ASPECTS OF HIGH ENERGY SCATTERING IN ABELIAN AND NON ABELIAN GAUGE THEORIES

By

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BONAFIDE CERTIFICATE

Certified that this thesis titled " ASPECTS OF HIGH ENERGY SCATTERING IN ABELIAN AND NON ABELIAN GAUGE THEORIES " is the bonafide work of Mr. TAPOBRATA SARKAR who carried out the research under our supervision. Certified further, that to the best of our knowledge the work reported herein does not form part of any other thesis or dissertation on the basis of which a degree or award was conferred on a earlier occasion on this or any other candidate.

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ABSTRACT

In this thesis, we review certain aspects of high energy scattering in abelian and non abelian gauge theories. This subject has been extensively studied over the last few decades, using perturbative techniques. Recently it has been pointed out that by a scaling argument, one can decouple certain degrees of freedom from the theory so that the resulting effective theory is simpler and of lower dimensions, and hopefully reproduces all known physics in this kinematic regime. We review the standard results as well as the scaling argument in the context of QED and QCD, but show that whereas in QED the scaling argument reproduces the standard results, there is as yet no sign of the well known reggeization of the gluon propagator in QCD.

திட்டப்பணியின் சுருக்கம்

இந்த ஆராய்ச்சிக் கட்டுரையில், அபிலியன் அபிலியனில்லாத அளவு கோட்பாடுகளிலுள்ள உயர் சக்தி சிதறலின் சில தன்மைகளை ஆய்வு செய்கிறோம். இந்தக் கருத்து (பொருள்) பல ஆண்டுகளாக Perturbative உத்திகளைப் பயன்படுத்தி மிக விரிவாக படிக்கப்பட்டு (ஆராயப்பட்டு) வருகின்றது. சமீப காலமாக, scaling வாதத்தின் மூலமாக, கோட்பாடுகளிலிருந்து தன்னுரிமையின் சில பாகைகளை (இணை முறிக்கலாம்) பிரிக்கலாம். இதனால் விளையும் கோட்பாடு எளிமையாகவும், தாழ்வான பருமனுள்ளதாயும், இந்த kinematic அனைத்து தெரிந்த பெளதிகத்தை மறு ஆக்கம் செய்வதாயும் இருக்கிறது. நாம் தரமான முடிவுகளையும், QED & QCD கட்டத்தின் scaling வாதத்தையும் ஆய்வு செய்து, QED-யில் scaling வாதம் தரமான முடிவுகளை மறு ஆக்கம் செய்யும்பொழுது, இன்னமும் QCD-யில் Reggeization of the gluon propagator பற்றிய அறிகுறி ஒன்றும் இல்லை என்பதைக் காட்டுகிறோம்.

CHAPTER 1

INTRODUCTION

High energy scattering in abelian and non abelian gauge theories have been extensively studied in the recent past. One motivation for doing this is that physics gets considerably simplified at these kinematical regimes and one has a better theoretical understanding of the situation. Indeed it has been shown that QCD at such high energies has a remarkable resemblance to a two dimensional field theory. Systematic procedures have been developed to extract the high energy behaviour of the scattering amplitudes in the framework of perturbation theory.

Formally we may consider the near forward scattering of two particles moving in the "longitudinal" direction i.e in the $z - t$ plane, which are widely separated in the "transverse" direction i.e in the $x - y$ plane. This is the situation we shall be mostly concerned with, and in this case one might expect intuitively that the transverse degrees of freedom get decoupled from the system, and non trivial physics takes place in the longitudinal direction. Thus on physical grounds we may hope to write the full four dimensional theory describing the scattering as some effective theory in two dimensions.

Recently an interesting attempt in this direction has been made by E.Verlinde and H.Verlinde. They have advanced a unique scaling argument that naturally results in the decoupling of the longitudinal and transverse degrees of freedom and gives from the QCD action a two dimensional topological field theory, which one might then attempt to solve in the framework of perturbation theory.

This thesis aims to study some aspects of high energy two particle scattering processes in gauge theories that are charactarised by large centre of mass energy and fixed low momentum transfer. To this effect we consider a process like $A + B = C + D$ and define the usual Mandelstam variables $s = (P_A + P_B)^2$, the square of the centre of mass energy and $t = (P_A - P_C)^2$, the square of the transverse momentum transfer. We shall be interested in the kinematical regime where $s/t \to \infty$ although in the asymptotically free non abelian gauge theories we shall keep $t > 1GeV$ (the QCD scale) so that perturbation theory is still valid.

The thesis is organized as follows. In chapter 2 examine high energy scattering in the case of abelian gauge theories, namely QED. We briefly review Jackiw's semiclassical way of computing the scattering amplitu de.Next we present Verlindes' scaling argument in the context of abelian gauge theories and show how to recover the old results for the amplitude in this approach.We also briefly review the essential features of the scaling argument.

Chapter 3 is concerned with aspects of high energy scattering in Quantum Chromodynamics. We first review the work of Nachtmann, wherein he obtains an expression for the scattering amplitude for two fast moving quarks, by solving the Dirac equation for one quark in the background gauge field produced by the other. We show how this amplitude reduces to the expectation value of the two Wilson lines representing the two quarks. Next, we discuss Verlindes' scaling argument in QCD and compute the expression for the effective action in this case, from which the scattering amplitude at high energies may be computed.

In chapter 4 we continue with the discussion of quark-quark scattering in the extremely high energy limit, in the framework of perturbation theory. We first summarize the standard results in this area, obtained by computation of Feynman diagrams, which shows the *reggeization* of the gluon propagator. Next we review the work done in order to compute this amplitude using the effective action at high energies. First we discuss the case of $2+1$ dimensional QCD, where using the Verlindes' scaling argument, one gets an effective one dimensional action, but the gluon propagator does not reggeize, as can be shown by diagrammatic calculations. Next, we propose a non diagrammatic method of calculating the amplitude in $3+1$ dimensions using a new parametrization of the effective action, and show that the scattering amplitude does not show regge behaviour. Further, we make a few comments regarding this computation and the reggeization of the gluon propagator.

Finally we summarize the results in the concluding chapter and make a few observations regarding future directions and open problems in this area.

This thesis also contains an appendix, that is intended to outline the machinery required to compute Feynman diagrams in the high energy limit.

CHAPTER 2

HIGH ENERGY SCATTERING IN QUANTUM ELECTRODYNAMICS

In this chapter we deal with ultra high energy scattering in the context of Quantum Electrodynamics. In the energy regime of our interest, it is possible to compute the scattering amplitude for a two particle elastic scattering process semi-classically, as done by Jackiw et al (Jackiw et al, 1992). We review this calculation first, and then discuss Verlindes' scaling argument as applied to QED. We show that scaling produces an effective action, which reproduces the result for the scattering amplitude obtained semi classically. This shows that the scaling argument is correct at least in QED. In all the analysis to follow, we shall use natural units, $\hbar = c = 1$. Also, we do not take gravitational effects into account, and for all our purposes, we shall take the flat metric, $g_{\mu\nu} = (1, -1, -1, -1,)$.

2.1 Shock Wave Picture

Let us consider the scattering of two electrically charged particles, assumed spinless for the time being, having very small rest masses, and the system having very large centre of mass energy. We go to a frame of reference in which one of the particles is almost at rest, and consider the scattering of this particle in the background electromagnetic field produced by the other, which is moving almost at the speed of light. This essentially takes us to the "shock wave" picture. Consider a charged particle moving in the z direction. When the particle is at rest, the vector potential due to it has the spherically symmetric form,

$$A^0 = \frac{e'}{r}, \quad A^i = 0; \quad i = 1, 2, 3. \tag{2.1}$$

where e' is its charge. Now when it moves, the spherical symmetry of the potential is lost, and the fields get contracted in the z direction. Finally when the speed of the particle has almost reached the speed of light, the electric and magnetic fields in the z direction become zero, and the fields reside entirely in the $x - y$ plane, which we shall call the shock plane. Hence when such a fast moving particle scatters off a slow moving one, the effect is *instantaneous*, the interaction being there at the instant the shock plane hits the charge at rest. To actually compute the scattering amplitude for such a process, the idea is as follows (Jackiw et al, 1992): we take the electromagnetic potential due to a charge at rest and then we give it a Lorentz boost along the positive z axis, with a boost parameter β. The gauge potentials A^μ transform according to the laws of special relativity. On taking the limit, $\beta \longrightarrow 1$, the potentials due to the lightlike particle may be found. Next we consider the quantum mechanical wave function of the slow moving particle. Given the boosted form of the gauge potential, we can easily calculate the phase shift produced due to this and the scattering amplitude can thus be evaluated. Let us suppose that the charge e' moves with a relative velocity β along the positive z direction with respect to a stationary frame. Then the boosted form of the potential as seen by the latter are:

$$\begin{aligned} {}^\beta A^0 &= \frac{A^0}{\sqrt{1-\beta^2}} \\ {}^\beta A^3 &= \frac{\beta A^0}{\sqrt{1-\beta^2}} \end{aligned} \tag{2.2}$$

The other components being unchanged. Now, when we put in the expressions for A^0 and express the coordinates in terms of their Lorentz transformed version, the above can be written as

$${}^\beta A^\mu = \eta^\mu_\beta \frac{e'}{R_\beta}, \qquad \eta^\mu_\beta = (1,0,0,\beta) \tag{2.3}$$

where

$$R_\beta = [(z-\beta t)^2 + (1-\beta^2) r_\perp^2]^2. \tag{2.4}$$

Now if we take the limit $\beta \longrightarrow 1$, and we use the result

$$Lim_{\beta\to 1}\frac{1}{R_\beta} = \frac{1}{|x^-|} - \delta(x^-)\, ln\left(\mu^2 r_\perp^2\right). \tag{2.5}$$

Where μ is an arbitrary parameter introduced to make the argument of the logarithm dimensionless. It may be noted that this form of the boosted potential can be attributed to a source current of the form

$${}^\beta j^\mu = e'\eta^\mu_\beta\, \delta^2(\mathbf{r}_\perp)\delta(z-vt), \quad \eta^\mu_\beta = (1,0,0,\beta) \tag{2.6}$$

as is to be expected for a particle, moving along the positive z direction. We note that in the limit $\beta \longrightarrow 1$ we get the form of the current

$${}^\beta j^\mu = e'\eta^\mu_\beta\, \delta^2(\mathbf{r}_\perp)\,\delta(x^-), \tag{2.7}$$

that is we have particle moving at the speed of light along the trajectory $z = t$. Now we substitute Equation (2.5) into Equation (2.3) and note that the term $e'/|x^-|$ is a pure gauge and does not contribute to the electromagnetic fields, and hence it may be conveniently dropped. Hence we finally obtain for the boosted potential

$$A_I^0 = A_I^3 = -2e' ln\,(\mu r_\perp)\,\delta(x^-); \qquad A_{\perp I}^i = 0, i = 1, 2. \tag{2.8}$$

An equivalent form of the potential which is obtained by a gauge transformation of the above is given by

$$\begin{aligned} A_{II}^0 &= A_{II}^3 = 0, \\ A_{\perp II}^i &= -\frac{e'}{2\pi}\theta(t - z)\nabla ln\ \mu r_\perp \end{aligned} \tag{2.9}$$

While Equation (2.8) clearly shows that the gauge potential is zero everywhere except the "shock plane" ($x^- = 0$) the form in Equation (2.9) is more useful as it is of the form of a total derivative on the transverse plane. For completeness, we give the form of the electric and the magnetic field strengths

$$E^i = \frac{2e' r_\perp^i}{r_\perp^2}\delta(x^-); \qquad E^z = 0. \tag{2.10}$$

$$B^i = -\frac{2e' \epsilon_{ij} r_\perp^j}{r_\perp^2}\delta(x^-); \qquad B^z = 0. \tag{2.11}$$

Thus as commented earlier, we see that the electric and magnetic fields are zero along the direction of boosting and reside entirely on the transverse plane, conforming to the shock wave picture. Now, we look at the charge we took to be at rest to begin with. Say it has a charge e. From the shock wave form of the vector potential it is clear that for times $t < z$, the particle is free and its quantum mechanical wave function is given by

$$\psi_< \left(x^\pm, \mathbf{r}_\perp\right) = \psi_0 = exp[ikx], \qquad for\ x^- = 0. \tag{2.12}$$

As the shock wave interacts instantaneously, the wave function picks up a phase factor, $exp(ie \int dx^\mu A_\mu)$. Inserting the form of the boosted potential in this expression, we obtain for the phase shifted wave function:

$$\psi_> \left(x^\pm, \mathbf{r}_\perp\right) = exp(-iee' ln\left(\mu^2 r_\perp^2\right)\psi_0', \qquad for\ x^- > 0. \tag{2.13}$$

