}(k)$ to the ~~right~~ left of a_{p_1} . There should now be another bracket, say $[j]$ with $t_j > t_k$ containing an $a_{p'}$ which is shifted through all the intervening brackets to the left of $a^{\dagger}_{p'}$ which again leads to a numerical factor $+1$ and we have

$$[n] \cdots [j+1]\, a^{\dagger}_{p''}(j)\, [j-1] \cdots [k+1][k-1] \cdots a_{p'}(j)\, a^{\dagger}_{p'}(k) \rangle_0 \qquad 35.$$

or

$$[n] \cdots [j+1]\, b_{p''}(j)\, [j-1] \cdots [k+1]\, [k-1] \cdots a_{p'}(j)\, a^{\dagger}_{p'}(k) \rangle_0 \qquad 36.$$

Thus we have given a complete prescription for the rearrangement of the operators a and $a^{\dagger}$. Turning to the wave functions corresponding to the above typical realisation, we can arrange them as

$$[n] \cdots [\bar{u}_{p''}(j)\, u_{p'}(j)] [\bar{u}_{p'}(k)\, u_{p_1}(k)] \qquad 37$$

and $u_{p'}(j)\, \bar{u}_{p'}(k)$ can be identified to be a single element of the positive energy part of Feynman kernel for energy. + E.

The sum of all such terms for all possible realisations will give the positive part of the Feynman kernel $K_+(j,k)$ with $t_j > t_k$.

If, on the other hand $[k]$ had a $b_{p'}$ to the left of $\hat{a}_{p_1}$ then there should be a $\hat{b}^+_{p'}$ in a bracket $[m]$ with $t_m < t_k$ since the positron should be created before it can be annihilated. Thus if we have

$$[n] \ldots [\hat{b}_{p'}(k)\, \hat{a}_{p_1}(k)] \ldots [\hat{a}^+_{p''}(m)\, \hat{b}^+_{p'}(m)] \ldots a^+_{p_1}\rangle_0 \qquad 38.$$

we move $\hat{a}_{p_1}$ to the left of $\hat{a}^+_{p_1}$ so that

$$a_{p_1}(k) a^+_{p_1}\rangle_0 = \rangle_0$$

with a numerical factor (+ 1) and if we now move $\hat{b}_{p'}(k)$ to the left of $\hat{b}^+_{p'}(m)$ (i.e.)

$$[n] \ldots [k+1][k-1] \ldots \hat{b}_{p'}(k) [\hat{a}^+_{p''}(m)\, \hat{b}^+_{p'}(m)] \ldots \hat{a}_{p_1}(m)\, a^+_{p_1}\rangle_0 \qquad 39.$$

or

$$[n] \ldots [k+1][k-1] \ldots \hat{b}_{p'}(k) [\hat{b}_{p''}(m)\, \hat{b}^+_{p'}(m)] \ldots a_{p_1} a^+_{p_1}\rangle_0 \qquad 40.$$

If we now shift $\hat{b}_{p'}(k)$ to the left of $\hat{b}^+_{p'}(m)$ we obtain a negative sign i.e.

$$(-)\ \hat{a}^+_{p''}(m)\, \hat{b}_{p'}(k)\, \hat{b}^+_{p'}(m) \ldots a_{p_1} a^+_{p_1}\rangle_0 \qquad 41.$$

or

$$(-)\ \hat{b}_{p''}(m)\, \hat{b}_{p'}(k)\, \hat{b}^+_{p'}(m) \ldots a_{p_1} a^+_{p_1}\rangle_0 \qquad 42.$$

We now shift the pair $-b_{p'}(k)\, b^+_{p'}(m)$ to the extreme right without any change in sign and identify $b_{p'} b^+_{p'}\rangle_0 = \rangle_0$.

This completes the prescription for all the operators.

As regards the wave functions, in this case we have

$$[n] \ldots [\bar{v}_{p'}(k)\, u_{p_1}(k)] [\bar{u}_{p''}(m)\, v_{p'}(m)] \ldots [1] \qquad 43.$$

which may be rewritten as

$$[n] \ldots [\bar{u}_{p''}(m) v_{p'}(m)][\bar{v}_{p'}(k) u_{p_1}(k)] \ldots [1] \quad 44.$$

It is to be noted that the positron operators occur in the order $b_{p'}(k) b^{\dagger}_{p'}(m)$ so that if they carried their wave functions with them we would have had $\bar{v}_{p'}(k) v_{p'}(m)$ i.e $\bar{v}_{p'} v_{p'} e^{-ip'.(x_m - x_k)}$ which cannot be identified with an element of the Feynman kernel for negative energies. However by shifting the product of wave function in a manner so as to arrange them in the Feynman sequence and taking over the negative ~~xxxxxx~~ sign obtained from the shifting of the operators we can recognise

$$-v_{p'}(m)\bar{v}_{p'}(k) = e^{+ip'.(x_m - x_k)} \times (-) u_{-E}\bar{u}_{-E} \quad 45.$$

to be an element of the Feynman kernel for negative energies, the sum of all such terms for all possible realisations giving $K_{-}(m,k)$. In this case we have detached the operators from the wave functions through the derivation of the field operators was based on the concept that the wave function was attached to the operator. This procedure while mathematically valid is not in consonance with the spirit of the derivation of field operators which require the operator to be attached to their wave functions. Also the sequence in which the operators occur is not the Feynman sequence. If we wish to satisfy this requirement of elegance we would insist that we move the operators and wave functions together. In such a case we will have to prescribe a different method.(1) A discussion on this method will be given in Chapter V.

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1) A. Ramakrishnan and N.R.Ranganathan, "Stochastic Methods in quantum mechanics", Jour. of Math. Analysis and Applications (in press) (1961).

5. Consequences of replacing particle by the corresponding antiparticle operators.

We now wish to point out an essential peculiarity in the handling of position operators. We have seen that in obtaining the expression for the field variable for the electron-positron field we have used the interpretation that the annihilation of a negative energy particle is equivalent to the creation of a positive energy antiparticle i.e.

$$a(-\vec{p}, -E) = b^{\dagger}(\vec{p}, +E) \qquad 46$$

But as far as the wave function was concerned, the transpose was not taken i.e. $u(-E)$ was replaced only by $V(+E)$ or in other words, while the electron annihilation operator a_p is attached to its wave function u_p it is the creation operator for the positron, $b^{\dagger}_p$ that goes with its wave function V_p. This we wish to stress implies the existence of the 'sea' of negative particles,

It is also equivalent to (1) the Feynman view point of negative energy particles travelling back in time so that the emergent positron becomes the initial state in a Feynman matrix element and (2) the existence of the 'sea' and the interpretation of the 'hole'.

We shall demonstrate this ~~ex~~ in the evaluation of the matrix element for positron scattering.

For electron scattering the initial state is $|a^{\dagger}_{p_1}\rangle_0$ $a^{\dagger}_{p_1}$ is matched with an a_{p_1} in the perturbation expansion which carries the wave function u_{p_1} and the final electron of momentum p_2 is represented in the interaction term of the S-matrix by $a^{\dagger}_{p_2}$ with the corresponding transposed wave function $\bar{u}_{p_2}$. The $a^{\dagger}_{p_2}$ is matched with a_{p_2} in the final state $\langle a_{p_2}|$ and therefore we have the matrix element in the Feynman form

$$\bar{u}_{p_2}[\text{propagators}]\,u_{p_1} \qquad 47.$$

If on the other hand we had started with a positron state initially there are two alternative view points depending on the definition of the vacuum.

(1) In the first method if we define the true vacuum as so that the initial state is $b_p\rangle_0 = 0$ $b^{\dagger}_{p_1}\rangle_0$, the corresponding annihilation operator in the perturbation sequence would have the transposed wave function $\bar{V}(\vec{p}_1 + E_1)$ attached to it. Thus we would have

$$\langle f|[n]\cdots[k+1][k-1]\cdots[j+1][j-1]\cdots\left(b_{p'}\,b^{\dagger}_{p''}(k)\right)\left[b_{p_1}\,b^{\dagger}_{p'}(j)\right]|b^{\dagger}_{p_1}\rangle_0$$
$$\times\left[\bar{V}_{p'}\,V_{p''}\right]\left[\bar{V}_{p_1}\,V_{p'}\right]$$

48

$$= \langle f|[n]\cdots[R+1][R-1]\cdots[j+1][j-1]\cdots\left[b^{\dagger}_{p''}\, b_{p'}(k)\right]\left[b^{\dagger}_{p'}\, b_{p_1}(j)\right]|b^{\dagger}_{p_1}\rangle_0$$

$$\times\left[\bar{V}_{p_1} V_{p'}\right]\left[\bar{V}_{p'} V_{p''}\right] \qquad 49$$

where since $[\bar{V}\,V]$ is scalar we have 'switched' the two brackets and from

$$\left[b^{\dagger}_{p''}\, b_{p'}(k)\right]\left[b^{\dagger}_{p'}\, b_{p_1}(j)\right] \qquad 50$$

we have for the propagator

$$-\, b^{\dagger}_{p'}(j)\, b_{p'}(k)\; V_{p'}(+E')\,\bar{V}_{p'}(+E')\, e^{ip'(x_j - x_k)}$$

$$= -a(-\vec{p}'-E')\, a^{\dagger}(-\vec{p}'-E')\, u(+E')\,\bar{u}(-E')\, e^{ip'\cdot(x_j - x_k)} \qquad 51$$

and therefore leads to the Feynman negative energy kernel. Thus the matrix element is in the form

$$\bar{V}_{p_1}\left[\text{propagators}\right] V_{p_2}$$

so that the initial state occurs at the left end in essential contrast to the case of electron scattering. This as we have now demonstrated is essentially due to the wave function V being attached to $b^{\dagger}_{p}$ and is consistent with (45).

(2) If we start with the concept of the sea and define the vacuum as $a^{\dagger}(-E)\rangle_0 = 0$ so that the initial positron state is represented as the creation of a 'hole' in the 'sea' of negative energy electrons and we have in this picture

$$\langle f|\,[n]\cdots[j+1][j-1]\cdots[k+1][k-1]\cdots\left[b_{p'}\,b^{\dagger}_{p'}(j)\right]\left[b_{p'}\,b^{\dagger}_{p'}(k)\right]\,|a(-E_1)\rangle_0 \times[\nabla V][\bar{\nabla} V]$$

$$= \langle f|\,[n]\cdots[j+1][j-1]\cdots[k+1][k-1]\cdots\left[a^{\dagger}(-E')\,a(-E'')(j)\right]a^{\dagger}(-E_1)\,a(-E',k)\,|a(-E_1)\rangle_0 \times \bar{u}_{-E_1}[u\bar{u}-E')\,u(-E'')$$

and we have for the propagator

52.

$$[a^{\dagger}_{-E'}(j)\,a_{-E''}(k)]\,u_{-E'}\bar{u}_{-E''}$$

$$= -\,a_{-E''}\,a^{\dagger}_{-E'}(u\bar{u}_{-E'})\,e^{ip'\cdot(x_j-x_k)} \qquad 53.$$

which again leads to the Feynman matrix element (48). Thus we emphasise that no matter how we may rearrange the operators as far as positron scattering is concerned, we shall always arrive at the matrix element with the initial wave function V_{p_1} occurring at the extreme left in the form of the transpose $\bar{V}_{p_1}$ i.e. the Feynman initial state is the wave function of the emergent particle V_{p_2} .

Till now we have interpreted only the annihilation of a negative energy particle as the creation of a positive energy antiparticle. The question naturally arises whether one can interpret quite generally the creation and annihilation of particles with a given energy as the annihilation and creation of antiparticles with opposite energy

i.e. $$a(-E) = b^{\dagger}(+E)$$
$$a^{\dagger}(-E) = b(+E)$$

We emphasise that there is no logical inconsistency in this view point. In fact, if we now follow the convention that in a matrix element all annihilation operators should be placed in the final state while all creation operators are in the initial state, then the 'physical' and 'non-physical' regions in dispersion theory can be easily understood from the S- matrix formalism.

If, for instance we have a positron in the initial state and replace the corresponding $b^{\dagger}(+E)$ by an $a(-E)$ then since $a(-E)$ is an annihilation operator we 'shift' it to the final state. [illegible] the final state $b(+E_2)$ is replaced by $a^{\dagger}(-E_2)$ and brought to the extreme right. Thus we have

$$\langle a(-p_1 - E_1) | S | a^{\dagger}(-p_2 - E_2) \rangle \qquad 54$$

In collision theory a process is characterized by the particles in the initial and final states and the above represents the scattering of an electron with negative energy - i.e. in the non-physical region. It is to be noted that we have not tampered with the photon operators. Thus if the first process represents an incoming electron (of four momentum p_1 $(\vec{p}_1 E_1)$ and photon of four momentum q_1 giving a final electron of four momentum p_2 $(\vec{p}_2\ E_2)$ then by our operation we have

arrived at a process with initial four momenta q_1 and $-p_2$ and final four momenta q_2 and $-p_1$. Defining the energy variable for the first process as the square of the sum of the initial four momenta, i.e.

$$s = (p_1 + q_1)^2 = (p_2 + q_2)^2 \qquad 55$$

we find that for the second process it is

$$\bar{s} = (q_1 - p_2)^2 = (q_2 - p_1)^2 \qquad 56$$

which is nothing but the momentum transfer squared of the first process. Thus for a given process in the physical region of energy, our operation leads us to an identical matrix element which represents the 'crossed' process in the non-physical region i.e. the fermions having negative energy. These two matrix elements actually represent the right and left hand cuts on the real axis of the complex energy plane. It should be stressed that on these arguments it is possible to establish the link between these cuts, only but one cannot bridge the actual unphysical gap between $-m$ and $+m$ (where m is the mass of the fermion). This requires the principle of analytical continuation which led to the now well-known Mandelstam representation.

The general procedure in a dispersion theoretic approach when we are interested in a given process is to arrive at the 'nearby' singularities on arguments based on the conservation of quantum numbers in initial, final and intermediate states which are now real. The singularities are poles in the case of single particle intermediate states and branch points for

two or more particle intermediate states. The former arise essentially due to the vanishing of the propagator for real intermediate states and our previous discussions on scattering has to be modified by the insertion of an $i\epsilon$ in the denominator which decomposes the propagator into a principal value and a δ function part is

$$\frac{1}{\not{p} - m - i\epsilon} = P \frac{1}{\not{p} - m} + i\pi \delta(\not{p} - m) \qquad 57$$

In determining the allowed intermediate state, we make use of the conservation laws of quantum numbers like the isospin and strangeness.

By "order of any perturbation expansion" we mean the number of vertices in a Feynman diagram and we have seen that in the particular case of Compton scattering in the n^{th} order, there are $(n-1)$ propagators and $n!$ field theoretic diagrams. In general the number of propagators will depend on the number and nature of the external lines and the number of lines that can be associated with a single vertex, the latter depending on the form of the interaction. In the most general case when there are N in-going and M out-going particles (~~for~~ usually $N = 2$ and $M \geq 3$ for free particles) the correspondence between vertices and propagator is not prescribed in any simple way if we assume that there are more than three lines at a single vertex. However there exists a method by which we can uniquely prescribe the sequence of intermediate states by grouping a class of propagators in such a manner that we can follow

through this sequence from the initial to the final state. We shall now demonstrate that in this sequence we shallnot be concerned with the order of the perturbation expansion but only with the possible intermediate states in a sense we are dealing with a non-perturbative approach leading to the threshold of the methods of dispersion theory.

We shall henceforth characterise a Feynman diagram not by the order but by the number of internal and external lines. While it is true that the sense of time is irrelevant for an internal line in a Feynman diagram the external lines fall into two classes - initial and final. Let there be N initial, M final and L intermediate lines. By this we mean that under the adiabatic hypothesis we have only the N particles at $t = -\infty$ and the M particles at $t = +\infty$. We shall characterise the initial, final and intermediate particles by the four momenta $(p_1, p_2, \cdots p_N)$ $(q_1, q_2, \cdots q_M)$ and $(s_1, s_2, \cdots s_L)$ respectively. The four-momenta $s_1 \cdots s_L$ represent virtual particles i.e. particles off the energy shell. We now state the following rule for the definition of an intermediate state.

Suppose we are able to clip K of the internal lines such that the Feynman diagram separates into two diagrams

(i) and (ii) such that $N \subset (i)$ and $M \subset (ii)$ i.e. the initial and final states get disconnected. (i) now consists of N initial and K outgoing lines. We shall now say that the particles represented by the K internal lines is an intermediate state. Of course all the particles represented by K are off the energy shell. The diagram (i) can now be decomposed into two diagrams by clipping say Q internal lines such that N and K get separated and a similar procedure can be adopted for (ii) and the sequence of intermediate states as we pass from (i) to (ii) uniquely determined. This of course is formal and does not give a detailed analysis of the classes of possible intermediate states. However the apparently trivial interpretation in the definition of the field operator ψ that creation operator of the antiparticle as equivalent to the annihilation operator of a particle of negative energy has non-trivial consequences by our prescription. In fact, even from the concept of elementary perturbation theory the switching of initial and final states has been shown to lead to the 'crossed' processes. Generalising this to our present case, by switching one of the M final lines to an initial one which now represents an antiparticle with opposite momenta we have a system with $N+1$ initial and $M-1$ final lines. By our prescription, the possible intermediate states in the two cases are mutually exclusive by very definition.

We shall attempt to express these ideas in the form of a "factorization theorem" relating to the dynamical variables of the initial, final and intermediate states. Writing the matrix element for N initial and M final particles as $\langle q_1 q_2 \cdots q_M | p_1 p_2 \cdots p_N \rangle$ 58

If a possible intermediate state is characterized by the four momenta $s_1 \cdots s_L$, a contribution occurs due to this to the total matrix element. This is of the form

$$\int \langle q_1 \cdots q_M | \{ s_1 \cdots s_L \} | p_1 \cdots p_N \rangle \quad 59$$

where the integration is over the intermediate variables.

If we now consider the 'switched' process with $N+1$ initial and $M-1$ final particles such that the matrix element is now

$$\langle q_1 \cdots q_{M-1} | p_1 \cdots p_N, -q_M \rangle \quad 60$$

In this form the contribution cannot be expressed as before since the former represents the intermediate state only for the process (M, N) and not for $(M-1, N+1)$. If for the second case the intermediate variables are $s_1' \cdots s_L'$ the contribution to the matrix element is

$$\int \langle q_1 \cdots q_{M-1} | \{ s_1' \cdots s_L' \} | p_1 \cdots p_N - q_M \rangle \quad 61$$

Some difficult questions do arise in explicitly understanding the dependence on the intermediate variables. For instance, we have assumed a sense of time in defining initial

and final states though when we decompose the diagram into (i) and (ii), the K internal lines now become 'final' for (i) and initial for (ii). We should therefore take into consideration the sense of time by suitably switching the lines of the original diagram.

It is to be noted that while the sequence of intermediate states is now uniquely determined, there is no restriction on the order of the Feynman diagram. Hence in spite of the seeming perturbative nature of the arguments the formalism is essentially non-perturbative. We hope to examine the precise meaning of this non-perturbative approach and its relation to the Mandelstam representation.

CHAPTER II

DENSITY CORRELATIONS IN QUANTUM MECHANICS

1. Introductory Remarks

In postulating interactions we have argued that the essential feature of the interaction term is the bilinearity in the fermion and linearity in the boson fields. In the definition of the field operator we have made use of the fact that the annihilation of a negative energy particle is equivalent to the creation of a positive energy antiparticle. The question arises whether this already implies the existence of the Dirac 'sea' or not. It was argued that the postulate of the 'sea' was equivalent to the additional postulate of conservation of fermions. This concept of the 'sea' has very large consequences. In fact, it was shown by Weisskopf that the existence of the 'sea' immediately implies a charge spread for the electron leading to a logarithmically divergent self energy.

It is felt that a deeper insight into the problem can be obtained by introducing different times into the equal time density functions defined by Weisskopf in 1939. This immediately leads to a derivation of the modified Feynman kernel. It is also worthwhile to extend this to multiparticle phenomena through the formalism of quantum field theory. The mathematical operations are of course similar to those of the single particle case,

since the creation and annihilation operators are independent of time. The extension is made in two distinct ways :

(1) through the electron-positron field with only positive energy states, and

(2) through the electron field with the 'sea' of negative energy states. It is found that in doing this, the concept of the Feynman kernel for a many particle system comes out as a natural consequence and therefore clarifies the role played by the Pauli principle.

The formalism has two definite applications:

(1) The study of scattering phenomena. This aspect is discussed in the last section of this chapter.

(2) Correlation problems. For instance, in pion nucleus scattering, the seat of the potential is not just the nucleus but the individual nucleons. Thus the double-scattering matrix element would obviously be weighted by the double correlation function . This approach has recently been initiated by Watson[1] and others.

Section 2 : In considering multiparticle systems, we shall first assume that there is no interaction between the particles of the system. In analogy with the concept of product density functions[2] in the study of evolutionary stochastic processes, we define the equal time quantum mechanical operator

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1) K.M.Watson and R.R.Johnston, Nucl.Phys.,28,583 (1961)

2) Alladi Ramakrishnan, Proc. Camb. Phil. Soc., 46 , 595 (1950).

$$\rho_n(r_1 \cdots r_n) = \rho(r_n) \cdots \rho(r_1) = \psi^*(r_n) \psi(r_n) \cdots \psi^*(r_1) \psi(r_1) \quad 1.$$

where the ψ-s are the field operators at the same time t and ρ-s the density operators. To get the density functions for a system in a given state we have to take the expectation value of this operator which will be shown to involve the correctly antisymmetrized (fermions) or symmetrized (bosons) wave functions. We now demonstrate this for sequent product densities f_n [1] at different times which is obtained by replacing the space points r_k by space-time points x_k in (1.).

The definition of the field operator ψ in the case of fermions can take two equivalent forms, one which involves the creation operator of the positron and the annihilation operator of the electron and the other with the annihilation operator of a negative energy electron replacing the creation operator of the positron. We now develop the theory of product densities in both these formalisms.

1 (a) Electron-positron field

The field variable at the space-time x_k is

$$\psi(x_k) = \sum_q \left[a_q u_q e^{-iq \cdot x_k} + b_q^+ v_q e^{iq \cdot x_k} \right] \quad 2.$$

where a_q and b_q^+ are the annihilation operator of an electron and the creation operator of a positron of four-momentum q respectively attached to u and v the

1) Alladi Ramakrishnan and T.K.Radha, Proc.Camb.Phil.Soc. 57 , 843 (1961)

corresponding spinors. The bilinear term in H_{int} at a space-time point x_k would read[1]

$$\sum_{q_i, q_i'} \left[\hat{a}^{\dagger}_{q_1} \hat{a}_{q_1'}(k) + \hat{b}_{q_2} \hat{a}_{q_3'}(k) + \hat{b}^{\dagger}_{q_2} \hat{a}^{\dagger}_{q_3'}(k) + \hat{b}^{\dagger}_{q_4} \hat{b}_{q_4'}(k) \right] = \sum_{\langle q \rangle} \sum_{\alpha} [k] \qquad 3.$$

where the symbol $\wedge$ indicates that the operators are accompanied by their corresponding wave functions.

We shall now apply the above considerations to determine the density correlation function $\mathcal{F}_m^n$ at m points due to n electrons for the following cases :-

(a) One electron and two space-time points ($n=1$, $m=2$)

(b) One electron and three space-time points ($n=1$, $m=3$)

(c) Two electrons and three space-time points ($n=2$, $m=3$)

(d) Two electrons and four space-time points ($n=2$, $m=4$)

a) $$\mathcal{F}_2^{(1)} = \rho(2)\rho(1) = \sum_{\langle q \rangle} \sum_{\alpha} [2][1] \quad \text{where } t_2 > t_1 \qquad 4.$$

The initial system consists of am electron of momentum q_0 and hence one of the brackets has to contain the annihilation operator $\hat{a}_{q_0}$. Associated with this $\hat{a}_{q_0}$ we can have either an $\hat{a}^{\dagger}_{q_1}$ or a $\hat{b}_{q_1}$. If it is an $\hat{a}^{\dagger}_{q_1}$, this will be annihilated at a later time (i.e. at t_2) by an $\hat{a}_{q_1}$ which has to contain an $\hat{a}^{\dagger}_{q_1}(2)$ to its left since the final system is an electron of momentum q_0. If the other hand $\hat{a}_{q_0}$ has a $\hat{b}_{q_1}$ associated with it, a positron should have existed before it could be annihilated so that $\hat{b}^{\dagger}_{q_1}$ should be associated with space-time point (1). Thus the only possible combinations are

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1. This ordering has been prescribed by A Ramakrishnan – a discussion on this is given in Chapter V.

$$\sum_{q'} \hat{a}^{\dagger}_{q_0}(2)\, \hat{a}_{q'}(2)\, \hat{a}^{\dagger}_{q'}(1)\, \hat{a}_{q_0}(1) \qquad 5.$$

and

$$\sum_{q'} \hat{b}_{q'}(2)\, \hat{a}_{q_0}(2)\, \hat{b}^{\dagger}_{q'}(1)\, \hat{a}_{q_0}(1) \qquad 6.$$

The second term can be rewritten as

$$-\sum_{q'} \hat{a}^{\dagger}_{q_0}(1)\, \hat{b}^{\dagger}_{q'}(1)\, \hat{b}_{q'}(2)\, \hat{a}_{q_0}(2) \qquad 7.$$

the negative sign resulting due to the anti-commutation of the operators in the rearrangement.

Thus we see that the contribution to the density consists of two parts:

(i) Starting with $\hat{a}_{q_0}$ at (1) and going to space-time point (2) and (ii) starting with $\hat{a}_{q_0}$ at (2) and going to (1). Since $t_2 > t_1$ (i) represents the electron going forward in time and the kernel to be used turns out to be $\sum_{+q'} \hat{a}_{q'}(2)\, \hat{a}^{\dagger}_{q'}(1)$ $(t_2 > t_1)$ while (ii) represents the electron "going" backward in time and the kernel for this is

$$-\sum_{q'} \hat{b}^{\dagger}_{q'}(1)\, \hat{b}_{q'}(2) = -\sum_{-q'} \hat{a}_{-q'}(1)\, \hat{a}^{\dagger}_{-q'}(2), \quad t_2 > t_1 \qquad 8.$$

Thus if we define

$$K_+(x, x') = \sum_{+q} \hat{a}_q(x)\hat{a}^{\dagger}_q(x') \text{ for } t > t'$$

$$= -\sum_{-q} \hat{a}_{-q}(x)\hat{a}^{\dagger}_{-q}(x') \text{ for } t < t' \qquad 9.$$

we recognise $K_+(x, x')$ to be the Feynman kernel and the product density can then be rewritten as

$$\mathcal{F}_2^{(1)} = \hat{a}^{\dagger}_{q_0}(2)\, K_+(2,1)\, \hat{a}_{q_0}(1) + \hat{a}^{\dagger}_{q_0}(1)\, K_+(1,2)\, \hat{a}_{q_0}(2) \qquad t_2 > t_1$$

This form is also valid for the case $t_2 < t_1$. 10.

(b) By arguments similar to those in (a) we find that when the initial and final system consists of an electron of momentum q_0 in the product

$$\mathcal{F}_3^{(1)} = f(3)f(2)f(1) = \sum_{\langle q \rangle} \sum_{\alpha} [3][2][1] \quad \text{for } t_3 > t_2 > t_1 \qquad 11.$$

only the following five terms give non-zero contributions.

(i) $$\sum_{\langle q \rangle} \hat{a}^+(3)\hat{a}(3)\hat{a}^+(2)\hat{a}(2)\hat{a}^+(1)\hat{a}(1)$$

$$= \sum_{q,q'} \hat{a}^+_{q_0}(3)\left[\hat{a}_{q'}(3)\hat{a}^+_{q'}(2)\hat{a}_{q}(2)\hat{a}^+_{q}(1)\right]\hat{a}_{q_0}(1)$$

$$= \hat{a}^+_{q_0}(3)\, K_+(3,2)\, K_+(2,1)\, \hat{a}_{q_0}(1) \qquad 12.$$

(ii) $$\sum_{\langle q \rangle} \hat{b}(3)\hat{a}(3)\hat{b}^+(2)\hat{a}^+(2)\hat{a}^+(1)\hat{a}(1)$$

$$= -\hat{a}_{q_0}(2)\left[\hat{b}^+_{q}(2)\hat{b}_{q}(3)\hat{a}_{q'}(3)\hat{a}^+_{q'}(1)\right]\hat{a}_{q_0}(1)$$

$$= \hat{a}^+_{q_0}(2)\left[K_+(2,3)\, K_+(3,1)\right]\hat{a}_{q_0}(1) \qquad 13.$$

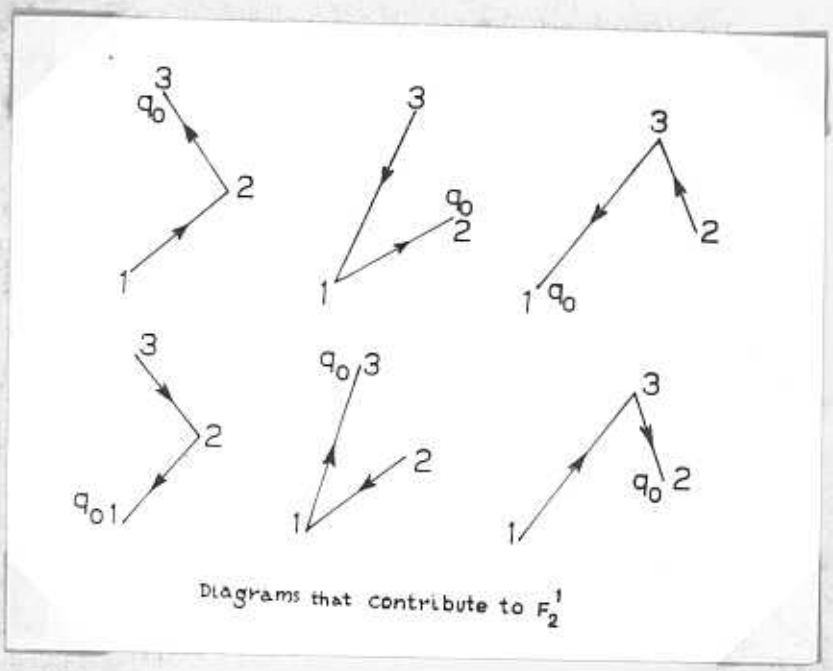
If the above term (ii) reads

$$\hat{b}(3)\hat{a}(3)\hat{a}^+(2)\hat{b}^+(2)\hat{a}^+(1)\hat{a}(1)$$

as it would if we merely use the definition $\bar{\psi}\psi$, then the rearrangement yields no negative sign and

$$\sum_{\langle q \rangle} \hat{b}^+_q \hat{b}_q(3) = +\sum_{-q} \hat{a}_{-q}(2)\hat{a}^+_{-q}(3)$$

cannot be identified with the Feynman kernel $K_+(2,3)$. Similarly the other terms are



Diagrams that contribute to F_2^1

(iii) $\sum_{\langle q \rangle} \hat{a}^{\dagger}(3)\,\hat{a}(3)\,\hat{b}(2)\,\hat{a}(2)\,\hat{b}^{\dagger}(1)\,\hat{a}^{\dagger}(1)$

$$= -\sum_{q,q'} \hat{a}^{\dagger}_{q_0}(3)\left[\hat{a}_q(3)\,\hat{a}^{\dagger}_q(1)\,\hat{b}^{\dagger}_{q'}(1)\,\hat{b}_{q'}(2)\right]\hat{a}_{q_0}(2)$$

$$= \hat{a}^{\dagger}_{q_0}(3)\,K_+(3,1)\,K_+(1,2)\,\hat{a}_{q_0}(2) \qquad 14$$

(iv) $\sum_{\langle q \rangle} \hat{b}(3)\,\hat{a}(3)\,\hat{a}^{\dagger}(2)\,\hat{a}(2)\,\hat{b}^{\dagger}(1)\,\hat{a}^{\dagger}(1)$ 15.

which gives rise to two diagrams

(iv (a)) $-\sum_{q,q'} \hat{a}^{\dagger}_{q_0}(2)\left[\hat{a}_q(2)\,\hat{a}^{\dagger}_q(1)\,\hat{b}^{\dagger}_{q'}(1)\,\hat{b}_{q'}(3)\right]\hat{a}_{q_0}(3)$

$$= \hat{a}^{\dagger}_{q_0}(2)\,K_+(2,1)\,K_+(1,3)\,\hat{a}_{q_0}(3)$$

(iv (b)) $-\sum_{q,q'} \hat{a}^{\dagger}_{q_0}(1)\left[\hat{b}^{\dagger}_q(1)\,\hat{b}_q(3)\,\hat{a}_q(3)\,\hat{a}^{\dagger}_{q'}(2)\right]\hat{a}_{q_0}(2)$

$$= \hat{a}^{\dagger}_{q_0}(1)\,K_+(1,3)\,K_+(3,2)\,\hat{a}_{q_0}(2)$$

(v) $\sum_{\langle q \rangle} \hat{b}(3)\,\hat{a}(3)\,\hat{b}^{\dagger}(2)\,\hat{b}(2)\,\hat{b}^{\dagger}(1)\,\hat{a}^{\dagger}(1)$

$$= \sum_{q,q'} \hat{a}^{\dagger}_{q_0}(1)\left[\hat{b}^{\dagger}_q(1)\,\hat{b}_q(2)\,\hat{b}^{\dagger}_{q'}(2)\,\hat{b}_{q'}(3)\right]\hat{a}_{q_0}(3)$$

$$= \hat{a}^{\dagger}_{q_0}(1)\,K_+(1,2)\,K_+(2,3)\,\hat{a}_{q_0}(3) \qquad 16.$$

(c) $\mathcal{T}_3^{(2)} = \rho(3)\rho(2)\rho(1)$ with $t_3 > t_2 > t_1$ 17.

But in this case we have a system of two electrons of momenta q_0 and q_1 and in addition to the six diagrams for q_0 in (b) we have identical diagrams for q_1. We also have the possibility that when one of the particles, say q_0, exists at a point, the other two points may be connected by a kernel which takes q_1 from one point to the other. Thus we have the terms

$$\sum_{q} \hat{a}^{+}_{q_1}(3)\left[\hat{a}_{q}(3)\,\hat{a}^{+}_{q}(1)\right] a_{q_1}(1)\left[\hat{a}^{+}_{q_0}(2)\,\hat{a}_{q_0}(2)\right]$$

$$= \hat{a}^{+}_{q_1}(3)\, K_{+}(3,1)\, \hat{a}_{q_1}(1)\, N_{q_0}(2) \qquad 18.$$

where N_{q_0} is the number operator for q_0. Three such terms arise for q_0 existing at the three points and similar terms when q_1 exists at the points and q_0 is connected by kernels. Since each kernel can occur in two ways corresponding to propagation forward and backward in time, we have on the whole twelve such terms.

(d) $\mathcal{F}_4^{(2)} = \rho(4)\rho(3)\rho(2)\rho(1)$ with $t_4 > t_3 > t_2 > t_1$ 19.

This is merely an extension of the previous results. We shall however take the terms which lead to the anti-symmetric Feynman kernel for a two particle state, i.e.

$$\sum_{\langle q \rangle} \hat{a}^{+}(4)\,\hat{a}(4)\,\hat{a}^{+}(3)\,\hat{a}(3)\,\hat{a}^{+}(2)\,\hat{a}(2)\,\hat{a}^{+}(1)\,\hat{a}(1)$$

which yields

$$\hat{a}^{+}_{q_0}(4)\,\hat{a}^{+}_{q_1}(3)\,K_+(3,2)K_+(4,1)\,\hat{a}_{q_1}(2)\hat{a}_{q_0}(1)$$

If we and $-\left(-\,\hat{a}^{+}_{q_0}(3)\hat{a}^{+}_{q_1}(4)\,K_+(4,2)\,K_+(3,1)\,\hat{a}_{q_1}(2)\hat{a}_{q_0}(1)\right)$ 20.

$$\sum_{\langle q\rangle}\hat{b}(4)\,\hat{a}(4)\hat{b}(3)\hat{a}(3)\hat{b}^{+}(2)\hat{a}^{+}(2)\hat{b}^{+}(1)\hat{a}^{+}(1)$$

which gives

$$\hat{a}^{+}_{q_0}(2)\hat{a}^{+}_{q_1}(1)K_+(1,4)\,K_+(2,3)\hat{a}_{q_1}(4)\hat{a}_{q_0}(3)$$

$$-\left(-\,a^{+}_{q_0}(1)\,a^{+}_{q_1}(2)\,K_+(2,4)\,K_+(1,3)\,a_{q_1}(4)a_{q_0}(3)\right) \quad 21.$$

If we require the final state to be antisymmetric, we find that the kernel $K_+(4,\,3\,;\,2,\,1)$ is also antisymmetric,

(i.e.)
$$K_+(4,3,2,1) = K_+(4,1)K_+(3,2) - K_+(4,2)K_+(3,1)$$

The extension to more than two particles can be made in the same manner and the n particle kernel can be expressed in the form of a Slater determinant of the single particle kernels.

2. (b) Electron field with negative energy states.

The results of the previous section are now derived by extending the instant product density functions defined by V.S.Weisskopf as early as 1939, while studying the charge distribution in the neighbourhood of an electron. It is interesting to note that a straightforward extension of this definition to sequent product densities leads to the natural deduction of the Feynman kernel not only for a single particle but for a system of n particles.[1)]

In this section we follow Weisskopf's notation and write the instant density operator as

$$\rho(\vec{r}_1) = \psi^*(\vec{r}_1)\,\psi(\vec{r}_1) - \sigma \qquad (3.3)$$

where $\psi(\vec{r}_1)$ is the field variable at the space point $\vec{r}_1$ and σ is the density of the unperturbed electrons in the negative energy 'sea' and has to be subtracted in the 'hole' theory. We can define sequent density operator as

$$\rho(k) = \psi^*(k)\,\psi(k) - \sigma \qquad (3.3)$$

The field variable $\psi(k)$ at the space-time point x_k is given by

$$\psi(k) = \sum_q a_q \phi_q(k)$$

where the sum over q is over both positive and negative energies and ϕ_q's are the electron-wave functions normalized such that

$$\{\phi_q^*(k), \phi_q(k)\} = \frac{1}{V} \qquad (3.4)$$

Thus we have for

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1) It is a curious historical fact that ten years had to elapse since Weisskopf's paper before the Feynman formulated his theory of the positron.

a) $\mathcal{F}_n^m$ for $n=2,\ m=1$

$$\mathcal{F}_2 = \rho(2)\rho(1) = \sum_q \sum_{q'} \sum_{q''} \sum_{q'''} \left(a_q^* \phi_q^*(2) a_{q'} \phi_{q'}(2) a_{q''}^* \phi_{q''}^*(1) a_{q'''} \phi_{q'''}(1) \right)$$
$$- \sigma \left(\sum_q \sum_{q'} a_q^* \phi_q^*(2) a_{q'} \phi_{q'}(2) + \sum_{q''} \sum_{q'''} a_{q''}^* \phi_{q''}^*(1) a_{q'''} \phi_{q'''}(1) \right) \qquad 25.$$
$$+ \sigma^2$$

Since a^* and a satisfy the usual anticommutation relations for creation and annihilation operators only combinations of the form

$$a_q^* a_q a_{q'}^* a_{q'} = N_q N_{q'}$$

and

$$a_q^* a_{q'} a_{q'}^* a_q = N_q (1 - N_{q'}) \qquad 26.$$

which are diagonal give contribution to the expectation value of $\mathcal{F}_2$. Thus

$$\rho(2)\rho(1) = \sum_q \sum_{q'} N_q N_{q'} + \sum_q \sum_{q'} N_q (1 - N_{q'}) \phi_q^*(2) \phi_{q'}(2) \phi_{q'}^*(1) \phi_q(1)$$
$$- 2\sigma \sum_q N_q + \sigma^2 \qquad 27.$$

In the one electron theory we have $\sigma = 0$ and assuming the single particle to be in the state q_0, we have $N_{q_0} = 1$ and $N_q = 0$ for $q \neq q_0$, then for $t_2 > t_1$,

$$\mathcal{F}_2^{(1)} = \rho(2)\rho(1) = \sum_q \phi_{q_0}^*(2) \phi_q(2) \phi_q^*(1) \phi_{q_0}(1) \qquad 28.$$

$\sum_q \phi_q(2) \phi_q^*(1)$ can be easily identified as the unmoidified Feynman kernel $K_0(2,1)$. Thus $\rho(2)\rho(1) = \mathcal{F}_2^{(1)}$ is given by

$$\mathcal{F}_2^{(1)} = \phi_{q_0}^*(2) K_0(2,1) \phi_{q_0}(1) \qquad 29.$$

In the Dirac hole theory $\sigma = \sum_{-q} N_{-q}$ and for the vacuum all $N_{-q} = 1$ and all $N_{+q} = 0$. So the product density function reduces to

$$\mathcal{F}_2^{Vac} = \sum_{+q} \sum_{-q} \phi_{-q}^*(2) \phi_{+q}(2) \phi_{+q}^*(1) \phi_{-q}(1) \qquad 30.$$

This can be interpreted as the vacuum fluctuation effect due

to the negative energy electrons in the 'sea' when there are no positive energy electrons present. Now for vacuum one positive energy electron we have all $N_{-q}=1$, $N_{q_0}=1$ and $N_{+q}=0$ for $q \neq q_0$. Thus

$$\mathcal{J}_2^{vac+1} = \sum_{+q} \phi_{q_0}^*(2)\,\phi_{+q}(2)\,\phi_{+q}^*(1)\,\phi_{q_0}(1) + \sum_{-q} \phi_{-q}^*(2)\left[\sum_{+q} \phi_{+q}(2)\phi_{+q}^*(1) - \phi_{q_0}(2)\phi_{q_0}^*(1)\right]\phi_{-q}(1) \qquad 31.$$

Thus

$$\mathcal{J}_2^{vac+1} - \mathcal{J}_2^{vac} = \sum_{+q} \phi_{q_0}^*(2)\left[\phi_{+q}(2)\,\phi_{+q}^*(1)\right]\phi_{q_0}(1) - \sum_{-q} \phi_{q_0}^*(1)\left[\phi_{-q}(1)\,\phi_{-q}^*(2)\right]\phi_{q_0}(2) \qquad 32.$$

Thus we see that the contribution to the density consists of two parts :

(i) starting with ϕ_{q_0} at (1) and going to space-time point (2) and

(ii) starting with ϕ_{q_0} at (2) and going to (1).

Since $t_2 > t_1$ (i) represents the electron going forward in time and the kernel to be used turns out to be $\sum_{+q} \phi_{+q}(2)\phi_{+q}^*(1)$ with summation over positive energies only for $t_2 > t_1$, while (ii) represents the electron 'going' backward in time and the kernel for this is

$$-\sum_{-q} \phi_{-q}(1)\,\phi_{-q}^*(2)$$

with the summation only over negative energies. Thus if we define

$$K_+(x',x) = \sum_{+q} \phi_{+q}(x')\,\phi^*_{+q}(x) \quad \text{for } t'>t$$
$$= -\sum_{-q} \phi_{-q}(x')\,\phi^*_{-q}(x) \quad \text{for } t'<t \qquad 33.$$

we recognize $K_+(x',x)$ to be the Feynman kernel and the product density can then be rewritten as

$$\mathcal{F}_2^{(1)} = \phi^*_{q_0}(1)\,K_+(1,2)\,\phi_{q_0}(2) + \phi^*_{q_0}(2)\,K_+(2,1)\,\phi_{q_0}(1) \qquad 34.$$

This form is also valid for $t_2 < t_1$.

(b) $\mathcal{F}_n^{n_1}$ for $n = 3$ and $m = 1$:

$$\mathcal{F}_3^{(1)} = P(3)P(2)P(1) \quad \text{for } t_3 > t_2 > t_1$$
$$= \phi^*_{q_0}(3)\,K_+(3,1)\,K_+(2,1)\,\phi_{q_0}(1) + \phi^*_{q_0}(2)\,K_+(2,1)\,K_+(1,3)\,\phi_{q_0}(3)$$
$$+ \phi^*_{q_0}(1)\,K_+(1,3)\,K_+(3,2)\,\phi_{q_0}(2) + \phi^*_{q_0}(1)\,K_+(1,2)\,K_+(2,3)\,\phi_{q_0}(3)$$
$$+ \phi^*_{q_0}(2)\,K_+(2,3)\,K_+(3,1)\,\phi_{q_0}(1) + \phi^*_{q_0}(3)\,K_+(3,1)\,K_+(1,2)\,\phi_{q_0}(2)$$
$$+ \sum_{-q} \phi^*_{-q}(3)\,K_+(3,2)\,\phi_{-q}(2)\,\phi^*_{q_0}(1)\,\phi_{q_0}(1)$$
$$+ \sum_{-q} \phi^*_{-q}(2)\,K_+(2,1)\,\phi_{-q}(1)\,\phi^*_{q_0}(3)\,\phi_{q_0}(3)$$
$$+ \sum_{-q} \phi^*_{-q}(3)\,K_+(3,1)\,\phi_{-q}(1)\,\phi^*_{q_0}(2)\,\phi_{q_0}(2). \qquad 35$$

The last three terms correspond to the contributions to the density from the negative energy electrons in the sea or the physical vacuum and are generally known as "vacuum fluctuation terms". These can be eliminated as should be done in any physically valid theory, by adopting the subtraction

procedure for each pair of points when the particle q_0 is at the third. This procedure can be generalized when there are more than three points.

(c) $\mathcal{F}_3^2 = P(3)P(2)P(1)$ for $t_3 > t_2 > t_1$ is given by

$$\mathcal{F}_3^2 = \phi_{q_0}^*(3) K_+(3,2) K_+(2,1) \phi_{q_0}(1) + \phi_{q_0}^*(2) K_+(2,1) K_+(1,3) \phi_{q_0}(3)$$
$$+ \phi_{q_0}^*(1) K_+(1,3) K_+(3,2) \phi_{q_0}(2) + \phi_{q_0}^*(1) K_+(1,2) K_+(2,3) \phi_{q_0}(3)$$
$$+ \phi_{q_0}^*(2) K_+(2,3) K_+(3,1) \phi_{q_0}(1) + \phi_{q_0}^*(3) K_+(3,1) K_+(1,2) \phi_{q_0}(2)$$
$$+ \phi_{q_1}^*(3) K_+(3,2) K_+(2,1) \phi_{q_1}(1) + \phi_{q_1}^*(2) K_+(2,1) K_+(1,3) \phi_{q_1}(3)$$
$$+ \phi_{q_1}^*(1) K_+(1,3) K_+(3,2) \phi_{q_1}(1) + \phi_{q_1}^*(1) K_+(1,2) K_+(2,3) \phi_{q_1}(3)$$
$$+ \phi_{q_1}^*(2) K_+(2,3) K_+(3,1) \phi_{q_1}(1) + \phi_{q_1}^*(3) K_+(3,1) K_+(1,2) \phi_{q_1}(2)$$
$$+ \phi_{q_0}^*(3) K_+(3,2) \phi_{q_0}(2) \phi_{q_1}^*(1) \phi_{q_1}(1) + \phi_{q_1}^*(3) K_+(3,2) \phi_{q_1}(2) \phi_{q_0}^*(1) \phi_{q_0}(1)$$
$$+ \phi_{q_0}^*(2) K_+(2,1) \phi_{q_0}(1) \phi_{q_1}^*(3) \phi_{q_1}(3) + \phi_{q_1}^*(2) K_+(2,1) \phi_{q_1}(1) \phi_{q_0}^*(3) \phi_{q_0}(3)$$
$$+ \phi_{q_0}^*(3) K_+(3,1) \phi_{q_0}(1) \phi_{q_1}^*(2) \phi_{q_1}(2) + \phi_{q_1}^*(3) K_+(3,1) \phi_{q_1}(1) \phi_{q_0}^*(2) \phi_{q_0}(2)$$
$$+ \phi_{q_0}^*(2) K_+(2,3) \phi_{q_0}(3) \phi_{q_1}^*(1) \phi_{q_1}(1) + \phi_{q_1}^*(2) K_+(2,3) \phi_{q_1}(3) \phi_{q_0}^*(1) \phi_{q_0}(1)$$
$$+ \phi_{q_0}^*(1) K_+(1,2) \phi_{q_0}(2) \phi_{q_1}^*(3) \phi_{q_1}(3) + \phi_{q_1}^*(1) K_+(1,2) \phi_{q_1}(2) \phi_{q_0}^*(3) \phi_{q_0}(3)$$
$$+ \phi_{q_0}^*(1) K_+(1,3) \phi_{q_0}(3) \phi_{q_1}^*(2) \phi_{q_1}(2) + \phi_{q_1}^*(1) K_+(1,3) \phi_{q_1}(3) \phi_{q_0}^*(2) \phi_{q_0}(2)$$
$$- \phi_{q_0}^*(3) K(3,1) \phi_{q_0}(1) \phi_{q_0}^*(2) \phi_{q_0}(2) - \phi_{q_1}^*(3) K_+(3,1) \phi_{q_1}(1) \phi_{q_1}^*(2) \phi_{q_1}(2)$$

36

$+$ Vacuum fluctuation terms.

Thus we have the six diagrams corresponding to those in (b) for the two electrons of energy q_0 and q_1 and in addition we also have the terms corresponding to the existence of the particle $q_0 (q_1)$ at one of the points while the other two points are connected by the kernel which takes $q_1 (q_0)$ from one to the other.

(d) $\mathcal{F}_n^m$ for $m = 2$ with $n = 4$.

