PARAMETRIZED FIELD THEORIES AND LOOP QUANTIZATION

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Alok Laddha

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Abstract

In this thesis we show how two dimensional Parametrized field theories constitute "perfect" toy models for Loop Quantum Gravity. We quantize two dimensional massless scalar field theories on a Minkowskian cylinder and on a Minkowskian plane, and show how various aspects of Loop quantization e.g. construction of quantum observables, determination of physical Hilbert space and emergence of discrete spacetime can be explicitly illustrated within these models. We also demonstrate how loop quantized parametrized field theories are quantum theories capturing non-perturbative aspects of two dimensional quantum Black-Holes.
The thesis is essentially divided into two parts. In the first part we present a polymer quantization of a parametrized scalar field theory on 2 dimensional flat cylinder. Both the matter fields as well as the embedding variables are quantized in LQG type 'polymer' representations. The quantum constraints are solved via group averaging techniques and, analogous to the case of spatial geometry in LQG, the smooth (flat) spacetime geometry is replaced by a discrete quantum structure. An overcomplete set of Dirac observables, consisting of (a) (exponentials of) the standard free scalar field creation- annihilation modes and (b) canonical transformations corresponding to conformal isometries, are represented as operators on the physical Hilbert space. None of these constructions suffer from any of the 'triangulation' dependent choices which arise in treatments of LQG. In contrast to the standard Fock quantization, the non- Fock nature of the representation ensures that the group of conformal isometries as well as that of the gauge transformations generated by the constraints are represented in an anomaly free manner. Semiclassical states can be analysed at the gauge invariant level. It is shown that 'physical weaves' necessarily underly such states and that such states display semiclassicality with respect to, at most, a countable subset of the (uncountably large) set of observables of type (a).
In the second part we present a polymer(loop) quantization of a two dimensional theory of dilatonic gravity known as the CGHS model. We recast the theory as a parametrized free field theory on a flat 2-dimensional spacetime and quantize the resulting phase space using techniques of loop quantization. The resulting (kinematical) Hilbert space admits a unitary representation of the spacetime diffeomorphism group. We obtain the complete spectrum of the theory using a technique

known as group averaging and perform quantization of Dirac observables on the resulting Hilbert space. Finally we argue that the algebra of Dirac observables get deformed in the quantum theory. We than tackle the problem of time. Combining the ideas from parametrized field theory with certain relational observables, evolution is defined in the quantum theory in the Heisenberg picture. Finally the dilaton field is quantized on the physical Hilbert space which carries information about quantum geometry.

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Introduction

The program of canonical quantization of gravity underwent a revolution in the early 80's When Ashtekar re-expressed the fundamental geometrodynamical variable $(g_{ab}, \tilde{p}^{ab})$ in terms of a complex $Sl(2, \mathbf{C})$ connection, $A^{i}_{C,a}$ and a canonically conjugate $Sl(2, \mathbf{C})$ densitized triad $\tilde{E}^{C,b}_{j}$ [7]. The constraints of the canonical gravity took on enormously simple form as certain lower order polynomials in the connection and the triad. This led to a renewed hope that the constraint quantization of canonical gravity when expressed in terms of the complexified Asthekar variables could be carried out successfully. The real advancement in the quantum theory came when Rovelli and Smolin quantized the algebra generated by Wilson loops (Path ordered exponential of connection integrated around a loop) and triads on certain square-integrable functions of loops [41]. Even formal solutions to the Hamiltonian constraint were computed in [13].

However the program of canonical quantization using complex Ashtekar variables suffered a major setback on account of what are known as reality conditions. In order to recover real theory from the complexified phase space one has to impose certain constraints known as reality conditions on the canonical data. These constraints are second class and non-polynomial in basic variables. Whence it seemed like one was replacing the non-polynomiality of constraints in geometrodynamics by equally complicated reality conditions in connection dynamics.

One way to sidestep the issue of reality conditions was suggested by Barbero who introduced real SU(2) valued connection and its conjugate triad as canonical variables on the real section of the complexified Ashtekar phase space [9]. Although the (density one) Hamiltonian constraint when expressed in terms of these real variables was once again non-polynomial and as complicated as the geometrodynamical Hamiltonian constraint, one could now use the ideas from Gauge theories to quantize gravity.

The key idea underlying Loop Quantum Gravity (LQG) ([54], [5]) is to quantize the Poisson *-algebra generated by holonomies of real Ashtekar-Barbero connections along paths and surface integrals of real densitized triads called fluxes. As the definitions of these "elementary variables" do not require any fixed spatial geometry, spatial diffeomorphisms act as outer automorphisms on this algebra. Also note that, given any functional on the phase space it can be approximated up to arbitrary precision by holonomies around infinitesimally small loops and fluxes through infinitesimally small surfaces.

Quantization of the resulting Lie-algebra, known as ACZ (Ashtekar-Corichi-Zapata) algebra proceeds via GNS construction where the GNS state ω is a spatial diffeomorphism invariant state. One of the remarkable facts about this state is it's uniqueness under certain technical but physically well-motivated assumptions [36]. The states of the resulting Hilbert space $\mathcal{H}_\omega$ are labelled by graphs whose transformation under a diffeomorphism results in unitary implementation of the spatial diffeomorphisms on the Hilbert space. The flux fields are represented as unbounded self-adjoint operators on $\mathcal{H}_\omega$ and have a pure point spectrum. As a result of which the spectra of geometric operators like area, volume, and length are discrete. At least at the kinematical level (i.e. before implementing the constraints) this is a concrete realization of the idea of Quantum geometry [4].

Thus the kinematical structure of LQG is tight and well under control, however it is at the level of dynamics and representation of observables that need for radical ideas have emerged.

• As the unitary operators representing finite spatial diffeomorphisms are not weakly continuous due to the non-separability of the Hilbert space, the generators of infinitesimal diffeomorphisms, which would correspond to the quantization of the diffeomorphism constraint is not well defined on $\mathcal{H}_\omega$. Whence although there is a proposed definition of quantized Hamiltonian constraint $\hat{H}[N]$ on $\mathcal{H}_\omega$, the full Dirac algebra of constraints cannot be realized on $\mathcal{H}_\omega$.[1] Thus it is not clear whether LQG is generally-covariant in the sense of admitting a representation of

[1] It certainly cannot be realized on the space of diffeomorphism invariant states $\mathcal{H}_{diff}$ as Hamiltonian constraint is not well defined on it.

the algebra of constraints.

• Although $\hat{H}[N]$ is not well defined on diffeomorphism-invariant states (i.e. it maps a state in $\mathcal{H}_{diff}$ to a distribution not belonging to $\mathcal{H}_{diff}$), the commutator $[\hat{H}[N], \hat{H}[M]]$ can be shown to vanish on $\mathcal{H}_{diff}$. However checking whether the classical limit of $\hat{H}[N]$ is the classical Hamiltonian constraint has remained out of reach. (See [54] for details.)

In order to cure the above problems, a Master constraint program [17] which replaces the Dirac algebra by a true Lie algebra consisting of certain quadratic combination of constraints has been proposed. However unlike the Dirac algebra which has a clear spacetime interpretation as the algebra of hypersurface deformations [28], the Master constraint algebra admits no such interpretation. We emphasize that one of the major obstacle in realizing Dirac algebra (or the associated group thereof) on $\mathcal{H}_{\omega}$ is its non-Lie algebraic nature.

• Now we come to the issue of defining observables. It is a generic problem in canonical gravity that (in spatially compact case) there aren't any observables in the theory. However even at the kinematical level, trying to promote a generic functional on phasespace to operator on $\mathcal{H}_{\omega}$ is ambiguous due to the following reason. In LQG only certain non- local functionals of the connection, namely the holonomies around spatial loops, can be promoted to quantum operators rather than the connection itself. [2] As a result, all questions of interest (including that of the quantum dynamics defined by the Hamiltonian constraint which is a *local* function of the connection and triad,) need to be phrased in terms of holonomy operators. Since holonomy operators associated with close by loops have actions unrelated by any sort of continuity, this leads to a situation where a *choice* of a subset of the uncountable set of all holonomy operators (or equivalently, the spatial loops labelling them) becomes necessary. We shall loosely refer to such choices as "triangulation" choices since, often, the family of loops is chosen to lie on some set of triangulations of the spatial manifold. Since there seems to be no natural choice independent of the intuition of the researcher, this leads to proposals which may be seen as radical or ad- hoc depending on ones taste.

[2] The reason for this is the lack of regularity in the action of the holonomy operators: while, classically, the connection at a point can be obtained from the holonomy of a loop containing the point in the limit that the loop is infinitesimally small, the limit of the corresponding operators does not exist in the LQG representation.

Even if the issues stated above were resolved, one would be led to the following speculative questions.
• Even though at the kinematical level, quantum geometry is discrete, is there any sense in which the physical states of LQG define a "discrete" spacetime, and if so, how does a smooth continuum structure emerge from it?
• Canonical quantum gravity is known to suffer from the well-known problem of time. Even if the solution to the Hamiltonian constraints (or the Master constraint) were found, how would one extract notion of evolution from it. Is the notion of "time" evolution discrete as observed in certain mini-superspace models or is it continuous.?
• One of the generic predictions of quantum (semi-classical) gravity is Hawking radiation emitted by Black Holes. Any theory of quantum gravity is expected to reproduce the thermal spectrum in the appropriate limit. Can one ask questions pertaining to Hawking radiation starting from the state space of LQG?

In this thesis we show that two-dimensional Parametrized field theories(PFTs) [30] offer 'perfect' toy models in which the issues related to the constraint algebra and the triangulation dependence of the observables can be resolved and questions stated above can be asked in a precise fashion.

Let us briefly outline the reasons as to why we call two-dimensional PFTs perfect 'toy' models.

1. In two-dimensional parametrized field theory, the Dirac algebra of constraints is a true-Lie algebra. Whence one can aspire to find a loop-type representation of an appropriate algebra which also admits the representation of the Dirac algebra or the associated Lie-group. This directly leads to a construction of physical Hilbert space which is so far unavailable in LQG.

2. Unlike gravity, it is straightforward to isolate the true degrees of freedom of a PFT from the pure gauge degrees of freedom. Whence one has a complete set of observables available at one's disposal. Thus one can try to quantize these observables on the physical Hilbert space, and see if they suffer from

the triangulation ambiguities mentioned above.

3. The problem of time shows up even in PFT once one has solved all the constraints. Thus one can try to define a notion of evolution using certain Relational observables and check if this evolution is discrete or continuous.

4. A state in the physical Hilbert space of PFT encodes information about the spacetime geometry. Thus the questions pertaining to discreteness of (spacetime) geometry can be posed precisely in this model.

5. Two-dimensional PFT on $\mathbf{R}^2$ is canonically equivalent to a two-dimensional dilatonic theory of gravity known as the CGHS model. This model, at its classical level admits 2-d Black holes as solutions. Also at semi-classical level, it is known that these black holes Hawking radiate. So if one quantizes the CGHS model (or the corresponding PFT) using LQG techniques, one potentially has a framework where non-perturbative issues like singularity resolution as well as semi-classical issues such as Hawking radiation can be analyzed in loop quantized field theories.

