SCALAR-TENSOR THEORY FOR GRAVITY IN D DIMENSIONS

By

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BONAFIDE CERTIFICATE

Certified that this dissertation titled **SCALAR-TENSOR THEORY FOR GRAVITY IN D DIMENSIONS** is the bonafide work of **Mr. Arjun Bagchi** who carried out the project under my supervision. Certified further, that to the best of my knowledge the work reported herein does not form part of any other dissertation on the basis of which a degree or award was conferred on an earlier occasion on this or any other candidate.

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Abstract

We study the general theory of relativity and the scalar-tensor theory in general D dimensions stressing on cosmology and the static spherically symmetric Schwarzschild solution. First we deal in brief with general relativity. Different aspects in the scalar-tensor theory like conformal transformation and geodesic equation in different conformal frames are then discussed. We take a specific form of the generalised Brans-Dicke action and workout the solutions to the field equations in the cases of cosmology and Schwarzschild metric. A set of constraints are imposed on the free function $\psi(\varphi)$ of the action under consideration. We look at the implications of these constraints on the two cases studied.

பணிச்சுருக்கம்

நாம் இங்கு பொது சார்பியல் கோட்பாடு மற்றும் ஸ்கேலார்-டென்சார் கோட்பாடின் D-பரிமானங்களை, அண்டவியல் மற்றும் ஸ்டேடிக் ஸ்பெரிக்கலி-சிம்மட்ரிக் ஸ்வார்ச்சைல்ட் தீர்வுகளுக்காகப் படிக்கின்றோம்.

முதலில் பொது சார்பியலைப்பற்றி சுருக்கமாகக் காண்கிறோம். ஸ்கேலார்-டென்சார் கோட்பாட்டின் பல்வேறு கூறுகளை(உதாரணமாக, கன்பார்மல் டிரான்ஸ்பர்மேசன் மற்றும் ஜியோடெசிக் சமன்பாடு) ஆராய்கிறோம். பொது பிரேன்ஸ்-டெக்கே செயலின் ஒரு குறிப்பிட்ட வடிவம் எடுத்துக் கொண்டு பீள்ட் சமன்பாடுகளுக்கானத் தீர்வுகளை(அண்டவியல் மற்றும் ஸ்வார்ச்சைல்ட் மெட்ரிக்) பெறுகிறொம். ஒரு குறிப்பிட்ட கட்டுப்பாடுகளை $\psi(\phi)$-ன் மீது விதிக்கிறொம். மேற்கண்ட இரண்டு நிலைகளில், இந்தக் கட்டுப்பாடுகளின் மூலம் பெறப்படுகின்ற முடிவுகள் ஆராயப்படுகின்றன.

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Arjun Bagchi

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Chapter 1

INTRODUCTION

1.1 Theories of Gravity

The General Theory of Relativity proposed by Albert Einstein is a theory dipped in aesthetic beauty. It is a geometrical theory of space time that has changed the very nature of our thinking. The theory is also successful in describing the present day observed universe. But there is no reason to believe that it is the one and only theory of gravitation. Many alternative theories of gravity have been proposed after the formulation of the general theory of relativity. And many of those have been discarded on grounds on feasibility.

For a gravitational theory to be viable it must satisfy a set of basic criteria.

- *Completeness*

  The theory under consideration must be capable of predicting from first principles the outcome of any experiment of interest.

- *Self consistency*

  The prediction of every experiment should be unique in the theory. That is to say that the results of the same experiment calculated in two different yet equivalent methods should always yield the same answer. The system of mathematical equations proposed should also be consistent.

- *Relativistic Theory*

  The laws of physics (non gravitational) in the theory should reduce to the laws of special relativity when we consider gravitational effects "turned off" in comparison to other types of interactions.

- *Correct Newtonian limit*

  In the limit of weak gravitational fields and non relativistic motion the theory under consideration must reproduce Newton's laws.

The above set of fundamental criteria are obtained from the Dicke framework for analyzing experimental tests (Dicke, 1964). Apart from the mathematical constraints imposed by the set of criteria above Dicke also imposed two physical constraints.

- Gravity must be associated with one or more fields of tensorial structure.

- The equations of gravitation must be derivable from an action principle.

The most successful theories of gravity satisfy these two constraints along with (obviously) the first set. The scalar-tensor theory ,the theory under consideration in this thesis, is among those successful theories of gravity.

1.2 What is the Scalar-Tensor theory?

The fundamental building block of the general theory of relativity is the metric tensor field $g_{\mu\nu}$ which is a tensor of rank two. So the theory can be called a tensor theory. The so called Scalar-Tensor theories contain a metric tensor field along with another dynamical scalar field (as the name suggests) coupled to it. So far there is no experiment which reveals or rejects the existence of the scalar field but there are several which place limits on the theory.

This sort of a theory was first proposed by Jordan (Jordan, 1949) who showed that the scalar field introduced could describe a time varying gravitational "constant" which was in keeping with Dirac's argument that the gravitational constant should be time dependent (Dirac, 1937). But the theory involved a non conservation of energy momentum tensor and was severely criticized on those and other grounds. A subsequent reformulation removed most of these objections (Jordan, 1959) but the theory still could not incorporate non relativistic matter successfully.

Brans and Dicke continued the work in the field (Brans and Dicke, 1961; Dicke, 1961; Dicke, 1962) and gave a complete and interesting version of a scalar-tensor theory. The motivation for their work was the lack of experimental evidence for the Strong Principle of Equivalence(SPE) of general relativity. The Brans-Dicke theory, as it came to be known, was constructed without taking SPE into account. In this theory the gravitational constant was replaced with the reciprocal of a scalar field. It was a generalization of general relativity (in the limit where the dimensionless Brans Dicke parameter goes to infinity we get back general relativity as we shall see in the main body of the thesis) and its predictions matched with the existing experimental results.

The very existence of a viable alternative to general theory of relativity led to more work in the field (Bergmann, 1968; Wagoner, 1970; Nordtvedt, 1970; Bekenstein, 1977) and the generalisation of the BD theory. This generalised theory has been applied to various cosmological and astrophysical aspects in (Bekenstein and Meisels, 1978) and many other papers.

In current context, the generalized BD theory appears naturally in supergravity , Kluza-Klein theories and in all the known effective low energy string theory actions.

1.3 What is the thesis about?

In this thesis we look at the generalised Brans-Dicke theory or the Scalar Tensor theory in general D dimensions. We are particularly interested in the case of cosmology and the static spherically symmetric Schwarzschild solution.

In order to give a picture of the differences between standard general theory of relativity and the scalar-tensor theory we present at first GTR in D dimensions with focus on cosmology and the Schwarzschild solution.

In the next chapter, we go on to give a brief description of the BD theory and its generalised version. Conformal transformations and the aspect of physics in different conformal frames are briefly visited. The particular framework we are working in is explicitly stated and the relations with the standard generalized BD model are established. We then take a passing glance at low energy string theory.

In Chapter 4, we refocus our attention on cosmology and the Schwarzschild solution this time in the scalar-tensor theory. We see that the Big Bang singularity persists in BD theories as well as in low energy string theory. In the Schwarzschild case there is the new feature of an extra essential singularity which is absent in GTR. We propose a set of constraints on the free function of the system $\psi(\varphi)$ and study the two systems under consideration. Some interesting results are obtained.

We conclude with a brief summary. An Appendix is included which contains various useful formulae and a few of the skipped steps in various calculations performed in the main body of the text.

Chapter 2

GENERAL RELATIVITY

We would be presenting in later chapters changes that are seen to Einstein's General Theory of Relativity, when we include the effects of the scalar field Φ,in a few particular cases.In this chapter we give an overview of those examples in GTR.The theory is considered in general D dimensions.

2.1 The Action Formulation of Gravity

2.1.1 The Action and Einstien's Equation

To arrive at the equations determining the gravitational field we need to determine first the action S_g of the field. The required equations can then be obtained by varying the sum of the field plus the material particles. It can be shown (Landau and Lifshitz, 1951) that the form of this action would be

$$S_g \quad = \quad -\frac{1}{16\pi G}\int d^dx\sqrt{-g}R. \tag{2.1}$$

Here R is the Ricci scalar, G is the gravitational constant and g is the determinant of the metric. The above action is called the *Einstein-Hilbert action.* Together with the matter action the total action of a system can be written as

$$S = S_g + S_m = -\frac{1}{16\pi G}\int d^dx\sqrt{-g}R + \int d^dx\sqrt{-g}\ \ \mathcal{L}_m \tag{2.2}$$

It will be useful to write the matter action in the following form (for use later).

$$S_m = S_m(\mathcal{M}, g_{\mu\nu}) \tag{2.3}$$

where $\mathcal{M}$ represents the matter fields, minimally coupled to the metric. An example to make the notation clear:

Scalar field: $\mathcal{M} = \Phi_m$

$$S_m(\Phi_m, g_{\mu\nu}) = \int d^d x \sqrt{-g}\ \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi_m \partial_\nu \Phi_m - V(\Phi_m)\right]$$

In this thesis we choose the signature of the metric to be $(- + + + ..)$ and define the Ricci tensor to be

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^{la}_{\mu\lambda} + \Gamma^\eta_{\mu\nu} \Gamma_\lambda \eta^\lambda - \Gamma^\eta_{\mu\lambda} \Gamma_\nu \eta^\lambda \tag{2.4}$$

The equations of motion are obtained from the least action principle $\delta(S_g + S_m) = 0$. The variations with the gravitational field i.e. $g_{\mu\nu}$ will generate the gravitational field equations and the variation with the other field variables present in the matter action gives the other field equations.