This is merely an extension of the previous results, and ~~since~~ no new principle is involved, we ~~relegate the expression to the Appendix.~~ We shall however group those terms which lead to the antisymmetric Feynman kernel for a two particle state, i.e.

$$\phi_{q_0}^*(4)\,\phi_{q_1}^*(3)\,K_+(3,2)\,K_+(4,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)$$
$$+\,\phi_{q_0}^*(3)\,\phi_{q_1}^*(4)\,K_+(4,2)\,K_+(3,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)$$
$$=\,\phi_{q_0}^*(4)\,\phi_{q_1}^*(3)\,K_+(3,2)\,K_+(4,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)$$
$$-\left[-\,\phi_{q_0}^*(3)\,\phi_{q_1}^*(4)\,K_+(4,2)\,K_+(3,1)\,\phi_{q_1}(2)\,\phi_{q_0}(1)\right]$$

If we require the final state to be antisymmetric, we find that the kernel $K_+(4, 3; 2, 1)$ is also antisymmetric (i.e.)

$$K_+(4,3;2,1) = K_+(4,1)K_+(3,2) - K_+(4,2)K_+(3,1)$$

The extension to more than two particles follows immediately and the n particle kernel can be expressed in the form of a Slater determinant of the single particle kernels.

It is relevant to draw attention to some observations made by Feynman[1] in this connection. In the fundamental paper in which he introduced the kernel formalism he made a special reference to the connection between the minus sign occurring before the sum over the negative energy states in the definition of $K_+(2, 1)$ (when $t_2 < t_1$) and the Pauli principle. Later in introducing the kernel corresponding to two particles, the antisymmetrical form was used by calling into aid the Pauli principle. Therefore it was clear that the existence of the minus sign in the negative energy part of K_+ while consistent with the Pauli principle, is not coequal with it. In spite of this Feynman seems to have attempted to deduce the Pauli principle from the single particle kernel itself by studying the probability for an electron to go from (1) to (2) with any number of virtual pairs occurring and proving that an inconsistency occurs unless the Pauli principle is invoked.

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1) R.P. Feynman, Phys. Rev., 76, (1949). See discussions on pages 753 and 755.

Our considerations however lead to the conclusion that the kernels of single particle, two parties and for n, can be deduced by once assuming the Pauli principle and therefore it is not possible to speak of the single particle kernel itself as implying the principle in its entirety. Or in other words, the Pauli principle implies at once that the kernel for the single particle is K_+ and for two particles is antisymmetric.

2 (c) Density correlations in the presence of interaction

Till now we have been considering the case when there was no interaction, (i.e.) the state of the system was the same from $t=-\infty$ to $t=+\infty$ and to get the density function we had to take the expectation value of the density operator between the same state $|i\rangle$. However if we "switch on" the interaction, the final state is not the same as the initial state and therefore the transition matrix element from a definite $|i\rangle$ to $|f\rangle$ is given by $\langle f|S|i\rangle$. The S matrix is an integral over space and time given by

$$S = P \exp(-i) \int \mathcal{H}^{int}(x)\, d^4x \qquad 39.$$

where P is the Dyson operator and $\mathcal{H}$ is the interaction Hamiltonian density. Since density operators are functions of space and time we have to take the expectation value when the interaction is on, i.e. $T(\mathcal{F}_2 S)$

where T is Wick's time operator with the understanding that we have to write the set of operators in $\mathcal{F}_2$, then feed in the S matrix and use the time ordered product in the integrand. Then we have

$$T\{f(2)\rho(1)S\} = T\{\bar{\psi}(2)T[\bar{\psi}(2)\bar{\psi}(1)]\psi(1)\}$$
$$= T\{\psi^*(2)\, G(2,1)\, \psi(1)\} \qquad 40.$$

where $G(2,1)$ is recognised to be the propagator in the interaction representation defined as

$$G(2,1) = \frac{\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^4x_{m_n} d^4x_{m_{n-1}} \cdots d^4x_{m_1} \langle T\{\bar{\psi}(2)\psi(1) \times \mathcal{H}^{int}(x_{m_n})\mathcal{H}^{int}(x_{m_{n-1}}) \cdots \mathcal{H}^{int}(x_{m_1})\}\rangle}{\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d^4x_{m_n} d^4x_{m_{n-1}} \cdots d^4x_{m_1} \langle T\{\mathcal{H}^{int}(x_{m_n}) \times \mathcal{H}^{int}(x_{m_{n-1}}) \cdots \mathcal{H}^{int}(x_{m_1})\}\rangle}$$

41.

Taking the simplest case of an electron being scattered from an initial state p to a final state p' in a potential field which is treated as classical and not quantized, we find that

$$\langle a_{p'}, \mathcal{F}_2 S\, a_p^{\dagger}\rangle_0 = \int_{x_1} \int_{x_2} \bar{u}_{p'} e^{ip'\cdot x_2} G(2,1)\, u_p e^{-ip\cdot x_1} d^4x_2 d^4x_1$$

$$= \bar{u}_{p'} G(p',p) u_p \qquad 42.$$

where $\langle a_{p'}, \mathcal{F}_2 S\, a_p^{\dagger}\rangle_0$ is the expectation value of $\mathcal{F}_2 S$ obtained by integrating over x_1 and x_2.

It is expected that the above considerations when applied to scattering processes will be useful in understanding the structure of the phenomenon. Also the correlation functions are of some importance in nuclear problems. For instance, in the study of pion nucleus scattering the pion-nucleon interaction is modified by the presence of the remaining nucleons in the nucleus and in the second order matrix element the effect may be included through the introduction of double correlation functions.[1)]

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1) R.R.Johnsen and K.M.Watson, Nuclear Physics, 28 , 583 (1961)

CHAPTER III

ON THE DECOMPOSITION OF THE FEYNMAN PROPAGATOR[1]

1. Introductory Remarks

It is well-known that with the advent of the Dirac theory, problems in quantum electrodynamics had to include the effect of negative energy in intermediate states. But the Feynman formulation of quantum electrodynamics characterized by its "inherent covariance" made the distinction between positive and negative energy in intermediate states unnecessary. This formalism of course guarantees covariance under proper Lorentz transformations which however cannot produce a change from positive to negative energy. It is therefore felt by the author that a better understanding of virtual processes is possible if we can study the separate contributions to the matrix element from positive and negative energy intermediate states.

This is possible if, in a perturbation expansion the space integration is ~~perhaps~~ performed first and the time integration subsequently in the "old-fasnioned" manner unlike as in the Feynman formulation where the integrations over space and time are performed together.

This method gives a clearer picture of the contributions due to the different types of virtual processes. It leads to the splitting of the Feynman propagator and while not affecting the elegance of the inherently relativistic approach, reveals the structure of the Feynman propagator in a manner which facilitates the computation of the relative contributions from transitions

1) A. Ramakrishnan, T. K. Radha and R. Thunga, Proc. Ind. Acad. Sc. LII 228 (1960)

to positive and negative energy intermediate states. In particular, calculations lead to the apparently paradoxical result that the negative energy states do contribute even if the electron in Compton scattering is non-relativistic. It is conventionally accepted that when the initial particles are non-relativistic, the energy denominators become large for negative menergy states and therefore the contribution in such cases can be neglected. This assumption may lead to erroneous results since the numerators may also become large and this can only be studied by the use of the decomposed propagator[1].

2a Decomposition of the Feynman Propagator

In the Feynman picture since a negative energy electron travels back in time, the positive and negative energy parts of the intermediate state can be separated

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1) The paradoxical result that the negative energy states should contribute even in the non-relativistic case seems to have been well recognized on the ground that the operator γ_μ connects the positive and negative energy states. What we wish to emphasise here is that the relative contributions can be studied by decomposing the Feynman propagator, the energy denominators in this case being naturally different from the field theoretic case. This is so because the definition of the intermediate states in field theory refers to systems of electrons, positrons and photons while in the Feynman formalism it relates only to the electron.

by splitting the time integration in the perturbation expansion (i.e. from $t = -\infty$ to $t = +\infty$) into two parts corresponding to the ranges 0 to ∞ with energy $+E$ and $-\infty$ to 0 with energy $-E$ respectively. The integration over space variables alone in a perturbation expansion say for a process in which an electron of momentum $\vec{p}$ absorbs a photon of momentum $\vec{q}$, would amount to picking out from the kernel, terms corresponding to momentum $\vec{p}+\vec{q}$ and energy E equal to

$$\pm E_{\vec{p}+\vec{q}} = \pm\sqrt{(\vec{p}+\vec{q})^2 + m^2} \qquad 1.$$

If we now perform the time integrations separately corresponding to $+E_{\vec{p}+\vec{q}}$ and $-E_{\vec{p}+\vec{q}}$ we have, for the Feynman propagator in momentum representation,

$$\frac{1}{\not{p}+\not{q}-m} = \frac{1}{2}\left[\frac{\overset{+}{\not{P}}+m}{(E_{\vec{p}+\vec{q}})(E_{\vec{p}}+E_{\vec{q}}-E_{\vec{p}+\vec{q}})} - \frac{\overset{-}{\not{P}}+m}{(E_{\vec{p}+\vec{q}})(E_{\vec{p}}+E_{\vec{q}}+E_{\vec{p}+\vec{q}})}\right] \qquad 2.$$

where $P = p+q$ and $\overset{+}{\not{P}}$ is the Feynman four vector with energy $+E_{\vec{p}+\vec{q}}$ and $\overset{-}{\not{P}}$ has the fourth component equal to $-E_{\vec{p}+\vec{q}}$.

We shall now make use of the above expression for the propagator and calculate the cross-sections for Compton scattering taking into account intermediate states of positive energy only, forbidding intermediate states of negative energy and <u>vice-versa</u>.

The matrix element representing the scattering of an incident photon of four momentum q_1 by an electron (at rest) to momentum q_2 the electron having a momentum p_2 is obtained by considering the two possibilities (1) the photon q_1 being absorbed and q_2 being emitted subsequently in the Feynman sense and (2) q_2 being emitted and q_1 absorbed subsequently.

The propagators for the two cases are given

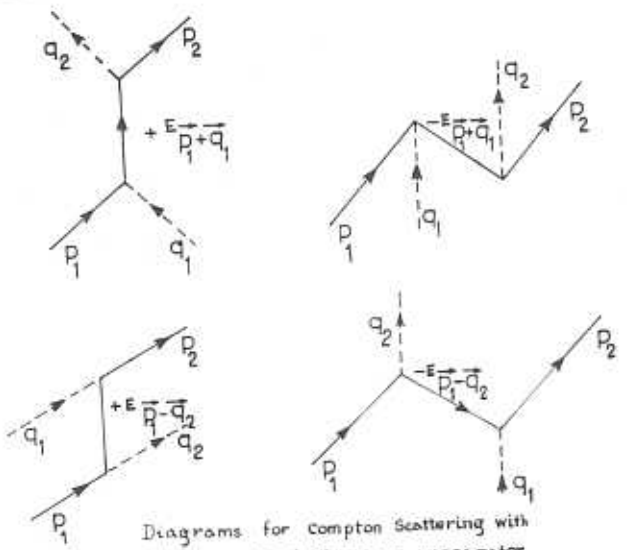
$$\frac{1}{\not{p}_1 + \not{q}_1 - m} = \frac{1}{2}\left[\frac{\not{\vec{p}}_1 + m}{(E_{\vec{p}_1+\vec{q}_1})(E_{\vec{p}_1} + E_{\vec{q}_1} - E_{\vec{p}_1+\vec{q}_1})} - \frac{\not{\bar{p}}_1 + m}{(E_{\vec{p}_1+\vec{q}_1})(E_{\vec{p}_1} + E_{\vec{q}_1} + E_{\vec{p}_1+\vec{q}_1})} \right] \quad 3.$$

where $\not{p}_1$ has energy component $E_{\vec{p}_1+\vec{q}_1}$ and

$$\frac{1}{\not{p}_1 - \not{q}_2 - m} = \frac{1}{2}\left[\frac{\not{\vec{p}}_2 + m}{(E_{\vec{p}_1-\vec{q}_2})\{E_{\vec{p}_1} - E_{\vec{q}_2} - E_{\vec{p}_1-\vec{q}_2}\}} - \frac{\not{\bar{p}}_2 + m}{(E_{\vec{p}_1-\vec{q}_2})(E_{\vec{p}_1} - E_{\vec{q}_2} + E_{\vec{p}_1-\vec{q}_2})} \right] \quad 4$$

where $\not{p}_2$ has energy component $E_{\vec{p}_1-\vec{q}_2}$.

The cross-sections for the entire process taking the effect of positive energy alone, ignoring the negative energy part in the intermediate state can be calculated by using the sum of the first terms of (3) and (4) as the propagator. Thus the matrix element after substituting for $\not{\vec{p}}_1$ etc., is



Diagrams for compton Scattering with the decomposed Feynman propagator

$$M^{(+)} = \left(\frac{-i}{2}\right)(4\pi e^2)\,\bar{u}_2\left[\not{\epsilon}_2\,\frac{\gamma_t\sqrt{\omega_1^2+m^2}-\gamma_x\omega_1+m}{\sqrt{\omega_1^2+m^2}\left\{m+\omega_1-\sqrt{\omega_1^2+m^2}\right\}}\not{k}_1\right.$$

$$\left.+\not{\epsilon}_1\,\frac{\gamma_t\sqrt{\omega_2^2+m^2}+\gamma_x\omega_2\cos\theta+\gamma_y\omega_2\sin\theta+m}{\sqrt{\omega_2^2+m^2}\left\{m-\omega_2-\sqrt{\omega_2^2+m^2}\right\}}\not{k}_2\right]u_1. \qquad 5.$$

$$M^{(-)} = \left(\frac{+i}{2}\right)(4\pi e^2)\,\bar{u}_2\left[\not{\epsilon}_2\,\frac{-\gamma_t\sqrt{\omega_1^2+m^2}-\gamma_x\omega_1+m}{\sqrt{\omega_1^2+m^2}\left\{m+\omega_1+\sqrt{\omega_1^2+m^2}\right\}}\not{k}_1\right.$$

$$\left.+\not{\epsilon}_1\,\frac{-\gamma_t\sqrt{\omega_2^2+m^2}+\gamma_x\omega_2\cos\theta+\gamma_y\omega_2\sin\theta+m}{\sqrt{\omega_2^2+m^2}\left\{m-\omega_2+\sqrt{\omega_2^2+m^2}\right\}}\not{k}_2\right]u_1 \qquad 6.$$

Taking the incoming photon to be along the $+x$ direction we have for polarisation

(A) $\not{\epsilon}_1 = \gamma_z$ or (B) $\not{\epsilon}_1 = \gamma_y$

and for the out-going photon

(A′) $\not{\epsilon}_2 = \gamma_z$ or (B′) $\not{\epsilon}_2 = \gamma_y\cos\theta - \gamma_x\sin\theta$

Thus the matrix elements M^+ and M^- for various polarisation combinations are given by

$$M^+(AB' \text{ or } BA') = \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i}{2}\right)(4\pi e^2)\left[\frac{F_1F_2\left\{2\sqrt{\omega_1^2+m^2}-im\right\}e^{i\theta}+i\omega_1F_1p_{2+}e^{i\theta}}{\sqrt{\omega_1^2+m^2}\left\{m+\omega_1-\sqrt{\omega_1^2+m^2}\right\}}\right.$$

$$\left.+\frac{F_1F_2\left\{-i\sqrt{\omega_2^2+m^2}+im\right\}e^{i\theta}+i\omega_2F_1p_{2+}}{\sqrt{\omega_2^2+m^2}\left\{m-\omega_2-\sqrt{\omega_2^2+m^2}\right\}}\right] \qquad 7.$$

where $p_{2+} = p_{2x} + ip_{2y}$.

$$M^{(-)}(AB' \text{ or } BA') = \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i4\pi e^2}{2}\right)\left[\frac{F_1F_2(i\sqrt{\omega_1^2+m^2}+im)e^{i\theta} - i\omega_1 F_1 p_{2+} e^{i\theta}}{\sqrt{\omega_1^2+m^2}\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left. + \frac{F_1F_2\{-i\sqrt{\omega_2^2+m^2}-im\}e^{i\theta} - i\omega_2 F_1 p_{2+}}{\sqrt{\omega_2^2+m^2}\{m-\omega_2+\sqrt{\omega_2^2+m^2}\}}\right] \quad 8.$$

$$M^{(+)}(AA') \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i4\pi e^2}{2}\right)\left[\frac{F_1F_2\{\sqrt{\omega_1^2+m^2}-m\} - \omega_1 F_1 p_{2-}}{\sqrt{\omega_1^2+m^2}\{m+\omega_1-\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left. + \frac{F_1F_2\{\sqrt{\omega_1^2+m^2}-m\} + \omega_2 F_1 p_{2-} e^{i\theta}}{\sqrt{\omega_2^2+m^2}\{m-\omega_2-\sqrt{\omega_2^2+m^2}\}}\right] \quad 9.$$

$$M^{(-)}(AA') = \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i4\pi e^2}{2}\right)\left[\frac{F_1F_2\{\sqrt{\omega_1^2+m^2}+m\} + \omega_1 F_1 p_{2-}}{\sqrt{\omega_1^2+m^2}\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left. + \frac{F_1F_2\{\sqrt{\omega_2^2+m^2}+m\} - \omega_2 F_1 p_{2-} e^{i\theta}}{\sqrt{\omega_2^2+m^2}\{m-\omega_2+\sqrt{\omega_2^2+m^2}\}}\right] \quad 10.$$

$$M^{+}(BB') = \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i4\pi e^2}{2}\right)\left[\frac{F_1F_2\{\sqrt{\omega_1^2+m^2}e^{-i\theta} - me^{-i\theta}\} - \omega_1 F_1 p_{2-} e^{i\theta}}{\sqrt{\omega_1^2+m^2}\{m+\omega_1-\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left. + \frac{F_1F_2\{\sqrt{\omega_2^2+m^2}e^{i\theta} - me^{i\theta}\} + \omega_2 F_1 p_{2-}}{\sqrt{\omega_2^2+m^2}\{m-\omega_2-\sqrt{\omega_2^2+m^2}\}}\right] \quad 11.$$

and

$$M^{(-)}(BB') = \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i4\pi e^2}{2}\right)\left[\frac{F_1F_2\{\sqrt{\omega_1^2+m^2}e^{-i\theta} + me^{-i\theta}\} + \omega_1 F_1 p_{2-} e^{i\theta}}{\sqrt{\omega_1^2+m^2}\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left. + \frac{F_1F_2\{\sqrt{\omega_2^2+m^2}e^{i\theta} + me^{i\theta}\} - \omega_2 F_1 p_{2-}}{\sqrt{\omega_2^2+m^2}\{m-\omega_2+\sqrt{\omega_2^2+m^2}\}}\right] \quad 12.$$

where $F_2 = E_2+m$, $F_1 = E_1+m$.

For any one of the polarization cases considered above, we

Sum over the initial spin states of the electron and average over the final spin states to obtain the cross-section. We shall now list $|M|^2$ for the various combinations of polarization.

$$|M^{(+)}(AB' \text{ or } BA')|^2 = \frac{1}{4F_1F_2}(4\pi e^2)^2 \left[\frac{F_1^2F_2^2(\omega_1^2+2m^2-2m\sqrt{\omega_1^2+m^2}) + F_1^2\omega_1^2\vec{p}_2^{\,2} + (\sqrt{\omega_1^2+m^2}-m)\,2F_1^2F_2\omega_1 p_{2x}}{(\omega_1^2+m^2)\{2m^2+2\omega_1^2+2m\omega_1-2(m+\omega_1)\sqrt{\omega_1^2+m^2}\}} \right.$$

$$- \frac{F_1^2F_2^2(\omega_2^2+2m^2-2m\sqrt{\omega_2^2+m^2}) + F_1^2\omega_2^2\vec{p}_2^{\,2} - 2F_1^2F_2\omega_2(p_{2x}\cos\theta + p_{2y}\sin\theta)(\sqrt{\omega_2^2+m^2}-m)}{(\omega_2^2+m^2)\{2m^2+2\omega_2^2-2m\omega_2-2(m-\omega_2)\sqrt{\omega_2^2+m^2}\}}$$

$$\left. + \frac{-2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}-m)(\sqrt{\omega_2^2+m^2}-m) + 2F_1^2\omega_1\omega_2\vec{p}_2^{\,2}\cos\theta + 2F_1^2F_2\omega_2(p_{2x}\cos\theta + p_{2y}\sin\theta)(\sqrt{\omega_1^2+m^2}-m) - 2F_1^2F_2p_{2x}\omega_1(\sqrt{\omega_2^2+m^2}-m)}{\sqrt{\omega_1^2+m^2}\sqrt{\omega_2^2+m^2}\{m+\omega_1-\sqrt{\omega_1^2+m^2}\}\{m-\omega_2-\sqrt{\omega_2^2+m^2}\}} \right] \quad 13.$$

Denoting the denominators of the three terms as $D_1^{(+)}$, $D_2^{(+)}$ and $D_3^{(+)}$ respectively, we have

$$|M^{(+)}(AA')|^2 = \frac{1}{4F_1F_2}(4\pi e^2)^2 \left[\frac{1}{D_1^{(+)}}\{F_1^2F_2^2(\omega_1^2+2m^2-2m\sqrt{\omega_1^2+m^2}) - 2F_1^2F_2\omega_1 p_{2x}(\sqrt{\omega_1^2+m^2}-m) + F_1^2\omega_1^2\vec{p}_2^{\,2}\} \right.$$

$$+ \frac{1}{D_2^{(+)}}\{F_1^2F_2^2(\omega_2^2+2m^2-2m\sqrt{\omega_2^2+m^2}) + F_1^2\omega_2^2\vec{p}_2^{\,2} + 2F_1^2F_2\omega_2(p_{2x}\cos\theta + p_{2y}\sin\theta)(\sqrt{\omega_2^2+m^2}-m)\}$$

$$\left. + \frac{1}{D_3^{(+)}}\{2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}-m)(\sqrt{\omega_2^2+m^2}-m) - 2\omega_1\omega_2F_1^2\vec{p}_2^{\,2}\cos\theta + 2F_1^2F_2\omega_2(\sqrt{\omega_1^2+m^2}-m)(p_{2x}\cos\theta + p_{2y}\sin\theta) - 2F_1^2F_2\omega_1p_{2x}(\sqrt{\omega_2^2+m^2}-m)\}\right] \quad 14$$

$$|M^{(+)}(BB')|^2 = \frac{1}{4F_1F_2}(4\pi e^2)^2\Big[\frac{1}{D_1^{(+)}}\Big\{F_1^2F_2^2(\omega_1^2+2m^2-2m\sqrt{\omega_1^2+m^2})+F_1^2\omega_1^2$$
$$-2F_1^2F_2\omega_1(\sqrt{\omega_1^2+m^2}-m)(p_{2x}\cos 2\theta+p_{2y}\sin 2\theta$$
$$+\frac{1}{D_2^{(+)}}\Big\{F_1^2F_2^2(\omega_2^2+2m^2-2m\sqrt{\omega_2^2+m^2})+F_1^2\vec{p}_2^{\,2}\omega_2^2$$
$$+2F_1^2F_2\omega_2(\sqrt{\omega_2^2+m^2}-m)(p_{2x}\cos\theta-p_{2y}\sin\theta)\Big\}$$
$$+\frac{1}{D_3^{(+)}}\Big\{2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}-m)(\sqrt{\omega_2^2+m^2}-m)\cos 2\theta-2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta$$
$$+2F_1^2F_2\omega_2(\sqrt{\omega_1^2+m^2}-m)(p_{2x}\cos\theta+p_{2y}\sin\theta)$$
$$-2F_1^2F_2\omega_1(\sqrt{\omega_2^2+m^2}-m)p_{2x}\Big\}\Big] \qquad 15.$$

Similarly in the case of negative energy, we have

$$|M^{(-)}(AB' \text{ or } BA')|^2 =$$
$$\frac{(4\pi e^2)^2}{4F_1F_2}\Bigg[\frac{F_1^2F_2^2(\omega_1^2+2m^2+2m\sqrt{\omega_1^2+m^2})-2F_1^2F_2\omega_1p_{2x}(\sqrt{\omega_1^2+m^2}+m)+F_1^2\omega_1^2\vec{p}_2^{\,2}}{(\omega_1^2+m^2)(2m^2+2\omega_1^2+2m\omega_1+2\sqrt{\omega_1^2+m^2}(m+\omega_1)}$$
$$+\frac{F_1^2F_2^2(\omega_2^2+2m^2+2m\sqrt{\omega_2^2+m^2}+F_1^2\omega_2^2\vec{p}_2^{\,2}+2F_1^2F_2\omega_2(p_{2x}\cos\theta+p_{2y}\sin\theta)(\sqrt{\omega_2^2+m^2}+\cdots}{(\omega_2^2+m^2)\{2m^2+2\omega_2^2-2m\omega_2+2\sqrt{\omega_2^2+m^2}(m-\omega_2)\}}$$
$$-2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}+m)(\sqrt{\omega_2^2+m^2}+m)+2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta$$
$$-2F_1^2F_2(p_{2x}\cos\theta+p_{2y}\sin\theta)(\sqrt{\omega_1^2+m^2}+m)+2F_1^2F_2\omega_1p_{2x}(\sqrt{\omega_2^2+m^2}+\cdots$$
$$+\frac{}{\sqrt{\omega_1^2+m^2}\,\sqrt{\omega_2^2+m^2}\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}\{m-\omega_2+\sqrt{\omega_2^2+m_2^2}\}}\Bigg] \qquad 16$$

Denoting the denominators in (15) by $D_1^{(-)}$, $D_2^{(-)}$ and $D_3^{(-)}$ respectively we have

$$|M^{(-)}(AA')|^2$$

$$= \frac{(4\pi e^2)^2}{4F_1F_2}\left[\frac{1}{D_1^{(-)}}\left\{F_1^2F_2^2(\omega_1^2+2m^2+2m\sqrt{\omega_1^2+m^2})+F_1^2\omega_1^2\vec{p}_2^{\,2}\right.\right.$$
$$\left.+2F_1^2F_2p_{2x}\omega_1(\sqrt{\omega_1^2+m^2}+m)\right\}$$
$$+\frac{1}{D_2^{(-)}}\left\{F_1^2F_2^2(\omega_2^2+2m^2+2m\sqrt{\omega_2^2+m^2}+F_1^2\omega_2^2\vec{p}_2^{\,2}\right.$$
$$\left.-2F_1^2F_2\omega_2(p_{2x}\cos\theta+p_{2y}\sin\theta)(\sqrt{\omega_2^2+m^2}+m)\right\}$$
$$+\frac{1}{D_3^{(-)}}\left\{2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}+m)(\sqrt{\omega_2^2+m^2}-m)-2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta\right.$$
$$-2F_1^2F_2\omega_2(p_{2x}\cos\theta+p_{2y}\sin\theta)(\sqrt{\omega_1^2+m^2}+m)$$
$$\left.\left.+2F_1^2F_2p_{2x}\omega_1(\sqrt{\omega_2^2+m^2}+m)\right\}\right] \quad 17.$$

$$|M^{-}(BB')|^2$$

$$= \frac{(4\pi e^2)^2}{4F_1F_2}\left[\frac{1}{D_1^{(-)}}\left\{F_1^2F_2^2(\omega_1^2+2m^2+2m\sqrt{\omega_1^2+m^2}+F_1^2\omega_1^2\vec{p}_2^{\,2}\right.\right.$$
$$\left.+2F_1^2F_2\omega_1(p_{2x}\cos2\theta+p_{2y}\sin2\theta)(\sqrt{\omega_1^2+m^2}+m)\right\}$$
$$+\frac{1}{D_2^{(-)}}\left\{F_1^2F_2^2(\omega_2^2+2m^2+2m\sqrt{\omega_2^2+m^2})+F_1^2\omega_2^2\vec{p}_2^{\,2}\right.$$
$$\left.-2F_1^2F_2\omega_2(p_{2x}\cos\theta-p_{2y}\sin\theta)(\sqrt{\omega_2^2+m^2}+m)\right\}$$
$$+\frac{1}{D_3^{(-)}}\left\{2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}+m)(\sqrt{\omega_2^2+m^2}+m)\cos2\theta\right.$$
$$-2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta+2F_1^2F_2\omega_1p_{2x}(\sqrt{\omega_2^2+m^2}+m)$$
$$\left.\left.-2F_1^2F_2\omega_2(\sqrt{\omega_1^2+m^2}+m)(p_{2x}\cos\theta+p_{2y}\sin\theta)\right\}\right] \quad 18$$

The differential cross-sections $d\sigma$ are given by $2\pi |M|^2_{\kappa}$ (density of states). After substituting for p_2^2 etc. in terms of ω_1 and ω_2, from the following relations we may compare the cross-sections for positive and negative intermediate energies

$$E_2 = 2m + \omega_1 - \omega_2 \quad \text{(from energy conservation)}$$

$$\vec{p}_2^{\,2} = E_2^2 - m^2 = (2m + \omega_1 - \omega_2)(\omega_1 - \omega_2)$$

$$E_1^2 = 4m^2$$

$$\left.\begin{aligned} p_2 &= \omega_1 - \omega_2 \cos\theta \\ p_{2x} &= -\omega_2 \sin\theta \end{aligned}\right\} \text{(from momentum conservation)}$$

19.

2b Discussion

Since the expressions are rather complicated, it is difficult to compare the cross-sections for the positive and negative energy intermediate states directly. However we shall take up some special cases to get an idea of the relative contributions:

(1) Non-relativistic case, i.e. $\omega_1 \ll m$ and $\omega_1 \sim \omega_2$. In the case of positive energy intermediate states

$$d\sigma^+(AA') = \frac{e^4}{16m^4}\Big[(-2\omega_1)(\omega_1 - \omega_2\cos\theta)\left(1 + \frac{\omega_1}{m}\right) + (2\omega_2)(\omega_2 - \omega_1\cos\theta)(1 - \omega_2/m) - (2\omega_2)(\omega_2\cos\theta - \omega_1) - 2\omega_1^2\cos\theta\Big]$$

20.

(neglecting terms of order ω_1^2/m^2)

$$\simeq \frac{e^4}{16m^2}\frac{\omega_1^2}{m^2} \sim 0$$ 21.

Similarly

$$d\sigma^+(BB' \text{ or } AB' \text{ or } BA') \sim 0$$

Now we shall calculate the cross-section for the intermediate negative energies

$$d\sigma^-(AA') = \frac{e^4}{16m^2}\Bigg[\frac{1}{4m^4\left(1+\frac{\omega_1}{m}\right)}\Big\{2m\Big[2m(4m^2+2\omega_1^2) + 2\omega_1\left(2m+\frac{\omega_1^2}{2m}\right)(\omega_1-\omega_2\cos\theta)\Big]\Big\}$$

$$+\frac{1}{4m^4\left(1-\frac{\omega_2}{m}\right)}\Big\{2m\Big[2m(4m^2+2\omega_2^2) + 2\omega_2\left(2m+\frac{\omega_2^2}{2m}\right)(\omega_2-\omega_1\cos\theta)\Big]\Big\}$$

$$+\frac{1}{4m^4}\Big\{2m\Big[4m\left(2m+\frac{\omega_1^2}{2m}\right)\left(2m+\frac{\omega_2^2}{2m}\right) + 2(\omega_1-\omega_2\cos\theta)\omega_1\left(2m+\frac{\omega_1^2}{2m}\right) + 2(\omega_2-\omega_1\cos\theta)\omega_2\left(2m+\frac{\omega_2}{2m}\right)\Big]\Big\}\Bigg]$$ 22.

Neglecting ω_1^2/m^2, we have

$$d\sigma^-(AA') \simeq \frac{e^4}{16m^2}\left[\frac{16m^4+16m^4+32m^4}{4m^4}\right] \qquad 23.$$

$$\simeq e^4/m^2$$

Similarly

$$d\sigma^-(BB') \simeq e^4/m^2\,(\cos^2\theta) \qquad 24$$

$$d\sigma^-(AB' \text{ or } BA') \simeq 0 \qquad 25.$$

(ii) Extreme relativistic case (a) small angles, i.e. $\omega_1 \sim \omega_2 >> m$. The results for different polarization are tabulated below, (b) large angles $\omega_1 >> m \sim \omega_2$. In this case the results are similar to the previous ones excepting that here

$$d\sigma(AB' \text{ or } BA') = d\sigma(BB') \neq d\sigma(AA') \qquad 26$$

This is to be expected since $e_1 \cdot e_2$ (for BB') ~ 0 since $\cos\theta \sim 0$

Differential Cross-sections for Compton effect.

Energy / Polarisation	Non-relativistic ($\omega_1 \sim \omega_2 \ll m$)		Extreme relativistic ($\omega_1 \gg m, \omega_1 \sim \omega_2$) Small angles	
	Intermediate energy		Intermediate energy	
	Positive	Negative	Positive	negative.
AA'	$\frac{e^4}{m^2}\left(\frac{\omega_1^2}{m^2}\right) \sim 0$	e^4/m^2	$\frac{e^4}{4m^2}\left(1-\frac{m}{\omega_1}-\frac{m}{\omega_2}\right)$	$\frac{e^4}{4m^2}\left(1+\frac{m}{\omega_1}+\frac{m}{\omega_2}\right)$
BB'	»	$e^4/m^2 \cos^2\theta$	$\frac{e^4}{4m^2}\left(1-\frac{m}{\omega_1}\cos 2\theta-\frac{m}{\omega_2}\right)$	$\frac{e^4}{4m^2}\left(1+\frac{m}{\omega_2}+\frac{m}{\omega_1}\cos 2\theta\right)$
AB'	»	$\frac{e^4}{m^2}\left(\frac{\omega_1^2}{m^2}\right) \sim 0$	$e^4/4m^2$	$e^4/4m^2$
BA'	»	»	»	»

It is interesting to note that the contributions arising from the negative energy intermediate states dominates the cross-sections in the non-relativistic limit for the electron. In fact the contribution from positive energy intermediate states is almost zero so that

$$\tfrac{1}{2}\, d\sigma^{-}(AA' + BB') = \frac{1}{2}\frac{e^4}{m^2}(1+\cos 2\theta) = \frac{1}{2}\, d\sigma(AA'+BB') \qquad (27.)$$

since $d\sigma^{+} = 0$

which is the actual cross-section (i.e. χ including both positive and negative energy intermediate states).

In the extreme relativistic case, we find that contributions arise from both positive and negative energy intermediate states. We also find that in case (a) $d\sigma^{+}(AB') = d\sigma^{-}(AB')$ though $d\sigma(AB' \text{ or } BA')$ is zero. This can be easily seen from the fact $M^{+}(AB') = -M^{-}(AB')$ and hence $|M^{+} + M^{-}|^2 = 0$. Similarly for $d\sigma^{+}(AA') + d\sigma^{-}(AA') = \frac{e^4}{2m^2}$ though $d\sigma(AA') = e^4/m^2$. This is again obvious since in this case $M^{+}(AA') \simeq +M^{-}(AA')$ so that $d\sigma = |M^{+} - M^{-}|^2 = |2M^{+}(AA')|^2 = e^4/m^2$.

We tabulate below the results. ~~for bremsstrahlung also.~~

CHAPTER IV

ENERGY DENOMINATORS IN THE FEYNMAN FORMALISM

1. Introductory Remarks

One of the most important characteristic feature of a matrix element obtained by the use of conventional field theory is the occurrence of "energy denominators" which has been described as 'awkward' since it does not have a covariant form. On the other hand this has been achieved in the Feynman formalism "which enormously simplifies the calculations particularly the higher order terms in the perturbation series"[1] The splitting of the Feynman propagator as we have shown introduces energy denominators which still preserve the relativistic covariance. It is therefore worthwhile to put the energy denominators in one formalism into correspondence with the other and demonstrate that by a suitable grouping of the terms containing energy denominators we get the corresponding set in the other.[2] The moment the grouping is identified we see no reason for the "awkwardness" of conventional field theoretic methods. It turns out remarkably enough that the numerators of the terms which are to be grouped are identical, a fact which makes the simplification and interpretation easy and elegant.--

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1) A survey of field theory - III by P.T.Matthews. Reports on the progress of physics, Vol.1, 443 (1955).

2) A. Ramakrishnan, T.K. Radha and R. Thunga, Jour. of Math. Anal. (1962) (in press)

The nth order matrix element for the scattering of an electron from momentum p_1' to momentum p_2' is given by

$$M = \int d^4x_n \cdots \int d^4x_1 \; \bar{\psi}_{p_2'}(x_n) K(x_n, x_{n-1}) \cdots \psi(x_1) \qquad 1.$$

In the Feynman formalism, the matrix element in momentum representation obtained by performing the 4-dimensional integration is given by

$$M = \bar{u}(p_2') \not{e}_n \frac{1}{\not{p}_n - m} \not{e}_{n-1} \cdots \not{e}_2 \frac{1}{\not{p}_1 - m} \not{e}_1 u(p_1) \qquad 2.$$

where the $\not{p}_i$ refer to intermediate virtual states with energy $p_4 \neq E_{p_i} = [p_i^2 + m^2]^{1/2}$ for a given order in the sequence of perturbations $\not{e}_n \cdots \not{e}_1$ (i.e. for a single Feynman diagram), the above can be decomposed into 2^{n-1} terms which are individually covariant as follows: The space and time integrations of (1) are separately performed, the former leading to conservation of three momentum at every vertex and the time integration which is subsequently performed is split into two parts corresponding to the ranges $t = -\infty$ to 0 with energy $-E$ and $t = 0$ to $+\infty$ with energy $+E$ respectively. As was shown in the previous chapter this leads to the decomposed Feynman propagator

$$\frac{1}{\not{p} + \not{q} - m} = \frac{1}{2E_{p+q}} \left\{ \frac{\overset{+}{\not{p}} + m}{E_p + E_q - E_{p+q}} - \frac{\bar{\not{p}} + m}{E_p + E_q + E_{p+q}} \right\} \qquad 3.$$

where $E_{p+q} = [(p+q)^2 + m^2]^{1/2}$ and $\overset{+}{\not{p}}$ in the Feynman four vector with energy $+E_{p+q}$ and $\bar{\not{p}}$ has the fourth component $-E_{p+q}$. In fact, the first term corresponding to positive energy is nothing but the transform of

$$\left(\frac{1}{2\pi}\right)^4 \int_{C_+} \frac{u\bar{u}\, e^{ip\cdot x}}{p^2 - m^2} d^4p \quad \text{for } x_4 > 0 \qquad 4.$$

and can be obtained from the contour C_+ omitting the pole at $p_4 = -E$: in a similar way, the second term can be represented by

$$\left(\frac{1}{2\pi}\right)^4 \int_{C_-} \frac{u\bar{u}\, e^{-ip\cdot x}}{p^2 - m^2} d^4p \quad \text{for } x_4 < 0 \qquad 5.$$

If we use this propagator it is more convenient to think of the energy of the 'virtual' particle to be $+E_{p+q}$ with momentum $\vec{p}+\vec{q}$. It is virtual in the sense that its energy does not correspond to $E_p + E_q$, the energy of the system before its creation. In a similar way $-E_{p+q}$ corresponding to a negative energy 'virtual' particle. These two parts are taken together in the usual Feynman formalism and we attribute an energy p_4 to the virtual particle.

The main advantage of this decomposed propagator is that it lends itself to an easy method of comparison with field theory due to the presence of the energy denominators so that the equivalence between Feynman and field theoretic formalism can be established even in the old fashioned manner, that is after space integration.

3. In a field theoretic picture for a given order in the sequence of perturbation, the nth order term has $n!$ diagrams each of which will give different energy denominators. This is because the position of every vertex relative to all other $n-1$ vertices is important since the time integration

is performed in a temporarily ordered way. Thus every new complexion gives a different energy denominator and a sum over intermediate states implies a sum over all such diagrams.

If on the other hand, we employ the method of the decomposed Feynman propagator, the position of every vertex on the time axis with respect to the previous (in the Feynman sense) one is relevant since the integration over interval $t_n - t_{n-1}$ is divided into two ranges, positive and negative, whether the n th vertex lies 'above' or 'below' the $n-1$ th vertex. Thus, since for an n th order process there are $n-1$ propagators each of which can be split into two parts, it follows that we can have 2^{n-1} diagrams. It now remains to be shown that the $n!$ diagrams of field theory are equivalent to the 2^{n-1} such diagrams (i.e.) $n! \geq 2^{n-1}$ for all $n \geq 2$).

2. Calculations.

We here demonstrate explicitly the equivalence upto the fourth order. We have for definiteness considered the Compton scattering of an electron.

In the second order, since $2^{n-1} = n!$ for $n = 2$ the correspondence is immediate. For the third order, let us consider the sequence in which an initial photon of four momentum $q_1(\omega, \vec{q})$ is absorbed by an electron ~~is absorbed by an electron~~ at rest and two photons of four momenta $q_{11}(\omega_1, \vec{q}_1)$ and $q_{12}(\omega_2, \vec{q}_2)$ are emitted in this order along the Feynman path, the final electron having momentum $p_2(E_2, \vec{p}_2)$. The calculation of the entire matrix element would of course involve all permutations of the above sequence. For our present purpose, it is sufficient

to consider this particular sequence only.

A. Field theoretic formalism

The matrix element in field theory for this third order process is

$$M_3 = \sum \frac{H_{f\,II}\, H_{II\,I}\, H_{I\,i}}{(E_0 - E_I)(E_0 - E_{II})} \qquad 6.$$

where f and i refer to final and initial states respectively and the summation is over all intermediate states . II and I ,

$$H_{f\,II} = (\Phi_f , H_{int}\, \Phi_{II}) \quad etc \qquad 7.$$

and

$$H_{int} = e \int \bar{\psi}\, \gamma_\mu\, \psi\, \phi\, d^3x \qquad 8.$$

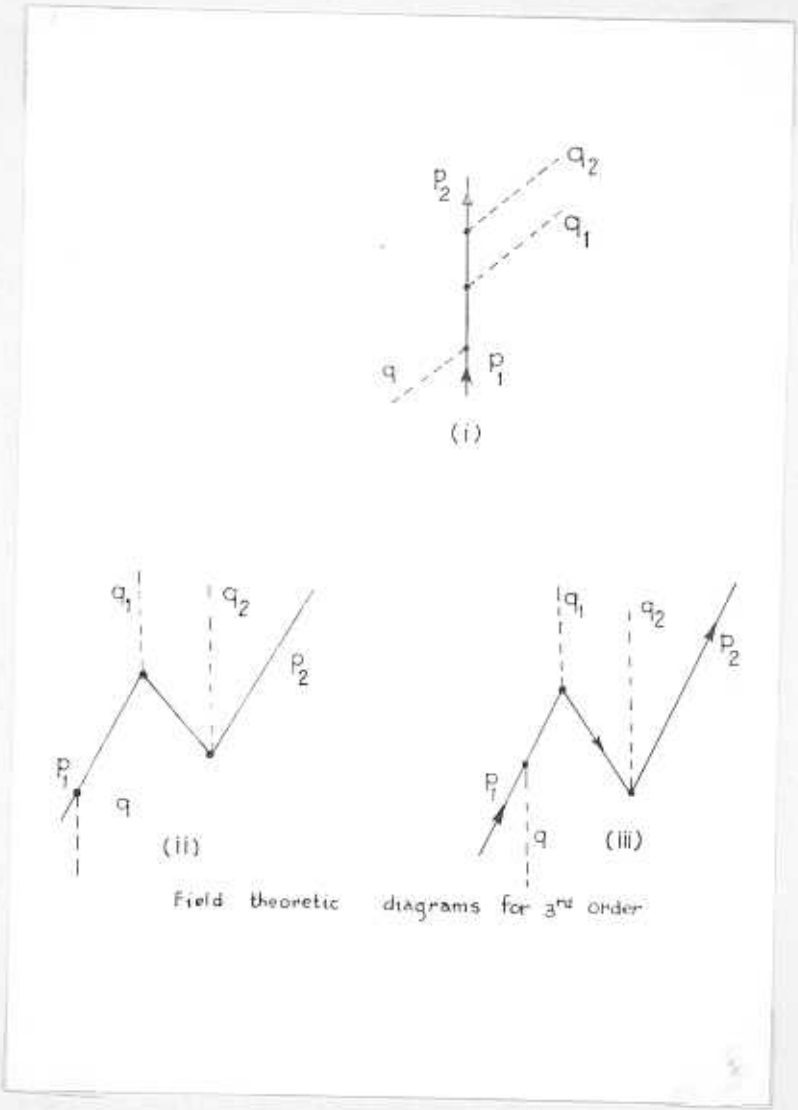
where ψ and ϕ are the electron and photon field operators respectively.

We now evaluate M_3 for each of the $\frac{n!}{a} = 3$ diagrams. The other half is just am exact counterpart with all positive energy intermediate states replaced by negative energies and vice versa. Thus corresponding to diagram (i) we have

$$M_{3,i} = \frac{\sum H_{f\,II}\, H_{II\,I}\, H_{I\,i}}{(m+\omega-E_{p_1+q})(m+\omega-\omega_1-E_{p_2+q_2})}$$

$$= \frac{\langle \Phi_f | H_{int}(x_3) | \Phi_{II} \rangle \langle \Phi_{II} | H_{int}(x_2) | \Phi_I \rangle \langle \Phi_I | H_{int}(x_1) | b^+_{p_1} \rangle_0}{(m+\omega-E_{p+q})(m+\omega-\omega_1-E_{p_2+q_2})} \qquad 9.$$

where $b^+_{p_1}$ is the creation operator of the electron with momentum $p_1(\omega, m)$. The photon field operators are all emitted in what follows since they always commute and are



Field theoretic diagrams for 3rd order

hence not relevant for our arguments. We shall also omit numerical factors for convenience. Conservation of energy implies

$$m + \omega = E_3 + \omega_1 + \omega_2 \qquad 10.$$

and the energy denominators in (9) correspond to diagram (i). Expanding $H_{int} - S$, inserting the photon operators and integrating over the space variables $x_1 \dots x_3$ leads ultimately to an overall function which implies momentum conservation. Thus the electron operators in the numerator of (9) reduce to

$$\sum_{\text{all mom. indices}} \langle \Phi_f | (b^+ b)(b^+ b)(b^+ b) | b^+_{p_1} \rangle_0 \bar{u}_{p_2} [u \bar{u} u \bar{u}] u_{p_1} \qquad 11.$$

where the b-s are the annihilation operators for electrons. Making use of

$$b_{p'} b^+_p \rangle_0 = \delta(p - p') \qquad 12.$$

and the over all δ function resulting from the space integration, then (9) becomes

$$M_{3,i} = \frac{\bar{u}_{p_3} [u \bar{u}_{p_3 + q_2}] [u \bar{u}_{p_1 + q}] u_{p_1}}{(m + \omega - E_{p_1 + q})(m + \omega - \omega_1 - E_{p_3 + q_2})} \qquad 13.$$

(ii) Similarly for the diagram (ii) we have

$$M_{3,ii} = \frac{\langle \Phi_f | H_{int}(x_3) | \Phi_{II} \rangle \langle \Phi_{II} | H_{int}(x_2) | \Phi_I \rangle \langle \Phi_I | H_{int}(x_1) | b^+_{p_1} \rangle_0}{(m_1 + \omega - E_{p_1 + q})(\omega_1 - E_{p_1 + q} - E_{p_2 + q_2})} \qquad 14.$$

and the numerator

$$= \sum \langle \Phi_f | (db)(b^\dagger d^\dagger)(b^\dagger b) | b^\dagger_{p_1} \rangle_0 \, \bar{u}_{p_2} [v \bar{v} \, u \bar{u}] u_{p_1} \qquad 15.$$

where d and $d^\dagger$ are the annihilation and creation operators of the positron; this is now rearranged as

$$- \sum \langle \Phi_f | b^\dagger (dd^\dagger)(bb^\dagger) b | b^\dagger_{p_1} \rangle_0 \, \bar{u}_{p_2} [v \bar{v} \, u \bar{u}] u_{p_1} \qquad 16$$

so that we can now ~~xx~~ apply (12) and we have

$$M_{3,ii} = \frac{- \bar{u}_{p_2} [v \bar{v}_{p_2 + q_2}] [u \bar{u}_{p_1 + q}] u_{p_1}}{(m + \omega - E_{p_1 + q})(\omega_1 - E_{p_1 + q} - E_{p_2 + q_2})} \qquad 17$$

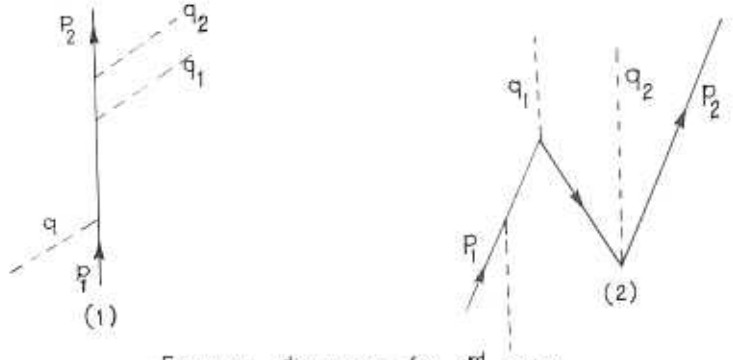
(iii) In the case of diagram (iii) which differs from (ii) in that the vertex at which ω_2 is emitted is 'below' the vertex at which ω_1 is absorbed, the energy denominators are obviously different. We have

$$M_{3,iii} = \frac{\sum \langle \Phi_f | (db)(b^\dagger b)(b^\dagger d^\dagger) | b^\dagger_{p_1} \rangle_0 \, \bar{u}_{p_2} [v \bar{v} \, u \bar{u}] u_{p_1}}{(-E_2 - \omega_2 - E_{p_2 + q_2})(\omega_1 - E_{p_1 + q} - E_{p_2 + q_2})} \qquad 18.$$

the numerator when arranged is $- \langle \Phi | b^\dagger_{p_2} (dd^\dagger)(bb^\dagger) b | b^\dagger_{p_1} \rangle_0 (\;)$ and again

$$M_{3,iii} = \frac{- \bar{u}_{p_2} [v \bar{v}_{p_2 + q_2}] [u \bar{u}_{p_1 + q}] u_{p_1}}{(-E_2 - \omega_2 - E_{p_2 + q_2})(\omega_1 - E_{p_1 + q} - E_{p_2 + q_2})} \qquad 19.$$

It is interesting to note that the numerators in (18) and (19) are identical though the denominators are different so that they are equivalent to a single Feynman diagram.