Where ψ_0 and ψ_0' are related by the continuity requirement

$$\psi_< = \psi_>, \qquad at\ x^- = 0. \tag{2.14}$$

Now it is straightforward to expand the phase shifted wave function in terms of complete set of momentum eigenstates, in the form,

$$\psi_{>} = \int dp_{+}dp_{-}d^{2}p_{\perp}A\,(p_{+},\mathbf{p}_{\perp})\,exp\;i[\mathbf{p}_{\perp}.\mathbf{r}_{\perp} - p_{+}x^{-} - p_{-}x^{+}] \qquad (2.15)$$

with the on shell condition $p_{+} = (p_{\perp}^{2} + m^{2})/k_{-}$. Clearly, the A's are the amplitude for scattering from an initial momentum k to a final momentum p. We proceed to calculate it by multiplying both sides of the above equation by a plane wave and integrating over x^{-}.Using the orthonormality of wave functions, we get

$$A\,(p_{+},p_{\perp}) = \frac{\delta\,(p_{+} - k_{+})}{4\pi^{2}}\int d^{2}r_{\perp}\;exp\;i\left(-2ee'\;ln(\mu r_{\perp}) + \mathbf{q}.\mathbf{r}_{\perp}\right), \qquad (2.16)$$

where we have denoted by $\mathbf{q} = \mathbf{k}_{\perp} - \mathbf{p}_{\perp}$ as the transverse momentum transfer. Now, putting $x = r_{\perp}\;cos\theta; \quad y = r_{\perp}\;sin\theta$, The above integral reduces to

$$A\,(p_{+},p_{\perp}) = \frac{\delta\,(p_{+} - k_{+})}{4\pi^{2}}\int dr_{\perp}d\phi(r_{\perp})^{1\,-\,2ee'}\;e^{qr_{\perp}cos\phi}, \qquad (2.17)$$

Here we have set the irrelevant dimensional scale parameter μ to 1. This integral is standard and finally we get for the scattering amplitude the result

$$f\,(s,t) = \frac{k_{+}}{4\pi k_{0}}\delta\,(p_{+} - k_{+})\frac{\Gamma\left(1 - iee'\right)}{\Gamma\,(iee')}\left(\frac{4\mu^{2}}{-t}\right)^{1\,-\,iee'} \qquad (2.18)$$

It is to be noted that we have computed the scattering amplitude semi classically, using the prescription of Jackiw et al. This type of semiclassical computation of scattering amplitude in the very high energy limit was first done in the case of gravity by 't Hooft. ('t Hooft, 1987). It should also be mentioned that the whole analysis above has been carried out for scalar particles. It has been shown (Jackiw et al, 1992) that with the inclusion of spin the formula for the scattering amplitude remains essentially the same apart for a kinematic factor of $m^{2}/2\pi^{2}s$ in front.

2.2 Verlindes' Scaling Argument

In this section, we review the scaling argument due to Verlinde and Verlindes (E.Verlinde and H.Verlinde, 1993) which gives an alternative way to look at the problem of high energy scattering.The basic idea is to consider the action formulation of the theory (QED in this case) and see what simplification can be made at the level of the action itself in the high energy limit. Let us assume that the particles that scatter move in the positive z direction, with four momentum $p^\mu = (p^0, \mathbf{p})$. For particles that move close to the speed of light, we have thus, $p^0 = p^3 = E$ where $E = \sqrt{s}$, and since the scattering is almost forward, the transverse momentum transfer $\sqrt{t}$ is extremely small. Here s and t are the usual Mandelstam variables. Thus we see that the square roots of s and t measure the typical momenta associated with the longitudinal and the transverse directions respectively. Now in natural units, momentum has the same dimensions as the inverse of length. Hence, it seems reasonable to associate *two* length scales, along these two directions. The charactaristic length scale along the longitudinal direction being much bigger that that along the transverse direction. Hence we perform a rescaling of the coordinates,

$$x^\alpha \rightarrow \lambda x^\alpha \tag{2.19}$$

$$x^i \rightarrow x^i \tag{2.20}$$

where α , β runs over the light cone coordinates $+$, $-$ and i signifies the space coordinates x, y, and λ is some scale parameter that we will ultimately take to be very small. Note that we have conveniently kept the scale parameter of the transverse direction to be 1, without any loss of generality. The gauge potentials A^μ have the dimension of $(length)^{-1}$ and hence their transformation property under the rescaling is given by

$$A^\alpha \rightarrow \lambda^{-1} A^\alpha, \qquad A^i \rightarrow A^i. \tag{2.21}$$

Now let us consider the usual Maxwell action for electrodynamics

$$S = -\frac{1}{4}\int d^4x \; (F_{\mu\nu}F^{\mu\nu}) \tag{2.22}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field tensor and $A^\mu = (A^0, \mathbf{A})$ is the four potential. If we substitute the rescaled form of the coordinates and the potential back in the action then we get a *scaled* action,

$$S = -\frac{1}{4}\int d^4x \left(\lambda^{-2} F_{\alpha\beta}F^{\alpha\beta} + 2F_{\alpha i}F^{\alpha i} + \lambda^2 F_{ij}F^{ij}\right) \tag{2.23}$$

Now we choose the scale factor in a way to incorporate the high energy limit in the action, namely we put

$$\lambda = \frac{k}{\sqrt{s}} \rightarrow 0 \tag{2.24}$$

where k is a constant having the dimensions of energy which we may put to be $= 1$ Thus, the limit $s \rightarrow \infty$ becomes equivalent to $\lambda \rightarrow 0$. From the scaled form of the action now it is clear that in this limit, the purely transverse part of the action in fact drops out and we are left with

$$S = -\frac{1}{4}\int d^4x \left(\lambda^{-2} F_{\alpha\beta}F^{\alpha\beta} + 2F_{\alpha i}F^{\alpha i}\right) \tag{2.25}$$

Thus we see that the scaling corresponds to a truncation of the full theory, i.e the contribution of the modes removed by the truncation is subleading for $s >> t$. Consider now the partition function

$$Z = \int DA\ exp(iS)\,. \tag{2.26}$$

Due to the smallness of the term λ it is clear that the term $\lambda^{-2}F_{\alpha\beta}F^{\alpha\beta}$ on exponentiation will be extremely oscillatory and one expects that all non zero field configurations will average out to zero. The only contribution comes from the configuration for which

$$F_{\alpha\beta} = 0, \tag{2.27}$$

i.e, $F_{+-} = 0$ which implies that $E_z = 0$. Similarly, if we write the action in the dual formalism, the scaling would imply $\tilde{F}_{+-} = B_z = 0$, where $\tilde{F}$ denotes the dual field strength tensor. Thus we are naturally led to the shock wave picture by the Verlinde scaling, in which as already mentioned, the electric and magnetic fields are zero in the z direction. Now, $F_{\alpha\beta} = 0$ means that

$$\partial_\alpha A_\beta - \partial_\beta A_\alpha = 0 \tag{2.28}$$

This in turn implies that

$$A_\alpha = \partial_\alpha\, \Omega \tag{2.29}$$

Thus,

$$F_{\alpha i} = [\partial_\alpha (A_i - \partial_i A_\alpha)] \tag{2.30}$$

Hence in this case, the action simplifies to

$$S = \frac{1}{2}\int d^4x\, [(\partial_\alpha A_i - \partial_\alpha \partial_i \Omega)]^2 \tag{2.31}$$

The Euler Lagrange equation for A_i calculated from the action gives

$$\partial_+\partial_- (A_i - \partial_i\Omega) = 0 \tag{2.32}$$

Which means that $(A_i - \partial_i\Omega)$ is a harmonic function, i.e

$$(A_i - \partial_i\Omega) = (A_i - \partial_i\Omega)_+ + (A_i - \partial_i\Omega)_- \tag{2.33}$$

Where the $+$ and the $-$ subscripts denote functions of x^+ and x^- respectively. At this point, we refer to the argument of Nachtmann (Nachtmann, 1991), wherein he has solved the Dirac equation of non abelian gauge particles in the extremely high energy limit, and has shown that the amplitude for scattering of two such particles in this limit reduces to the correlation of two light like Wilson lines. We shall review his arguments in detail in the next chapter, but for the time being we state that the same results are valid in this (abelian) case, and hence we shall not be concerned with the fermionic part of the action, namely $\bar{\psi}\,(i\,\not{D}\;-\;m)\,\psi$, but we shall simply replace the particles by their classical trajectories. This fact is clear if we recall that in the energy regime under consideration, the scattering is almost forward, and the particles hardly deviate from their classical trajectories, as the transverse momentum transfer is very small. Also, the mass term $m\bar{\psi}\psi$ drops out in the scaling limit, a point we shall come back to later. Hence, if we substitute the classical solutions for A_i in the action, the latter reduces to

$$S \;=\; -\frac{1}{4}\int d^2z \int_{-\infty}^{\infty} dx^+\partial_+\,(A_i\;-\;\partial_i\Omega)\int_{-\infty}^{\infty} dx^-\partial_-\,(A_i\;-\;\partial_i\Omega) \tag{2.34}$$

Hence, denoting the asymptotic values of Ω at the end points of the particle trajectories as g_1, g_2 and h_1, h_2 respectively, we have

$$\int_{-\infty}^{\infty} dx^-\partial_-\,(A_i\;-\;\partial_i\Omega)\;=\;\partial_i\,(g_2\;-\;g_1) \tag{2.35}$$

$$\int_{-\infty}^{\infty} dx^+\partial_+\,(A_i\;-\;\partial_i\Omega)\;=\;\partial_i\,(h_2\;-\;h_1) \tag{2.36}$$

$$\tag{2.37}$$

Where we have imposed the condition that the value of A_i is the same at the two end points of each Wilson line. Thus, denoting $(g_2\;-\;g_1)\;=\;g$ and $(h_2\;-\;h_1)\;=\;h$, The electrodynamics action finally reduces in the scaling limit to

$$S\;=\;-\frac{1}{4}\int d^2z\;\partial_i g\partial_i h \tag{2.38}$$

Now, following the work of Nachtmann, we define the scattering amplitude to be

$$f(s,t)\;=\;\int d^2z e^{iq.z}\;<\;V_+\,(z)\,V_-\,(z)\;> \tag{2.39}$$

Where V_+ and V_- are the Wilson line operators,

$$V_\pm\;=\;exp\Big(\int_{-\infty}^{\infty} dx^\pm A_\pm(z)\Big) \tag{2.40}$$

Here we see that since $A_\pm\;=\;\partial_\pm\Omega$ hence,

$$\int dx^+A_+\;=\;g_2\;-g_1\;=\;g \tag{2.41}$$

$$\int dx^-A_-\;=\;h_2\;-h_1\;=\;h \tag{2.42}$$

Hence the scattering amplitude reduces to

$$f(s,t) = \int d^2ze^{iq.z} < e^g\ e^h > \tag{2.43}$$

Now,we note that $< e^g\ e^h > = e^{<gh>}$. Further from the action we see that the $g-h$ propagator is given by $< g(x)\ h(y) > = ln|x - y|$,so that the amplitude is

$$f(s,t) = \int d^2ze^{iq.z}\ e^{ln|\ z\ |} \tag{2.44}$$

modulo some arbitrary constants. This integral has been computed ('tHooft, 1987) and, putting back all factors we see that the scattering amplitude reduces to the previous shock wave calculation of Jackiw et al, Equation (2.18).

Thus we see that the scattering amplitude computed using the Verlindes' scaling argument exactly corresponds to that calculated in the shock wave limit. This is of course expected as we have already seen how the scaling leads in a natural way to the shock wave picture. This gives strong evidence that the scaling approximation is indeed a valid way of tackling the problem of high energy scattering. It now remains to apply the scaling procedure in the context of non abelian gauge theories as we shall do shortly.

However at this stage, we must mention that in the Verlindes' scaling approximation, we have assumed exactly light like Wilson lines, that is, we have taken the particles to be massless. Presumably if we include a small mass for the particles, it is expected to give corrections to the exact result. Indeed, as we shall show in the non-abelian case, there arises non trivial propagator singularities if we take this massless limit, and we are in fact compelled to add a small mass term correction to the Wilson lines, to regulate the divergences. But as it turns out, in electrodynamics, these singularities are not manifestly present. The other point to note is that in the scaling process, we have taken the limit $\lambda \to 0$, and have dropped all the subleading terms in the action.These terms may provide corrections to our results and one possible way to implement this is to take the $\lambda \to 0$ limit at the end. It must also be mentioned that the same amplitude for high energy scattering in quantum electrodynamics can be computed diagramatically, (Itzykson and Abarbanel, 1969), wherein we sum all the Feynman diagrams of the ladder type in the high energy limit and essentially get the same result as obtained in the semi classical computation. This is the so called eikonal approximation in QED.

CHAPTER 3

ASPECTS OF HIGH ENERGY SCATTERING IN QUANTUM CHROMODYNAMICS

In this chapter we first review some results on high energy scattering of non abelian gauge particles, in particular, quarks. The problem of quark-quark scattering at extremely large centre of mass energies is important in the following sense: to the first approximation we may treat high energy hadron hadron scattering as a sum of incoherrent quark-quark scatterings. Of course the secondary and tertiary collisions of the scattered quarks will make the problem much more complicated but one may attempt to calculate these corrections once the primary amplitude has been computed. We shall do this in two ways, first we discuss the work of Nachtmann (Nachtmann, 1991) wherein he attempts to solve the Dirac equation for one of the quarks in the high energy limit in the background gauge field produced by the other quark. We then look at the Verlindes' scaling argument as applied to this non abelian gauge theory, and indicate how the quark-quark scattering amplitude may be computed in this picture. We also make a few comments regarding the calculation of this in the shock wave picture.