The outline of this thesis is as follows.
After giving a brief introduction to PFT in arbitrary dimensions and Loop quantization in chapter 2, In chapter 3 we study massless parametrized scalar field theory on $S^1 \times \mathbf{R}$. Points 1,2 and 4 mentioned above are addressed in this chapter.
In chapter 4, we polymer quantize the CGHS model re-casted as a PFT on a Minkowskian plane. To draw contrast from the previous chapter, where the Dirac observables are quantized in a triangulation independent manner, we perform a more "traditional" (triangulation-dependent) quantization of Dirac observables in chapter 4. However we show that, contrary to the naive expectation, such operators can still be well defined on physical Hilbert space. In chapter 5, we define the notion of evolution of certain relational observables on physical Hilbert space and address the point 3. Finally we define an operator corresponding to the dilaton field and complete the framework in which the question raised in point 5 above can be asked. We end the thesis with discussion of certain open issues and conclusions.

2

Introduction to Parametrized field theory and Polymer quantization

2.1 introduction

In this chapter we review classical parametrized scalar field theory(PFT) on $d+1$ dimensional flat space-time [26]. We will first define the Lagrangian for PFT and than pass onto the Hamiltonian framework. After introducing the constraint surface and the reduced phase space, we will define a complete set of Dirac observables for the theory.

We then review basics of Loop Quantum Gravity(LQG) and loop quantization of (non-gravitational) field theories. After reviewing the kinematical structure of LQG, we extract out certain bare essentials which underlie loop quantization of any field theory.

2.2 Basics of PFT

2.2.1 Lagrangian formulation

Consider a free massive scalar field theory on $(d+1)$-dimensional Minkowski space-time $(\mathcal{M}, \eta)$. The action is given by,

$$S_0[\varphi] \;=\; \frac{1}{2}\int d^{d+1}X[\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2] \tag{2.1}$$

where $(X^0, ..., X^d)$ are inertial co-ordinates on $\mathcal{M}$. Clearly, the space of solutions of this theory does not admit an action of the group of diffeomorphisms of $\mathcal{M}$, Diff($\mathcal{M}$). i.e. if φ_0 is a solution to,

$$(\partial_\mu \partial^\mu - m^2)\varphi = 0 \tag{2.2}$$

then for an arbitrary diffeomorphism $f \in Diff(\mathcal{M})$, $\varphi_0 \circ f$ is not a solution to (2.2), unless f is an isometry of η. The basic idea behind PFT is to enlarge the space of solutions of the given field theory, so that the enlarged solution space admits an action of Diff($\mathcal{M}$). This can be achieved as follows.
Parameterize X^μ by arbitrary co-ordinates x^α on $\mathcal{M}$. In terms of these arbitrary co-ordinates, the action $S_0[\varphi]$ becomes,

$$S_0[\varphi] \;=\; \frac{1}{2}\int d^dx\sqrt{g(X(x))}[g^{\alpha\beta}(X(x))\partial_\alpha(\varphi\circ X)(x)\partial_\beta(\varphi\circ X)(x) \;-\; m^2(\varphi\circ X)^2(x)] \tag{2.3}$$

where $g_{\alpha\beta}(X(x)) = \frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial X^\nu}{x^\beta}\eta_{\mu\nu}$.
The action for PFT is obtained by considering the inertial co-ordinates X^μ as dynamical variables (along with the scalar field $\psi = \varphi \circ X$).

$$S_{PFT}[\psi, X^\alpha] \;=\; \frac{1}{2}d^dx\sqrt{g(X(x))}[g^{\alpha\beta}(X(x))\partial_\alpha\psi(x)\partial_\beta\psi(x) \;-\; m^2\psi^2(x)] \tag{2.4}$$

This action is clearly invariant under arbitrary change of the co-ordinates x^α. More precisely, we can show that $\forall f \in Diff(\mathcal{M})$,

$$S_{PFT}[f^*\psi, X^\mu \circ f] \;=\; S_{PFT}[\psi, X^\mu] \tag{2.5}$$

(This is simply because the original action is unchanged by a diffeomorphism acting on both the metric and the scalar field). Whence the solution space of PFT will admit an action of Diff($\mathcal{M}$).
The equation of motion for ψ is simply the KG equation in arbitrary co-ordinates.

$$\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta\psi) - m^2\psi = 0 \tag{2.6}$$

The equations of motion obtained by varying X^{μ} are given by,

$$\nabla_{\mu} T^{\mu\nu} = 0 \tag{2.7}$$

where $T^{\mu\nu} = -2g^{-\frac{1}{2}}\frac{\delta S_{PFT}}{\delta g_{\mu\nu}}$.
As is well known, (2.7) is automatically satisfied when ψ satisfies (2.6). This implies that the d+1 scalars X^{μ} are undetermined functions on $\mathcal{M}$. This d+1-functions worth of gauge is a consequence of diffeomorphism invariance of S_{PFT}.

2.2.2 Hamiltonian formulation

Set $x^0 = t$ and $\{x^{\alpha}\} = \{t, x^a; a = 1, ..., d\}$. In order to derive the canonical action for PFT, we restrict our attention to those $X^{\alpha}(t, x^a)$'s such that for a fixed t, X^{α} define a smooth space-like (w.r.t η) embedding of a 3-manifold Σ in $\mathcal{M}$.[1]
This means that for a fixed t, the functions $\{X^{\alpha}(x^a)\}$ are such that the symmetric form,

$$q_{ab} = \partial_a X^{\mu} \partial_b X^{\nu} \eta_{\mu\nu} \tag{2.8}$$

is a Riemann metric on Σ. Also let n^{μ} denote the unit time-like normal to $X(\Sigma)$. A d+1 decomposition of S_{PFT} w..r.t "time" t leads to the following canonical form of the action

$$S_{PFT}[\phi, X^{\mu}, \pi_{\phi}, \Pi_{\mu}; N^{\mu}] = \int dt \int d^d(x)[\pi_{\phi}\dot{\phi} + \Pi_{\mu}\dot{X}^{\mu} - N^{\mu}H_{\mu}] \tag{2.9}$$

where,

$$\begin{aligned} \phi(x) &= \varphi(X(x)) \\ \pi_{\phi}(x) &= \sqrt{q}n^{\mu}\partial_{\mu}\phi(x) \end{aligned} \tag{2.10}$$

and the quadruple of scalar densities $\Pi_{\mu}(x)$ on Σ are canonically conjugate to X^{μ}. N^{μ} are a quadruple of scalar fields on Σ and are Lagrange multipliers associated with the constraints H_{μ}.

$$H_{\mu} = \Pi_{\mu} + H_{\mu}^{\phi} \tag{2.11}$$

[1] In what follows, we do not assume any topological restrictions on Σ. Such specifications are necessary to rigorously derive the symplectic structure of PFT [26]. However, as we will be eventually dealing with much simpler two-dimensional theories, we will completely ignore the analytic and topological subtleties associated with higher dimensional PFTs in this chapter.

with

$$H^{\phi}_{\mu} := -\frac{1}{2}\sqrt{q}(\frac{\pi_{\phi}^{2}}{q} + q^{ab}\partial_{a}\phi\partial_{b}\phi + m^{2}\phi^{2})n_{\mu} + \pi_{\phi}\partial_{a}\phi X^{a}_{\mu} \tag{2.12}$$

where $X^{a}_{\mu} = q^{ab}X^{\nu}_{b}\eta_{\mu\nu}$. We now describe the resulting phase space of PFT in some detail.

2.2.3 The extended phase space

As is well known, the phase space Γ_{ϕ} of scalar field theory on $(\mathcal{M}, \eta)$ is a linear space co-ordinatized by the solutions φ to the Klein-Gordon equation. It can be equivalently characterized as the vector space of all the Cauchy data (ϕ, π_{ϕ}) defined in (2.10) on any Cauchy slice Σ through $\mathcal{M}$. The symplectic form on Γ_{ϕ} is given by,

$$\Omega_{\phi} = \int_{\Sigma} \delta\phi \wedge \delta\pi_{\phi} \tag{2.13}$$

The phase space of PFT can be obtained by adding all space-like embeddings X and their conjugate momenta Π to Γ_{ϕ} and by imposing suitable constraints which ensure that the embedding degrees of freedom are pure gauge [26].
More precisely, denoting the space of all space-like embeddings along with their conjugate momenta by $T^{*}\mathcal{E}$ [2] The phase-space of PFT is given by,

$$\Gamma = \Gamma_{\phi} \times T^{*}\mathcal{E} \tag{2.14}$$

The symplectic structure on Γ is defined as,

$$\Omega_{PFT} = \int_{\Sigma}(\delta X^{\mu} \wedge \delta\Pi_{\mu} + \delta\phi \wedge \delta\pi_{\phi}) \tag{2.15}$$

The constraints that must be imposed on Γ are given in (2.12). The constraint surface $\tilde{\Gamma}$ defined by $H_{\mu}(x) \approx 0\, \forall x \in \Sigma$ is co-ordinatized by $(\phi, \pi_{\phi}, X^{\mu}, -H^{\phi}_{\mu}(\phi, \pi_{\phi}, X))$. It can be shown that [26] under certain mild restrictions, $\tilde{\Gamma}$ is a sub-manifold of Γ. We now list several facts about $\tilde{\Gamma}$ without proving any of them. Details can be found in ([26], [23], [24], [27]).

- Let φ satisfy Klein-Gordon equation on $\mathcal{M}$ and let $\lambda \rightarrow X_{\lambda}$ be a curve in the

[2] There are good reasons for this notation. The space of all space-like embeddings $\mathcal{E}$ is an infinite dimensional manifold modeled on some Banach space. The conjugate momenta Π are sections of the cotangent bundle $T^{*}\mathcal{E}$. Details can be found in [26].

space of all embeddings. Then the initial data $(\phi_\lambda, \pi_{\phi,\lambda})$ induced by φ on X_λ satisfy

$$(\dot{\phi}, \dot{\pi_\phi}, \dot{X}) \;=\; X_{H[N]} \tag{2.16}$$

where dot denotes derivative w.r.t λ.

• Conversely if $(\dot{\phi}, \dot{\pi_\phi}, \dot{X})$ satisfies (2.16) then it defines a unique solution to the Klein-Gordon equation.

• The Hamiltonian vector fields $X_{H[N]}$ of the smeared constraints $H[N] = \int H_\mu(x)N^\mu(x)$ is tangential to $\tilde{\Gamma}$ $\forall$ N^μ.

• The constraints form a closed Poisson algebra,

$$\{H[N], H[M]\} \;=\; 0 \tag{2.17}$$

We now introduce the notion of reduced phase space. But before that, we will define the gauge orbits in the constraint surface. Consider the following automorphism $\rho_{XX'}$ on the linear space Γ_ϕ. Start with an initial data (ϕ, π_ϕ) on a Cauchy slice $X(\Sigma)$, determine the corresponding space-time solution φ, and determine the corresponding Cauchy data on $X'(\Sigma)$. One can use this automorphism to determine gauge orbits as follows.