The variations give

$$\delta S_g = -\frac{1}{16\pi G} \int d^d x \sqrt{-g}\ (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \delta g^{\mu\nu} \tag{2.5}$$

$$\delta S_m = \frac{1}{2} \int d^d x \sqrt{-g}\ T_{\mu\nu} \delta g^{\mu\nu} \tag{2.6}$$

where $T_{\mu\nu}$ is called the *stress energy tensor* of the matter fields and is defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{2.7}$$

The gravitational field equations thus read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.8}$$

This is the famous *Einstein's Equation*, the basic equation of the general theory of relativity.

In empty space $T_{\mu\nu} = 0$, the equations of the gravitational field reduce to

$$R_{\mu\nu} = 0 \tag{2.9}$$

This does not mean that in vacuum spacetime is flat,- for that we would require the stronger condition $R^{\lambda}_{\mu\nu\sigma} = 0$. The divergence of the energy momentum tensor is zero.[1]

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{2.10}$$

Thus the divergence of the left side of eq (2.8) must be zero. This is actually the case because the Reimann tensors obey an identity called the *Bianchi identity*

$$R^{\alpha}_{\mu\nu\sigma;\lambda} + R^{\alpha}_{\mu\sigma\lambda;\nu} + R^{\alpha}_{\mu\lambda\nu;\sigma} \quad = \quad 0. \tag{2.11}$$

where ; implies covariant differention. This equation under two contractions with the metric can be brought to the form

$$(2R^{\mu\nu} - g^{\mu\nu} R)_{;\nu} \quad = \quad 0$$

which is called the *contrancted Bianchi identity.*

So eq.(2.10) is essentially contained in the field equations (2.8). Eq.(2.10) is an expression of the law of conservation of energy and momentum. This contains the equations of motion of the system which the particular $T_{\mu\nu}$ describes. Thus the gravitational field equations also contain the equations of matter that produce the field.

For complete determination of the distribution and motion of matter it is necessary to add to them the equation of state of the matter, i.e. a relation between the pressure and energy-density. This must be given along with the field equations.

2.1.2 The Stress-Energy Tensor

In this section we present a few explicit examples of stress tensors for particular matter fields.

- **Scalar field**

  Action:

$$S_{\phi} \quad = \quad \int d^d x \sqrt{-g} \ \left[\frac{1}{2} g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi - V(\phi)\right] \tag{2.12}$$

[1] The matter action is invariant under diffeomorphisms. For such actions $T_{\mu\nu}$ is always conserved by the virtue of the matter field equations.

Stress Energy Tensor:

$$T^{\phi}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(g^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi)g_{\mu\nu} - V(\phi)g_{\mu\nu} \tag{2.13}$$

- **Electromagnetic field**

  Action:

$$S_{em} = -\frac{1}{16\pi}\int d^dx\sqrt{-g}\;\; F_{\alpha\beta}F^{\alpha\beta} \tag{2.14}$$

  Stress Energy Tensor:

$$T^{em}_{\mu\nu} = \frac{1}{4\pi}(-F_{\mu\alpha}F^{\alpha}_{\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}) \tag{2.15}$$

- **Perfect Fluid**

  We would also require a stress energy tensor for macroscopic bodies (treated to be continuous). This is what is given by the Perfect fluid model. The perfect fluid model is largely a phenomenological model. So attempts to derive it from an action principle are not physical. The energy momentum tensor in this case is defined as

$$T^a_b = \begin{pmatrix} -\rho & & & & & \\ & p & & & & \\ & & p & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & p \end{pmatrix} \tag{2.16}$$

  here ρ and p are the energy density and pressure of the perfect fluid respectively. It is useful to write the above in the form

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} \tag{2.17}$$

  where u_a is a "velocity" 4-vector ($u^t = 1, u^i = 0$) and $u_a u^a = -1$.

2.1.3 Geodesic Equation

Consider a relativistic point particle. The action for it is given by

$$S_{PP} = -m \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} dt \tag{2.18}$$

In general relativity its equation of motion derived from the variation of the above action is given by a geodesic. The variation yields

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{dt} \frac{dx^\lambda}{dt} = 0 \tag{2.19}$$

This is the *geodesic equation.* Note that the integrand in the action is the measure of the distance between any two spacetime points. A geodesic thus connects two points on a curved geometry by the shortest (or longest) possible path. A particle in free fall will follow the geodesics of the spacetime. The importance of eq.(2.19) lies in the fact that it tells us how the trajectory of a test particle would be when we consider general relativity.

Newtonian Limit:

In the limit of weak fields and non relativistic particles the geodesic equation should yield Newton's gravitational laws.

Now at first we quantify our assumptions

(1) Weak Field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.20}$$

where $\eta_{\mu\nu}$ is the Minkowskian metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & . & & \\ & & & & . & \\ & & & & & 1 \end{pmatrix}$$

and $h_{\mu\nu}$ is small.

(2) Non relativistic limit

$$|\frac{dx^i}{dt}| \ll 1 \tag{2.21}$$

$$\Rightarrow \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} \approx \Gamma^i_{tt} \tag{2.22}$$

With these assumptions eq.(2.19) reduces to

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{tt} = 0 \tag{2.23}$$

Now

$$\Gamma^i_{tt} = -\frac{1}{2}\eta^{i\alpha}h_{tt,\alpha}$$

So eq.(2.23) becomes

$$\frac{d^2x^i}{dt^2} - \frac{1}{2}\eta^{ii}h_{tt,i} = 0$$

$$\frac{d^2x^i}{dt^2} = \partial^i(\frac{1}{2}h_{tt}) \tag{2.24}$$

With the identification

$$h_{tt} = -2V_n \quad \text{or} \quad g_{tt} = -(1+2V_n) \tag{2.25}$$

where V_n is the newtonian potential , we get back Newton's law

$$\ddot{x}^i = -\nabla^i V_n \tag{2.26}$$

Let us turn our attention briefly to **D=4**. For spacetime outside a spherically symmetric body of mass M the metric (this is the Schwarzchild solution which will be discussed later) is given by

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where G is the gravitational constant.

So, $V_n = \frac{-GM}{r}$ and this gives

$$\ddot{r} = -\frac{GM}{r^2}$$

$$\Rightarrow F = -\frac{GMm}{r^2} \tag{2.27}$$

where F is the radial force and m the mass of the test particle. This is the very familiar form of the Newton's law of Gravitation between two bodies of mass m and M separated by a distance of r.

2.2 Cosmology

2.2.1 The Metric

All study of Cosmology is based on the hypothesis called the Cosmological Principle which says that our Universe is spatially homogeneous and isotropic. On a large scale (typically a distance of thousand Megaparsecs) our observed universe is so to a very high accuracy. The formulation of the Cosmological Principle allows us to write down the most general form of the metric as

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu \\ &= -\,dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2_{D-2}\right), \end{aligned} \tag{2.28}$$

where $a(t)$ is an unknown function of time known as the *cosmic scale factor.* k is a constant which by a suitable choice of the units of r can be chosen to have the values +1, 0, -1. This constant determines the spatial curvature and the values +1, 0, -1 correspond to open, flat and closed universes.This metric is known in cosmology as the *Friedmann Robertson Walker metric* . There is overwhelming experimental evidence that our universe is spatially flat. In view of that, throughout this thesis we would be working with $k = 0$.

2.2.2 Einstein's Equations

Einstein's equation (2.8) can be rewritten to give

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{(D-2)} T g_{\mu\nu}) = 8\pi G S_{\mu\nu}. \tag{2.29}$$

where $S_{\mu\nu}$ is the source term. If we assume the universe to have a stress tensor of the ideal fluid form, then

$$T_{tt} = \rho \tag{2.30}$$

$$T_{ij} = p g_{ij} \tag{2.31}$$

$$S_{tt} = \frac{1}{(D-2)}[(D-3)\rho + (D-1)p] \tag{2.32}$$

$$S_{ij} = \frac{g_{ij}(\rho - p)}{(D-2)} \tag{2.33}$$

Using the above equations and the forms of the Ricci tensor components given in the appendix, - eqs(6.9) and (6.10) we arrive at

$$(D-1)(D-2)\frac{\ddot{a}}{a} = -8\pi G[(D-3)\rho + (D-1)p] \tag{2.34}$$

$$(D-1)(D-2)\frac{\dot{a}^2}{a^2} = 16\pi G\rho \tag{2.35}$$

As stated before along with this we would need another equation ,- the equation of state,to solve the system of equations. It is useful to have an equation for evolution of the mass density. This can be obtained directly from the above equations or from the fact that the stress energy tensor $T_{\mu\nu}$ is covariantly conserved. The equation we get is

$$\dot{\rho} + (D-1)(\rho + p)\frac{\dot{a}}{a} = 0. \tag{2.36}$$

2.2.3 The Equation of State

In general, the energy density ρ will include contributions from various distinct components. In cosmology the important aspect is how ρ evolves as the universe expands. Often the individual components (labelled here by i) have very simple equations of state of the form

$$p_i = \omega_i \rho_i \tag{2.37}$$

with ω_i as a constant.

Plugging in the equation of state into the evolution equation of ρ , - i.e. eq.(2.36) we find that the energy density has a power law dependence on the scale factor,

$$\rho_i \propto a^{-n_i} \tag{2.38}$$

where the exponent in the equation is related to the equation of state parameter by

$$n_i = (D-1)(1+\omega_i) \tag{2.39}$$

The simplest example of a component of this type is a set of massive particles with negligible relative velocities, known in cosmology as *dust*. For this form

$$\begin{aligned} p_d &= 0 \\ \Rightarrow \rho_d &\propto a^{-(D-1)}. \end{aligned}$$

For relativistic particles, known in cosmology as *radiation* we have

$$\begin{aligned} p &= \frac{\rho}{(D-3)} \\ \Rightarrow \rho_r &\propto a^{-D}. \end{aligned}$$

(For a derivation of the equations of state for dust and radiation see the Appendix.)