3.1 Quark-Quark Scattering in the Functional Integral Approach

We start with the QCD Lagrangian,

$$L(x) \;=\; -\frac{1}{4}Tr\,(F_{\mu\nu}F^{\mu\nu}) \;+\; \bar{\psi}\,(i\,\not{D} \;-\; m)\,\psi, \qquad (3.1)$$

where ψ is the quark field for mass m. We denote the gauge fields by A_μ, and as usual, we have,

$$A_\mu \;=\; A^a_\mu\frac{\lambda^a}{2}, \qquad (3.2)$$

$$F_{\mu\nu} = \partial_\mu A\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu], \tag{3.3}$$
$$D_\mu = \partial_\mu + gA_\mu \tag{3.4}$$

Now let us consider the scattering of two quarks. We work in the Feynman gauge, and assume that the incoming quark states are described by free particle wave functions, which carries an extra colour index that we do not write explicitly. Let us consider the quark two point function. This is given by the functional integral

$$< 0|T(\psi(x_1)\psi(x_2))|0 > = Z^{-1}\int D(A,\psi,\bar{\psi})\;\psi(x_1)\;\psi(x_2)\;exp\left[\,i\int dx\;L\right] \tag{3.5}$$

where,

$$Z = < 0\;out\;|0\;in > = \int D(A,\psi,\bar{\psi})exp\left[\,i\int dxL\,\right] \tag{3.6}$$

We then integrate over the fermionic modes to obtain the fermionic determinant, and putting back in the above equation, it can be shown to reduce to

$$< 0|T(\psi(x_1\bar{\psi}(x_2))|0 > = \left\langle \frac{1}{i}S_F(x_1,x_2;A)\right\rangle \tag{3.7}$$

Where S_F is the Green's function for a quark in an external gluon field which has been averaged over all field configurations. The Green's function of a quark in an external gluon field, satisfies

$$(i\gamma^\mu D_\mu - m)\,S_I(x,y;A) = -\delta(x - y) \tag{3.8}$$

The index I here stands for inhomogeneous, and may be Feynman ($I = F$) or retarded ($I = r$) boundary conditions. The free Green's functions for these two boundary conditions are given by:

$$S_F^0(x,y) = -\int\frac{dk}{2\pi^4}e^{-ik(x - y)}\frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \tag{3.9}$$
$$S_F^0(x,y) = -\int\frac{dk}{2\pi^4}e^{-ik(x - y)}\frac{\not{k} + m}{k^2 - m^2 + i\epsilon k^0} \tag{3.10}$$

We also have the following standard results for the Green's functions

$$S_I = S_I^0 - S_I^0(\,g\,\not{A})S_I \tag{3.11}$$
$$= S_I^0 - S_I^0(\,g\,\not{A})S_I^0 \tag{3.12}$$

The Green's function is needed in order to compute the scattering amplitude via the reduction formula. Using this formula, we can show that if S_{fi} denotes the scattering amplitude for the process $\psi(p_1) + \psi(p_2) = \psi(p_3) + \psi(p_4)$ with p_1, p_2, and p_3, p_4 denoting the momenta of the incoming and outgoing quarks respectively, then,

$$
\begin{aligned}
<\psi(p_3)\psi(p_4)|S|\psi(p_1)\psi(p_2)> \quad &\equiv \quad S_{fi} \\
&= \quad Z_\psi^{-2} < (p_3|\ (i\vec{\not\partial}\ -\ m)\ S_F\ (i\vec{\not\partial}\ +\ m)|p_1) \times \\
&\quad\ (p_4|\ (i\vec{\not\partial}\ -\ m)\ S_F\ (i\vec{\not\partial}\ +\ m)|p_2) \\
&\quad\ -\ (p_3\ \leftrightarrow\ p_4) > \qquad (3.13)
\end{aligned}
$$

Where,

$$|p_j)\ \simeq\ u(p_j)e^{-ip_jx} \qquad (3.14)$$

And Z_ψ is the usual quark wave renormalization constant. Using Equation (3.11), we can write the scattering amplitude as

$$S_{fi}\ =\ -Z_\psi^{-2} < (p_3|(g\ \not A)|\psi^F_{p_1})(p_4|(g\ \not A)|\psi^F_{p_2})\ -\ (p_3\ \leftrightarrow\ p_4) > \qquad (3.15)$$

In the above equation, for convenience, we have defined

$$|\psi^F_{p_j})\ =\ S_F(i\vec{\not\partial}\ +\ m)|p_j) \qquad (j\ =\ 1,2) \qquad (3.16)$$

Now in the above equation, the first term corresponds to all the t- channel exchange graphs, and the term$(p_3 \leftrightarrow\ p_4)$ corresponds to all the u channel exchanges. In the energy regime that we are interested in, all the u channel exchange diagrams are suppressed by a power of s compared to the t channel exchange ones. Hence we neglect the term given by $(p_3 \leftrightarrow\ p_4)$ in the analysis to follow. It may be see from the form of $|\psi^F_{p_j})$ that it satisfies the Dirac equation in the presence of an external gluon field, i.e

$$(i\gamma^\mu D_\mu\ -\ m)|\psi^F_{pj})\ =\ 0 \quad (j\ =\ 1,2) \qquad (3.17)$$

Defining

$$M^F_{kj}(A)\ =\ (p_k|(g\ \not A)|\psi^F_{p_j}) \qquad (3.18)$$

we see that these quantities with $j\ =\ 1,\ k\ =\ 3$, and $j\ =\ 2,\ k\ =\ 4$ may be considered as the probability amplitude for a free quark in an external gluon field to evolve into a state ψ, that is they are some sort of scattering amplitudes, for the individual quarks. Hence, it is clear that we may determine the full scattering amplitude by taking these individual M's and averaging their product over all possible gluon field configurations. In this way, we get the amplitude for the two quarks starting from some initial free state to scatter and finally evolve into some other state. To determine this, we first evaluate the quantities $M^F_{31}(A)$ and $M^F_{42}(A)$. We note that the wave functions with Feynman boundary condition do not have a simple behaviour at large times, which is obviously needed in order to impose the boundary condition that the quark states are free in the distant past. However, the retarded Green's functions do satisfy such a condition, namely,

$$|\psi^r_{p_j}) \;\rightarrow\; |p_j) \quad for\ x^0 \;\rightarrow\; -\infty \tag{3.19}$$

Thus the question is whether we can work with the relatively simpler $|\psi^r_{p_j})$ rather than with the $|\psi^F_{p_j})$. Nachtmann (Nachtmann, 1991) has shown that in the case of a slowly varying gluon potential in this energy regime, we can indeed replace wave functions with the Feynman boundary conditions with those with the retarded boundary conditions. We do not give the explicit proof here, but simply state the result, $|\psi^F_{p_j}) = |\psi^r_{p_j})$ Thus we arrive at the final result of this section, that is the quark-quark scattering amplitude is given by

$$S_{fi} \;=\; -Z_\psi^{-2} < \; M^r_{31}(A)\, M^r_{42}(A) > \tag{3.20}$$

With Z_ψ as the usual quark wavefunction renormalization constant. In the next section we take a look at how to compute the functions $M^r_{31}(A)$ and $M^r_{42}(A)$ by solving the Dirac equation for a quark in an external gauge field. Then we shall use those results to compute the scattering amplitude.

3.2 Solution of the Dirac Equation in the High Energy Limit

In this section, we shall review the solution to the Dirac equation in the high energy limit as given by Nachtmann (Nachtmann, 1991). This is essential in order to compute the quark-quark scattering amplitude which we shall do subsequently. The idea is to find the Dirac wave functions of a quark in the background gauge field produced by the other, in the high energy limit. We shall then use these in getting amplitudes $M^r_{31}(A)$ and $M^r_{42}(A)$, expressions for which were given in the last section and finally from these we may calculate the scattering amplitude of two quarks, as explained before. We thus begin writing the Dirac equation in an external gluon potential,

$$\begin{aligned} & (i\gamma^\mu D_\mu \;-\; m)\psi^r_{p_j} \;=\; 0 \\ where \qquad & D_\mu \;=\; \partial_\mu \;-\; iA_\mu, \qquad j \;=\; 1,2. \end{aligned} \tag{3.21}$$

and we have to impose the boundary condition that at distant past, the wave function was a plane wave, i.e,

$$\psi^r_{p_j} \;\rightarrow\; 0, \qquad for \quad x^0 \;\rightarrow\; -\infty. \tag{3.22}$$

Here we take the time to run from $-\infty$ to ∞. Now, we work in centre of mass frame of the reaction. Our coordinates are such that the z axis lies symmetrically with respect to the incoming and the outgoing quarks. The advantage of this frame is that the momentum transfer is purely transverse. We may write the four momenta of the two quarks (which we call as particle 1 and 2) as

$$p_{1,2} = \begin{pmatrix} E_+ + \frac{\mu^2}{4E_+} \\ \pm\frac{1}{2}\mathbf{q}_T \\ \pm E+ \mp \frac{\mu^2}{4E_+} \end{pmatrix} \tag{3.23}$$

Where

$$\begin{aligned} E_+ &= \frac{1}{2}(p_1^0 + p_1^3) \\ \mu^2 &= m^2 + \frac{1}{4}\mathbf{q}^2 \end{aligned} \tag{3.24}$$

Here $\mathbf{q}_T$ denotes the two component momentum transfer vector that lies entirely in the transverse plane. It is also naturally advantageous to use light cone variables $x^{\pm} = x^0 \pm x^3$, in this high energy computation and all other four vectors we shall denote by their transverse and light cone components. To solve the Dirac equation, we make use of its $SU(3)$ gauge invariance. (The matrices ψ have a colour index that we have not written explicitly). Indeed gauge invariance implies that the Dirac equation is invariant under the gauge transformation

$$\begin{aligned} \psi'^{r}_{p_j} &= U(x)\psi^{r}_{p_j} \\ A'_{\mu}(x) &= U(x)A_{\mu}(x)U^{\dagger}(x) - \frac{i}{g}U(x)\partial_{\mu}U^{\dagger}(x) \end{aligned} \tag{3.25}$$

Now we attempt to solve this gauge transformed Dirac equation. The reason why we are doing this is that we are treating a Dirac particle in a background field, which is supposed to be fixed, and hence we cannot fix them arbitrarily. Gauge invariance of the Dirac equation however assures us that we can always make a gauge transformation on the fields along with a corresponding transformation on the wave function, and since the gauge transformation is arbitrary, we can choose any one of the transformed fields arbitrarily. This simplifies the situation as we shall see. We solve the equation in the usual way of converting into a second order scalar equation by the following transformation

$$\psi'^{r}_{p_j}(x) = (i\gamma^{\mu}D'_{\mu} + m)\phi'_j(x) \tag{3.26}$$

This form of ϕ when we substitute in the original Dirac equation reduces it to

$$\left(i\gamma^{\mu}D'_{\mu} - m\right)\left(i\gamma^{\mu}D'_{\mu} + m\right)\phi'_j(x) = 0 \tag{3.27}$$

The equation for ϕ is a scalar field equation and is not difficult to solve. And, once we can determine the ϕ's we may put it back in the original equation to determine the ψ's. The boundary condition on $\psi^r_{p_j}$ implies that

$$\phi'_j(x) \rightarrow \frac{1}{p^0_j + m}\frac{1 + \gamma^0}{2}u(p_j)e^{-ip_jx} \tag{3.28}$$

Now, to solve for ϕ, we take a trial solution of the form

$$\phi'_j(x) = e^{-ip_jx}\tilde{\phi}_j(x) \tag{3.29}$$

When we substitute this trial solution in Equation (3.27), we get a second order differential equation for the scalar field ϕ,

$$\left[\Box - 2i\left(p^\mu_j - gA'^\mu\right)\partial_\mu + 2gp^\mu_j A'^\mu + ig\gamma^\mu\left(\partial_\mu \not{A}'\right) - g^2 \not{A}'^2\right]\phi'_j(x) = 0 \tag{3.30}$$

We now consider the particle 1, by which we shall mean the quark coming from the left, as seen by a centre of mass frame observer. At this point we exploit the gauge invariance of the Dirac equation and we choose,

$$A'^+ \equiv A'^0 + A'^3 = 0 \tag{3.31}$$

Now, if we put in the form of the four momentum and use the gauge condition, a little algebra shows that the Equation (3.30) reduces in the high energy limit to the very simple form

$$\left[\frac{\partial}{\partial x^-} + \frac{i}{2}gA'_-(x) + O(E_+^{-1})\right]\tilde{\phi}_1(x) = 0 \tag{3.32}$$