Given a point (ϕ, π_ϕ, X) in the constraint surface, the gauge orbit $\gamma_{(\phi,\pi_\phi,X)}$ passing through that point, is given by

$$\gamma_{(\phi,\pi_\phi,X)} = \{(\rho_{X,X'}(\phi, \pi_\phi), X')|\forall X' \in \mathcal{E}\} \tag{2.18}$$

The tangent space at any point on $\gamma_{(\phi,\pi_\phi,X)}$ is spanned by the Hamiltonian vector fields of $H[N]$ $\forall N$, whence they are the gauge orbits. Note how the dynamical trajectory of a given Cauchy data along any foliation of $\mathcal{M}$(by space-like slices) is contained in γ. Thus each gauge orbit corresponds to a distinct solution to the Klein-Gordon equation. This is one of the primary reasons why PFTs are such good toy models for gravity for as in the case of gravity notion of gauge and evolution become intertwined.

These orbits define an obvious equivalence relation between different points of $\tilde{\Gamma}$. The space of equivalence classes define the reduced phase space of the theory.

2.2.4 Dirac observables

We can associate an observable o_φ with any solution to the Klein-Gordon equation as follows [26]. Let $(\zeta, \eta, X, \Pi) \in \Gamma$. Let (ϕ, π_ϕ) be the Cauchy data corresponding to φ on X, then

$$o_\varphi(\sigma, \pi_\sigma, X, \Pi) := \int_\Sigma (\phi\pi_\sigma - \sigma\pi_\phi) \;=\; (\varphi, \varsigma)_{KG} \tag{2.19}$$

where ς is the solution corresponding to the cauchy data (σ, π_σ) on X.
This is an observable as the Klein-Gordon inner product is invariant under deformations of the hyper-surface. Unlike an observable in ordinary gauge theories where there exists a true Hamiltonian, the Dirac observables of PFT do not "evolve". Whence in they are more commonly known as perennials. Note that the observables associated with $exp(ik \cdot X)|\ k = (k_0 = \sqrt{\vec{k}^2 + m^2}, \vec{k})$ are nothing but Fourier modes a_k evaluated at a given point in phase-space. The Dirac observables satisfy the following properties which are straightforward to prove.

• Given two observables o_φ, o_ς, $ao_\varphi + bo_\varsigma$ is an observable for any real numbers a, b.

• Poisson bracket between two observables in an observable, whence the set of all such observables form a closed Poisson algebra.

• Given two observables o_φ, o_ς corresponding to two distinct solutions φ, ς, there exists a solution ϱ such that $(\varrho, \varphi - \varsigma) \neq 0$, whence o_ϱ will take distinct values on the two gauge orbits which correspond to φ and ς. Thus the set of all observables separate the points of reduced phase space.

2.3 Basics of Polymer quantization

In the remaining part of this chapter we review basics of Loop Quantum Gravity(LQG) and loop quantization of (non-gravitational) field theories, and extract out certain bare essentials which underlie loop quantization of any field theory.

2.3.1 Kinematic structure of LQG

We start with classical canonical theory of gravity formulated in terms of smooth connections A on a principal SU(2)-bundle P, over a 3-dimensional manifold Σ and smooth sections E of associated vector bundle of su(2)-valued vector densities of weight one. The pair (A, E) co-ordinatizes the phase-space of general relativity with the fundamental Poisson bracket being,

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \delta_a^b \delta_j^i \delta^3(x, y) \tag{2.20}$$

One then constructs a Poisson *-algebra $\mathcal{U}$ (defined below) generated by holonomies and fluxes given by,

$$\begin{aligned} h_e[A] &= \mathcal{P}exp(-\textstyle\int_e A) \\ E_{S,n}[E] &= \textstyle\int_S (*E)_j n^j \end{aligned} \tag{2.21}$$

where n^j is su(2)-valued scalar of compact support on S.
Two properties of these elementary variables are worth noting.
• Holonomies are covariant objects under the action of local SU(2)-gauge transformations and spatial diffeomorphisms. The reason for latter being, one has not used any background structure (e.g. some fiducial spatial metric) for its construction.
• Although $E_{S,n}[E]$ are not SU(2) gauge-covariant, interesting gauge-invariant geometrical objects like length, area and volume of compact sub-manifolds can be constructed out of them. Also as we will see shortly, they do behave covariantly under action of spatial diffeomorphisms.
An arbitrary element of $\mathcal{U}$ generated only by holonomies is known as cylindrical function and is based on (piecewise-analytic) graph γ with finite number of edges and vertices.

$$F_\gamma[A] = f_\gamma(h_{e_1}[A], ..., h_{e_2}[A]) \tag{2.22}$$

where f_γ is a complex valued function on $SU(2)^{|E(\gamma)|}$. The space of all cylindrical functions based on a graph γ is denoted as Cyl_γ, and the *-algebra of cylindrical functions is denoted by $Cyl = \bigcup_\gamma Cyl_\gamma$, where the union is over all piece-wise analytic graphs with finite number of edges and vertices. Note that involution in this case is simply complex conjugation.

Each of the fluxes, $X_{S,n}$ act as derivations on Cyl via the Poisson bracket relations,

$$X_{S,n}F_\gamma[A] \;=\; \{E_{S,n}[E], F_\gamma[A]\} \tag{2.23}$$

The space of all such derivations will be denoted by Vec. The property which ensures that $X_{S,n}$ is a derivation is the fact that number of points in which γ transversally intersects S is finite. This requires S to be semi-analytic and orientable.[3].

Finally we are in a position to define the Poisson algebra which is quantized in LQG.

$\mathcal{U} := Cyl \times Vec$ with a Lie bracket defined by,

$$[(F_\gamma[A], X_{S,n}), (F'_{\gamma'}[A], X_{S',n'})] \;=\; (X_{S,n}F'_{\gamma'}[A] - X_{S',n'}F_\gamma[A], [X_{S,n}, X_{S',n'}]) \tag{2.24}$$

The involution on $\mathcal{U}$ is as follows.

$$(F_\gamma[A], X_{S,n})^* = (\overline{F}_\gamma[A], \overline{X}_{S,n}) \tag{2.25}$$

with $\overline{F}_\gamma[A] \;=\; F_\gamma[A]^*$ and $\overline{X}_{S,n}F'[A] \;=\; \overline{X_{S,n}F[A]}$.

Now consider the free tensor algebra $\bigoplus_n T^n(\mathcal{U})$ modulo the 2-sided ideal generated by elements of the type $u \otimes v - v \otimes u - [u, v]$ for any u,v in $\mathcal{U}$. We denote this algebra by $\hat{\mathcal{U}}$. An abstract algebra isomorphic to thus constructed associative algebra is the quantum algebra which is to be represented on some Hilbert space. With slight abuse of notation, we will denote this abstract algebra also as $\hat{\mathcal{U}}$ and its generators by $\hat{F}_\gamma$, $\hat{X}_{S,n}$.

[3]Roughly speaking semi-analyticity of a surface means analyticity except for on some curves which in turn have to be piecewise analytic

2.3.2 Action of fibrewise automorphisms and base-space automorphisms on $\mathcal{U}$

The group $Diff(\Sigma)$ of semi-analytic diffeomorphisms on Σ can be naturally represented as outer automorphisms on $\mathcal{U}$. Let $\phi \in Diff(\Sigma)$,

$$\begin{aligned} \alpha_\phi \hat{F}_\gamma &= \hat{F}_{\phi^{-1}(\gamma)} \\ \alpha_\phi \hat{X}_{S,n} &= \hat{X}_{\phi^{-1}(S), n o \phi} \end{aligned} \tag{2.26}$$

Similarly the group $Fun(\Sigma, SU(2))$ (this is a group under point-wise multiplication) of local gauge transformations act as outer-automorphisms on $\hat{\mathcal{U}}$. For the sake of simplicity, we display its action on holonomy operators $\hat{h}_e$ rather then an arbitrary cylindrical function. Let $g \in Fun(\Sigma, SU(2))$,

$$\begin{aligned} \alpha_g \hat{h}_e &= g(e(0))\hat{h}_e g(e(1)) \\ \alpha_g \hat{X}_{S,n} &= \hat{X}_{s,n^g} \end{aligned} \tag{2.27}$$

where $n^g(x) = Ad_{g(x)}(n^i(x)\tau_i)$.
Although in this thesis our main focus will be on the space-time-diffeomorphisms, it is important to note that in loop formulation, when dealing with a generally covariant theory of connections, the entire group of bundle automorphisms (by which we mean the semi-direct product of $Diff(\Sigma)$, $Fun(\Sigma, SU(2))$) can be represented as outer automorphisms on the quantum algebra. This group is the kinematical gauge group of the canonical gravity.

2.3.3 GNS construction

Before passing onto the representation theory of $\hat{\mathcal{U}}$, we note the following.
• The Abelian *-algebra Cyl is naturally embedded into $\hat{\mathcal{U}}$ via $F_\gamma[A] \to (\hat{F}_\gamma, 0)$. We will denote the image of this embedding by $\widehat{Cyl}$.
• Cyl can be completed (in the sup-norm) to a C*-algebra $\overline{Cyl}$ whose spectrum is known as the space of generalized connections, $\overline{\mathcal{A}}$, and is the space of groupoid homomorphisms from $\mathcal{P}$ to SU(2). ($\mathcal{P}$ is the path groupoid of Σ.) Note that any smooth connection is an element of $\overline{\mathcal{A}}$ via holonomy-functional.

• Any cylindrical function on the space of smooth connections can be uniquely extended to a function on $\overline{\mathcal{A}}$. Whence any $\hat{F}_\gamma$ can be identified with a functional on the space of generalized connections.

The Hilbert space on which $\hat{\mathcal{U}}$ is represented faithfully is obtained using the GNS construction. The underlying idea of GNS construction is that, given a state[4] on $\hat{\mathcal{U}}$, one obtains $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$. Here $\mathcal{H}_\omega$ is the Hilbert space on which π_ω is the representation of $\hat{\mathcal{U}}$, and Ω_ω is a vector in $\mathcal{H}_\omega$, which coincides with ω when viewed as a state(via expectation value) on $\hat{\mathcal{U}}$.

The key idea underlying loop representation is to work with a state which is invariant under the action of the kinematical gauge group . This ensures that the gauge group is represented unitarily on $\mathcal{H}_\omega$. In the case of theory of connections on a spatial manifold of dimension ≥ 3, such a state is unique [36], and is given by,

$$\begin{aligned} \omega(\hat{F}_\gamma) &= \int_{\overline{\mathcal{A}}} d\mu_0(A) F_\gamma(A) \\ \omega(\hat{X}_{S,n}) &= 0 \end{aligned} \tag{2.28}$$

where we have identified $\hat{F}_\gamma$ with a cylindrical function (of the generalized connections).

μ_0 is the measure on $\overline{\mathcal{A}}$ which can be identified with product of a finite number of copies of Haar measure. Owing to the translation-invariance of the Haar measure one can show that ω is invariant under the action of bundle automorphisms.

