A COMPARITIVE STUDY OF STRING THEORY AND LOOP QUANTUM GRAVITY IN COSMOLOGICAL CONTEXT

By

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BONAFIDE CERTIFICATE

Certified that this dissertation titled **A COMPARITIVE STUDY OF STRING THEORY AND LOOP QUANTUM GRAVITY IN COSMOLOGICAL CONTEXT** is the bonafide work of **Mr. Kinjal Banerjee** who carried out the project under my supervision. Certified further, that to the best of my knowledge the work reported herein does not form part of any other dissertation of the basis of which a degree or award was conferred on an earlier occasion for this or any other candidate.

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Abstract

One of the central problems in theoretical physics today is combining the principle of quantum mechanics and general relativity into a quantum theory of gravity. It is believed that quantum gravity can address many problems in physics, especially Black Hole (BH) entropy and classical singularities.

There is no clear consensus on what a quantum theory of gravity should be, except that it should at least address the two problems mentioned above. At present there are two main candidates - String Theory and Loop Quantum Gravity (LQG).

In String theory, the quantum modifications arise from stringy corrections due to genus expansion as well as the α' corrections from sigma model description of first quantized strings. In the cosmological context of LQG, the quantum modifications have a non-perturbative as well as perturbative component in terms of the Immirizi parameter γ^2. The non-perturbative effects are responsible for singularity avoidance as well as bounded growth of curvature near classical singularity.

In both approaches one is able to reproduce the Bekenstein-Hawking entropy formula from microstate counting, implying a statistical mechanical cause of BH radiation, i.e. the entropy measures the number of microstates and they are quantum mechanical in nature. Another area where the quantum effects of gravity are expected to show up is in the modification in the behaviour of the universe near big bang singularity.

Given the very different premises and frameworks used in String Theory and LQG, a comparison of their implication even in the highly simplified context of homogeneous and isotropic cosmology is quite non trivial. At present a possible comparison between these two theories is conceivable only at perturbative level in the large volume (in

Planck units) limit. As a first attempt we look at the simplest context of homogeneous isotropic cosmology models and concentrate only on the perturbative modifications of the gravity sector alone. This is done only in the spirit of comparison between the two theories and not with an effort to relate to observations, because homogeneity and isotropy are themselves approximations to the real world. We look at the second order term in α' coming from string theory and γ^2 coming from LQC and compare the two theories in terms of effective classical equations of motion of the underlying theory. Prima facie the equations appear to be very different. However the solutions to both sets of equations predict that at least to the first order in perturbation theory the Minkowski geometry is stable.

பணிச்சுருக்கம்

தற்காலத்தில் கணிதம் சார்ந்த இயற்பியலின் முதன்மையான பணியானது குவாண்டம் எந்திரவியல் தத்துவத்தையும் பொது சார்பியலையும் ஒருங்கிணைத்து குவாண்டம் கிராவிட்டி என்னும் கோட்பாட்டை பெறுவதாக அமைகிறது. குவாண்டம் கிராவிட்டி கோட்பாடு இயற்பியலில் காணக்கிடைக்கும் பல சிக்கலான கணக்கிடுகளை கணக்கிட துணை செய்யும் என்று நம்பப்படுகிறது, முக்கியமாக கருந்துளை என்ட்ரோபி, கிளாசிகல் சிங்குளாரிட்டி என்ற இரு முக்கிய நிகழ்வுகளை பற்றி அறிவதில் பயன்படுகிறது. இந்த கோட்பாட்டின் மூலம் நாம், ஸ்டிரிங் கோட்பாடு மற்றும் லூப்-குவாண்டம்-கிராவிட்டி(LQG) கணக்கிடலாம்.

ஸ்டிரிங் கோட்பாட்டில் குவாண்டம் கொள்கையானது குவாண்டம் ஸ்டிரிங்கின் சிக்மா மாட்லின் α' திருத்தத்தின் போது பெறப்படுகிறது. அண்டம் சார்ந்த LQG-ல் குவாண்டம் மாறுபாடுகள் γ^2-ன் கூறுகளில் தோன்றுகின்றன.

கருந்துளை கதிரியக்கம் பற்றிய நிகழ்வினை விளக்குவதில் ஸ்டிரிங் கோட்பாடும், LQG-யும் ஒத்து போகின்றன. பெரு வெடிப்பு நிகழ்வினை விளக்கும் நிலையிலும் குவாண்டம் கிராவிட்டி கோட்பாடு தேவைப்படுகிறது.

மிக அதிக பரும எல்லையில் ஒரு சிறிய இடைவெளியில் இரு கோட்பாடுகளையும் ஒப்பு நோக்க முயற்சி மேற்கொள்ளப்படுகிறது. முதல் முயற்சியாக ஹோமோஜினியஸ் மற்றும் ஐசோட்ராபிக் காஸ்மாலஜியின் மாடலில் இவற்றை படிப்போம். இந்த முயற்சியானது இரு கோட்பாடுகளையும் ஒப்பு நோக்கவே அன்றி நிகழ்வுகளில் கண்ணுறபவைகளை ஒப்பு நோக்குவதல்ல.

ஸ்டிரிங் கோட்பாட்டில் பெறப்படும் α'-ன் இரண்டாம் படி உறுப்பையும் LQC-ல் பெறப்படும் γ^2-யையும் ஒப்பு நோக்குவோம். பின் இயக்க சமன்பாடுகள் மூலம் இரு கோட்பாடுகளையும் ஒப்பு நோக்கலாம். இப்படி செய்த முயற்சியில் திருத்தங்கள் மிகவும் வேறுபடுகின்றன, இருப்பினும் விளைவுகள் ஒத்துப்போகின்றன. மேலும் முடிவானக் கருத்தினை கூறுவதற்கு முன்னால் இன்னும் நிறைய சோதனைகள் செய்ய வேண்டி உள்ளது.

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Chapter 1

INTRODUCTION

1.1 Why a Quantum Theory of Gravity

General Theory of Relativity and Quantum Mechanics are the two principles on which modern physics is based. These two theories have profoundly changed the way we look at nature. Quantum Mechanics has put paid to the determinism of Newton's Equations of Motion and has replaced it by probabilities consistent with Heisenberg's Uncertainty Principle at the observational level. The concept of unchanging (static) background space and time has been replaced in the General Theory of Relativity by a spacetime geometry which is dynamic. However we are still some way away from finding a satisfactory physical theory that would combine both these fundamental principles in a consistent way to give a theory of Quantum Gravity.

There are reasons why we need a quantum theory of gravity.

- Classical Einstein's equations predict spacetime singularities, i.e. points at which the theory itself breaks down. A theory which predicts its own breakdown cannot be a complete theory and hence needs to be extended.

- There is a view that if we keep the geometry classical while matter fields are quantum it leads to inconsistencies. Hence the gravitational field must be quantized as well.

- It is hoped that the U-V divergences of quantum field theories will be avoided if we can incorporate gravity. This is because the quantum theory of gravitation will have an inbuilt length scale, the Planck length ($l_p^2 \sim G\hbar$) which will provide the short distance cut-off.

- From Black Hole (BH) physics we can see that the laws of Black Hole mechanics can formally be mapped to laws of thermodynamics. The Bekenstein proposal of identifying area with entropy (Bekenstein, 1973) and the demonstration by Hawking of BH radiation (Hawking, 1975) takes this analogy to a deeper level. This implies that there is a micro structure to the event horizon which is sought to be understood as a manifestation of quantum gravity.