The solution to the above is found by a simple integration,

$$\begin{aligned}\tilde{\phi}_1(x) &= P\left(exp[-\frac{i}{2}g\int_{-\infty}^{x^-} dx'^- A'_-(x'^+, x^-, \mathrm{x}_T)]\right) \times (p^0_1 + m)^{-1}\frac{1}{2}(1 + \gamma^0)u(p_1) \\ &+ O(E_+^{-1})\end{aligned} \tag{3.33}$$

Also, in Equation (3.25) if we put $A'_+ = 0$, we may easily integrate the equation and find a solution for $U^\dagger$ which turns out to be

$$U^\dagger(x) = P\left(exp\left[-\frac{i}{2}g\int_{-\infty}^{x} dx'^+ A_+(x^-, x'^+, \mathrm{x}_T)\right]\right). \tag{3.34}$$

In the above, P denotes path ordering as here we are dealing with non abelian matrix valued gauge fields that are non commutative. Now, we have to put everything together. From Equation (3.25), we have,

$$\psi^r_{p_1}(x) = U^\dagger(x)\psi'^r_{p_1}(x) \tag{3.35}$$

Substituting the expression for $\psi'^{r}_{p_1}(x)$ from Equation (3.26), we get

$$\psi^{r}_{p_1}(x) = U^{\dagger}(x)\left[i\gamma^{\mu}D'_{\mu} + m\right].e^{-ip_1x}\tilde{\phi}_1(x) \tag{3.36}$$

Now, writing D'_{μ} in terms of the gauge transformed field and and putting in the values of $\tilde{\phi}_1(x)$ and $U^{\dagger}(x)$ as obtained from Equation (3.33) and Equation (3.34), we obtain the following expression

$$\psi^{r}_{p_1}(x) = V_{-}(x^{+}, x^{-}, \mathrm{x}_T)e^{-ip_1x}u(p_1) \tag{3.37}$$

Where

$$V_{-} = P\left(exp\left[-\frac{i}{2}g\int_{-\infty}^{x^{-}} dx'^{-}A_{-}(x'^{-}, x^{+}, \mathrm{x}_T)\right]\right) \tag{3.38}$$

. We may similarly solve the equation for the particle with incoming momentum p_2, and the same steps will all follow, and we can easily guess the result,

$$\psi^{r}_{p_2}(x) = V_{+}(x^{+}, x^{-}, \mathrm{x}_T)e^{-ip_2x}u(p_2) \tag{3.39}$$

Where

$$V_{+} = P\left(exp\left[-\frac{i}{2}g\int_{-\infty}^{x^{+}} dx'^{+}A_{+}(x^{-}, x'^{+}, \mathrm{x}_T)\right]\right) \tag{3.40}$$

. It should be noted that V_{+} and V_{-} are $SU(3)$ matrices acting on colour space. They satisfy the differential equation and boundary condition as follows

$$\begin{aligned}\frac{\partial}{\partial x^{\pm}}V_{\pm} &= -\frac{i}{2}gA_{\pm}V_{\pm}\\ V_{\pm} \rightarrow 1 \quad &for\ x^{\pm} \rightarrow -\infty\end{aligned} \tag{3.41}$$

This solution may be interpreted as follows. Since we are in the regime of ultra high energy scattering, and consider near forward scattering of two quarks, the transverse momentum transfer in the process is extremely small, and the quarks are hardly deviated from their light cone trajectories. Thus these travel essentially along the rays $x^{+} = constant$ and $x^{-} = constant$. In the process however the wave function of the quark travelling along $x^{-} = constant$ picks up a non abelian phase factor of V_{-}. Similarly the wave function of the other quark will pick up a phase factor V_{+}. It is interesting to compare this with the shock wave picture that we developed in the last chapter. There also we treated one particle (abelian in that case) travelling in the background gauge field of the other, and it picked up a phase factor due to the instantaneous interaction with the shock front. The net phase picked up in the process if calculated in these two ways i.e by solving the Dirac equation in the high energy limit and by calculating the phase factor in the shock wave method of Jackiw et al (Jackiw et al, 1992), gives the same result. This is not surprising as the two methods are essentially looking at the same phenomenon from different reference frames. Whereas in solving the Dirac equation we have used the centre of mass frame, where before and after the reaction the particles move with equal but opposite velocities, in the method of computation of Jackiw et al, we use the frame of reference in which the scattered particle is at rest or moving very slowly. Lorentz invariance assures that whichever frame we look at the reaction from, the end result (i.e the scattering amplitude) is bound to be the same, hence it is expected that the two methods give the same result. However, we must comment that a full shock wave picture of scattering of non abelian particles is yet to be developed. This is mainly because the non linearity of the Yang Mills equation makes it impossible to find a unique solution. Efforts are being made to boost some of the known solutions of these equations exactly like the abelian case, and to see what type of shock wave, if any, emerge from this.

3.3 Quark-Quark Scattering Amplitude

In this section, we give the explicit form of the scattering amplitude of two quarks in the high energy limit. We shall show that it reduces to the expectation value of two Wilson lines in this limit. This will justify the assumption that we made in the last chapter regarding the same. Let us recall that we wrote the quark-quark scattering amplitude in the form $S_{fi} = -Z_{\psi}^{-2} < M_{31}^r(A)\ M_{42}^r(A) >$ where, $M_{31}^r(A) = (p_3|\ g\ \not{A}\ |\psi_{p1}^r)$ and similarly $M_{42}^r(A) = (p_4|\ g\ \not{A}\ |\psi_{p2}^r)$. Now that we have computed an explicit form of $|\psi_{p1}^r)$ and $|\psi_{p2}^r)$ we may attempt to put it back in the expression for the amplitude. We briefly outline the calculation here. If we insert the approximate solution for ψ_{p1}^r in $M_{31}^r(A)$, we obtain

$$M_{31}^r(A) = \int dx e^{i\ (p_3\ -\ p_1\ )} \bar{u}\,(p_3)\,(g\ \not{A})\,V_- u\,(p_1) \tag{3.42}$$

Which may be shown to reduce to

$$M_{31}^r(A) \simeq \int dx e^{i\ (p_3\ -\ p_1\ )} g A_- V_- \frac{\bar{u}\,(p_3)\ \not{p}'\, u\,(p_1)}{\sqrt{2\nu}} \tag{3.43}$$

Where A and V are functions of all four coordinates. In the above, we have defined

$$\begin{aligned} p &= \frac{1}{2}\,(p_1\ +\ p_3) \\ p' &= \frac{1}{2}\,(p_2\ +\ p_4) \end{aligned}$$

$$\nu \;=\; p\,.p' \;=\; 2E_{+}^{2} \;+\; \frac{1}{2}\left(\frac{\mu^{2}}{2E_{+}}\right)^{2} \tag{3.44}$$

And we have taken the limit $\nu \longrightarrow \infty$ Similarly it can be shown that

$$M_{31}^{r}(A) \simeq \int dx e^{i\,(p_3\; -\; p_1\;)} g A_{-} V_{-} \frac{\bar{u}\,(p_3)\; \not{p}'u\,(p_1)}{\sqrt{2\nu}} \tag{3.45}$$

Inserting these in Equation (3.20), after some algebra, we see that the quark-quark scattering amplitude in the high energy limit reduces to

$$S_{fi} \;=\; i\; 2\pi^{4}\; \delta\,(p_3 \;+\; p_4 \;-\; p_1 \;-\; p_2) \times T_{fi} \tag{3.46}$$

Where

$$T_{fi}^{\bullet} \;=\; i\bar{u}\,(p_3)\,\gamma^{\mu}u\,(p_1)\,\bar{u}\,(p_4)\,\gamma_{\mu}u\,(p_2)\,J\left(q^{2}\right)\delta_{A_3A_1}\delta_{A_4A_2} \tag{3.47}$$

J being given by

$$J\left(q^{2}\right) \;=\; -Z_{\psi}^{-2}\int d^{2}z e^{iq.z}\frac{1}{9}\left\langle Tr\left[V_{-}\left(z\right) \;-\; 1\right]Tr\left[V_{+}\left(z\right) \;-\; 1\right]\right\rangle \tag{3.48}$$

It must be noted that the indices A refer to the colour indices of the quarks which we had till now suppressed. The above equation makes it clear that in the high energy limit, the scattering amplitude reduces to the expectation value of the two Wilson lines that replaces the two quarks. It might be interesting to compute the corrections to the above picture possibly by keeping the first order correction to the $\psi_{p_j}^{r}$'s. It is this definition of the amplitude that we shall be using in all high energy approximations.

3.4 Scaling Argument in Quantum Chromodynamics

In this section, we apply the Verlindes' scaling argument to the case of non abelian gauge theories, namely QCD. Starting from the full four dimensional Yang Mills action, we implement the high energy limit via the same scaling argument as we have discussed before. Doing the scaling directly at the level of the action, we reduce the full theory to a much simpler effective theory. The resulting action is still four dimensional, but the interactions can be summarised in terms of a two dimensional sigma model residing on the transverse plane. Let us start directly with the pure Yang Mills action,

$$S \;=\; -\frac{1}{4}\int d^{4}x\; tr\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{3.49}$$

Where the field strength is as usual,

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu \;-\; \partial_\nu A_\mu \;+\; g\,[A_\mu,\, A_\nu] \tag{3.50}$$

and the gauge fields transform under the fundamental representation of $SU(3)$. We do the same Verlinde scaling that we had described in details in the last chapter, namely $x^\alpha \;\rightarrow\; \lambda x^\alpha, \quad x^i \;\rightarrow\; x^i$ the scaled Yang Mills action now reads,

$$S \;=\; -\frac{1}{4}\int d^4x\; tr\left(\lambda^{-2} F_{\alpha\beta}F^{\alpha\beta} \;+\; 2F_{\alpha i}F^{\alpha i}\right) \tag{3.51}$$

Again we see that if we want to calculate the Green's function etc and consider the quantum partition function of the scaled theory, namely $Z \;=\; \int DA_\mu\; exp(iS)$, fluctuations of the term $\lambda^{-2}F_{\alpha\beta}F^{\alpha\beta}$ will be damped out due to the smallness of λ and we retain our flat gauge condition,

$$F_{\alpha\beta} \;=\; 0, \tag{3.52}$$

which again implies that $E_z \;=\; 0$ and $B_z \;=\; 0$ in the dual formalism. Thus the Yang Mills action simplifies to

$$S \;=\; -\frac{1}{2}\int d^4x\; tr\left(F_{\alpha i}F^{\alpha i}\right) \tag{3.53}$$

with the pure gauge condition implying that

$$A_\alpha \;=\; \frac{1}{g}\partial_\alpha U U^{-1} \tag{3.54}$$

We shall return to the action in a moment but before that, let us look at the fermionic part,

$$S_{fermionic} \;=\; \int d^4x\; \bar{\psi}\,(i\;\not{D} \;-\; m)\,\psi \tag{3.55}$$

In the scaling limit, the mass term automatically becomes subleading and drops out, and we are left with,

$$S_{fermionic} \;=\; \int \bar{\psi}^\alpha\,(\partial_\alpha \;+\; gA_\alpha)\,\psi \tag{3.56}$$

Thus we see clearly that the quarks propagate in the longitudinal direction only and they couple to the longitudinal components A_α of the gauge fields. That the mass term drops out in the scaling limit suggests that in the energy regime that we are interested in, we have essentially a theory of chiral fermions coupled to gauge fields in the longitudinal direction. However it must be kept in mind that we still have a full four dimensional theory, as the ψ's and the A's are functions of all four coordinates. Now, from the action, it is possible to calculate the two point function of two right moving quarks, and this quantity turns out to be

$$< T\left(\bar{\psi}(x_1, z_1)\psi(x_2, z_2)\right) > \; = \; \delta^{(2)}(Z_{12}) \left(\delta(x_{12}^-)\theta(x_{12})^+ \; + \; \frac{1}{x_{12}^- \; + \; i\epsilon}\right) \times P \; exp\left(g\int_1^2 dx^+ A_+\right) \quad (3.57)$$

Thus, we are led to a description of the quark-quark scattering amplitude as proposed by Nachtmann, in which as we have already seen, the quarks are represented by light like Wilson lines. It must be mentioned that in this derivation, the Wilson lines essentially arise due to the flat gauge condition on the A_α's, which in turn can be traced back to the scaling argument. In Nachtmann's derivation, however these came because we had neglected the recoil terms and had taken the gauge fields produced by the scatterer as background gauge fields. So, following Nachtmann, in this picture also, we shall define the quark-quark scattering amplitude as

$$\begin{aligned} f(s,t) &= \frac{is}{2m^2}\int d^2ze^{-iq.z}\left\langle V_+(0)V_-(z)\right\rangle \\ V_\pm(z) &= tr\left[P\; exp\left(g\int_{-\infty}^{\infty} dx^\pm A_\pm(z)\right)\right] \end{aligned} \quad (3.58)$$