The corresponding pre-Hilbert space is the space of finite linear span of elements of $\widehat{Cyl}$ (or equivalently cylindrical functions on $\overline{\mathcal{A}}$.) The inner-product is given by,

$$< \hat{F}_\gamma, \hat{F'}_{\gamma'} > = \omega(\hat{F}^*_\gamma \hat{F'}_{\gamma'}) \tag{2.29}$$

The Cauchy completion of the pre-Hilbert space w.r.t the above inner product is $\mathcal{H}_\omega$. A useful set of basis-states in $\mathcal{H}_\omega$ are known as spin-network states and are defined as follows. Let $s = (\gamma, \vec{j}, \vec{m}, \vec{n})$ be a triple containing a graph γ, a set $\vec{j} = \{j_e\}_{e \in E(\gamma)}$ of irreducible representation of SU(2) with one representation assigned to each edge, and the sets $\vec{m} = \{m_e\}_{e\in E(\gamma)}$, $\vec{n} = \{n_e\}_{e \in E(\gamma)}$ labeling matrix elements of the representation. s is known as a spin-network and there is

[4] A state on a unital *-algebra A is linear functional $\omega : A \rightarrow \mathbf{C}$ which satisfies, $\omega(a^*) = \overline{\omega(a)}$, $\omega(a^* a) \geq 0$, and $\omega(I) = 1$ where I is the unit of A.

an injection from the set of all spin-networks to a set of orthonormal basis-states in $\mathcal{H}_\omega$

$$\begin{aligned} s \rightarrow T_s(\overline{A}) &= \prod_{e\in E(\gamma)} \sqrt{d_{j_e}}(j_e(\overline{A}(e))_{m_e n_e}) \\ < T_s, T_{s'} >_{\mathcal{H}_\omega} &= \delta_{s,s'} \end{aligned} \tag{2.30}$$

As the set of all graphs embedded in Σ is uncountable, so is the set of all spin-networks which implies $\mathcal{H}_\omega$ is non-separable.[5]
Finally the representation of $\hat{\mathcal{U}}$ is as follows.

$$\begin{aligned} \pi_\omega(\hat{F}_\gamma)F'_{\gamma'}(A) &= F_\gamma A F'_{\gamma'}(A) \\ \pi_\omega(\hat{X}_{S,n})F'_{\gamma'}(A) &= X_{S,n}\hat{F}'_{\gamma'}(A) \end{aligned} \tag{2.31}$$

The unitary representation of spatial diffeomorphisms and local gauge transformations is,

$$\begin{aligned} \hat{U}(\phi)F_\gamma(A) &= F_{\phi^{-1}(\gamma)}(A)\ \forall\phi \in Diff(\Sigma) \\ \hat{V}(g)F_\gamma(A) &= F_\gamma(A_g)\forall\ g \in Fun(\Sigma, SU(2)) \end{aligned} \tag{2.32}$$

where $A_g(e) \;=\; g(e(0)A(e)g(e(1))^{-1}\ \forall e \in \mathcal{P}$.

This in a nutshell is the kinematical setup underlying LQG. That the requirement of (spatial)-diffeomorphism invariance leads to a quantum configuration space $\overline{\mathcal{A}}$ with surprisingly simpler topological and measure theoretic properties (as compared to the configuration space $\mathcal{A}$ of gauge theories on a fixed background) is certainly remarkable. One can now build operators corresponding to geometrical quantities, such as area, volume and length of sub-manifolds from the self-adjoint operators $\hat{X}_{S,n}$. These operators turn out to be densely-defined self-adjoint operators on $\mathcal{H}_\omega$. Their spectra are discrete(pure point) and provide the first hints of the underlying discrete nature of quantum geometry [4].
We now pass over to the construction of diffeomorphism-invariant Hilbert space.[6] The method used to obtain the Hilbert space of diffeomorphism-invariant states ($\mathcal{H}_{diff}$) is known as Refined algebraic quantization(RAQ) [37], which we summarise

[5] A Hilbert space is separable iff it admits a countable, orthonormal basis.
[6] The local SU(2)-gauge transformations can be moded out in a variety of ways [54]. The resulting Hilbert space is a subspace of $\mathcal{H}_\omega$ spanned by gauge-invariant cylindrical functions. But as this gauge group is irrelevant from the point of view of PFT, we refrain from giving any details here.

below.
Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}_\omega$ generated by finite span of spin-network states. Let $\mathcal{D}^*$ be its algebraic dual. The idea is to look for elements of $\mathcal{D}^{*}$[7] which satisfy,

$$f(\hat{U}(\phi)T_s) \;=\; f(T_s)\forall s, \forall\phi \in Diff(\Sigma) \tag{2.33}$$

where $f \in \mathcal{D}^*$.
In RAQ framework, the space of diffeomorphism invariant-states is the image of an anti-linear map [6],

$$\eta : \mathcal{D} \to \mathcal{D}^* \tag{2.34}$$

known as the rigging map. Rigging map is required to satisfy the following properties.

$$\begin{aligned} (\eta(\Psi))[\Psi] &\geq 0 \;\forall\Psi \in \mathcal{D} \\ \hat{O}'\eta(\Psi) &= \eta(\hat{O}\Psi) \end{aligned} \tag{2.35}$$

for any operator $\hat{O}$ acting on $\mathcal{H}_\omega$ that satisfies $\hat{U}(\phi)\hat{O}\hat{U}(\phi^{-1}) \;=\; \hat{O}$. $\hat{O}'$ is the dual representation of $\hat{O}$ on $\mathcal{D}^*$.
The inner product on $Im(\eta) \subset \mathcal{D}^*$ is given by,

$$< \eta(\Psi_1)|\eta(\Psi_2) >_{diff} \;:=\; \eta(\Psi_2)[\Psi_1] \tag{2.36}$$

A simple analysis shows that the image of the Rigging map contains distributions of the following form

$$\eta(|s>) \;=\; \eta_{[s]}\sum_{s' \in s} < s'| \tag{2.37}$$

where we have denoted a spin-network state T_s by a ket $|s>$. The sum on the R.H.S is over all spin-networks which can be obtained from s via an action of some $\phi \in Diff(\Sigma)$ are diff-invariant. Here $\eta_{[s]}$ is any positive real number which is constant along the orbit [s]. However in order to show that η satisfies eq.2 in (2.35), one needs a more explicit form of the Rigging map.
In the present case, one avenue for constructing the Rigging map is offered by a technique called group averaging. The idea is to start with a state (say a spin-network state $|s>$) in $\mathcal{H}_\omega$ and sum over all the states obtained by applying the

[7]Note that the reason we are not looking for diffeomorphism-invariant states in $\mathcal{H}_\omega$ is because there aren't any (except the constant vector 1)

entire diffeomorphism group to the given state. The distribution so obtained will be trivially diffeomorphism-invariant. However averaging over $Diff(\Sigma)$ is subtle due to the following reason.

Given a graph γ , there are infinitely many diffeomorphisms of Σ which will keep γ invariant, and the sum $\sum_{\phi \in Diff(\Sigma)} < \phi \cdot s|$ diverges. Whence one has to divide out the isotropy group of each graph.

The renormalized Rigging map (derived in [6]) is given by,

$$\eta(|s>) = \eta_{[s]} \sum_{\phi \in Diff_{[\tilde{\gamma}(s)]}(\Sigma)} \sum_{\phi' \in GS(\tilde{(\gamma)})} < s|\hat{U}(\phi')\hat{U}(\phi) \tag{2.38}$$

where, $\gamma(s)$ is a graph associated to s and $\tilde{\gamma}(s)$ is its maximal analytic extension.[8] $Diff_{[\tilde{\gamma}(s)]}(\Sigma)$ is the set of diffeomorphisms, each element of which maps $\gamma(s)$ to some distinct graph.

$GS(\gamma)$ is the symmetry group of γ which keeps γ invariant but necessarily permutes the edges.

Several comments are in order.

• The rigging map defined above, satisfies eq.2 in (2.35) only for the so-called strongly diffeomorphism-invariant observables, i.e. those operators on $\mathcal{H}_\omega$ which satisfy $\hat{U}(\phi)\hat{O}\hat{U}(\phi^{-1}) = \hat{O}$. This is a huge drawback as the only densely-defined strongly diff-invariant observable is the volume operator. (As such it can be easily shown that strongly diff-invariant operators cannot depend on connection).

• The numbers $\eta_{[s]}$ are completely arbitrary and thus there is an infinite parameter worth of ambiguity in the construction of the rigging map.

• Only working with strongly diffeomorphism-invariant operators lead to super-selection sectors in $\mathcal{H}_\omega$ (with each sector being labeled by a given maximally analytically extended graph). It is not clear if all the interesting physics can be recovered by only working in a given super-selected sector.

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[8] A maximal analytic extension of a graph γ is a graph generated by maximal analytic extension of the edges of γ.

2.3.4 Quantum Dynamics

We now turn to a very brief and qualitative discussion of quantization of Hamiltonian constraint in LQG. Our purpose here is to only note certain important features of the Hamiltonian constraint and so we do not provide any detailed expressions in this section. All the details can be found in references.
In a remarkable series of papers titled quantum spin dynamics ([51], [52], [47], [48], [49], [50]), Thiemann proposed a definition of Hamiltonian constraint operator ($\hat{H}[N]$). This operator turned out to be a densely-defined diffeomorphism-covariant operator on $\mathcal{H}_\omega$. One of the remarkable properties of this operator is that it is independent of any choice triangulation of the spatial manifold. This is certainly apriori unexpected as any classical functional of connection, will be promoted to an operator in quantum theory by expressing connection in terms of holonomies around small loops. In the case of integrated functions, like the Hamiltonian constraint, this would imply a choice of the triangulation of Σ. Generically this triangulation dependence can not be removed by any limiting procedure. However in the case of the Hamiltonian constraint, it is possible to define a limiting procedure and obtain a manifestly regularization independent operator.
• $\hat{H}[N]$ is necessarily graph-changing. Given a spin-network state $|s>$ it maps it to a linear combination of spin-network states with corresponding spin-networks having a different graph then $\gamma(s)$.
• $\hat{H}[N]$ (or more appropriately its dual) does not preserve $\mathcal{H}_{diff}$. However the commutator $[\ \hat{H}[N], \hat{H}[M]\ ]$ vanishes on diffeomorphism-invariant states. This hints at the anomaly freeness of the quantum constraint algebra. Graph-changing nature of the operator is crucial to this property.
• Precisely due to the graph-changing nature of the constraint, its semi-classical limit has remained elusive. As all the semi-classical states constructed so far in LQG are only suitable for graph-preserving operators. This also means that it is not at all clear if the quantum constraint algebra reduces to the Dirac-algebra in classical limit.
• The space of solutions to the Wheeler-Dewitt equation $\hat{H}[N]\Psi = 0\ \forall N$ has so far remained elusive.
Some recent progress in defining quantum dynamics in LQG involves replacing the infinite number of Hamiltonian constraints ($\hat{H}[N]\ \forall\ N$) with a single constraint

called the Master constraint such that the algebra generated by the Master constraint and the diffeomorphism constraints is a true Lie algebra [17]. In this thesis, we will be working with two-dimensional generally covariant field theories, where the constraint algebra is already a true Lie-algebra. So we will not use the Master constraint ideas to solve the quantum constraints.