1.2 Approaches to Quantum Gravity

There is no clear consensus as what exactly is meant by a quantum theory of gravity. Intuitively, it is supposed to incorporate the ideas of the "fluctuating spacetimes" and "sum over spacetime metrics". While there have been several approaches at constructing a quantum theory of gravity, the two most developed ones are *String Theory* and *Loop quantum Gravity.*

String Theory treats the gravitational field on the same footing as the other fundamental interactions namely the electromagnetic, the weak and the strong and tries to incorporate all four into one fundamental unified theory. Gravitational interaction is sought to be understood as exchange of gravitons in analogy with other interactions. The spacetime metric is split into two parts,a background metric and a fluctuating one. Usually one chooses the background to be a flat Minkowskian metric $\eta_{\mu\nu}$. Then $g_{\mu\nu} = \eta_{\mu\nu} + Gh_{\mu\nu}$ where $h_{\mu\nu}$ is the dynamical variable with Newton's constant G acting as the coupling constant. The field $h_{\mu\nu}$ is then quantized on the $\eta_{\mu\nu}$ background and the perturbative machinery of QFT is applied to the Einstein-Hilbert action. Since the background enters manifestly, the background independence is to be obtained by summing over all possible backgrounds. Since the notion of graviton is a perturbative one, this approach is perturbative in nature with non-perturbative effects sought subsequently.

String Theory is the only known consistent perturbative approach to quantum gravity. In this theory the point particles are replaced by one-dimensional extended objects which sweep out two-dimensional world sheets embedded in a D-dimensional manifold which represents the spacetime of the physical world. Matter and the mediators of the various interactions are all thought of as excitation modes of the strings.

The other approach is a relativist's approach viewing gravitational interactions as manifestations of curvature of a dynamical spacetime. In this view the metric plays the dual role of a mathematical object that defines spacetime geometry and encodes the physical gravitational field as well. Hence background independence lies at the heart of this approach. If we are going to quantize the gravitational field , i.e. the metric itself, it would make sense if the scheme does not explicitly depend on some background metric. Minkowski metric is not an externally prescribed eternal background structure but only one possible example. In this formalism the ordinary Quantum Field Theory(QFT) approach is no longer possible because it is defined when the background metric is fixed and hence it cannot handle variations of the background metric. New mathematical techniques have to be invented to go beyond the framework of perturbation theory and QFT.

Loop quantum gravity (LQG) is an attempt to construct a mathematically rigorous, non-perturbative, background independent formulation of quantum General Relativity in terms of Sen-Ashtekar connection variables. General Relativity is reformulated in terms of the connection variables and its dynamics is treated in a canonical framework. In a canonical framework, spacetime is viewed as an evolution of a 3 dimensional geometry. The 4 dimensional diffeomorphism of general relativity is manifest as spatial diffeomorphism on the 3 dimensional slice together with the Hamiltonian constraint generating time evolution.

1.3 Possible method of comparison

A priori the two approaches are very different and it is reasonable to ask whether they can be compared. For comparison we need to look at cases where the quantum effects should play a role (e.g. BH Entropy). Another important factor in making such comparison is that there must be areas where both the theories have made enough progress to have made some definitive predictions.

Comparison between these two theories can be made in the sector of Black Hole Entropy. String theory reproduces the Bekenstein-Hawking entropy formula for the so called "(near) extremal" black holes (Strominger and Vafa, 1996). LQG reproduces the entropy formula for "generic" black holes (including the astrophysical ones) modulo the Barbero-Immirizi parameter (Ashtekar et al., 1998a). In recent years the progress made in the cosmology sector of LQG has increased the scope of comparing the results in the cosmological context. In this thesis we will attempt to make a comparison of the results obtained from both the theories in cosmology.

In cosmological models, all but a finite number of degrees of freedom are frozen by imposing spatial homogeneity and isotropy. As a result of these mathematical simplifications it becomes tractable to compare the predictions of these two candidates of the quantum theory of gravity. Moreover cosmology is physically relevant and is perhaps the only area where experimental tests can even be thought of as of now. It must be noted though, homogeneity and isotropy themselves are approximations of the physical world. So such a comparison may not directly translate into results that can be confronted with observations, it will at least serve as a test whether and how these two theories differ.

Loop Quantum Cosmology (LQC) has both perturbative as well as non-perturbative features. The non-perturbative features are those which are related to avoidance of classical singularities (Bojowald and Morales-Tecotl, 2003). These non-perturbative effects ensure there is no breakdown of the evolution equation or there is no unbounded growth in curvature. Such non-perturbative results are not *generically* available in string theory context. In this thesis we will therefore look only at the perturbative corrections that are available in the two approaches. We will focus on the comparison of α' corrections in string theory and γ^2 corrections in LQC in the context of Friedmann-Robertson-Walker (FRW) metric.

Since both approaches, to leading order, must reproduce equations which match with those obtained in classical general relativity, we recall these equations. Gravity coupled with a massless scaler field is described by Einstein-Klein-Gordon equation.

The homogeneity and isotropy implies that the scaler be spatially constant and that the metric take the form

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\Big] \qquad \text{(FRW metric)} \qquad (1.1)$$

Then the field equations (with the cosmological constant set to zero) are ordinary, coupled nonlinear differential equations:

$$\frac{3\ddot{a}}{a} + \kappa\dot{\phi}^2 = 0 \qquad \text{(Raychaudhuri equation)} \qquad (1.2)$$

$$\ddot{\phi} + \frac{3\dot{\phi}\dot{a}}{a} = 0 \qquad \text{(Klein-Gordon equation)} \qquad (1.3)$$

$$\frac{3k}{a^2} + \frac{3\dot{a}^2}{a^2} - \frac{\kappa}{2}\dot{\phi}^2 = 0 \qquad \text{(Friedmann equation)} \qquad (1.4)$$

Differences, if any, will occur at the higher orders. We will focus on trying to compare the leading corrections coming from both the theories. The comparison between string theory and the LQC approach can be done at the level of the effective action or at the level of the equations of motion. Here we shall compare the equations of motion because these are more easily obtainable in both of these theories. To begin with, in this thesis we will focus only on the gravitational sector. Note that in the absence of matter and without the cosmological constant, k can only take values 0 and -1. However $k = -1$ cannot be handled in a Hamiltonian formalism. So for the sake of comparison with LQC equations we shall restrict ourselves to $k = 0$. Then the above equations reduce to Minkowskian geometry in the absence of matter with cosmological constant set to zero. Hence our calculations will basically show whether the Minkowskian geometry is stable under string and LQC corrections.

The thesis is organized as follows. In Chapter 2 we give a brief review of bosonic string theory. We discuss the sigma model approach to string theory and what the β functions mean in that context. This formalism has a parameter α' with dimensions of square of length. The perturbation corrections are organized in terms of this parameter. Then specializing to the FRW metric we measure what are the corrections to Einstein's equation upto 2nd order in α' of the corresponding field theory. We limit ourselves to bosonic strings for simplicity and to facilitate easier comparison of the results.

In Chapter 3 we discuss both LQG and its restriction to Loop Quantum Cosmology (LQC) and look at the Hamiltonian constraint equation in the context of isotropic models (FRW metric). Here the perturbation series is obtained in terms of a constant γ which is known as Barbero-Immirizi parameter. We obtain the corrections to the Einstein's equations in the large volume limit upto leading order in γ^2 in the constraint equation as well as in the equations of motion.

As a consistency check we verify that the "tree-level" equations of motions obtained from both the theories with (1.2, 1.4). Once the equations have been obtained in terms of the scale factor (a) and its derivatives from both the theories we compare them in Chapter 4 and discuss the similarities and differences in the two sets of equations obtained. We briefly mention further directions in extending the comparison.