The factor $\frac{is}{2m^2}$ is a conventional normalization. Let us now return to the simplification of the action in Equation (3.53). If we substitute in this the form for A_α in Equation (3.54), dictated by the flat gauge condition, then we have

$$S \; = \; \frac{1}{2g^2}\int d^4x\left(\partial_\alpha\left(U^{-1}D_iU\right)\right)^2 \qquad where \; D_i \; = \; \partial_i \; + \; gA_i \quad (3.59)$$

The next step is to find a classical solution for the gauge fields A_i and put it back into the action. This may be done by computing the equations of motion for the A_i's and putting them back in the action. Indeed by the equations of motion of the A_i's turn out to be

$$\partial_+\partial_-\left(U^{-1}D_iU\right) \; = \; 0 \quad (3.60)$$

which implies that the solution is harmonic, i.e

$$U^{-1}D_iU \; = \; U^{-1}D_iU(x^+) \; + \; U^{-1}D_iU(x^-) \quad (3.61)$$

Clearly, if we put this form of the solution back in the action, the latter takes the form

$$S \; = \; \frac{1}{2g^2}\int d^2ztr\left[\int_{-\infty}^{\infty} dx^+\partial_+\left(U^{-1}D_iU\right) \times \int_{-\infty}^{\infty} dx^-\partial_-\left(U^{-1}D_iU\right)\right] \quad (3.62)$$

Now, as in the case of Quantum Electrodynamics, we can easily do the above two integrals. Let us denote the end point values of the field U as g_1 and g_2 in the x^- direction and as h_1 and h_2 in the x^+ direction. Then, we have the following relations

$$\begin{aligned} \int_{-\infty}^{\infty} dx^- \partial_- \left(U^{-1} D_i U\right) &= g_2^{-1} D_i^+ g_2 - g_1^{-1} D_i^+ g_1 \\ \int_{-\infty}^{\infty} dx^+ \partial_+ \left(U^{-1} D_i U\right) &= h_2^{-1} D_i^- h_2 - h_1^{-1} D_i^- h_1 \end{aligned} \tag{3.63}$$

Where we have imposed the restriction that the transverse gauge fields A_i take the same values at the end points of the two Wilson lines. The values at the end points of the x^+ and x_- directions are denoted by a_i^+ and a_i^- respectively. The covariant derivatives in the above equation are with respect to $a_i^{\pm}$. Putting this back in the action of Equation (3.62), we see that it can be written in a very compact form as

$$S = \frac{1}{2g^2} \int d^2 z M^{AB} \, tr \left(g_A^{-1} D_i^+ g_A \; h_B^{-1} D_i^- h_B\right) \tag{3.64}$$

where M^{AB} is the 2×2 matrix,

$$M^{AB} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{3.65}$$

It should be noted that this matrix M^{AB} represents a discrete version of the operator $\partial_+ \partial_-$. It can actually be identified with the inverse propagator of the gluon field between the end points of the Wilson lines, a comment we shall come back to later. The action of Equation (3.64) thus gives a two dimensional model in which the longitudinal plane has been replaced by the four end points of the Wilson lines. However the fields etc. depend on all four coordinates. This is the simplified form of the action as obtained from the Verlinde's scaling argument. It is however a semiclassical action, and one may attempt to compute the one loop effective action by computing the quark and gluon loops. Since we have replaced the longitudinal plane by the four end points of the Wilson lines, interactions between the fields g and h are purely transverse in nature. There is however one major problem with the action of Equation (3.64). As has already been remarked, the inverse M_{AB}^{-1} of the matrix M^{AB} can be identified with the A_i propagators between the end points of the Wilson lines. However this inverse does not exist at all ! This problem may be traced back to the fact that we have taken exactly light like Wilson lines in our picture, i.e essentially massless quarks. This is the cause of the divergence and to avoid it we have to give our Wilson lines a small timelike component, that is, we have to give a small mass to the quarks. In this case, the

Wilson lines are no longer light like, and they begin and end after a certain finite amount of time. This regulates the divergence in the gluon propagator. We shall not go through the entire derivation of the Verlindes' but we simply state that the proposed regulated form of the matrix M^{AB} is,

$$M^{AB}_{reg} = \begin{pmatrix} 1+\epsilon & -1+\epsilon \\ -1+\epsilon & 1+\epsilon \end{pmatrix} \tag{3.66}$$

Where

$$\epsilon^{-1} = 1 - \frac{2i}{\pi} log\ s \tag{3.67}$$

Thus, the final form of the Yang Mills action obtained by the Verlindes' scaling argument is

$$S = \frac{1}{2g^2} \int d^2 z M^{AB}_{reg}\ tr\left(g_A^{-1} D_i^+ g_A\ h_B^{-1} D_i^- h_B\right) \tag{3.68}$$

3.5 Verlindes' Parametrization and the Quark Quark Scattering Amplitude

In the last section we saw how the scaling argument leads to a simplified effective two dimensional action out of the full four dimensional Yang Mills action for QCD. We also saw how the $log\ s$ entered in the effective action through the regularization of the gluon propagator. Now the issue is to compute the scattering amplitude of two quarks using this effective two dimensional action. To this end, we shall discuss a parametrization of the action as proposed by the Verlindes'. Let us recall that the scattering amplitude had been defined as $f(s,t) = \frac{is}{2m^2} \int d^2 z e^{-iq.z} \langle V_+(0) V_-(z) \rangle$, where the Wilson lines were defined as $V_\pm(z) = tr\left[P\ exp\left(g \int_{-\infty}^{\infty} dx^\pm A_\pm(z)\right)\right]$.Now, we see that since the longitudinal components of the gauge fields, A_α are flat, hence the Wilson lines turn out to be very simple functions of the variables g_1, g_2, h_1, h_2, introduced earlier.

$$\begin{aligned} P\ exp\left(g \int_{-\infty}^{\infty} dx^+ A_+(z)\right) &= g_2 g_1^{-1} \\ P\ exp\left(g \int_{-\infty}^{\infty} dx^- A_-(z)\right) &= h_2 h_1^{-1} \end{aligned} \tag{3.69}$$

Thus the expectation value of the two Wilson lines representing the elastic scattering amplitude of two quarks is given by

$$\langle V_-(z_L) V_+(z_R) \rangle = \int \left[dg_A\ dh_B\ da_i^\pm\right] e^{iS[g_A,\ h_B,\ a_i^\pm]} tr\left(g_2 g_1^{-1}(z_L)\right)\ tr\left(h_2 h_1^{-1}(z_R)\right) \tag{3.70}$$

Where z_L and z_R denote the transverse coordinates of the Wilson lines denoting the left and the right moving quarks respectively. Hence the scattering amplitude can be given a simple form if we introduce the following parametrization

$$g_2 g_1^{-1} \;=\; exp(\; ig\theta \;) \qquad h_2 h_1^{-1} \;=\; exp(\; ig\phi \;) \tag{3.71}$$

Using this, the quark-quark scattering amplitude reduces to

$$f(s,\; t) \;=\; \frac{is}{2m^2} \int d^2z\; e^{-iq.z} \left\langle e^{ig\theta}(0)\; e^{ig\phi}(0) \right\rangle \tag{3.72}$$

Now, we make a gauge choice, namely the Lorentz gauge, $\partial_i a_i^{\pm} \;=\; 0$. This implies that

$$a_i^{\pm} \;=\; \frac{e}{2}\; \epsilon_i^j \partial_j \alpha^{\pm} \tag{3.73}$$

Where we find it convenient to choose the constant e such that

$$e^2 \;=\; \frac{2g^2}{\pi} log\; s \tag{3.74}$$

Further, to parametrize the action fully, we also choose,

$$g_1 h_1^{-1} \;=\; exp(\; ie\chi) \tag{3.75}$$

Now, we have to insert the parametrizations, Equation (3.71), Equation (3.73), Equation (3.75) into the action in order to get the effective parametrized action. Doing this, we find that the action reduces to

$$\begin{aligned} S \;=\; & \frac{1}{2} \int d^2z\; tr \left(\partial_i \theta \partial_i \phi \;+\; i\,(\partial_i \chi)^2 \;+\; i\partial_i \alpha^+ \partial_i \alpha^- \right) \\ & + \frac{e}{2} \int d^2z\; tr \left(\chi\, [\partial_i \theta,\; \partial_i \phi] \;+\; \frac{1}{2} \left(\alpha^+ \;+\; \alpha^- \right) [\partial_i \theta,\; \partial_j \phi]\, \epsilon^{ij} \right) \\ & + \frac{e^2}{4} \int d^2z\; tr \left([\chi,\; \partial_i \theta]\, [\chi,\; \partial_i \phi] \;+\; \frac{1}{2} \left[\partial_i \alpha^+,\; \theta \right] \left[\partial_i \alpha^+,\; \theta \right] \right) \end{aligned} \tag{3.76}$$

Where we have expanded upto the first two orders in e. In the next chapter, we shall come back to the computation of the actual scattering amplitude of the quark-quark scattering process.

CHAPTER 4

BEHAVIOUR OF SCATTERING AMPLITUDES IN QCD AT HIGH ENERGIES

In this chapter, we take a look at the high energy behaviour of two particle elastic scattering amplitudes in the context of Quantum Chromodynamics. To be specific, we shall first review the existing formalism in which one calculates these amplitudes by making suitable approximations in the relevant Feynman graphs. We shall show how the scattering amplitude exhibits the so called *regge behaviour* in the limit of extremely large centre of mass energies. We then come back to the effective theory obtained out of the full four dimensional Yang Mills theory by the Verlindes' scaling formalism, and try to compute the quark quark scattering amplitude in this picture. We first develop this latter formalism in the simpler case of (2 + 1) dimensional QCD, following Li and Tan (Li and Tan, 1993) and show how the diagrammatic calculations indicate that the gluon propagator fails to *reggeize*. We then propose a new parametrization of the Verlindes' effective action and calculate an approximate form for the scattering amplitude in this picture in (3 + 1) dimensions, in a non diagrammatic way. We then show how to calculate the scattering amplitude in the Verlindes' framework.

4.1 Diagrammatic Calculation of the High Energy Elastic Scattering Amplitude in QCD

In this section we outline the calculation of the quark quark scattering amplitude at extremely high energies by means of Feynman diagrams. Since the calculations are lengthy, but standard, we do not present the full details here, but we would rather quote the results (Cheng and Wu, 1987). In the appendix, however, we outline the method of computing Feynman graphs in the high energy approximation. Quark quark scattering amplitudes are more complicated that the fermion

fermion amplitudes in abelian gauge theories because the quarks, in addition to electric charge, carry colour. However, as we shall see, there is an amazing pattern of regularity in the perturbative calculation of amplitudes at high energies. Before we begin however, it is worth noting that as the gluon is massless, the quark quark scattering amplitude has infrared divergences. We could made our results rigorous by giving the gluons a small mass via the Higgs mechanism, however, for the sake of simplicity, we keep the gluons massless and ignore the infrared divergences. Let us consider the second order diagram for quark quark scattering as given in Figure 4.1. At very large centre of mass energies when the scattering is almost forward, the amplitude corresponding to this diagram can be shown to approach

$$A_2 \simeq -\frac{s}{2m^2}\frac{g^2}{\Delta^2}\,T_a^{(1)}T_a^{(2)} \tag{4.1}$$

Δ is the momentum transfer in the elastic scattering process, and $T_a = \lambda_a/2$ are the usual $SU(3)$ generators with λ's being the Gell Mann matrices. The indices 1 and 2 refer to the two quarks. It can be shown that the above expression for the amplitude differs from that in abelian fermion-fermion scattering only by the factor $T_a^{(1)}T_a^{(2)}$, which we call the *group factor*. It is this group factor that is crucial in the computation of amplitudes in QCD. It is to be noted that there is another second order diagram that contributes to the amplitude in this case, the so called "crossed diagram " obtained be interchanging the two incident particles, but this can be shown to be suppressed by a factor of u/t in the high energy limit (u and t being the usual Mandelstam variables), and hence is ignored. Since the group factor plays such a crucial role in computation of these amplitudes, we shall shortly develop a systematic procedure for treating these, but before that, let us look at the fourth order diagrams in quark quark scattering given in Figure 4.2. The amplitudes for Figure 4.2(a) and Figure 4.2(b) respectively can be shown to approach in the high energy limit

$$\begin{aligned} & M \quad \left(T_b^{(1)}T_a^{(1)}T_b^{(2)}T_a^{(2)}\right) \\ \textit{and} \quad & M' \quad \left(T_a^{(1)}T_b^{(1)}T_b^{(2)}T_a^{(2)}\right) \end{aligned} \tag{4.2}$$

Where,

$$\begin{aligned} M &\simeq -\frac{1}{2}g^4\frac{s}{m^2}\frac{ln\left(se^{-i\pi}\right)}{2\pi}I\left(\Delta\right) \\ M' &\simeq \frac{1}{2}g^4\frac{s}{m^2}\frac{ln\left(s\right)}{2\pi}I\left(\Delta\right) \end{aligned} \tag{4.3}$$

and

$$I\left(\Delta\right) \equiv \int \frac{d^2q_\perp}{(2\pi)^2}\frac{1}{\left(\mathbf{q}_\perp^2\right)\left[\left(\mathbf{\Delta} - \mathbf{q}_\perp\right)^2\right]} \tag{4.4}$$