2.3.5 Polymer quantization

We now turn to loop or what is commonly known as polymer quantization of non-gravitational field theories. As we saw in last section, the central idea behind LQG is to choose a suitable sub-algebra $\mathcal{U}$ of the Poisson-algebra of functions on the phase space such that spatial diffeomorphisms (as well as SU(2) gauge transformations) act as automorphisms on $\mathcal{U}$. Then there exists a unique representation of this algebra on a Hilbert space $\mathcal{H}_\omega$ such that spatial diffeomorphisms are represented unitarily on $\mathcal{H}_\omega$. The states of this Hilbert space are labeled by graphs. Whence the fundamental excitations of geometry are one dimensional (polymer like), and the underlying manifold Σ is replaced by arbitrary graphs. Thus from the quantum geometry point of view, matter fields can have support only on graphs.

Polymer quantization of a matter field theory is a background independent (in the sense that the Hilbert space admits a unitary action of spatial diffeomorphisms of Σ) quantization such that the fundamental excitations of the matter field have support only on graphs. For gauge fields this means, working with the holonomy-flux algebra similar to the holonomy-flux algebra of gravity. In the case of a theory of real scalar field f, the quantization proceeds as follows [7].

Let V be a set consisting of finite number of points on Σ, $V = (p_1, ..., p_n)$. Consider the vector space Cyl_V generated by finite linear combinations of functions of the following type,

$$\mathcal{N}_{V,\vec{\lambda}}(f) = e^{i\sum_{p_j\in V}\lambda_j f(p_j)} \tag{2.39}$$

where $\vec{\lambda} = (\lambda_1, ..., \lambda_n)$ is an n-tuple of arbitrary real numbers. Cyl_V is an Abelian *-algebra. Finally we can define the vector space of all cylindrical functions as,

$$Cyl = \oplus_V Cyl_V \tag{2.40}$$

Let π_ϕ be momentum conjugate to ϕ. As π_ϕ are scalar densities on weight one, following functionals do not require any background structure in their definition,

$$\pi_f[g] \;=\; \int_\Sigma \pi_f(x)g(x) \tag{2.41}$$

where g is any test-function with suitable fall-off or boundary conditions.
$\pi_f[g]$ acts as derivation on Cyl via,

$$X_{\pi_f[g]}\mathcal{N}_{V,\vec{\lambda}}(f) \;=\; i\sum_{p_j\in V}(\lambda(p_j)g(p_j))\mathcal{N}_{V,\vec{\lambda}}(f) \tag{2.42}$$

Note that, spatial diffeomorphisms act as outer automorphisms on the Poisson algebra generated by cylindrical functions and smeared momenta, as

$$\begin{aligned}\alpha_\phi(\mathcal{N}_{V,\vec{\lambda}}(f)) &= \mathcal{N}_{\phi^{-1}(V),\vec{\lambda}}(f)\\ \alpha_\phi(\pi_f[g]) &= \pi_f[g\circ\phi]\end{aligned} \tag{2.43}$$

We can construct an abstract *-algebra $\hat{\mathcal{U}}_f$ similar to the one constructed for gravity in the previous section. The background independent quantum field theory is obtained by using following positive linear functional for GNS construction.

$$\begin{aligned}\omega_f(N_{V,\vec{\lambda}}) &= 1\; if\; \lambda_j \;=\; 0\;\forall j\\ &= 0\; otherwise\\ \omega_f(\pi_f[g]) &= 0\;\forall\; g\end{aligned} \tag{2.44}$$

ω_f is clearly background independent in the sense that $\omega_f(\alpha_\phi(a)) \;=\; \omega_f(a)\;\forall\phi\in Diff(\Sigma)$.
The resulting GNS Hilbert space is a cauchy completion of finite linear span of the cylindrical functions $\mathcal{N}_{V,\vec{\lambda}}$. The inner product being given by,

$$<\mathcal{N}_{V,\vec{\lambda}}|\mathcal{N}_{V',\vec{\lambda}'}> \;=\; \delta_{V,V'}\delta_{\vec{\lambda},\vec{\lambda}'} \tag{2.45}$$

and $\widehat{\pi_f[g]}$ are self-adjoint unbounded operators for all test functions g.

3

Polymer quantization of PFT on $S^1 \times \mathbf{R}$

3.1 Introduction

This chapter is devoted to an application of canonical Loop Quantum Gravity (LQG) techniques to the quantization of a two dimensional Parametrized field theory on a flat two-dimensional spacetime. Let us briefly recall the idea behind parametrizing a field theory from the previous chapter. PFT offers an elegant description of free scalar field evolution on *arbitrary* (and in general curved) foliations of the background spacetime by treating the 'embedding variables' which describe the foliation as dynamical variables to be varied in the action in addition to the scalar field. Specifically, let $X^A = (T, X)$ denote embedding coordinates on 2 dimensional flat spacetime. In PFT, X^A are parametrized by a new set of arbitrary coordinates $x^\alpha = (t, x)$ such that for fixed t, the embedding variables $X^A(t, x)$ define a spacelike Cauchy slice of flat spacetime. General covariance of PFT ensues from the arbitrary choice of x^α and implies that in its canonical description, evolution from one slice of an arbitrary foliation to another is generated by constraints. While 2 dimensional PFT has been quantized in a Fock representation for the matter fields in References [29, 58], here we are interested in the construction of an LQG type representation for both the embedding as well as the matter fields.

As we noticed in the introduction, one of the major open issues in LQG is the arbitrary choices of triangulation of the spatial manifold, which generically underlies any operator of interest that depends on the connection. In this chapter we present a 'perfect' toy model in which an LQG type of quantization can be constructed which is free from any triangulation ambiguities. Specifically, we construct, in a triangulation independent manner: an appropriate kinematic 'holonomy' algebra and its LQG type 'polymer' representation on a kinematic Hilbert space $\mathcal{H}_{kin}$, a representation on $\mathcal{H}_{kin}$ of both (the finite transformations generated by) the constraints and an over- complete set of gauge invariant observables, the group averaging map [37, 6] and the physical state space $\mathcal{H}_{phys}$ which naturally inherits a representation of the Dirac observables from that on $\mathcal{H}_{kin}$.

The above quantization of PFT offers an arena in which proposals for quantum dynamics developed for LQG may be tested against the manifestly triangulation/regularization free group averaging techniques used in this work. Further, semiclassical issues can be examined at the physical state level since both $\mathcal{H}_{phys}$ and representation of an overcomplete set of Dirac observables thereon, are available. This is in contrast to LQG wherein most current proposals are defined on $\mathcal{H}_{kin}$ with the hope that they may still be useful at the physical state level. Again, since the quantization here admits a representation of Dirac observables on $\mathcal{H}_{kin}$ as well as $\mathcal{H}_{phys}$, it offers a useful testing ground for proposed constructions of semiclassical states in LQG. Finally, since PFT also admits the usual Fock space quantization of the scalar field [29, 58], this can be compared with the "polymer" quantization presented here. This comparison is useful for similar 'graviton from LQG' issues [61] in canonical LQG.

The layout of the chapter is as follows. Section 3.2 contains a brief review of classical PFT on $S^1 \times R$. Details may be found in [30]. In section 3.3, $\mathcal{H}_{kin}$ is constructed as the tensor product of Hilbert spaces for the matter and embedding sectors, each of which supports a polymer representation of suitably defined LQG-type operators. It is shown that $\mathcal{H}_{kin}$ also supports a unitary representation of the finite canonical transformations generated by the constraints. In section 3.4 an overcomplete set of gauge invariant (Dirac) observables corresponding to (a) exponentials of the standard mode functions of the free scalar field on flat spacetime and (b) conformal isometries, are promoted to operators on $\mathcal{H}_{kin}$. These operators commute with those corresponding to finite gauge transformations. In section 3.5, the physical state space, $\mathcal{H}_{phys}$, is constructed through group averaging techniques [37, 6]. Ambiguities in the group averaging map are systematically reduced by requiring commutativity with the Dirac observables and superselection sectors are described, each of which provide a cyclic, *non- seperable* representation of the algebra generated by the gauge invariant operators of section 3.4. Section 3.6 is devoted to a preliminary discussion of semiclassical issues. It is shown that, at most, only a countable subset of the overcomplete (and uncountable) set of Dirac observables of type (a) can be approximated by semiclassical states in $\mathcal{H}_{phys}$. Further, it is shown that any such state must be characterized by a suitably defined "physical" weave. Two issues (connected with the S^1 spatial topology and the treatment of zero modes) are addressed in section 3.7 Section 3.8 contains a discussion of our results as well as of open issues.

In the interests of brevity, we shall refrain from providing detailed proofs where such proofs are straightforward. Some Lemmas are proved in the Appendices A-D. The dimensions of various quantities and our choice of units are displayed in Appendix E.

3.2 Classical PFT on $S^1 \times R$.

We provide a brief review of classical 2 dimensional PFT. In sections 3.2.1 and 3.2.2 we shall implicitly assume that the spatial topology is that of a circle. The consequences of this non-trivial spatial topology on the formalism will be made explicit in section 3.2.3.

3.2.1 The Action for PFT.

The action for a free scalar field f on a fixed flat 2 dimensional spacetime in terms of global inertial coordinates X^A, $A = 0, 1$ is

$$S_0[f] = -\frac{1}{2}\int d^2X \eta^{AB}\partial_A f \partial_B f, \tag{3.1}$$

where the Minkowski metric in inertial coordinates, η_{AB}, is diagonal with entries $(-1, 1)$. If instead, we use coordinates x^α , $\alpha = 0, 1$ (so that X^A are 'parameterized' by x^α, $X^A = X^A(x^\alpha)$), we have

$$S_0[f] = -\frac{1}{2}\int d^2x \sqrt{g} g^{\alpha\beta}\partial_\alpha f \partial_\beta f, \tag{3.2}$$

where $g_{\alpha\beta} = \eta_{AB}\partial_\alpha X^A \partial_\beta X^B$ and g denotes the determinant of $g_{\alpha\beta}$. The action for PFT is obtained by considering the right hand side of (3.2) as a functional, not only of ϕ, but also of $X^A(x)$ i.e. $X^A(x)$ are considered as 2 new scalar fields to be varied in the action ($g_{\alpha\beta}$ is a function of $X^A(x)$). Thus

$$S_{PFT}[f, X^A] = -\frac{1}{2}\int d^2x \sqrt{g(X)} g^{\alpha\beta}(X)\partial_\alpha f \partial_\beta f. \tag{3.3}$$

Note that S_{PFT} is a diffeomorphism invariant functional of the scalar fields f, X^A. Variation of f yields the equation of motion $\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta f) = 0$, which is just the flat spacetime equation $\eta^{AB}\partial_A\partial_B f = 0$ written in the coordinates x^α. On varying X^A, one obtains equations which are satisfied if $\eta^{AB}\partial_A\partial_B f = 0$. This implies that $X^A(x)$ are undetermined functions (subject to the condition that determinant of $\partial_\alpha X^A$ is non- vanishing). This 2 functions- worth of gauge is a reflection of the 2 dimensional diffeomorphism invariance of S_{PFT}. Clearly the dynamical content of S_{PFT} is the same as that of S_0; it is only that the diffeomorphism invariance of S_{PFT} naturally allows a description of the standard free field dynamics dictated by S_0 on *arbitrary* foliations of the fixed flat spacetime.

3.2.2 Hamiltonian Formulation of PFT.

In the previous subsection, $X^A(x)$ had a dual interpretation - one as dynamical variables to be varied in the action, and the other as inertial coordinates on a flat spacetime. In what follows we shall freely go between these two interpretations.