A *cosmological constant* which is equivalent to vacuum energy would not change as the universe expands, so

$$\rho_\Lambda \propto a^0$$

This implies a negative pressure when the vacuum energy is positive.

Restrictions on energy density:

The ranges of values of ρ_i that are allowed will obviously depend on the theory of the included matter fields. Still if we wish to make any statement about it we could look to the energy conditions,- especially the dominant energy condition(DEC).

DEC states that $T_{\mu\nu}l_\mu l_\nu \geq 0$ and $T_\mu^\nu l_\mu$ is nonspacelike for any null vector l_μ. This essentially implies that energy flow does not occur faster than the speed of light (Hawking and Ellis, 1973). For a perfect fluid $T_{\mu\nu}$ these two requirements imply that

$$\begin{aligned} \rho + p &\geq 0 \quad \text{and} \\ |\rho| &\geq |p| \end{aligned}$$

Type of Matter	ω_i	n_i
dust	0	D-3
radiation	$\frac{1}{D-3}$	D
cosm const.	-1	0

Table 2.1: Table for equation of state for different energy sources

In terms of the equation of state parameter ω this would imply that either

$$\begin{aligned} \rho &\geq 0 \quad \text{and} \quad |\omega| \leq 1 \\ \text{or} \quad \rho &\leq 0 \quad \text{and} \quad \omega = -1. \end{aligned}$$

So, a negative energy density is only allowed if it is in the form of a cosmological constant.

Here, in the above discussion the conventional DEC has been somewhat modified (Carroll, 2001) by using null vectors l_μ rather than the conventional null and timelike vectors which rules out , without any physical reason, a negative cosmological constant.

2.2.4 Solution of Einstein's Equations

- **Qualitative Aspects**

  Before going into the exact solutions let us first examine a few important qualitative features of the solutions of the equations. The first striking result is that the universe cannot be static provided $\rho > 0$ and $p \geq 0$. This follows from eq.(2.34) which tells us that $\ddot{a} < 0$. Thus the universe must always be expanding ($\dot{a} > 0$) or contracting ($\dot{a} < 0$) with the possible exception when the expansion goes over to a contraction. The distance between all isotropic observers changes with time but there is no preferred centre of this expansion or contraction. The expansion of the universe has been confirmed by experimental observation (red shift of distant galaxies).

Given that the universe is expanding and that $\ddot{a} < 0$ according to the theory it must have been expanding faster as one goes back in time. If the universe had always expanded at its present rate, then at a time $T = a/\dot{a}$ ago we would have $a = 0$. Since its expansion was faster the time taken was even less. Thus, under the assumption of homogeneity and isotropy GTR makes the remarkable prediction that at a time less than $T = a/\dot{a}$ the universe was in a singular state, - the distance between all space time points was zero and the density of matter and curvature of space time was infinite. This is referred to as the *Big Bang Singularity* .

Near this extreme state the theory of general relativity breaks down on account of strong quantum phenomenon. Theories like String theory and Quantum gravity may resolve these difficulties.

- **Explicit solutions**

  Now for the explicit solutions of the field equations . For a general case, where $\rho_i \propto a^{-n_i}$,the dependence of the scale factor and Ricci scalar on time is

$$a = Ct^{2/n} \tag{2.40}$$

$$R = 4\frac{(1-\omega)}{(1+\omega)^2}\, t^{-2} \tag{2.41}$$

  As $t \to 0$ which in turn implies $a \to 0$, we encounter a curvature singularity. The coordinate invariant quantities ,- the scalars created out of the various curvature tensors and their derivatives all blow up at at this point. (Exception: for radiation the Ricci scalar is zero.)

2.3 The Schwarzschild Solution

The Einstein's equations even for empty space being non linear are notoriously difficult to solve. The case of the static spherically symmetric field produced by a spherically symmetric body at rest can however be solved without too much difficulty. This solution was first arrived at by K.Schwarzschild and is named after him.

2.3.1 The Metric

The spherical symmetry of the field means that the metric must be same for all the points located at equal distances from the centre. In Euclidean space this distance is the radius vector but in a non-Euclidean spacetime which is the case in presence of a gravitational field,- there is no such quantity with all the properties of the Euclidean radius vector. The static condition means that with a static coordinate system, the $g_{\mu\nu}$ are independent of time and $g_{ti} = 0$. The most general form for ds^2 compatible with spherical symmetry is

$$ds^2 = -Udt^2 + Vdr^2 + Wr^2 d\Omega^2_{D-2} \tag{2.42}$$

where U,V,W are all functions of r alone.

Here we choose the form to be

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + h^2(r)d\Omega^2_{D-2} \tag{2.43}$$

2.3.2 The Equations and the solutions

The nonzero connection terms and the relevant Ricci tensor components are listed in the appendix. There are only three independent components of the Ricci tensor [eqs (6.12),(6.13),(6.14)]. The $(D-2)$ components in the solid angle $d\Omega_{D-2}$ are related to one another. This is again gievn in the appendix explicitly (see eqs(6.15)). We may replace r by any function of r without disturbing the spherical symmetry. Using this freedom we can effectively set $h = r$. We wish to find the vacuum solutions which in this case is known as the *Exterior Schwarzschild Solutions*. The equations to solve are (2.9). The relevant field equations are

$$\frac{f'^2(r)}{4f^2(r)} - \frac{f''(r)}{2f(r)} - \frac{f'(r)g'(r)}{4f(r)g(r)} - \frac{(D-2)g'(r)}{2rg(r)} = 0 \tag{2.44}$$

$$\frac{2f''(r)}{f'(r)} - \frac{f'(r)}{f(r)} + \frac{g'(r)}{g(r)} + 2(D-2)\frac{1}{r} = 0 \tag{2.45}$$

$$(D-3)\frac{1-g(r)}{rg(r)} - \frac{g'(r)}{2g(r)} - \frac{f'(r)}{2f(r)} = 0 \tag{2.46}$$

Solving the equations we find

$$f(r) = 1 - (\frac{r_0}{r})^{D-3} \quad (2.47)$$
$$g(r) = 1 - (\frac{r_0}{r})^{D-3} \quad (2.48)$$

In the weak field limit,- i.e. for large values of r , we saw previously, one gets the Newtonian approximation. The Newtonian limit gives

$$g_{tt} = -(1 + 2V_n) \quad (2.49)$$

where V is the potential energy. When we consider the case of potential far away from a spherical body of mass m the limit will give

$$g_{tt} = -[1 + \frac{2Gm}{r^{D-3}}] \quad (2.50)$$

So comparing with equation(2.47) we see that $r_0^{D-3} = 2Gm$.

The complete solution is

$$ds^2 = -(1 - \frac{2Gm}{r^{D-3}})dt^2 + (1 - \frac{2Gm}{r^{D-3}})^{-1}dr^2 + r^2 d\Omega_{D-2}^2 \quad (2.51)$$

The solution holds outside the surface of the body producing the field where there is no matter. Thus it holds fairly accurately outside the surface of a star.

The solution at first glance seems to have two singular points,- $r = 0$ and $r^{D-3} = 2Gm$. Of these the latter is a coordinate singularity , i.e. choosing suitable coordinates makes the singularity disappear.[2] The $r = 0$ singularity however cannot be avoided by any such coordinate transformation and is an essential singularity in the geometry of spacetime.

[2]The coordinates are the Kruskal coordinates. See for example (Wald, 1984)

Chapter 3

SCALAR TENSOR THEORY OF GRAVITY

In this chapter we look at the novel features of the scalar tensor theory also known as the generalised Brans Dicke theory. Many new features arise with the inclusion of the scalar field ϕ.

3.1 The Brans-Dicke Model

3.1.1 Original Brans Dicke Formalism

Inspired probably by the work of Paul Dirac on the time varying gravitational fields (Dirac, 1937), a graduate student, Brans and his advisor Dicke (Brans and Dicke, 1961) proposed their now famous model. They replaced the gravitational constant G with the reciprocal of a scalar field ϕ. The action that they wrote down for their theory was

$$S_{BD} = -\frac{1}{16\pi}\int d^dx\sqrt{-g}(\phi R - \quad \omega\frac{(\nabla\phi)^2}{\phi}) + \int d^dx\sqrt{-g} \ \ \mathcal{L}_m \tag{3.1}$$

where ω is a constant. The primary role of the scalar field introduced according to the authors was the determination of the local value of the gravitational constant. The terms in the Lagrangian density involving the matter fields in Eq.(2.2) and Eq.(3.1) are identical. So, the equation of motion of the matter fields in a given predetermined metric field are one and the same. The difference arises in the gravitational field equations.