Chapter 2

STRING THEORY CALCULATION

2.1 Basics

In String Theory we assume that the basic entities are not point particles but one dimensional extended objects called strings. A point particle will sweep out a world line as it moves in spacetime. The simplest Poincare invariant action, that does not depend on paramatrization, that can be written for it will be proportional to the proper time along the world line.

$$S_{PP} = -m \int d\tau (-\dot{X}_M \dot{X}^M)^{1/2}$$

Similarly a 1-dimensional object will sweep out a world-sheet embedded in a D-dimensional target space with coordinates X^M which can be described in terms of the embedding functions $X^M(\sigma, \tau)$ where (σ, τ) paramatrize the world sheet. Again the simplest Poincare invariant action that does not depend on paramatrization of the world sheet will depend on the area of the world sheet. That action known as Nambu-Goto action is given by

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau [-(\dot{X}\dot{X})(X'X') - (\dot{X}X')^2]^{1/2} \tag{2.1}$$

where $\dot{X} = \frac{\partial X^M}{\partial \tau}$ and $X' = \frac{\partial X^M}{\partial \sigma}$ for $M = 0$ to d. The constant α' must have dimensions of length squared from dimensional considerations (Polchinski, 1998). Denoting the induced metric $h_{ab} = \partial_a X^M \partial_b X_M$, we can write the Nambu-Goto action as

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau (-\det\ h_{ab})^{1/2} \tag{2.2}$$

Here a,b takes the values σ and τ .

This action can be put into a more convenient form by introducing the world sheet metric $\gamma_{ab}(\sigma\tau)$. This is commonly known as the Polyakov action.

$$S_P = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau (-\gamma)^{1/2} \gamma^{ab} \partial_a X^M \partial_b X^N \eta_{MN} \tag{2.3}$$

where η_{MN} is the standard Minkowski metric, diag (- + + ...). This action has the following symmetries:

1. D dimensional Poincare invariance.

$$\begin{aligned} \acute{X}^M(\sigma,\tau) &= \Lambda^M_N X^N(\sigma,\tau) + a^M \\ \acute{\gamma}_{ab}(\sigma,\tau) &= \gamma_{ab}(\sigma,\tau) \end{aligned}$$

2. 2 dimensional diffeomorphism invariance.

$$\begin{aligned} \acute{X}^M(\acute{\sigma},\acute{\tau}) &= X^M(\sigma,\tau) \\ \frac{\partial \acute{\sigma}^c}{\partial \sigma^a}\frac{\partial \acute{\sigma}^d}{\partial \sigma^b}\acute{\gamma}_{cd}(\acute{\sigma},\acute{\tau}) &= \gamma_{ab}(\sigma,\tau) \end{aligned}$$

3. 2 dimensional Weyl invariance

$$\begin{aligned} \acute{X}^M(\sigma,\tau) &= X^M(\sigma,\tau) \\ \acute{\gamma}_{ab}(\sigma,\tau) &= exp(2\omega(\sigma,\tau))\gamma_{ab}(\sigma,\tau) \end{aligned}$$

The Weyl symmetry is a new symmetry which is not present in Nambu-Goto action. It implies that the Polyakov action remains the same under a local rescaling of the world sheet metric. Diffeomorphism and Weyl invariance is together known as Conformal invariance. Note the Poincare group is the group of isometries for Minkowskian spacetime only. For a different target space this symmetry will be absent.

One way in which the Weyl symmetry manifests itself is in the vanishing of the energy-momentum tensor T^{ab}. In infinitesimal form the Weyl symmetry can be expressed as:

$$\delta\gamma^{ab} = -(\delta\phi)\gamma^{ab}$$

Therefore,

$$\gamma^{ab}\frac{\delta S}{\delta\gamma^{ab}} = -\frac{\delta S}{\delta\phi}$$

Now Weyl invariance of the action implies that $\delta S/\delta\phi = 0$

$$\Rightarrow \gamma^{ab}\frac{\delta S}{\delta\gamma^{ab}} = 0$$

$$or,\ \gamma^{ab}T_{ab} = 0.$$

$$i.e,\ T^a_a = 0 \tag{2.4}$$

where $T_{ab} = -2\pi(-\gamma)^{-1/2}(\delta S/\delta\gamma^{ab})$ It can be easily seen from (2.3) that the Weyl invariance holds for the Polyakov action, at least in the classical level. If the symmetry holds even in the quantum level (i.e. the symmetry is not anomalous) first quantized string theory can be considered to be a *conformally invariant field theory (CFT)* in 2-dimensions.

It turns out that to ensure Weyl invariance at the quantum level, the dimension of the target space D must be 26. To get back the 4 dimensional picture there are compactification schemes (e.g. Kaluza-Klein compactification) which compactify the extra dimensions and reduce it to the observed 4 dimensions along with a set of massless scalars and vectors, etc.

The spectrum of the above string theory contains tachyon. Also one of its massless modes is spin two state which is identified to be the *graviton* whose interactions at low energy reduce to general relativity. Thus we see the emergence of quantum theory of gravity with gravity being treated on the same footing as other excitations. To incorporate fermionic matter one has to consider fermionic strings, in particular superstrings. In superstring theory, the critical dimension turns out to be 10 and the spectrum is free of tachyon. In this thesis we will be concerned only with bosonic string theory.

2.2 Sigma Model Approach

The above theory of strings propagating in a non trivial d-dimensional spacetime M with an arbitrary metric G_{MN} may also be formulated as a 2-dimensional *non-linear* σ-model with M as its target space. The conditions of conformal invariance of the σ-model puts restrictions on the target space background fields. In general a non-linear σ-model action has the form

$$S = \int d\sigma d\tau f_{MN}(X^M)\sqrt{-\gamma}\ \Gamma_{ab}\partial_a X^M \partial_b X^N$$

The Polyakov action (2.3) corresponds to Γ_{ab} being γ_{ab} and f_{MN} being η_{MN}. To consider more general target space background metrics we can write the Polyakov action as

$$S_P = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau (-\gamma)^{1/2}\gamma^{ab}\partial_a X^M \partial_b X^N G_{MN}(X) \tag{2.5}$$

where G_{MN} is any arbitrary metric. However from dimensional point of view it is not the only action that can be written respecting the diffeomorphism symmetry (Callan and L.Thorlacius, 1989). We can have an antisymmetric tensor field which will have the following action:

$$S_{AS} = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \epsilon^{ab}\partial_a X^M \partial_b X^N B_{MN}(X) \tag{2.6}$$

where, ϵ^{ab} is the 2-D antisymmetric tensor density i.e. $\epsilon^{ab}/\sqrt{\gamma}$ transforms as a tensor. Another term we can incorporate is the massless scaler field:

$$S_D = \frac{1}{8\pi}\int d\sigma d\tau \sqrt{-\gamma} R^{(2)}\phi(X) \tag{2.7}$$

where ϕ is a massless scalar(dilaton), $R^{(2)}$ is the Ricci scalar corresponding to the world sheet metric. Since $\phi(X)$ is dimensionless we do not need the factor of α' in the normalization of its action. Also note that (2.7) is not Weyl invariant. However if we were to do perturbation series expansion in α' this term would enter at the first loop level rather than the classical level. Hence this term may be used to cancel the Weyl anomalous terms arising from quantum corrections in (2.3) and from (2.6). We can have one more such term.

$$S_T = \frac{1}{4\pi}\int d\sigma d\tau \sqrt{-\gamma}\, T(X) \tag{2.8}$$

This will describe coupling to a background tachyon field. The coupling functions $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\phi(X)$ are all dimensionless and correspond to the massless states of closed bosonic string theory: graviton, antisymmetric tensor and dilaton. However for simplicity as a first step we will neglect the antisymmetric tensor field and the tachyon. The sigma model that we will consider will involve (2.3) and (2.7):

$$S_X = \frac{1}{4\pi\alpha'}\int d\sigma d\tau\{\sqrt{-\gamma}\gamma^{ab}\partial_a X^M \partial_b X^N G_{MN} + \alpha'\sqrt{-\gamma}R^{(2)}\phi(X)\} \tag{2.9}$$

(Tseytlin, 1987)

There are two ways in which we can think of a perturbation series expansion in string theory. We can consider α' to be the perturbation parameter and do perturbation theory in the σ model level on a fixed world sheet topology. Else we can consider different world sheet topologies by constructing a genus expansion. In 2-D any compact topological surface can be characterized by a single constant known as *Euler number*(χ). The Euler number of compact, connected, oriented or unoriented surfaces is given by (Polchinski, 1998):

$$\chi = 2 - 2g - b - c$$

where g = number of handles(genus); b = number of boundaries; c = number of crosscaps. If we add a term λ_χ to the action, in the path integral formalism it will contribute a factor $e^{-\lambda_\chi}$ which will affect only the relative weighing of different world-sheet topologies. Now the world-sheet topology can be interpreted in terms of string interactions. Considering higher order of interactions of strings is equivalent to adding handles or boundaries to the topology i.e. making a genus expansion. Higher order of interactions are therefore suppressed by factors of

$$g_{string} \sim e^{\lambda_\chi}$$

This provides another parameter for perturbation expansion. To consider "stringy" corrections we need to do perturbation theory in g_{string}. In this thesis we will consider only sigma model (i.e. α') corrections with the world-sheet topology of a sphere. These corrections lead to the perturbative modifications of the target space background fields.