Feynman diagrams for 3rd order

B. Feynman formalism.

The Feynman matrix elements for a given type of diagram can be immediately written down with the use of the decomposed propagator (3). Thus we have for the Feynman diagram equivalent to the first

$$M^F_{3,1} = \frac{\bar{u}_{p_2}\left[(\not{p}_2+\not{q}_2)+m\right]\left[(\not{p}_1+\not{q})+m\right]u_{p_1}}{\left(m+\omega-\omega_1-E_{p_2+q_2}\right)\left(m+\omega-E_{p_1+q}\right)} \qquad 20$$

and corresponding to the other Feynman diagram equivalent to (ii) and (iii) we have

$$M^F_{3,2} = +\frac{\bar{u}_{p_2}\left[(\not{p}_2+\not{q}_2)+m\right]\left[(\not{p}_1+\not{q})+m\right]u_{p_1}}{\left(E_2+\omega_2+E_{p_2+q_2}\right)\left(m+\omega-E_{p_1+q}\right)} \qquad 21$$

C. Equivalence

We now have to show that (19) and (17) together reduces to (21).

1) It is seen that the expression (20) is the same as ( ) i.e. $M_{3,i} \equiv M^F_{3,1}$; since $\sum_{\text{spins}} u\bar{u}_{p_2+q_2} = \left[(\not{p}_2+\not{q}_2)+m\right]$ etc.

2) To show

$$M_{3,ii} + M_{3,iii} \equiv M^F_{3,2}$$

From (17) and (19)

$$M_{3,ii} + M_{3,iii} = N \Big/ \left(\omega_1 - E_{p_1+q} - E_{p_2+q_2}\right) \times \left[\frac{1}{m+\omega - E_{p_1+q}} + \frac{1}{E_2 + \omega_2 + E_{p_2+q_2}} \right]$$

$$= - N \Big/ \left(m + \omega - E_{p_1+q}\right)\left(E_2 + \omega_2 + E_{p_2+q_2}\right) \qquad 22.$$

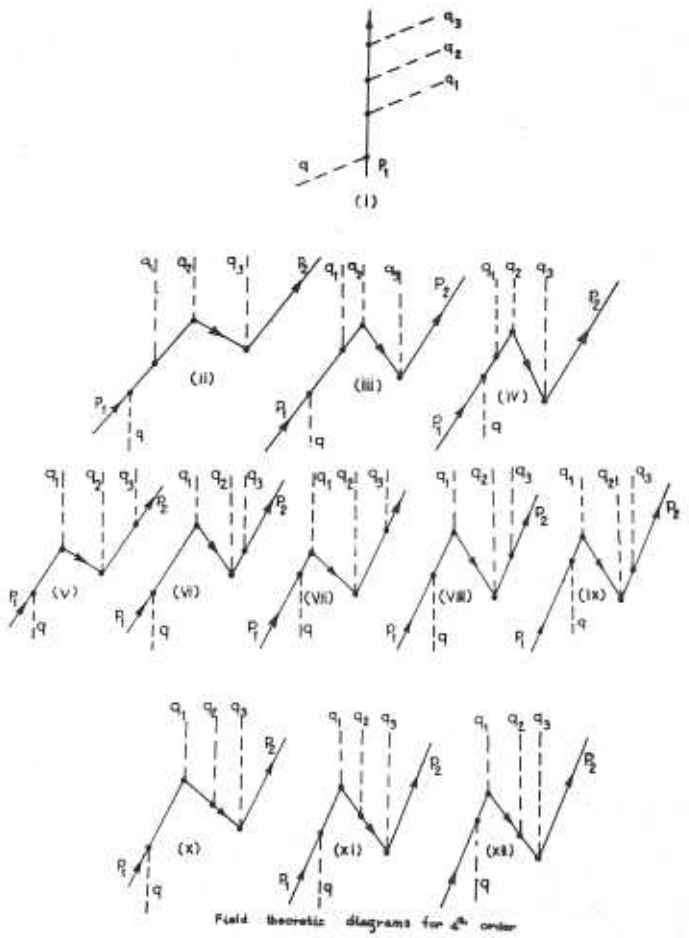
and since

$$-\sum_{\text{spins}} \left[V \bar{V}_{p_2+q_2} \right] = -\sum_{\text{spins}} u_{-E}\, \bar{u}_{-E}\, (p_2+q_2)$$

$$= \left[(\bar{p}_2 + \bar{q}_2) + m \right] \qquad 23$$

$$M_{3,ii} + M_{3,iii} = \frac{\bar{u}_{p_2} \left[(\bar{p}_2 + \bar{q}_2) + m \right] \left[(\bar{p}_1 + \bar{q}) + m \right] u_{p_1}}{\left(m + \omega - E_{p_1+q}\right)\left(E_2 + \omega_2 + E_{p_2+q_2}\right)} \qquad 24$$

$$\equiv M^F_{3,2}$$



Field theoretic diagrams for 4th order

We shall now establish the equivalence for the fourth order process when three photons $q_1(\omega_1, \vec{q}_1)$, $q_2(\omega_2, \vec{q}_2)$ and $q_3(\omega_3, \vec{q}_3)$ are emitted in this order along the Feynman path, the final electron having momentum $p_2(E_2, \vec{p}_2)$. Again we concern ourselves only with the particular sequence since we are not interested in the evaluation of the entire matrix element.

A. Field theoretic formalism

(i) The matrix element in field theory for the diagram (i) i.e.

$$M_{4i} = \frac{\sum H_{fIII} H_{IIII} H_{IIII} H_{III} H_{Ii}}{(m+\omega-E_{p_1+q})(m+\omega-\omega_1-E_{p_1+q-q_1})(E_2+\omega_3-E_{p_2+q_3})}$$

$$= \frac{\langle \Phi_f | H_{int}(x_4) | \Phi_{III} \rangle \langle \Phi_{III} | H_{int}(x_3) | \Phi_{II} \rangle \langle \Phi_{II} | H_{int}(x_2) | \Phi_I \rangle \langle \Phi_I | H_{int}(x_1) | b^\dagger_{p_1} \rangle}{(m+\omega-E_{p_1+q})(m+\omega-\omega_1-E_{p_1+q-q_1})(E_2+\omega_3-E_{p_2+q_3})} \quad 25$$

The numerator as before can be reduced * as

$$\sum_{\substack{\text{all mom.} \\ \text{indices}}} \langle \Phi_f | (b^\dagger b)(b^\dagger b)(b^\dagger b)(b^\dagger b) | b^\dagger_{p_1} \rangle_c \, \bar{u}_{p_2} [u\bar{u}\, u\bar{u}\, u\bar{u}] u_{p_1}$$

$$= \bar{u}_{p_2} [u\bar{u}_{p_2+q_3}][u\bar{u}_{p_1+q-q_1}][u\bar{u}_{p_1+q}] u_{p_1} \quad 26.$$

$$H_{4i} = \frac{\bar{u}_{p_2} [u\bar{u}_{p_2+q_3}][u\bar{u}_{p_1+q-q_1}][u\bar{u}_{p_1+q}] u_{p_1}}{(m+\omega-E_{p_1+q})(m+\omega-\omega_1-E_{p_1+q-q_1})(E_2+\omega_3-E_{p_2+q_3})} \quad 27$$

where we have made use of the energy conservation i.e.

$$m+\omega = E_2+\omega_1+\omega_2+\omega_3 \quad 28$$

(ii) Similarly for diagram (ii) we have

$$M_{4,ii} = \frac{\langle \Phi_f | H_{int}(x_1) | \Phi_{II} \rangle \langle \Phi_{III} | H_{int}(x_3) | \Phi_{II} \rangle \langle \Phi_{II} | H_{int}(x_2) | \Phi_I \rangle \times \langle \Phi_I | H_{int}(x_1) | i \rangle}{(m+\omega - E_{p_1+q})(m+\omega-\omega_1 - E_{p_1+q-q_1})(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})} \quad 29$$

and the numerator

$$-\sum \langle \Phi_f | b^\dagger [dd^\dagger][bb^\dagger][bb^\dagger] b | b^\dagger_{p_1} \rangle_0 \bar{u}_{p_2} [v\bar{v}\, u\bar{u}\, u\bar{u}] u_{p_1} \quad 30$$

$$M_{4,ii} = \frac{-\bar{u}_{p_2}[v\bar{v}_{p_2+q_3}][u\bar{u}_{p+q-q_1}][u\bar{u}_{p_1+q}] u_{p_1}}{(m+\omega - E_{p_1+q})(m+\omega-\omega_1 - E_{p+q-q_1})(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})} \quad 31$$

(iii) In the case of diagram (iii) which differs from (ii) in that the vertex at which ω_3 is emitted is 'below' the vertex at which ω_1 is emitted, the energy denominators are obviously different. We have

$$M_{4,iii} = \frac{\sum \langle \Phi_f | [dd^\dagger][b^\dagger b][b^\dagger b][b^\dagger d^\dagger][b^\dagger b] | b^\dagger_{p_1} \rangle_0 \bar{u}_{p_2} [v\bar{v}\, u\bar{u}\, u\bar{u}] u_{p_1}}{(m+\omega - E_{p_1+q})(\omega_1+\omega_2 - E_{p_1+q} - E_{p_2+q_3})(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})} \quad 32$$

and the numerator when rearranged is $-\langle \Phi_f | b^\dagger_{p_1} [dd^\dagger][bb^\dagger][bb^\dagger] \times b_{p_1} | b^\dagger_{p_1} \rangle_0 (\,)$ and again

$$M_{4,iii} = -\frac{\bar{u}_{p_2}[v\bar{v}_{p_2+q_3}][u\bar{u}_{p_1+q-q_1}][u\bar{u}_{p_1+q}] u_{p_1}}{(m+\omega - E_{p_1+q})(\omega_1+\omega_2 - E_{p_1+q} - E_{p_2+q_3})(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})} \quad 33$$

We note that the numerators in $M_{4,ii}$ and $M_{4,iii}$ are identical though the denominators are different.

(iv) In a similar way the vertex at which ω_3 is emitted can be still further 'lowered' which gives rise to the diagram, (iv) and it can be easily verified that the numerator is the same as in the previous two cases. We thus have

$$M_{4,iv} = \frac{-\bar{u}_{p_2}[v\bar{v}_{p_2+q_3}][u\bar{u}_{p_1+q-q_1}][u\bar{u}_{p_1+q}]u_{p_1}}{(-E_{p_2}-\omega_3-E_{p_2+q_3})(\omega_1+\omega_2-E_{p_1+q}-E_{p_2+q_3})(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})} \quad 34$$

The diagrams (ii), (iii) and (iv) form a group which will later be shown to be equivalent to one Feynman diagram.

(v) A group of diagrams can now be obtained by having the second Feynman intermediate state of negative energy while the first and third have positive energies. The procedure for the reduction of the numerator is quite similar to the previous ones and in future we shall merely write down the matrix element with the appropriate energy denominators. We have for diagram(v)

$$M_{4,v} = \frac{-\bar{u}_{p_2}[u\bar{u}_{p_2+q_3}][v\bar{v}_{p_1+q-q_1}][u\bar{u}_{p_1+q}]u_{p_1}}{(m+\omega-E_{p_1+q})(m+\omega-\omega_2-E_{p_1+q}-E_{p_2+q_3}-E_{p_1+q-q_1})(E_2+\omega_3-E_{p_2+q_3})} \quad 35$$

(vi) This diagram can be obtained from (v) by lowering the last vertex below the second (on the time axis) and we have

$$M_{4,vi} = \frac{-\bar{u}_{p_2}[u\bar{u}_{p_2+q_3}][v\bar{v}_{p_1+q-q_1}][u\bar{u}_{p_1+q}]u_{p_1}}{(m+\omega-E_{p_1+q})(m+\omega-\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3}-E_{p_1+q})(\omega_1-E_{p_1+q}-E_{p_1+q_3})} \quad 36$$

(vii) In this the second vertex in (v) becomes the first and vice versa so that

$$M_{4,\,vii} = \frac{-\bar{u}_{p_2}\left[u\bar{u}_{p_2+q_3}\right]\left[v\bar{v}_{p_1+q-q_1}\right]\left[u\bar{u}_{p_1+q}\right]u_{p_1}}{\left(m+\omega-\omega_2-E_{p_1+q}-E_{p_1+q-q_1}-E_{p_2+q_3}\right)\left(-E_{p_1+q-q_1}-\omega_2-E_{p_3+q_3}\right)\left(E_3+\omega_3-E_{p_3+q_3}\right)} \quad 37.$$

(viii) Here again the last vertex of (vii) becomes the third and

$$M_{4,\,viii} = \frac{-\bar{u}_{p_2}\left[u\bar{u}_{p_2+q_3}\right]\left[v\bar{v}_{p_1+q-q_1}\right]\left[u\bar{u}_{p_1+q}\right]u_{p_1}}{\left(-E_{p_1+q-q_1}-\omega_2-E_{p_3+q_3}\right)\left(m+\omega-E_{p_1+q-q_1}-E_{p_3+q}-\omega_2-E_{p_3+q_3}\right)\left(\omega_1-E_{p_1+q}-E_{p_1+q_1}\right)} \quad 38.$$

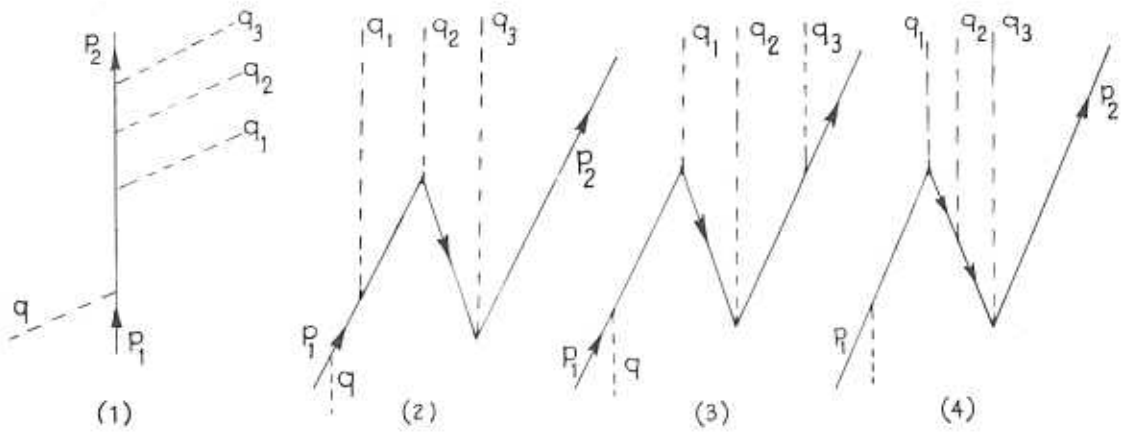
(ix) The last diagram of the class (v ..... viii) is got by making the second vertex in (viii) the third and vice versa so that

$$M_{4,\,ix} = \frac{-\bar{u}_{p_2}\left[u\bar{u}_{p_2+q_3}\right]\left[v\bar{v}_{p_1+q-q_1}\right]\left[u\bar{u}_{p_1+q}\right]u_{p_1}}{\left(-E_{p_1+q-q_1}-\omega_3-E_{p_3+q_3}\right)\left(-E_{p_1+q-q_1}-\omega_2\;\omega_3-E_2\right)\left(\omega_1-E_{p_1+q}-E_{p_1+q-q_1}\right)} \quad 39.$$

This exhausts all such possible diagrams and the five diagrams can be shown to be equivalent to a single Feynman diagram.

Another class of diagrams will be with the first Feynman intermediate state of positive energy and the other two of negative energy. It is easily seen that we have three such diagram (x .... xii) which can be shown to be equivalent to one Feynman diagram. We have

$$M_{4,\,x} = \frac{\bar{u}_{p_2}\left[v\bar{v}_{p_3+q_3}\right]\left[v\bar{v}_{p_1+q-q_1}\right]\left[u\bar{u}_{p_1+q}\right]u_{p_1}}{\left(m+\omega-E_{p_1+q}\right)\left(\omega_1+\omega_2-E_{p_2+q_3}-E_{p_1+q}\right)\left(\omega_1-E_{p_1+q-q_1}-E_{p_1+q}\right)} \quad 40.$$



Feynman diagrams for 4th order

$$M_{4\,xi} = \frac{\bar{u}_{p_2}\left[V\bar{V}_{p_2+q_3}\right]\left[V\bar{V}_{p_1+q-q_1}\right]\left[u\bar{u}_{p_1+q}\right]u_{p_1}}{\left(-E_2-\omega_3-E_{p_2+q_3}\right)\left(\omega_1+\omega_2-E_{p_1+q}-E_{p_2+q_3}\right)\left(\omega_1-E_{p_1+q-q_1}-E_{p_1+q}\right)} \quad 41$$

$$M_{4\,xii} = \frac{\bar{u}_{p_2}\left[V\bar{V}_{p_2+q_3}\right]\left[V\bar{V}_{p_1+q-q_1}\right]\left[u\bar{u}_{p_1+q}\right]u_{p_1}}{\left(-E_2-\omega_3-E_{p_2+q_3}\right)\left(-E_2-\omega_3-\omega_2-E_{p_1+q-q_1}\right)\left(\omega_1-E_{p_1+q-q_1}-E_{p_1+q}\right)} \quad 42.$$

B. Feynman formalism

The Feynman matrix elements for a given type of diagram can be immediately written down with the use of the decomposed propagator (S) and we have for the four equivalent diagrams in the order discussed in Section A the following :

$$M^F_{4,1} = \frac{\bar{u}_{p_2}\left[(\not{p}_2+\not{q}_3)+m\right]\left[(\not{p}_1+\not{q}-\not{q}_1)+m\right]\left[(\not{p}_1+\not{q})+m\right]u_{p_1}}{\left(m+\omega-E_{p_1+q}\right)\left(m+\omega-\omega_1-E_{p_1+q-q_1}\right)\left(E_2+\omega_3-E_{p_2+q_3}\right)} \quad 43$$

$$M^F_{4,2} = \frac{\bar{u}_{p_2}\left[(\not{p}_2+\not{q}_3)+m\right]\left[(\not{p}_1+\not{q}-\not{q}_1)+m\right]\left[(\not{p}_1+\not{q})+m\right]u_{p_1}}{\left(E_2+\omega_3+E_{p_2+q_3}\right)\left(m+\omega-\omega_1-E_{p_1+q-q_1}\right)\left(m+\omega-E_{p_1+q}\right)} \quad 44.$$

$$M^F_{4,3} = \frac{\bar{u}_{p_2}\left[(\not{p}_2+\not{q}_3)+m\right]\left[(\not{p}_1+\not{q}-\not{q}_1)+m\right]\left[(\not{p}_1+\not{q})+m\right]u_{p_1}}{\left(E_2+\omega_3-E_{p_2+q_3}\right)\left(m+\omega-\omega_1+E_{p_1+q-q_1}\right)\left(m+\omega-E_{p_1+q}\right)} \quad 45.$$

$$M^F_{4,2} = \frac{\bar{u}_{p_2}\left[(\not{p}_2+\not{q}_3)+m\right]\left[(\not{p}_1+\not{q}-\not{q}_1)+m\right]\left[(\not{p}_1+\not{q})+m\right]u_{p_1}}{\left(E_2+\omega_3+E_{p_2+q_3}\right)\left(m+\omega-\omega_1+E_{p_1+q-q_1}\right)\left(m+\omega-E_{p_1+q}\right)} \quad 46.$$

C. Equivalence

It now remains to be shown that the sum of the individual expressions in Section A for a given type of diagram reduces to the corresponding expression in (B).

1. $M_{4,i} \equiv M^{F}_{4,1}$ as is seen from (27) and (43.).
since $\sum_{spins} u\bar{u}_{p_2+q_3} = [(p_2+q_3)+m]$ etc

2. To show

$$M_{4,ii} + M_{4,iii} + M_{4,iv} = M^{F}_{4,2}$$

Now from (33) and (34)

$$M_{4,iii} + M_{4,iv} = N\Big/(\omega_1+\omega_2-E_{p_1+q}-E_{p_2+q_3})(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})$$
$$\times\left[\frac{1}{m_1+\omega-E_{p_1+q}} - \frac{1}{E_2+\omega_3+E_{p_2+q_3}}\right]$$
$$= -N\Big/(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})(m+\omega-E_{p_1+q})(E_2+\omega_3+E_{p_2+q_3}) \quad 47.$$

And

$$M_{4,ii}+M_{4,iii}+M_{4,iv} = N\Big/(m+\omega-E_{p_1+q})(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})$$
$$\times\left[\frac{1}{m+\omega-\omega_1-E_{p_1+q-q_1}} - \frac{1}{E_2+\omega_3+E_{p_2+q_3}}\right]$$
$$= -N\Big/(m+\omega-E_{p_1+q})(m+\omega-\omega_1-E_{p_1+q-q_1})(E_2+\omega_3+E_{p_2+q_3}) \quad 48.$$

Since $\equiv M^{F}_{4,2}$

$$-\sum_{Spins} u\bar{v}_{p_2+q_3} = -\sum_{Spins} u_{-E}\bar{u}_{-E}(p_2+q_3) = [(p_2+q_3)+m]$$

3. We now demonstrate

$$M_{4,v} + M_{4,vi} + M_{4,vii} + M_{4,viii} + M_{4,ix} = M^{F}_{4,3}$$

The numerators can similarly be shown to be the same as that of (45)

$$\alpha = M_{4,v} + M_{4,vi} = N\Big/\left(m+\omega-E_{p_1+q}\right)\left(E_2+\omega_3-E_{p_2+q_3}\right)\left(\omega_1-E_{p_1+q}-E_{p_1+q-q_1}\right) \quad 49.$$

$$\beta = M_{4,vii} + M_{4,viii} = N\Big/\left(-\omega_3-E_{p_1+q-q_1}-E_{p_2+q_3}\right)\left(\omega_3+E_2-E_{p_2+q_3}\right)\left(\omega_1-E_{p_1+q}-E_{p_1+q-q_1}\right) \quad 50.$$

$$\gamma = M_{4,vii} + M_{4,viii} + M_{4,ix} = -N\Big/\left(\omega_1-E_{p_1+q}-E_{p_1+q-q_1}\right)\left(\omega_3+E_2-E_{p_2+q_3}\right)\left(\omega_2+E_2+\omega_3+E_{p_1+q-q_1}\right) \quad 51.$$

Therefore

$$\alpha+\gamma = -N\Big/\left(E_2+\omega_3-E_{p_2+q_3}\right)\left(m+\omega-E_{p_1+q}\right)\left(\omega_2+E_2+\omega_3+E_{p_1+q-q_1}\right)$$

$$= -N\Big/\left(E_2+\omega_3-E_{p_2+q_3}\right)\left(m+\omega-E_{p_1+q}\right)\left(m+\omega-\omega_1+E_{p_1+q-q_1}\right)$$

$$\equiv M^F_{4,3}$$

4. Similarly

$$M_{4,x} + M_{4,xi} + M_{4,xii} = M^F_{4,4}$$

$$M_x + M_{xi} = N\Big/\left(\omega_1-E_{p_1+q-q_1}-E_{p_1+q}\right)\left(-E_2-\omega_3-E_{p_2+q_3}\right)\left(m+\omega-E_{p_1+q}\right) \quad 52.$$

and

$$M_x + M_{xi} + M_{xii} = N\Big/\left(E_2+\omega_3+E_{p_2+q_3}\right)\left(m+\omega-E_{p_1+q}\right)\left(E_2+\omega_3+\omega_2+E_{p_1+q-q_1}\right)$$

53.

$$= N \Big/ (E_{p_2} + \omega_2 + E_{p_2+q_3})\,(m+\omega-E_{p_1+q})\,(m+\omega-\omega_1+E_{p_1+q-q_1})$$

$$\equiv M^{F}_{4,4}$$

We have thus demonstrated that with the use of the decomposed propagator the equivalence between Feynman and the field theory can be established in a simple and straightforward manner.

CHAPTER V.

APPLICATION OF STOCHASTIC METHODS TO QUANTUM MECHANICS.

In this chapter, we shall establish the correspondence between the Feynman and field theoretic formalisms from a view point based on stochastic theory.

The essential feature of this method is that we move the wave functions along with the operators and redefine the interaction Hamiltonian as

$$H(n) \supset \sum_{p,p'} \hat{a}^{+}_{p'} \hat{a}_{p} + \hat{b}^{+}_{p'} \hat{a}^{+}_{p} + \hat{b}^{+}_{p'} \hat{b}_{p} + \hat{b}_{p'} \hat{a}_{p} \qquad 1.$$

where the creation operator b^{+} always occurs to the left of either a^{+} or b, unlike the ordering obtained from $\bar{\psi}\psi$. At any vertex one of the four fundamental processes can occur and the operators should occur in pairs but not necessarily in the order prescribed by $\bar{\psi}\psi$. A unique prescription can be given for the choice of the correct order based on the following arguments.

The process of pair annihilation at t represents the transition of a positive energy electron at t to a negative energy state at $t+\Delta$ the perturbation acting in the interval Δ and hence the electron destruction operators should be placed to the right of the positron destruction operator. In the case of pair creation in the interval between $t-\Delta$ and t, we trace the negative energy state of the electron at $t+\Delta$ back to a positive energy state at t so that in

this case $b^{\dagger}_{p'}$ should be placed to the left of $a^{\dagger}_{p}$. For electron and positron scattering the creation operators will be to the left of the annihilation operators. Thus the interaction Hamiltonian reads as (1.).

Having postulated the interaction Hamiltonian in the above form we now proceed to derive the matrix element for n-th order scattering of a single electron. The integrand will be given by

$$\langle f | [n][n-1] \cdots [k] \cdots [1] | i \rangle \qquad 2.$$

where $[n]$ is now given by ( ).

If we use this interaction term at each vertex, we can ignore the commutation relations of operators corresponding to different time points. We know that this commutation relation was used only when we were moving the positron operators in the previous method. Ignoring the commutation relations corresponding to different times amounts to viewing the process abinitio in the Feynman sequence. Thus for a typical realisation, we move the entire bracket $[k]$ containing an a_{p_1} to the left of $a^{\dagger}_{p_1}\rangle_0$ (i.e.)

$$\langle f | [n] \cdots [k+1][k-1] \cdots [1][k] | a^{\dagger}_{p_1} \rangle_0 \qquad 3.$$

(2.45)

The rearrangement is identical to that in method I, in this case when the operator attached to a_{p_1} in $[k]$ is an $a^{\dagger}_{p'}$. If on the other hand it is a $b_{p'}$, we place the bracket $[m]$ with $t_m < t_k$ containing $b^{\dagger}_{p'}$ to the left of $[k]$ (i.e.)

$$\langle f | [m] \cdots [k+1][k-1] \cdots [m+][m-1] \cdots [i][m][k] | i \rangle \qquad 4.$$

so that we have either

$$\left[\hat{b}^{\dagger}_{p'}(m)\ \hat{b}_{p'}(m)\right]\left[\hat{b}_{p'}(k)\ \hat{a}_{p_1}(k)\right] a^{\dagger}_{p_1} \rangle_0 \qquad 5.$$

or

$$\left[\hat{b}_{p'}(m)\ \hat{a}^{\dagger}_{p'}(m)\right]\left[\hat{b}_{p'}(k)\ \hat{a}_{p_1}(k)\right] a^{\dagger}_{p_1} \rangle_0 \qquad 6.$$

Thus in shifting the operator $b^{\dagger}_{p'}(m)$ to the left of $b_{p'}(k)$ we ~~re~~acquire a negative sign i.e.

$$-\hat{b}_{p'}(m)\hat{b}^{\dagger}_{p'}(m)\hat{b}_{p'}(k) \rangle_0 \qquad 7.$$

or

$$-\hat{b}^{\dagger}_{p'}(m)\hat{b}_{p'}(k) = -b^{\dagger}_{p'} b_{p'} v_{p'}(m)\bar{v}_{p'}(k) = -u_{-E}(m)\,\bar{u}_{-E}(k)\,e^{-ip'\cdot(x_m - x_k)} \qquad 8.$$

and we can straightaway identify

$$-u_{-E}\bar{u}_{-E}e^{-ip'\cdot(x_m - x_k)} \quad \text{for} \quad t_m < t_k \qquad 9$$

with the negative energy part of the Feynman kernel.

It is to be noted that the negative sign in Method I comes from the switching of adjacent operators in two successive brackets while in the present method we obtain the negative sign by commuting the operators within a single bracket.

The choice between the two procedures is therefore a matter of taste. It is claimed that the above constitutes a rigorous proof of the Feynman and field theoretic formalisms as

complete as those given in previous treatments. The simplicity of our approach is essentially due to two factors:

(1) We have taken the "elements" of the field operator,

(2) We have moved the terms within a bracket together.

If we have moved the ~~xxxxx xxxx~~ operators separately their partners in the brackets get "lost" and their "recovery" demands complicated algebraic operations.

Thus in the Feynman point of view the interaction takes place at n space-time vertices which connect the initial and final states in a sequence prescribed by the following rule :

If a particle makes a transition to a positive energy state it is followed to the next vertex at a later time when it makes a further transition. If on the other hand it makes a transition to a negative energy state this state is considered to be the result of <u>another</u> transition made earlier in time and we trace the negative energy particle back to that time point. In the language of the hole theory this is easily understandable. A particle can make a transition to a negative energy state only if there is a hole in the negative energy 'sea' and we then ask "When was the hole created"? Since the equation of the hole is the same as that of the negative energy particle, tracing the hole back to the origin is the same as tracing the negative energy electron. The interactions in space and time may be diagramatically represented in a two-dimensional diagram. One of the axes referring to space and the other to time and the arrows indicating the path to be followed in a Feynman sequence. Taking n such interactions the matrix element is obtained by an integration over all space and time from $-\infty$ to $+\infty$. This diagram

with the same sense of the arrows can be used even after integration provided we do not attach any space time coordinates to the vertices but merely ordinal numbers to indicate the Feynman sequence in which they are connected. It is this that is usually termed the Feynman diagram. By external line is meant a line which terminates only at one vertex and represents a particle either in the initial or final state with energy and momentum corresponding to a real particle. A line which connects two vertices is called an internal line or a propagator characterised by a four momentum p such that $p_4^2 \neq \vec{p}^2 + m^2$. It is customary to call states with such four momenta, virtual intermediate states though more accurately they should be called the transform of a propagator corresponding to virtual momenta.

The intermediate states here are therefore not attributed to any particular time but to the intermediate position in the Feynman sequence. If however we wish to retain time ordering in a strict sense including even the intermediate states, we have to do the space integration first and time interaction subsequently In this case, only the time axis has a definite meaning while the other axis is necessary only for diagramatic representation and has no significance as a space axis. If we further split the time integration into two parts corresponding to the intervals 0 to ∞ and $-\infty$ to 0 respectively we obtain two parts (A^+ and A^-) of propagator corresponding to ve and ve energy virtual intermediate states

$$A^{+} = \frac{1}{2E_{p+q}}\left[\frac{\not{p}+m}{E_p + E_q - E_{p+q}}\right] \qquad 10$$

$$A^{-} = \frac{1}{2E_{p+q}}\left[\frac{\not{p}+m}{E_p + E_q + E_{p+q}}\right] \qquad 11.$$

Thus the vertex (say) at the n th time point τ_n can be above or below the $n-1$ vertex (i.e.) $\tau_n > \tau_{n-1}$ corresponding to the part A^{+} or $\tau_n < \tau_{n-1}$ corresponding to the part A^{-}. Hence for an n-th order process there are $n-1$ Feynman propagators and since each propagator splits into two corresponding to + ve and − ve energy intermediate states we have 2^{n-1} "patterns". A pattern is characterised by the position of a vertex τ_i relative to the previous one τ_{i-1} in the Feynman sequence.

The picture of intermediate states in field theory is different. In this case we have real states only at $t = -\infty$ and $t = +\infty$ and at any finite time t we have a system, the momentum of which is identical with that of the initial and final states while the energy corresponding to this momentum is not the same as in initial or final energy. The sequence of events of an interaction diagram is obtained by performing the integration in a time ordered way so that the position of each vertex relative to all others is significant. Thus for an n th order process we have on the whole $n!$ diagrams corresponding to all possible time ordering. The energy of these intermediate states do not correspond to those of the intermediate states of the 2^{n-1} patterns of the Feynman diagram.

We know that if we want a matrix element we should be able to express it with a sequence of propagators $\frac{1}{\not{p}-m}$ or as a sum of $n!$ terms involving energy denominators representing the deviation of the energy of the intermediate states from that of the initial state. This though true is not easy to establish explicitly in the general case n . It is more interesting and instructive to compare the matrix element corresponding to one of the 2^{n-1} 'patterns'. In any one of the 2^{n-1} 'patterns' the order τ_n with respect to τ_{n-1} only should be preserved while in field theory, the order of the entire sequence should be preserved. It is interesting that the overall sign is as should be and the numerators of the field theoretic diagrams corresponding to a single "pattern" are the same.

We shall now make some remarks about the meaning of the kernel.

It is a sum over the three momenta of free particle wave functions or an integral with a weight factor $\frac{1}{2E_p}$ i.e.

$$K(2,1) = \int \frac{1}{2E_p} u_p \bar{u}_p e^{-ip\cdot(x_2-x_1)} d^3p, \quad t_2 > t_1$$

$$= -\int \frac{1}{2E_p} u_p \bar{u}_p e^{-ip\cdot(x_2-x_1)} d^3p \quad t_2 < t_1 \qquad (2.$$

Equally well it can be treated as a sum over all particles with four momenta $p_4, \vec{p}$ with the weight factor $\frac{1}{p^2-m^2}$ (energy momentum relation does not hold for p). Thus we can say that the amplitude for the creation and annihilation of a particle of four momenta is $\frac{1}{p^2-m^2} e^{-ip\cdot(x_2-x_1)}$. This

interpretation is not possible if we are not summing over all the momenta.

We have now completely established correspondence between the Feynman and field theoretic formalism in the case when we have only one electron in the initial state. There can exist of course a boson but while this forms part of field theory it does not in Feynman. If we have more than one electron in the initial state in field theory we will have more than one creation operator operating on the vacuum. But in Feynman since we are following the track of an electron, what should we do if we have more than one? Feynman postulated we should use anti-symmetric kernel for two particles, the antisymmetrisation necessitated by the Pauli principle. Actually if we used two particle initial states and rearranged - - - - - - - - - - the terms in the integrand of the field theoretic matrix element, we should have got the antisymmetric kernel in the place of K_+. The particular form of K_+ with the negative sign attached to the negative energy kernel and the form for the antisymmetrised kernel for two particles, both result from the anticommutation relations which expresses the Pauli principle in field theory. Thus it is not possible to derive the Pauli principle just by the use of K_+ for a single electron as was attempted by Feynman.

Symmetry operations on field variables

We have tillnow attempted to give the motivation for a field theoretic formalism. From what has been hitherto described its advantages are threefold:

1) The description of initial and final states is simplified by representing them through a series of creation and annihilation ~~xxxxxxxxx~~ operators operating on the vacuum respectively.

2) The interaction term is characteristic only of the process and not of the initial and final states.

Were these the only advantages, field theory would just be regarded as a convenient formalism and we can dispense with it especially if we are working only upto a few orders in the perturbation expansion and the possible states can be enumerated. In fact, many of the fundamental formulae in quantum electrodynamics like the Bethe-Heitler and Klein-Nishina cross-sections were derived ~~in~~ from wave mechanics ~~only~~ long before this formalism ~~formulated by~~ Feynman in 1949. In short, it may look as if we have not travelled far since the initial formulation of wave and matrix mechanics by Schrodinger and Heisenberg and the relativistic wave equation by Dirac; we have only devised a more elegant and perhaps a more sophisticated formalism, than the q-number theory.

This seems so if the only operations we perform with field operators are integrations over space and time and proper Lorentz transformations. With the discovery of new particles, the presence or absence of interaction between various particles had to be explained. The interaction that has been postulated

merely gives the amplitude for the annihilation and creation of particles and is therefore proportional to their wave functions. But this does not explain why the particles should interact at all, i.e. we have not given any prescription which determines the types of particles that should interact beyond the well-known conservation laws.

With the discovery of the new particles three new operations had to be defined.

(1) Space-inversion or the parity operation (P) :

(2) Time reversal (T) &,

(3) Particle-antiparticle conjugation (C).

The simplest interpretation that we can give to P is that the position vector $\vec{x}$ in the wave function should be replaced by $-\vec{x}$. An interesting feature of the probabilistic interpretation of a wave function in this interpretation occurs if we attach a phase factor of modulus unity to the wave function. We may therefore postulate that the wave function in the 'inverted' world is not obtained just by replacing x by $-x$ but by attaching an arbitrary phase factor η_P. But this is not logical in the case of ordinary wave functions since the only freedom we have is in changing $\psi(x)$ to $-\psi(x)$. But we may on the other hand postulate a generalised wave function

$$\Psi(x) \equiv \psi(x)$$

in one "world" and $\equiv \eta_p \psi(x')$

where $x' = -x$

in the inverted "world". This difficulty will not arise where we work with operators. It has become customary to define the parity operation on a one-particle state as

$$P a^\dagger_k \rangle_0 = P a^\dagger_k P^{-1} P \rangle_0 = \eta_p a^\dagger_{-k} \rangle_0 \qquad 13.$$

Even though by describing η_p as defined arising through

$$P a^\dagger_k P^{-1} = \eta_p a^\dagger_{-k} \qquad 14.$$

This freedom we have in obtaining phase factors in operators is a great advantage over the wave function formalism.

The introduction of these phase factors is of trivial consequence in the absence of interaction. This is not so if interaction exists since the phase factors may be different for the various particles created or destroyed in an interaction and the superposition of these phases, becomes consequential and leads to 'selection' rules of fundamental importance. Similar considerations apply to operators like C and T .

Interactions involving bound states

It is to be noted that in collision processes described till now, at any time t we have only a system of free particles and the interaction changes the aggregate when particles are destroyed and created at any particular vertex. If we are not interested in a particular type of particle say A of ~~particle~~ and assume does not exist in the initial or final state but is only created and annihilated in the collision process defined by the amplitude ~~is~~ $R_{\alpha\beta}$, it is obvious

that we can define $\rho_{\gamma\xi}$ as an integral over more than one space-time vertex involving $R_{\alpha\beta}$ where aggregate γ and δ do not contain particles of type A. We may then regard the process as a "non-local" interaction (i.e. both in space and time) defined in terms of $\rho_{\gamma\delta}$ or as a 'local' interaction in terms of $R_{\alpha\beta}$. In fact from this point of view any collision process can be treated as non-local space-time interaction between initial and final states.

As an example of $\rho_{\gamma\xi}$ we can have a non-local interaction between two fermions composed of two local interactions each involving two fermions and an intermediate boson. In some cases, assuming some of the particles, b_λ have infinite mass, the non-locality becomes purely spatial and in such a case we speak of "form factors" and "potentials". If we do not wish to include the particles of infinite mass in the description, then we have a system interacting with an "external" potential "due to those particles of infinite mass".

If we are only envisaging interaction through creation or annihilation of particles, as already stated, by the energy of the system is meant the energy ~~of the energy~~ of the free particle but the moment we exclude parts of infinite mass and suppress one type of particle by introducing an external potential ~~together~~, energy and momentum will be conserved though this is not true when the potential is excluded since the system can impart or draw energy from the potential. In the case of an external potential, there

is an extra freedom in that the wave functions need not be those of free particles, (i.e.) they may be "bound" such that only for the system of particles plus the potential, the total energy is conserved. This amounts to saying that the system of particles can exist as an eigen state of $H_0 + H_I$ (where H_I includes the external potential) from $t = -\infty$ to $t = +\infty$. Thus with an external potential concept of interaction enters in two ways.

1. A system of free particles at $t = -\infty$ going over to ~~$t = +\infty$~~ another system of free particles at $t = +\infty$ and

2. A system of particles bound to the external potential such that for all t and this system exists in an eigen state of $H_0 + H_I$. We can conceive of a situation when the potential is not external but mutual (i.e.) we are suppressing one type of particle without assuming that some particles have infinite mass. We can still have a bound system under this potential but since the origins of these potentials are subject to annihilation and creation due to the change of momenta, the situation becomes very complicated. This indeed is the case in nuclear many body problems. No one has ever dared to eliminate potentials and apply local field theory by introducing suitable quanta in the place of the potentials.

The structure is perhaps simpler in collision processes since the initial system from physical considerations, can never consist of more than two particles. As long as we assume α and β to be small in $R_{\alpha\beta}$ we can use perturbation approaches in regard to the creation and annihilation of some particles and phenomenological potentials if we wish to suppress particles for which perturbation concepts fail. Calculations using Chew's theory is one example of this approach.

Our description of the collision process therefore amounts to a mere rearrangement of momenta and energy among the various types of particles consistent only with the over all conservation of momentum and energy. This led many theoretical physicists to believe that it is possible to study the matrix elements without any reference to the details of interactions as envisaged by the "happenings" at the vertices.

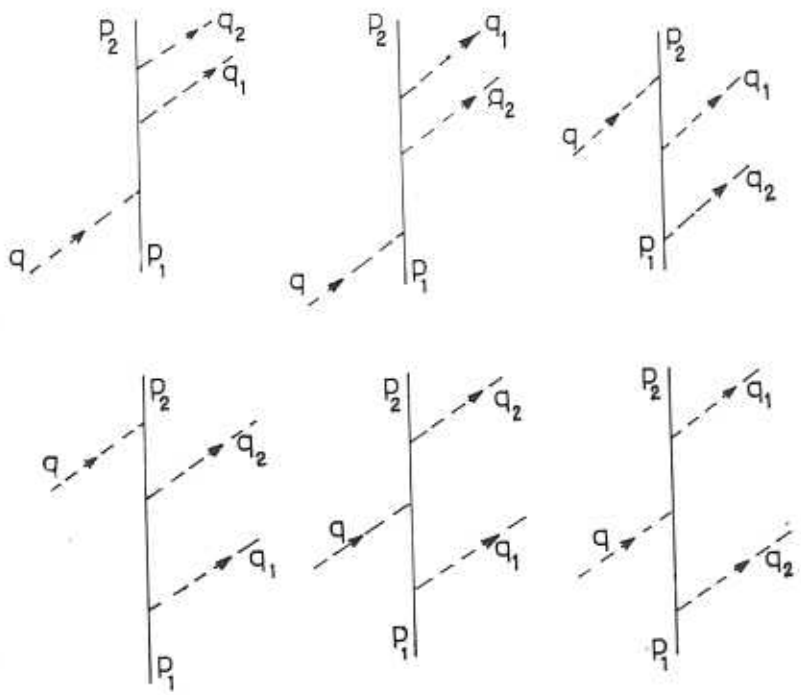
CHAPTER VI

PHOTON-ELECTRON CORRELATIONS IN DOUBLE COMPTON SCATTERING.[1]

1. Introductory Remarks

Recently considerable interest has been evinced in the theoretical aspect of double Compton scattering since it seems possible to get accurate experimental verification of this phenomemon. We here present systematic calculations for the differential cross-section when two circularly polarised photons of given energy are produced in the collision between a polarised electron at rest and a circularly polarised photon. It is well known that the cross-section for double scattering increases with energy (unlike in the case of single scattering) and also that there is a marked preference for small angle scattering at higher energies. The exact expression for the matrix element taking the photons to be circularly polarised can be calculated using the familiar Feynman methods. For a particular direction of polarisation of the initial electron, there are eight possible combinations of photon polarisations (i.e.) four +complex conjugate) so that when the final spin states of the electron are summed over, we have sixteen matrix elements on the whole. Taking the special case when the photons are emitted in the near-forward direction, it is found that due to the property of the γ matrices, four of the above matrix elements vanish. It should be pointed out that this is not a consequence of interference between the

1) A.K. Radha and R. Thunga, Zeit. Fur. Physik 161 20 (1961)



Feynman diagram for double Compton effect

six possible Feynman diagrams since the matrix elements for each of them vanishes independently.

2. Calculation of the matrix element

Denoting the matrix element for the process by $M = \bar{u}_2 N u_1$ (where u_2 and u_1 are the spinors for the initial and final electron states of four momenta $p_1 (m, \vec{0})$ and $p_2 (E_2, \vec{p_2})$ respectively) we have

$$N = \not{e}_2 \frac{1}{\not{p}_2 + \not{q}_2 - m} \not{e}_1 \frac{1}{\not{p}_1 + \not{q} - m} \not{e} +$$
$$+ \not{e}_2 \frac{1}{\not{p}_2 + \not{q}_2 - m} \not{e} \frac{1}{\not{p}_1 - \not{q}_1 - m} \not{e}_1 +$$
$$+ \not{e} \frac{1}{\not{p}_2 - \not{q} - m} \not{e}_2 \frac{1}{\not{p}_1 - \not{q}_1 - m} \not{e}_1 + \qquad 1.$$
$$+ \quad 1 \longleftrightarrow 2$$

corresponding to the six possible Feynman diagrams. The four momenta q, q_1 and q_2 refer to the incident and two outgoing photons with polarisation vectors $\vec{e}$, $\vec{e_1}$ and $\vec{e_2}$ respectively. Making use of the relations

$$\not{p}_1 u_1 = m u_1 ; \quad \text{and} \quad \not{q}^2 = 0 \quad \text{and} \quad \not{a}\not{b} = -\not{b}\not{a} + 2a \cdot b \text{ etc}$$

N can be conveniently written as

$$N = (4\pi)^{3/2} \frac{e^3}{4} \Big\{ \frac{1}{p_2 \cdot q} \Big[\not{q}_2 \frac{\not{e} \not{e}_2 \not{e}_1 \not{q}_1}{p_1 \cdot q_1} + \not{q}_1 \frac{\not{e} \not{e}_1 \not{e}_2 \not{q}_2}{p_1 \cdot q_2} \Big] +$$
$$+ \frac{1}{p_2 \cdot q_2} \Big[\frac{\not{q} \not{e}_2 \not{e} \not{e}_1 \not{q}_1}{p_1 \cdot q_1} - \not{q}_1 \frac{\not{e}_2 \not{e}_1 \not{e} \not{q}}{p_1 \cdot q} \Big] +$$
$$+ \frac{1}{p_2 \cdot q_1} \Big[\frac{\not{q}_1 \not{e}_1 \not{e} \not{e}_2 \not{q}_2}{p_1 \cdot q_2} - \frac{\not{q}_2 \not{e}_1 \not{e}_2 \not{e} \not{q}}{p_1 \cdot q} \Big] +$$

$$+\frac{2k_2 q\epsilon}{p_1\cdot q}\left[\frac{e_1\cdot q}{p_2 q_2}-\frac{e_1\cdot q_2}{p_2\cdot q_1}\right]+\frac{2k_1 q\epsilon}{p_1\cdot q}\left[\frac{e_2\cdot q}{p_2\cdot q_1}-\frac{e_2\cdot q_1}{p_2\cdot q_2}\right]+$$

$$+\frac{2\epsilon q_1 k_1}{p_1\cdot q_1}\left[\frac{e_2\cdot q_1}{p_2\cdot q}+\frac{e_2\cdot q}{p_2\cdot q_2}\right]+\frac{2\epsilon q_2 \epsilon_2}{p_1\cdot q_2}\left[\frac{e_1\cdot q_2}{p_2\cdot q}+\frac{e_1\cdot q}{p_2\cdot q_1}\right]+$$

$$+\frac{2k_2 q_1 \epsilon_1}{p_1\cdot q_1}\left[\frac{e\cdot q_2}{p_2\cdot q}-\frac{e\cdot q_1}{p_2\cdot q_2}\right]+\frac{2\epsilon_1 q_2 \epsilon_2}{p_1\cdot q_2}\left[\frac{e\cdot q_1}{p_2 q}-\frac{e\cdot q_2}{p_2\cdot q_1}\right]+$$

$$+4k_1\left[\frac{-e_2\cdot e}{p_2\cdot q_1}\right]+4k_2\left[\frac{-e_1\cdot e}{p_2\cdot q_2}\right]+4\epsilon\left[\frac{e_1\cdot e_2}{p_2 q}\right]\Big\}$$

$$=(4\pi)^{3/2}\frac{e^3}{4}\Big\{A_1 q_2\epsilon e_2 k_1 q_1+A_2 q_1\epsilon\epsilon_1 k_2 q_2+A_3 q\epsilon_2\epsilon k_1 q_1-$$

$$-A_4 q_1 k_2 e_1\epsilon q+A_5 q k_1\epsilon k_2 q_2-A_6 q_2 k_1 e_2\epsilon q+$$

$$+A_7 k_2 q\epsilon+A_8 k_1 q\epsilon+A_9\epsilon q_1 k_1+A_{10}\epsilon q_2 e_2$$

$$+A_{11} e_2 q_1 k_1+A_{12} e_1 q_2 k_2+A_{13} k_1+A_{14} e_2+A_{15}\epsilon\Big\}_{2.}$$

where $A_1=\frac{1}{(p_1\cdot q_1)(p_2\cdot q)}$ etc.