3.1.2 Generalised Brans Dicke Theory

A variety of theories of gravity have been proposed in which like in the above case in addition to the metric a dynamical scalar field is introduced (Bergmann, 1968; Wagoner, 1970; Nordtvedt, 1970; Bekenstein, 1977). The generalisation of the Brans Dicke model is achieved by the transformation of the constant ω into a function of the scalar field variable χ. In a further generalisation a cosmological function $\lambda(\chi)$ can also be added. The reformulated action now looks like

$$S_{gBD} = -\frac{1}{16\pi}\int d^dx\sqrt{-g}(\chi R - \frac{\omega(\chi)}{\chi}(\nabla\chi)^2 + 2\chi\lambda(\chi)) + \int d^dx\sqrt{-g}\mathcal{L}_m \tag{3.2}$$

Varying the action with respect to $g_{\mu\nu}$ and χ we arrive at the field equations which read

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \lambda(\chi)g_{\mu\nu} - \chi^{-2}\omega(\chi)(\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}(\nabla\chi)^2) \\ +\chi^{-1}(\nabla_\mu\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}\nabla^2\chi) = 8\pi\chi^{-1}T_{\mu\nu} \end{aligned} \tag{3.3}$$

$$\nabla^2\chi + \frac{1}{2}(\nabla\chi)^2\frac{d}{d\chi}\ln(\frac{\omega(\chi)}{\chi}) + \frac{\chi}{2\omega(\chi)}[R + 2\frac{d}{d\chi}\chi\lambda(\chi)] = 0 \tag{3.4}$$

where $T_{\mu\nu}$ is given by eq(2.7). In this thesis we would not be interested in the cosmological function and would proceed setting it to zero. The somewhat simplified field equations now take the form

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \chi^{-2}\omega(\chi)(\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}(\nabla\chi)^2) \\ +\chi^{-1}(\nabla_\mu\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}\nabla^2\chi) = 8\pi\chi^{-1}T_{\mu\nu} \end{aligned} \tag{3.5}$$

$$\nabla^2\chi + \frac{1}{2}(\nabla\chi)^2\frac{d}{d\chi}\ln(\frac{\omega(\chi)}{\chi}) + \frac{\chi}{2\omega(\chi)}R = 0 \tag{3.6}$$

For the original Brans-Dicke theory (i.e. where ω is a constant), in the limit $\omega \to \infty$, the theory reduces to general relativity. In particular, for large current values of ω, the theory make predictions for all gravitational situations differ from the general relativity values at most by corrections of $O(1/\omega)$. The same limit $\omega \to \infty$ yields GR in the generalized theories too, but under a few conditions. For details see (Will, 1992).

3.2 Conformal Transformations

By putting in the scalar field into the action as a replacement for the gravitational constant we have nonminimaly coupled it to gravity. If we wish to put this nonminimal coupling term into another form we need to apply a *conformal transformation.* This comes about from the fact that general relativity is not invariant under conformal transformations. [1] Among the infinitely many conformal frames we would be interested in the Einstein frame and the so called Physical frame.

3.2.1 A bit about conformal transformations

We saw in the previous section that the scalar field which was not coupled to the initial matter Lagrangian couple to matter finally through the field equations. By applying conformal transformations and moving from one conformal frame to another the nature of initial coupling of the scalar field to both the gravitational and matter Lagrangians would change, as would the equations of motion governing the system. From a completely technical point of view solving these equations in some particular frame often becomes simpler and by applying the inverse transform we could always get back to the original frame of interest. A conformal transformation transforms a metric $g_{\mu\nu}$ to another metric $g_{*\mu\nu}$ according to the rule

$$g_{*\mu\nu} = \Delta^2(x) g_{\mu\nu} \tag{3.7}$$

[1] If two metrics are related by a conformal transformation their causal structures are identical (Wald, 1984).

where $\Delta(x)$ is an arbitrary function of the spacetime coordinate x. This is equivalent to the transformation applied to a line element

$$ds_*^2 = \Delta^2(x) ds^2 \tag{3.8}$$

Here we limit ourselves to transformations which preserve the positive sign of the line element and hence use $\Delta^2(x)$ instead of any arbitrary function $\Delta(x)$ of the spacetime variable. This transformation is essentially different from a general coordinate transformation. The above transformation (3.8) implies that under it the change of distance of any two spacetime points differs from point to point on the spacetime manifold. The change is also independent of direction and hence isotropic. As a result, the transformation preserves the angle between any two vectors. This is what is meant by "conformal". The relevant formulae for the transformation of a few metric dependent quantities are given below

$$\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{*\nu\lambda} - (f_\nu \delta^\mu_\lambda + f_\lambda \delta^\mu_\nu - f^\mu_* g_{*\nu\lambda}) \tag{3.9}$$

$$R = \Delta^2 (R_* + 2(D-1)\nabla^2_* f - (D-1)(D-2) g_*{}^{\mu\nu} f_\mu f_\nu) \tag{3.10}$$

where

$$\begin{aligned} f &= \ln \Delta \ , \quad f_\mu = \frac{\partial_\mu \Delta}{\Delta} = \partial_\mu f \ , \quad f^\mu_* = g^{\mu\nu}_* f_\nu \ ; \\ \nabla^2_* f &= \frac{1}{\sqrt{-g_*}} \partial_\mu (\sqrt{-g_*} g^{\mu\nu}_* \partial_\nu f) \end{aligned}$$

We attach a symbol * to remind ourselves that we are in a new conformal frame.

3.2.2 The Physical and Einstein frames

The two conformal frames of interest as mentioned before are the so called Physical(P) and Einstein(E) frames.

The action (3.2) that we have been dealing with so far is the action in the P frame. The scalar field is nonminimally coupled with gravity and the matter action couples to system only through the metric. In general relativity the matter couples to gravity in the same way and that is the reason behind calling the frame Physical. Now let us take the action in the P frame and perform a conformal transformation on it such that $\chi\Delta^2 = 1$. So the first term in the action becomes

$$\mathcal{L}_{*g} = \sqrt{-g}R_* \tag{3.11}$$

So as in the Einstein-Hilbert action, the scalar field is no longer nonminimally coupled to gravity. By E frame, we would be referring to this particular frame. It is worth noting here that the matter Lagrangian no longer couples via only the metric,- the scalar field is now present in the coupling in a specific manner which reflects what the theory looks like in other conformal frames. So the theory in this frame is not a trivial case of general relativity in the presence of a scalar field. The action in the E frame looks like the following

$$S_E = -\frac{1}{16\pi}\int d^dx\sqrt{-g_*}(R_* - \frac{1}{2}(\nabla_*\phi)^2) + S_m(\mathcal{M}, e^{-2f}g_{*\mu\nu}) \tag{3.12}$$

where the notation of the matter action is the one used in eq.(2.3)

The geodesic equations in the two frames

The geodesic equations in the two frames give us a clear picture of the difference of the two frames and also tell us why the P frame is so called.

In the P frame the coupling of the matter to the system as seen above is only through the metric. So the test particle couples to gravity only via $g_{\mu\nu}$ as in general relativity. The geodesic equation thus remains the same. We use $u^\mu = \frac{dx^\mu}{dt}$ and rewrite eq(2.19) as

$$\frac{du^\mu}{dt} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0 \tag{3.13}$$

We now look at the E frame. From eq.(3.8) we have

$$\frac{dt_*}{dt} = \Delta \tag{3.14}$$

$$\Rightarrow u^\mu = \Delta u^\mu_* \tag{3.15}$$

Therefore

$$\frac{du^\mu}{dt} = \frac{dt_*}{dt}\frac{d}{dt_*}(\Delta u^\mu_*) = \Delta^2(\frac{du^\mu_*}{dt_*} + f_\nu u^\nu_* u^\mu_*) \tag{3.16}$$

by using

$$\frac{d\Delta}{dt_*} = \frac{\partial\Delta}{\partial x^\sigma}\frac{dx^\sigma}{dt_*} = \Delta^2 f_\sigma u_*^\sigma$$

So putting in eqs(3.16) and (3.9) into eq (3.13) gives us the geodesic equation in the E frame which now reads

$$\frac{du_*^\mu}{dt_*} + \Gamma^\mu_{*\nu\lambda}u_*^\nu u_*^\lambda = f_\nu u_*^\nu u_*^\mu - f^\mu u_{*\nu}u_*^\nu \tag{3.17}$$

The right hand side of the eq (3.17) is no longer zero and that is an indicator to the fact that the test particle in the E frame follows a path different to the P frame. The effect of the scalar field on the test particle manifests itself in the E frame whereas there is no departure from the standard GR path of the test particle in the P frame.

Newtonian limit:

We do an analysis similar in lines to that in sec(2.1.3) to get at the Newtonian limit in the two frames. As in sec(2.1.3) we will do the analysis in $D = 4$.

In the P frame the previous analysis goes through identically.

Let us see what changes take place in the E frame. With the restrictions imposed as before eq(3.17) reduces to

$$\frac{du_*^i}{dt_*} + \Gamma^i_{*tt} = f_t u_*^i - f^i \tag{3.18}$$

$$\Rightarrow \frac{d^2x_*^i}{dt_*^2} - \nabla^i(\frac{1}{2}h_{t_*t_*}) = k\dot{\varphi}\frac{dx_*^i}{dt_*} - k\nabla^i\varphi \tag{3.19}$$

where we choose $f = k\varphi$ (This is the case in low energy String theory which will be discussed later).

As before we take the 4 dimensional Schwarzchild solution . Here $\varphi = \varphi(r)$. So in this context in the presence of a scalar field in the E frame Newton's law in modified to

$$F = -\frac{GMm_0}{r^2} - km_0\frac{d\varphi}{dr} \tag{3.20}$$

where m_0 is the mass of the test particle. We see that in the absence of the scalar field ($k = 0$) the Newton's law in the two frames would look the same. The same is true if the scalar field was a constant.

3.2.3 A specific form of the action

In this thesis we would use a particular form for the action in the physical frame. (The motivation for this is to reach the precise for of the action(3.12)after transformation to the E frame which might otherwise involve some function of φ before the kinetic term). In this section we state that action and the relations with the previous action (3.2) for the P frame. Using a few examples we try to give a clear picture of the mapping. The equations of motion for this particular action are presented after that.

- **The Action**

$$\begin{aligned} S_P = \quad & -\frac{1}{16\pi}\int d^dx\sqrt{-g}e^{(\frac{D-2}{2})\psi(\varphi)} \\ & [R - \frac{1}{2}\{1 - \frac{(D-1)(D-2)}{2}\psi_\varphi^2\}(\nabla\varphi)^2] \end{aligned} \tag{3.21}$$

Here $\psi_\varphi = \frac{d\psi}{d\varphi}$

- **The mapping from one action to the other**

We have two actions for the P frame viz. Eq(3.2) and Eq(3.21). We compare the two equations to get the following results

$$\chi = \exp(\frac{D-2}{2})\psi \tag{3.22}$$

$$\frac{\omega(\chi)}{\chi^2}\chi_\varphi^2 = \frac{1}{2}\{1 - \frac{(D-1)(D-2)}{2}\psi_\varphi^2\} \tag{3.23}$$

$$\omega(\chi) = \frac{2}{(D-2)^2\psi_\varphi^2}\{1 - \frac{(D-1)(D-2)}{2}\psi_\varphi^2\} \tag{3.24}$$

$$\Omega(\chi)\frac{\chi_\varphi^2}{\chi^2} = 1 \tag{3.25}$$

$$\varphi = \int d\chi\frac{\sqrt{\Omega(\chi)}}{\chi} \tag{3.26}$$

where $\Omega(\chi) = 2(\omega(\chi) + \frac{(D-1)}{(D-2)})$.