2.3 Weyl Invariance and β function

In this section we will summarize the properties of a perturbatively renormalizable bosonic σ model (2.9) defined on a 2-D world-sheet with spherical topology. As we have seen earlier in equation (2.4), to ensure Weyl invariance is maintained T_a^a must be equal to 0 (*Weyl invariance condition*). It can be shown that the Weyl invariance condition and the Renormalization group β function are same upto a total derivative (Tseytlin, 1989). We shall denote the RG beta functions by $\bar{\beta}$ and the Weyl anomaly coefficients by β. Let us define the partition function of the sigma model action (2.9)

$$Z = \int [DX] e^{-S_X}$$

As we have defined earlier

$$T_a^a = -2\pi(-\gamma)^{-1/2}\gamma^{ab}(\delta S/\delta\gamma^{ab})$$

For any field theory with a coupling constant λ the renormalization group beta functions are defined to be

$$\bar{\beta} = \frac{\partial\lambda}{\partial(ln\mu)}$$

where μ is the renormalization scale. For the 2-D non-linear sigma model upto the 2nd order:

$$\beta_{MN} = \bar{\beta}_{MN} + 2\alpha'\nabla_M\nabla_N\phi \tag{2.10}$$

$$\beta_\phi = \bar{\beta}_\phi + \alpha'(\nabla_M\phi)(\nabla^M\phi) \tag{2.11}$$

Again the RG beta functions upto 2nd order are given by (Tseytlin, 1987)

$$\bar{\beta}_{MN} = \alpha' R_{MN} + \left(\frac{\alpha'^2}{2}\right) R_M^{\ IJK} R_{NIJK} \tag{2.12}$$

$$\bar{\beta}_\phi = \frac{D-26}{6} - \alpha'(\frac{1}{2}\nabla^2\phi + \left(\frac{\alpha'^2}{16}\right) R^{MIJK} R_{MIJK} \tag{2.13}$$

Putting these together we have, the Weyl anomaly equation for bosonic sigma model without matter fields is:

$$\beta_{MN} = \alpha' R_{MN} + \left(\frac{\alpha'^2}{2}\right) R_M^{\ IJK} R_{NIJK} + O(\alpha'^3) = 0 \tag{2.14}$$

The Weyl anomaly function equation for bosonic sigma model with a scalar matter field is

$$\begin{aligned}
\beta_{MN} &= \alpha' R_{MN} + \left(\frac{\alpha'^2}{2}\right) R_M^{IJK} R_{NIJK} + 2\alpha' \nabla_M \nabla_N \phi + O(\alpha'^3) = 0 \quad (2.15) \\
\beta_\phi &= \frac{D-26}{6} + \alpha'(-\frac{1}{2}\nabla^2\phi + (\nabla\phi)^2) \\
&+ \left(\frac{\alpha'^2}{16}\right) R^{MIJK} R_{MIJK} + O\alpha'^3) = 0 \quad (2.16)
\end{aligned}$$

These are the equations that must be satisfied to have a conformally invariant field theory. We can see that at the one loop level in the absence of matter fields

$$\beta_{MN} = 0 \Rightarrow \alpha' R_{MN} = 0$$

Hence in the first approximation, string theory also predicts that the metric must satisfy Einstein's equation. The higher order terms to the (2.14) will give the corrections to Einstein's equation.

We would like to get our results in 4 dimensions while (2.15, 2.16) shows clearly that $D = 26$. To get 4 dimensional equations of motion from the 26 dimensional string theory we had two approaches open before us.

We need not have taken the "β function approach". We could have assumed that the target space metric $G_{MN} = \eta_{MN}$ for $M, N = 0, 1, \ldots, 25$. Then we could have taken the field theory limit, calculated the effective action from string scattering amplitudes. The higher order terms in the scattering amplitude would be weighed by powers of g_{string}. The effective action in 26 dimensions could be reduced to 4 dimensions using Kaluza-Klein reduction. This could then have been specialized to FRW metric. However this would give us stringy corrections in terms of g_{string}.

We are interested in α' corrections. Hence we will follow a different strategy. We assume that the target space metric was some arbitrary G_{MN} to begin with. There we obtained the Weyl anomaly equations using the 2-D renormalizable σ model. Now we will specialize to a metric which is of the following form:

$$\begin{aligned}
G_{MN} &= g_{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3 \qquad (2.17) \\
&= g_{\mu a} = 0 \qquad \mu = 0, 1, 2, 3 \qquad a = 4, 5, \ldots, 26
\end{aligned}$$

$$= \quad g_{aa} = 1 \qquad a = 4, 5, \ldots, 26$$
$$= \quad g_{ab} = 0 \qquad a, b = 4, 5, \ldots, 26 \quad a \neq b$$

In this metric we will specialize the $g_{\mu\nu}$ to be the FRW metric. The Weyl invariance conditions then give equations of motion which are in 4 dimensions. Henceforth we will concentrate only on these 4 dimensional equations of motion. The other Weyl invariance conditions $\beta_{ab} = 0 = \beta_{0a}$ are identically satisfied. Since we are concerned only with the gravity sector, we will set the dilaton field to be equal to zero. We will therefore be using equation 2.14 only.

2.4 Equations of Motion in the FRW Metric

Recall that the FRW metric is given by (1.1):

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\Big] \tag{2.18}$$

where $k = 0$ or ± 1. For comparison with the LQC results we will restrict ourselves to $k = 0$ and 1 only because only those values of k can be handled in LQC context. Consequently:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{(1-kr^2)} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2 sin^2\theta \end{pmatrix}$$

And the inverse metric is:

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{(1-kr^2)}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2(t)r^2 sin^2\theta} \end{pmatrix}$$

The non -zero components of the Reimann tensor are :

$$R_{1001} = \frac{\ddot{a}a}{(1 - kr^2)} \tag{2.19}$$

$$
\begin{aligned}
R_{2002} &= a\ddot{a}rr^2 \\
R_{3003} &= a\ddot{a}r^2 sin^2\theta \\
R_{2121} &= \frac{a^2r^2(\dot{a}^2+k)}{(1-kr^2)} \\
R_{3131} &= \frac{a^2r^2sin^2\theta(\dot{a}^2+k)}{(1-kr^2)} \\
R_{2323} &= a^2r^4sin^2\theta(\dot{a}^2+k)
\end{aligned}
$$

The non zero components of the Ricci tensor are :

$$
\begin{aligned}
R_{00} &= \frac{-3\ddot{a}}{a} \qquad (2.20)\\
R_{11} &= \frac{2k+2\dot{a}^2+a\ddot{a}}{(1-kr^2)} \\
R_{22} &= 2kr^2+2r^2\dot{a}^2+r^2a\ddot{a} \\
R_{33} &= 2kr^2sin^2\theta+2r^2\dot{a}^2sin^2\theta+r^2a\ddot{a}sin^2\theta
\end{aligned}
$$