Taking the frame of reference in which the initial electron is at rest and the incident photon moves along the $+Z$ direction, the azimuth angles for the two outgoing photons are $(\theta_1, \phi_1\equiv 0)$ and (θ_2, ϕ_2). From the conservation laws we have

$$p_2=p_1+q-q_1-q_2 \qquad 3.$$

where

$$p_1 = m\gamma_t$$
$$q = \omega(\gamma_t - \gamma_z)$$
$$q_1 = \omega_1(\gamma_t - \gamma_x \sin\theta_1 - \gamma_z \cos\theta_1) \qquad 4.$$
$$q_2 = \omega_2(\gamma_t - \gamma_x \sin\theta_2 \cos\varphi_2 - \gamma_y \sin\theta_2 \sin\varphi_2 - \gamma_z \cos\theta_2)$$

which yields the relation

$$m(\omega - \omega_1 - \omega_2) = \omega\omega_1(1 - \cos\theta_1) + \omega\omega_2(1 - \cos\theta_2) -$$
$$- \omega_1\omega_2(1 - \sin\theta_1 \sin\theta_2 \cos\varphi_2 - \cos\theta_1 \cos\theta_2) \qquad 5.$$

When the photons are taken to be circularly polarised, we have for right circular polarization

$$k = \frac{1}{\sqrt{2}}[\gamma_x + i\gamma_y] = k_+$$
$$k_1 = \frac{1}{\sqrt{2}}[\gamma_x \cos\theta_1 - \gamma_z \sin\theta_1 + i\gamma_y] = k_{1+}$$
$$k_2 = \frac{1}{\sqrt{2}}[\gamma_x(\cos\theta_2 \cos\varphi_2 - i\sin\varphi_2) + i\gamma_y(\cos\varphi_2 - i\cos\theta_2 \sin\varphi_2)$$
$$- \gamma_z \sin\theta_2] = k_{2+} \qquad 6.$$

while the complex conjugate expressions refer to left circular polarization e_-, k_{1-} and k_{2-} respectively.

Substituting the above values for e, q, q_1 etc. we have (for a particular combination of polarisation say, $e_+ e_{1+} e_{2-}$ the following terms

where

$$p_1 = m\gamma_t$$
$$q = \omega(\gamma_t - \gamma_z)$$
$$q_1 = \omega_1(\gamma_t - \gamma_x \sin\theta_1 - \gamma_z \cos\theta_1) \qquad 4.$$
$$q_2 = \omega_2(\gamma_t - \gamma_x \sin\theta_2 \cos\phi_2 - \gamma_y \sin\theta_2 \sin\phi_2 - \gamma_z \cos\theta_2)$$

which yields the relation

$$m(\omega - \omega_1 - \omega_2) = \omega\omega_1(1 - \cos\theta_1) + \omega\omega_2(1 - \cos\theta_2) -$$
$$- \omega_1\omega_2(1 - \sin\theta_1 \sin\theta_2 \cos\phi_2 - \cos\theta_1 \cos\theta_2) \qquad 5.$$

When the photons are taken to be circularly polarised, we have for right circular polarization

$$\epsilon = \frac{1}{\sqrt{2}}[\gamma_x + i\gamma_y] = \epsilon_+$$
$$\epsilon_1 = \frac{1}{\sqrt{2}}[\gamma_x \cos\theta_1 - \gamma_z \sin\theta_1 + i\gamma_y] = \epsilon_{1+}$$
$$\epsilon_2 = \frac{1}{\sqrt{2}}[\gamma_x(\cos\theta_2\cos\phi_2 - i\sin\phi_2) + i\gamma_y(\cos\phi_2 - i\cos\theta_2 \sin\phi_2) - \gamma_z \sin\theta_2] = \epsilon_{2+} \qquad 6.$$

while the complex conjugate expressions refer to left circular polarization ϵ_-, ϵ_{1-} and ϵ_{2-} respectively.

Substituting the above values for ϵ, q, q_1 etc. we have (for a particular combination of polarisation say, $\epsilon_+ \epsilon_{1+} \epsilon_{2-}$ the following terms

$$A_1 \frac{w_1 w_2}{2\sqrt{2}} \{ (i\gamma_y\gamma_z\gamma_t + \gamma_x)[\sin\theta_1 \sin\theta_2 (\cos\theta_2 \sin\varphi_2 - i\cos\varphi_2) e^{i\varphi_2}] +$$
$$+ (\gamma_x\gamma_z\gamma_t + i\gamma_y)[(1-\cos\theta_2) B + A \sin\theta_2 e^{i\varphi_2}] +$$
$$+ (i\gamma_x\gamma_y\gamma_t + \gamma_z)[(1-\cos\theta_2) A + B \sin\theta_2 e^{i\varphi_2}] +$$
$$+ (\gamma_x\gamma_y\gamma_z + \gamma_t)[(1-\cos\theta_2) A - B \sin\theta_2 e^{i\varphi_2}] \},$$

where $A = \sin\theta_2 (\cos\theta_1 - 1) - e^{i\varphi_2} \sin\theta_1 (1 + \cos\theta_2)$

and $B = (1+\cos\theta_2)(1+\cos\theta_1) e^{i\varphi_2} + \sin\theta_1 \sin\theta_2 .$

$$A_2 \frac{w_1 w_2}{2\sqrt{2}} \{ (-i\gamma_y\gamma_z\gamma_t + \gamma_x)[-B(1+\cos\theta_1) - A \sin\theta_1] +$$
$$+ (\gamma_x\gamma_z\gamma_t - i\gamma_y)[B(1+\cos\theta_1) - A \sin\theta_1] +$$
$$+ (-i\gamma_x\gamma_y\gamma_t + \gamma_z)[-A(1+\cos\theta_1) + B \sin\theta_1] +$$
$$+ (-i\gamma_x\gamma_y\gamma_z + \gamma_t)[A(1+\cos\theta_1) + B \sin\theta_1] \}$$

where $A = \sin\theta_2 (\cos\theta_1 - 1) - (1+\cos\theta_2) \sin\theta_1 e^{i\varphi_2}$

and $B = (\cos\theta_2 - 1)(1 - \cos\theta_1) e^{-i\varphi_2} - \sin\theta_1 \sin\theta_2,$

$$A_3 \frac{w w_1}{2\sqrt{2}} \{ 2(1-\cos\theta_1)(1+\cos\theta_2) e^{i\varphi_2} (-\gamma_x + i\gamma_y - i\gamma_y\gamma_z\gamma_t + \gamma_x\gamma_z\gamma_t) +$$
$$+ \sin\theta_1 (1+\cos\theta_2) e^{i\varphi_2} (\gamma_t - \gamma_z + i\gamma_x\gamma_y\gamma_z - i\gamma_x\gamma_y\gamma_t) \}$$

$$-A_4 \frac{w w_1}{2\sqrt{2}} \{ (\gamma_x + i\gamma_y + \gamma_x\gamma_z\gamma_t + i\gamma_y\gamma_z\gamma_t)[B(1-\cos\theta_1) - A \sin\theta_1] +$$
$$+ \gamma_z - \gamma_t + i\gamma_x\gamma_y\gamma_t + i\gamma_y\gamma_x\gamma_z)[-A(1+\cos\theta_1) - B \sin\theta_1] \}$$

where $A = (\cos\theta_1 - 1)\sin\theta_2 - \sin\theta_1 (1+\cos\theta_2) e^{i\varphi_2}$

and $B = -\sin\theta_1 \sin\theta_2 - (\cos\theta_1 - 1)(\cos\theta_2 - 1) e^{i\varphi_2},$

$$A_5 \frac{\omega\omega_2}{2\sqrt{2}}\Big\{2\sin\theta_1\sin\theta_2\left(-\gamma_x - i\gamma_y + i\gamma_y\gamma_2\gamma_t + \gamma_x\gamma_2\gamma_t\right) + \\ + 2\sin\theta_1(1+\cos\theta_2)e^{i\varphi_2}\left(\gamma_t - \gamma_2 - i\gamma_x\gamma_y\gamma_2 + i\gamma_x\gamma_y\gamma_t\right)\Big\},$$

$$A_6 \frac{\omega\omega_2}{2\sqrt{2}}\Big\{\left(\gamma_x + i\gamma_y + \gamma_x\gamma_2\gamma_t + i\gamma_y\gamma_2\gamma_t\right)\left[B(1-\cos\theta_2) - A\sin\theta_2 e^{-i\varphi_2}\right] \\ + \left(\gamma_t - \gamma_2 - i\gamma_x\gamma_y\gamma_t + i\gamma_x\gamma_y\gamma_2\right)\left[A(1+\cos\theta_2) - B\sin\theta_2 e^{i\varphi_2}\right]\Big\},$$

where $A = \sin\theta_2(1-\cos\theta_1) + \sin\theta_1(1+\cos\theta_2)e^{i\varphi_2}$

and $B = (1+\cos\theta_1)(1+\cos\theta_2)e^{i\varphi_2} + \sin\theta_1\sin\theta_2,$

$$A_7 \frac{\omega}{2}\Big\{\sin\theta_2\left(-\gamma_x - i\gamma_y - \gamma_x\gamma_2\gamma_t - i\gamma_y\gamma_2\gamma_t\right) + \\ + (1+\cos\theta_2)e^{i\varphi_2}\left(\gamma_t - \gamma_2 + i\gamma_x\gamma_y\gamma_2 - i\gamma_x\gamma_y\gamma_t\right)\Big\}$$

$$A_8 \frac{\omega}{2}\Big\{\sin\theta_1\left(-\gamma_x - i\gamma_y - i\gamma_y\gamma_2\gamma_t - \gamma_x\gamma_2\gamma_t\right) + \\ + (\cos\theta_1 - 1)\left(\gamma_t - \gamma_2 + i\gamma_x\gamma_y\gamma_2 - i\gamma_x\gamma_y\gamma_t\right)\Big\}$$

$$A_9 \frac{\omega_1}{2}\Big\{\sin\theta_1\left(\gamma_x + i\gamma_y + \gamma_x\gamma_2\gamma_t + i\gamma_y\gamma_2\gamma_t\right) + \\ + (\cos\theta_1 - 1)\left(\gamma_t + \gamma_2 + i\gamma_x\gamma_y\gamma_2 + i\gamma_x\gamma_y\gamma_t\right)\Big\},$$

$$A_{10} \frac{\omega_2}{2}\Big\{\sin\theta_2\left(-\gamma_x - i\gamma_y + \gamma_x\gamma_y\gamma_t + i\gamma_x\gamma_y\gamma_2\right) + \\ + (\cos\theta_2 + 1)\left(\gamma_t - \gamma_2 - i\gamma_x\gamma_y\gamma_2 + i\gamma_x\gamma_y\gamma_t\right)e^{i\varphi_2}\Big\},$$

$$A_{11} \frac{\omega_1}{2}\Big\{\left(\gamma_x + i\gamma_y\gamma_2\gamma_t\right)\left(-i\sin\theta_1\cos\theta_2\sin\varphi_2 - \sin\theta_1\cos\varphi_2 - \sin\theta_2\right) \\ + \left(\gamma_x\gamma_2\gamma_t + i\gamma_y\right)\left(\sin\theta_1\cos\theta_2\cos\varphi_2 + i\sin\theta_1\sin\varphi_2 - \sin\theta_2\cos\theta_1\right) \\ + \left(i\gamma_x\gamma_y\gamma_t + \gamma_2\right)\left(-\cos\theta_2\cos\varphi_2 - i\sin\varphi_2 - i\cos\theta_1\cos\theta_2\sin\varphi_2 \right. \\ \left. - \cos\theta_1\cos\varphi_2\right) \\ + \left(i\gamma_x\gamma_y\gamma_2 + \gamma_t\right)\left(i\cos\theta_1\cos\theta_2\cos\varphi_2 + i\cos\theta_1\sin\varphi_2 \right. \\ \left. + i\cos\theta_2\sin\varphi_2 + \cos\varphi_2 + \sin\theta_1\sin\theta_2\right)\Big\}$$

$$A_{12}\frac{\omega_2}{2}\Big\{(\gamma_x - i\gamma_y\gamma_z\gamma_t)(-\sin\theta_2 - i\sin\theta_1\cos\theta_2\sin\phi_2 - \sin\theta_1\cos\phi_2) +$$
$$+(\gamma_x\gamma_z\gamma_t - i\gamma_y)(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2\cos\phi_2 - i\sin\theta_2\sin\phi_2)$$
$$+(-i\gamma_x\gamma_y\gamma_t + \gamma_z)(-\cos\theta_2\cos\phi_2 - i\sin\phi_2 - i\cos\theta_1\cos\theta_2\sin\phi_2$$
$$-\cos\theta_1\cos\theta_2) +$$
$$+(-i\gamma_x\gamma_y\gamma_z + \gamma_t)(i\sin\phi_2\cos\theta_1 + \cos\phi_2 + i\cos\theta_2\sin\phi_2 +$$
$$\cos\theta_1\cos\theta_2\cos\phi_2 + \sin\theta_1\sin\theta_2)\Big\}$$

$$A_{13}\frac{1}{\sqrt{2}}\left[\gamma_x\cos\theta_1 - \gamma_z\sin\theta_1 + i\gamma_y\right],$$

$$A_{14}\frac{1}{\sqrt{2}}\left[\gamma_x(\cos\theta_2\cos\phi_2 + i\sin\phi_2) - i\gamma_y(\cos\phi_2 + i\cos\theta_2\sin\phi_2)\right.$$
$$\left.-\gamma_z\sin\theta_2\right]$$

$$A_{15}\frac{1}{\sqrt{2}}\left[\gamma_x + i\gamma_y\right] \qquad 7.$$

where $\frac{(4\pi)^{3/2}e^3}{4}\times$ (the sum of the above 15 terms) constitutes N. 8.

This expression is exact. Similar expressions for N for other combination may be calculated. The differential cross-section is given by $d\sigma = 2\pi \times |\bar{u}_2 N u_1|^2 \times$ density of states (D) / normalization factors (η) , where D is the density of states, for the three particle final state and

$$\frac{D}{\eta} = \frac{1}{(2\pi)^6} \frac{E_2 \omega_2^2 \omega_1^2 d\omega_1 d\Omega_{\omega_1} d\Omega_{\omega_2} \left(2E_1 2E_2 2\omega 2\omega_1 2\omega_2\right)^{-1}}{m + \omega(1-\cos\theta_2) - \omega_1(1-\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos\varphi_2)} \quad 9.$$

To evaluate the cross-section, we shall assume that the two photons are produced in the near forward direction[1] (i.e. θ_1 and θ_2 are small so that $O(\theta_{1,2}^4)$ may be neglected). From energy conservation (4) it follows that $\omega - \omega_1 - \omega_2 = E_2 - m \sim \theta_{1,2}^2$. Similarly from momentum conservation, it is seen that $|\vec{p}_2| \sim \theta_{1,2}$ and $p_z \sim \theta_{1,2}^2$ and $p_{2,y} \sim \theta_{1,2}$ i.e. since the momentum transfer is very small, it is a "grazing collision" and the recoil electron moves nearly at right angles to the incident photon direction.

The values for $\sum_{\text{Spins}2} |\bar{u}_2 N u_1|^2$ for various polarisation combinations are listed below.

It is interesting to note that the cross-sections for $e_+ e_{1+} e_{2+}$ and $e_- e_{1-} e_{2-}$ are found to be zero under this approximation irrespective of the spin of the electron.

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1) Mandl, F., and T.H.R.Skyrme: Proc. Roy. Soc. Lond. 215, 497 (1952). However note that we cannot set $\omega - \omega_1 - \omega_2 = 0$ (as has been stated in the above paper (since this would imply that $E_2 - m = 0$ (from energy conservation).

Polarisation of outgoing γ-s	Spin of initial e^-	$\sum_2 \lvert \bar{u}_2 N u_1 \rvert^2 \frac{4}{\pi^3}$ for e_+	$\sum_2 \lvert \bar{u}_2 N u_1 \rvert^2 \frac{4}{\pi^3}$ for e_-
$e_{1+}\, e_{2+}$	$\uparrow$	0	$64m(E_2-m)\left(\frac{1}{p_2\cdot q_1}+\frac{1}{p_2\cdot q_2}\right)^2$
	$\downarrow$	0	$\frac{32E_2}{m}\left\{\omega^2\left[\frac{\sin^2\theta_2}{(p_2\cdot q_1)^2}+\frac{\sin^2\theta_1}{(p_2\cdot q_2)^2}+\frac{2\sin\theta_1\sin\theta_2\cos\varphi_2}{(p_2\cdot q_1)(p_2\cdot q_2)}\right]\right.$ $\left.+\frac{\omega_1^2\sin^2\theta_1}{(p_2\cdot q_2)^2}+\frac{2\omega\omega_1\sin^2\theta_1}{(p_2\cdot q_2)^2}+\frac{2\omega\omega_1\sin\theta_1\sin\theta_2\cos\varphi_2}{(p_2\cdot q_1)(p_2\cdot q_2)}\right\}$
$e_{1-}\, e_{2-}$	$\uparrow$	$\equiv e_{1+}\, e_{2+}\, e_- \downarrow$	0
	$\downarrow$	$\equiv e_-\, e_{1+}\, e_{2+} \uparrow$	0
$e_{1+}\, e_{2-}$	$\uparrow$	$\frac{32E_2}{m}\left\{\omega_2^2\sin^2\theta_2\left[\frac{1}{p_2\cdot q}-\frac{1}{p_2\cdot q_1}\right]^2+\omega_2^2\sin^2\theta_1\frac{1}{(p_2\cdot q)^2}\right.$ $\left.-2\omega_2^2\sin\theta_1\sin\theta_2\cos\varphi_2\left(\frac{1}{p_2\cdot q}-\frac{1}{p_2\cdot q_1}\right)\frac{1}{p_2\cdot q}\right\}$	$64m(E_2-m)\left(\frac{1}{p_2\cdot q}-\frac{1}{p_2\cdot q_2}\right)^2$
	$\downarrow$	$64m(E_2-m)\left(\frac{1}{p_2\cdot q}-\frac{1}{p_2\cdot q_1}\right)^2$	$\frac{32E_2}{m}\left\{\omega_1^2\sin^2\theta_1\left(\frac{1}{p_2\cdot q}-\frac{1}{p_2\cdot q_2}\right)^2+\frac{\omega_1^2\sin^2\theta_2}{(p_2\cdot q)^2}\right.$ $\left.-2\omega_1^2\sin\theta_1\sin\theta_2\cos\varphi_2\left(\frac{1}{p_2\cdot q}-\frac{1}{p_2\cdot q_2}\right)\frac{1}{p_2\cdot q}\right\}$
$e_{1-}\, e_{2+}$	$\uparrow$	$\equiv e_-\, e_{1+}\, e_{2-} \downarrow$	$\equiv e_+\, e_{1+}\, e_{2-} \downarrow$
	$\downarrow$	$\equiv e_-\, e_{1+}\, e_{2-} \uparrow$	$\equiv e_+\, e_{1+}\, e_{2-} \uparrow$

PART II.

CHAPTER I.

ON THE Y^* RESONANCE.

Introductory remarks

The experimental discovery of the theoretically predicted ρ resonance, a two pion system in the $T=1, J=1$ state, was the first indication that the dispersion theoretic approach could lead to results directly susceptible to experimental tests. Since then various resonant states of strongly interaction particles have been discovered,[1] and have provided one of the main sources of approximations necessary in solving single variable dispersion relations.[2] It was shown in the πN problem that an effective range analysis of partial wave amplitudes led rather elegantly to the original

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1) Attempts have been made to consider these resonant states themselves as 'elementary' and associate fields with them. (see for example J.J.Sakurai, Annals of Physics, 10) However according to Feynman the correct theory should be such that it does not allow one to say which particles are "elementary" and this has been interpreted by Chew to mean that the singularities of the S-matrix corresponding to elementary particle states, bound states and resonant states should all have a common and equivalent basis. (see G.F.Chew, UCRL Report 9701)

2) G.F. Chew, M.L. Goldberger, F.E. Low and Y. Nambu, Phys. Rev. 106 1337 (1957)

Table 1. Some of the observed resonances.

Name	Mass (MeV)	Width (MeV)	I	J	Decay products
ρ^0	725	80	1	1^-	$\pi^+\pi^-$; $\pi^\pm\pi^0$
$\rho^\pm$	750	"	"	"	
ω	780	≤ 25	0	ϕ^-	$\pi^+\pi^-\pi^0$
η	546	< 25	0	0^-	neutrals
Y_1^*	1385	50	1	$3/2$?	" $\Lambda+\pi$ / $\Sigma+\pi$
Y_0^*	1405	40	0	?	$\Sigma+\pi$ / $\Lambda+2\pi$
Y_0^*	1525	16	0	$3/2$	$\Sigma+\pi$ / $\Lambda+2\pi$

results of the static theory. The essential spirit of the approximation is that since the strange of force is inversly proportional to the distance from the origin in the complex momentum plane of the associated singularity, scattering in the centrifugally excluded higher orbital waves should be dominated only by the nearby singularities. For low angular momentum states where short-range interaction comes into play, the effect of multiparticle configurations is approximated by arbitrary parameters to be fitted by experiment. It is in this respect that the data on resonant states have been most useful.

In this chapter we wish to extend this procedure to pion-hyperon scattering. We set up two parameter effective range formulae for $\Lambda\pi$ and $\Sigma\pi$ scattering amplitudes for both even and odd $\Sigma\Lambda$ relative parities. These are evaluated assuming the mass of the recently observed Y_1^* and Y_0^* resonances (see table 1). Block et. al. tentatively conclude that the resonance has total angular momentum $J = 1/2$ and odd parity. The spin and parity of the Y^* resonances are not yet known. Hence we set up effective range formulae for $\Lambda\pi$ and $\Sigma\pi$ scattering amplitudes in the resonance channels with different spin and parity

assignments. A point of special interest in the $\Lambda\pi$ scattering is the role of the relative $\Sigma\Lambda$ parity which determines the sign and magnitude of the residues of the poles. This enables one to investigate the possibility of $\Lambda\pi$ resonance in different partial wave amplitudes. Further it is possible to give an order of magnitude estimate of the renormalized $\Sigma\Lambda\pi$ coupling constant from a knowledge of the width of the resonance.

Though a direct experimental verification of the scattering cross-sections may look impossible, it is interesting to note that information on this scattering may be obtained from an analysis of the experimentally observed reaction

$$\gamma + N \longrightarrow Y + N + \pi$$

by the extrapolation method suggested by Chew and Low. The residue at the (π) pole in this reaction is the product of the meson nucleon coupling constant and the $\Lambda\pi$ scattering cross-section with all mesons on the mass-shell. This will be discussed in detail in chapters IV .

Thus in section (2) we derive an effective range formulae assuming the Y^* resonance to be in the $\ell = 0$, $J = 1/2$ state and evaluate the total cross-sections for different coupling constants and present the conclusions in section (2 b) . In Section (3 a) we develop analogous effective range formulae for $\Lambda\pi$ scattering for the $J = 3/2$ and $J = 5/2$ channels in the $\ell = 1$, 2 and 3 states and present the conclusions in section (3 b).

2. Partial wave analysis for $\Sigma\pi$ scattering.

2 a. S wave scattering.

We shall illustrate the derivation of the effective range formula for odd $\Sigma\Lambda$ parity. The derivation in the other casesis similar. Let p_1 and q_1 denote the four-vector momenta of the incident Λ and π and p_2 and q_2 those of the outgoing Λ and π respectively. The T-matrix for the reaction can as usual be written as

$$T^I = -A^I + i\gamma\cdot\frac{(q_1+q_2)}{2} B^I \qquad 1.$$

where the superscripts denote the isotopic spin of the channel* , a notation which we shall use throughout the chapter.

The Σ pole contributions to A and B for odd $\Sigma\Lambda$ parity are then found to be

$$A^0 = g^2_{\Sigma\Lambda\pi}\frac{m_\Lambda + m_\Sigma}{m_\Sigma^2 - s} + g^2_{\Sigma\Lambda\pi}\frac{m_\Lambda + m_\Sigma}{m_\Sigma^2 - \bar{s}} \qquad 2.$$

$$B^0 = g^2_{\Sigma\Lambda\pi}\frac{1}{m_\Sigma^2 - s} + g^2_{\Sigma\Lambda\pi}\frac{1}{m_\Sigma^2 - \bar{s}} \qquad 3.$$

where $g_{\Sigma\Lambda\pi}$ denotes the renormalized $\Sigma\Lambda\pi$ coupling constant, m_Λ and m_Σ denote the masses of Λ and Σ and

$$s = (p_1 + q_1)^2 = W^2 \qquad 4.$$

$$\bar{s} = (p_1 - q_2)^2 = -2k^2(1-\cos\theta) \qquad 5.$$

---

* These superscripts are really not necessary for $\Lambda\pi$ scattering since only one isotopic spin channel is available in this case. For $\Sigma\pi$ case we have to use this notation.

W being the total centre of mass energy, k and θ the corresponding centre of mass momentum and scattering angle respectively.

Following Frazer and Fulco, we define the partial wave amplitude as

$$h_{\ell\pm} = \frac{W}{E+m_\Lambda} \frac{e^{i\delta_{\ell\pm}} \sin\delta_{\ell\pm}}{k^{2\ell+1}}$$

$$= \frac{1}{16\pi} \left\{ \frac{A_\ell}{k^{2\ell}} + (W-m_\Lambda)\frac{B_\ell}{k^{2\ell}} + (E-m_\Lambda)^2 \left[\frac{-A_{\ell\pm1}}{k^{2\ell+2}} + \frac{(W+m_\Lambda)B_{\ell\pm1}}{k^{2\ell+2}} \right] \right\} \quad 6$$

where

$$\left[A_\ell^{(s)}; B_\ell^{(s)}\right] = \int_{-1}^{1} d\cos\theta \; P_\ell(\cos\theta) \left[A(s,\cos\theta); B(s,\cos\theta)\right] \quad 7.$$

and $\delta_{\ell\pm}$ is the phase shift for $J = \ell \pm \frac{1}{2}$.

For S waves, (6) becomes

$$h_0(W) = \frac{1}{16\pi} \left\{ A_0 + (W-m_\Lambda)B_0 + (E-m_\Lambda)^2 \left[\frac{-A_1}{k^2} + (W+m_\Lambda)\frac{B_1}{k^2} \right] \right\}, \quad 8$$

where $h_\ell(W)$ is related to the S -wave phase-shift through

$$h_0(W) = \frac{W}{E+m_\Lambda} \frac{e^{i\delta_{0+}} \sin\delta_{0+}}{k} \quad 9.$$

In developing the effective range formula, we shall completely neglect the cuts associated with the crossed processes. Similarly the Σ -pole associated with the crossed $\Lambda\pi$ scattering will also be neglected. This is consistent with the spirit of the effective range approximation which retains only the nearest singularities. In the Chew-Low

formula, the "crossed" nucleon pole gives rise to branch cuts inthe W -plane for the partial wave amplitudes of which the part lying in the region

$$m_N - \frac{1}{m_N} \leq W \leq \left(m_N^2 + 2\right)^{1/2} \qquad 10$$

is retained and becomes the static pole of this theory.[1] The analogous region in our case is

$$\frac{m_\Lambda^2 - 1}{m_\Sigma} \leq W \leq \left(2 m_\Lambda^2 + 2 - m_\Sigma^2\right)^{1/2} \qquad 11$$

Estimates show that the contribution from this cut is small compared to the contribution from the "direct" Σ -pole and hence is neglected. With these approximations, the theory requires only single variable dispersion relations. For odd $\Sigma\Lambda$ -parity the part of $h_c(W)$ arising from the latter reads

$$\bar{J}_c(W) = g^2_{\Sigma\Lambda\pi} \frac{1}{8\pi} \frac{1}{m_\Sigma - W} \qquad 12$$

Thus we may write for $h_0(W)$ with the pion mass set equal to unity

$$h_c(W) = \bar{J}_0(W) + \frac{1}{\pi} \int_{m_\Lambda + 1}^{\infty} dW' \frac{\mathrm{Im}\, h_0(W')}{W' - W} \qquad 13$$

where we have as yet made no subtractions. The N/D solution of equation (13) with two subtractions for D reads

$$\frac{g^2_{\Sigma\Lambda\pi}}{8\pi} \frac{1}{m_\Sigma - W} \left(E + m_\Lambda\right) \frac{k}{W} \cot \delta_{c+} = 1 - \frac{W - m_\Sigma}{\omega_\Lambda} + \frac{(W - m_\Sigma)^2}{\pi} P \int_{m_\Lambda + 1}^{\infty} \frac{dW' \mathrm{Im}\, J(W')}{(W' - m_\Sigma)(W' - W)} \qquad 14$$

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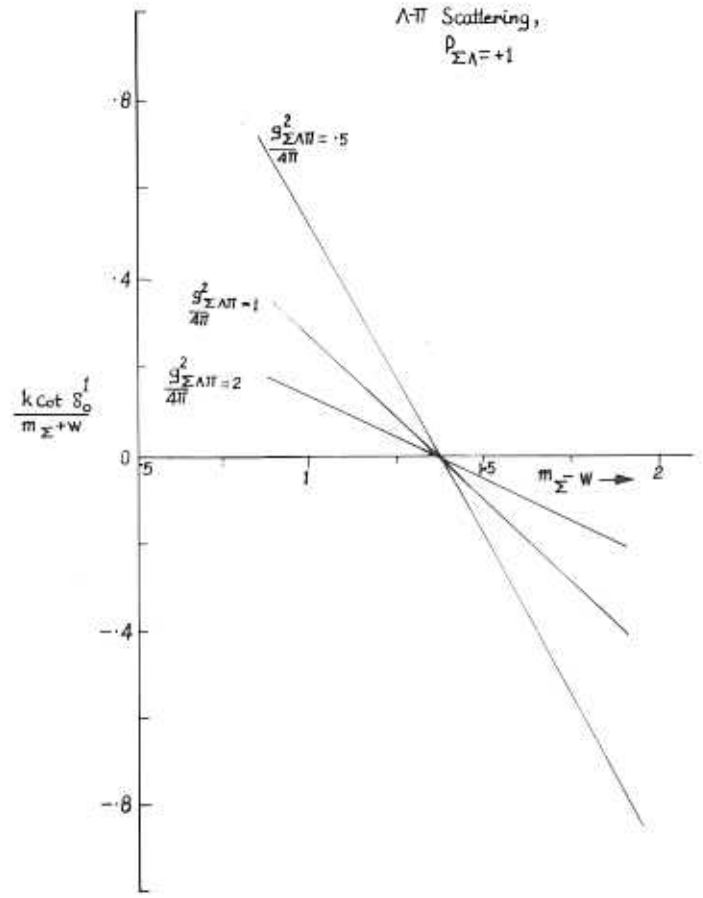
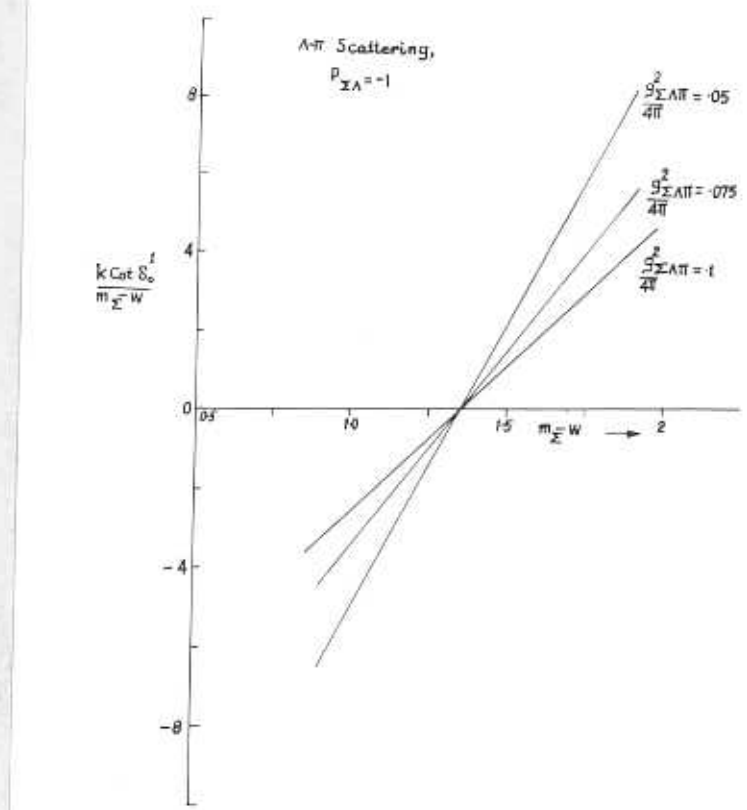
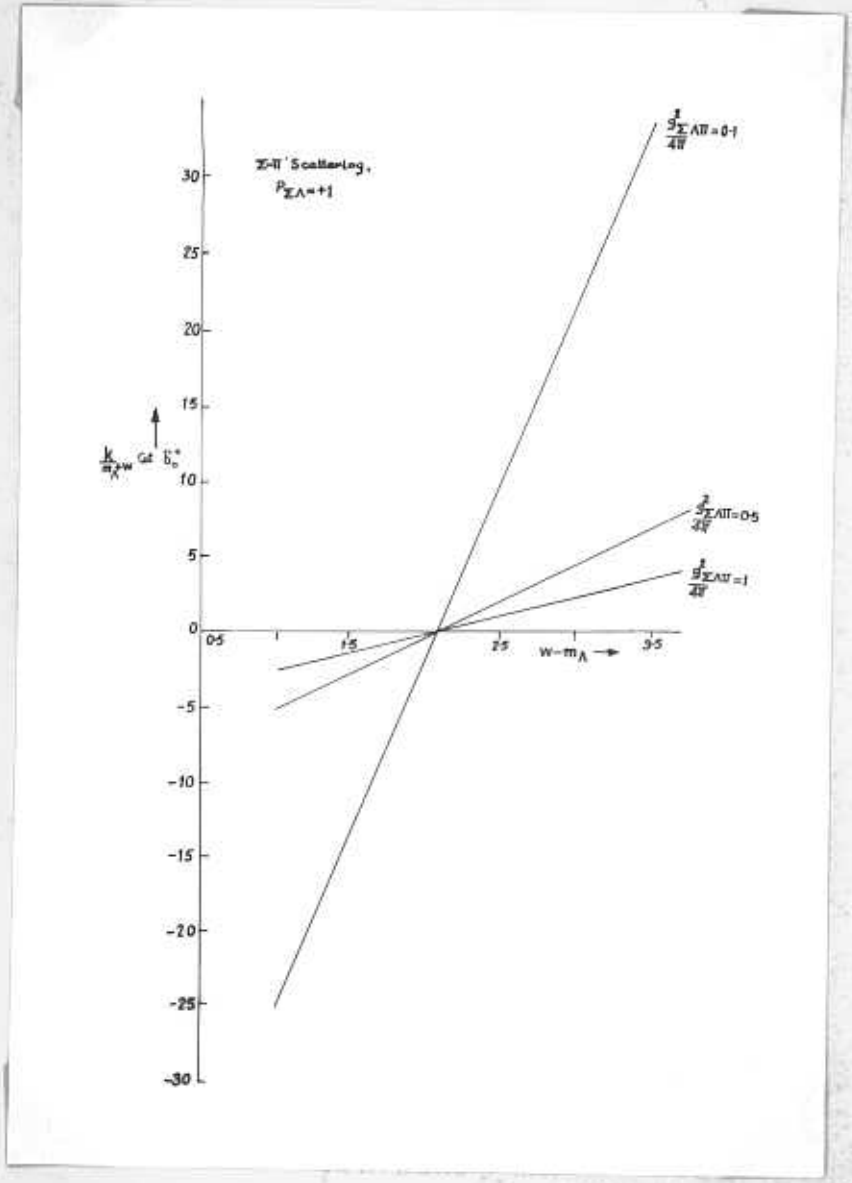
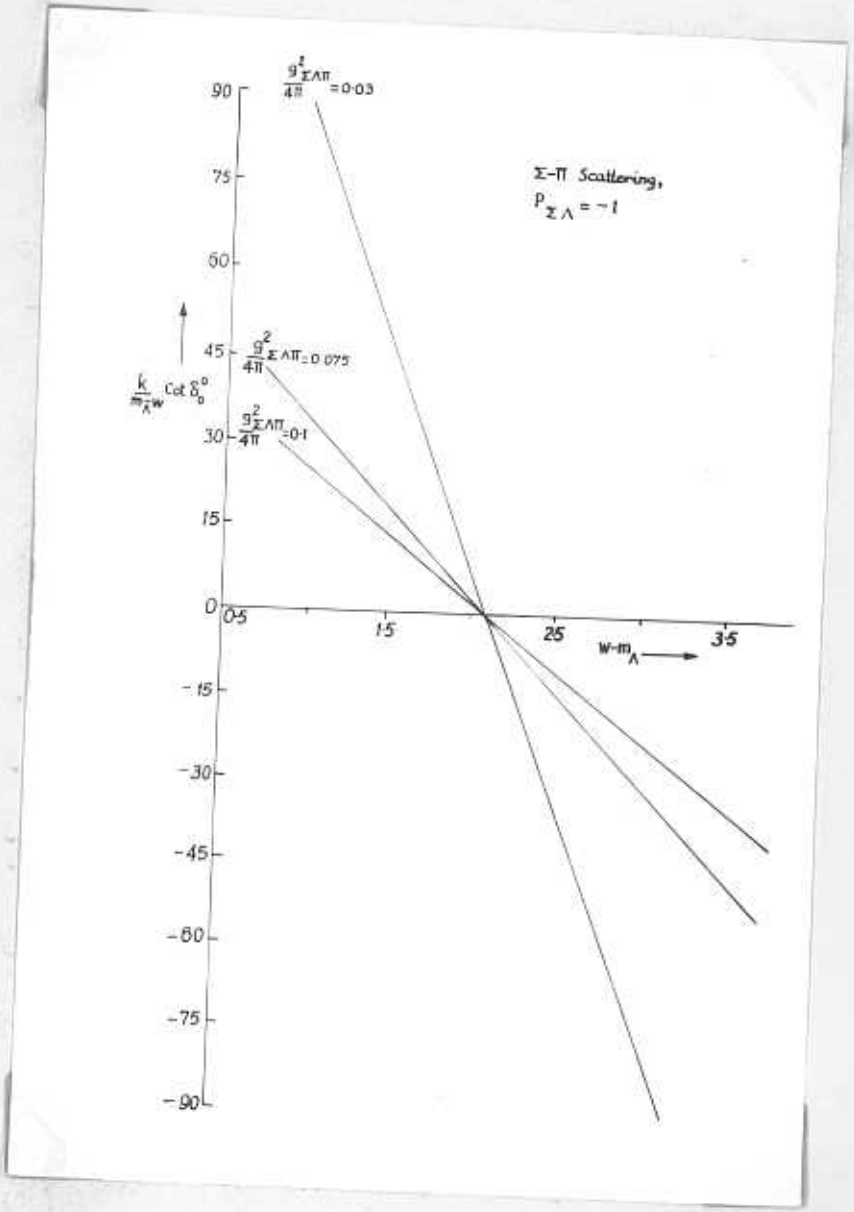
1) Chew, Goldberger, Low and Nambu, Phys.Rev. 106, 1337 (1957)


Λ-π Scattering,
$p_{\Sigma\Lambda} = +1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = \cdot 5$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 2$
$\frac{k \cot \delta_0^1}{m_{\Sigma^+} w}$
$m_{\Sigma^-} w \rightarrow$
·8
·4
0
-·4
-·8
·5
1
1·5
2


Λ-π Scattering,
$p_{\Sigma\Lambda} = -1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = \cdot 05$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = \cdot 075$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = \cdot 1$
$\frac{k \cot \delta_0^1}{m_{\Sigma^-} w}$
$m_{\Sigma^-} w \rightarrow$
8
4
0
-4
-8
0·5
1·0
1·5
2


Σ-π Scattering,
$P_{\Sigma\Lambda} = +1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.5$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 1$
$\frac{k}{m_\Lambda w}$ Cot δ_0°
$w - m_\Lambda$
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
0·5
1·5
2·5
3·5


Σ-π Scattering,
$P_{\Sigma\Lambda} = -1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.03$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.075$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.1$
$\frac{k}{m_\Lambda w}$ Cot δ_0°
$w - m_\Lambda$
90
75
60
45
30
15
0
-15
-30
-45
-60
-75
-90
0·5
1·5
2·5
3·5

where ω_Λ is a subtraction constant. We have normalized D such that $D(m_\Sigma) = 1$. This implies the definition of the renormalized $\Sigma\Lambda\pi$ coupling constant by an analogue of the Lepore-Watson convention for the $\pi-N$ coupling constant.[10] In a static approximation, (14) becomes

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{k}{m_\Sigma - W}\cot\delta_{0+} = 1 - \frac{W-m_\Sigma}{\omega_\Lambda} \qquad [P_{\Sigma\Lambda} = -1] \qquad 15$$

Similarly one derives the corresponding formula for even $\Sigma\Lambda$ parity :

$$-\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{k}{m_\Sigma + W}\cot\delta_{0+} = 1 - \frac{W-m_\Sigma}{\omega_\Lambda} \qquad [P_{\Sigma\Lambda} = +1] \qquad 16$$

ω_Λ is fixed by the requirement that $\delta_{0+} = \pi/2$ at resonance. This gives

$$\omega_\Lambda = 1.357$$

The formulae for $\Sigma-\pi$ scattering in the resonance channel can now be written down. We have

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{k}{m_\Lambda - W}\cot\delta_{0+} = 1 - \frac{W-m_\Lambda}{\omega_\Sigma} \qquad [P_{\Sigma\Lambda} = -1] \qquad 17$$

and

$$-\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{k}{m_\Lambda + W}\cot\delta_{0+} = 1 - \frac{W-m_\Lambda}{\omega_\Sigma} \qquad [P_{\Sigma\Lambda} = +1] \qquad 18.$$

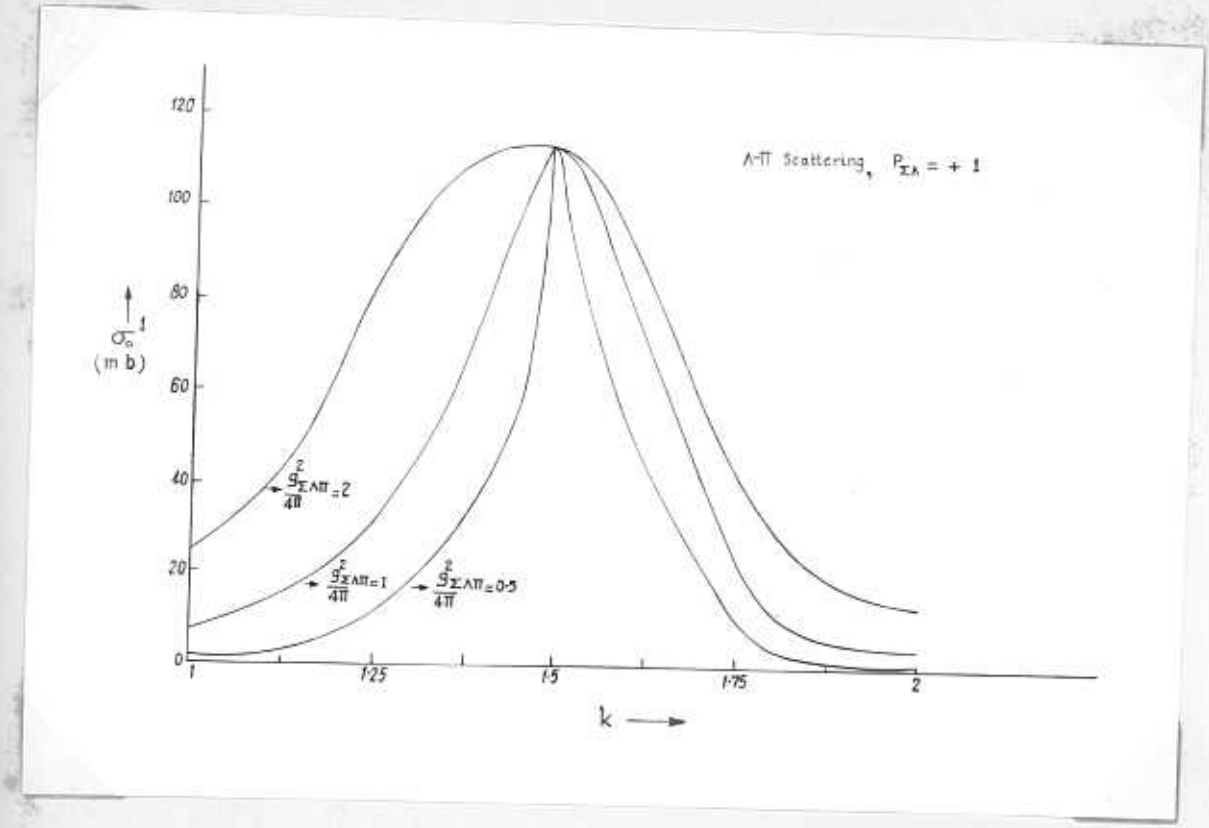
with

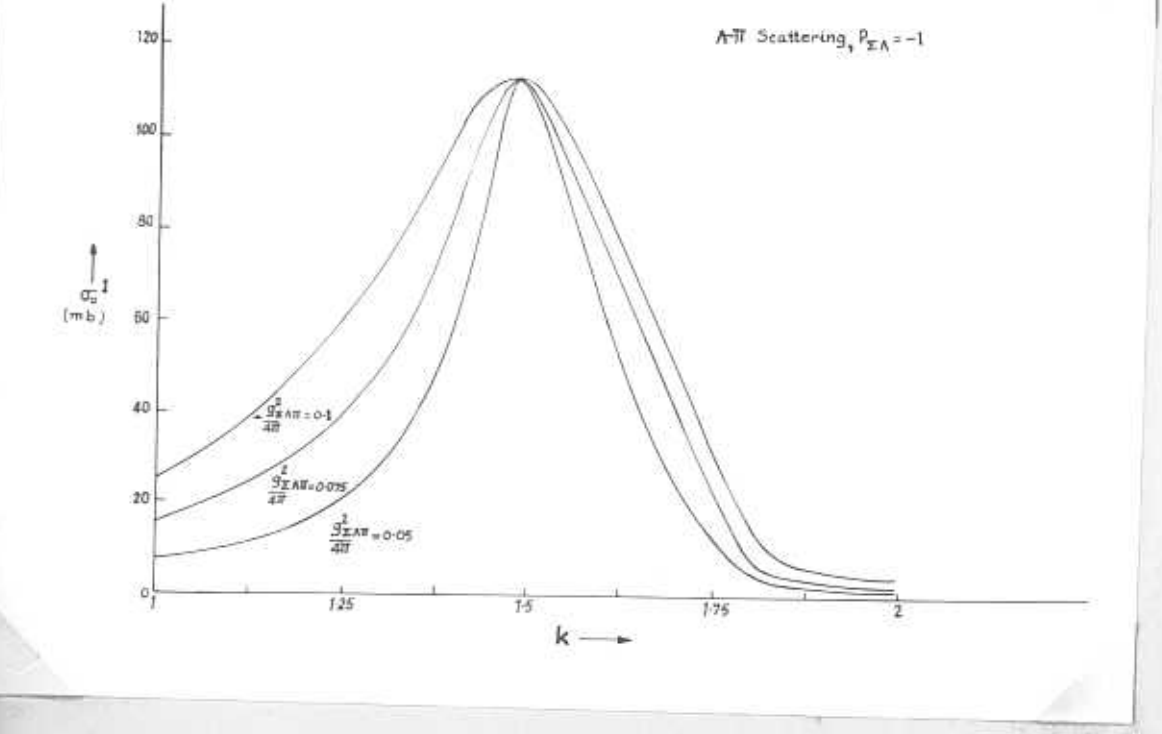
$$\omega_\Sigma = 2.076$$

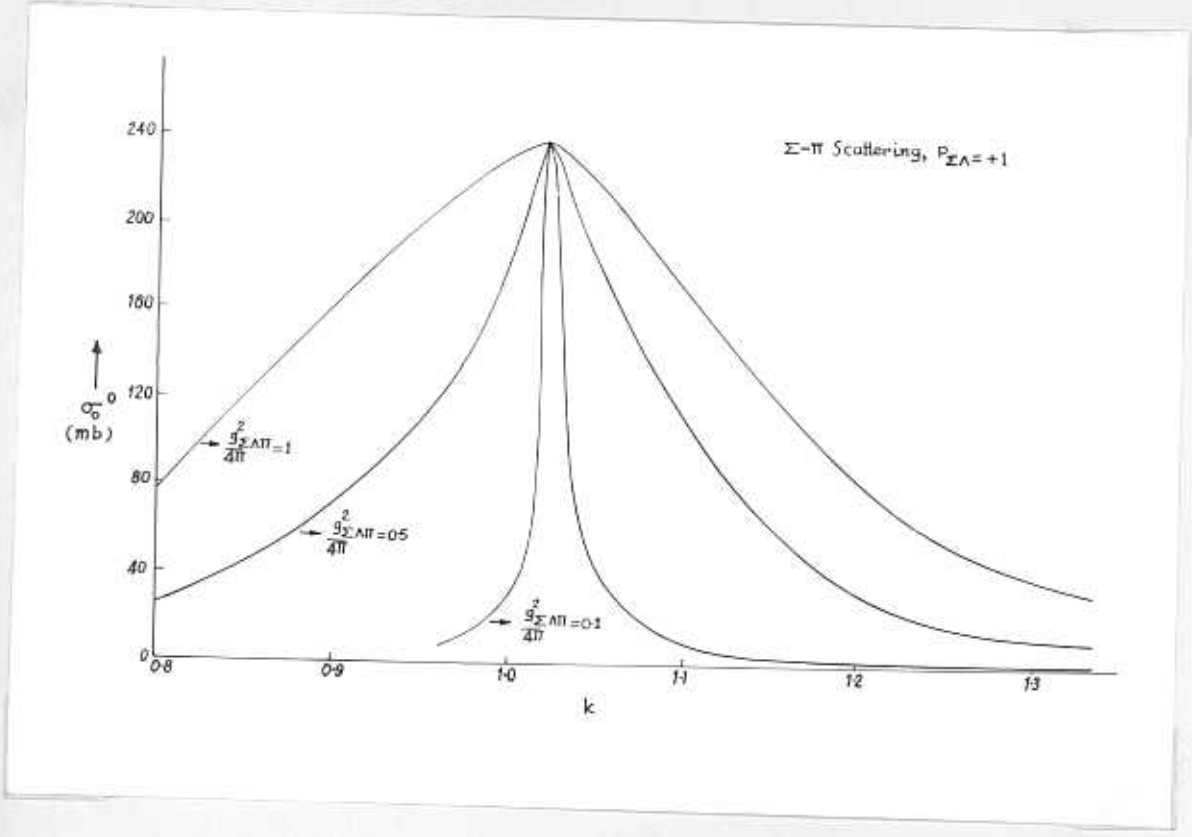
where the variables now are those of $\Sigma-\pi$ scattering.