- **Examples**

1.We first consider a functional form of $\omega(\chi)$ and find the function $\psi(\varphi)$

$$\begin{aligned}
\omega(\chi) &= \frac{1}{2}\chi^2 - \frac{(D-1)}{(D-2)} \\
\varphi &= \chi + c \\
\psi(\varphi) &= \frac{2}{(D-2)}\ln(\varphi - c)
\end{aligned}$$

Here c is a constant of integration.

2. Now one the other way around. We start off with $\psi(\varphi)$ and arrive at $\omega(\chi)$

$$\begin{aligned}
\psi(\varphi) &= k\varphi \\
\chi &= e^{(D-2)k\varphi/2} \\
\omega(\chi) &= \frac{-(D-1)(D-2)k^2 + 2}{(D-2)^2k^2}
\end{aligned}$$

This is the form of $\psi(\varphi)$ for the prototype Brans-Dicke model. Putting $k = 1$ we get

$$\omega(\chi) = -\frac{D(D-3)}{(D-2)^2}$$

which reduces to $\omega(\chi) = -1$ for D=4.

Putting $k = 0$ we see that $\omega(\chi)$ blows up. This is what we expect as for $k = 0$ the scalar field decouples from gravity and we get the case of general relativity in the presence of a scalar field. And $\omega \to \infty$ is the limit where the Brans-Dicke theory reduces to general relativity.

3. This one would be of importance later.

$$\begin{aligned}
\psi(\varphi) &= -\lambda\sqrt{\varphi^2 + c} \\
\chi &= e^{-\lambda(D-2)\sqrt{\varphi^2+c}/2} \\
\Omega(\chi) &= \frac{c_1(\ln\chi)^2}{(\ln\chi)^2 + c_2} \\
\omega(\chi) &= \frac{c_1(\ln\chi)^2}{(\ln\chi)^2 + c_2} - \frac{(D-1)}{(D-2)}
\end{aligned}$$

Again here c,c_1 and c_2 are constants with $c_1 = 4/\lambda^2(D-2)^2$ and $c_2 = -cc_1$.

- **Equations of Motion**

Varying the actions (3.21) and (3.12) with the metric and the scalar field we get the equations of motion of the system. We consider only the vacuum solutions in this thesis.

The Physical Frame:

In the P frame the equations of motion take the following form (see Appendix for details)

$$\begin{aligned} R_{\mu\nu} = & \quad (\frac{D-2}{2})\psi_\varphi \nabla_\mu \nabla_\nu \varphi + \frac{1}{2}\psi_{\varphi\varphi} g_{\mu\nu} (\nabla\varphi)^2 \\ & + \quad [(\frac{D-2}{2})(\psi_{\varphi\varphi} - \frac{1}{2}\psi_\varphi^2) + \frac{1}{2}]\nabla_\mu \varphi \nabla_\nu \varphi \qquad (3.27) \\ \nabla^2 \varphi & + \quad (\frac{D-2}{2})\psi_\varphi (\nabla\varphi)^2 = 0 \qquad (3.28) \end{aligned}$$

The Einstein Frame:

In the E frame the equations of motion simplify significantly when we consider the vacuum solutions.

First the transformation that gets us there from the P frame

$$g_{*\mu\nu} = e^{\psi(\varphi)} g_{\mu\nu} \qquad (3.29)$$

$$\varphi_* = \varphi \qquad (3.30)$$

The equations of motion

$$\begin{aligned} 2R_{*\mu\nu} - \nabla_{*\mu}\varphi \nabla_{*\nu}\varphi &= 0 \qquad (3.31) \\ \nabla_*^2 \varphi &= 0 \qquad (3.32) \end{aligned}$$

- **Low Energy String Theory**

In low energy String Theory, the graviton dilaton part of the theory can be derived from the spacetime action (Polchinski, 1998) given below

$$S_{string} = \int d^d x \sqrt{-g} \;\; e^{-2\varphi} \{R + 4(\nabla\varphi)^2\} \qquad (3.33)$$

Comparing with our action (3.21) we see

$$(\frac{D-2}{2})\psi = -2\varphi$$
$$-\frac{1}{2}\{1 - \frac{(D-1)(D-2)}{2} = 4$$

Using these we get the field equations as

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \varphi = 0 \tag{3.34}$$
$$\nabla^2\varphi - 2(\nabla\varphi)^2 = 0 \tag{3.35}$$

and as a spurious coincidence

$$D = 10 \tag{3.36}$$

The field equations match the string equations as expected. $D = 10$ the prediction of Super-symmetric String theory that manifests itself here. We are unaware of the significance of this result,- if there are any.

Chapter 4

COSMOLOGY AND SCHWARZSCHILD SOLUTION IN SCALAR TENSOR THEORY

We have so far looked at the Scalar Tensor theory in general. In this chapter we look at the two particular examples that were presented in the chapter on general relativity viz., cosmology and the Schwarzschild solution. We would as stated before only be interested in the vacuum solutions in this thesis. Some interesting results are obtained as we look at the properties of these solutions. The analysis of the novel properties given here are a higher dimensional extension of (Kalyana Rama and Ghosh, 1996; Kalyana Rama, 1996).

4.1 Cosmology in Scalar-Tensor Theory

In this section we investigate how cosmology changes from the usual general relativistic case with the inclusion of the coupled scalar field. First we look at the equations of motion in the P frame. Then we move to the more convenient E frame and find solutions for the differential equations. The Ricci scalar is calculated in both the E and P frames. Then we impose certain conditions on the form of the function $\psi(\varphi)$ and study the features of the resulting solution.

4.1.1 Equations of motion in the P frame

We already have the form of the Ricci tensor for the FRW metric (2.28) in D dimensions. The Appendix gives the explicit derivation and the nonzero components of $R_{\mu\nu}$ are given by Eqs (6.9) and (6.10). From the previous chapter we have the form of the equations of motion in the P frame i.e. Eqs (3.27) and (3.28). Putting all these together we get the equations for cosmology. Here the scalar field is taken to be a function of time alone.

$$\begin{aligned}\ddot{\varphi} &+ (D-1)\frac{\dot{a}\dot{\varphi}}{a} + (\frac{D-2}{2})\psi_\varphi \dot{\varphi}^2 = 0 & (4.1)\\ (D-1)\frac{\ddot{a}}{a} &= \frac{1}{2}(D-1)(D-2)\psi_\varphi \frac{\dot{a}\dot{\varphi}}{a} \\ &- \frac{1}{2}\{(D-1)\psi_{\varphi\varphi} + \{1 - \frac{(D-1)(D-2)}{2}\psi_\varphi^2\}\}\dot{\varphi}^2 & (4.2)\\ \frac{\ddot{a}}{a} + (D-2)\frac{\dot{a}^2}{a^2} &= -(\frac{D-2}{2})\psi_\varphi \frac{\dot{a}\dot{\varphi}}{a} - \frac{1}{2}\psi_{\varphi\varphi}\dot{\varphi}^2 & (4.3)\end{aligned}$$

Using the above eqs(4.2) and (4.3) we can arrive at

$$\frac{\dot{a}^2}{a^2} + \psi_\varphi \frac{\dot{a}\dot{\varphi}}{a} - \frac{1}{2}\{\frac{1}{(D-1)(D-2)} - \frac{1}{2}\psi_\varphi^2\}\dot{\varphi}^2 = 0 \tag{4.4}$$

We here have three equations and two unknowns. The Bianchi identity links these equations. So, in effect we can use any two equations to get at the solutions. The third becomes a constraining equation which determines the system uniquely. The fact that the Bianchi identities link the equations can be clearly seen if we differentiate the left hand side of say eq(4.4) and use any two of the other equations to explicitly calculate the value,- which obviously is zero as the eq (4.4) suggests.

4.1.2 The equations and solutions in the E frame

The equations of motion for vacuum get simplified in the E frame. Using the eqs(3.31) and (3.32) and the form of the Ricci tensor components or equivalently setting $\psi(\varphi) = 0$ in (4.1),(4.3) and (4.4) we get

$$\ddot{\varphi} + (D-1)\frac{\dot{a}_*\dot{\varphi}}{a_*} = 0 \tag{4.5}$$

$$\frac{\ddot{a_*}}{a_*} + (D-2)\frac{\dot{a_*}^2}{a_*^2} = 0 \tag{4.6}$$

$$2(D-1)(D-2)\frac{\dot{a_*}^2}{a_*^2} - \dot{\varphi}^2 = 0 \tag{4.7}$$

We solve the first two equations to get

$$a_*(t_*) = At_*^{1/(D-1)} \tag{4.8}$$

$$\varphi(t_*) = \epsilon m \ln t_* + \varphi_0 \tag{4.9}$$

where A,m,φ_0 are all constants and $\epsilon = \pm 1$. The third equation behaves as a constraint and gives

$$m = \sqrt{2\frac{(D-1)}{(D-2)}}$$

Note that by definition m is always positive.