$I(\Delta)$ is clearly infrared divergent, but we shall ignore this and keep all our results in terms of I. The other fourth order diagrams for quark quark scattering can be shown to be suppressed by a factor of s compared to these two, hence are ignored. Now, the group factor associated with a diagram is the matrix in colour space obtained by associating with each vertex the appropriate factor (Cheng and Wu, 1987). We represent the group factor of a diagram by the diagram itself. For example, the factor associated with $T_a^{(1)}T_a^{(2)}$ is represented simply by Figure 4.1, while $T_b^{(1)}T_a^{(1)}T_b^{(2)}T_a^{(2)}$ and $T_a^{(1)}T_b^{(1)}T_b^{(2)}T_a^{(2)}$ is given in Figure 4.2. Writing the group factors explicitly, we may obtain relations between them, which when converted to diagrammatic equations, show that upto sixth order in the scattering amplitude, all factors may be expressed in terms of only four, which we call G_1, G_2, G_3, and G_4. If we define the group factors in this way, the fourth order expression for the scattering amplitude becomes

$$A_4 \simeq \left(M + M'\right) G_2 + \frac{3}{2} M' G_1 \tag{4.5}$$

Now from Equation (4.3), we get

$$\begin{aligned} M + M' &\simeq \frac{ig^4}{4}\frac{s}{m^2} I(\Delta) \\ \textit{and} \qquad \frac{3}{2}M' &\simeq \frac{3}{4} g^4 \frac{s}{m^2}\frac{ln\ s}{2\pi} I(\Delta) \end{aligned} \tag{4.6}$$

Hence by adding the second to the fourth order amplitudes, we find that it asymptotically approaches as $s \to \infty$

$$A_2 + A_4 \simeq -\frac{sg^2}{2m^2\Delta^2}\left[1 - \bar{\alpha}(\Delta)\, ln\ s\right] G_1 + \frac{ig^4 s}{4m^2} I(\Delta)\, G_2 \tag{4.7}$$

where

$$\bar{\alpha}(\Delta) = \frac{3g^2}{4\pi}\Delta^2 I(\Delta) \tag{4.8}$$

and $I(\Delta)$ being given by Equation (4.4). In the same way, one may calculate the sixth order contribution to the scattering amplitude. There are 21 Feynman graphs that contribute to this order, we do not list all of them, but simply state the result that the sixth order amplitude approaches asymptotically in the extreme high energy limit

$$\begin{aligned} A_6 \simeq \frac{sg^6}{2m^2}\Bigg[& -\frac{1}{\Delta^2}\frac{\left(g^{-2}\bar{\alpha}(\Delta)\, ln\ s\right)^2}{2!} G_1 - \frac{3i}{4\pi} s\, I_1(\Delta)\, G_2 \\ & + \frac{i}{4\pi} ln\ s \left(2I_1(\Delta) - \Delta^2 I^2(\Delta)\right) G_3 + \frac{1}{3!} I_1(\Delta)\, G_4 \Bigg] \end{aligned} \tag{4.9}$$

Hence, from Equation (4.1), Equation (4.5) and Equation (4.9), we may deduce the behaviour of the amplitude upto the sixth order, which is seen to be

$$M \simeq -\frac{sg^2}{2m^2\Delta^2}\left[1 - \bar{\alpha}(\Delta)\, ln\ s + \frac{1}{2}\bar{\alpha}^2(\Delta)\, ln^2 s\right] G_1$$

$$
\begin{aligned}
&+ \frac{is}{4m^2}\left[g^4 I(\Delta) - \frac{3g^6}{2\pi} ln\ s I_1(\Delta)\right] G_2 \\
&+ \frac{isg^6 ln\ s}{8\pi m^2}\left[2I_1(\Delta) - \Delta^2 I^2(\Delta)\right] G_3 \\
&+ \frac{sg^6}{12m^2} I_1(\Delta) G_4
\end{aligned} \tag{4.10}
$$

The G_1 term in the above equation is seen to come in an exponential series. Since the G_1 term is associated with one gluon exchange, we see that the propagator for a gluon of momentum $\mathbf{q}_\perp$ is modified by a factor of $s^{-\bar{\alpha}}$. This term thus gives the amplitude for exchange of one *reggeon*. The G_2 term may be shown to be the amplitude for exchange of two reggeized gluons.Thus in this picture the gluons are reggeized, as we had commented earlier. Thus, diagrammatic calculations show that the gluon propagator is reggeized, and we expect the same picture to emerge in any effective theory of high energy scattering in QCD, in particular in the Verlinde framework, which we shall be interested in. In the next section, we look at very high energy scattering in $2+1$ dimensional QCD in the Verlinde picture, and show that the gluon propagator *does not* reggeize, before we move on to the case of full $3+1$ dimensional QCD.

4.2 High Energy Scattering in $2+1$ Dimensional QCD

In this section, we apply the Verlinde scaling argument to a simpler situation, namely $2 + 1$ dimensional QCD, following Li and Tan (Li and Tan, 1994). We show how the full theory can be reduced to an effective 1 dimensional theory, and we try to see if the gluon propagator reggeizes in this picture. To begin with, we recall Verlindes' effective action for $3 + 1$ dimensional QCD, namely

$$
S = \frac{1}{2g^2}\int d^2 z M^{AB}_{reg}\ tr\left(g_A^{-1} D_i^+ g_A\ h_B^{-1} D_i^- h_B\right) \tag{4.11}
$$

where

$$
M^{AB}_{reg} = \begin{pmatrix} 1+\epsilon & -1+\epsilon \\ -1+\epsilon & 1+\epsilon \end{pmatrix} \tag{4.12}
$$

Expanding out this action, the $a_i^\pm$ independent part of it can be written as

$$
\begin{aligned}
S &= \frac{1}{g^2}\int d^2 z \left[(1+\epsilon) tr\left(g_2^{-1}\partial_i g_2 - g_1^{-1}\partial_i g_1\right)\left(h_2^{-1}\partial_i h_2 - h_1^{-1}\partial_i h_1\right)\right] \\
&+ \frac{1}{g^2}\int d^2 z\ 2\epsilon\ tr\left[g_1^{-1}\partial_i g_1\left(h_2^{-1}\partial_i h_2 - h_1^{-1}\partial_i h_1\right)\right]
\end{aligned}
$$

$$+ \quad \frac{1}{g^2} \int d^2z \; 2\epsilon \; tr \left[h_1^{-1}\partial_i h_1 \left(g_2^{-1}\partial_i g_2 \; - \; g_1^{-1}\partial_i g_1 \right) \right]$$

$$+ \quad \frac{1}{g^2} \int d^2z \; 4\epsilon \; tr \left[g_1^{-1}\partial_i g_1 h_1^{-1}\partial_i h_1 \right] \tag{4.13}$$

From the above form of the action, it can be seen that when we write down the partition function e^{iS}, fluctuations of the terms $g_2^{-1}\partial_i g_2 \; - \; g_1^{-1}\partial_i g_1$ and $h_2^{-1}\partial_i h_2 \; - \; h_1^{-1}\partial_i h_1$ are controlled by g^2 and the fluctuations in $g_1^{-1}\partial_i g_1$ and $h_1^{-1}\partial_i h_1$ are controlled by the parameter $g^2\epsilon^{-1}$, a parameter which in the high energy limit is much greater than g^2. Hence the second term in the action is negligible compared to the other two terms and we may ignore this. Further, we note that the third term in the action may be written as

$$4\epsilon \; tr \left(g_1^{-1}\partial_i g_1 h_1^{-1}\partial_i h_1 \right) \;=\; \epsilon \; tr \left[\left(g_1^{-1}\partial_i g_1 \; + \; h_1^{-1}\partial_i h_1 \right)^2 - \left(g_1^{-1}\partial_i g_1 \; - \; h_1^{-1}\partial_i h_1 \right)^2 \right] \tag{4.14}$$

Hence, the term $g_1^{-1}\partial_i g_1 \; + \; h_1^{-1}\partial_i h_1$ essentially drops out of the action by the same argument that we have dropped the second term. To simplify the action further, we define

$$g \;=\; g_2 g_1^{-1}; \qquad h \;=\; h_2 h_1^{-1}; \qquad G \;=\; g_1 h_1^{-1} \tag{4.15}$$

which we shall ultimately parametrize by θ, ϕ and χ as in the earlier case. Putting these in the action, it is readily seen that since there is only one transverse degree of freedom in 2 + 1 dimensions, hence in this case, the $a_i^{\pm}$ independent part of the action is

$$S \;=\; \frac{1}{g^2} \int dx \; tr \left[g^{-1}\dot{g} \; G \; h^{-1}\dot{h} G^{-1} \; - \; i\frac{1}{e^2} \left(G^{-1}\dot{G} \right)^2 \right] \tag{4.16}$$

Where the . denotes differentiation with respect to the only transverse coordinate, x, and $e^2 \;=\; 2g^2 ln \; s/\pi$. In 2 + 1 dimensions, we can in fact choose both $a_i^{\pm}$ to be zero. Hence the above is the effective action in this case. Now, we are interested in seeing if the gluon propagator reggeizes. In other words, we have to calculate the quantity $< \theta(q) \; \phi(q) >$ which we shall do upto two loops. In order to do this, we introduce the following parametrizations

$$g \;=\; e^{g\theta}; \qquad h \;=\; e^{g\phi}; \qquad G \;=\; e^{e\chi} \tag{4.17}$$

In terms of this parametrization, we see that the action may be written upto the lowest order in g as

$$S \;=\; \int dx \; tr \left(\; \dot{\theta} \; G \; \dot{\phi} \; G^{-1} \right) \tag{4.18}$$

Expanding G in terms of χ, we get the necessary interaction terms to calculate $< \theta(q)\ \phi(q) >$ upto two loops. Here, $\theta = \theta^a T^a$ and $\phi = \phi^b T^b$, where T^a, T^b are generators of $SU(3)$, for $i = 1$ to 8. At tree level, the $\theta - \phi$ correlator is just $-(i/q^2)\,\delta_{ab}$. At one loop level, there are two distinct Feynman diagrams as shown in Figure 4.3. The solid lines denote the $\theta - \phi$ propagator, while the dotted lines denote the χ propagators. These diagrams can be easily evaluated, and they give

$$\left\langle\ \theta^a(q)\ \phi^b(-q)\ \right\rangle \ = \ -\frac{i}{q^2}\left[1 \ + \ \frac{3e^2}{2} I\right]\ \delta_{ab} \tag{4.19}$$

Where

$$I \ = \ \int \frac{dk}{2\pi}\frac{1}{k^2} \tag{4.20}$$

Upto the fourth order, there are seventeen topologically distinct Feynman graphs, (Li and Tan, 1994) which we do not list here. These diagrams are also evaluated in the standard way, and after putting everything together, the $\theta - \phi$ correlator upto two loops is given by

$$\left\langle \theta^a(q)\phi^b(-q)\right\rangle \ = \ -\frac{i}{q^2}\left[1 \ - \ \frac{3e^2}{2} I \ + \ \frac{99e^4}{48} I^2\right]\ \delta_{ab} \tag{4.21}$$

It is clear from the above equation that the series in brackets do not exponentiate, and hence reggeization fails to occur in 2 + 1 dimensional QCD in the Verlinde framework.