We set $x^0 = t$ and $\{x^\alpha\} = \{t, x\}$. We restrict attention to $X^A(x^\alpha)$ such that for any fixed t, $X^A(t, x^a)$ describe an embedded spacelike hypersurface in the 2 dimensional flat spacetime (it is for this reason that $X^A(x)$ are called embedding variables in the literature). This means that, for fixed t, the functions $X^A(x)$ must be such that the symmmetric form q_{ab} defined by

$$q_{ab}(x) := \eta_{AB}\frac{\partial X^A(x)}{\partial x}\frac{\partial X^B(x)}{\partial x} \tag{3.4}$$

is an 1 dimensional Riemannian metric. This follows from the fact that $q_{ab}(x)$ is the induced metric on the hypersurface in the flat spacetime defined by $X^A(x)$ at fixed t.

A 1+1 decomposition of S_{PFT} with respect to the time 't', leads to its Hamiltonian form:

$$S_{PFT}[f, X^A; \pi, \Pi_A; N^A] = \int dt \int d^2x (\Pi_A \dot{X}^A + \pi_f \dot{f} - N^A H_A). \tag{3.5}$$

Here π_f is the momentum conjugate to the scalar field f, Π_A are the momenta conjugate to the embedding variables X^A, N^A are Lagrange multipliers for the first class constraints H_A. It turns out that the motions on phase space generated by the 'smeared' constraints, $\int d^2x(N^A H_A)$ correspond to scalar field evolution along arbitrary foliations of the flat spacetime, each choice of foliation being in correspondence with a choice of multipliers N^A. Since the constraints are first class they also generate gauge transformations, and as in General Relativity, the notions of gauge and evolution are intertwined.

Since free scalar field theory in 2 dimensions finds its simplest expression in terms of left and right movers, it is useful to make a point canonical transformation to light cone embedding variables $X^{\pm}(x) := T(x) \pm X(x)$ (here we have set $X^0 = T, X^1 = X$). Denoting the conjugate embedding momenta by $\Pi_{\pm}(x)$, and setting $H_{\pm} = H_0 \pm H_1$, the action takes the form

$$S = \int dt \int dx \, [\, \pi_f \dot{f} + \Pi_+ \dot{X}^+ + \Pi_- \dot{X}^- - N^+ H_+ - N^- H_- \,]. \tag{3.6}$$

where $N^{\pm}$ are the new Lagrange multipliers appropriate to $H_{\pm}$. Explicitly, the constraints $H_{\pm}$ are given by

$$H_{\pm}(x) = [\, \Pi_{\pm}(x) X^{\pm'}(x) \pm \frac{1}{4}(\pi_f \pm f')(x)(\pi_f \pm f')(x) \,]. \tag{3.7}$$

Note that while $X^{\pm}(x), f(x)$ transform as scalars under spatial coordinate transformations, $\Pi_{\pm}, \pi_f, N^{\pm}$ transform as scalar densities (or equivalently as spatial vector fields).

The Poisson brackets between various fields are given by,

$$\begin{aligned} \{f(x), \pi_f(x')\} &= \delta(x, x'), \\ \{X^{\pm}(x), \Pi_{\pm}(x')\} &= \delta(x, x'), \end{aligned} \tag{3.8}$$

and the remaining brackets are zero. Here $\delta(x, x')$ is the delta-function on S^1.

To complete the transition to variables closely related to the left and right movers of free scalar field theory [30], we perform a canonical transformation on the matter variables. $(f, \pi_f) \rightarrow (Y^+, Y^-)$. Here $Y^{\pm}(x) = \pi_f(x) \pm f'(x)$ (strictly speaking this transformation is not invertible when the spatial topology is S^1 due to the existence of zero modes; we shall return to this issue in section 3.3). The Poisson brackets between the scalar densities, $Y^{\pm}$, are given by,

$$\begin{aligned} \{Y^{\pm}(x), Y^{\pm}(x')\} &= \pm[\partial_x \delta(x,x') - \partial_{x'}\delta(x',x)] \\ \{Y^{\mp}(x), Y^{\pm}(x')\} &= 0. \end{aligned} \tag{3.9}$$

The constraints are now

$$H^{\pm}(x) = [\, \Pi_{\pm}(x) X^{\pm'}(x) \pm \frac{1}{4} Y^{\pm}(x)^2 \,]. \tag{3.10}$$

and the constraint algebra is

$$\begin{aligned} \{H_{\pm}[N^{\pm}], H_{\pm}[M^{\pm}]\} &= H_{\pm}[\mathcal{L}_{N^{\pm}} M^{\pm}] \\ \{H_{\pm}[N^{\pm}], H_{\mp}[M^{\mp}]\} &= 0 \end{aligned} \tag{3.11}$$

Here $\mathcal{L}_N$ denotes the Lie derivative with respect to the 1 dimensional spatial vector field with component $N(x)$ in the coordinate system 'x'. The action of the constraints on the phase space variables can be expressed as follows. Let $\Phi^{\pm} = (Y^{\pm}, \Pi_{\pm})$, we have

$$\begin{aligned} \{\Phi^{\pm}(x),\ H_{\pm}[N^{\pm}]\,\} &= \mathcal{L}_{N^{\pm}} \Phi^{\pm}(x) \\ \{\Phi^{\mp}(x),\ H_{\pm}[N^{\pm}]\,\} &= 0, \end{aligned} \tag{3.12}$$

Thus, on the set of variables $\Phi^{\pm}$, infintesmal gauge transformations act as diffeomorphisms on S^1 and there is a split of the constraints and the phase space variables into commuting '+' and '-' parts which correspond to the usual right and left moving sectors of free scalar field theory. The action of the constraints on the embedding variables $X^{\pm}(x)$ preserves this split:

$$\{X^{\pm}(x), H_{\pm}[N^{\pm}]\} = N^{\pm}(X^{\pm})', \tag{3.13}$$

$$\{X^{\mp}(x), H_{\pm}[N^{\pm}]\} = 0. \tag{3.14}$$

Indeed, the above equations seem to indicate that infinitesimal gauge transformations, once again, act as diffeomorphisms on S^1; however, as we shall see in the next subsection, this interpretation is not strictly true for equations (3.13), (3.14) due to the non- existence of global, single valued coordinates on S^1.

3.2.3 Consequences of spatial topology $= S^1$.

Conditions on the canonical variables.

S^1 does not admit a global single valued coordinate system. However, at the cost of introducing appropriate periodic/quasiperiodic boundary conditions on the fields we may choose x to be the standard angular coordinate, $x \in [0, 2\pi]$ with the identification $x = 0 \sim x = 2\pi$. The Minkowskian coordinates $X^A = (T, X)$ in the action (3.1) are chosen so that $T \in (-\infty, \infty)$,$X \in (-\infty, \infty)$ with the identifications $X \sim X + 2\pi$. The above specifications on x, X imply the following conditions on the canonical embedding variables and the Lagrange multipliers:
(i) $X^{\pm}(2\pi) - X^{\pm}(0) = \pm 2\pi$.
(ii) Any two sets of embedding data $(X_1^+(x), X_1^-(x))$ and $(X_2^+(x), X_2^-(x))$ are to be identified if there exists an integer m such that $X_1^+(x) = X_2^+(x) + 2m\pi \;\forall\, x \in \; [0, 2\pi]$ and $X_1^-(x) = X_2^-(x) - 2m\pi \;\forall\, x \in \; [0, 2\pi]$.
(iii) $\Pi_{\pm}(x), N^{\pm}(x)$ and their spatial derivatives to all orders, as well as the spatial derivatives to all orders of the embedding coordinates $X^{\pm}(x)$ are periodic on $[0, 2\pi]$ with period 2π. This follows from the 1+1 Hamiltonian decomposition of (3.3) and the fact that $\frac{\partial X^A}{\partial x^\alpha}$ in equation (3.4) is single valued on $S^1 \times R$.
An additional "non-degeneracy" condition arises from (3.4):
(iv)$\pm(X^{\pm})' > 0$.

Since f in (3.1) is a single valued function on $S^1 \times R$, it follows that the matter phase space variables, (f, π_f) and their spatial derivatives to all orders are also periodic functions on $[0, 2\pi]$. Note also that the delta function $\delta(x, y)$ in (3.8), (3.9) is periodic in both its arguments.

Finite gauge transformations.

Whereas equation (3.12) implies that finite gauge transformations act on $(\Pi_{\pm}, Y^{\pm})$ as spatial diffeomorphisms on S^1, as remarked earlier the case of the embedding variables $X^{\pm}$ is more subtle as $X^{\pm}$ are not single valued fields on S^1 by virtue of (**i**), section 3.2.3. Therefore, evolution of $X^{\pm}$ under the flow generated by the constraints is better understood in terms of transformations on the universal cover of S^1 as follows.

Unwind S^1 to its universal cover $\mathbf{R}$. Quasi-periodic boundary conditions obeyed by the embeddings suggest that their extension to $\mathbf{R}$ satisfies:

$$X^{\pm}_{ext}(x \pm 2n\pi) \;:=\; X^{\pm}(x) \;\pm\; 2n\pi \tag{3.15}$$

where $x \in [0, 2\pi]$ and $n \in \mathbf{Z}$. The vector fields $N^{\pm}(x)$ on S^1 extend to periodic vector fields $N^{\pm}_{ext}$ on $\mathbf{R}$ so that $N^{\pm}_{ext}(x + 2n\pi) \;=\; N^{\pm}(x),\; x \in [0, 2\pi]$. Let the 1 parameter family of (periodic) diffeomorphisms of $\mathbf{R}$ generated by $N^{\pm}_{ext}$ be denoted by $\phi[N^{\pm}_{ext}, t]$. And let $\phi[N^{\pm}_{ext}, t](x) \in \mathbf{R}$ be the image of $x \in [0, 2\pi]$ under $\phi[N^{\pm}_{ext}, t]$. Then it is straightforward to check that the finite transformations generated by the constraints on $X^{\pm}(x)$ are labelled by $\phi[N^{\pm}_{ext}, t]$ and act as follows:

$$\begin{aligned} (\alpha_{\phi[N^{\pm}_{ext},t]} X^{\pm})(x) &= X^{\pm}_{ext}(\phi[N^{\pm}_{ext}, t](x)) \;\forall\; x \in [0, 2\pi] \\ (\alpha_{\phi[N^{\pm}_{ext},t]} X^{\mp})(x) &= X^{\mp}(x) \;\forall\; x \in [0, 2\pi] \end{aligned} \tag{3.16}$$

Here $\alpha_{\phi[N^{\pm}_{ext},t]}$ is the flow generated by Hamiltonian vector field of $H_{\pm}[N^{\pm}]$.