The solutions thus are

$$a_*(t_*) = At_*^n \tag{4.10}$$

$$\varphi(t_*) - \varphi_0 = \epsilon m \ln t_*$$

$$\Rightarrow e^{\varphi - \varphi_0} = t_*^{\epsilon m} \tag{4.11}$$

where we have replaced $n = 1/(D-1)$.

4.1.3 The Big Bang singularity

The metric in the E frame is

$$ds_*^2 = -dt_*^2 + a_*^2(dr^2 + r^2 d\Omega_{D-2}^2) \tag{4.12}$$

Again the metric in the P frame is

$$ds^2 = -dt^2 + a^2(dr^2 + r^2 d\Omega_{D-2}^2) \tag{4.13}$$

The conformal transformation thus relates the cosmic times and scale factors of the two frames. The relations are

$$\frac{dt}{dt_*} = e^{-\psi/2} \quad , \quad a = a_* e^{-\psi/2}; \tag{4.14}$$

Using the formula (3.10) from the previous chapter and putting in the fact that $f = 2\psi(\varphi)$, we get

$$\begin{aligned} R &= e^{\psi(\varphi)}(R_* + (D-1)\nabla_*^2\psi - \frac{1}{4}(D-1)(D-2)g_*{}^{\mu\nu}\nabla_\mu\psi\nabla_\nu) \\ \Rightarrow R &= e^{\psi(\varphi)}R_*\{1 + 2(D-1)\psi_{\varphi\varphi} - \frac{1}{2}(D-1)(D-2)\psi_\varphi^2\} \end{aligned} \quad (4.15)$$

here we also use the equations of motion of the E frame.

Again the form of R for cosmology (see Appendix (6.11)) together with (4.10) gives us the form of the Ricci scalar in the E frame.

$$R_* = -(\frac{D-2}{D-1})t_*^{-2} \quad (4.16)$$

Low Energy String Theory

As seen before we can go to the picture of the low energy string theory where

$$\psi(\varphi) = k\varphi. \quad (4.17)$$

with $k = -4/(D-2)$

With the above choice the cosmic time and scale factors in the two frames can now be explicitly related

$$a = a_* e^{-k\phi/2} \quad (4.18)$$

$$t = \frac{t_*^{1-\epsilon km/2}}{1-\epsilon km/2} + c \quad (4.19)$$

where c is a constant. So now we can have an explicit solution for the Ricci scalar in the P frame

$$R = -\{1 - \frac{1}{2}(D-1)(D-2)k^2\}(\frac{D-2}{D-1})t_*^{-2(1-\epsilon km/2)} \quad (4.20)$$

So we see that at $t_* \to 0$ the physical curvature scalar diverges (for positive values of the exponent which implies $|km| \leq 2$). In the same limit the physical time reaches a constant value. Here we encounter a physical curvature singularity.This is the same as the singularity encountered in general relativistic cosmology,- the *Big Bang Singularity.* The physical cosmic time cannot be extended beyond this. The reason for this is that t_* cannot be extended to negative values which is not allowed.

4.1.4 Constraints

The constraints we impose on the on the function $\psi(\varphi)$ are (the motivation of this particular choice will become clear in (Kalyana Rama and Ghosh, 1996))

$$(1) \qquad \psi_{(n)}(\varphi) \equiv \frac{d^n \psi}{d\varphi^n} \quad \text{is finite } \forall n \geq 1$$

$$(2) \qquad \lim_{\varphi \to \pm\infty} \psi(\varphi) = -\lambda|\varphi|$$

Here λ is a positive constant.

The two constraints mean that there is a finite upper bound on $\psi(\varphi)$.

$$\psi(\varphi) \leq \psi_{max}(\varphi) < \infty \tag{4.21}$$

There are many functions which have the above properties. For example in Sec(3.2.3) the third example ($\psi(\varphi) = -\lambda\sqrt{\varphi^2 + c}$) dealt with such a function. We would be interested in the general properties of any such function.

4.1.5 Novel features of the solution

We will look at how the constraints imposed on $\psi(\varphi)$ lead to interesting results in cosmology. Putting in the constraint (2) into (4.11) we get As $t \to \infty$

$$\epsilon = -1; \qquad e^{\psi(\varphi)} \to t^{-\lambda m} \tag{4.22}$$

and as $t \to 0$

$$\epsilon = 1; \qquad e^{\psi(\varphi)} \to t^{\lambda m} \tag{4.23}$$

Again eq(4.21) tells us that $e^{-\psi/2} \geq e^{-\psi_{max}/2} > 0$. So invoking eq(4.14) we see that the physical time t is a strictly increasing function of t_*.

We look at the two limiting cases now.

The limit $t_* \to \infty$

Using eqs we get

$$t = \frac{2}{2 + \lambda m} t_*^{1+\lambda m/2} \tag{4.24}$$

$$a = A t_*^{n+\lambda m/2} \tag{4.25}$$

$$R = -\{1 - \frac{1}{2}(D-1)(D-2)\lambda^2\}(\frac{D-2}{D-1}) t_*^{-(2+\lambda m)} \tag{4.26}$$

Remembering that λ is a positive constant, as $t_* \to \infty$

$$t \to +\infty \ , \ a \to \infty \ , \ R \to 0. \tag{4.27}$$

The limit $t_* \to 0$

Using eqs we get

$$t = \frac{2}{2-\lambda m} t_*^{1-\lambda m/2} \quad \text{for } \lambda m \neq 2 \tag{4.28}$$

$$t = \ln t_* \qquad \text{for } \lambda m = 2 \tag{4.29}$$

$$a = A t_*^{n-\lambda m/2} \tag{4.30}$$

$$R = -\{1 - \frac{1}{2}(D-1)(D-2)\lambda^2\}(\frac{D-2}{D-1}) t_*^{-(2-\lambda m)} \tag{4.31}$$

So, for $\lambda m \geq 2$ which in turn implies $\lambda \geq \sqrt{\frac{2(D-1)}{(D-2)}}$, as $t_* \to 0$ we get

$$t \to -\infty \ , \ a \to \infty \ , \ R \to 0 \quad \text{or constant (for } \lambda m = 2) \quad . \tag{4.32}$$

As $t_* \to 0$ we can now continue the physical time indefinitely into the past. At this limit as in the $t_* \to \infty$ limit the physical scale factor becomes infinite . Keeping in mind the upper bound imposed on $\psi(\varphi)$ from (4.14) we can see that the scale factor never hits zero for any value of the physical time. ψ_φ and $\psi_{\varphi\varphi}$are also finite. The curvature scalar in the P frame ,thus also remains finite for all t. This result is interesting and should be studied further.

4.2 The Schwarzschild solution

Here we go straight to the equations of motion in the E frame and solve them. The Ricci scalar is then calculated for both the E and P frames and we see the novel feature of the theory.

- **The equations and their solutions in the E frame**

  Referring to the Appendix eqns(6.12), (6.13) and (6.14) and also invoking (3.31) and (3.32) we can arrive at the required field equations.

$$\frac{2f''(r)}{f'(r)} - \frac{f'(r)}{f(r)} + \frac{g'(r)}{g(r)} + 2(D-2)\frac{h'(r)}{h(r)} = 0 \tag{4.33}$$

$$\begin{aligned}
\frac{f'^2(r)}{f^2(r)} - 2\frac{f''(r)}{f(r)} - \frac{f'(r)g'(r)}{f(r)g(r)} & \\
-2(D-2)\frac{g'(r)h'(r)}{h(r)g(r)} - 4\frac{h''(r)}{h(r)} &= 2\varphi'^2 \qquad (4.34)\\
(D-3)(1-g(r)h'^2(r)) - g(r)h(r)h''(r) & \\
-\frac{1}{2}g'(r)h'(r) - \frac{g(r)h(r)h'(r)f'(r)}{2f(r)} &= 0 \qquad (4.35)\\
\varphi'' + \frac{1}{2}\{\frac{f'(r)}{f(r)} + \frac{g'(r)}{g(r)} + 2(D-2)\frac{h'(r)}{h(r)}\}\varphi' &= 0 \qquad (4.36)
\end{aligned}$$

One can simplify (4.34) by using (4.33). This gives

$$-(D-2)\{\frac{h'(r)}{h(r)}(\frac{g'(r)}{g(r)} - \frac{f'(r)}{f(r)}) + 2\frac{h''(r)}{h(r)}\} = \varphi'^2 \qquad (4.37)$$

Consider the ansatz for a solution

$$\begin{aligned}
f(r) &= Z^a \qquad (4.38)\\
g(r) &= Z^b \qquad (4.39)\\
h^2(r) &= r^2 Z^q \qquad (4.40)\\
e^{\varphi-\varphi_0} &= Z^p \qquad (4.41)
\end{aligned}$$

where

$$Z = 1 - (\frac{r_0}{r})^{D-3} \qquad (4.42)$$

Solving then we get

$$a = 1-(D-3)q \qquad (4.43)$$

$$b = 1-q \qquad (4.44)$$

$$\frac{2p^2}{D-2} + \{(D-3)q-2\}q = 0; \qquad (4.45)$$

As before it suffices to solve any three of the four differential equations. The remaining one is dependent through the Bianchi identities. The relation between p and q obtained from that equation acts as the constraining equation of the system. Note that if $p = 0$ then

$$\varphi = \text{const}\ ,\ q = 0\ ,\ a = b = 1$$

which is the Schwarzschild solution in D dimensional general relativity.