The Ricci scalar is :

$$
R = \frac{6(k+\dot{a}^2+a\ddot{a})}{a^2} \qquad (2.21)
$$

Then the β function equations (2.14) can be written as :

$$
\beta_{00} = \frac{-3\ddot{a}}{a}(1+\frac{\alpha'}{2}\frac{\ddot{a}}{a}) = 0 \qquad (2.22)
$$

$$
\beta_{ii} = (2(k+\dot{a}^2)+a\ddot{a})+\frac{\alpha'}{2}[2((k+\dot{a}^2)^2)+(a\ddot{a})^2] = 0 \qquad (2.23)
$$

These equations to the $O(\alpha'^0)$ should reproduce the known classical equations from standard cosmology. As these equations occur in a perturbation series of α', the higher order terms will vanish when α' goes to 0. Hence these equations can be considered order by order and even when we are considering higher order terms we can be sure that the classical level the equations remain the same. To the first order O(α'^0) the equations of motion can be written in the form

$$
\frac{3\ddot{a}}{a} = 0 \qquad (2.24)
$$

$$
\frac{3k}{a^2}+\frac{3\dot{a}^2}{a^2} = 0 \qquad (2.25)
$$

These are the same equations that we get from general relativity (1.2, 1.4) with the matter stress-energy tensor set to zero. To summarize, in this chapter we looked at the basic formalism of string theory and have obtained the modifications to the (1.2, 1.4) predicted by string theory In the next chapter we will discuss how we get the LQC equations and try to match the variables using the same classical equations to compare the tree level predictions.

Chapter 3

LQG CALCULATION

3.1 Basics

The approach of LQG is completely different to that of string theory. The central idea here is that of background independence. We do not begin with a background metric and try to incorporate quantum effects of gravity perturbatively. Here we begin with a differentiable manifold structure with no specified background metric or any other background physical field. In that sense it is attempt to describe spacetime in a background independent (non-perturbative) manner. The mathematical techniques used are new and somewhat unfamiliar. The quantization route followed is that of *canonical quantization.*

The first attempt in this direction was the ADM Formalism (Arnowitt et al., 1962). General Relativity can for formulated as a constrained Hamiltonian theory (Wald, 1984) beginning with the Einstein-Hilbert action

$$S = \frac{1}{2\kappa}\int_M d^4X(-G)^{1/2}R$$

where κ is $8\pi G$. The basic variables are g_{ij} and K_{ij} which are symmetric tensor fields on a three manifold Σ. From the spacetime perspective Σ is a spatial slice of M and g_{ij} is the induced metric on Σ while K_{ij} is the extrinsic curvature of Σ. This is a highly complicated constrained Hamiltonian system with infinite degrees of freedom. To gain some insight Wheeler and De-Witt (De-Witt, 1967) looked at highly symmetric class of spacetimes, did the symmetry reduction classically and quantized the finite number of degrees of freedom left over. The evolution equation obtained (Wheeler-De Witt equation) gives the correct classical limit for large values of the scale factor (a). However when the scale factor vanishes, the inverse scale factor and curvatures blow up resulting in persistence of classical singularities. However despite years of efforts a well-defined quantum formulation based on the ADM variables is not yet available

Loop Quantum Gravity is another (and more successful attempt) in canonical quantization. Here the basic variables chosen are the *Ashtekar variables* (Ashtekar, 1987). Classical general relativity can be formulated as a gauge theory in phase space form. At each point on a three dimensional compact manifold M we define a smooth real gauge potential (i.e. covariant vector field) with the $SU(2)$ gauge group and call it $A^i_a(x)$. Again at each point we can define a set of triad vectors $\{e^a_i\}$. Here a, b ...=1, 2, 3 refer to spatial indices while i, j ...=1, 2, 3 refer to the internal labeling of the triad axes. Then we form a set of densitized triad $\{E^a_i(x)\} = \{|det(e^i_a)|e^a_i(x)\}$. The SU(2) connections are related to the ADM variables by:

$$A^i_a(x) = \Gamma^i_a(x) - \gamma K^i_a(x) \tag{3.1}$$

where Γ^i_a is the spin connection associated to the triad, K^i_a is the extrinsic curvature while γ is a constant known as Barbero-Immirizi parameter. A^i_a is called the Sen-Ashtekar-Immirizi-Barbero connection. The symplectic structure satisfied by these variables is:

$$\begin{aligned} \{E^a_i(x), E^b_j(y)\} &= \{K^i_a(x), K^j_b(y)\} = 0 \\ \{A^j_b(y), E^a_i(x)\} &= \gamma\kappa\delta^a_b\delta^j_i\delta(x,y) \quad \text{which imply} \\ \{E^a_i(x), K^j_b(y)\} &= \kappa\delta^a_b\delta^j_i\delta(x,y) \end{aligned} \tag{3.2}$$

The theory is invariant under local SU(2) gauge transformations, three dimensional diffeomorphism of the manifold, as well as under time translation generated by Hamiltonian constraint. It can be shown (Thiemann, 2001) that the dynamical content of general relativity is captured by 3 constraints, Gauss, Diffeomorphism, and Hamiltonian that generate these invariances. These are respectively

$$G[\Lambda^i] = \int_\sigma d^3x \Lambda^i(\partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k) \tag{3.3}$$

$$\mathcal{V}[N^a] = \int_\sigma d^3x N^a F^i_{ab} E^b_i \quad (3.4)$$

$$\mathcal{H}[N] = \int_\sigma d^3x N (det(e^i_a))^{-\frac{1}{2}} (\epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab} - 2(1+\gamma^2) E^a_{[i} E^b_{j]} K^i_a K^j_b) \quad (3.5)$$

where $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$ is the curvature of the Ashtekar connection, N is the lapse function and N^a is the shift vector.

The δ functions in the Poisson brackets (3.2) actually imply that one should use 3-dimensional smeared fields and this is the place where background dependence sneaks in, in the measure of the integral. Since the connection is a 1-form and the triad is a dual of a 2-form, these can be smeared naturally without introducing the background structure. These smearing leads to "holonomies" and "fluxes" defined below.

Holonomies can be formed as functions of connections for all curves $e : [0, 1] \rightarrow \Sigma$,

$$h_e(A) = \mathcal{P}\ exp \int_e A^i_a(e(t)) \dot{e}^a(t) \tau_i dt \quad (3.6)$$

where $\dot{e}^a$ is the tangent vector to the curve e and $\tau_i = -i\sigma_i/2$ are the generators of the gauge group SU(2) in terms of Pauli matrices. The symbol $\mathcal{P}$ denotes path ordering which means that the non commuting su(2) elements in the exponential are ordered along the curve. Similarly, given a surface $S : [0, 1] \times [0, 1] \rightarrow \Sigma$ we can form a flux as a function of triads,

$$E(S) = \int_S E^a_i(y) n_a(y) \tau^i d^2y \quad (3.7)$$

where n_a is the co-normal to the surface S defined as $n_a = \frac{1}{2}\epsilon_{abc}\epsilon^{de}(\partial x^b/\partial y^d)(\partial x^c/\partial y^e)$. The holonomies and fluxes are used as the basic variables in this formulation of gravity. Note that these are labelled by curves and surfaces. Their Poisson bracket structure is bit complicated (Ashtekar et al., 1998b). In a quantum theory these variables are promoted to operators on a suitable kinematic Hilbert space with Poisson bracket structure going into the commutator relations. The $\hbar$ along with the κ present in the symplectic structure will lead to the appearance of Planck length ($\hbar\kappa = l_p^2$) in the relations.

3.2 Quantum Cosmology

Loop Quantum Cosmology (LQC) is obtained from LQG by a process called "symmetry reduction". This means restricting the basic variables to those consistent with the symmetry of the system. In cosmological context, the symmetry is homogeneity and isotropy. In terms of the metric variables this leads to the FRW metric.