It is also straightforward to see that the action of finite gauge transformations on the phase space variables $\Phi^{\pm} \in \{Y^{\pm}, \Pi_{\pm}\}$ can equally well be written in terms of the action of the periodic diffeomorphisms $\phi[N^{\pm}_{ext}, t]$ on the periodic extensions $\Phi^{\pm}_{ext}$ as

$$\begin{aligned} (\alpha_{\phi[N^{\pm}_{ext},t]} \Phi^{\pm})(x) &= \Phi^{\pm}_{ext}(\phi[N^{\pm}_{ext}, t](x)) \;\forall\; x \in [0, 2\pi] \\ (\alpha_{\phi[N^{\pm}_{ext},t]} \Phi^{\mp})(x) &= \Phi^{\mp}(x) \;\forall\; x \in [0, 2\pi] \end{aligned} \tag{3.17}$$

Here $\Phi^{\pm}_{ext}(x + 2n\pi) \;=\; \Phi^{\pm}(x)\; \forall x \in [0, 2\pi], n \in \mathbf{Z}$.

Since $\phi[N^{\pm}_{ext}, t]$, $\forall (N^{\pm}_{ext}, t)$ range over all periodic diffeomorphisms of $\mathbf{R}$ connected to identity, we label every finite gauge transformation by a pair of such diffeomorphisms (ϕ^{+}, ϕ^{-}) so that the Hamiltonian flows generated by $H_{\pm}$ are denoted by $\alpha_{\phi^{\pm}}$. To summarise: Let $\Psi^{\pm}(x) \in (X^{\pm}(x), \Pi_{\pm}(x), Y^{\pm}(x))$ and let its appropriate quasiperiodic/periodic extension on $\mathbf{R}$ be $\Psi^{\pm}_{ext}$. Then we have that,

$\forall x \in [0, 2\pi]$,

$$\begin{aligned} (\alpha_{\phi^\pm}\Psi^\pm)(x) &= \Psi^\pm_{ext}(\phi^\pm(x)) \\ (\alpha_{\phi^\pm}\Psi^\mp)(x) &= \Psi^\mp(x). \end{aligned} \tag{3.18}$$

Equations (3.18) imply a left representation of the group of periodic diffeomorphisms of $\mathbf{R}$ by the Hamiltionian flows corresponding to finite gauge transformations:

$$\alpha_{\phi_1^\pm}\alpha_{\phi_2^\pm} = \alpha_{\phi_1^\pm\circ\phi_2^\pm} \tag{3.19}$$

$$\alpha_{\phi_1^\pm}\alpha_{\phi_2^\mp} = \alpha_{\phi_2^\mp}\alpha_{\phi_1^\pm}. \tag{3.20}$$

We emphasize that the extended fields are only formal constructs which are useful for interpreting gauge transformations in terms periodic diffeomorphisms of $\mathbf{R}$. The spatial slice is always S^1 coordinatized by $x \in [0, 2\pi]$ with boundary points identified.

3.2.4 Dirac Observables

Since finite gauge transformations act as periodic diffeomorphisms of $\mathbf{R}$, it follows, directly, that the integral over $x \in [0, 2\pi]$ of any periodic scalar density constructed solely from the phase space variables, is an observable.

An analysis of the Hamiltonian equations [30] shows that the relation between solutions $f(X^+, X^-)$ of the flat spacetime wave equation and canonical data $(Y^\pm, X^\pm)$ on the constraint surface is

$$\pm 2\frac{\partial f}{\partial X^\pm} = \frac{Y^\pm}{(X^\pm)'}. \tag{3.21}$$

Here f is evaluated at the spacetime point (X^+, X^-) defined by the canonical data. Recall that any solution $f(X^+, X^-)$ to the free scalar field equation is of the form

$$f(X^+, X^-) = \frac{\mathbf{q}}{\sqrt{2\pi}} + \frac{\mathbf{p}}{\sqrt{2\pi}}\frac{(X^+ + X^-)}{2} + \sum_{n=1}^{\infty}\frac{i}{4\pi n}(\mathbf{a}_{(+)\mathbf{n}}e^{-inX^+} + \mathbf{a}_{(-)\mathbf{n}}e^{-inX^-} - \mathrm{c.c}), \tag{3.22}$$

where c.c. stands for 'complex conjugate'. Equations (3.21) and (3.22) yield an interpretation for the Dirac observables constructed below.

Mode functions.

From (3.21) and (3.22) and the remarks above, it follows that

$$a_{(\pm)n} = \int_{S^1} dx Y^{\pm}(x) e^{inX^{\pm}(x)}, \; n \in \mathbf{Z}, \; n > 0 \tag{3.23}$$

(and their complex conjugates, $a^*_{(\pm)n}$,) are Dirac observables which correspond to the mode functions $\mathbf{a}_{(\pm)\mathbf{n}}$ of equation (3.22). These observables form the (Poisson) algebra,

$$\begin{aligned} \{a_{(\pm)n}, a_{(\pm)m}*\} &= -4\pi i n \delta_{n,m}, \\ \{a_{(\pm)n}, a_{(\pm)m}\} &= 0, \\ \{a_{(\pm)n}*, a_{(\pm)m}*\} &= 0. \end{aligned} \tag{3.24}$$

The Dirac observables corresponding to right-moving sector $(a_{(+)m}, a_{(+)n}*)$ Poisson commute with the observables corresponding to the left moving sector $(a_{(-)m}, a_{(-)n}*)$.

Zero modes.

The quantities $\mathbf{q}, \mathbf{p}$ in equation (3.22) are referred to as zero modes of the scalar field and are also realizable as Dirac observables which are canonically conjugate to each other [30]. Indeed, it is straightforward to see from (3.21), (3.22) that $\mathbf{p}$ corresponds to $p := \int_{S^1} dx Y^+(x) = \int_{S^1} dx Y^-(x)$. However, the degree of freedom corresponding to $\mathbf{q}$ is absent in the phase space coordinates $(X^{\pm}, \Pi_{\pm}, Y^{\pm})$ as a result of $Y^{\pm}$ containing only derivatives of f (see equation (3.21)).

Our aim in this work is to construct a triangulation independent polymer quantization of a generally covariant field theoretic model. Issues related to the construction of zero modes (which are anyway mechanical as opposed to field theoretic degrees of freedom) as Dirac observables serve to distract from this aim. Hence we shall switch off the zero modes by setting $\mathbf{q} = \mathbf{p} = 0$. Since $\mathbf{q}$ and $\mathbf{p}$ are canonically conjugate, this can be done consistently. In the free scalar field action (3.1) this corresponds to limiting the space of all scalar fields by the conditions $\mathbf{q} = \frac{1}{\sqrt{2\pi}} \int_{S^1} dX f(T=0, X) = 0$ and $\mathbf{p} = \frac{1}{\sqrt{2\pi}} \int_{S^1} dX \frac{\partial f(T,X)}{\partial T} = 0$. In the canonical description of PFT in terms of $(\Pi_{\pm}, X^{\pm}, Y^{\pm})$, since $\mathbf{q}$ does not appear, we only need to set the quantity

$$p := \frac{1}{\sqrt{2\pi}} \int_{S^1} dx Y^+(x) = \frac{1}{\sqrt{2\pi}} \int_{S^1} dx Y^-(x) = 0. \quad (3.25)$$

Since, as can easily be checked, p commutes with $(\Pi_\pm, X^\pm, Y^\pm)$ as well as the constraints (3.10), it is consistent to impose (3.25).

To summarize: The system we consider in this work is PFT on $S^1 \times R$ with the zero modes switched off. The phase space variables are $(\Pi_\pm, X^\pm, Y^\pm)$ subject to the conditions of section 3.2.3. The symplectic structure is given by (3.8) and (3.9) and the constraints by (3.10). The degrees of freedom of the theory reside entirely in the mode coefficients $\mathbf{a}_{(\pm)\mathbf{n}}, \mathbf{a}_{(\pm)\mathbf{n}}{}^*$ (3.22) which are expressed as the functions $a_{(\pm)n}, a^*_{(\pm)n}$ on phase space via (3.23).

Conformal Isometries.

Free scalar field theory in 1+1 dimensions (3.1) is conformally invariant. It turns out that the generators of conformal isometries in free scalar field theory are expressible as Dirac observables in PFT(for details, see Reference [30]). Consider the conformal isometry generated by the conformal Killing field $\vec{U}$ on the Minkowskian cylinder. Let $\vec{U}$ have the components $(U^+(X^+), U^-(X^-))$ in the (X^+, X^-) coordinate system. $U^\pm$ are periodic functions of $X^\pm$ by virtue of the fact that $\vec{U}$ is smooth vector field on the flat spacetime $S^1 \times R$ These components of $\vec{U}$ naturally correspond to the functions $(U^+(X^+(x)), U^-(X^-(x)))$ on the phase space of PFT. The Dirac observable in PFT corresponding to the generator of conformal transformations in free scalar field theory associated with $\vec{U}$ is given by

$$\Pi_\pm[U^\pm] \ = \ \int_{S^1} \Pi_\pm(x) U^\pm(X^\pm(x)) \quad (3.26)$$

These observables generate a Poisson algebra isomorphic to that of the commutator algebra of conformal Killing fields:

$$\begin{aligned} \{\Pi_\pm[U^\pm], \Pi_\pm[V^\pm]\} \ &= \ \Pi[[V, U]^\pm] \\ \{\Pi_\pm[U^\pm], \Pi_\mp[V^\mp]\} \ &= 0. \end{aligned} \quad (3.27)$$

Here $[V,U]^{\pm}$ refer to the $\pm$ components of the commutator of the spacetime vector fields $\vec{U}, \vec{V}$, i.e. $[V,U]^{\pm} = V^{\pm}\frac{\partial U^{\pm}}{\partial X^{\pm}} - U^{\pm}\frac{\partial V^{\pm}}{\partial X^{\pm}}$. $[V,U]^{\pm}$ define functions of the embedding variables $X^{\pm}(x)$ in the manner described above.

Note that these observables are weakly equivalent, via the constraints (3.10) to quadratic combinations of the mode functions [30]. In the standard Fock representation of quantum theory (see for e.g. Reference [29]), these quadratic combinations are nothing but the generators of the Virasoro algebra.

As we shall see, the polymer quantization of PFT provides a representation for the finite canonical transformations generated by $\Pi^{\pm}[U^{\pm}]$. For future reference, it is straightforward to check that the Hamiltonian flow, $\alpha_{(\Pi_{\pm}[U^{\pm}],t)}$ generated by $\Pi_{\pm}[U^{\pm}]$ leaves the matter sector of phase space untouched and acts on the embedding variables $X^{\pm}$ as

$$\alpha_{(\Pi_{\pm}[U^{\pm}],t)}X^{\pm}(x) = (\phi_{(\vec{U},t)}X^{\pm})(x). \tag{3.28}$$

Here $\phi_{(\vec{U},t)}$ denotes the one parameter family of conformal isometries generated by the conformal Killing field $\vec{U}$ on spacetime. $\phi_{(\vec{U},t)}$ maps the spacetime point (X^+, X^-) to $\phi_{(\vec{U},t)}X^{\pm}$ and hence maps the spatial slice defined by the canonical data $X^{\pm}(x)$ to the new slice (and hence the new canonical data) $(\phi_{(\vec{U},t)}X^{\pm})(x)$. $\phi_{(\vec{U},t)}$ ranges over all conformal isometries connected to identity. Any such conformal isometry ϕ_c is specified by a pair of functions $\phi_c^{\pm}$ so that $\phi_c(X^+, X^-) := (\phi_c^+(X^+), \phi_c^-(X^-))$. Invertibilty of ϕ_c together with connectedness with identity implies that

$$\frac{d\phi_c^{\pm}}{dX^{\pm}} > 0, \tag{3.29}$$

and the cylindrical topology of spacetime implies that

$$\phi_c^{\pm}(X^{\pm} \pm 2\pi) = \phi_c^{\pm}(X^{\pm}) \pm 2\pi. \tag{3.30}$$

Thus, we may denote the Hamiltonian flows which generate conformal isometries by α_{ϕ_c} or, without loss of generality, by $\alpha_{\phi_c^{\pm}}$ with $\alpha_{\phi_c^{\pm}}$ acting trivially on the $\mp$ sector.