From eq(4.45) the bounds on q and hence on b are

$$0 \leq \quad q \quad \leq \frac{2}{D-3} \tag{4.46}$$

$$1 - \frac{2}{D-3} \leq \quad b \quad \leq 1 \tag{4.47}$$

$$-1 \leq \quad a \quad \leq 1 \tag{4.48}$$

- **The Ricci Scalar**

For the case of the Schwarzschild solution in the scalar-tensor theory in the E frame all the Ricci tensor components except R_{*rr} are zero. The Ricci scalar is given by

$$R_* \;=\; g_*^{\mu\nu} R_{*\mu\nu} = g_*^{rr} R_{*rr} \tag{4.49}$$

$$\;=\; g(r)\frac{1}{2}\varphi'^2 \tag{4.50}$$

Putting in values from (4.39) and (4.41) we get

$$R_* = \frac{1}{2}p^2 Z'^2 Z^{b-2} \tag{4.51}$$

This diverges at $r \to 0$ and $r \to r_0$. So for the scalar tensor theories of gravitation we get an extra essential singularity for the Schwarzchild metric , which is absent for the standard Schwarzschild solution.

- **Imposing the constraints**

We concentrate on the singularity at $r = r_{00}$. Referring to (4.41) we see that :

At $r = r_0$

$$p > 0 \Rightarrow Z^p \to 0 \Rightarrow \varphi \to -\infty \tag{4.52}$$

$$p < 0 \Rightarrow Z^p \to \infty \Rightarrow \varphi \to +\infty \tag{4.53}$$

The Ricci scalar in the P frame invoking eq(4.15) and (4.51) is

$$R = e^{\psi(\varphi)}\frac{1}{2}p^2 Z'^2 Z^{b-2}\{1 + 2(D-1)\psi_{\varphi\varphi} - \frac{1}{2}(D-1)(D-2)\psi_\varphi^2\} \tag{4.54}$$

The singular behaviour of the scalar is due to the piece $e^{\psi(\varphi)}Z^{b-2}$. So in order that there is no singularity we must have

$$\lim_{|\varphi|\to\infty} e^{\psi(\varphi)}Z^{b-2} = finite \tag{4.55}$$

and ψ_φ , $\psi_{\varphi\varphi}$ must also be finite.

Now let us impose the constraints set in the previous section. (The first constraint in this particular case could have been limited to $n = 2$. However their necessisity is obvious if one considers higher curvature invariants like $(nabla_\mu R)(nabla^\mu R)$, $[nabla_\mu(nablaR)^2][nabla^\mu(nablaR)^2]$ etc. and require that they are not divergent at $r = r_0$.)

At $r \to r_0$

$$e^{|\varphi|} = Z^{-|p|} \tag{4.56}$$

$$\Rightarrow Z^{b-2} = e^{-\frac{|\varphi|}{|p|}(b-2)} \tag{4.57}$$

Now constraint (2) implies

$$\lim_{|\varphi|\to\infty} e^{\psi(\varphi)}Z^{b-2} = e^{-|\varphi|(\lambda-\frac{2-b}{|p|})} \tag{4.58}$$

This is finite (zero or constant) for

$$\lambda \geq \frac{2-b}{|p|} \tag{4.59}$$

$$\Rightarrow \lambda \geq \frac{1+q}{\sqrt{(\frac{D-2}{2})q\{2-(D-3)q\}}} \tag{4.60}$$

The right hand side of λ (4.60) minimizes for $q = 1/(D-2)$. Putting this minimum value in we get

$$\lambda \geq \sqrt{\frac{2(D-1)}{(D-2)}} \tag{4.61}$$

This is the same as the limit on λ obtained from the cosmological case.

4.3 Another interesting result

As we have seen the constraints impose a finite upper bound ψ_{max} on $\psi(\varphi)$. So for at least one finite value of φ, say φ_c, where $\psi(\varphi)$ is maximum and $\frac{d\psi}{d\varphi} = 0$. We have seen in section(3.2.3) how the various quantities are related to the original form of the generalized Brans-Dicke theory. We had seen the Brans-Dicke function is given by

$$\omega_{BD}(\chi) = \frac{2}{(D-2)^2\psi_\varphi^2}\{1 - \frac{(D-1)(D-2)}{2}\psi_\varphi^2\} \tag{4.62}$$

In the limit $\varphi \to \varphi_c$ we have

$$\psi_\varphi(\varphi) \to 0 \quad \text{and} \quad \omega_{BD} \to \frac{2}{(D-2)^2\psi_\varphi^2} \to \infty \tag{4.63}$$

In this limit the general theory of relativity emerges as the theory of gravitation in the present epoch. For the case $D = 4$ i.e. for our observed universe, recent experiments set the lower bound on the Brans-Dicke function at $\omega > 3.6 \times 10^3$ (Eubanks, 1999). So eq.(4.63) is a desireable feature.

This interesting feature arises naturally from the constraints imposed and is an added bonus. But issue of how the present day corresponds to the limit $\varphi \to \varphi_c$ requires further study.

Chapter 5

SUMMARY AND CONCLUSIONS

In this thesis we have looked at the Scalar-Tensor theory of gravity in general dimensions. The general theory of relativity which is the standard theory of gravity was first examined. We developed the Einstein equation from the invariant D dimensional Einstein Hilbert action. Cosmology and the static spherically symmetric Schwarzchild solutions were dealt with in detail.

Next we turned our attention to Scalar tensor theories and discussed the prototype Brans-Dicke theory and its generalisation. Conformal transformations were dealt with in brief and we spoke of the Physical and Einstein conformal frames and looked at the geodesic equations in them. Then the specific form of the action we work with was given along with the relations of the our parameters with the standard case illustrated with examples. The equations of motion in both the P and E frames were derived .

The next chapter contained the main ingredients of the thesis. We first as in the chapter on general relativity looked at the specific cases of cosmology and the Schwarchild solution.In the case of cosmology the Big bang singularity persisted in the theory . We imposed two constraints on the function $\psi(\varphi)$ and investigated the consequence. We saw that with these the scale factor does not vanish for any value of the physical time which in itself can be extended to $-\infty$. The Ricci scalar also remained finite for all time.

In the Schwarzchild solution the Ricci scalar was found to have an extra singularity at the Schwarzchild radius ($r = r_0$) as compared with the GR case. The constraints imposed got rid of that extra singularity as well and the lower limits imposed on λ in both cases were found to be the same.

The fact that the Ricci scalar remains finite and the cosmic time can be extended without bound is indeed a remarkable result and the indications are that the Big bang singularity in the limited context of the theory discussed may be absent. But, that requires all curvature invariants to be finite. We have not taken up this issue in the thesis. It is also essential to note that the cases considered were vacuum solutions. Matter in the form of dust and radiation needs to be incorporated to make the analysis realistic. We are hoping to work on these aspects in the very near future. Another issue that needs to be resolved is the point discussed at the end of the fourth chapter. We saw that the high value of the BD function ($\omega(\chi)$) in D=4 as predicted by current experiments arises naturally out of our theory. But why the value of the scalar field in the current epoch should be such that the function $\psi(\varphi)$ maximises is not resolved and requires more analysis.

Chapter 6

APPENDIX

Throughout the thesis various steps have been omitted in view of making reading easier. In this chapter the calculations of those skipped portions are included.

6.1 Chapter 2

6.1.1 Action Formulation

- The explicit forms of $\Gamma^{\lambda}_{\mu\nu}$, $R^{\lambda}_{\mu\nu\kappa}$ and $R_{\mu\nu}$

$$\begin{aligned} \Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2} g^{\lambda\sigma}(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) \\ R^{\lambda}_{\mu\nu\kappa} &= \partial_\nu \Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} \\ R_{\mu\nu} &= \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} \end{aligned}$$

- The formulae useful for performing variation

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} \tag{6.1}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu} \tag{6.2}$$

- The explicit form of $T_{\mu\nu}$ from the matter action:

$$\delta S_m = \delta \int d^d x \sqrt{-g}\ \mathcal{L}_m$$

$$
\begin{aligned}
&= \int d^d x \left(\frac{\partial\sqrt{-g}\mathcal{L}_m}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\lambda}\frac{\partial\sqrt{-g}\mathcal{L}_m}{\partial(\partial g^{\mu\nu}/\partial x^\lambda)}\right)\delta g^{\mu\nu} \\
&= \int d^d x\, \frac{1}{2}(\sqrt{-g}\, T_{\mu\nu})\, \delta g^{\mu\nu}
\end{aligned}
$$

So we get

$$
T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{6.3}
$$

6.1.2 Cosmology

- The Derivation of $\Gamma^\lambda_{\mu\nu}$ and $R_{\mu\nu}$ for FRW

We have seen that the metric can be chosen of the form

$$
\begin{aligned}
g_{tt} &= -1 \\
g_{it} &= 0 \\
g_{ij} &= a^2(t)\tilde{g}_{ij}
\end{aligned} \tag{6.4}
$$

Here t is a cosmic time coordinate; i and j run over the $D-1$ space coordinates and $\tilde{g}_{ij}$ is the metric for a $D-1$ dimensional maximally symmetric space.