4.3 An Approximate Calculation of The Scattering Amplitude In 3 + 1 QCD

In this section, we propose a new parametrization of the Verlindes' action in the context of 3 + 1 dimensional QCD. In the approximation where we put the transverse gauge fields to zero, we try to compute an approximate expression for the scattering amplitude and see how far it agrees with the results of Cheng and Wu. Let us recall that we have defined the scattering amplitude at high energies of two quarks to be

$$f(s,\ t) \ = \ \frac{is}{2m^2}\int d^2z\ e^{-iq.z}\left\langle\ tr\left(g_2 g_1^{-1}\right)\ tr\left(h_2 h_1^{-1}\right)\right\rangle \tag{4.22}$$

The idea is to introduce the parametrization

$$\begin{aligned} g_1 \ &= \ e^{g\theta_1}, \quad & g_2 \ &= \ e^{g\theta_2} \\ h_1 \ &= \ e^{g\phi_1}, \quad & h_2 \ &= \ e^{g\phi_2} \end{aligned} \tag{4.23}$$

Now, let us take for example the term $e^{g\theta_2}\, e^{-g\theta_1}$ corresponding to the term $g_2 g_1^{-1}$. Using the Baker Cambell Hausdorf identity, we can write the above term as $exp\,\left(g(\theta_2 \;-\; \theta_1) \;-\; \frac{g^2}{2}\,[\theta_2,\,\theta_1] \;-\; \frac{g^3}{12}\,([\theta_2,\,[\theta_2,\,\theta_1]] \;-\; [\theta_1,\,[\theta_1,\,\theta_2]]) \;+\; ...\right)$ and hence, expanding the exponentials on both $g_2 g_1^{-1}$ and $h_2 h_1^{-1}$, we get an expression for the scattering amplitude, which may be evaluated by standard Wick contraction techniques using the effective two dimensional action that we have derived earlier. To match our results with the calculations of Cheng and Wu, we take a trace at the end to get the scattering amplitude. We note that the longitudinal propagator between the end points of the Wilson lines that we have parametrized by the θ's and the ϕ's are just the elements of the inverse of the matrix M, M^{-1}. This inverse is

$$M^{-1}_{AB} \;=\; \frac{1}{4\epsilon}\begin{pmatrix} 1+\epsilon & 1-\epsilon \\ 1-\epsilon & 1+\epsilon \end{pmatrix} \tag{4.24}$$

Hence, in calculating the expectation values using Wick contraction techniques, we may write the longitudinal propagators between the θ's and ϕ's in terms of the ϵ's which we finally write in terms of $ln\ s$. The transverse propagator will be two dimensional and will generate integrals of the type I as given in Equation (4.4). Upto order g^2, we see that only the terms of order g from both the exponents contribute, and the net term is $g^2\,(\theta_2 \;-\; \theta_1)\,(\phi_2 \;-\; \phi_1)$. Since we have the longitudinal propagators

$$\begin{aligned} &\langle\, \theta_2\, \phi_2 \,\rangle \;=\; \langle\, \theta_1\, \phi_1 \,\rangle \;=\; \frac{1}{4\epsilon}(1 \;+\; \epsilon) \\ \textit{and,}\qquad &\langle\, \theta_2\, \phi_1 \,\rangle \;=\; \langle\, \theta_1\, \phi_2 \,\rangle \;=\; \frac{1}{4\epsilon}(1 \;-\; \epsilon) \end{aligned} \tag{4.25}$$

Hence we may compute the longitudinal propagator corresponding to this term, which is simply a constant piece, $=\ 4$. The transverse component gives $\frac{1}{\Delta^2}$ where $\Delta \;=\; p_2 \;-\; p_1$ is the momentum transfer in the transverse direction. Hence the net contribution of this term is

$$A_2 \;=\; \frac{g^2}{\Delta^2}G_1 \tag{4.26}$$

The appearance of the group factor G_1 is clear since we are contracting $\theta^a T^a$ with $\theta^b T^b$ which yields a factor $\delta^{ab} T^a T^b$, that is precisely G_1 as we have defined before. In the same way, one may compute the fourth and the sixth order terms in g. We shall outline the calculation of the fourth order term. It is clear from the parametrized form of $g_2 g_1^{-1}$ and $h_2 h_1^{-1}$ that the terms to order g^2 are $\frac{-g^2}{2}\,[\theta_1, \theta_2]$ and $\frac{g^2}{2}\,(\theta_1 \;-\; \theta_2)^2$ from $g_2 g_1^{-1}$ and $\frac{-g^2}{2}\,[\phi_1, \phi_2]$ and $\frac{g^2}{2}\,(\theta_1 \;-\; \theta_2)^2$ from $h_2 h_1^{-1}$. Various combination of these terms will give all the g^4 terms in the amplitude. We saw that the cross terms add up to a net zero contribution, while the direct terms give non trivial contribution to this order. It may indeed be seen that for the longitudinal part,

$$\frac{g^2}{4}\langle\, ([\theta_1,\ \theta_2]\,[\phi_1,\ \phi_2])\,\rangle \;=\; \frac{g^2}{4}\langle\, (\theta_1\theta_2\phi_1\phi_2 - \theta_1\theta_2\phi_2\phi_1 - \theta_2\theta_1\phi_1\phi_2 + \theta_2\theta_1\phi_2\phi_1)\,\rangle$$
$$= \frac{g^4}{4}\left(-\frac{3}{4}G_1 + \frac{3i}{2\pi}ln\ s\ G_1\right) \tag{4.27}$$

In the same way, we show that

$$\frac{g^2}{4}\left\langle\,(\theta_1 - \theta_2)^2(\phi_1 - \phi_2)^2\right\rangle = \frac{g^4}{4}\left(2G_2 + \frac{3}{2}G_1\right) \tag{4.28}$$

Thus we see that the fourth order terms add up to give a contribution to the

$$A_4 = g^4\left[\frac{3}{16} - \frac{3i}{8\pi}ln\ s\right]G_1 I + \frac{g^4}{2}G_2 I \tag{4.29}$$

Where I is the integral defined as in Equation (4.4). It comes from the transverse part of the propagators. In the same way, we may compute the g^6 term in the scattering amplitude. The calculation is lengthy but straightforward, and we present only the final result, that the sixth order amplitude approaches, in the high energy limit

$$A_6 = g^6\left(\frac{6i}{128\pi}ln\ s - \frac{3}{32}ln^2 s\right)G_1\Delta^2 I(\Delta) + g^6\left(\frac{3i}{8\pi}ln\ s\right)G_2 I_1$$
$$- g^6\left(\frac{i}{8\pi}ln\ s\right)G_3 I_1 + \frac{g^6}{6}G_4 I_1 \tag{4.30}$$

Where I_1 is defined as

$$I_1 = \frac{1}{i}\int\frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{1}{k_1^2\,k_2^2\,(\Delta - k_1 - k_2)^2} \tag{4.31}$$

Now, if we add up all the contributions, we see that the scattering amplitude is given by

$$M = \frac{is}{2m^2}\left[\frac{g^2}{\Delta^2} + g^4\left(\frac{3}{16} - \frac{3i}{8\pi}ln\ s\right)I + g^6\left(\frac{6i}{128\pi}ln\ s - \frac{3}{32}ln^2 s\right)\Delta^2 I(\Delta)\right]G_1$$
$$+ \frac{is}{2m^2}\left[\frac{g^4}{2}I + g^6\frac{3}{8\pi}ln\ sI_1\right]G_2$$
$$- \frac{is}{2m^2}\left[g^6\frac{ln\ s}{8\pi}I_1\right]G_3 + \frac{s}{2m^2}\left[\frac{g^6}{6}I_1\right]G_4 \tag{4.32}$$

Hence, we see that the scattering amplitude in this case is quite different from the amplitude for 3 + 1 QCD, as computed by Cheng and Wu. However, if we compute the amplitude using the $\langle trV_+\ trV_-\rangle$ form, it effectively gives only a G_2 and a G_4 contribution, and we see that our expression for the amplitude is essentially in broad agreement with Cheng and Wu's result, upto the sixth order, apart from a a negative sign with the g^6 term in G_2. Using this non diagrammatic method, one may try to compute the scattering amplitude in the Verlindes' formalism. The calculation becomes complicated once we include the gauge fields. Till fourth order, the expression for the gluon propagator is seen to agree with the standard form, but the sixth order calculation has not shown any sign of regge behaviour.

However, it would be wrong at this point to say that the gluon propagator does not reggeize in the Verlindes' approach. we point out a recent work by Korchemsky (Korchemsky, 1994), the formalism of which, we think, may be applied to this problem. Korchemsky has treated the renormalization group properties of the so called cross divergences of the Wilson lines, and shown that the asymptotics of the scattering amplitude are indeed governed by these divergences. More specifically, after renormalization of the so called cross singularities of the Wilson lines, there appear 2×2 matrices of anomalous dimension that control the high energy behaviour of the scattering amplitude.

Instead of trying to incorporate this approach directly in the framework of the effective action obtained by the Verlinde scaling, we propose to look at the problem in a slightly different way, which is due to the calculation of Polyakov (Polyakov, 1979). Polyakov has calculated the kink divergence of a Wilson loop, which is the divergence obtained in the expectation value of a loop which has a sharp spike. It seems to us that one can take the viewpoint that in the shock wave picture, where one treats the scattering amplitude of two fast moving particles as the expectation value of the Wilson lines corresponding to those particles, one of the loops can effectively be understood as producing a kink in the other. This is justified in the shock wave approximation in which the interaction between the particles is indeed instantaneous. So, we may think of a particle travelling along its classical trajectory with very high energy, being acted on instantaneouly by the gauge field due to the other, and hence it undergoing an instantaneous interaction after which it travels along its classical trajectory again, and we effectively have a Wilson loop with a kink corresponding to this particle. The argument is further justified as due to a suitable choice of gauge (called the contour gauge), one of the Wilson lines, in the expression for the expectation value of two lines that gave the scattering amplitude, may be taken to be 1, in which case we indeed have a single Wilson line. Heuristic arguments show that in this approach the regge behaviour of the scattering amplitude does follow, but at this point we are unable to justify it further, and hope to report on this in future.

CHAPTER 5

CONCLUSIONS

In this chapter, we summarize all the results we have reviewed till now. Then, we shall make a few comments about the open problems in this area of high energy scattering in gauge theories, and the possible means to address them.

We saw that in the regime of ultra high energies, the two particle elastic scattering amplitude can be computed semi classically, as shown by Jackiw et al (Jackiw et al, 1992), which tallies with the earlier results obtained by diagrammatic calculations. This semi classical computation was possible essentially because in this high energy regime, one can treat the scattering of one particle in the classical background gauge field produced by the other, by going to a frame of reference in which the first (scattered) particle is almost at rest, while the other (scatterer) moves with almost the speed of light. The computation was done in the framework of the shock wave picture, where the interaction is via a "shock wave", and is instantaneous. The scattering amplitude was obtained by considering the phase shift produced in the wave function of the scattered particle by the background gauge field of the other. We then reviewed Verlindes' scaling argument (Verlinde and Verlinde, 1993). This scaling argument consisted of associating two typical length scales with the problem, along the longitudinal and the transverse directions. Since the momentum along the longitudinal direction is by far greater than that in the transverse direction, the Verlindes' have argued that since length scales inversely as the momentum, we can associate a length scale along the longitudinal direction that extremely small compared with that in the transverse direction. When one incorporates this scaling at the level of the action itself, it splits up and one can retain only some effective terms, at high energies. This gave a simplified two dimensional action out of the full four dimensional Maxwell action for electrodynamics. Next we showed that this effective action for QED reproduced the result obtained by Jackiw et al for the scattering amplitude at ultra high energies.

Next, we reviewed the work done in context of non abelian gauge theories, namely QCD. We saw how Nachtmann (Nachtmann, 1991) has computed an expression for the scattering amplitude for two quarks by solving the Dirac equation for one quark in the background gauge field produced by the other in the high energy limit. We saw that in his expression, the scattering amplitude for two quarks basically reduced to the expectation value of the two Wilson lines associated with the two. This of course had a physical explanation in the sense that the Wilson line is precisely the non abelian phase factor picked up by the quark in the background gauge field produced by the other. Then, by applying Verlindes' scaling argument in the context of QCD, we saw that it gave the same Wilson line picture of the quarks. We then reviewed the parametrization and the computation of the effective action as done by the Verlindes. We proposed a non diagrammatic calculation of the scattering amplitude, using a new parametrization of the Verlindes' action, but showed that it did not show any sign of regge behaviour. The same result was obtained by Li and Tan (Li and Tan, 1994) in the case of 2 + 1 dimensional QCD.

Verlindes' scaling argument is an extremely physical argument that incorporates the high energy limit elegantly in the theory at the level of the action itself, and thus produces from the full four dimensional action for QED or QCD an effective two dimensional action. In the case of QED, since this effective action gave back the standard result for the scattering amplitude, it means that the scaling argument is correct at least in this framework. We must also mention here that scaling in the context of the Einstein action in gravity also produces back the results for gravitational scattering amplitudes obtained by other standard means. Also the splitting up of the action in the longitudinal and transverse planes is intuitively clear as in ultra high energy scattering, we expect the physics to be dictated by the centre of mass energy (which is governed by the longitudinal momenta) and we expect the transverse momentum transfer to add corrections to leading order results. However, when one applies the scaling argument to the QCD action, an effective simplified two dimensional action is indeed obtained, but one has to do a further perturbative calculation in order to extract the scattering amplitude, with its complicated $ln\ s$ behaviour. This is indeed a peculiar feature of QCD, and efforts are being made to see if one can extract the scattering amplitude semi classically in this case as in the case of QED, via the shock wave picture.

We have not seen in particular the regge behaviour of the scattering amplitude, which is peculiar to QCD. However, this does not mean that the scaling argument is wrong, and we believe that the correct way of computing the amplitude is via the renormalization group equation analysis of the Wilson lines, as done by Korchemsky (Korchemsky, 1994).

The most outstanding problem in this area is to prove or disprove the Regge behaviour of the scattering amplitude in the Verlinde framework, and once this is done, the model can be studied further in order to extract more information on Physics at high energy scales. One may also try large-N expansion techniques on this model to see what new feature we can obtain. Finally the challenge would be to apply this simple model to extract phenomenological data for hadron hadron scattering at high energies.