Spatial homogeneity implies invariance under the group of translations acting on the spatial manifold. When the action is simply transitive (i.e. there exists an unique element connecting two distinct point) we can identify the spatial manifold with the group manifold. All such 3 dimensional groups are classified into the so-called *Bianchi I-IX groups.*

In connection variables homogeneity leads to A and E having the form (Bojowald et al., 2003)

$$A_a^i(x,t) = \Phi_I^i(t)\omega_a^I(x), \quad E_i^a(t,x) = P_i^I(t)X_I^a(x) \tag{3.8}$$

where ω^i are left invariant 1-forms and X^i are corresponding densitized invariant dual vector fields; $\int_\Sigma \omega^i X_j = \delta^i_j$. Here a b ... =1, 2, 3 refer to spatial indices while i j ... =1, 2, 3 refer to the internal SU(2) indices while I J ... =1, 2, 3 refer to the indices of the Lie algebra of the Bianchi group. They satisfy the Poisson bracket relation

$$\{\Phi_I^i, P_j^J\} = \kappa\gamma\delta_j^i\delta_I^J \tag{3.9}$$

Now if in addition to homogeneity we also have isotropy, an additional SO(3) action, these variables are further simplified into

$$\Phi_I^i(t) = c(t)\Lambda_I^i, \quad P_i^I(t) = p(t)\Lambda_i^I \tag{3.10}$$

where Λ_I^i forms an orthonormal triad on which the SO(3) group of isotropy acts. With this form of connection and triad the diffeomorphism and Gauss constraints are automatically satisfied and we are only left with the Hamiltonian constraint. The Poisson bracket relation satisfied by c, p are

$$\{c,p\} = \frac{1}{3}\gamma\kappa \tag{3.11}$$

This is the classical symmetry reduction. Note that p coming from the definition of the triad is related to the scale factor as $p = a^2$. For the quantum system the Poisson bracket relation (3.11) becomes

$$\begin{aligned} [c, p] &= \frac{i\hbar}{3}\gamma\kappa \\ &= \frac{il_p^2\gamma}{3} \end{aligned} \tag{3.12}$$

As in (3.1) c can be written as $\Gamma - \gamma K$. Putting this in(3.12) we have

$$-\gamma[K, p] = \frac{il_p^2\gamma}{3}$$

$$\text{Hence} \qquad [p, K] = \frac{l_p^2}{3} \tag{3.13}$$

$$\text{Therefore} \qquad K = \frac{\kappa\hbar}{3i}\frac{\partial}{\partial p} \tag{3.14}$$

Quantization in the traditional way leads to a Hilbert space of square integrable functions with the usual Lebesgue measure. This is what is used in the Wheeler De-Witt quantization. Loop quantization proceeds in a different way. Since the Hilbert space of the full theory was based on connections, the Hilbert space here turns out to be made of square integrable functions of c with respect to the SU(2) Haar measure.

We can use the eigenfunctions of the flux operators to obtain an explicit basis for the Hilbert space.

$$\hat{p}|\mu> = \frac{1}{6}\gamma l_P^2 \mu|\mu> \equiv p|\mu> \tag{3.15}$$

and as functions of c are given explicitly by

$$f_\mu(c) = \frac{exp(i\mu c/2)}{\sqrt{2}sin(c/2)}$$

Now the holonomies can be expressed as:

$$h_i(c) = exp(c\tau_i) = cos(c/2) + 2\tau_i sin(c/2) \tag{3.16}$$

The action of the holonomies on the states are given by

$$\hat{h}_i(c)|\mu> = \left(\frac{|\mu+1> + |\mu-1>}{2}\right) + 2\tau_i\left(\frac{|\mu+1> + |\mu-1>}{2i}\right) \tag{3.17}$$

Holonomies can therefore be used as creation annihilation operators. All states we get by acting with $h_i(c)$ on the ground state will be functions of c.

Note that unlike in Wheeler De-Witt quantization c is not a well defined operator. Here the basic operators are $h(c)$ which are functions of $exp(c)$. Now we have a quantum theory of cosmology which is different from the old Wheeler De-Witt theory especially when the system approaches classical singularities. LQC manages to avoid these. However for large values of scale factor it should give back the results of classical general relativity in tree level. In the next section we shall verify that and find out the leading order corrections in LQC.

3.3 LQC Predictions for FRW Metric

Hamiltonian formalism of General Relativity can only be carried out for the so-called *Bianchi Class A models* which in the homogeneous isotropic case allow only k = 0 or 1 (Ashtekar and Samuel, 1991). For FRW cosmology the dynamical law can be written as a difference equation (Bojowald and Vandersloot, 2003)

$$\begin{aligned}(V_{\mu+5} - V_{\mu+3})e^{ik}\psi_{\mu+4}(\phi) - (2+\gamma^2k^2)(V_{\mu+1} - V_{\mu-1})\psi_\mu(\phi) \\ +(V_{\mu-3} - V_{\mu-5})e^{ik}\psi_{\mu-4}(\phi) = -\frac{1}{6}\gamma^3\kappa l_p^2 H_{matter}(\mu)\psi_\mu(\phi)\end{aligned} \tag{3.18}$$

where the volume eigenvalues $V_\mu = (\gamma l_p^2|\mu|/6)^{3/2}$

$$\text{and } \psi_\mu = e^{\frac{-i\mu\Gamma(\mu)}{2}}\tilde{\psi}_\mu$$

where $\Gamma(\mu)$ is the spin connection

$$\begin{aligned}\Gamma(\mu) &= 0 \qquad \text{if k=0} \\ &= 1/2 \qquad \text{if k=1}\end{aligned}$$

It can be easily seen that for both values of k the phase part drop out and we are left with a difference equation in $\tilde{\psi}_\mu$. Let $\tilde{t_\mu} = (V_{\mu+1} - V_{\mu-1})\tilde{\psi}_\mu$ Writing $V_{\mu+5} - V_{\mu+3}$ as $V_{\mu+1+4} - V_{\mu-1+4}$ and so on, the difference equation (3.18) can be written as:

$$\tilde{t}_{\mu+4} - (2+\gamma^2k^2)\tilde{t_\mu} + \tilde{t}_{\mu-4} = -\frac{1}{6}\gamma^3\kappa l_p^2\frac{H_{matter}}{(V_{\mu+1} - V_{\mu-1})}\tilde{t_\mu} \tag{3.19}$$

Hence from definition of V_μ we can write

$$\begin{aligned} V_{\mu+i} &= (\gamma l_p^2/6)^{3/2}(\mu+i)^{3/2} \\ &= (\gamma l_p^2/6)^{3/2}(\mu^{3/2}+\frac{3i\mu^{1/2}}{2}) \end{aligned} \quad (3.20)$$

$$\Rightarrow 1/(V_{\mu+1}-V_{\mu-1}) = \frac{\mu}{3}(V_\mu^{-1}) = \frac{\mu}{3}(\gamma l_p^2\mu/6)^{-3/2} \quad (3.21)$$

Now we want to obtain the continuum description. For that we need to make the continuum approximation and replace the difference equation with a differential equation. For that we introduce a continuous variables $p(\mu)$ and a function $T(p)$ such that

$$\begin{aligned} \tilde{t}_\mu &= T(p(\mu+m)) \\ p(\mu) &= \frac{1}{6}\gamma l_P^2\mu \end{aligned} \quad (3.22)$$

$$\begin{aligned} \tilde{t}_{\mu+m} &= T(p(\mu+m)) \\ &= T(p)+\frac{\partial T}{\partial p}\delta p+\frac{\partial^2 T}{\partial p^2}(\delta p)^2+\frac{\partial^3 T}{\partial p^3}(\delta p)^3+\frac{\partial^4 T}{\partial p^4}(\delta p)^4 \end{aligned} \quad (3.23)$$