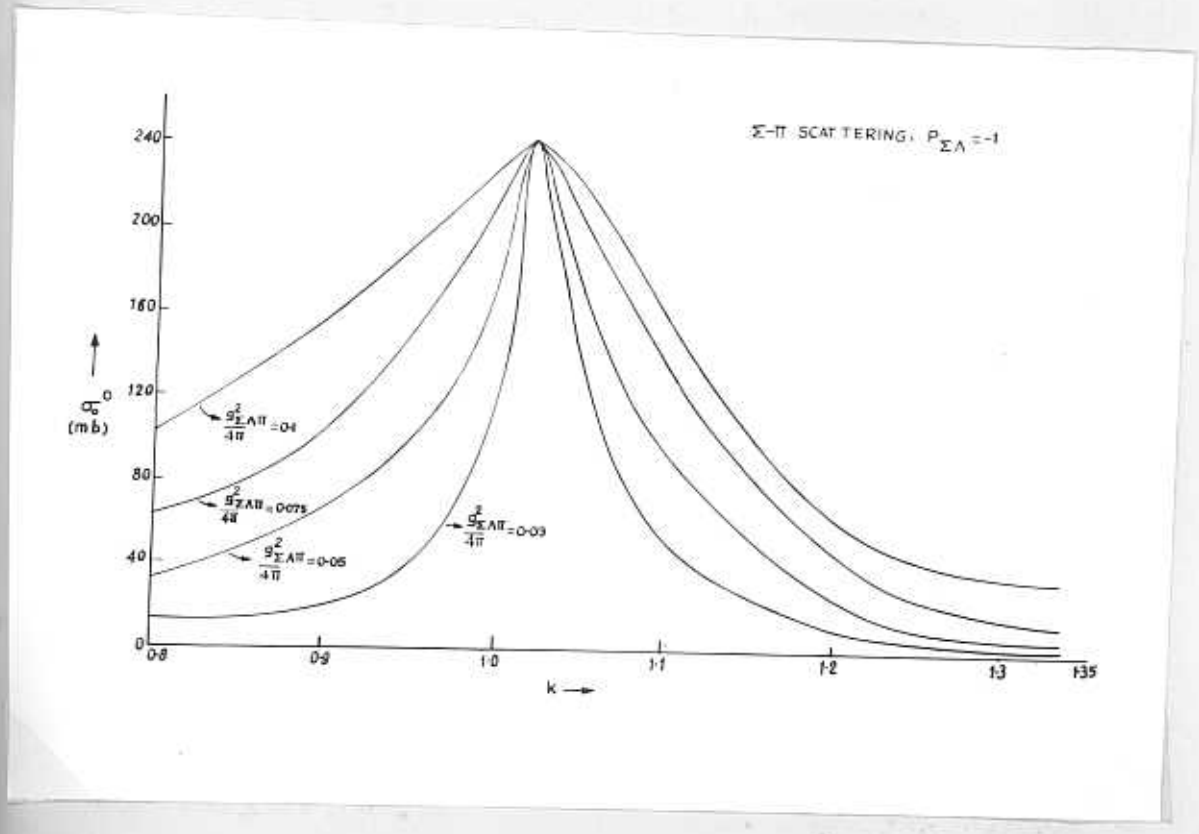
We give below curves showing the variation of the S -wave cross-section with the centre-of-mass momentum in natural units for various values of $g^2_{\Sigma\Lambda\pi}/4\pi$ for both $\Lambda-\pi$ and $\Sigma-\pi$ scattering. We also give the plots of $k/m_\Sigma \mp W \cot\delta^0_{0+}$ and $k/m_\Sigma \mp W \cot\delta^1_{0+}$ with $W-m_\Lambda$ and $W-m_\Sigma$

1) Watson and Lepore, Phys. Rev., 76, 1157 (1949).


Λ-π Scattering, $P_{\Sigma\Lambda} = +1$
σ_0^1 (mb)
120
100
80
60
40
20
0
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 2$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.5$
1
1·25
1·5
1·75
2
k


Λ-π Scattering, $P_{\Sigma\Lambda} = -1$
σ_0^1 (mb)
120
100
80
60
40
20
0
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.075$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} = 0.05$
1
1·25
1·5
1·75
2
k


Σ-π Scattering, $P_{\Sigma\Lambda}=+1$
240
200
160
120
80
40
0
σ_0 (mb)
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=1$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.5$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.1$
0·8
0·9
1·0
1·1
1·2
1·3
k


Σ-π SCATTERING, $P_{\Sigma\Lambda}=-1$
240
200
160
120
80
40
0
σ_0 (mb)
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.4$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.075$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.05$
$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}=0.03$
0·8
0·9
1·0
1·1
1·2
1·3
1·35
k

respectively as given by the equations (15), (16), (17) and (18).

3b. Conclusion

From the experimental results on the widths of the Y_1^* and Y_0^* resonances, it is possible to deduce the value of $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$. Thus taking the half-width of the Y_1^* resonance to be 25 Mev[1], we find $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} \approx 0.075$ for $P_{\Sigma\Lambda} = -1$ and $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} \approx 1$ for the $P_{\Sigma\Lambda} = +1$. The former gives a Y_0^* resonance with a half-width of about 5 Mev and the latter with a half-width of about 10.3 Mev. Clear-cut data on the half-width of Y_0^* can thus prove or disprove the theory, especially since the theory predicts that the Y_0^* resonance should be much narrower than the Y_1^* resonance. Present evidence suggests a half-width of about 10 Mev which fits in with the theory with even $\Sigma\Lambda$ parity.

3 a. Effective range formulae for higher orbital angular momentum states.

In the previous section, we have investigated the possibility that the recently observed Λ-π resonance is an S-wave resonance using the static model and the effective range approximation. Since there seems to be some experimental evidence for Y^* having $J \geq 3/2$[2], we shall here develop analogous effective range formulae for Λ-π[3]

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1) Dalitz and Miller, Phys. Rev. Letters, 6, 562 (1961)
2) Introductory talk by M.G.K.Menon at the Cosmic Ray Symposium, Madras (~~1961~~ 1961).
3) A. Ramakrishnan, T.K. Radha, R. Tiruniga and A.P. Balachandra Nu. Cim. (1962) (in press)

scattering for the $J = 3/2$ and $J = 5/2$ channels in the orbital angular momentum states $\ell = 2$ and 3 . The possibility of $J = 3/2$, $\ell = 1$ has also been the subject of extensive investigations by other workers in the context of the global symmetry model[1]. For orbital angular momentum states other than $\ell = 0$ we know that the direct pole does not contribute and hence we have to take the residue of the crossed pole only. In the static limit for $P_{\Sigma\Lambda} = +1$

$$h^{U}_{1+} = \frac{1}{16\pi k^2}\left[A_1 + (W - m_\Lambda) B_1 + \frac{(E - m_\Lambda)^2}{k^2}\left(-A_2 + (W - m_\Lambda) B_2\right)\right]$$

$$= \frac{g^2_{\Sigma\Lambda\pi}}{16\pi}\,\frac{2}{(8.5 + 16\omega)^3}\left[3.2 + 11.8\omega + 10.7\omega^2\right] \qquad 19.$$

where we have used the values

$$A_1 = -g^2_{\Sigma\Lambda\pi}\left(m_\Lambda - m_\Sigma\right)\left(\tfrac{2}{3}\, a/b^2\right) \qquad 20$$

$$B_1 = g^2_{\Sigma\Lambda\pi}\left(\frac{2}{3}\, a/b^2\right) \qquad 21$$

$$A_2 = g^2_{\Sigma\Lambda\pi}\left(m_\Lambda - m_\Sigma\right)\left(\frac{4}{15}\, a^2/b^3\right) \qquad 22$$

$$B_2 = -g^2_{\Sigma\Lambda\pi}\left(\frac{4}{15}\, a^2/b^3\right) \qquad 23$$

and

$$a = m_\Sigma^2 - m_\Lambda^2 - m_\pi^2 + 2E_\Lambda\omega \qquad 24$$

$$= 8.5 + 16\omega \quad (\text{in the static limit with } m_\pi = 1),$$

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1) Alston et. al., Phys. Rev. Letters, 6, 520 (1961)
Berge et. al., Phys. Rev. Letters, 6, 557 (1961)
Block et. al., Phys. Rev. Letters, Nuovo Cimento, 20, 724 (1961).
Dahl et. al., Phys. Rev. Letters, 6 , 142 (1961)
Alston and Peno-Luzzi, Rev. Mod. Phys., 33, 416 (1961)

$$\ell = 2k^2 = 2(\omega^2-1) \quad \text{and} \quad \omega = W - m_\Lambda \qquad 25.$$

It follows that h_{1+} satisfies the approximate dispersion relation

$$h_{1+} = h^0_{1+} + \frac{1}{\pi}\int_{m_\Lambda+1}^{\infty} dW' \, \frac{\mathrm{Im}\, h_{1+}(W')}{W'-W} \qquad 26$$

h^0_{1+} being given by (19). This may be solved by the N/D method and the solution in the one meson approximation reads

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{3\cdot2+11\cdot8\omega+10\cdot7\omega^2}{(8\cdot5+16\omega)^3}\right]\frac{8}{(\omega+8)}\,k^3\cot\delta_{1+}$$
$$= 1 - \frac{16\cdot2+8\cdot5}{\pi}\,P\int_1^\infty d\omega' \, \frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\,\frac{1}{(8\cdot5+16\omega')^4}\left[3\cdot2+11\cdot8\omega'+10\cdot7\omega'^2\right]\frac{8\,k'^3 v^2(k'^2)}{(\omega'+8)(\omega'-\omega)} \qquad 27.$$

where we have chosen the subtraction point such that renormalised $\Sigma\Lambda\pi$ coupling constant gets defined by an analogue of the Lepore-Watson convention for the coupling constant and $v^2(k'^2)$ is the usual cut-off function. As in the Chew-Low theory we may now approximately write

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{3\cdot2+11\cdot8\omega+10\cdot7\omega^2}{(8\cdot5+16\omega)^3}\right]\frac{8}{\omega+8}\,k^3\cot\delta_{1+} = 1 - \frac{16\omega+8\cdot5}{\omega_\gamma} \qquad 28.$$

where the constant ω_γ is positive. The right hand side of equation (28) now shows the possibility of a resonance inthe system corresponding to the vanishing of $\cot\delta_{1+}$ for an appropriate value of ω_γ . A similar calculation yields for even $\Sigma\Lambda$ parity for $J = 3/2$ and $\ell = 2$,

$$\frac{g^2_{\Sigma\Lambda\pi}}{16\pi}\left[\frac{\cdot072+\cdot26\omega+\cdot054\omega^2}{(8\cdot5+16\omega)^3}\right]\frac{16}{\omega+8}(\omega^2-1)^{5/2}\cot\delta_{3-} = 1 - \frac{16\omega+8\cdot5}{\omega_\gamma} \qquad 29.$$

which can also give rise to a resonance for a suitable ω_r.

The results for $J = 5/2$ and even $\Sigma\Lambda$ parity are

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{-1.28-4.7\omega-4.2\omega^2}{(8.5+16\omega)^4}\right]\frac{16}{(\omega+8)}(\omega^2-1)^{5/2}\cot\delta_{2+}=1+\frac{16\omega+8.5}{\omega_r} \quad 30$$

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{-.003-.03\omega-.01\omega^2}{(8.5+16\omega)^4}\right]\frac{4}{(\omega+8)}(\omega^2-1)^{7/2}\cot\delta_{3-}=1+\frac{16\omega+8.5}{\omega_r} \quad 31$$

where we have used

$$A_3 = -g^2_{\Sigma\Lambda\pi}(m_\Lambda - m_\Sigma)\left(\frac{4}{35}b^3/a^4\right) \quad 32.$$

$$B_3 = g^2_{\Sigma\Lambda\pi}\left(\frac{4}{35}b^3/a^4\right) \quad 33.$$

For odd $\Sigma\Lambda$ parity, the corresponding formulae are

$$-\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{95+16.3\omega-11\omega^2}{(8.5+16\omega)^3}\right]\frac{8}{\omega+8}(\omega^2-1)^{3/2}\cot\delta_{1+}=1+\frac{16\omega+8.5}{\omega_r} \quad 34.$$

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{4+.4\omega+.02\omega^2}{(8.5+16\omega)^3}\right]\frac{16}{\omega+8}(\omega^2-1)^{5/2}\cot\delta_{2-}=1-\frac{16\omega+8.5}{\omega_r} \quad 35.$$

$$\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{38+72\omega-4.25\omega^2}{(8.5+16\omega)^4}\right]\frac{16}{\omega+8}(\omega^2-1)^{5/2}\cot\delta_{2+}=1-\frac{16\omega+8.5}{\omega_r} \quad 36.$$

$$-\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}\left[\frac{2+.16\omega+.01\omega^2}{(8.5+16\omega)^4}\right]\frac{16}{\omega+8}(\omega^2-1)^{7/2}\cot\delta_{3-}=1+\frac{16\omega+8.5}{\omega_r} \quad 37.$$

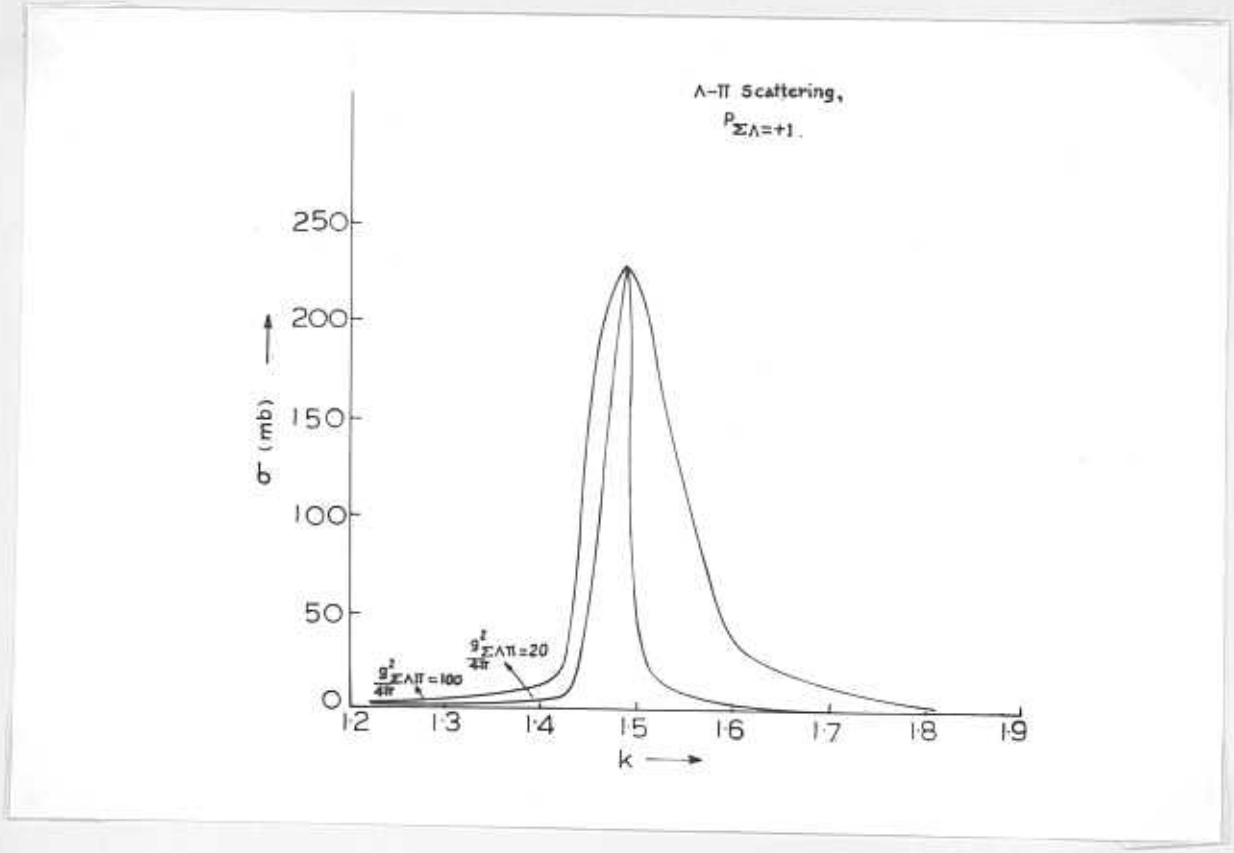
In all these ω_γ denotes an appropriate positive constant. We have also written down the P wave, $J = 3/2$ effective range formula for odd $\Sigma\Lambda$ parity in equation (34).

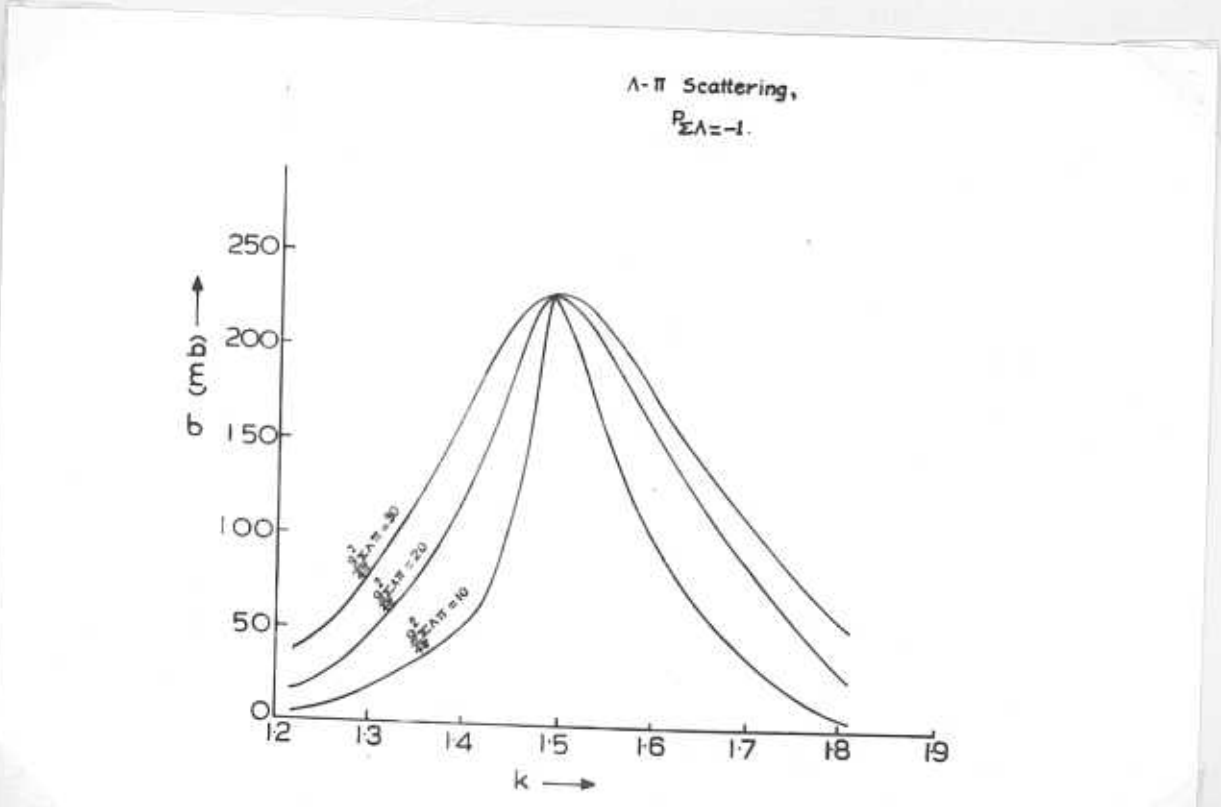
3 b. Conclusions

Equations (30) and (31) now indicate that within the scope of our approximations a $J = 5/2$ resonance is forbidden for even $\Sigma\Lambda$ parity. Similarly we find that if the $\Sigma\Lambda$ parity is odd there can be no P -wave resonance with $J = 3/2$ or F wave resonance with $J = 5/2$. However, a P or a D wave resonance with $J = 3/2$ is possible for even $\Sigma\Lambda$ parity while D wave resonance with $J = 3/2$ or $5/2$ is possible for odd $\Sigma\Lambda$ parity.

We give below curves showing the variation of the scattering cross-section in the $J = 3/2$, $\ell = 2$ channel with the centre-of-mass energy for various values for the $\Sigma\Lambda\pi$ coupling constant and for both even and odd $\Sigma\Lambda$ parities.

The curves show that we can preproduce the observed half-width of 25 Mev for the $\Lambda\pi$ resonance with $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi} \approx 0$ for odd $\Sigma\Lambda$ parity. However for even $\Sigma\Lambda$ parity $\frac{g^2_{\Sigma\Lambda\pi}}{4\pi}$ has to be of the order of 100 if we are to obtain the 25 Mev half-width for the Y^*.


Λ-π Scattering,
$P_{\Sigma\Lambda}=+1$.
250
200
150
100
50
0
σ (mb)
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
k
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=100$
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=20$


Λ-π Scattering,
$P_{\Sigma\Lambda}=-1$.
250
200
150
100
50
0
σ (mb)
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
k
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=30$
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=20$
$\frac{g^2}{4\pi}_{\Sigma\Lambda\pi}=10$

The very large value of the coupling constant required to obtain a D wave $J = 3/2$, Y^* resonance with the observed width for even $\Sigma\Lambda$ parity indicates that such a resonance is precluded within the scope of our model. It is worthwhile noticing that the one-meson approximation which we have made implies that we have completely neglected the presence of the $\bar{K}N$ channel and its influence on the scattering. If it should subsequently turn out that the resonance has the parameters $J = 3/2$ and $\ell = 2$ and the parity is even it may well be an indication that this resonance can be understood only if we approximately take into account the $\bar{K}N$ intermediate states.

CHAPTER II.

PHOTO PRODUCTION OF PIONS ON Λ HYPERONS.

1. Introductory Remarks

In this chapter we extend our previous results on $\Lambda\pi$ scattering to include photoproduction of pions from Λ hyperons. Since the process $\gamma + \Lambda \rightarrow \Lambda + \pi$ is related by the unitarity condition to the $\Lambda\pi$ elastic scattering amplitude, a solution is possible in terms of the phase shifts of the corresponding scattering amplitudes. The main departure from the problem of photo-production on nucleons arise due to the following reasons:

(i) Since Λ is unchanged, amplitudes generated by the change e are zero and the only non-vanishing amplitudes are due to the magnetic moment.

(ii) Since the single particle real intermediate state is that of a Σ particle, it is the $\Sigma\Lambda$ transition magnetic moment that is involved.

(iii) Since the mass difference $m_\Sigma - m_\Lambda$ does not vanish unlike in the nucleon case when the pole occurs at the nucleon mass itself, we have contributions from terms proportional to $m_\Sigma - m_\Lambda$.

(iv) The relative $\Sigma\Lambda$ parity not being experimentally fixed, we give amplitudes for both even and odd parities.

It is to be noted that though this process cannot be directly observed experimentally it would be of importance in the study of reactions like $\gamma + N \rightarrow K + \Lambda + \pi$ and also in the study of hyperon form factors.

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1) Alladi Ramakrishnan, T.K.Radha, R.Thunga and A.P.Balachandran, Letter to Nuovo Cimento (in press).

2. The calculation

Let P_1 and k denote the four-vector momenta of the incoming Λ and γ and P_2 and Q those of the outgoing Λ and π and ϵ the polarization vector of the photon. Then with Chew, Goldberger, Low and Nambu[1] we write the transition matrix for the process in the form

$$T = M_A A + M_B B + M_C C + M_D D \qquad 1.$$

where

$$M_A = i \gamma_5 \gamma \cdot \epsilon \gamma \cdot K \qquad 2.$$

$$M_B = 2i \gamma_5 [P \cdot \epsilon Q \cdot K - P \cdot K Q \cdot \epsilon] \qquad 3.$$

$$M_C = \gamma_5 [\gamma \cdot \epsilon Q \cdot K - \gamma \cdot K Q \cdot \epsilon] \qquad 4.$$

$$M_D = 2 \gamma_5 [\gamma \cdot \epsilon P \cdot K - \gamma \cdot K P \cdot \epsilon - i m_\Lambda \gamma \cdot \epsilon \gamma \cdot K] \qquad 5.$$

Here $P = \frac{1}{2}[P_1 + P_2]$ and m_Λ ~~and~~ denotes the mass of the Λ. As in ref.(1), we may also introduce am matrix related to the differential cross-section in the centre-of-mass system through

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} |\langle 2 | \mathcal{F} | 1 \rangle|^2 \qquad 6.$$

where q and k denote the magnitude of the centre-of-mass three-vector momenta of the incident γ and outgoing π and the matrix element of $\mathcal{F}$ is to be taken between initial and final Pauli spinors. $\mathcal{F}$ may be written as

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1) Chew, Goldberger, Low and Nambu, Phys.Rev., 106, 1345 (1957).

$$\mathcal{F} = i\vec{\sigma}\cdot\vec{\epsilon}\,\mathcal{F}_1 + \frac{\vec{\sigma}\cdot\vec{q}\,\vec{\sigma}\cdot(\vec{k}\times\vec{\epsilon})}{qk}\mathcal{F}_2 + \frac{i\vec{\sigma}\cdot\vec{k}\,\vec{q}\cdot\vec{\epsilon}}{qk}\mathcal{F}_3$$

$$+ i\frac{\vec{\sigma}\cdot\vec{q}\,\vec{q}\cdot\vec{\epsilon}}{q^2}\mathcal{F}_4 \qquad 7.$$

where the σ 's are Pauli matrices and $\vec{q}$ and $\vec{k}$ are the meson and photon three-momenta. The $\mathcal{F}_i$'s are related to transitions involving multipole radiations through

$$\mathcal{F}_1 = \sum_{\ell=0}^{\infty}\left\{\left[\ell M_{\ell+} + E_{\ell+}\right]P'_{\ell+1}(\cos\theta) + \left[(\ell+1)M_{\ell-} + E_{\ell-}\right]P'_{\ell-1}(\cos\theta)\right. \qquad 8.$$

$$\mathcal{F}_2 = \sum_{\ell=1}^{\infty}\left\{(\ell+1)M_{\ell+} + \ell M_{\ell-}\right\}P'_{\ell}(\cos\theta) \qquad 9.$$

$$\mathcal{F}_3 = \sum_{\ell=1}^{\infty}\left\{\left[E_{\ell+} - M_{\ell+}\right]P''_{\ell+1}(\cos\theta) + \left[E_{\ell-} + M_{\ell-}\right]P''_{\ell-1}(\cos\theta)\right. \qquad 10.$$

$$\mathcal{F}_4 = \sum_{\ell=2}^{\infty}\left[M_{\ell+} - E_{\ell+} - M_{\ell-} - E_{\ell-}\right]P''_{\ell}(\cos\theta) \qquad 11.$$

Here θ is the scattering angle in the centre-of-mass system and $M_{\ell\pm}$ and $E_{\ell\pm}$ refer to transitions initiated by magnetic and electric radiation respectively leading to final states of orbital angular momentum ℓ and total angular momentum $J = \ell \pm \frac{1}{2}$.

With these preliminaries which are to be found in ref. and which we have reproduced for completeness, let us proceed to the calculation of the matrix elements for our process. We shall illustrate the procedure first for odd $\Sigma\Lambda$ parity. In a static approximation for odd $\Sigma\Lambda$ relative parity, we find we can conveniently ignore the contributions arising from the energy variables of the crossed channels since in this case,

these only modify the scattering amplitude by corrections of the order of $1/m_\Lambda$ compared to the terms which are retained. Thus if μ denotes the $\Sigma\Lambda$ transition magnetic moment*, the pole terms on A, B, C and D for odd $\Sigma\Lambda$ parity read

$$A^B = -\mu g \frac{m_\Lambda + m_\Sigma}{m_\Sigma^2 - s} \qquad 12.$$

$$B^B = 0 \qquad 13.$$

$$C^B = -\mu g / m_\Sigma^2 - s \qquad 14.$$

$$D^B = -\mu g / m_\Sigma^2 - s = C^B \qquad 15.$$

Here $s = -(P_1 + k)^2$, m_Σ denotes the mass of the Σ hyperon and g is the renormalized $\Sigma\Lambda\pi$ coupling constant. A straightforward calculation then reveals the contributions of (12-15) to the individual multipole amplitudes to be

$$E_{0+}^B = \frac{\mu g}{2\pi} \frac{W - m_\Lambda}{W - m_\Sigma} \frac{m_\Lambda}{W} \qquad 16.$$

$$M_{1+}^B = \frac{1}{32\pi} \frac{q}{W} \mu g \frac{W - m_\Lambda}{W + m_\Sigma} \qquad 17.$$

$$M_{1-}^B = \frac{1}{16\pi} \frac{q}{W} \mu g \frac{W - m_\Lambda}{W + m_\Sigma} \qquad 18.$$

$$E_{1+}^B = 0 \qquad 19.$$

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* It is interesting to note that in global symmetry even though f_Λ and μ_{Σ^0} are approximately zero, the $\Sigma\Lambda$ transition magnetic moment is equal to the minus neutron magnetic moment ~ 1.9. It is to be noted that according to Sakurai's symmetry model, the transition moment would vanish within the limit that $(m_\Sigma - m_N)$ mass difference is zero.

whereas in the rest of the calculation, we retain only those multipole amplitudes leading to final states with $\ell = 0$ or 1. In equations (16-19), we have denoted the total centre-of-mass energy by W. We notice that in a static approximation, the leading Born term is E_{0+}^{B}, the others being of the order of $1/m_\Lambda$ compared to it. Therefore, keeping only E_{0+}, we may write

$$E_{0+} = E_{0+}^{B} + \frac{\omega}{\pi}\int_{1}^{\infty} d\omega' \; \frac{Im\, E_{0+}(\omega')}{\omega'(\omega' - \omega)} \qquad 20.$$

where $\omega = W - m_\Lambda$ and we have set the pion mass equal to unity. $Im\, E_{0+}$ can as usual be calculated using the unitarity condition in which only the Λ-π intermediate state is retained. We find, in a static approximation

$$Im\, E_{0+}(\omega) = e^{-i\delta_{0+}} \sin\delta_{0+}\; E_{0+}(\omega) \qquad 21$$

where we have explicitly assumed that the $\Lambda\pi$ interaction proceeds dominantly through S-wave and denoted the S-wave scattering phase shift by δ_{0+}. Equation (20) now reduces to the familiar mapping problem discussed by Omnes [1] whose solution reads

$$E_{0+}(\omega) = \left[\frac{f^2 g}{2\pi} \frac{\omega}{\omega - \delta m} \cos\delta(\omega) + \frac{\omega}{\pi} \exp[P(\omega)] \times \right.$$
$$\left. \times P\int_{1}^{\infty} \frac{f^2 g}{2\pi} \frac{\sin\delta(\omega') \exp[P(\omega')]}{(\omega' - \delta m)(\omega' - \omega)} d\omega' \right] e^{i\delta(\omega)} \qquad 22$$

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1) R. Omnes, Nuovo Cimento, 8, 316 (1958).

where

$$P(\omega) = \omega/\pi \, \mathcal{P} \int_1^\infty \frac{\delta(\omega')}{\omega'(\omega' - \omega)} \, d\omega' \qquad 23$$

Here $\delta m = m_\Sigma - m_\Lambda$ and E_{0+}^B has also been replaced by its static form.

For even $\Sigma\Lambda$ relative parity, it is found that the contribution from the crossed channel are of the same order of magnitude as that from the direct channel. Bearing in mind that under crossing $A \to A$; $C \to -C$ and $D \to D$, the Born terms for the multipole in the static approximation turn out to be

$$E_{0+}^B = \frac{\mu g}{2\pi} \omega \qquad 24$$

$$E_{1+}^B = 0 \qquad 25$$

$$M_{1+}^B = -\frac{\mu g}{6\pi} \frac{kq}{\omega + \delta m} \qquad 26$$

$$M_{1-}^B = -\frac{\mu g}{2\pi} \frac{1}{\omega + \delta m} \qquad 27$$

Since we are retaining only the S-wave $\Lambda\pi$ scattering amplitude in the unitarity condition, it is easy to see that the different multipole amplitudes do not get coupled by it so that for each of them we have a relation of the form (21). Finally the solutions of the dispersion integrals for these amplitudes read

$$E_{0+}(\omega) = \left[\frac{\mu g}{2\pi} \omega \cos\delta(\omega) + \frac{\omega}{\pi} \exp[P(\omega)] \times \right.$$
$$\left. \times \mathcal{P} \int_1^\infty \frac{\mu g}{2\pi} \frac{\sin\delta(\omega') \exp[P(\omega')]}{(\omega' - \omega)} \, d\omega' \right] e^{i\delta(\omega)} \qquad 28$$

$$E_{1+}(\omega) = 0 \qquad 29.$$

$$M_{1+}(\omega) = -\left[\frac{\mu g}{6\pi} \frac{kq}{\omega + \delta m} \cos\delta(\omega) + \frac{\omega}{\pi} \exp[P(\omega)] \times \right.$$

$$\left. \times \mathcal{P} \int_1^\infty \frac{\mu g}{6\pi} \frac{k'q' \sin\delta(\omega') \exp[P(\omega')]}{\omega'(\omega' + \delta m)(\omega' - \omega)} d\omega' \right] e^{i\delta(\omega)} \qquad 30$$

$$M_{1-}(\omega) = -\left[\frac{\mu g}{2\pi} \frac{1}{\omega + \delta m} \cos\delta(\omega) + \frac{\omega}{\pi} \exp[P(\omega)] \times \right.$$

$$\left. \times \mathcal{P} \int_1^\infty \frac{\mu g}{2\pi} \frac{\sin\delta(\omega') \exp[P(\omega')]}{\omega'(\omega' + \delta m)(\omega' - \omega)} d\omega' \right] e^{i\delta(\omega)} \qquad 31$$

where $f(\omega)$ is again given by (23.).

3. Conclusion.

We have calculated the multipole amplitudes for the process $\gamma + \Lambda \to \Lambda + \overline{\pi}$ for low values of ℓ in terms of the $\Lambda\overline{\pi}$ scattering phase shifts on the assumption that the $\Lambda\overline{\pi}$ scattering is dominated by S -wave interaction. Since we have already presented effective range formulae for S -wave $\Lambda\overline{\pi}$ scattering and we can easily calculate the actual values of the appropriate cross-sections from these formulae. The influence of the process $\gamma + \Lambda \to \Lambda + \overline{\pi}$ on processes like $\gamma + N \to \Lambda + K + \overline{\pi}$ on the basis of our effective range theory can be studied again by the extrapolation method of Chew and Low which will be discussed in ~~the next~~ chapter IV

CHAPTER III.

DISPERSION ANALYSIS OF Ξ PRODUCTION IN $\bar{K}N$ COLLISIONS.

1. Introductory remarks

With the recent laboratory production of Ξ particles in $\bar{K}N$ collisions[1] it seems worthwhile to make a dispersion theoretic analysis of the process

$$\bar{K} + N \rightarrow \Xi + K$$

In spite of the complications involved in solving the dispersion relations for production processes one major simplification arises due to the existence of crossing symmetry. We here show that simple solutions may be obtained under certain reasonable approximations.[2]

In section (2), the kinematics of the problem is discussed. In section (3) single variable partial wave dispersion relations are set up for the $I=0$ and $I=1$ channels separately and the solutions are given assuming the possible resonances to dominate. The effect of anomalous thresholds are also taken into account. In section (4) graphs are given for the differential cross-sections in the case of relative parity $P(\Xi N)=+1$ and $P(\Sigma N) = -1$.

2. Calculations

Let q_1 and p_1 denote the four momenta of the incident $\bar{K}$ and N and q_2 and p_2 those of the outgoing K and Ξ respectively. The invariant square of the energy and momentum transfer variables of the process

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1. L.W. Alvarez et. al. UCRL (9943).

2) A. Ramakrishnan, R. Thunga, T.K Radha, Gr. Bhamathi, and S Indumathi Nucl. Phys. (in press) (1962)

$$\bar{K}_1 + N \rightarrow \Xi + K_2 \qquad 1.$$

are

$$s = (p_1 + q_1)^2 = (E_{\bar{K}} + E_N)^2 = W^2 \qquad 2a$$

$$\bar{s} = (p_1 - q_2)^2 = m_N^2 + m_K^2 - 2E_N E_K - 2kq\cos\theta \qquad 2b$$

$$t = (p_1 - p_2)^2 = 2(m_K^2 - E_{\bar{K}} E_K) + 2kq\cos\theta \qquad 2c.$$

with $s + \bar{s} + t = 2m_K^2 + m_N^2 + m_\Xi^2$ 3.

where $\vec{k}$, $\vec{q}$ are the initial and final centre of mass momenta and θ the scattering angle. The centre of mass energies and momenta of the particles are given in terms of s by

$$k^2 = [s - (m_N + m_K)^2][s - (m_N - m_K)^2]/4s$$

$$q^2 = [s - (m_\Xi + m_K)^2][s - (m_\Xi - m_K)^2]/4s$$

$$E_N = \frac{s + m_N^2 - m_K^2}{2\sqrt{s}}, \quad E_{\bar{K}} = \frac{s - (m_N^2 - m_K^2)}{2\sqrt{s}}$$

$$E_\Xi = \frac{s + m_\Xi^2 - m_K^2}{2\sqrt{s}}, \quad E_K = \frac{s - (m_\Xi^2 - m_K^2)}{2\sqrt{s}} \qquad 4$$

The T matrix is given by

$$T = \left[-A + i\gamma \cdot \frac{q_1+q_2}{2} B\right] \Gamma \qquad 5.$$

where $\Gamma = 1$ or γ_5 depending upon the combination of the intrinsic parity of the Ξ and Y which we will discuss presently. A satisfies the following one dimensional integration

$$A = \frac{R}{m_Y^2 - s} + \frac{R'}{m_Y^2 - \bar{s}} + \frac{1}{\pi}\int_{(m_\Lambda+1)^2}^{\infty} \frac{ds'\, a_1(s',t)}{(s'-s)} + \int_{(m_\Lambda+1)^2}^{\infty} \frac{d\bar{s}'\, a_2(\bar{s}',t)}{(\bar{s}'-\bar{s})} \qquad 6.$$

where a_1 and a_2 are the absorptive parts (i.e. $\mathrm{Im}\, A$) when s, $\bar{s}$ and t are in the physical regions of the processes

$$\bar{K}_1 + N \longrightarrow \Xi + K_2 \qquad 7.$$

and

$$\bar{K}_2 + N \longrightarrow \Xi + K_1$$

respectively. R and R' are the residues at the poles in the direct and crossed processes respectively. B also has a similar one dimensional dispersion representation. There are no poles in the t variable which represents the $(\text{energy})^2$ variable for the process

$$\bar{\Xi} + N \longrightarrow K + K$$

since there is neither a single particle state nor a resonance with strangeness $S = +2$ and baryon number $N = 0$.

It is interesting to note that we have crossing symmetry which is unusual in production processes so that

$$A(s, \bar{s}, t) = A(\bar{s}, s, t)$$
$$B(s, \bar{s}, t) = -B(\bar{s}, s, t) \qquad 8$$

and the corresponding relations for a_1 and a_2 can be deduced. Since neither the ΞN nor the γN relative parity has been fixed the Born terms of A and B for different combinations of parity are given below.

Case (i) : $P(\Xi N) = +1$; $P(\gamma N) = +1$; $\Gamma = 1$

$$B^B = g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{1}{m_\gamma^2 - s} - \frac{1}{m_\gamma^2 - \bar{s}}\right]$$

$$A^B = g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{m_\Xi + m_N}{2} - m_\gamma\right]\left[\frac{1}{m_\gamma^2 - s} + \frac{1}{m_\gamma^2 - \bar{s}}\right] \qquad 9.$$

where we have already included crossing symmetry by writing $R = R'$

Case (II) : $P(\Xi N) = -1$; $P(\gamma N) = +1$; $\Gamma = \gamma_5$

$$A^B = -g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{m_N - m_\Xi}{2} - m_\gamma\right]\left[\frac{1}{m_\gamma^2 - s} + \frac{1}{m_\gamma^2 - \bar{s}}\right]$$

$$B^B = -g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{1}{m_\gamma^2 - s} - \frac{1}{m_\gamma^2 - \bar{s}}\right] \qquad 10.$$

Case (iii) $P(\Xi N) = -1$; $P(\gamma N) = -1$; $\Gamma = 1$

$$A^B = -g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{m_\Xi + m_N}{2} + m_\gamma\right]\left[\frac{1}{m_\gamma^2 - s} + \frac{1}{m_\gamma^2 - \bar{s}}\right]$$

$$B^B = g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{1}{m_\gamma^2 - s} - \frac{1}{m_\gamma^2 - \bar{s}}\right] \qquad 11.$$

Case (iv):

$$P(\Xi N) = +1 \; ; \; P(\gamma N) = -1 \; ; \; \Gamma = \gamma_5$$

and ~~are the same as in case (ii)~~.

$$A^B = -g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{m_\Xi - m_N}{2} - m_N\right]\left[\frac{1}{m_\gamma^2 - s} + \frac{1}{m_\gamma^2 - \bar{s}}\right]$$

$$B^B = g_{\Xi\gamma K}\, g_{\gamma N K}\left[\frac{1}{m_\gamma^2 - s} - \frac{1}{m_\gamma^2 - \bar{s}}\right] \qquad 12.$$

3. Partial wave analysis

Following Frazer and Fulco[2], we define the amplitude

$$h_{l\pm}(W) = \frac{W}{\sqrt{(E_N+m_N)(E_\Xi+m_\Xi)}}\,\frac{f_{l\pm}}{(kq)^l}$$

$$= \frac{1}{16\pi}\left\{\frac{A_l}{(kq)^l} + \left[W-(m_N+m_\Xi)\right]\frac{B_l}{(kq)^l}\right.$$

$$\left. + (E_\Xi - m_\Xi)(E_N - m_N)\left[\frac{-A_{l\pm1}}{(kq)^{l+1}} + \left(W+\frac{m_N+m_\Xi}{2}\right)\frac{B_{l\pm1}}{(kq)^{l+1}}\right]\right\}, \qquad (13)$$

and

$$f_{l\pm} = e^{i\delta_{l\pm}}\sin\delta_{l\pm}/(kq)^{1/2} \qquad (14)$$

where the phase shift δ is complex since we are considering a production process. It is therefore important to note that we cannot obtain the simple relation

$$\mathrm{Im}\,\frac{1}{f_{l\pm}} = -\sqrt{kq}$$

A_l and B_l in equation (13) are obtained by projecting the l^{th} partial wave amplitude from A and B by

$$[A_l; B_l] = \int_{-1}^{1} [A; B]\, P_l(\cos\theta)\, d(\cos\theta) \qquad (15)$$

We have so far considered only the pole contributions. The singularities in the form of branch cuts start at $s = (m_\gamma+1)^2, (m_\gamma+2)^2, (m_K+m_N)^2 \ldots (m_\Xi+m_N)^2$ However it is well known that the cross-sections for $\gamma\pi\pi$ productions are very small compared to the $\bar{K}N$ scattering cross-sections. Now we make the reasonable assumption that only nearby singularities dominate and hence take into account only the poles and the $\gamma\pi$ and $\bar{K}N$ cuts. Further simplification is achieved by writing the dispersion relations

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2. W.R. Frazer and J. Fulco, Phys. Rev., 117, 1609(1960).

separately for the $I = 0$ and $I = 1$ channels. Thus the contributions from the Λ and Σ poles get separated as they occur in different isotopic spin channels. In view of the existence of the well-established Y^* resonance we assume that the amplitude $\langle Y\pi | \bar{K}N \rangle$ in the $J = 3/2$, $l = 1$, $I = 1$, channel is dominated by the Y^* resonance and that it gives negligible contribution in all other channels. In such a case it is found that the integral equation reduces to one for which the Omnès solution can be immediately written down. We first give below the integral equation satisfied by the different partial wave amplitudes in the $I = 0$ and $I = 1$ channels taking into account the possibility of anomalous thresholds[3] in the case of $\Sigma\pi$ intermediate states. It is found that there is no anomalous threshold for any of the other two particle intermediate states. Thus we have for

$$l = 0 \quad J = 1/2 \quad \text{and} \quad I = 0$$

$$h^{\circ}_{0+}(W) = C - \frac{g_{\Xi\Lambda K}\, g_{KN\Lambda}}{16\pi}\left\{\frac{2}{W+m_\Lambda} + \frac{(W + m_\Lambda - m_\Xi - m_N)}{K + E_N E_{\bar{K}}}\right\}$$
$$+ \frac{1}{\pi}\int_{(m_\Sigma+1)-\lambda^2}^{\infty} \frac{dW'\, h^{\circ}_{0+}(W')\, e^{i\delta^{\circ}_{0+}(W')} \sin\delta^{\circ}_{0+}(W')}{W' - W} \qquad 16.$$

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* The spin and parity of the Y^* resonance are not yet fixed though it is known that the spin $J_{Y^*} \geq 3/2$. We here assume $J = 3/2$.

3) R. Karplus, C.M. Sommerfeld and E. Wichmann, Phys. Rev., 114, 376 (1959).

and $2k^2 = m_N^2 + m_K^2 - m_\Lambda^2$

where $h^I_{l+}(W)$ represents the production amplitude for the process (1) in a channel with isotopic spin I, orbital angular momentum l and total angular momentum $l + 1/2$. C is the unknown subtraction constant which will represent high energy contributions and has to be determined from experimental data. δ^I_{0+} is the $\bar{K}N$ scattering phase shift in the $I=0, J=\frac{1}{2}$ channel and we have represented $\mathrm{Im}\, h^I_{l+}(W)$ in the integral by

$$\langle \Xi K | \bar{K}N \rangle \langle \bar{K}N | \bar{K}N \rangle = h^I_{l+}(W)\, e^{i\delta^I_{l+}} \sin\delta^I_{l+} \qquad 17.$$

using the unitarity condition. λ is the quantity to be subtracted from the lower limit due to the effect of anomalous threshold and is found to be

$$\lambda = (m_\Sigma + 1) - \sqrt{(m_\Sigma + 1)^2 - \cdot 006 \times 2 m_\Sigma m_\pi} = 1 \text{ Mev} \qquad 18.$$

(2) $l = 0$, $J = \frac{1}{2}$ and $I = 1$

$$h'_{0+}(W) = C - g_{\Xi K} g_{KN\Sigma} \left[\frac{2}{N + m_\Sigma} + \frac{W + m_\Sigma - m_\Xi - m_N}{K' + E_N E_{\bar{K}}} - \right.$$

$$\left. + \frac{1}{\pi} \int_{(m_\Lambda + 1)}^{\infty} \frac{h'_{0+}(W')\, e^{i\delta'_{0+}(W')} \sin\delta'_{0+}(W')}{W' - W}\, dW' \right] \qquad 19$$

where $2K' = m_N^2 + m_K^2 - m_\Sigma^2$

(3) $\ell = 1$, $J = 1/2$ and $I = 0$

$$h^{\circ}_{1-}(W) = -\frac{g_{\Xi\Lambda K} g_{KN\Lambda}}{16\pi}\left[\frac{2}{W-m_\Lambda} + \frac{W-m_\Lambda+m_N+m_\Xi}{2(K+E_N E_K)} - \frac{1}{3}\frac{(W+m_\Lambda-m_\Xi-m_N)}{(K+E_N E_K)^2}\right] + \frac{1}{\pi}\int_{m_\Sigma+1-\lambda}^{\infty} dW' \frac{h^{\circ}_{1-}(W') e^{i\delta^{\circ}_{1-}(W')} \sin\delta^{\circ}_{1-}(W')}{W'-W} \quad (20.)$$

(4) $\ell = 1$, $J = 1/2$ and $I = 1$

$$h'_{1-}(W) = -\frac{g_{\Xi\Sigma K} g_{KN\Sigma}}{16\pi}\left[\frac{2}{W-m_\Sigma} + \frac{W-m_\Sigma+m_N+m_\Xi}{2(K'+E_N E_K)} - \frac{1}{3}\frac{W+m_\Sigma-m_\Xi-m_N}{(K'+E_N E_K)^2}\right] + \frac{1}{\pi}\int_{(m_\Lambda+1)}^{\infty} dW' \frac{h'_{1-}(W') e^{i\delta'_{1-}(W')} \sin\delta'_{1-}(W')}{W'-W} \quad (21)$$

(5) $\ell = 1$, $J = 3/2$ and $I = 0$

$$h^{\circ}_{1+}(W) = \frac{1}{16\pi}\frac{g_{\Xi\Lambda K} g_{KN\Lambda}}{3(K+E_N E_K)^2}\left[W+m_\Lambda-m_\Xi-m_N\right]$$

$$+\frac{1}{\pi}\int_{m_\Sigma+1-\lambda}^{\infty} dW' \frac{h^{0}_{1+}(W')\, e^{i\delta^{0}_{1+}(W')} \sin\delta^{0}_{1+}(W')}{W'-W} \qquad 22.$$

(6) $\ell=1$, $T=3/2$ and $I=1$

$$h^{1}_{1+}(W) = \frac{1}{16\pi}\frac{g_{\Xi K} g_{KN\Sigma}}{3(K'+E_N E_{\bar K})^2}\left[W+m_\Sigma-m_\Xi-m_N\right]$$

$$+\frac{1}{\pi}\int_{m_{\Sigma^-}}^{\infty} dW'\left[\frac{h^{1}_{1+}(W')\, e^{i\delta^{1}_{1+}(W')} \sin\delta^{1}_{1+}(W')}{W'-W} + \frac{\mathrm{Im}\, h^{(\Lambda\pi)}_{1+}(W')}{W'-W}\right] \qquad 23.$$

The ~~first~~ second term in Eq. integral represents the process proceeding through $Y\pi$ intermediate state and in view of the existence of the Y^* resonance can be written as [1]

$$\lambda(s,\theta)\,\frac{\Gamma_0^2}{\pi}\int_{(m_2+\mu)^2=\lambda'}^{\infty} ds'\Big/\left[(s'-s_0)^2+\Gamma_0^2\right](s'-s) \qquad 24$$

where $s_0^{1/2}$ and $\Gamma_0^{1/2}$ represent the mass and half width of the Y^* resonance and $\lambda(s,\theta)$ a slowly varying function of s.