APPENDIX

CALCULATION OF FEYNMAN DIAGRAMS IN THE HIGH ENERGY LIMIT

In this appendix, we outline the calculation of Feynman diagrams in the high energy limit. This is extremely important in order to extract the high energy behaviour of scattering amplitudes in both abelian and non abelian gauge theories. It may be recalled that a Feynman diagram may be written as an integral over momentum variables, or, alternatively we may write it by introducing the Feynman parameters, in which case it finally reduces to doing integrations over these parameters. The high energy limit of a Feynman diagram may be computed in either of these two ways, by making suitable approximations. We shall be concerned with the computation of the diagrams using momentum variables. To illustrate the procedure, we shall use the simple example of a box diagram, that appears in the process of elastic scattering of two scalar mesons. Finally we make some comments regarding the amplitude corresponding to this diagram in the case of elastic fermion- fermion scattering at extremely energies.

To perform the calculation, we have to first fix up the frame of reference that we shall be working in. Now, it is possible to work in various frames. To be specific, let us choose a system of coordinates in which the spatial momenta of the two incident particles are along the positive and the negative z direction having magnitude ω and ω' respectively. The system in which $\omega = \omega'$ is the centre of mass frame, the system in which $\omega' = 0$ is the laboratory frame and that in which $\omega' = 0$ is the projectile frame. These three frames are related by Lorentz boosts in the z direction. It is also convenient in the high energy limit to work in light cone coordinates, and the four momentum of a particle is labelled by

$$p^{\mu} = (p_+, p_-, \mathbf{p}_\perp) \tag{1}$$

In terms of these coordinates then, if p^{μ} and p'^{μ} are the components of p in two Lorentz frames with relative velocity v, then, we can use the following transformation law,

$$p_+' = \sqrt{\frac{1 + v}{1 - v}} p_+$$

$$p_-^{'} = \sqrt{\frac{1 - v}{1 + v}} p_-$$
$$\mathbf{p}_\perp^{'} = \mathbf{p}_\perp \qquad (2)$$

The calculation of the diagrams can be carried out in any frame of reference, but we shall always do it the centre of mass frame. This has an obvious advantage that the momentum transfer is always in the transverse plane as we had commented in an earlier chapter.

Now, we specify the method of computing Feynman diagrams in the high energy limit. To make matters simple, we have chosen the box diagram of Figure A 1 arising in elastic scattering of two scalar mesons. Our strategy will be as follows. We will do the contour integration over all the + momentum variables, then as we shall see, the integration over the − variables becomes very simple under suitable approximations. All the lines in the box diagram represent mesons of mass μ. This diagram has only one loop, hence we need only one loop momentum, q. Let us look at the singularities in the Feynman amplitude. Considering first the meson with momentum $p_1 + q$. The denominator for this propagator is

$$(p_1 + q)^2 = (p_{1+} + q_+)(p_{1-} + q_-) - \mathbf{q}_\perp^2 - \mu^2 + i\epsilon \qquad (3)$$

This is linear in q_+. As can be seen from the above, this propagator has exactly one pole located at

$$q_+ = -p_{1+} + \frac{\mathbf{q}_\perp^2 + \mu^2 - i\epsilon}{p_{1-} + q_-}. \qquad (4)$$

It is clear that according as $p_{1-} + q_-$ is positive or negative, the pole in the q_+ plane lies in the lower half or the upper half respectively. Now, there may be different graphs corresponding to different directions of momentum flow, but energy momentum conservation at the vertices restricts us to only one diagram, namely that of Figure A 2. As we have already stated, our strategy will be to do the integration over the q_+ variable. From the direction of the arrows on the lines it is clear that the poles on the propagators for lines 1,2 and 3 lie on the same side of the real q_+ axis, while that of line 4 lies on the side opposite to the others. Hence while doing the q_+ integration, we can choose a contour that encloses either the three poles on one side or the other pole. It is obviously simpler to do the latter. That is why we have drawn the figure with a cross on line 4 indicating that it is the pole on this line that we are enclosing. Now, the variable q_- can take on any value between 0 and 2ω. Hence, we find it convenient to parametrize it in terms of a variable y, such that

$$q_- = 2\omega y \qquad (5)$$

where,

$$0 < y < 1 \tag{6}$$

We also define the variables

$$a_i = \mathrm{k}_{i\perp}^2 + \mu_i^2 \tag{7}$$

Where k_i and μ_i denote the momentum and the mass respectively of the scalar meson represented by the i th line. From the relation $k^2 = \mu^2$ at the pole, we also get the relation

$$k_{i+} = \frac{a_i}{k_{i-}} \tag{8}$$

At the pole corresponding to the i th line In the high energy regime that we are interested in, it is justified to set $p_{2+} = 0$, since the particle with momentum p_2 travels essentially along the x^- direction. Now, we see that, since $p_{2+} = 0$ and $p_{2-} = 2\omega$, hence,

$$-q_+ = \frac{a_4}{2\omega(1 - y)} \tag{9}$$

at the pole of the fourth line. At this pole, the values of the denominator for the lines 1, 2, and 3 are

$$\begin{aligned}
D_1 &= 2\omega y\left[-\frac{a_4}{2\omega(1 - y)}\right] - a_1 = -\frac{ya_4}{1 - y} - a_1 \\
D_2 &= 2\omega(2\omega y) - a_2 \simeq sy \\
D_3 &= 2\omega y\left[-\frac{a_4}{2\omega(1 - y)}\right] - a_3 = -\frac{ya_4}{1 - y} - a_3
\end{aligned} \tag{10}$$

Thus from Cauchy's Residue Theorem, we have for the q_+ integration, the result

$$\int_{-\infty}^{\infty} \frac{dq_+}{\prod_{i=1}^4 (k_i^2 - \mu_i^2)} = \frac{-2\pi i}{2\omega(1 - y) D_1 D_2 D_3} \tag{11}$$

Now the full integral corresponding to this box diagram is given by

$$\begin{aligned}
M &= -ig^4 \int \frac{d^4q}{(2\pi)^4} \frac{1}{\prod_{i=1}^4 (k_i^2 - \mu_i^2)} \\
&= -ig^4 \int \frac{d^2q_\perp}{(2\pi)^2} \int \frac{dq_0 dq_3}{(2\pi)^2} \frac{1}{\prod_{i=1}^4 (k_i^2 - \mu_i^2)} \\
&= \int \frac{d^2q_\perp}{(2\pi)^2} \times I
\end{aligned} \tag{12}$$

To evaluate the integral I, we note that the integration measure may be written as

$$\frac{dq_0 dq_3}{(2\pi)^2} = \frac{1}{2} \frac{dq_+ dq_-}{(2\pi)^2} = \omega \frac{dq_+ dy}{(2\pi)^2} \tag{13}$$

Hence from Equation (12), we get

$$I_{\cdot} = \int \frac{dq_0 dq_3}{(2\pi)^2} \frac{1}{\prod_{i=1}^{4}(k_i^2 - \mu_i^2)} = \frac{-i}{4\pi}\int_0^1 \frac{dy}{(1 - y) D_1 D_2 D_3} \tag{14}$$

Now, the main contribution to the integral comes from the region $y \rightarrow 0$ for which it is divergent. Hence to the first approximation we may set $y = 0$ and thus we obtain

$$I = -\frac{i}{4\pi a_1 a_3 s}\int_0^1 \frac{dy}{y} \simeq -\frac{i}{4\pi a_1 a_3}\frac{ln\ s}{s} \tag{15}$$

Where we have put a lower cutoff $\frac{1}{s}$ on y. Where s is the usual Mandelstam variable whose square gives the centre of mass energy. This cutoff can be justified as follows. We have approximated p_-, which is of the order of ω^{-1}, to zero. Hence our approximation breaks down if q_- is of the order of ω^{-1}.From Equation (5), we see that since $q_+ q_- = s$, hence the cutoff on y should be of the order of $\frac{1}{s}$. Thus, putting everything together, the amplitude for the box diagram is found from Equation (12) as

$$M \simeq -\frac{g^4\ ln\ s}{2s\ 2\pi}\int \frac{d^2 q_\perp}{(2\pi)^2}\frac{1}{a_1 a_3} \tag{16}$$

It should be noted that this amplitude is purely real. If we want the leading imaginary part of it, we must observe that there is a term $i\epsilon$ in the denominator of the propagator, and D_2 is approximately

$$D_2 = sy - a_2 + i\epsilon \tag{17}$$

If we put in this form of D_2 in M, we see that its only effect is to add a small imaginary component to M that is down by a factor of $ln\ s$ compared to the real part, and Equation (16) takes the form

$$\begin{aligned} M &\simeq -\frac{g^4\ ln\ (se^{i\pi})}{2s\quad 2\pi}\int \frac{d^2 q_\perp}{(2\pi)^2}\frac{1}{a_1 a_3} \\ &= -\frac{g^4\ ln\ (se^{i\pi})}{2s\quad 2\pi}\int \frac{d^2 q_\perp}{(2\pi)^2}\frac{1}{(\mathbf{q}_\perp^2 + \lambda^2)\left[(\mathbf{\Delta} - \mathbf{q}_\perp)^2 + \lambda^2\right]} \end{aligned} \tag{18}$$

So far, we have been concerned with the elastic scattering of scalar mesons only. Now say we want to compute the scattering amplitude of two fermions interacting via exchange of a vector meson, there will essentially be an additional numerical factor arising due to the incoming and outgoing fermionic wave functions that one has to put in this case. This numerical factor may be evaluated in the high energy limit. We shall not go into this calculation, but simply state the result that the numerical factor arising due to the wave functions is in the high energy limit $\simeq \frac{s^2}{m^2}$. Thus for the elastic scattering of fermions, the box diagram gives the result

$$M_{fermionic} = -\frac{g^4 s}{2m^2}\frac{ln\ (se^{i\pi})}{2\pi}\int\frac{d^2q_\perp}{(2\pi)^2}\frac{1}{(\mathbf{q}_\perp^2 + \lambda^2)\left[(\mathbf{\Delta} - \mathbf{q}_\perp)^2 + \lambda^2\right]} \tag{19}$$

It should be mentioned here that we have been so far concerned with abelian gauge theories. In non abelian gauge theories, the amplitudes have to be modified because the gluons carry colour, hence we have to take into account the colour changing amplitude as well as the non colour changing amplitude. But the approximations and the integrals are done in the same spirit as was illustrated here, the difference being in the group factor as we have discussed before.

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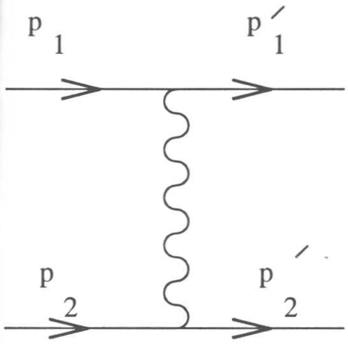


Figure 4.1

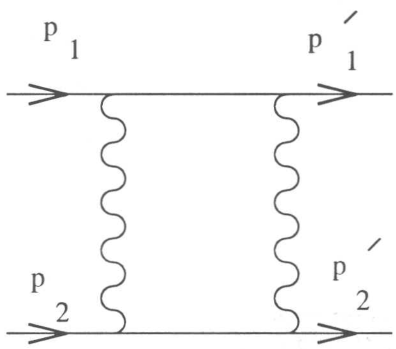


Figure 4.2 (a)

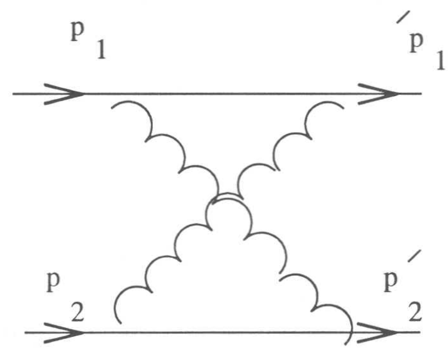
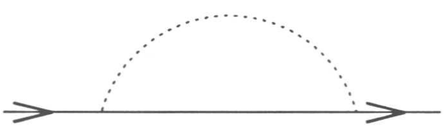


Figure 4.2(b)



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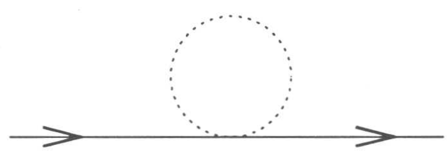

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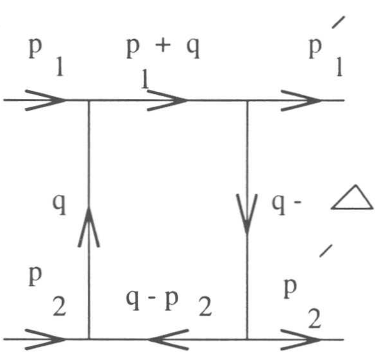

Figure A 1

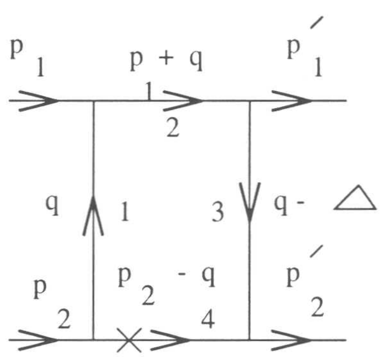

Figure A 2