Now from the definition of $p(\mu)$

$$\begin{aligned} p(\mu+m) &= \frac{1}{6}\gamma l_P^2(\mu+m) = \frac{1}{6}\gamma l_P^2\mu+\frac{1}{6}\gamma l_P^2 m \\ &= p(\mu)+\delta p \end{aligned} \quad (3.24)$$

Putting this in (3.23) we get

$$\begin{aligned} \tilde{t}_{\mu+m} &= T(p(\mu))+m\left(\frac{1}{6}\gamma l_P^2\right)\frac{\partial T}{\partial p}+\frac{m^2}{2}\left(\frac{1}{6}\gamma l_P^2\right)^2\frac{\partial^2 T}{\partial p^2} \\ &+ \frac{m^3}{6}\left(\frac{1}{6}\gamma l_P^2\right)^3\frac{\partial^3 T}{\partial p^3}+\frac{m^4}{24}\left(\frac{1}{6}\gamma l_P^2\right)^4\frac{\partial^4 T}{\partial p^3 4} \end{aligned} \quad (3.25)$$

Using this we can write the difference equation (3.18) as a differential equation:

$$3p^{1/2}\left(\frac{4}{9}l_P^4 T''-k^2T+\frac{4}{243}\gamma^2 l_P^8 T''''\right) = -2\kappa H_{matter}T \quad (3.26)$$

[Notation $T' = \partial T/\partial p$]

We shall determine the properties of the wave function using *WKB Approximation* to take the semi-classical limit. The conditions for the validity of the WKB approximation is that the quantum corrections must be small compared to the classical solution. In the following derivation we will begin with explicit factors of $\hbar$. However at the end we will obtain the answers in terms of l_P^2 and then we can go back to $\hbar = 1$ units.

Consider a quantum mechanical system satisfying $[X, P] = i\alpha\hbar$.Therefore

$$[X, \frac{P}{\alpha}] = i\hbar$$

Now consider the time independent Schroedinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$$

Then we will have $P/\alpha = \sqrt{2mE}$.
In WKB approximation we make the ansatz

$$\psi = Ae^{\frac{iB}{\hbar}}$$

Putting in the time independent Schroedinger equation and looking at the real part we have

$$A\left[\frac{\partial^2 A}{\partial X^2} - \frac{1}{\hbar^2}\left(\frac{\partial B}{\partial X}\right)^2 + \frac{2mE}{\hbar^2}\right] = 0 \tag{3.27}$$

When $\hbar \to 0$ we have

$$\frac{1}{\hbar^2}\left(\frac{\partial B}{\partial X}\right)^2 = \frac{2mE}{\hbar^2} = \frac{P^2}{\hbar^2\alpha^2} \tag{3.28}$$

Coming back to our system which satisfies the commutation relation given by (3.13). It is clear that for our system:

$$\begin{aligned} X &\to p \\ P &\to K \\ \alpha &\to \kappa/3 \end{aligned} \tag{3.29}$$

So if we use the WKB approximation with the same ansatz in our system we would be able to directly replace

$$\frac{1}{\hbar^2}\left(\frac{\partial B}{\partial p}\right)^2 \rightarrow \frac{9K^2}{\kappa^2\hbar^2}$$

For this we need to prove that when the WKB ansatz is put into our differential equation(3.26), in the limit $\hbar \rightarrow 0$ we will have only terms of the form $(\partial B/\partial x)^n$.

Looking at the structure of initial difference equation (3.18) and at the way in which we derived the differential equation (3.26) it is clear that to all orders

- odd derivatives of T will be absent
- even derivatives of T will be present multiplied by the corresponding power of l_P^2

Let us try to derive a recursion relation between the coefficients of the real part of the WKB expansion:

$$\begin{aligned} T &= Ae^{\frac{iB}{\hbar}} \\ \text{Let} \quad T_k &= \frac{\partial^k T}{\partial p^k} = (A_k + iB_k)e^{\frac{iB}{\hbar}} \\ T_{k+1} &= \left[\left(\frac{\partial A_k}{\partial p} - \frac{B_k}{\hbar}\frac{\partial B}{\partial p}\right) + i\left(\frac{A}{\hbar}\frac{\partial B}{\partial p} + \frac{\partial B_k}{\partial p}\right)\right]e^{\frac{iB}{\hbar}} \\ &= (A_{k+1} + iB_{k+1})e^{\frac{iB}{\hbar}} \end{aligned} \tag{3.30}$$

The real part of T_{k+2} is:

$$\left[\frac{\partial^2 A}{\partial p^2} - \frac{1}{\hbar}\frac{\partial B}{\partial p}\frac{\partial B_k}{\partial p} - \frac{B_k}{\hbar}\frac{\partial^2 B}{\partial p^2} - \frac{A}{\hbar^2}\left(\frac{\partial B}{\partial p}\right)^2 - \frac{1}{\hbar}\frac{\partial B}{\partial p}\frac{\partial B_k}{\partial p}\right] \tag{3.31}$$

From this we see that in the limit $\hbar \rightarrow 0$ the most dominant term in every even derivative of T will be of the form:

$$-\frac{A}{\hbar^{2n}}\left(\frac{\partial B}{\partial p}\right)^{2n}$$

Now we are free to make the following replacements in the differential equation

$$T'' \rightarrow \frac{9K^2}{\kappa^2\hbar^2} = \frac{9K^2}{l_P^4} \tag{3.32}$$

$$T'''' \rightarrow \frac{81K^4}{l_P^8} \tag{3.33}$$

Making these replacements in (3.26) we get our Hamiltonian constraint equation

$$\frac{3p^{1/2}}{2\kappa}[-(4K^2+k^2)+\frac{4}{3}\gamma^2K^4]=-H_{matter} \tag{3.34}$$

Again, as we did in string theory, as a first step we will set the matter Hamiltonian to be zero and look only at the gravity sector. The equations of motion coming from this constraint are:

$$\dot{p} = \frac{\kappa}{3}\frac{\partial H}{\partial K}=\frac{1}{2}p^{1/2}[-8K+\frac{16}{3}\gamma^2K^3] \tag{3.35}$$

$$\dot{K} = -\frac{\kappa}{3}\frac{\partial H}{\partial p}=\frac{1}{4}p^{-1/2}[(4K^2+k^2)-\frac{4}{3}\gamma^2K^4] \tag{3.36}$$

Putting p as a^2 into (3.35) we get $K=(-1/2)\dot{a}$ for the leading order i.e. $\gamma\to 0$. When these variables are put in the equations (3.34, 3.36) in the limit $\gamma\to 0$ we get the following :

$$\dot{a}^2+k^2 = 0$$

$$\ddot{a} = 0$$

These are the same equations as obtained from classical gravity (1.2, 1.3, 1.4) when the matter field is set to zero. So LQC gives the classical results in the leading approximation as expected.

To summarize, LQC implied the equation system of three equations (3.34, 3.35, 3.36) of which the equation of motion involving $\dot{p}$ gives the value of K as a function of a and $\dot{a}$. The constraint equation is the "Friedmann equation" while the equation of motion of $\dot{K}$ is the "Raychaudhuri equation". In the next chapter we will attempt to compare the corrections to the gravitational sector as predicted by string theory and LQC.

Chapter 4

COMPARISON OF RESULTS AND CONCLUSIONS

4.1 Problems and comparison

It is very difficult to compare two theories which are formulated differently. Even when some common ground is found it is a hard task to match conventions and variables so as to make a reasonable comparison.