Thus the above equations (19 to 23) can be written in the form

$$h(w) = f(w) + \int_1^{\infty} \frac{h(w')\,e^{i\delta(w')}\sin\delta(w')\,dw'}{w'-w} \qquad 25$$

where $w = W-m_1$ or $W-(m_2-\lambda)$ as the case may be for which the Omnes [2] solution can be immediately written.

$$h(w) = \left[f(w)\cos\delta(w) + \frac{4\Gamma}{\pi}\exp P(w)\,P\int_1^{\infty} \frac{f(w')\sin\delta(w')\exp.P(w')}{w'\,(w'-w)}\,dw'\right] \times \exp P(w) \qquad 26$$

where

$$P(w) = \frac{w}{\pi}\,P\int_1^{\infty} \frac{\delta(w')}{w'(w'-w)}\,dw' \qquad 27.$$

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1) A. Kanazawa, Phys. Rev., 123 , 998 (1961)
2) R. Omnes, Nuovo Cimento, 8 , 316 (1958).

4. Numerical results for differential cross-section for case (iv).

The differential cross-section can be obtained in form

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi)^2}\frac{m_N m_\Xi}{W^2}\frac{q}{k}\left\{|f_1|^2 + |f_2|^2 \sin^2\theta\right\} \quad \text{for } P(\Xi N)=+1 \qquad (28)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\frac{m_N m_\Xi}{W^2}\frac{q}{k}\left\{|f_1|^2 + |f_2|^2 + 2\,\mathrm{Re}(f_1 f_2^*)\cos\theta\right\} \quad \text{for } P(\Xi N)=-1 \qquad (29)$$

when the T-matrix is expressed as

$$T = \langle \Xi | f_1 + \sigma.k\,\sigma.q\, f_2 | N\rangle \quad \text{for } P(\Xi N)=+1 \qquad (30)$$

and $\quad = \langle \Xi | \sigma.k\, f_1 + \sigma.q\, f_2 | N\rangle \quad \text{for } P(\Xi N)=-1 \qquad (31)$

f_1 and f_2 turn out to be

$$f_1 = \left(\frac{E_N+m_N}{2m_N}\right)^{1/2}\left(\frac{E_\Xi+m_\Xi}{2m_\Xi}\right)^{1/2}\left[A - \left(W - \tfrac{1}{2}(m_N+m_\Xi)\right)B\right] + f_2\cos\theta \qquad (32)$$

$$f_2 = -\left(\frac{E_N+m_N}{2m_N}\right)^{1/2}\left(\frac{E_\Xi+m_\Xi}{2m_\Xi}\right)^{1/2}\left[\frac{k}{E_N+m_N}\right]\left[\frac{q}{E_\Xi+m_\Xi}\right] \times\left\{A + \left[W + \tfrac{1}{2}(m_N+m_\Xi)\right]B\right\} \qquad (33)$$

for $P(\Xi N)=+1$

and

$$f_1 = \left(\frac{E_N+m_N}{2m_N}\right)^{1/2}\left(\frac{E_\Xi+m_\Xi}{2m_\Xi}\right)^{1/2}\left(\frac{k}{E_N+m_N}\right)\left\{A+\left[W-\tfrac{1}{2}(m_\Xi-m_N)\right]B\right\} \quad (34)$$

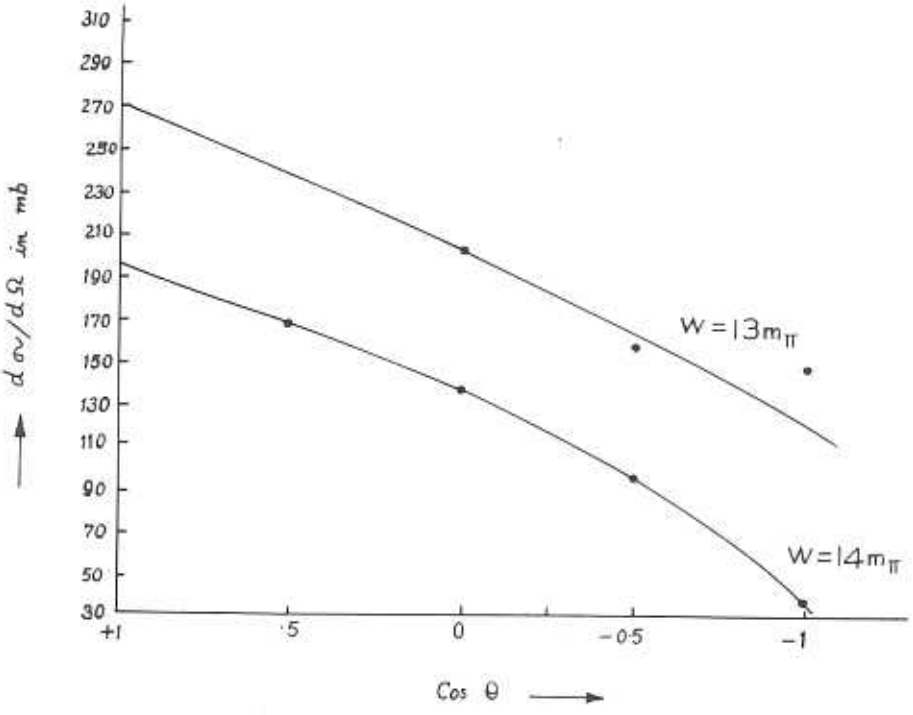
$$f_2 = \left(\frac{E_N+m_N}{2m_N}\right)^{1/2}\left(\frac{E_\Xi+m_\Xi}{2m_\Xi}\right)^{1/2}\left(\frac{q}{E_\Xi+m_\Xi}\right) \times\left\{-A+\left[W+\tfrac{1}{2}(m_\Xi-m_N)\right]B\right\} \quad \text{for } P(\Xi N)=-1 \quad (35)$$

Assuming that only the nearby singularities dominate we can write A and B as

$$A = A^B + \alpha(s,\theta)\frac{\Gamma_0^2}{\pi}\int_{(m_Y+m_\pi)^2}^{\infty}\frac{ds'}{(s'-s)\left[(s'-s_0)^2+\Gamma_0^2\right]} \quad (36)$$

$$B = B^B + \beta(s,\theta)\frac{\Gamma_0^2}{\pi}\int_{(m_Y+m_\pi)^2}^{\infty}\frac{ds'}{(s'-s)\left[(s'-s_0)^2+\Gamma_0^2\right]} \quad (37)$$

where A^B and B^B correspond to the pole terms and the integral represents the contribution of the Y^* resonance which is assumed to dominate the integral terms of the double dispersion representation of A and B. $\alpha(s,\theta)$ and $\beta(s,\theta)$ are slowly varying functions of s and Γ_0 is the half width of the Y^* resonance. Substituting these expressions of A and B in f_1 and f_2 we find that


310
290
270
250
230
210
190
170
150
130
110
90
70
50
30
dσ/dΩ in mb
W=13m_π
W=14m_π
+1
·5
0
-0·5
-1
Cos θ

$$f_1 = f_{p_1} + a_1(s,\theta)\, J(s, s_0, \Gamma_0)$$

$$f_2 = f_{p_2} + a_2(s,\theta)\, J(s, s_0, \Gamma_0) \quad \text{for} \quad P(\Xi N) = -1, \quad P(\Upsilon N) = +1$$

where 38.

$$J(s, s_0, \Gamma_0) = \frac{\Gamma_0^2}{\pi} \int \frac{ds'}{(s'-s)\left[(s'-s_0)^2 + \Gamma_0^2\right]} \qquad 39.$$

$$f_{p_1} = \left(\frac{E_N + m_N}{2m_N}\right)^{1/2} \left(\frac{E_\Xi + m_\Xi}{2m_\Xi}\right)^{1/2} \left(\frac{k}{E_N + m_N}\right) \left[A\beta + \left[W - \tfrac{1}{2}(m_\Xi - m_N)\right]\beta\right] \qquad 40$$

$$f_{p_2} = \left(\frac{E_N + m_N}{2m_N}\right)^{1/2} \left(\frac{E_\Xi + m_\Xi}{2m_\Xi}\right)^{1/2} \left(\frac{q}{E_\Xi + m_\Xi}\right) \left[A\beta + \left[W + \tfrac{1}{2}(m_\Xi - m_N)\right]\beta\right] \qquad 41$$

$$a_1(s,\theta) = \left(\frac{E_N + m_N}{2m_N}\right)^{1/2} \left(\frac{E_\Xi + m_\Xi}{2m_\Xi}\right)^{1/2} \left(\frac{k}{E_N + m_N}\right) \left\{\alpha + \left[W - \tfrac{1}{2}(m_\Xi - m_N)\right]\beta\right\} \qquad 42$$

$$a_2(s,\theta) = \left(\frac{E_N + m_N}{2m_N}\right)^{1/2} \left(\frac{E_\Xi + m_\Xi}{2m_\Xi}\right)^{1/2} \left(\frac{q}{(E_\Xi + m_\Xi)}\right) \left\{\alpha + \left[W + \tfrac{1}{2}(m_\Xi - m_N)\right]\beta\right\} \qquad 43$$

Similar expressions can be arrived at when the parity $P(\Xi N)$ is even. Graphs have been given for $d\sigma/d\Omega$ against $\cos\theta$ for odd parity of $P(\Xi N)$ assuming the established half width (25 Mev) and the spin $J = 3/2$ ($\ell = 1$) of the Y^* resonance.

CHAPTER IV

EXTRAPOLATION METHODS

1. Introductory remarks

In the previous chapters, we have used dispersion theoretic methods to calculate the cross-sections for the collision between two strongly interacting particles resulting in a two particle final states. In some cases, a direct experimental study of such processes is not feasible. For instance, $\Lambda\pi$ scattering cannot be observed directly since neither pions nor Λ-s are available as targets at present. However since hyperons are produced in a bubble chamber with say a proton as the target, the reaction that one might expect to observe is

$$Y + p \longrightarrow Y + N + \pi \qquad A$$

It is naturally to be expected that any information regarding the partial process $\Lambda + \pi \longrightarrow \Lambda + \pi$ or $N + \pi \longrightarrow N + \pi$ will be useful in the study of the above reaction. Or conversely experiments with 'complex' targets can be used to obtain the cross-sections of target constituents.

A general method for the analysis of two particle collisions leading to three particle final state has been proposed by Chew and Low.[1] The basic principle involved is the analytical continuation of the scattering amplitude for the process to a pole, the existence of which is conjectured from field theory. The connection of the residue

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1. G. F. Chew and F. E. Low, Phys Rev. 113 p 1640 (1959)

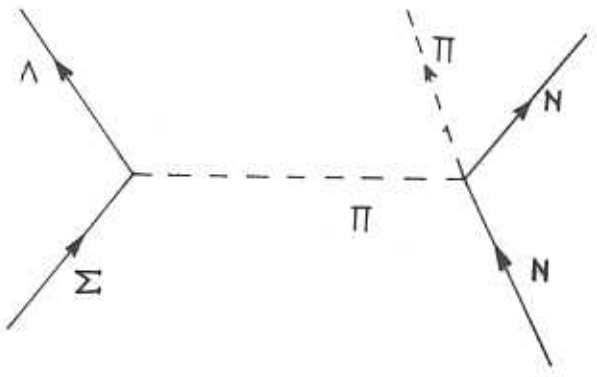


Diagram for $\Sigma + N \rightarrow \Lambda + N + \pi$

at this pole to physically measurable quantities is based on plausibility arguments which are intuitive rather than rigorous. It is assumed that one of the particles in the initial state is 'unstable' i.e. it is composed of two constituents of which only one interacts while the other remains a 'spectator'. The amplitude for the process is studied as a function of Δ^2, the invariant square of the four-momentum transfer between the incident and the spectator particles. It is conjectured that the amplitude is dominated by the contribution from a pole which occurs when Δ^2 is equal to the square of the mass of a single particle with the same quantum numbers as the final state. That this pole lies outside the physical region is nothing more than a stability criterion for the initial target. Under these conditions, the differential cross-section for the entire region can be related to the total cross-section for the partial process in the physical region.

With these preliminary remarks, we shallnow study the process $\bar{\Sigma} + p \rightarrow Y^0 + p + \pi^-$ with the Σ as a composite particle. Our analysis has a two-fold objective:

(a) We include the effect of final state interactions, the importance of which will be emphasized presently, and

(b) We study the cases when Y is either a Σ or a Λ with a view to determining the relative $\Sigma\Lambda$ parity.

2 a. Calculations with final $Y\pi$ interaction [1]

In general the interaction between pairs of particles emitted in a reaction is called the final state interaction. This imprecisely defined separation of the interaction of two particles with each other from the general production is justified by its utility and is useful only if significant features of the whole process can be interpreted simply in terms of the interaction of two particles with each other. This situation is likely to occur if two particles can be emitted with relative momenta such that a resonance scattering takes place or, in other words, an isobar is formed. We now emphasise that this is not merely possible in our case but very probable since both $\pi^- p$ and $\Sigma\pi$ resonances occur very close to the threshold of (A) and hence final state interactions can by no means be neglected. The πp interaction can be taken into account by feeding the total cross-section for $\pi^- p$ scattering at resonance. We shall assume that the dominant $\Sigma\pi$ resonance is in the $I = 1$ and $J = 1/2$ state and incorporate this Y^* resonance by constructing the projection operator P_{11} which picks out the final $I = 1$ $J = 1/2$ state.

From the Feynman diagram we see that the pole would occur due to the exchange of a single pion at $\Delta^2 = (p_{\Sigma^-} - p_{\Sigma^0})^2 = m_\pi^2$ and the branch cuts start at the square of the masses of all multiparticle states, the nearest one in this case being the 2π state. Taking only the contribution from the pole, the residue is the product of the $\Sigma\Sigma\pi$ coupling constant and the

1) G Bhamathy, S Indumathy, T K Radha and R Thunga, Nuovo Cimento (in press).

pion-nucleon scattering amplitude with all particles on the mass shell. Experimentally we can of course never reach the pole, but the dominance could make itself felt in the low Σ^0-recoil region. Also, in our reaction, the composite particle Σ^- is the incident particle and the proton is the target at rest so that in the absence of final state interactions, the differential cross-section for (A) is given by

$$\frac{\partial^2\sigma}{\partial\Delta^2\partial\omega^2} \xrightarrow{\Delta^2=-m_\pi^2} \frac{\Gamma^2}{2\pi} \frac{\left[\frac{1}{4}\omega^4 - \frac{1}{2}\omega^2\left(M_p^2+m_\pi^2\right) + \frac{1}{4}\left(M_p^2-m_\pi^2\right)^2\right]}{\left\{\left[\frac{W^2-M_{\Sigma^-}^2-M_p^2}{2M_{\Sigma^-}}\right]^2 - M_p^2\right\}\left(\Delta^2+m_\pi^2\right)^2} \times \sigma_{\pi p}(\omega) \qquad 1.$$

$$= \frac{\Gamma^2}{2\pi} \frac{K(\omega)}{(\Delta^2+m_\pi^2)^2}$$

where W is the total energy in the barycentric system of Σ^- and p,

Δ is the momentum transfer between Σ^- and Σ^0,

ω the total energy of the target and the intermediate pion

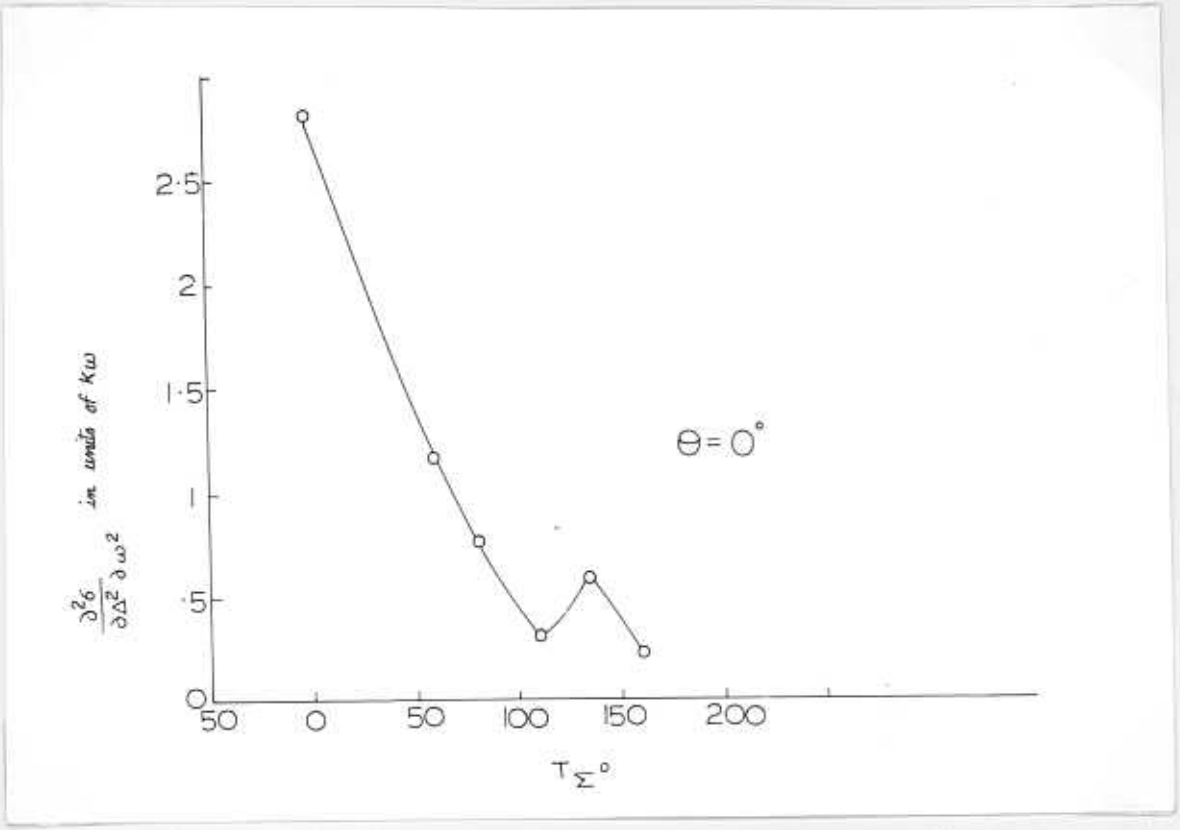
and Γ^2 is the square of the $\Sigma\Sigma\pi$ vertex in the Born approximation.

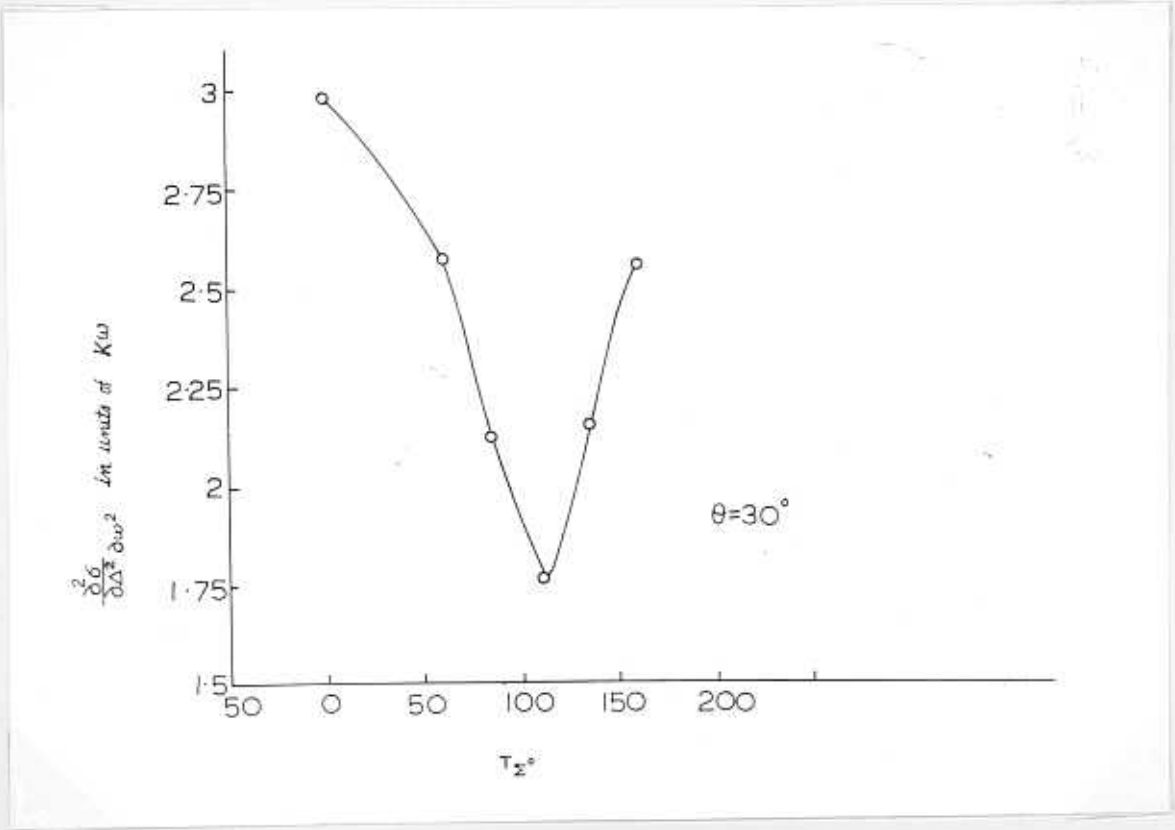
Then

$$\Delta^2 = 2M_{\Sigma^-} T_{\Sigma^0 lab} - \left(M_{\Sigma^-} - M_{\Sigma^0}\right)^2 \qquad 2.$$

so that the recoil kinetic energy of Σ^0 in the Σ^- rest system at the pole is

$$T_{\Sigma^0} = -\frac{\left[m_\pi^2 - \left(M_{\Sigma^-} - M_{\Sigma^0}\right)^2\right]}{2M_{\Sigma^-}} = -8.1\,\text{Mev} \qquad 3.$$


$\frac{\partial^2 \delta}{\partial \Delta^2 \partial \omega^2}$ in units of $K\omega$
2·5
2
1·5
1
·5
0
50
0
50
100
150
200
$T_\Sigma°$
$\Theta = 0°$


$\frac{\partial^2 \delta}{\partial \Delta^2 \partial \omega^2}$ in units of $K\omega$
3
2·75
2·5
2·25
2
1·75
1·5
50
0
50
100
150
200
$T_\Sigma°$
$\theta = 30°$

which though unphysical is small enough to admit extrapolation into the physical region

$$\Gamma^2 = \frac{g^2_{\Sigma\Sigma\pi}}{2M_\Sigma}\left[-2p_\Sigma - p_{\Sigma^0} + M_\Sigma^2\right] \qquad 4.$$

where $M_\Sigma - -M_{\Sigma^0} \sim 0$

The graph (p. 178) shows the variation of $\left(\Delta^2 + m_\pi^2\right)^2 \frac{\partial^2\sigma}{\partial\Delta^2\partial\omega^2}$ for various recoil kinetic energies. T_{Σ^0} lab for given recoil angles assuming $g^2_{\Sigma\Sigma\pi} = 1$.

To include the final state $\gamma\pi$ interaction we may write

$$d\sigma \sim |\langle p_p p_\pi p_{\Sigma^0} | M | p_\Sigma - M_p\rangle|^2 = R \qquad 5.$$

omitting constants and the over-all δ function, where

$$R = R_0 + (S-1)P_{11}R_0 \qquad 6.$$

R_0 indicates the matrix element uncorrected for final state interaction. For the production of particles in the states $J \neq 1/2$ (where $I - P_{11}$ is used) the outgoing particles are represented by plane waves. In the remaining state the factor S takes into account the distortion of the pion wave function by the hyperon. It is possible to estimate S from a knowledge of pion-hyperon phase shifts in $J = 1/2$, $I = 1$ state.

We now evaluate the three terms in

$$|\langle f|M|i\rangle|^2 = |\langle f|M_0|i\rangle|^2 + (S-1)^2|\langle f|P_{11}M_0|i\rangle|^2 + 2\,Re\,(S-1)\langle f|P_{11}M_0|i\rangle\langle f|M_0|i\rangle^* \quad 7.$$

(i) The projection operator P_{11} is given by

$$P_{11} = \left(\frac{1}{2\pi}\right)^3 \sum_{M=-\frac{1}{2}}^{M=+\frac{1}{2}} \int dk\, |\tfrac{1}{2}|M\rangle\langle M|\tfrac{1}{2}| \quad 8.$$

and

$$\langle q\sigma|P_{11}M_0|i\rangle = \sum_M V_\sigma^* \Omega_{\frac{1}{2}|M}^{(\hat{n})} \sum_\mu \int d\sigma' V_\mu \Omega_{\frac{1}{2}|M}^{(\hat{n}')} \langle q'; 1\cdots 3|k\rangle \quad 9.$$

where for convenience we work in the c.m. system of the resonanting particles $\Sigma\pi$ their momentum being q. The V_σ-s are two component spinors of Σ in the spin states σ and

$$\Omega = C(\ell\, s\, j;\ M-\mu\ \ \mu\ M)\, Y^{\ell}_{M-\mu}(\hat{n}) \quad 10$$

Thus we have

$$|\langle q\sigma|P_{11}M_0|i\rangle|^2 = K(\omega) \sum_M |\Omega_{\frac{1}{2}|M}^{(n)}|^2 \sum_\mu \int d\sigma' C^{1\ M-\mu\ \mu}_{\frac{1}{2}\ M} Y^*_{1,M-\mu}(\sigma')\times C^{1\ M-\mu\ \mu}_{\frac{1}{2}\ M} Y_{1,M-\mu}(\sigma') \left[\frac{-E_\Sigma - E_{\Sigma_0} + \vec{p}_\Sigma - \vec{p}_{\Sigma_0}\cos\theta + M_\Sigma^2}{2M_c}\right] \quad 11$$

$$= K(\omega) \sum_M |\Omega_{\frac{1}{2},M}|^2 \frac{[M_\Sigma^2 - E_\Sigma - E_{\Sigma^0}]}{2M_\Sigma} \quad 11.$$

(ii) The term

$$\langle P_{11} M_0 \rangle \langle M_0 \rangle^* = \frac{K(\omega)}{2} \sum_M \sum_{\sigma,\mu} C^{1,M-\sigma,\sigma}_{\frac{1}{2},M} C^{1,M-\mu,\mu}_{\frac{1}{2},M} \int d\omega' Y^*_{1,M-\mu}(\hat{n}) \times Y^{(n)}_{1,M-\sigma} \frac{[-E_\Sigma - E_{\Sigma^0} + p_\Sigma - p_\Sigma \cos\theta + M_\Sigma^2]}{2M_\Sigma}$$

$$= \frac{K(\omega)}{6M_\Sigma} [p_\Sigma - p_{\Sigma^0}(\cos\theta - \sin\theta\, e^{i\phi})] \quad 12$$

(iii) Finally the term

$$\frac{1}{2}\sum_\sigma |\langle M_0 \rangle|^2 = \frac{\Gamma^2}{2\pi} \frac{\left[\frac{1}{4}\omega^4 - \frac{1}{2}\omega^2(M_p^2 + m_\pi^2) + \frac{1}{4}(M_p^2 - m_\pi^2)^2\right]}{\left\{\left[\frac{W^2 - M_\Sigma^2 - M_p^2}{2M_\Sigma}\right]^2 - M_p^2\right\}(\Delta^2 + m_\pi^2)^2} \times \frac{\sigma(\omega)}{\pi p} \quad 13.$$

$$= \frac{\Gamma^2}{2\pi} \frac{k(\omega)}{(\Delta^2 + m_\pi^2)^2}$$

Thus the differential cross-section at the pole when there is a Y^* resonance in the final state is

$$K(\omega)\left\{\sum_M |\Omega_{1,\frac{1}{2},M}|^2 (M_\Sigma^2 - E_\Sigma - E_{\Sigma^0}) + \frac{1}{6M_\Sigma}[p_\Sigma - p_{\Sigma^0}(\cos\theta - \sin\theta e^{i\phi})] + \frac{\Gamma^2}{2\pi}\right\} \quad 14$$

The corrected cross-section cannot be plotted since the $Y\pi$ phase shifts are not known.

2 b. The $\Sigma\Lambda$ relative parity

We shall now consider

$$\Sigma^- + p \longrightarrow \Sigma^0(\Lambda) + n + \pi^0$$

Depending on whether the final state has a Σ or a Λ, the complex system is taken to be composed of either $\Sigma^0\pi^-$ or a $\Lambda\pi^-$. For the latter, the differential cross-section is obtained from (1) by replacing M_{Σ^0} by M_Λ We now determine the ratio of the cross-sections for the Σ^0 and Λ^0 production. For this we make use of the experimental cross-section for the charge exchange scattering of $\pi^- p$ for a particular value of incident pion energy which fixes the value of ω. Let this be say ω_1 for the Σ^0 production. Then the value of W which is most feasible for extrapolation is given by the relation

$$\omega_1^2 = W^2 + M_{\Sigma^0} - \frac{M_{\Sigma^0}}{M_{\Sigma^-}}\left[W^2 + M_{\Sigma^0}^2 - \frac{M_{\Sigma^0}}{M_{\Sigma^-}}\left(W^2 + M_{\Sigma^-}^2 - M_p^2\right)\right] \quad 15$$

This fixes the value of the incident Σ^- energy and with this value of (W) we find the corresponding value of ω_2 for the case when Λ^0 is the spectator.

The matrix element Γ for the first vertex depends on the relative parities of the composite particle and spectator and their coupling constants. When Σ^0 is the

spectator, the parity is ~~xxxxxxxx~~ necessarily even while in the case of Λ it can be either even or odd. If we assume $\Sigma\Lambda$ and $\Sigma\Sigma$ couplings to be of the same order of magnitude we can find the $\frac{\Gamma^2(\Sigma')}{\Gamma^2(\Lambda)}$ for both even and ~~xxx~~ odd $\Sigma\Lambda$ parities at the pole. It has been shown by Muraskin that the ratio of Γ^{-2} for even $\Sigma\Lambda$ parity to odd is of the order of $\frac{1}{382}$. The ratio of the differential cross-sections for Σ^- and Λ^0 production are seen to be as follows:

T_{π_1}	T_{π_2}	$\sigma(\omega_2)$	$\sigma(\omega_1)$	R
210 Mev	148 Mev	28.5	42.5	0.59
194 Mev	157.1 Mev	39.6	4.2	0.52

It is interesting to note that the values of the ratio obtained for even $\Sigma\Lambda$ parity is close to the experimental value obtained in the hydrogen bubble chamber which is 0.5. It seems plausible to assume that the ratio remains the same when the differential cross-sections are extrapolated into the physical region. This clearly indicates even $\Sigma\Lambda$ parity.

Proceeding in a similar way for the capture of Σ^- by a dueteron and treating the deuteron as the composite particle, we find that the value of the ratio for the Σ^- and Λ production decreases to about 0.25, although we find that this is not sufficient to explain the latest experimental value ~~xx~~ of 0.43. However the experimental

value seems to be fluctuating since the original value given by Hortwitz, Miller et. al., is about 0.14. Perhaps in this case, the K -meson interactions play a considerable part of the final state interactions have to be taken into account since two nucleons are available in the final state.

PART III.

CHAPTER I

ON THE POSSIBLE RESONANCES IN Ξ^- - p COLLISIONS[1]

1. Introductory remarks

It seems very probable that cascade particles may be produced in large numbers in the laboratory in the near future, the most likely reaction being the $\bar{K}N$ collisions resulting in the production of Ξ along with a K particle. The threshold for this reaction is 1.52 Mev while the $\bar{K}$ beams now available have an energy of Mev . It therefore seems worthwhile to investigate the possible reactions of Ξ particles of which the most relevant from an experimental point of view would be the $\Xi^- p$ and $\Xi^- n$ reactions which may be studied in the hydrogen and deuterium bubble chambers respectively.

In this chapter, we study cascade-nucleon reactions resulting in three particle final states under the assumption of charge independence. The main emphasis in our discussion is on the possibility of resonances in the final state. Besides the now well-established Y^* resonance, it has been suggested that a YN resonance with strangeness $S = -1$ and a YY resonance with $S = -2$

1) A. Ramakrishnan, T. K. Radha and R. Thunga, Nucl. Phys. 32 517 (1962)

may also exist.[1] It is suggested that a study of the ΞN reactions can throw light on all these resonances. In order to get information regarding the spin and parity of these resonances, we give an analysis of the angular distribution of the meson produced in the reactions.

This chapter is thus sub-divided into two sections. Section 1 contains an isotopic spin analysis of the reactions

$$\Xi + N \longrightarrow \Xi + N + \pi \qquad A.$$
$$\longrightarrow \Lambda + N + K \qquad B.$$
$$\longrightarrow \Lambda + \Lambda + \pi \qquad C.$$

on the basis of charge independence with a discussion on the possible resonances in the final state. In Section 2, we give the angular distribution for the omitted pion in (A+C). For completeness each section is preceded by an account of the mathematical preliminaries and basic physical concepts necessary for our analysis. A critical discussion of the implications of the assumptions made are presented at the end of each section.

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1) R.H.Dalitz, Lectures delivered at the Summer School for Theoretical Physics at Bangalore (1961).

2. Isotopic Spin Analysis

a) Preliminary comments

The introduction of isotopic spin to nucleons was stimulated by the experimentally observed fact of charge independence of nuclear forces, i.e. if the electromagnetic and weak interactions are switched off, the neutron and proton would be indistinguishable. This situation was recognised to be similar to that of ordinary spin in a magnetic field and the isotopic spin formalism is based on this analogy. It is also well-known that the assignment of isotopic spin and strangeness quantum numbers to particles resulted in the famous Gell-Mann-Nishijima scheme for the classification of baryons and mesons. The relation between charge Q, isospin I and strangeness S is given by

$$Q = I_z + \frac{N+S}{2} \qquad (1)$$

where I_z is the third component of I and N is the baryon number. According to this scheme the π and Σ are isotopic triplets with $S=0$ and $S=-1$ respectively and $N(p,n)$, $K(K^+,K^0)$ and $\Xi(\Xi^-,\Xi^0)$ are isotopic doublets with $S=0$, -1 and -2 respectively, so that by analogy with spin, their isospin wave functions can be immediately constructed.

b) Iso spin wave functions for two and three particle systems

The addition of two or more isospins follows the same rules as in the case of ordinary angular momenta. The eigen function $|I I_z\rangle$ of $\vec{I} = \vec{I}_1 + \vec{I}_2$ with I_z

as third component of I can be written as a product of the eigen functions $\psi(I_1, I_{1z})$ and $\psi(I_2, I_{2z})$ of $\vec{I}_1$ and $\vec{I}_2$ i.e.

$$|I\, I_z\rangle = \sum_{I_{1z}} C(I_1 I_2 I_{1z} I_{2z} | I_1 I_2 I\, I_z)\psi(I_1 I_{1z})\psi(I_2 I_{2z})$$

where

$$C(I_1 I_2 I_{1z} I_{2z} | I_1 I_2 I\, I_z)$$

are the usual Clebsh-Gordon coefficients. In the case of three particle systems, three distinct modes of coupling between the individual isospins would naturally arise. Thus for a final isospin I resulting from $\vec{I}_1$, $\vec{I}_2$ and $\vec{I}_3$ we may have

$$\vec{I}_1 + \vec{I}_2 = \vec{I}', \quad \vec{I}' + \vec{I}_3 = \vec{I}$$

or

$$\vec{I}_1 + \vec{I}_3 = \vec{I}', \quad \vec{I}' + \vec{I}_2 = \vec{I}$$

or

$$\vec{I}_2 + \vec{I}_3 = \vec{I}', \quad \vec{I}' + \vec{I}_1 = \vec{I}$$

Out of these three combinations the one that is most suitable will be chosen from the nature of the problem at hand.

c) Charge Independence

With the introduction of the isomultiplet structure for elementary particles the concept of charge independence when extended to the other particles would imply conservation of isospin in strong interactions. More explicitly, when

~~extended to the other particles would imply conservation of isospin in strong interactions~~. More explicitly, when electromagnetic and weak interactions are neglected a physical situation does not depend on a choice of coordinates in isospin space or in other words the interaction has to be a scalar in isospin space. Thus charge independence ~~se-~~ implies quences that all physical processes should be invariant under rotations in isospin space. This leads us to relationships between the cross-sections for the different channels in a given process. In fact this can be achieved by using charge wave functions only even without the knowledge of the exact matrix element. This method of approach is particularly useful in the study of strong interactions where there is no consistent theory for the evaluation of matrix elements.

3. Calculations

We now present a systematic analysis of the following processes under the assumption of charge independence.

$$\Xi + p \longrightarrow \Xi + N + \pi \quad (1115\ \text{MeV}) \qquad (a)$$

$$\longrightarrow \Sigma + N + K \quad (970\ \text{"}) \qquad (b)$$

$$\longrightarrow \Lambda + N + K \quad (755\ \text{"}) \qquad (c)$$

$$\longrightarrow \Sigma + \Sigma + \pi \quad (689\ \text{"}) \qquad (d)$$

$$\longrightarrow \Sigma + \Lambda + \pi \quad (480\ \text{"}) \qquad (e)$$

$$\longrightarrow \Lambda + \Lambda + \pi \quad (280\ \text{"}) \qquad (f)$$

The threshold for these processes are given within brackets.[1)]

1) A detailed discussion on thresholds is given in Chapter IV, Part III.

We now illustrate the procedure for one of the reactions (i.e. 2 a) in detail. We feel that this is of particular importance since if for the moment we take the global symmetry hypothesis for granted so that the πN, πY and $\pi \Xi$ interactions are identical, we should naturally expect to find a $\Xi\pi$ resonance in the $J=3/2$, $I=3/2$ state in analogy with the π-N resonance. Even if global symmetry were not valid, we should expect such a resonance on the assumption that pionic interactions of baryons are similar (though not identical). In fact it is felt that such a $\Xi\pi$ resonance would be aesthetically satisfying, particularly since the $Y\pi$ resonance is now well-established.

Since our object is to study the effects of a $\Xi\pi$ resonance in the final state, the choice of the iso-spin coupling to be employed is automatically fixed. Thus we couple $I_\Xi = 1/2$ with $I_\pi = 1$ to give $I' = 3/2$ or $1/2$ which is then coupled to $I_N = 1/2$ to give the final I. By conservation of isospin, this should be equal to the initial I of the Ξp system. Thus taking a typical example

$$\Xi^- + p \rightarrow \Xi^- + p + \pi^0$$

we have for the matrix element

$$M = \left[C\left(1\,\tfrac{1}{2}\,\tfrac{3}{2}, -\tfrac{1}{2}\,-\tfrac{1}{2}\right) C\left(\tfrac{3}{2}\,\tfrac{1}{2}\,1, +\tfrac{1}{2}\,0\right) A_{13}\right] C\left(\tfrac{1}{2}\,\tfrac{1}{2}\,1, +\tfrac{1}{2}\,0\right)$$

$$+\left[C\left(1\,\tfrac{1}{2}\,\tfrac{1}{2}, -\tfrac{1}{2}\,-\tfrac{1}{2}\right) C\left(\tfrac{1}{2}\,\tfrac{1}{2}\,1, +\tfrac{1}{2}\,0\right) A_{11}\right] C\left(\tfrac{1}{2}\,\tfrac{1}{2}\,1, +\tfrac{1}{2}\,0\right)$$

$$+\left[C\left(1\tfrac{1}{2}\tfrac{1}{2}, -\tfrac{1}{2}-\tfrac{1}{3}\right) C\left(\tfrac{1}{3}\tfrac{1}{2}0, +\tfrac{1}{2}0\right) A_{01}\right] C\left(\tfrac{1}{2}\tfrac{1}{2}0, -\tfrac{1}{2}0\right) \quad 3.$$

The C-S are the usual Clebsch-Gordon coefficients. The first two coefficients arise since we are dealing with three particle states while the last one is due to the possibility of two distinct iso-spin values for the initial system. The $A_{I,2I'}$ are the amplitudes for the process i.e. A_{01} denotes the matrix element for the transition from an initial $I = 0$ state to a final state in which the Ξ and π are in an $I' = 1/2$ state. If on the other hand we had considered an initial $\Xi^0 p$ system with $I_z = 1$, then the $I = 0$ initial state is forbidden so that we would have had the amplitudes A_{11} and A_{01} only. Thus we have for (3)

$$M = \left[-\frac{1}{\sqrt{6}} A_{13} + \frac{1}{2\sqrt{3}} A_{11} - \frac{1}{2\sqrt{3}} A_{01}\right] \quad 4$$

and the cross-section for the process is

$$|M|^2 = \left| -\frac{1}{\sqrt{6}} A_{13} + \frac{1}{2\sqrt{3}} A_{11} - \frac{1}{2\sqrt{3}} A_{01} \right|^2 \quad 5.$$

We thus evaluate the total cross-section for all possible channels of (a) for both the $\Xi^- p$ and $\Xi^0 p$ initial systems. If we now assume that there is a dominant resonance in the $\Xi\pi$ final system in the $I' = 3/2$ state, we may set A_{01} and A_{11} amplitudes to be zero and the resulting cross-sections are evaluated. There develop interesting equalities and in equalities among the cross-sections in the various channels which lend themselves to easy experimental verification.

Table 1. Cross-sections for (6) and (e) without and with Y_N resonance.

Process	Cross-sections σ	$I = 3/2$ Resonance			$I = 1/2$ Resonance.		
		σ^R	Σ^+/Σ^-	$\frac{\Sigma^+ + \Sigma^-}{\Sigma^0}$	σ^R	$\Sigma^+/\Sigma^- \quad \frac{\Sigma^+ + \Sigma^-}{\Sigma^0}$	
$\Xi^- + p \to \Lambda + n + \bar{K}^0$	$\sigma_1 = \frac{1}{4}\lvert K^{\Lambda}_{11} + K^{\Lambda}_{01}\rvert^2$	0			$\frac{1}{4}\lvert K^{\Lambda}_{11} + K^{\Lambda}_{01}\rvert^2$		
$\to \Lambda + p + K^-$	$\sigma_2 = \frac{1}{4}\lvert K^{\Lambda}_{11} - K^{\Lambda}_{01}\rvert^2$	0			$\frac{1}{4}\lvert K^{\Lambda}_{11} - K^{\Lambda}_{01}\rvert^2$		
$\to \Sigma^0 + n + \bar{K}^0$	$\sigma_3 = \frac{1}{12}\lvert K^{\Sigma}_{11} + K^{\Sigma}_{01} - \sqrt{2}K^{\Sigma}_{13}\rvert^2$	$\frac{1}{6}\lvert K^{\Sigma}_{13}\rvert^2$	1	1/2	$\frac{1}{12}\lvert K^{\Sigma}_{11} + K^{\Sigma}_{01}\rvert^2$	$\frac{\sigma_6}{\sigma_5}$	2
$\to \Sigma^0 + p + K^-$	$\sigma_4 = \frac{1}{12}\lvert\sqrt{2}K^{\Sigma}_{13} - K^{\Sigma}_{11} + K^{\Sigma}_{01}\rvert^2$	$\equiv \sigma_3^R$			$\frac{1}{12}\lvert K^{\Sigma}_{01} + K^{\Sigma}_{01}\rvert^2$		
$\to \Sigma^- + p + \bar{K}^0$	$\sigma_5 = \frac{1}{12}\lvert K^{\Sigma}_{13} + \sqrt{2}K^{\Sigma}_{11} + \sqrt{2}K^{\Sigma}_{01}\rvert^2$	$\equiv \frac{1}{3}\sigma_3^R$			$\equiv 2\sigma_3^R$		
$\to \Sigma^+ + n + K^-$	$\sigma_6 = \frac{1}{12}\lvert K^{\Sigma}_{13} + \sqrt{2}K^{\Sigma}_{11} - \sqrt{2}K^{\Sigma}_{01}\rvert^2$	$\equiv \frac{1}{2}\sigma_3^R$			$\equiv 2\sigma_4^R$		
$\Xi^0 + p \to \Lambda + p + \bar{K}^0$	$\sigma_7 = \lvert K^{\Lambda}_{11}\rvert^2$	0			$\lvert K^{\Lambda}_{11}\rvert^2$		
$\to \Sigma^0 + p + \bar{K}^0$	$\sigma_8 = \frac{1}{6}\lvert K^{\Sigma}_{13} + \sqrt{2}K^{\Sigma}_{11}\rvert^2$	$\equiv \sigma_3^R$	2	1	$\frac{1}{3}\lvert K^{\Sigma}_{11}\rvert^2$		
$\to \Sigma^+ + n + \bar{K}^0$	$\sigma_9 = \frac{1}{12}\lvert 2\sqrt{2}K^{\Sigma}_{11} - K^{\Sigma}_{13}\rvert^2$	$\equiv \frac{1}{2}\sigma_3^R$			$\equiv 2\sigma_8^R$		
$\to \Sigma^- + p + K^-$	$\sigma_{10} = 3/4\lvert K^{\Sigma}_{13}\rvert^2$	$\equiv 9/2\,\sigma_3^R$			0		

Table 2: Cross-sections for (b) and (c) without and with KN resonance

Cross-section σ	$I=1$ Resonance			$I=0$ Resonance		
	σ_R	Σ^+/Σ^-	$\Sigma^+ + \Sigma^-/\Sigma^0$	σ_R	Σ^+/Σ^-	$\Sigma^+ + \Sigma^-/\Sigma^0$
$\sigma_1 = \frac{1}{4}\lvert T_{11}^{\wedge} + T_{00}^{\wedge}\rvert^2$	$\frac{1}{4}\lvert T_{11}^{\wedge}\rvert^2$			$\frac{1}{4}\lvert T_{00}^{\wedge}\rvert^2$		
$\sigma_2 = \frac{1}{4}\lvert T_{11}^{\wedge} - T_{00}^{\wedge}\rvert^2$	$\equiv \sigma_1^R$			$\equiv \sigma_1^R$		
$\sigma_3 = \frac{1}{12}\lvert T_{01}^{\Sigma} - \sqrt{3}\, T_{10}^{\Sigma}\rvert^2$	$\frac{1}{12}\lvert T_{01}^{\Sigma}\rvert^2$	∞	1	$\frac{1}{4}\lvert T_{10}\rvert^2$		0
$\sigma_4 = \frac{1}{12}\lvert T_{01}^{\Sigma} + \sqrt{3}\, T_{10}^{\Sigma}\rvert^2$	$\equiv \sigma_3^R$			$\equiv \sigma_3^R$		
$\sigma_5 = \frac{1}{12}\lvert \sqrt{3}\, T_{11}^{\Sigma} - \sqrt{2}\, T_{01}^{\Sigma}\rvert^2$	$\frac{1}{12}\lvert \sqrt{3}\, T_{11}^{\Sigma} - \sqrt{2}\, T_{01}^{\Sigma}\rvert^2$			0		
$\sigma_6 = \frac{1}{12}\lvert \sqrt{3}\, T_{11}^{\Sigma} + \sqrt{2}\, T_{01}^{\Sigma}\rvert^2$	$\frac{1}{12}\lvert \sqrt{3}\, T_{11}^{\Sigma} + \sqrt{2}\, T_{01}^{\Sigma}\rvert^2$			0		
$\sigma_7 = \lvert T_{11}^{\wedge}\rvert^2$	$\equiv 4\sigma_1^R$	$\frac{\sigma_5}{\sigma_6}$	$\frac{\sigma_5+\sigma_6}{2\sigma_3}$	0		
$\sigma_8 = \frac{1}{2}\lvert T_{11}^{\Sigma}\rvert^2$	$\frac{1}{2}\lvert T_{11}^{\Sigma}\rvert^2$			0	∞	
$\sigma_9 = \frac{1}{4}\lvert T_{11}^{\Sigma} + \sqrt{2}\, T_{10}^{\Sigma}\rvert^2$	$\equiv \frac{1}{2}\sigma_8^R$			$\frac{1}{2}\lvert T_{10}^{\Sigma}\rvert^2$		
$\sigma_{10} = \frac{1}{4}\lvert \sqrt{2}\, T_{10}^{\Sigma} - T_{11}^{\Sigma}\rvert^2$	$\equiv \frac{1}{2}\sigma_8^R$			$\frac{1}{2}\lvert T_{10}^{\Sigma}\rvert^2$		

Similar calculations are carried out for the reaction (b) and (c) and we first envisage the possibility of a YN resonance. The amplitudes in this case are denoted by $T^{Y}_{I,I'}$ where Y is either Σ or Λ and $I' = 1/2$ or $3/2$ for Σ and $I' = 1/2$ for Λ. It is obvious that the results would be identical with that in (a) since the YN system is similar to the $\Xi\pi$ system as far as the isotopic spin is concerned, but of course the amplitude would be different. The cross-sections for the possible reactions with and without resonances in the final state are tabulated in table (I). The triangular inequalities that follow are given below the table

The Table (II) contains the results with and without a $\bar{K}N$ resonance in the final state of reactions (b) and (c). For this, the coupling scheme is chosen to be $\vec{I}' = \vec{I}_{\bar{K}} + \vec{I}_{N}$ and the amplitudes are denoted by $T_{I,I'}$ where $I' = 1$ or 0. In the absence of any resonance, the triangular inequalities are the same as in (I) since the processes are the same. However at resonance, they are naturally different.