The non vanishing terms of the Affine connection are

$$
\begin{aligned}
\Gamma^t_{ij} &= a\dot{a}\tilde{g}_{ij} \\
\Gamma^i_{tj} &= \frac{\dot{a}}{a}\delta_{ij} \\
\Gamma^k_{ij} &= \tilde{\Gamma}^k_{ij}
\end{aligned} \tag{6.5}
$$

The corresponding components of the Ricci tensor then are

$$
\begin{aligned}
R_{tt} &= -(D-1)\frac{\ddot{a}}{a} \qquad (6.6) \\
R_{ti} &= 0 \\
R_{ij} &= \tilde{R}_{ij} + (a\ddot{a} + (D-2)\dot{a}^2)\tilde{g}_{ij}
\end{aligned}
$$

Now, $\tilde{R}_{ij}$ is the Ricci tensor for the $(D-1)$ dimensional maximally symmetric space (Weinberg, 1972). For such spaces the curvature tensors take simplified forms and can be written as

$$\begin{aligned}
R_{\lambda\mu\sigma\nu} &= k(g_{\lambda\sigma}g_{\mu\nu} \; - \; g_{\mu\sigma}g_{\lambda\nu}) \\
R_{\mu\nu} &= g^{\lambda\sigma}R_{\lambda\mu\sigma\nu} \\
&= k(D-1)g_{\mu\nu}
\end{aligned}$$

So in our case

$$\begin{aligned}
\tilde{R}_{ij} &= k(D-1)\tilde{g}_{ij} \\
R_{ij} &= [k(D-1) \; + \; (a\ddot{a} + (D-2)\dot{a}^2)]\tilde{g}_{ij} \qquad (6.7)
\end{aligned}$$

The Ricci scalar takes the following form

$$\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} = g^{tt}R_{tt} + g^{ij}R_{ij} \\
&= (D-1)[2\frac{\ddot{a}}{a} + (D-2)\frac{\dot{a}}{a}^2 + (D-2)\frac{k}{a}^2] \qquad (6.8)
\end{aligned}$$

With the choice of $k=0$, the above equations reduce to

$$\begin{aligned}
R_{tt} &= -(D-1)\frac{\ddot{a}}{a} \qquad (6.9) \\
R_{ij} &= [(a\ddot{a} + (D-2)\dot{a}^2)]\tilde{g}_{ij} \qquad (6.10) \\
R &= (D-1)[2\frac{\ddot{a}}{a} + (D-2)\frac{\dot{a}^2}{a^2}] \qquad (6.11)
\end{aligned}$$

- Derivation of equation of state

  Let us consider in $(D-1)$ spatial dimensions, a container enclosing a perfect gas such that one of its surfaces is perpendicular to one of the axes say X_1. Velocity of a gas molecule in space is given by

  $$v^2 = v_{x_1}^2 + v_{x_2}^2 + ... + v_{x_{D-1}}^2.$$

  where v_{x_i} are the components in the respective directions. For the sake of simplifying the symbols,- let us call $v_{x_i} = u$ The change in u is brought about by reflections on the surface perpendicular to X_1. Change of momentum after reflection for a molecule of mass m is $2mu$. If there are n_u such molecules striking unit area of the surface per unit time, - the total change of momentum per unit area per unit time i.e. the average pressure is

$$p = 2mun_u$$

Now the velocities in this particular direction need to be summed over. So if n_{u_i} particles move with velocities u_i where we only consider particles moving in positive X_1 axis, then the pressure is given by

$$\begin{aligned} p &= 2m \sum n_{u_i} {u_i}^2 \\ &= 2m \times \frac{1}{2} n\bar{u^2} \end{aligned}$$

where $\bar{u^2}$ is the mean square velocity in the X_1 direction is given by

$$\bar{u^2} = \frac{\sum n_{u_i} {u_i}^2}{\sum n_{u_i}} = \frac{\sum n_{u_i} {u_i}^2}{n/2}$$

Note that the factor of $\frac{1}{2}$ comes in because we consider only the molecules moving along the positive X_1 axis. So the pressure along this axis is given by

$$p_{x_1} = mn\bar{u^2}$$

which will be the same as the pressure along all other axis and hence the pressure in general is given by

$$\begin{aligned} p &= mn\bar{u^2} \\ &= \frac{1}{(D-1)} mn\bar{v^2} \end{aligned}$$

where $\bar{v_{x_1}}^2 = \bar{v_{x_2}}^2 = ... = \bar{v_{x_{D-1}}}^2 = \frac{1}{(D-1)}\bar{v^2}$ with $\bar{v^2}$ the mean square of the total velocity. So,

$$\begin{aligned} p &= \frac{1}{(D-1)} mn\bar{v^2} \\ \Rightarrow p &= \frac{1}{(D-1)} \rho_m \bar{v^2} \end{aligned}$$

where $\rho_m = mn$ is the mass density of the gas.

Now velocity of light,c=1. So for non relativistic particles or *dust*, $\bar{v^2} \ll 1$ or $\bar{v^2} \sim 0$. So,

$$p_{\text{dust}} = 0$$

For relativistic particles or *radiation*, the energy density (ρ) is the mass density in the Natural units (i.e here with c=1). And $\bar{v^2} = 1$.

So,

$$p_{\text{rad}} = \frac{1}{(D-1)}\rho$$

6.1.3 Schwarzchild Solution

- Form of the D dimensional line and volume elements

$$\begin{aligned}
ds_n^2 &= dr^2 + r^2 d\Omega_{D-2}^2 \\
d\Omega_{D-2}^2 &= d\theta_1{}^2 + \sin^2\theta_1 d\theta_2{}^2 + ... \\
&+ \sin^2\theta_1 \sin^2\theta_2 .. \sin^2\theta_{n-3} d\theta_{n-2} \\
&+ \sin^2\theta_1 .. \sin^2\theta_{n-2} d\phi^2 \\
dV_n &= r^{n-1} dr \sin\theta_1 \sin^2\theta_2 ... \sin^{n-2}\theta_{n-2} d\theta_1 d\theta_2 d\theta_{n-2} d\phi
\end{aligned}$$

- Non-zero Connection terms

$$\begin{aligned}
\Gamma^t_{rt} &= \frac{f'(r)}{2f(r)} \\
\Gamma^r_{tt} &= \frac{1}{2} g(r) f'(r) \\
\Gamma^r_{rr} &= \frac{g'(r)}{2g(r)} \\
\Gamma^r_{\theta\theta} &= -g(r)h(r)h'(r) \\
\Gamma^r_{\phi\phi} &= -g(r)h(r)h'(r)\sin^2\theta \\
\Gamma^\theta_{r\theta} &= \frac{h'(r)}{h(r)} \\
\Gamma^\phi_{r\phi} &= \frac{h'(r)}{h(r)} \\
\Gamma^\theta_{\phi\phi} &= -\cos\theta\sin\theta \\
\Gamma^\phi_{\theta\phi} &= \cot\theta
\end{aligned}$$

The other non zero terms are in the ones which come from the symmetric solid angle part and will not be needed in calculation as they would be depend on $\Gamma^r_{\theta\theta}$ or some term of that type and would be taken care of by the inverse metric.

- The Independent Components of the Ricci Tensor

$$\begin{aligned} R_{tt} &= \frac{1}{4}[2g(r)f''(r) - \frac{g(r)f'^2(r)}{f(r)} \\ &+ f'(r)(g'(r) + 2(D-2)g(r)\frac{h'(r)}{h(r)})] \qquad (6.12) \\ R_{rr} &= \frac{f'^2(r)}{4f^2(r)} - \frac{f'(r)g'(r)}{4f(r)g(r)} - \frac{f''(r)}{2f(r)} \\ &- (D-2)[\frac{h'(r)g'(r)}{2h(r)g(r)} - \frac{h''(r)}{h(r)}] \qquad (6.13) \\ R_{\theta\theta} &= (D-3)(1 - g(r)h'^2(r)) - g(r)h(r)h''(r) \\ &- \frac{1}{2}g'(r)h'(r) - \frac{g(r)h(r)h'(r)f'(r)}{2f(r)} \qquad (6.14) \end{aligned}$$

The rest are related to $R_{\theta\theta}$. Below are the relations

$$\begin{aligned} R_{\theta_1\theta_1} &= \sin^2\theta R_{\theta\theta} \\ R_{\theta_2\theta_2} &= \sin^2\theta_1 R_{\theta_1\theta_1} \\ \dots\dots\dots & \quad \dots\dots\dots\dots\dots\dots \qquad (6.15) \\ R_{\theta_{D-4}\theta_{D-4}} &= \sin^2\theta_{D-5} R_{\theta_{D-5}\theta_{D-5}} \\ R_{\phi\phi} &= \sin^2\theta_{D-4} R_{\theta_{D-4}\theta_{D-4}} \end{aligned}$$

6.2 Chapter 3

- **Equations of motion in the P frame**

The action

$$\begin{aligned} S_P = & -\frac{1}{16\pi}\int d^dx\sqrt{-g}e^{(\frac{D-2}{2})\psi} \\ & [R - \frac{1}{2}\{1 - \frac{(D-1)(D-2)}{2}\psi_\phi^2\}(\nabla\phi)^2] \end{aligned}$$

Equation of motion for $g_{\mu\nu}$

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \quad & (\frac{D-2}{2})\psi_\phi(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) \\
+ \quad & [(\frac{D-2}{2})(\psi_{\phi\phi} - \frac{D-2}{2}\psi_\phi^2) + \frac{1}{2}\{1 - \frac{(D-1)(D-2)}{2}\psi_\phi^2\}]\nabla_\mu\phi\nabla \\
- \quad & [(\frac{D-2}{2})(\psi_{\phi\phi} - \frac{D-2}{2}\psi_\phi^2) + \frac{1}{4}\{1 - \frac{(D-1)(D-2)}{2}\psi_\phi^2\}]g_{\mu\nu}(\nabla
\end{aligned}$$

Taking the trace we get

$$\begin{aligned}
R = \quad & (D-1)\psi_\phi\nabla^2\phi \\
+ \quad & [(D-1)(\psi_{\phi\phi} - \frac{D-2}{2}\psi_\phi^2) + \frac{1}{2}\{1 - \frac{(D-1)(D-2)}{2}\psi_\phi^2\}](\nabla\phi)^2
\end{aligned}$$

Equation of motion for ϕ

$$\begin{aligned}
& \{1 - \frac{(D-1)(D-2)}{2}\psi_\phi^2\}\nabla^2\phi + (\frac{D-2}{2})\psi_\phi R \\
+ \quad & \frac{1}{2}[\{1 - \frac{(D-1)(D-2)}{2}\psi_\phi^2\}(\frac{D-2}{2})\psi_\phi - (D-1)(D-2)\psi_\phi\psi_{\phi\phi}](\nabla\phi)^2 = 0
\end{aligned}$$

Using the trace formula, the equations of motion simplify somewhat and become of the form (3.27) and (3.28).

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