Consider the string equations (2.22, 2.23)

$$\beta_{00} = \frac{-3\ddot{a}}{a}(1+\frac{\alpha'}{2}\frac{\ddot{a}}{a})+ = 0 \tag{4.1}$$

$$\beta_{ii} = (2(k+\dot{a}^2)+a\ddot{a})+\frac{\alpha'}{2}[2((k+\dot{a}^2)^2)+(a\ddot{a})^2] = 0 \tag{4.2}$$

$$\tag{4.3}$$

The LQC equations on the other hand are (3.34, 3.35, 3.36)

$$\frac{3p^{1/2}}{2\kappa}[-(4K^2+k^2)+\frac{4}{3}\gamma^2K^4] = 0 \tag{4.4}$$

$$\dot{p} = \frac{\kappa}{3}\frac{\partial H}{\partial K} = \frac{1}{2}p^{1/2}[-8K+\frac{16}{3}\gamma^2K^3] \tag{4.5}$$

$$\dot{K} = -\frac{\kappa}{3}\frac{\partial H}{\partial p} = \frac{1}{4}p^{-1/2}[(4K^2+k^2)-\frac{4}{3}\gamma^2K^4] \tag{4.6}$$

At the outset, we can see that the LQC equations are not even in terms of a and its derivatives. p is identified as a^2 at the kinematical level itself i.e. it is not changed by the dynamics. However the expression of K as a function of $\dot{a}$ is dependent on the dynamics and has to be obtained from these equations.

The most important thing we have to keep in mind is that the equations must be treated as perturbation expansions and not as ordinary algebraic equations. To illustrate the difference we give a small example below. Consider the equation

$$\epsilon x^2 + bx + c = 0 \Rightarrow x = -\frac{c}{b}, \text{ when } \epsilon = 0$$

where ϵ is a small real number. Solving as quadratic equation we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4\epsilon c}}{2\epsilon} \\ &= -\frac{b}{2\epsilon} \pm \frac{b}{2\epsilon}\left(1 - \frac{4\epsilon c}{b}\right)^{\frac{1}{2}} \\ &= -\frac{b}{2\epsilon}\left(1 \mp \left(1 + \frac{2\epsilon c}{b}\right)\right) \\ &= -\frac{b}{2\epsilon}\left(1 \mp 1 \mp \frac{2\epsilon c}{b}\right) \end{aligned}$$

As $\epsilon \to 0$ none of the solutions produce $-c/b$.

We can also solve it as perturbation series: $x = (x_0 + \epsilon x_1)$ upto $O(\epsilon)$

$$\begin{aligned} & & \epsilon(x_0^2 + 2\epsilon x_0 x_1) + bx_0 + b\epsilon x_1 + c = 0 \\ O(\epsilon^0) & \longrightarrow & bx_0 + c = 0 \Rightarrow x_0 = -\frac{c}{b} \\ O(\epsilon^1) & \longrightarrow & \epsilon x_0^2 + b\epsilon x_1 = 0 \\ & & x_1 = -\frac{x_0^2}{b} = -\frac{c^2}{b^3} \end{aligned}$$

Here we get back the original value of x as the perturbation parameter goes to zero. Any algebraic manipulation of the equations must be done in this spirit.

Keeping this in mind we expand $K = K_0 + \gamma^2 K_1$. Using this in equation 4.5 and keeping terms upto $O(\gamma^2)$ only we have

$$\begin{aligned} K &= K_0 + \gamma^2 K_1 \\ &= -\frac{1}{2}\dot{a} - \gamma^2 \frac{\dot{a}^3}{12} \end{aligned} \quad (4.7)$$

Now the equation (4.6) vanishes automatically because it is a multiple of the Hamiltonian constraint which is 0. Hence the two equations coming from LQC are the $H = 0$ equation and the $\dot{K} = 0$ equation. Again solving them perturbatively using the value of K we have found, we get:

$$a(\dot{a}^2 + k^2) - \gamma^2 \frac{a\dot{a}^4}{4} = 0 \tag{4.8}$$

$$\ddot{a} + \gamma^2 \frac{\dot{a}^2 \ddot{a}}{2} = 0 \tag{4.9}$$

Now we have put both the string and the LQC corrections in terms of the scale factor a and its derivatives. The equations still look different. LQC equations are in the form of a constraint equation (i.e. equation independent of second derivatives) and an evolution equation in terms of the second derivatives. By contrast both the string equations involve first and second derivatives. We will now try to see what the solutions of these equations imply. Again in the spirit of perturbation theory we will look for solutions of a of the form

$$a = a_0 + \epsilon a_1 \tag{4.10}$$

where ϵ will be α' in case of string theory and γ^2 in case of LQC. Using this we can see from both sets of equations (4.1, 4.2) and (4.8, 4.9) the solutions imply that if a_0 is not zero

$$\ddot{a}_0 = 0 \qquad \ddot{a}_1 = 0 \qquad \Rightarrow \ddot{a} = 0 \tag{4.11}$$

$$\dot{a}_0 = 0 \qquad \dot{a}_1 = 0 \qquad \Rightarrow \dot{a} = 0 \tag{4.12}$$

Thus we have the same implication from both the theories. In both the theories Minkowskian metric remains stable at least upto first order in perturbation theory. Whether this is a generic feature present to all orders in perturbation theory is a question we are working on. Another question we are addressing at the moment is whether inclusion of matter changes these predictions. However incorporating matter involves additional complications.

For example when incorporating matter an important point that must be kept in mind is that the Weyl anomaly coefficients that we calculated in Chapter 2 are actually done in what is called *String Frame.* There the string theory effective action has the form:

$$S = \frac{1}{2\kappa_0}\int d^DX(-G)^{1/2}e^{-2\phi}[-\frac{2(D-26)}{3\acute{\alpha}'} + R + 4\partial_\mu\phi\partial^\mu\phi + O(\acute{\alpha}')] \qquad (4.13)$$

To compare with equations of motion obtained from the classical $(-g)^{1/2}R$ actions we need to make a field redefinition and go to what is known as "*Einstein frame*". For that we need to make a spacetime Weyl transformation on our metric:

$$\tilde{G}_{\mu\nu}(X) = e^{2\phi(X)}G_{\mu\nu}(X)$$

Then we have the following action :

$$\acute{S} = \frac{1}{2\kappa}\int d^DX(-\tilde{G})^{1/2}[-\frac{2(D-26)}{3\acute{\alpha}'}e^{4\tilde{\phi}/(D-2)} + \tilde{R} - \frac{4}{D-2}\partial_\mu\tilde{\phi}\tilde{\partial}^\mu\tilde{\phi} + O(\acute{\alpha}')] \qquad (4.14)$$

In this action the tildes indicate that the raising of indices is done with $\tilde{G}_{\mu\nu}$. To do that we would have had to change our metric from

$$ds^2 = -dt^2 + a^2ds_3^2$$

to

$$ds'^2 = -\Omega^2dt^2 + a^2\Omega^2ds_3^2$$

where Ω is e^ϕ. Even then note that in General Relativity we usually write the action as

$$S \sim \frac{1}{\kappa}(-g)^{1/2}R - \partial_\mu\phi\partial^\mu\phi$$

Comparing this with the relative coefficients of the string theory effective action(4.15) at D=4 we see that to get the correct classical equations of motion:

$$\phi \;\rightarrow\; \kappa\phi/\sqrt{2} \qquad (4.15)$$

We are currently working on these issues.

4.2 Conclusion

In this work we have taken the first step to compare the two candidates of the theory of quantum gravity in the cosmological context. We have highlighted where and to what extent a comparison is possible. The Weyl anomaly results in string theory giving corrections to Einstein's equations are already available and we just had to specify the metric (FRW) and find the equations of motion. In the LQC context we had to derive the effective classical Hamiltonian from the quantum theory in the continuous approximation combined with the WKB ansatz. However general semi classical limit in terms of physical states is an open problem in LQG.

We have found that in cosmology without matter and with the cosmological constant zero, both the theories imply that the Minkowskian metric is stable to the leading order. However more work needs to be done before this can be confirmed as a generic result. Also matter in some form needs to be incorporated. We are working on these issues at present.

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