An $S = -2$ resonance with baryon number $N = 2$ may also be studied from these reactions. For this we consider the reactions (d), (e) and (f) and the table (3) gives a list of Ξ^{*} and Λ^{*} resonance, the effect of a $\Sigma\pi$ on these reactions is also listed in table (4). In this case each of the two hyperons can be separately coupled to the pion which leads to two

Process		
$\Xi^- + p \to \Lambda + \Lambda + \pi^0$	$\frac{1}{2}\lvert R_{10}^{\prime \Lambda\Lambda}\rvert^2$	$\frac{1}{2}\lvert R_{10}^{\prime \Lambda\Lambda}\rvert^2$
$\to \Lambda + \Sigma^+ + \pi^-$	$\frac{1}{12}\lvert \sqrt{3}\, R_{11}^{\prime \Lambda\Sigma} - \sqrt{2}\, R_{01}^{\prime \Lambda\Sigma}\rvert^2$	0
$\to \Lambda + \Sigma^- + \pi^+$	$\frac{1}{12}\lvert \sqrt{3}\, R_{11}^{\prime \Lambda\Sigma} + \sqrt{2}\, R_{01}^{\prime \Lambda\Sigma}\rvert^2$	0
$\to \Lambda + \Sigma^0 + \pi^0$	$\frac{1}{6}\lvert R_{01}^{\prime \Lambda\Sigma}\rvert^2$	0
$\to \Sigma^0 + \Sigma^+ + \pi^-$	$\frac{1}{120}\lvert 3R_{12}^{\prime \Sigma\Sigma} + \sqrt{15}\, R_{11}^{\prime \Sigma\Sigma} - \sqrt{10}\, R_{01}^{\prime \Sigma\Sigma}\rvert^2$	0
$\to \Sigma^0 + \Sigma^- + \pi^+$	$\frac{1}{120}\lvert 3R_{12}^{\prime \Sigma\Sigma} - \sqrt{15}\, R_{11}^{\prime \Sigma\Sigma} - \sqrt{10}\, R_{01}^{\prime \Sigma\Sigma}\rvert^2$	0
$\to \Sigma^0 + \Sigma^0 + \pi^0$	$\frac{1}{30}\lvert 2R_{12}^{\prime \Sigma\Sigma} + \sqrt{5}\, R_{10}^{\prime \Sigma\Sigma}\rvert^2$	$\frac{1}{6}\lvert R_{10}^{\prime \Sigma\Sigma}\rvert^2$
$\to \Sigma^+ + \Sigma^- + \pi^0$	$\frac{1}{60}\lvert -\sqrt{2}\, R_{12}^{\prime \Sigma\Sigma} - \sqrt{10}\, R_{10}^{\prime \Sigma\Sigma} + \sqrt{5}\, R_{01}^{\prime \Sigma\Sigma}\rvert^2$	$\equiv \sigma_7^R$
$\Xi^0 + p \to \Lambda + \Lambda + \pi^+$	$\lvert R_{01}^{\prime \Lambda\Lambda}\rvert^2$	$2\sigma_1^R$
$\to \Lambda + \Sigma^0 + \pi^+$	$\frac{1}{2}\lvert R_{11}^{\prime \Sigma\Lambda}\rvert^2$	0
$\to \Lambda + \Sigma^+ + \pi^0$	$\frac{1}{2}\lvert R_{11}^{\prime \Sigma\Lambda}\rvert^2$	0
$\to \Sigma^0 + \Sigma^0 + \pi^+$	$\frac{1}{15}\lvert R_{12}^{\prime \Sigma\Sigma} - \sqrt{5}\, R_{10}^{\prime \Sigma\Sigma}\rvert^2$	$2\sigma_7^R$
$\to \Sigma^+ + \Sigma^0 + \pi^0$	$\frac{1}{20}\lvert \sqrt{3}R_{12}^{\prime \Sigma\Sigma} + \sqrt{5}\, R_{11}^{\prime \Sigma\Sigma}\rvert^2$	0
$\to \Sigma^+ + \Sigma^- + \pi^+$	$\frac{1}{60}\lvert R_{12}^{\prime \Sigma\Sigma} - \sqrt{15}\, R_{11}^{\prime \Sigma\Sigma} + 2\sqrt{5}\, R_{10}^{\prime \Sigma\Sigma}\rvert^2$	$2\sigma_7^R$
$\to \Sigma^+ + \Sigma^+ + \pi^-$	$\frac{3}{5}\lvert R_{12}^{\prime \Sigma\Sigma}\rvert^2$	0

3. Cross sections for (d), (e) and (f) without and with YY resonance.

	for γ_1 resp.	II	for γ_1 resonance
$\sigma_1 = \frac{1}{2}\left\|R_{11}^{\Lambda\Lambda}\right\|^2$	0	$\left\|\frac{1}{\sqrt{2}} R_{11}^{\Lambda\Lambda}\right\|^2$	0
$\sigma_2 = \frac{1}{12}\left\|\sqrt{3} R_{11}^{\Lambda\Sigma} - \sqrt{2} R_{00}^{\Lambda\Sigma}\right\|^2$	0	$\frac{1}{12}\left\|\sqrt{3} R_{11}^{\Lambda\Sigma} + \sqrt{2} R_{01}^{\Lambda\Sigma}\right\|^2$	$\equiv \sigma_{2(II)}$
$\sigma_3 = \frac{1}{12}\left\|\sqrt{3} R_{11}^{\Lambda\Sigma} + \sqrt{2} R_{00}^{\Lambda\Sigma}\right\|^2$	0	$\frac{1}{12}\left\|\sqrt{3} R_{11}^{\Lambda\Sigma} - \sqrt{2} R_{01}^{\Lambda\Sigma}\right\|^2$	$\equiv \sigma_{3(II)}$
$\sigma_4 = \frac{1}{6}\left\|R_{00}^{\Lambda\Sigma}\right\|^2$	0	$\frac{1}{6}\left\|R_{01}^{\Lambda\Sigma}\right\|^2$	0
$\sigma_5 = \frac{1}{60}\left\|-\sqrt{2} R_{12}^{\Sigma\Sigma} - \sqrt{10} R_{10}^{\Sigma\Sigma} + \sqrt{5} R_{01}^{\Sigma\Sigma}\right\|^2$	0	$\frac{1}{120}\left\|3R_{12}^{\Sigma\Sigma} - \sqrt{15} R_{11}^{\Sigma\Sigma} - \sqrt{10} R_{01}^{\Sigma\Sigma}\right\|^2$	$\equiv \sigma^R_{8(I)}$
$\sigma_6 = \frac{1}{60}\left\|\sqrt{2} R_{12}^{\Sigma\Sigma} + \sqrt{10} R_{10}^{\Sigma\Sigma} + \sqrt{5} R_{01}^{\Sigma\Sigma}\right\|^2$	0	$\frac{1}{120}\left\|3R_{12}^{\Sigma\Sigma} - \sqrt{15} R_{11}^{\Sigma\Sigma} + \sqrt{10} R_{01}^{\Sigma\Sigma}\right\|^2$	$\frac{1}{24}\left\|\sqrt{3} R_{11}^{\Sigma\Sigma} - \sqrt{2} R_{01}^{\Sigma\Sigma}\right\|^2$
$\sigma_7 = \frac{1}{30}\left\|2R_{12}^{\Sigma\Sigma} + \sqrt{5} R_{10}^{\Sigma\Sigma}\right\|^2$	0	$\frac{1}{30}\left\|2R_{12}^{\Sigma\Sigma} + \sqrt{5} R_{10}^{\Sigma\Sigma}\right\|^2$	0
$\sigma_8 = \frac{1}{120}\left\|3R_{12}^{\Sigma\Sigma} + \sqrt{15} R_{11}^{\Sigma\Sigma} + \sqrt{10} R_{01}^{\Sigma\Sigma}\right\|^2$	$\frac{1}{24}\left\|\sqrt{3} R_{11}^{\Sigma\Sigma} + \sqrt{2} R_{01}^{\Sigma\Sigma}\right\|^2$	$\frac{1}{120}\left\|3R_{12}^{\Sigma\Sigma} + \sqrt{15} R_{11}^{\Sigma\Sigma} - \sqrt{10} R_{01}^{\Sigma\Sigma}\right\|^2$	$\sigma^R_{6,(II)}$
$\sigma_9 = \left\|R_{11}^{\Lambda\Lambda}\right\|^2$	$\left\|R_{11}^{\Lambda\Lambda}\right\|^2$	$\equiv \sigma_{9(I)}$	$\sigma^R_{9(I)}$
$\sigma_{10} = \frac{1}{2}\left\|R_{11}^{\Lambda\Sigma}\right\|^2$	$\frac{1}{2}\left\|R_{11}^{\Lambda\Sigma}\right\|^2$	$\frac{1}{2}\left\|R_{11}^{\Lambda\Sigma}\right\|^2$	$\sigma^R_{10(I)}$
$\sigma_{11} = \frac{1}{2}\left\|R_{11}^{\Lambda\Sigma}\right\|^2$	$\equiv \sigma^R_{10}$	$\frac{1}{2}\left\|R_{11}^{\Lambda\Sigma}\right\|^2$	0
$\sigma_{12} = \frac{1}{20}\left\|\sqrt{3} R_{12}^{\Sigma\Sigma} + \sqrt{5} R_{11}^{\Sigma\Sigma}\right\|^2$	$\frac{1}{4}\left\|R_{11}^{\Sigma\Sigma}\right\|^2$	$\frac{1}{20}\left\|\sqrt{3} R_{12}^{\Sigma\Sigma} + \sqrt{5} R_{11}^{\Sigma\Sigma}\right\|^2$	$\sigma^R_{12(I)}$
$\sigma_{13} = \frac{1}{15}\left\|R_{12}^{\Sigma\Sigma} - \sqrt{5} R_{10}^{\Sigma\Sigma}\right\|^2$	0	$\frac{1}{20}\left\|-\sqrt{3} R_{12}^{\Sigma\Sigma} + \sqrt{5} R_{11}^{\Sigma\Sigma}\right\|^2$	$\sigma^R_{12(I)}$
$\sigma_{14} = \frac{1}{60}\left\|R_{12}^{\Sigma\Sigma} + \sqrt{15} R_{11}^{\Sigma\Sigma} + 2\sqrt{5} R_{10}^{\Sigma\Sigma}\right\|^2$	0	$\frac{3}{5}\left\|R_{12}^{\Sigma\Sigma}\right\|^2$	0
$\sigma_{15} = \frac{1}{60}\left\|R_{12}^{\Sigma\Sigma} - \sqrt{15} R_{11}^{\Sigma\Sigma} + 2\sqrt{5} R_{10}^{\Sigma\Sigma}\right\|^2$	0	$\equiv \sigma_{15(I)}$	0

Table 4: Cross-sections for (d), (e) and (f) without and with Y^1 Resonances.

different resonances in the same process when the hyperons are not identical. We find from the above table that when Λ^{0*}_{1} (an $I = 1$ resonance with $Q = 0$) is not allowed most of the cross-sections vanish which, if experimentally verified, will confirm the absence of Y^{0*}_{1}. At resonance we also find that the cross-sections reduce to the corresponding ones in (4). However it is interesting to note that the multiplicities are different in the two cases as is seen from the columns in table (3).

Till now we have restricted ourselves to a purely isotopic spin analysis without specifying anything about the actual matrix elements. It is to be pointed out that the cross-sections may be calculated by the Chew-Low extrapolation method as was discussed in Part II. In this case, we may for example have diagram (b) with the pion pole for reaction (2 a) and diagram (a) with the K pole for the other reactions, where the Ξ is considered as a $Y\overline{K}$ system. The extrapolation energies in the two cases will be -87.5 Mev and -7.5 Mev respectively.

4. Angular distribution

a) Preliminary comments

One of the most fruitful application of the theory of angular momenta is the study of angular distributions and correlations.[1] The amount of information that can be obtained with a few perfectly reasonable assumptions is in fact quite remarkable. This approach has been of particular

1) See for example, "Elementary theory of Angular Momentum" by M.E.Rose, John Wiley & Sons (1957).

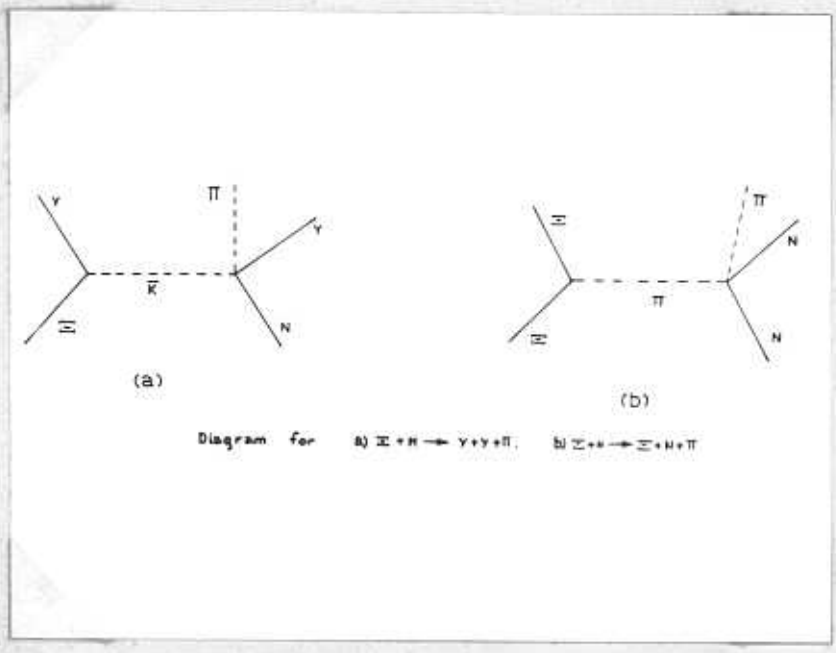

Diagram for a) $\Xi^- + n \rightarrow \gamma + \gamma + \Xi^0$, b) $\Sigma^+ + n \rightarrow \Xi^- + N + \pi$

importance in the study of nuclear reactions and has also been extended to reactions involving elementary particles. For instance, the radiation emitted from an oriented nucleus with angular momentum j will not be isotopic since in an oriented nucleus, the $2j+1$ substates are not equally populated. The situation is quite similar when a spin zero particle is emitted in a reaction involving particles with spin.

The basic assumption that is generally made is in restricting the orbital angular momenta to low values which is valid for small values of incident energies. This restriction, when made on a pair of 'identical' particles, fermions (in a given spin state) or bosons belonging to the same charge multiplet leads to a corresponding restriction on the iso-spin state of the system in accordance with the generalized Pauli principle. This requires that the total wave function - the product of orbital, spin and isospin wave functions - be symmetric in the case of bosons and antisymmetric in the case of fermions. Let us now consider an initial system of two particles with a relative angular momentum ℓ and spins s_i and s_t resulting in a three particle final state. Let us further assume that two of these belong to the same multiplet while the third is spinless. Then in analogy with the procedure adopted in nuclear physics, we may denote the two fermion state as being characterised by a total angular momentum J_a and the boson by the

relative angular momentum L_2 so that the total $J = L_2 + J_2$ Since J_2 is obtained by coupling the relative angular momentum of the fermions and their spins, we can naturally have two schemes, i.e. (a) the jj coupling where the individual ℓ-s and s-s are first coupled to give the individual j-s which are then coupled to give the final J and (b) the LS coupling scheme where the total L and S are found by the addition of individual ℓ-s and s-s and the final J computed from L. and S .

The angular distribution in the latter scheme is given by

$$W(LS) \sim \sum_{\nu} \eta_{\nu} \Gamma_{\nu}^{2} F_{\nu}(L_2 J_2 J) P_{\nu}(\cos\theta) \quad 1$$

where

$$\eta_{\nu} = (2J+1)(2j+1)(-1)^{J_t - J - 1/2} C(\nu j j; 0\,1/2) \times W(JJjj; \nu J_t) \quad 7$$

and

$$\Gamma_{\nu} = (2j+1)^{1/2} \times (J_t L_t S_t; j L 1/2; J L S) \quad 8$$

$$F_{\nu}(L_2 J_2 J) = (-1)^{J_2 - J - 1} (2J+1)^{1/2} (2L_2+1)\, C(L_2 L_2 \nu; 1-1) \times W(JJL_2L_2; \nu J_2) \quad 9$$

and $\times$ is the Wigner $9j$ symbol. The need for this is intutively clear since the situation is similar to that of compound nucleus formation in nuclear reactions - i.e.

in our case the system of two fermions. The W-s and C-s are the Racah and the Clebsch-Gordon coefficients respectively.

b) Calculations

With the above basic preliminaries we now turn to the problem of the angular distribution for reaction (e) or (f) . We shall first enumerate the allowed final states for the initial system with a specific angular momentum. For this we need additional information regarding the parity of the initial system. Since parity is a good quantum number in strong interactions, the allowed final states would depend on the initial parity - i.e. the relative Ξ N parity. Since there is no experimental evidence to favour either odd or even Ξ N parity, we shall consider both possibilities.

Let us restrict ourselves to initial S waves say in the singlet spin state i.e. $^1S_0$ state. The final system necessarily has $J=0$ which can result from either $L_2=0$ and $J_2=0$ or $L_2=1$ and $J_2=1$. Here J_2 is the total angular momentum of the $\Sigma\Sigma$ system. The isotopic spin states of this system can be either symmetric ($I=0$) or antisymmetric ($I=1$) . We shall further restrict the relative orbital angular momentum between the two fermions to S waves only so that their orbital wave function has even parity $(-1)^\ell = 1$. Since the particles belong to the same charge multiplet, their intrinsic relative parity is necessarily even. And since the pion is a pseudoscalar with respect to

the nucleon, it should be in a P state with $L_2 = 1$ for $P_{\Xi N} = +1$ and in an S state with $L_2 = 0$ for $P_{\Xi N} = -1$. Thus in the former case $J_2 = 1$ and in the latter $J_2 = 0$ since the final $\vec{J} = \vec{J}_2 + \vec{L}_2 = 0$

Thus for $P_{\Xi N} = +1$, since $J_2 = 1$ for the $\Sigma\Sigma$ system, so that they are in a symmetric triplet spin state, the isospin wave function should be antisymmetric. Thus only the $I = 1$ state is allowed and the symmetric $I = 0$ state is forbidden. Similarly for $P_{\Xi N} = -1$ since $J_2 = 0$, the $I = 0$ state is allowed while the $I = 1$ state is forbidden. Similar arguments can be applied for various initial states, i.e. $^3S_1$, $^1P_1$, $^3P_1$ etc. and the allowed final states are listed in table (5) .

In each case the angular distribution is calculated with the help of (7) . For initial S waves the distribution is naturally isotropic. In the case of P waves however, we should naturally expect a θ dependence in the distribution (θ being the angle of emission of the pion). Thus for example for the transition

$$^1P_1 \rightarrow {}^1S_0\, p_1$$

we have

$$s = \tfrac{1}{2} \ , \ \ell = 1 \quad j = \tfrac{3}{2} \text{ or } \tfrac{1}{2} \ ; \ J_t = \tfrac{1}{2} \ , \ L_2 = 1 \ \ J_2 = 0$$
$$\text{and } J = 1 \text{ with } \nu = 2 \text{ and } 0$$

and $$W(LS) \sim 5/3 - \cos^2\theta$$

The distributions in each case are given in table (5) within brackets.

Table 5: Angular distribution ($\Xi^- + p \to \Sigma + \Sigma + \pi$)

Initial State	(ΞN) Final states Parity-even		(ΞN) Final states Parity-odd	
	$I=0$	$I=1$	$I=0$	$I=1$
$^1S_0$	—	$^3S_1 P_0$ (1)	$^1S_0 S_0$ (1)	—
$^3S_1$	$^1S_0 P_1$ (1)	$^3S_1 P_1$ (1)	—	$^3S_1 S_1$ (1)
$^1P_1$	—	$^3S_1 S_1$ (1)	$^1S_0 P_1$ $(\frac{5}{3} - x^2)$	$^3S_1 P_1$ $(\sqrt{3} - x^2)$
$^3P_0$	$^1S_0 S_0$ (1)	—	—	$^3S_1 P_0$ (1)
$^3P_1$	—	$^3S_1 S_1$ (1)	$^1S_0 P_1$ $(3x^2+1)$	$^3S_1 P_1$ $(3x^2+1)$
$^3P_2$	—	—	—	$^3S_1 P_2$ $(21x^2+13)$

From the table we may draw the following conclusions. If the initial state is in an S state, the distributions are isotropic for either parity. In case we allow initial P wave also we find that for $P_{\Xi N} = +1$ the distribution is always isotropic while for $P_{\Xi N} = -1$ it is of the form $a + b\cos^2\theta$ for all states except the $^3S_1 P_0$ state. Thus an isotropic distribution automatically implies $P_{\Xi N} = -1$.

We point out that the same applies to (f), (a) and (e) also. Of course, in (e) and (a) since the Pauli principle cannot be applied, there is no restriction on the allowed states of the two fermion system. However for (f) though the Pauli principle has to be applied the $I=1$ state is not possible so that from table (5) we find that the distinction between even and odd ΞN parity is more pronounced since isotropy is impossible for $P_{\Xi N} = -1$ for P waves. For (e) however the relative $\Sigma \Lambda$ parity has to be taken into account and also there will be additional states possible since the restrictions of the Pauli principle is removed. The angular distributions can be similarly calculated. It is felt that the study of these angular distributions would be of importance in the determination of the spin and parity of the resonances.

CHAPTER II.

ON THE SPIN OF THE Ξ [1)]

1. Introductory remarks

In the last chapter we have discussed the implications of the assignment of even and odd parity to the cascade particle with respect to the nucleon. In the entire discussions, we have assumed that the cascade is a particle with spin 1/2. However there is no clear cut experimental evidence in support of this and therefore the possibility of higher spins for the Ξ cannot be ruled out. We now wish to investigate some of the consequences if the Ξ were to be a spin 3/2 particle. Of course this assignment would immediately imply that the Ξ is not an 'elementary' particle which would be consistent with Chew's view that the whole conception of elementary particles may be meaningless for baryons and mesons with all 'particles' being bound states of one another. We now present some of the consequences of attributing spin 3/2 for the Ξ in strong and weak interactions. In Section (2) we study the reaction $\Xi^- + p \rightarrow \Lambda + \Lambda$ and in (3) the possibility of the double hyperfragment and the distribution of the decay pions. In section (4) the reaction $\Xi + p \longrightarrow K + K$ is studied and in(5) the decay of the cascade analysed.

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1) Alladi Ramakrishnan, G.Bhamathi, S.Indumathi, T.K. Radha and R.Thunga, Nuovo Cimento, 22 604, (1961)

2. The strong interactions $\Xi^- + p \rightarrow \Lambda + \Lambda$.

The most probable reactions when a Ξ particle comes to rest and collides with a nucleon as are

$$\Xi^- + p \rightarrow \Lambda + \Lambda \quad (1)$$
$$\rightarrow \Xi^- + p \quad (2)$$
$$\rightarrow \Xi^0 + n \quad (3)$$

The first reaction has been analysed in great detail in the case of spin 1/2 for the Ξ particle.[1] It is well-known that the spins of the two Λ particles have to be correlated because of the Pauli principle and the parity of the Λ does not play any role in the analysis since two Λ's are produced. It is quite reasonable to assume that the Ξ is captured from S state since the interaction is subsequent to the stopping of the Ξ in the hydrogen bubble chamber. We list below the allowed and forbidden transitions for spin 1/2 and 3/2 for the Ξ.

Each of these final states is characterised by a distinct pattern of polarization of the Λ which will be reflected in the Λ decay. If in the final state we rule out all orbital angular momenta higher than $\ell = 1$ we find that in the case of even ΞN parity and spin 3/2 the reaction cannot occur. Of course, the initial state should have $I = 0$ and thus we see that the two body ΞN hyper-fragment may be stable against decay via strong

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1) L.B.Okun et.al. Jetp, 7 , 862 (1959).

Transitions for the process $\Xi^- + p \rightarrow \Lambda + \Lambda$.

Spin	Parity	Initial state	Final state.
1/2	+	$^1S_0$	$^1S_0$
1/2	+	$^3S_1$	forbidden
1/2	−	$^1S_0$	$^3P_0$
1/2	−	$^3S_1$	$^3P_1$
3/2	+	$^3S_1$	forbidden
3/2	+	$^5S_2$	$^1D_2$
3/2	−	$^3S_1$	$^3P_1$
3/2	−	$^5S_2$	$^3P_2$

interaction not only for $I=1$ but also for $I=0$ if the ΞN parity is even and the Ξ has spin 3/2 . This suggests the possibility of a bound system of $\Xi^- p$ corresponding to the deuteron in a metastable state since it has no allowed final state into which it can decay. In such a case the splitting of the $(\Xi^- + p)$ and $(\Xi^0 + n)$ levels which comes basically from the difference in masses of Ξ^- and Ξ^0 is small compared to the splitting of the levels with $I=1$ $\left(\frac{\Xi^- p + \Xi^0 n}{\sqrt{2}}\right)$ and $I=0$ $\left(\frac{\Xi^- p + \Xi^0 n}{\sqrt{2}}\right)$ and the ΞN systems will be in a well defined state of ~~well defined~~ isotopic spin I .

It is important to note that the possibility of a bound ΞN system is greater with spin 3/2 for Ξ than with spin 1/2 since in the latter case for ΞN parity +1 the reaction can proceed through the $^1S_0$ state. De-excitation through

$$\Xi^- + p \rightarrow \Lambda + \Lambda + \gamma \qquad (4)$$

will occur rapidly but as in the case of Σ^0 , it is expected that the fragment might live long enough to have a reasonably well defined mass.

The analysis of the masses of the Ξ^- and Ξ^0 particles produced does not definitely indicate as to which is the heavier particle between the two. This is of vital interest since it plays an important role in the cusp behaviour of the cross-section for the reaction (1) . Various ~~xxxx~~ theories have been put forward to predict

sign of this mass difference in comparison with that of nucleon mass differences.[1]

We note that if $m_{\Xi^0} > m_{\Xi^-}$ then the charge exchange reaction (3) will have a threshold while of course the reaction (1) has no threshold. A cusp in the cross-section for a particular process occurs at energies corresponding to the threshold for a new channel. Thus it is natural that we must expect a cusp in the $\Lambda+\Lambda$ production cross-section at the energy corresponding to the threshold for the charge exchange reaction ~~(assuming $m_{\Xi^0} = m_{\Xi^-}$)~~.

It was shown by Baz and Okun[2] that one can determine $\Sigma\Lambda$ relative parity by measuring the cusp contributions to the reaction

$$\pi^- + p \rightarrow \Lambda + K \qquad (5)$$

at the threshold for ΣK production[3]. In our case then the angular distribution of the $\Lambda\Lambda$ system at $\Xi^0 n$ threshold should be very sensitive to the relative ΞN parity. The final particles of reaction (3) for spin 1/2 and 3/2 are expected to be in the S state since it has a threshold and hence are the same as the initial states for odd and even

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1) J.J.Sakurai, Phys. Rev., 114 , 1152 (1959).
2) A.N.Baz and L.B.Okun, Jetp , 8 , 526.
3) Experimental evidence for this cusp was reported by Alston et. al. at the Rochester Conference (1960).

ΞN parity. The final states for reaction (1) have already been listed in the table. To determine the exact behaviour of the cross-section we have to write the total S-matrix of the channel $I=0$ ($\Lambda\Lambda$ system has $I=0$) and $J=0$(or 1) and for a given ΞN parity in the form

$$S = \begin{vmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & \cdots & \cdots & \cdot \\ \vdots & & & \\ S_{N1} & \cdots & \cdots & \cdot \end{vmatrix} \qquad 6.$$

where the matrix element at the intersection of the i th column and j th line is the transition from the i th channel into the j th one. Here say $i=1$ corresponds to $\Lambda\Lambda$ $i=2$ to $\Xi^- p$ and $i=3$, Ξ^0+n..etc. Then we have

$$S = \begin{vmatrix} \Xi^-+p\to\Lambda+\Lambda & \Xi^-+p\to\Xi^0+p & \Xi^-+p\to\Xi^0+n\ \cdots \\ \Xi^-+p\to\Xi^-+p & \cdots & \cdots \\ \Xi^-+p\to\Xi^0+n & \cdots & \cdots \\ \vdots & & \end{vmatrix} \qquad 7.$$

and we would require the form of the matrix elements for the process (1), (2) and (3) which of course is not known. However if one looks at the angular distribution as a function of energy one could hope to deduce ΞN parity from the appearance of the cusp in the $^1S_0$ or ($^3P_0$, $^3P_1$) state for the case of spin 1/2 for Ξ . It is interesting to note that in the case of spin 3/2 the cusp behaviour should be more marked. We find that a crisp in the angular distribution

~~fm~~ the ^3P state indicates odd ΞN parity and for even parity it should be in the $^1D_2$ state. Thus if it is experimentally found that there is a cusp in the $\Lambda\Lambda$ production cross-section in $\Xi^- + p$ reactions at Mev then it automatically fixes the sign of the mass difference $m_{\Xi^0} - m_{\Sigma^-}$. Further our considerations show that a cusp in the $^1S_0$ state implies spin 1/2 for the Ξ and even ΞN parity while if it is in the $^1D_2$ state it clearly indicates that the spin of the Ξ is 3/2 and $P_{\Xi N} = +1$. However a cusp in the $^3P_1$ state while fixing the ΞN parity as odd cannot distinguish between spin 1/2 or 3/2 for the Ξ particle.

3. The double hyperfragment analysis

~~The~~ It is well-known that Ξ^- capture in nuclei can lead to double hyperfragments. Assuming spin 1/2 for Ξ a detailed analysis of the binding energies of double hyperfragments where the two Λ's form a singlet S state, has been calculated by Iwao[1]. If however the Ξ has spin 3/2 the Λ's will be in a relative P or D state depending on the odd or even ΞN relative parity. It is unlikely that bound light hypernuclei occur with Λ's in a D-state. We shalltherefore consider only the case of ΞN odd parity. The Ξ^- is assumed to be captured from rest and the excess of energy ~~of~~ released in the process $\Xi^- + p \rightarrow \Lambda + \Lambda$ will be shared by all the nucleons present. Under these assumptions, the

Table : 3 Binding Energies of double hyperfragments.

System	Binding Energy in MeV	
	Parallel Spin	Ant-parallel Spin
$\Lambda\Lambda H^3 (\Lambda\Lambda n)$	− 12.54	− 10.13
$\Lambda\Lambda nn$	− 8.51	− 8.9
$\Lambda\Lambda H^4$	− 9.66	− 6.4
$\Lambda\Lambda H^5$	− 8.03	− 6.69
$\Lambda\Lambda He^6$	− 1.07	− 1.77
$\Lambda\Lambda H^6$	− 1.55	2.82
$\Lambda\Lambda He^7 ({}^7_{\Lambda}Li)$	2.39	4.10
$\Lambda\Lambda He^8 ({}^8_{\Lambda\Lambda}Be)$	6.47	5.3.

binding energies of the double hyperfragments are given below in the Table3)

As should be expected there is a correlation between the two charged pions arising from the decay of the two Λ 's. To calculate this we need the density matrix in the combined spin space of the two Λ 's. Before constructing this density matrix we shall give a brief descriptionof the method of determining the joint distribution of the pions using the density matrix[1].

Since a wave function ψ will always describe a particle completely polarised in some direction, a partially polarized beam has to be a statistical mixture of pure states. If a basic set of states ψ^α is chosen so that we may write $\psi^\alpha = \sum_i a_i^\alpha \psi_i^\alpha$ the expectation value of any observable A is given by

$$\bar{A} = \sum_\alpha P(\alpha) \left(\psi^{\alpha\dagger}, A \psi^\alpha \right) \qquad 8.$$

where $P(\alpha)$ is the relative frequency with which each is represented in the beam, or we have

$$\bar{A} = \sum_\alpha \sum_{ij} P(\alpha) \left(a_i^{\alpha *} a_j^\alpha \right) \left(\psi_i^{\alpha *} A \psi_j^\alpha \right)$$

$$= \sum_\alpha \sum_{i,j} P(\alpha)\, a_i^{\alpha *} a_j^\alpha A_{ij} \qquad 9$$

$$= Tr(UA)$$

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1) R.H.Dalitz, Proc. Phys. Soc., 65 (A) , 175.

where U is the matrix with elements $U_{ij} = \sum_{\alpha} P(\alpha)\, a_i^{\alpha *} a_j^{\alpha}$.

From the properties of the probabilities $P(\alpha)$ we have

$$Tr(U) = 1 \,. \qquad (10)$$

All information regarding the beam under consideration are contained in the matrix U. For example the matrix U for a linear beam of spin 1/2 particles is given by $U = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{p})$ and the polarization p of the beam will be given by

$$p = \text{Trace}\,(\sigma \cdot U) \qquad (11)$$

This U is called the density matrix. In the spin space the density matrix for the final state is given by

$$U_f = T U_i T^{\dagger} / Tr(T U_i T^{\dagger}) \qquad (12)$$

where U_i is the density matrix of the initial state and T is the transition operator. The polarization after the collision will be

$$p(\vec{\sigma}) = Tr(\sigma U_f) = Tr(\sigma T U_i T^{\dagger}) / Tr(T U_i T^{\dagger}) \qquad (13)$$

Since we have assumed spin 3/2 for Ξ the basic matrices are of order $(2.3/2 + 1) \times (2.3/2 + 1) = 4 \times 4$ and we can conveniently choose the sixteen linearly independent γ matrices as our base. The density matrix U_i will be then given by

$$U_i = \sum_{\mu} \langle S^{\mu} \rangle S^{\mu} / \left(2\cdot\tfrac{3}{2}+1\right)\left(2\cdot\tfrac{1}{2}+1\right)$$

$$= \sum_{\mu} \frac{\langle S^{\mu} \rangle S^{\mu}}{8} \qquad (14)$$

where S^μ is the direct product $S^\beta \times \sigma$, S^β and σ being the corresponding matrices for spin 3/2 and 1/2 respectively.

The distribution in angle of the pion in the Λ decay is given by

$$P(\vec{\sigma})\, d\Omega = (1 + \alpha \vec{\sigma} \cdot \vec{p})\, d\Omega \qquad 15$$

where α is the asymmetry parameter and $\vec{p}$ the unit vector in the direction of the decay pion in the Λ rest frame. In the present case the joint distribution of the two pions is

$$P(\vec{\sigma}_1, \vec{\sigma}_2)\, d\Omega_1 d\Omega_2 = (1 + \alpha \vec{\sigma}_1 \cdot \vec{p}_1)(1 + \alpha \vec{\sigma}_2 \cdot \vec{p}_2)\, d\Omega_1 d\Omega_2 \qquad 16$$

where σ_1 and σ_2 represent the spins of the two Λ's and p_1 and p_2 the momenta of the decay pions in the rest system of the Λ's. Then the distribution function in pion angles is given by

$$P(\theta_1, \theta_2) = \frac{Tr[U_f\, \rho(\sigma_1, \sigma_2)]}{Tr(U_f)}$$

$$= \frac{Tr[T U_i T^\dagger\, \rho(\sigma_1, \sigma_2)]}{Tr[T U_i T^\dagger]}$$

$$= 1 + \frac{39}{14} \alpha^2 \vec{p}_1 \cdot \vec{p}_2 \qquad 17.$$

4. The reaction $\bar{\Xi}^+ + p \rightarrow K^+ + K^+$.

With very energetic beams of antiprotons that are available now it should be possible to produce cascade pairs naturally through the reactions

$$p + \bar{p} \rightarrow \Xi + \bar{\Xi} \qquad 18$$

which requires at threshold 1.8 BeV antiprotons in the laboratory system. We are particularly interested in the process

$$\bar{\Xi}^+ + p \rightarrow K^+ + K^+ \qquad 19$$

which appears as the third process in the Mandelstam representation for the process

$$K^- + p \rightarrow \bar{\Xi} + K^+ \qquad 20$$

which we have already considered in Chapter III of Part II. Barshay[1] has studied this process assuming spin 1/2 for the Ξ. Extending this to the case of spin 3/2, we find that the angular distribution of the final pair of bosons is sensitive to the spin and parity of the Ξ.

The final two particles are identical bosons and we find by Pauli principle since $I = 1$, that odd orbital angular momentum is ruled out for the final state. Since the spin of the Ξ is 3/2, the initial system can have spin $S = 1$ or $S = 2$. If $S = 1$, the $\ell = 0$ state cannot occur. However if $S = 2$ the $\ell = 0$ state leads to the $J = 2$ state which is allowed. With $\ell = 1$, $S = 1$ leads to $J = 0$ or 2 while $S = 2$ leads only to $J = 2$. The angular distribution for the transition say $^3D_2 \rightarrow {}^1D_2$ is given by

---

1) S. Barshay, Phys. Rev., 120, 265 (1960).

$$P(x) = \sum_m |C(l, s, J | 0 m) P_J^m(x)|^{2*}$$
$$= \sum_m |C(2 1 2 | 0 m) P_2^m(x^*)|^2 \quad (21)$$

We list in the table the allowed and forbidden transitions with the corresponding angular distributions of the final two bosons.

As before if we restrict ourselves to S waves only in the initial state we find from Table I that the reaction occurs only for odd parity and a final isotopic distribution does not distinguish between spin 1/2 and spin 3/2 .

Including P waves also in the initial state the distribution when , $^3P_0$, $^3P_2$ and $^5D_2$ states participate is found to have the same form but different coefficients from that corresponding to spin 1/2 (i.e.)

$$F(x) = a' + b' x^2 \quad (23a)$$

with

$$a' = 18|A|^2 + |B|^2 + 3|C|^2 + 6\sqrt{2}\, Re\, A^* B$$
$$b' = 3|B|^2 - 3|C|^2 - 18\sqrt{2}\, Re\, A^* B \quad (23)$$

The corresponding values for a and b in the case of spin 1/2 are given by

$$a = 18|A|^2 + |B|^2 + 6\sqrt{2}\, Re\, A^* B$$
$$b = 3|B|^2 - 18\sqrt{2}\, Re\, A^* B \quad (24)$$

where A, B and C are the amplitudes describing the transitions from the $^3P_0$, $^3P_2$ and $^5P_2$ states respectively.

---

* A detailed discussion with spin has already been given in the previous chapter.

Table 2: Angular distribution for $\Xi^+ + p \rightarrow K^+ + K^+$

Spin	Parity	Initial State	Final ℓ	Angular Distribution $x = \cos\theta$
1/2	−	$^1S_0$	0	1
	−	$^1D_2$	2	$1-6x^2+9x^4$
	−	$^3D_2$	2	x^2-x^4
	+	$^3P_0$	0	1
	+	$^3P_2$	2	$1+3x^2$
3/2	+	$^3P_0$	0	1
	+	$^3P_2$	2	$1+3x^2$
	+	$^5P_2$	2	$1-x^2$
	−	$^5S_2$	2	1
	−	$^5D_0$	0	1
	−	$^5D_2$	2	$4-9x^2+9x^4$
	−	$^3D_2$	2	x^2-x^4
	−	$^5D_4$	4	$45x^4-10x^2+13$

when D waves are also included the distribution is for $P_{\Xi N} = -1$ is of the form

$$F(x) = c' + d'x^2 + e'x^4 + f'x^6 \quad \text{for spin } 3/2$$

25

and

$$F(x) = c + dx^2 + ex^4 \quad \text{for spin } 1/2$$

26

The x^6 dependence in $F(x)$ for spin 3/2 arises mainly because of the form $^5D_4 \rightarrow {}^1G_4$ reaction.

However if we consider the reaction

$$\bar{\Xi}^0 + p \rightarrow K^0 + K^+$$

27

since there is no restriction on the final state orbital angular momentum to be even from the angular distribution it is possible to distinguish between spin 3/2 and 1/2 for Ξ with initial only S and P waves.

5. Decay of the Ξ.

It is well-known that the up-down asymmetry in the decay of oriented nuclei established the non-conservation of parity in weak interactions. For the observation of such asymmetries even in the decays of strange particles all that is needed is that they should be polarized at the time of production. This happens to be the case for Σ and Λ resulting from associated production. An exhaustive study

of Λ decay seems to favour the postulate of a universal $V-A$ theory for weak interactions. Any additional information in this direction should naturally stem from the study of the Ξ decay. It is further expected that the Ξ produced in $K^- p$ reactions will be partially polarized in the plane of production so that the observation of an up-down asymmetry in the subsequent decay

$$\Xi^{o(-)} \longrightarrow \Lambda + \pi^{o(-)}$$

is possible. The further decay of Λ into $p + \pi^-$ would give a measure of the spin of the Λ and hence a determination of the decay parameters of the Ξ seems quite feasible.

We shall now analyse the decay distribution of the Ξ assuming that its spin is $3/2$. For the Ξ decaying from rest, the final states will be a linear combination of at most two states with opposite parity differing in orbital angular momentum by one unit. If the spin of Ξ is $3/2$, the allowed final states are P and D waves only. Let us assume that the Ξ is completely polarized along the $+Z$ direction and let $V^{1/2}$ and $V^{-1/2}$ be the spin part of the Λ wave function with spin up and spin down respectively. The space part of the Λ wave function is $Y_\ell^m(\theta, \phi)$ where θ is the angle that Λ makes with the Z direction and the corresponding projection. Thus the wave function of the decaying system is

$$\psi = A_p V^{1/2} Y_1^1 + A_D\left(-\frac{1}{\sqrt{5}} V^{1/2} Y_2^1 + \frac{2}{\sqrt{5}} Y_2^2\right)$$

28

where the coefficients are the Clebsch-Gordon coefficients for the combination of a spin 1/2 (Λ) and an angular momentum 2 (D waves) into a resultant spin 3/2 (Ξ). And the square of the modulus of ( ) gives

$$\frac{3}{8\pi} \sin^2\theta \left[|A_p|^2 + |A_D|^2 - 2 \mathrm{Re}\, A_p^* A_D \cos\theta \right] = a \sin^2\theta + b \sin^2\theta \cos\theta \qquad 29.$$

The experimentally measured quantity

$$X = N_{up} - N_{down} / \tfrac{1}{2}(N_{up} + N_{down}) \qquad 30$$

where

$$N_{up} = \int_0^{\pi/2} (a \sin^2\theta + b \sin^2\theta \cos\theta) \sin\theta \, d\theta = \tfrac{2}{3} a + b/4 \qquad 31$$

and

$$N_{down} = \int_{\pi/2}^{\pi} (a \sin^2\theta + b \sin^2\theta \cos\theta) \sin\theta \, d\theta = \tfrac{2}{3} a - b/4 \qquad 32$$

and hence

$$X = \tfrac{3}{4} b/a = -\tfrac{3}{2} \mathrm{Re}\, A_P^* A_D / |A_P|^2 + |A_D|^2 \qquad 33$$

is the asymmetry parameter. If initially the Ξ is only partially polarised, then the observed quantity will be where P is the polarisation of the Ξ .However, the Λ emitted in this decay again decays into a p and π^- and a measurement of the correlation between the directions of motion of Λ and the direction of motion of the proton, each measured in the rest system of the parent particle determines $\alpha\alpha_\Lambda$ where α_Λ is the asymmetry parameter of the Λ decay. A non-zero value for $\alpha\alpha_\Lambda$ indicates that parity is non-conserved in both the decays. The actual value of this quantity would also throw light on the nature of the couplings besides the spin of the cascade.

CHAPTER IV :

ON SOME DECAY MODES OF THE K^* RESONANCES[1]

1. Introductory remarks

One of the most important sources of information regarding the recently discovered resonant states of elementary particles is their decay modes. Besides the strong decay modes of these resonant states by which they are usually identified, it is also possible that they have other modes of decay with finite though small branching ratios. A theoretical investigation of such possible modes viz., a study of their decay rates and the angular correlations and energy spectra of the decay products is expected to provide a better understanding of these states. We have here made a systematic study of possible two and three particle decay modes of the two K^* resonances - a broad one at 880 Mev and width 60 Mev and a very narrow one at 730 MEv. ~~and w~~ The spin and parity assignments of these have not been experimentally established but it is expected that the former has 0^+ and the latter 1^-. However we shall consider both possibilities - scalar and vector for the two K^*.

2. Calculations

2 A. Two particle decay modes.

The dominant strong decay mode of K^* is

$$K^* \rightarrow K + \pi$$

The decay rate for vector K^* is calculated from the interaction term

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1. To be submitted to the Nuovo Cimento

$$\left(\phi_\pi \partial_\mu \varphi_K - \varphi_K \partial_\mu \phi_\pi\right) B_\mu \qquad 2.$$

which yields for the matrix element in momentum space

$$T = (q_1 - q_2)_\mu \cdot e_\mu \qquad 3.$$

where q_1 and q_2 are the momenta of the outgoing K and π and e_μ the polarization 4-vector of the initial K^*

Thus the decay rate

$$d\omega = \frac{1}{12}\left(\frac{g^2}{4\pi}\right)\left(\frac{1}{M^2 - m^2 + \mu^2}\right)\left[M^2 - 2(m^2+\mu^2) + \frac{(m^2-\mu^2)^2}{M^2}\right]^{3/2} \qquad 4$$

where $M = M_{K^*}$, $m = m_K$ and $\mu = m_\pi$. In the case of a scalar K^* the term is $\varphi_{K^*}\varphi_K\phi_\pi$ and the decay rate is given by

$$d\omega = m^2 \frac{g^2}{4\pi}\frac{1}{8M^2}\left[M^2 - 2(m^2+\mu^2) + \frac{(m^2-\mu^2)^2}{M^2}\right]^{1/2} \qquad 5$$

where m is a constant of the dimension of mass.
Besides this dominant mode, the K^* may also have the electromagnetic decay mode

$$K^* \longrightarrow K + \gamma \qquad 6$$

which is of course possible only if K^* has spin 1. A possible mechanism for this decay mode would be the decay of the K^* into a K and a vector meson V, presumably a ρ or ω (or ζ if this is 1^-), followed by a direct coupling of the V meson with a photon. We shall assume this to be the dominant mechanism. The V-γ coupling may be related to the nucleon form factors, the dominant contributions to which are expected to arise from such vector mesons V. The K^*KV coupling is unknown but is expected to play a role

in other decay modes of the K^* which can be compared with the mode $K^* \to K + \gamma$

The matrix element for $V \equiv \rho$ is

$$M = f_{K_V K\rho}\, \varepsilon_{\lambda\mu\nu\tau}\, e_\lambda(K_V)\, p_\mu\, e_\nu(\gamma)\, q_\tau(\rho) \frac{1}{q^2 - m_\rho^2}\left(\frac{e m_\rho^2}{2\gamma_\rho}\right) \quad 7.$$

where $\varepsilon_{\lambda\mu\nu\tau}$ is the antisymmetric tensor of rank 4 and the decay rate

$$\Gamma = \frac{1}{96}\left[\frac{f^2_{K_V K\rho}}{4\pi}\right]\left[\frac{e^2}{4\pi}\right]\left[\frac{\gamma_\rho^2}{4\pi}\right]^{-1} \frac{(M^2 - m^2)^3}{M^3(M^2 + m^2)} \quad 8.$$

$= 10$ kev if $\frac{\gamma_\rho^2}{4\pi} \sim \frac{1}{2}$ and $\frac{f^2}{4\pi} = \frac{g^2}{4\pi M^2} \sim \frac{1}{M^2}$

2-B. Three particle decay modes. The possible decay modes with three particle final state considered here are

$$K^* \to K + \pi + \pi \qquad (a) \qquad 9.$$
$$\to K + \pi + \gamma \qquad (b)$$

(a) is possible only for the higher mass K^* and (b) will be allowed for both the K^* resonances.

The phenomenological matrix element for a vector K^* for (a) is

$$M = f\, \varepsilon_{\lambda\mu\nu\rho}\, e_\lambda(K^*)\, q_{1\mu}\, q_{2\nu}\, q_{3\rho} \quad 10.$$

where q_1, q_2 and q_3 are the four-momenta of K and the two pions and the decay rate

$$\Gamma = f^2 \text{ const} \int d\omega_1\, d\omega_2\, |M|^2_{av}$$
$$= (\text{const})\, \frac{2}{3} M^2 \int d\omega_1\, I(\omega_1) \quad 11$$

where $I(\omega_1) = \int d\omega_2\, q_2^2\, q_3^2 \sin\theta_{23}$

It is found to be more convenient to evaluate the integral graphically, and $I(\omega_1)$ versus ω_1 is plotted and the energy spectrum of the K meson or the pions may be found. The diagram show the energy spectra that would result (i) from phase spare alone, (ii) with an effective point interaction. A possible mechanism for the decay could be with either a ρ or σ intermediate state or with the lower K^* as the intermediate state as seen in Fig. 2 (1).

It can be shown from a straight forward calculation that the energy spectrum from phase space is given by

$$\frac{\partial w}{\partial \omega_{\pi_2}} = \frac{2 q_2 \sqrt{a^2 - m^2(1+M^2-2M\omega_2}}{M^2+1-2M\omega_2} \; ; \quad a = \frac{1}{2}(M^2+m^2-2 \qquad (12)$$

where M is the mass of the K^*, m that of K and $(\vec{q}_1, \omega_1)$, $(\vec{q}_2, \omega_2)$ and $(\vec{q}_3, \omega_3)$ are the momenta and energies of the K and two pions respectively.

Again, the pion energy spectrum using a phenomenological point interaction is given by

$$\frac{\partial w}{\partial \omega_{\pi_{23}}} = q_2^2 \int d\omega_1 \, q_3^2 \sin^2 \theta_{23} \qquad (13)$$

which may be evaluated graphically. The results are shown in figure (2).

Assuming the f intermediate state as a possible mechanism, we find for the corresponding Feynman diagram, the decay rate

$$\omega = (\ ) \int_{m}^{W_\pi} q_2^2\, dw_2 \int \frac{dw_1 \quad q_3^2 \sin^2\theta_{23}}{\left[\frac{M^2+m^2-m_p^2}{2M} - w_1\right]^2}$$

And the pion energy spectrum 14

$$\frac{\partial w}{\partial w_{\pi_2}} = (\ ) \int dw_1 \frac{q_2^2\, q_3^2 \sin^2\theta_{23}}{\left[\frac{M^2+m^2-m_p^2}{2M} - w_1\right]^2} \qquad 15$$

The integral is evaluated graphically and the results plotted in figure (2). In these calculations, it has been considered worthwhile to focus our attention on energy spectra and angular correlations rather than decay rates since the couplings are unknown in most cases.

The angular correlation between the two pions is given by

$$\frac{\partial^2 w}{\partial w_{\pi_2}\, \partial(\cos\theta_{23})} = \frac{q_2^2\, q_3^2 \sin^2\theta_{23}}{w_1 \left[\frac{M^2+m^2-m_p^2}{2M} - w_1\right]^2} \qquad 16$$

~~and is shown in figure ( )~~

Similar calculations may be carried out with the lower mass K_V^* as the intermediate state. The corresponding Feynman diagram is given by interchanging π_2 and K in (1.i).

For the decay mode (b) we calculate the pion and photon energy spectra and the $\pi\gamma$ angular correlation. As before for K_S^*

$$\frac{\partial \omega}{\partial \omega_\pi} = \int d\omega_K \frac{q_K^2}{(W_\pi - \omega_\pi)^2} \qquad 17$$

in the case of diagram (9.i) and

$$W_\pi = \frac{M^2 + \mu^2 - m^2}{2M} = 2.26 \qquad 18$$

is the maximum energy of the pion. Similarly $\frac{\partial \omega}{\partial \omega_\gamma}$ is also found and the results are plotted in figure (11) . The angular correlation is given by

$$\frac{\partial^2 \omega}{\partial \omega_\pi \partial(\cos\theta_{\pi\gamma})} = ( ) \frac{q_\pi q_\gamma q_K^2}{\omega_K (2.26 - \omega_\pi)^2} \qquad 19$$

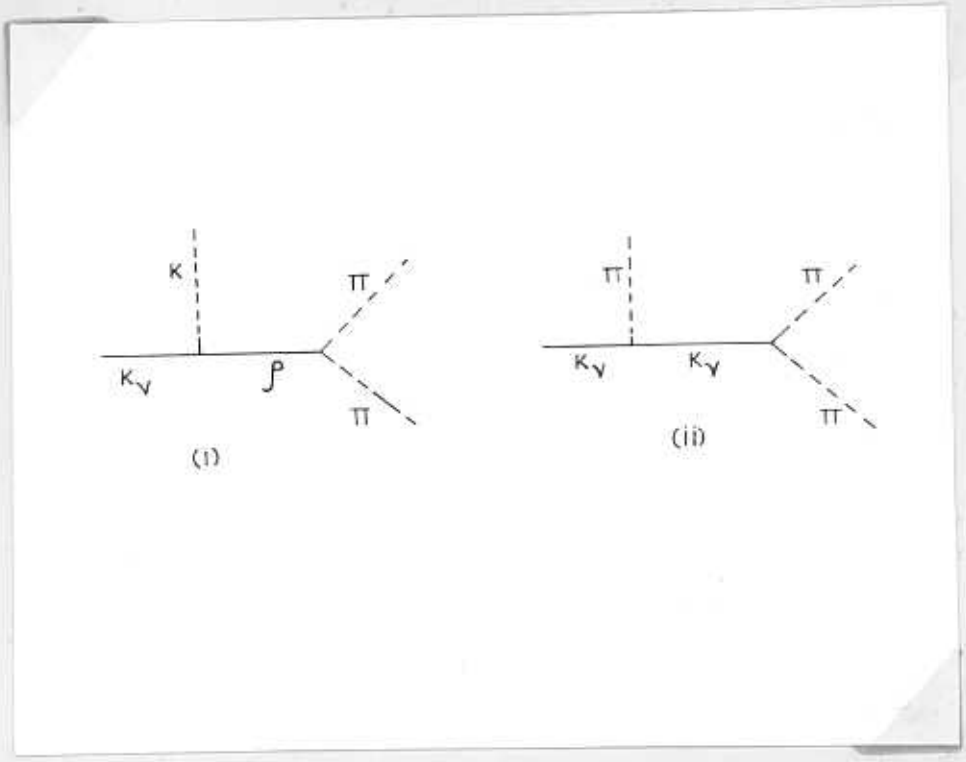
and

$$\cos\theta_{\pi\gamma} = \frac{q_K^2 - q_\gamma^2 - q_\pi^2}{2 q_\pi q_\gamma} \qquad 20$$

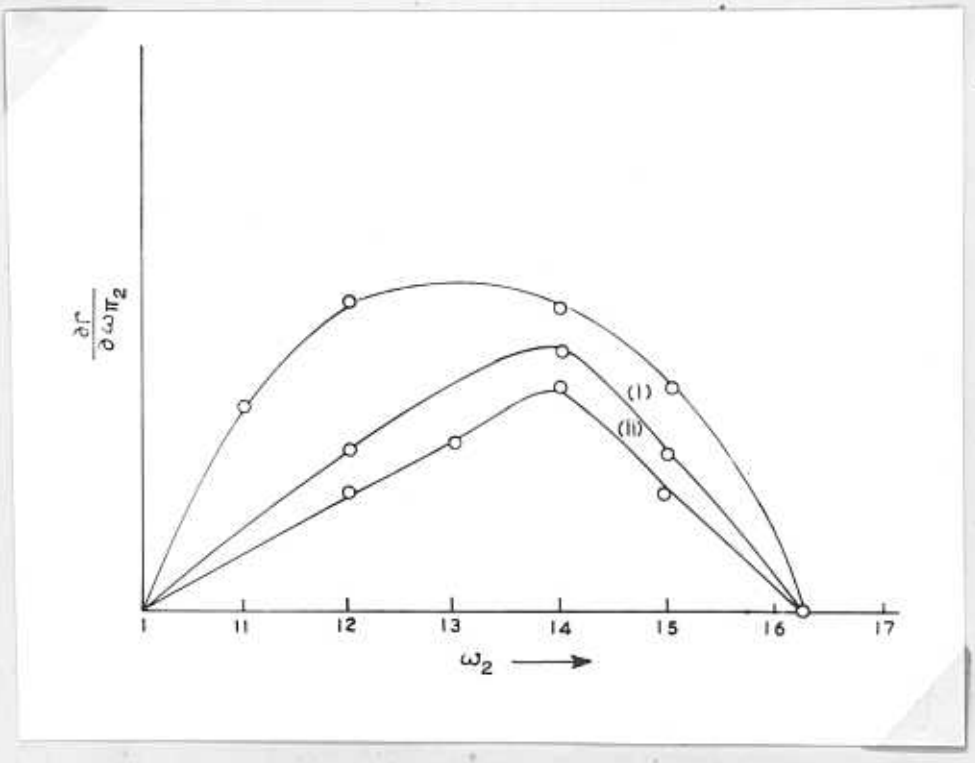
This may be calculated for specific values of ω_π .

The calculations can be repeated for a pion-intermediate state obtained by interchanging the π and K . The pion energy spectrum in this case has been shown in figure (13).

While it is true that these are of no quantitative significance, it is felt that the qualitative features of the curves will help in the understanding of the mechanism for the decay modes of these resonant states.



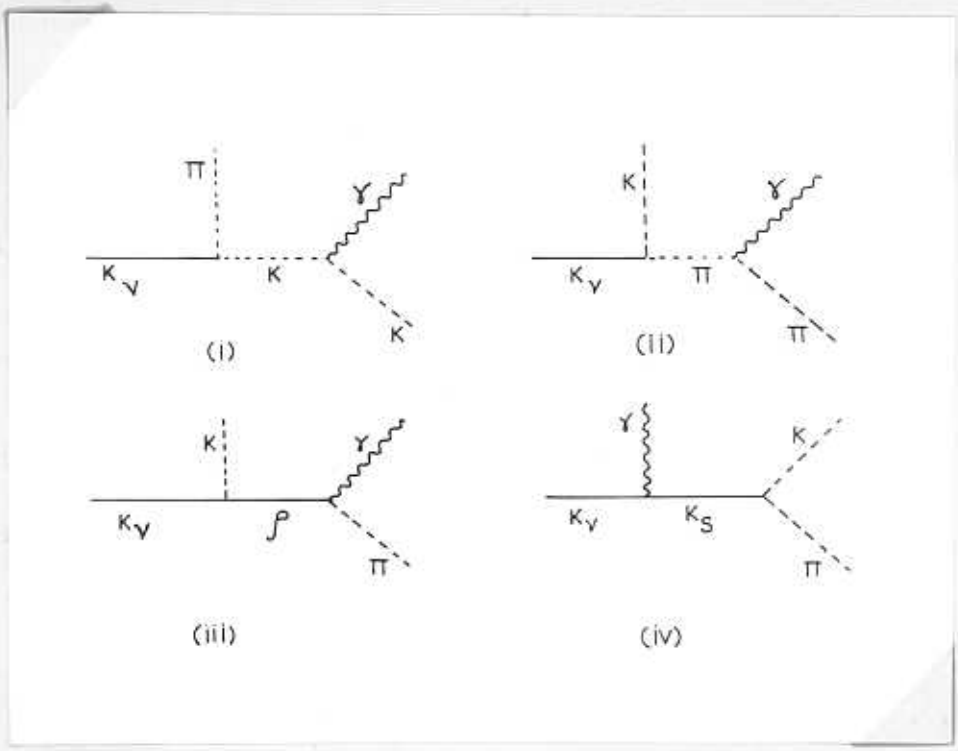
1

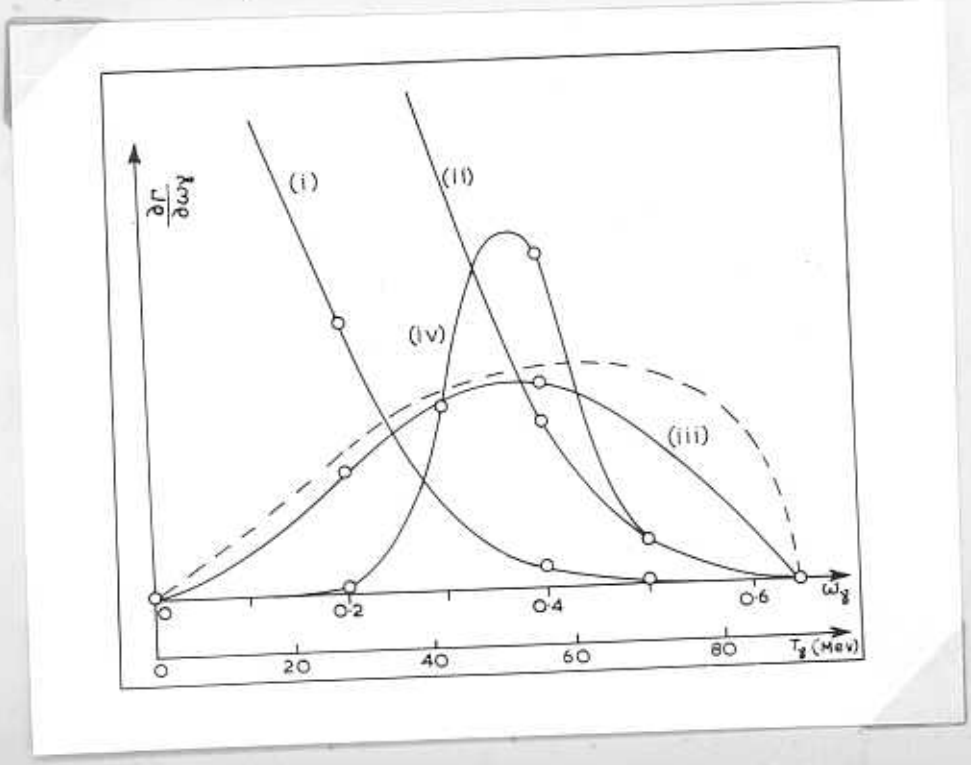


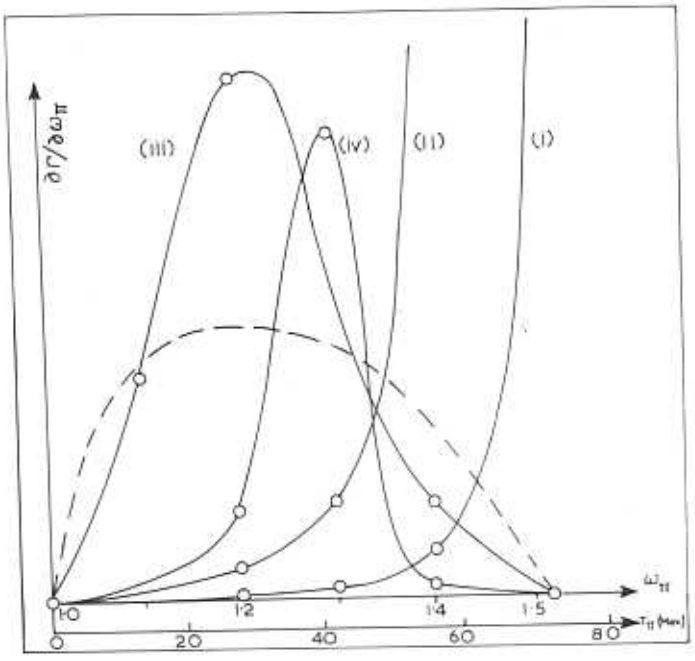
(ii) point interaction

9

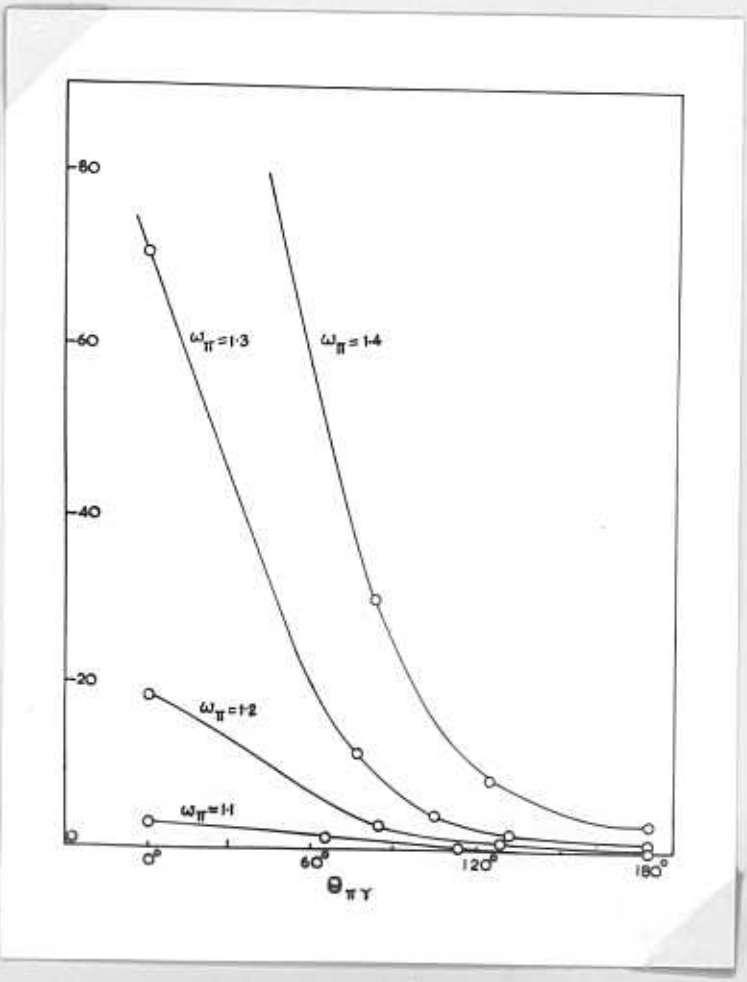
vectors K^*


π
γ
K_V
K
K
(i)
K
γ
K_V
π
π
(ii)
K
γ
K_V
ρ
π
(iii)
γ
K
K_V
K_S
π
(iv)

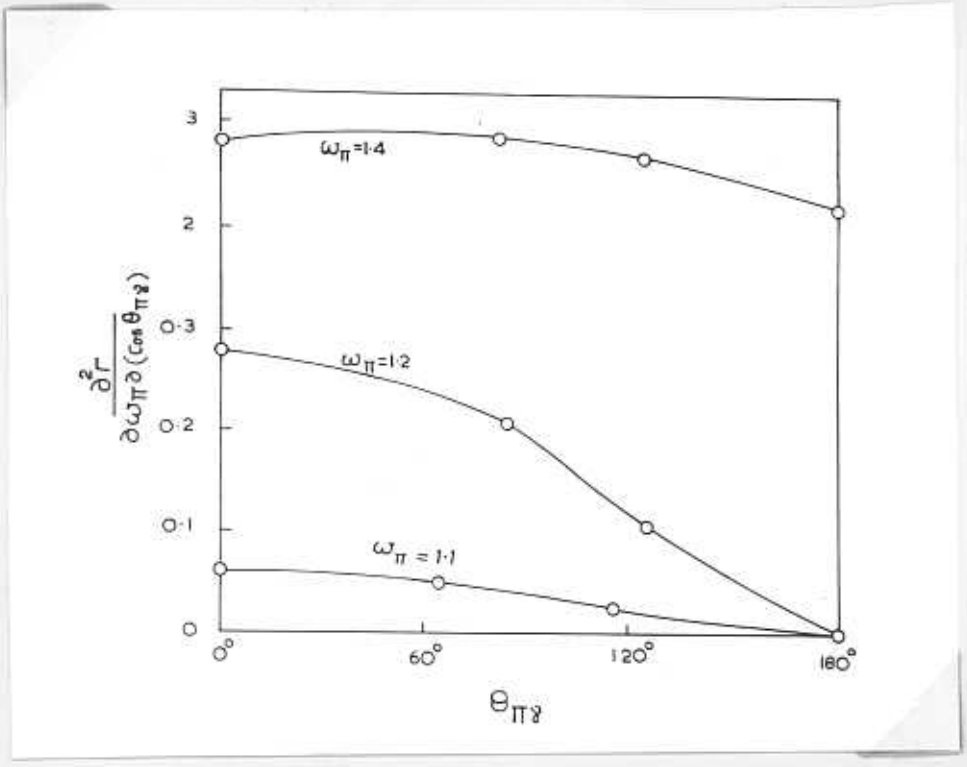

$\frac{\partial \Gamma}{\partial \omega_\gamma}$
(i)
(ii)
(iv)
(iii)
0·2
0·4
0·6
ω_γ
0
20
40
60
80
T_γ (Mev)


∂Γ/∂ω_π
(III)
(IV)
(II)
(I)
ω_π
1·0
1·2
1·4
1·5
τ_π (μsec)
20
40
60
80

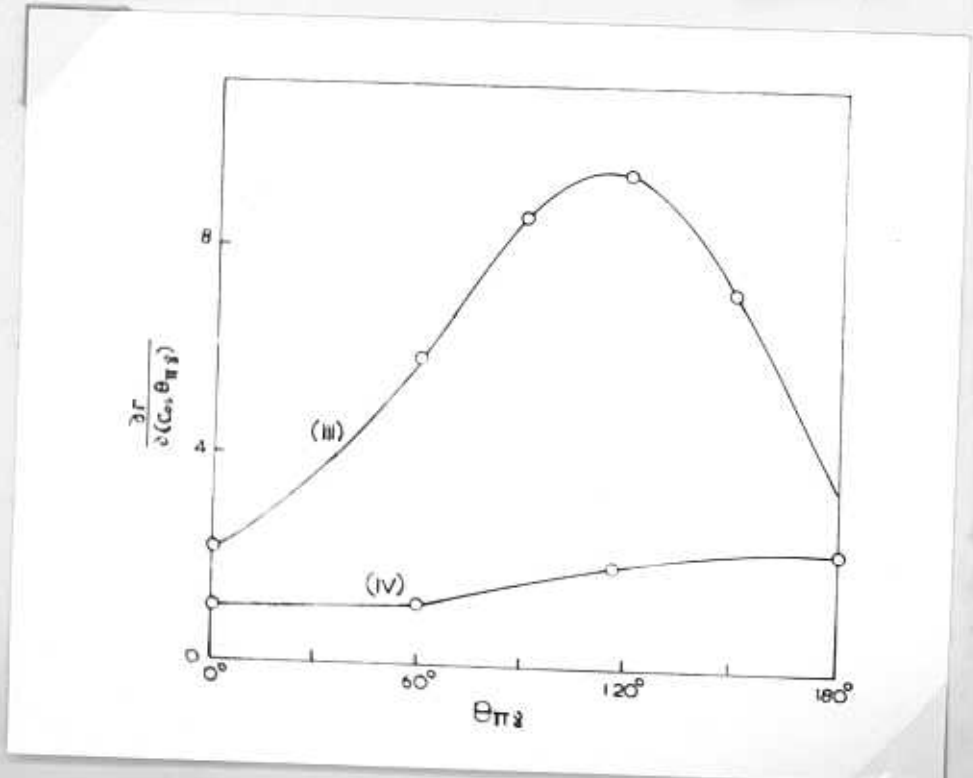
5

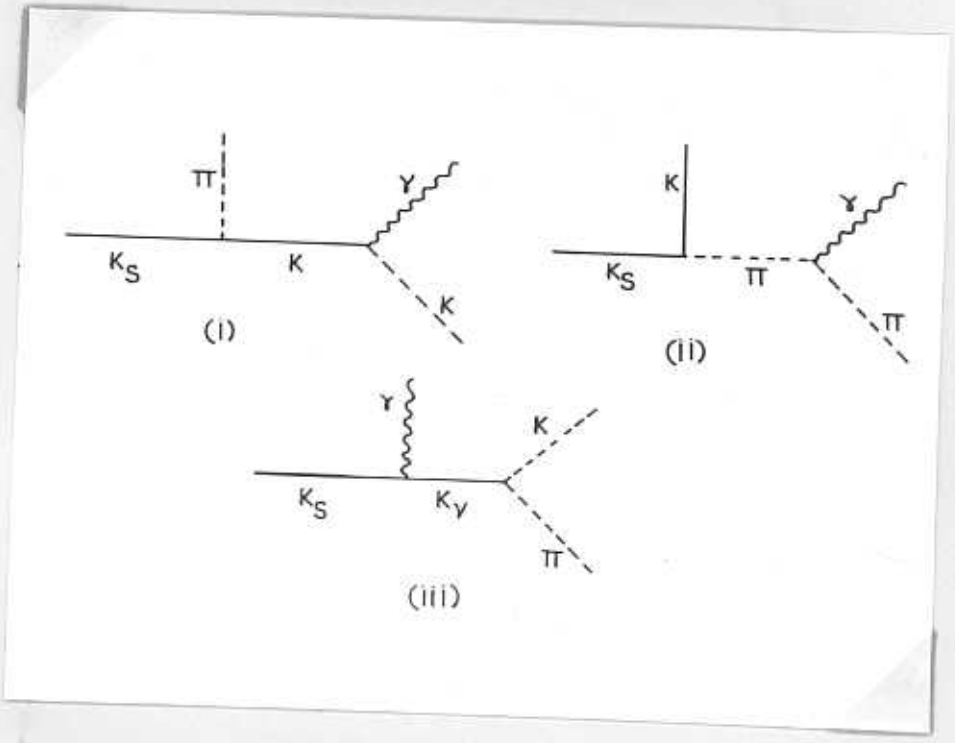

80
60
40
20
ω_π = 1·3
ω_π = 1·4
ω_π = 1·2
ω_π = 1·1
0°
60°
120°
180°
Θ_πγ

6

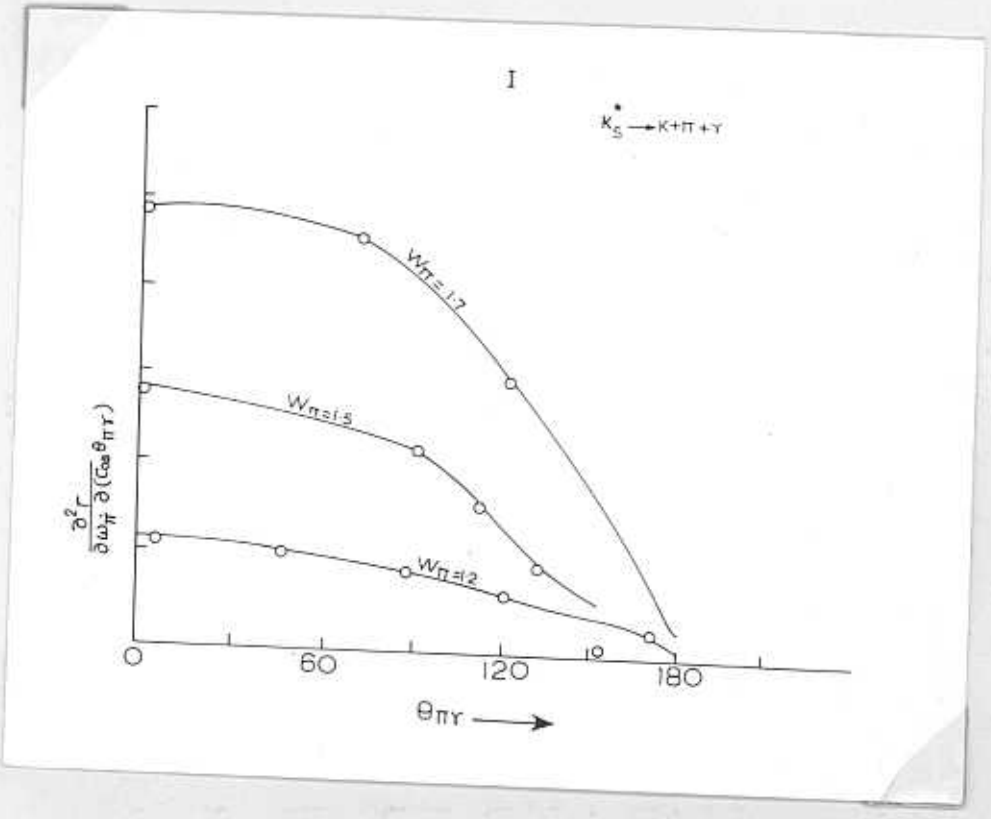

$\frac{\partial^2 r}{\partial \omega_\pi \partial (\cos\theta_{\pi\delta})}$
3
2
0·3
0·2
0·1
0
ω_π=1·4
ω_π=1·2
ω_π = 1·1
0°
60°
120°
180°
$\theta_{\pi\delta}$

7


$\frac{\partial r}{\partial(\cos\theta_{\pi\delta})}$
8
4
0
(III)
(IV)
0°
60°
120°
180°
$\theta_{\pi\delta}$

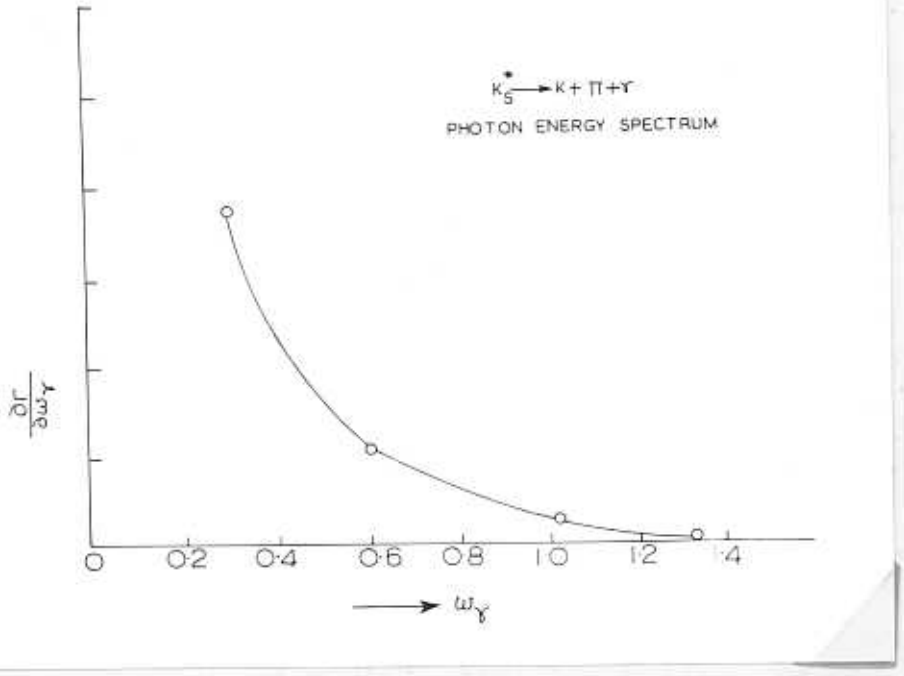

π
γ
K_S
K
K
(i)
K
γ
K_S
π
π
(ii)
γ
K
K_S
K_γ
π
(iii)

9


I
$K_S^* \rightarrow K+\pi+\gamma$
$W_{\pi\pi}=1.7$
$W_{\pi\pi}=1.5$
$W_{\pi\pi}=1.2$
$\frac{\partial^2 \Gamma}{\partial \omega_{\pi\gamma} \partial(\cos\theta_{\pi\gamma})}$
0
60
120
180
$\theta_{\pi\gamma}$

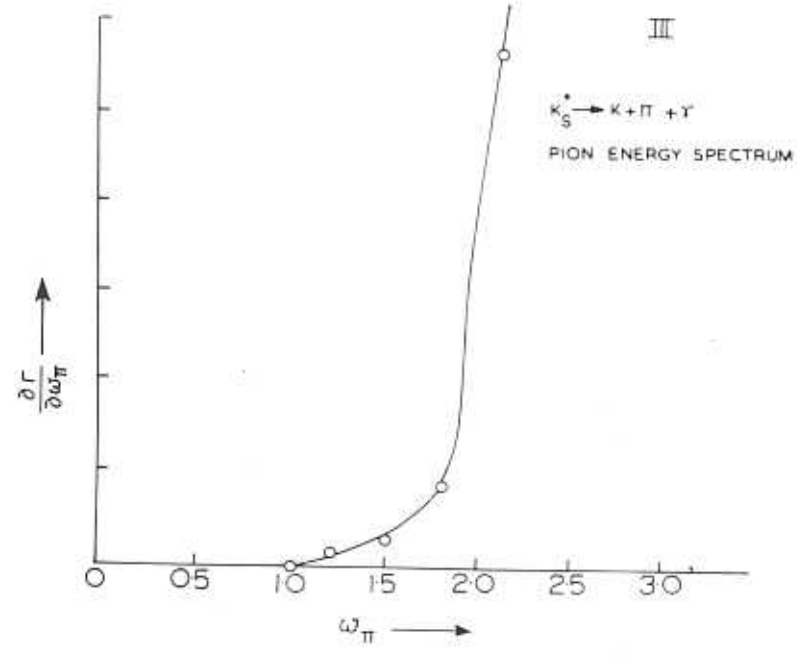
10

II

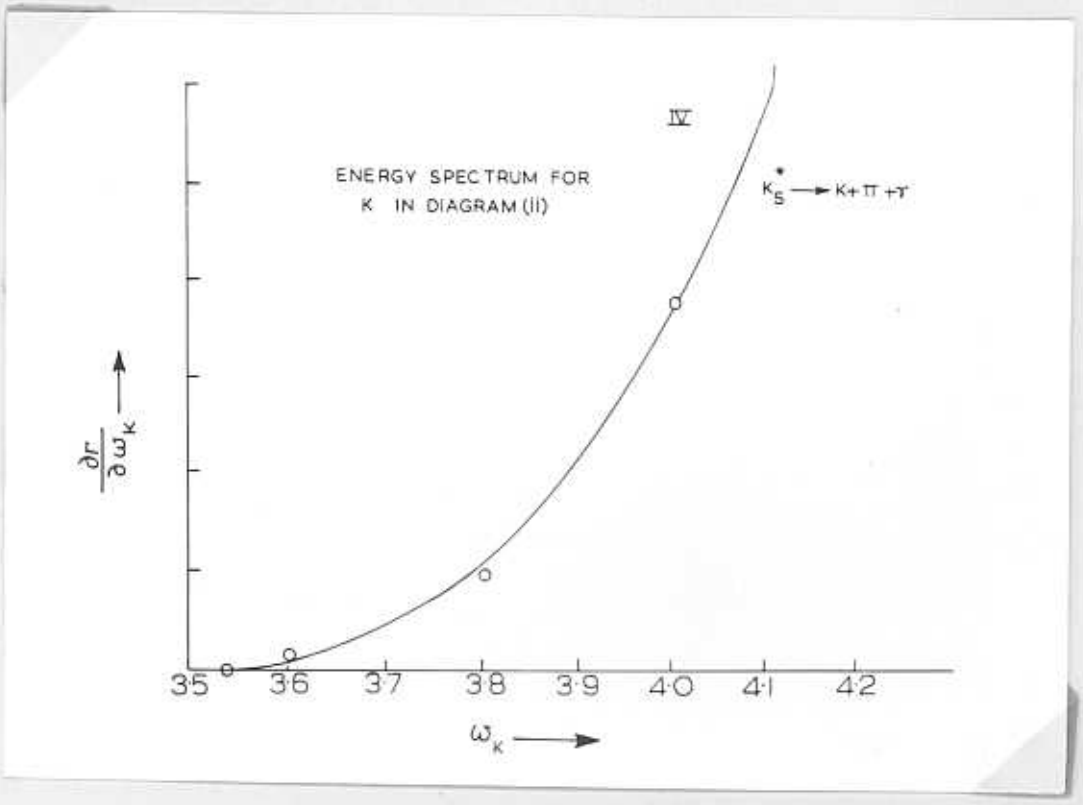

$K_S^* \rightarrow K + \pi + \gamma$
PHOTON ENERGY SPECTRUM
$\frac{\partial \Gamma}{\partial \omega_\gamma}$
0
0·2
0·4
0·6
0·8
1·0
1·2
·4
ω_γ

11

III


$K_S^* \rightarrow K + \pi + \gamma$
PION ENERGY SPECTRUM
$\frac{\partial \Gamma}{\partial \omega_\pi}$
0
05
10
1·5
2·0
25
30
ω_π

12


IV
ENERGY SPECTRUM FOR
K IN DIAGRAM (ii)
$K_S^* \longrightarrow K + \pi + \gamma$
$\frac{\partial r}{\partial \omega_K}$
3·5
3·6
3·7
3·8
3·9
4·0
4·1
4·2
ω_K

13

CHAPTER IV

A NOTE ON BARYON-BARYON INTERACTION*

Okun et. al.[1] have considered Ξ^- capture bya proton leading to a $\Lambda\Lambda$ system. In this chapter we study the process of a $(\Xi^- n)$ collision giving rise to hyperon systems, the possibility of a bound state in the final system being also envisaged. The implication arising from the various parity assignments for the hyperons is discussed and the decay distribution of the possible bound system is analysed.

We consider in particular the reactions. The threshold energy for the reaction (2) of

$$\Xi^- + n \rightarrow \Xi^- + n \quad (1)$$
$$\rightarrow \Sigma^- + \Lambda \quad (2)$$
$$\rightarrow \Lambda + \Lambda + \pi^- \quad (3)$$
$$\rightarrow \Sigma^- + \Sigma^0 \quad (4)$$

in $\Xi^- n$ collisions is given by equating the invariant four vector products in the laboratory and centre-of-mass system of the Ξ and n . Assuming the reaction to take place in a hydrogen bubble chamber, we have in the laboratory system

$$(\text{Total energy})^2 - (\text{total momentum})^2$$
$$= \left(E_\Xi + M_N\right)^2 - \left(p_\Xi\right)^2 = M_\Xi^2 + M_N^2 + 2E_\Xi M_N \quad 5$$

In the centre-of-mass system of the final particles we have at threshold

$$(\text{Total energy})^2 - (\text{total momentum})^2$$
$$= \left(M_\Xi + M_\Lambda\right)^2 \quad 6$$

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* G.Bhamathi, S.Indumathi, T.K.Radha and R. Thunga, Progress of Th. Physics, V. 25 , 870 (1961)

1) Okun et. al., JETP , 7, 862 (1958).

Thus we have the threshold kinetic energy of the Ξ given by

$$K_{\Xi} = E_{\Xi} - M_{\Xi} = \frac{(M_{\Sigma} + M_{\Lambda})^2 - (M_{\Xi} + M_N)^2}{2 M_N} - M_{\Xi}$$

$$= 52.5 \text{ Mev}$$

Similarly the thresholds for processes (3) and (4) are given by 110.8 , 134 Mev , respectively. The branching ratios cannot be directly calculated but on assuming charge independence, and the Sakata model for the final state particles the ratios for the processes (2) and (3) are found to be

$$\Sigma^- + \Lambda^0 : \Lambda^0 + \Lambda^0 + \pi^- :: 1 : 1$$

We assume the $(\Xi^- n)$ system is scattered from the S -state, the $(\Xi^- n)$ relative parity is even and the incident system has $J = s = 1$.

If $(\Lambda^0 \Lambda^0)$ is assumed to be in the S -state, this state is $^1S_0$ according to Pauli's exclusion principle, hence both angular momentum and parity cannot be conserved when the final π is in the S-state with respect to the centre of mass of the $(\Lambda^0 \Lambda^0)$ system.

However conservation of parity and angular momentum is possible when P wave π is emitted with respect to the centre of mass of the $(\Lambda^0 \Lambda^0)$ system.

We shall now consider the possibility of Σ^- and Σ^0 in (4) forming a bound state. The conservation of I spin leads to the $I = 1$ state for the $(\Sigma^- \Sigma^0)$ system which is antisymmetric. Thus the product of the space and spin wave functions should be symmetric, i.e. for $\ell = 0$ it is in the

triplet spin state. This system is analogous to a deuteron and 235 since the Σ's form a multiplet the force between them can be due to the exchange of a single pion.

Assuming the values for the depth of the phenomenological potential used in the deuteron theory, i.e. $V_0 = 25$ Mev, we obtain the binding energy of the ($\Sigma^-\Sigma^0$) system to be 4.66 Mev.

It is expected that the $(\Xi^0 p)$ collision leads to the bound system of ($\Sigma^+\Sigma^0$). Due to charge independence the $(\Sigma^+\Sigma^0)$ and ($\Sigma^-\Sigma^0$) systems will have the same energies and other parameters. However the distribution of the decay pions of the two systems will be entirely difference since Σ^+ has a decay mode, the ($p\pi^0$) which exhibits maximal asymmetry while the ($n\pi^+$) mode of Σ^+ and the ($n\pi^-$) mode of the Σ^- have no asymmetry.

In both systems ($\Sigma^+\Sigma^0$ and $\Sigma^-\Sigma^0$) the Σ^0 will decay instantaneously into $\Lambda+\gamma$ and the pions arising from the decay of the Λ^0 and the charged Σ will be correlated. Incidentally it would be of interest to determine the energy of the γ quanta arising from the Σ^0 decay since it would provide an estimate of the binding energy of the $\Sigma^\pm\Sigma^0$ system. In the case of the decay of the Λ particle the distribution of the decay pion is given by

$$P(\sigma)\,d\Omega = (1+\alpha\vec{\sigma}\cdot\vec{p})\,d\Omega \qquad (8)$$

where $\vec{\sigma}$ is the polarisation of the Λ, $\vec{p}$ the unit vector along the direction of the pion momentum in the rest system of the Λ and α the asymmetry parameter which is known to be $|\alpha| > 0.7$ for both the neutral and charged modes. In the case of a ($\Sigma^-\Sigma^0$) system the decay mode ($n\pi^-$) exhibits no asymmetry, hence, the joint distribution of the two points which

is given by

$$P(\vec{\sigma}_1 \vec{\sigma}_2)\, d\Omega_1 d\Omega_2 = (1+\alpha_1 \vec{\sigma}_1 \cdot \vec{p}_1)(1+\alpha_2 \vec{\sigma}_2 \cdot \vec{p}_2) d\Omega_1 d\Omega_2 \qquad 9.$$

reduces to

$$P(\vec{\sigma}_1 \vec{\sigma}_2)\, d\Omega_1 d\Omega_2 = (1+\alpha \vec{\sigma}_1 \cdot \vec{p}_1)\, d\Omega_1 d\Omega_2 \qquad 10.$$

since the asymmetry parameter α_2 for the $n\pi^-$ decay mode of Σ^- is zero. It is obvious that the distribution of the pions of the ($\Sigma^+\Sigma^0$) system for the ($n\pi^+$) mode of Σ^+ is the same as that of the ($\Sigma^-\Sigma^0$) system. For the $p\pi^0$ mode of the Σ^+ decay the distribution is given by (8) .

The joint distribution function ϕ can be evaluated by constructing the spin space density matrix $\rho(\vec{\sigma}_1, \vec{\sigma}_2)$ and

$$\phi = \frac{Tr[\rho P(\vec{\sigma}_1 \vec{\sigma}_2)]}{Tr\,\rho} \qquad 11$$

The polarization of Λ arising from a Σ^0 decay is given by[1)]

$$\langle \sigma \rangle_\Lambda = -(\langle \sigma \rangle_\Sigma \cdot \vec{u})\, \vec{u}$$

where $\vec{u}$ is the unit vector along the direction of Λ momentum in the Σ^0 rest system. Thus the Λ^0 and Σ^+ will have spins oriented in the opposite directions. Since the density matrix for the $^1S_0$ state is

$$\rho = 1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \qquad 12$$

the distribution function of the decay pions is

$$\phi = (1 - \alpha_1 \alpha_2 \vec{p}_1 \cdot \vec{p}_2) \qquad 13$$

This also enables the determination of the relative signs of α_1 and α_2 .

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1) R. Gatto, Phys. Rev., 109 , 610 (1958).

APPENDIX

A Note on the Theory of Geomagnetic Effects [1]

It is well-known that the question of calculating the intensity of cosmic rays at any point on the surface of the earth can be reduced after suitable modifications to that of finding the directions in which particles of given energy coming from infinity can reach that point. Taking the conventional notation inthis field for granted, the expression for the kinetic energy of a particle is

$$K.E. = Q = (1 - \cos^2 \omega) = \frac{\gamma^2}{4g^2} P$$

where

$$\cos \omega = \frac{2g}{\gamma \cos \lambda} - \frac{\cos \lambda}{\gamma^2}$$

and γ is γ in Stormer units and so depends on the energy. Thus for a particle of momentum $\vec{p}$, γ is assigned a value $\gamma_0(\vec{p})$.

Now since $\cos \omega$ can lie only between -1 and $+1$ the two hyperbolae

$$\left.\begin{aligned} \cos \omega &= +1 \\ \text{and} \quad &= -1 \end{aligned}\right\} = \frac{2g}{\gamma \cos \lambda} - \frac{\cos \lambda}{\gamma^2}$$

for a given latitude λ, divide the $\gamma - g$ plane into regions of positive and negative energy, the latter being forbidden.

A trajectory of a particle in the meridional plane is characterised by the co-ordinates γ and λ corresponding

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1. R. Thunga and T.K. Radha, Proceedings of the Cosmic Ray Symposium (at Ahmedabad) 1960

to a given γ and is represented in the $r-\gamma$ plane by a line parallel to the r axis. In order that a trajectory of a particle with momentum $\vec{p}$ may reach a point $(r_0 \lambda_c)$ on the surface of the earth, it must assume values from $+\infty$ to r_0 . If it were to assume values less than r_0 , it will lie within the earth and will be forbidden by the shadow effect of the earth. Ignoring this for the moment, the following geometrical features of the curves serve to define the Störmer and the main cones.

(i) $\gamma = \gamma_\lambda = \cos^{3/2}\lambda$ is a line tangential to the curve $\cos\omega = +1$ at $r_\lambda = \cos^{1/2}\lambda$ and cuts the curve $\cos\omega = -1$ at $r_m = (\sqrt{2}-1)\cos^{1/2}\lambda$. The maximum of both γ_λ and r_λ is unity and occurs for $\lambda = 0$. We shall consider the trajectory of a particle of momentum $\vec{p}$ to be absolutely forbidden if any part of it $(r > r_0)$ lies in the forbidden region. Since we do not restrict the λ values of the trajectory, we require $\gamma \geq 1$ to ensure that the trajectory of a particle with momentum $\vec{p}$ such that $r_0 < 1$ cuts the forbidden region whatever be the value of λ . Hence $\cos\omega$ varies from -1 to a value

$$\cos\omega_s(\lambda_0, p) = \frac{2}{r_0\cos\lambda_0} - \frac{\cos\lambda_0}{r_0^2}$$

so that all directions in a cone with angle ω_s are absolutely forbidden and this defines the Stormer Cone.

(ii) Trajectories with $\lambda \geq \lambda_0$ corresponding to particles with momentum $\vec{p}$ such that $r_c < \cos^{1/2}\lambda_0$ are forbidden if $q > \cos^{3/2}\lambda_c$ since such trajectories always pass through the forbidden region. The critical angle ω_c for $q_c = \cos^{3/2}\lambda_0$ is given by

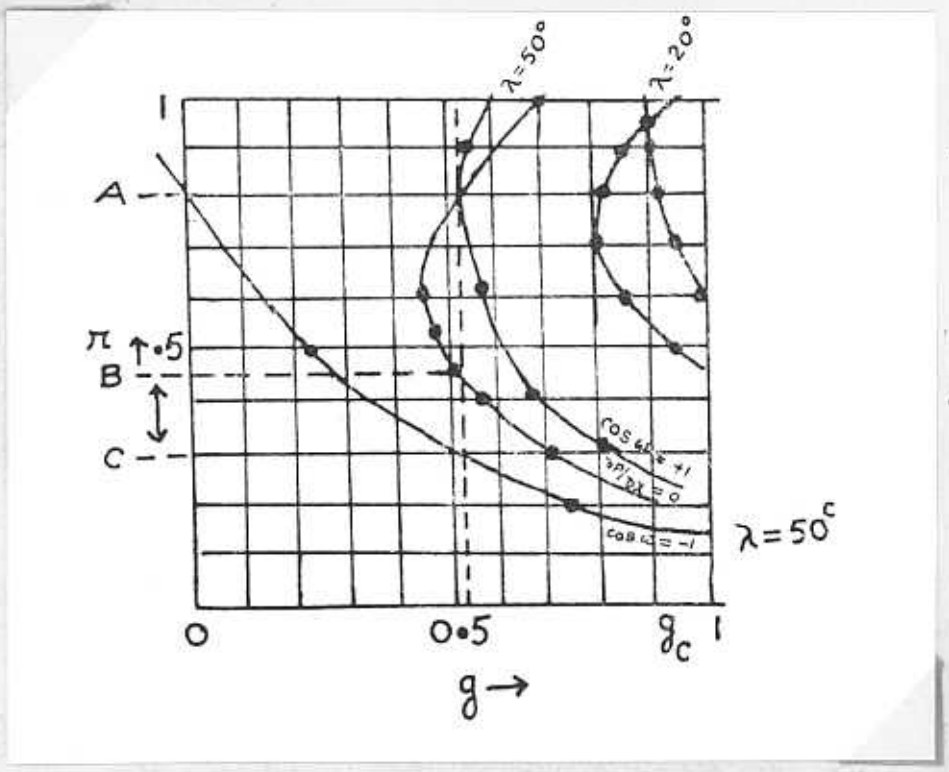
$$\cos\omega_c(\lambda_0, p) = \frac{2\cos^{1/2}\lambda_c}{r_c} - \frac{\cos\lambda_0}{r_0^2}$$

which defines the main cone within which all directions (i.e.) as ω varies from π to ω_c) are allowed for all $\lambda \leq \lambda_0$.

The only condition that we have till now imposed in determining the allowed directions is that the kinetic energy should never be negative. But if a particle were to be subjected to a negative acceleration it will execute oscillations and may cut the surface of the earth before reaching the observation point. This shadow effect of the earth gives rise to the so-called simple shadow cone and penumbra. Since the curve $\frac{\partial P}{\partial \lambda} = 0$ is analytically known, it is interesting to study the cruve in the λ-q plane. It is given by the relation

$$2q = r^3 + \frac{\cos^2\lambda}{r}$$

and the plot for a given λ is as seen in the figure. The arguments of the previous curve can again be applied and the basic features of the simple shadow cone may be qualitatively explained.


λ=50°
λ=20°
1
A
π
B
·5
C
cos φ = +1
cos φ = -1
λ=50°
0
0·5
g_c
1
g →

(i) The two branches correspond to P_{max} and P_{min} along which $\frac{\partial P}{\partial \sigma} = 0$ and the region between them represents a region of negative acceleration. The tangential value of g is $.8774 \cos^{3/2}\lambda$ and is hence always less than g_λ. Thus for particles with momentum $\vec{p}$ such that $p_c < \cos^{1/2}\lambda_0$ the cone with $g \leq .8774 \cos^{3/2}\lambda_0$ for ~~kkk~~ trajectories with $\lambda \leq \lambda_0$ all directions will certainly be allowed. The critical value of ω is given by

$$\cos\omega_0 = \frac{2 \times .8774 \cos^{1/2}\lambda_0}{p_0} - \frac{\cos\lambda_0}{p_0^2}$$

(ii) The $\cos\omega$ value at any point on this curve is $\frac{p_0^2}{\cos\lambda_0}$. For p_0 values between BC, $\cos\omega = p_0^2/\cos\lambda$ is greater than $\cos\omega_c$, the value on the tangent of the curve $\cos\omega = +1$. But for p_0 values between A and B $p_0^2/\cos\lambda_0 < \cos\omega_c$. Thus we further define the cone $\cos\omega = p_c^2/\cos\lambda_c$ which is either more or less restrictive than the main cone, depending upon the energy of the particle.