To summarise:$\alpha_{\phi_c^{\pm}}$ leave the matter variables untouched, so that

$$\alpha_{\phi_c^{\pm}}Y^{\pm}(x) = Y^{\pm}(x), \quad \alpha_{\phi_c^{\pm}}Y^{\mp}(x) = Y^{\mp}(x), \tag{3.31}$$

and act on $X^{\pm}(x)$ as

$$\alpha_{\phi_c^{\pm}} X^{\pm}(x) = \phi_c^{\pm}(X^{\pm}(x)), \quad \alpha_{\phi_c^{\pm}} X^{\mp}(x) = X^{\mp}(x). \tag{3.32}$$

Further, since $\Pi_{\pm}[U^{\pm}]$ are observables which commute strongly with the constraints, the corresponding Hamiltonian flows are gauge invariant. This translates to the condition that for all

$$\begin{aligned} \alpha_{\phi_c^{\pm}} \circ \alpha_{\phi^{+}} &= \alpha_{\phi^{+}} \circ \alpha_{\phi_c^{\pm}} \\ \alpha_{\phi_c^{\pm}} \circ \alpha_{\phi^{-}} &= \alpha_{\phi^{-}} \circ \alpha_{\phi_c^{\pm}} \end{aligned} \tag{3.33}$$

where as before $\phi^{\pm}$ label finite gauge transformations.

3.3 Polymer Quantum Kinematics.

3.3.1 Preliminaries.

As in LQG, the polymer quantization is based on suitably defined "holonomies" and the polymer Hilbert space is spanned by suitably defined "charge network" states. In view of the correspondence between finite gauge transformations and periodic diffeomorphisms of $\mathbf{R}$, it is useful to to define periodic and quasiperiodic extensions of charge network labels. Hence we define the following.

Definition 1 : A charge-network s is specified by the labels $(\gamma(s), (j_{e_1}, ..., j_{e_n}))$ consisting of a graph $\gamma(s)$ (by which we mean a finite collection of closed, non-overlapping(except in boundary points) intervals which cover $[0, 2\pi]$) and 'charges' $j_e \in \mathbf{R}$ assigned to each interval e. (Note that $j_e = 0$ is allowed.) Equivalence classes of charge- networks are defined as follows. The graph γ' is said to be finer than graph γ iff every edge of γ is identical to, or composed of, edges in γ'. The charge- network s' is said to be finer than s iff (a) $\gamma(s')$ is finer than $\gamma(s)$ (b) the charge labels of identical edges in $\gamma(s), \gamma(s')$ are identical and the charge labels of the edges of $\gamma(s')$ which compose to yield an edge of $\gamma(s)$ are identical and equal to that of their union in $\gamma(s)$. Two charge- networks are equivalent if there exists a charge- network finer than both. Hence we can represent each equivalence class by a unique representative s such that no two adjacent edges have the same charge. However, unless otherwise mentioned, s will not necessarily denote this unique choice.

Definition 2: The periodic extension of the charge- network s to $\mathbf{R}$ is denoted by s_{ext} and defined as follows.

Given a graph γ as in Definition 1 above, $T_N(\gamma)$ denotes the translation of γ by $2N\pi$, i.e. $T_N(\gamma)$ lies in $[2N\pi, 2(N+1)\pi]$. We define the *extension* of γ to $\mathbf{R}$ as $\gamma_{ext} = \cup_{N\in\mathbf{Z}} T_N(\gamma)$. The *restriction* of γ_{ext} to any interval $I \subset \mathbf{R}$ is denoted by $\gamma_{ext}|_I$ so that $\gamma_{ext}|_{[0,2\pi]} = \gamma$.

Given a charge network $s = (\gamma(s), (j_{e_1}, ..., j_{e_n}))$, s_{ext} is specified by the graph $\gamma(s_{ext}) := \gamma(s)_{ext}$ ($\gamma(s)_{ext}$ denotes the extension of $\gamma(s)$ to $\mathbf{R}$) and charge labels for each edge of $\gamma(s_{ext})$ which are such that $T_N(\gamma(s)) \subset \gamma(s_{ext})$ has the same set of charges which are on γ. Thus

1. On any closed interval $I_N = [2N\pi,\ 2(N+1)\pi]$, $N \in \mathbf{Z}$, $\gamma(s_{ext})|_{I_N}$ is naturally isomorphic to $\gamma(s)$.
2. The set of charges on $\gamma(s_{ext})|_{I_N}$ is $(j_{e_1}, ..., j_{e_n})$.

We refer to $s_{ext}|_{[0,2\pi]}$ as the restriction of s_{ext} to $[0, 2\pi]$ so that $s_{ext}|_{[0,2\pi]} = s$.

Definition 3: The quasi- periodic extension of the charge- network s to $\mathbf{R}$ is denoted by $\bar{s}_{ext}$ and defined as follows. Given a charge network $s = (\gamma(s), (j_{e_1}, ..., j_{e_n}))$, $\bar{s}_{ext}$ is specified by the graph $\gamma(\bar{s}_{ext}) := \gamma(s)_{ext}$ and charge labels for each edge of $\gamma(\bar{s}_{ext})$ which are such that $T_N(\gamma(s)) \subset \gamma(\bar{s}_{ext})$ has the set of charges which are on γ augmented by $2N\pi$. Thus

1. On any closed interval $I_N = [2N\pi,\ 2(N+1)\pi]$, $N \in \mathbf{Z}$, $\gamma(\bar{s}_{ext})|_{I_N}$ is naturally isomorphic to $\gamma(s)$.
2. The set of charges on $\gamma(\bar{s}_{ext})|_{I_N}$ is $(j_{e_1} + 2N\pi, ..., j_{e_n} + 2N\pi)$.

Definition 4: The action of periodic diffeomorphisms with period 2π on $\gamma_{ext}, s_{ext}, \bar{s}_{ext}$ may be defined as follows. Any periodic diffeomorphism ϕ of $\mathbf{R}$ commutes with the 2π translations, T_N. Hence its natural action $\phi(\gamma_{ext})$ on the extension γ_{ext} of graph γ preserves periodicity i.e. $(\phi(\gamma_{ext})|_{[0,2\pi]})_{ext} = \phi(\gamma_{ext})$. Let the edge $\phi(e) \in \phi(\gamma_{ext})$ be the image, by ϕ of the edge $e \in \gamma_{ext}$. The action of ϕ on the extensions $s_{ext}, \bar{s}_{ext}$ is defined by

(i) mapping the underlying graph $\gamma(s)_{ext}$ to $\phi(\gamma(s)_{ext})$

(ii) labelling the edge $\phi(e) \in \phi(\gamma(s)_{ext})$ by the same charge as the edge $e \in \gamma(s)_{ext}$ so that $k_{\phi(e)} = k_e$.

Denote the resulting periodic/quasiperiodic charge networks on $\mathbf{R}$ by $\phi(s_{ext})/\phi(\bar{s}_{ext})$

3.3.2 Embedding sector.

The *- Algebra

The elementary variables which generate the *-Poisson algebra are, $X^+(x), T_{s^+}[\Pi_+]$, $X^-(x), T_{s^-}[\Pi_-]$. Here $T_{s^\pm}[\Pi_\pm]$ are the holonomy- type functions associated with the charge networks $s^\pm$, and are given by

$$T_{s^\pm}[\Pi_\pm] = \prod_{e^\pm \in \gamma(s^\pm)} \exp[-ik^\pm_{e^\pm} \int_{e^\pm} \Pi_\pm]. \tag{3.34}$$

The only non- trivial Poisson brackets are:

$$\begin{aligned}
\{X^\pm(x), T_{s^\pm}[\Pi_\pm]\} &= -ik^\pm_{e^\pm} T_{s^\pm}[\Pi_\pm] \text{ if } x \in \text{Interior}(e^\pm) \\
&= -\tfrac{i}{2}(k^\pm_{e^\pm_{I^\pm}} + k^\pm_{e^\pm_{(I+1)^\pm}}) T^E_{s^\pm}[\Pi_\pm] \\
&\qquad \text{if } x \in e^\pm_{I^\pm} \cap e^\pm_{(I^\pm+1)} \; 1 \le I^\pm \le (n^\pm - 1) \\
\{X^\pm(0), T_{s^\pm}[\Pi_\pm]\} &= \{X^\pm(2\pi), T_{s^\pm}[\Pi_\pm]\} = -\tfrac{i}{2}(k^\pm_{e^\pm_1} + k^\pm_{e^\pm_{n^\pm}}) T_{s^\pm}[\Pi_\pm],
\end{aligned} \tag{3.35}$$

where the last Poisson bracket uses the periodicity of delta function. The *-relations are given by

$$\begin{aligned}
(X^\pm(x))^* &= X^\pm(x) \; \forall \; x \in \; [0, 2\pi] \\
T_{s^\pm}[\Pi_\pm]^* &= T_{-s^\pm}[\Pi_\pm], \; -s^\pm = (\gamma(s^\pm), (-k^\pm_{e^\pm_1}, ..., -k^\pm_{e^\pm_{n^\pm}}))
\end{aligned} \tag{3.36}$$

The action of finite gauge transformations on these elementary functions is as follows (we only analyze the right-moving sector; the analysis of the left moving sector is identical).

From equation (3.18) we have,

$$\alpha_{\phi^+} T_{s^+}[\Pi_+] = T_{s^+}[(\phi^+)_* \Pi_+]. \tag{3.37}$$

It is straightforward to check, using the periodicity of ϕ^+, Π_+, s^+_{ext} and the various definitions in section 3.3.1 that

$$T_{s^+}[(\phi^+)_* \Pi_+] = T_{\phi^+(s^+_{ext})|_{[0,2\pi]}}[\Pi_+]. \tag{3.38}$$

Finite gauge transformations act on $X^\pm$ as in equations (3.16), (3.18). To summarise, under finite gauge transformations the generators of the Poisson algebra transform as:

$$\begin{aligned}
\alpha_{\phi^\pm}(X^\pm)(x) &= X^\pm_{ext}((\phi^\pm)(x)) \;=\; X^\pm(y^\pm) \;\pm 2\pi\; N^\pm \\
&\qquad\qquad \text{if } (\phi^\pm)(x) = y^\pm + 2\pi N^\pm \; y^\pm \in \; [0, 2\pi] \\
\alpha_{\phi^\mp}(X^\pm)(x) &= X^\pm(x) \\
\alpha_{\phi^\pm}(T_{s^\pm}[\Pi^\pm]) &= T^\pm_{\phi^\pm(s^\pm_{ext})|_{[0,2\pi]}}[\Pi^\pm] \\
\alpha_{\phi^\mp}(T_{s^\pm}[\Pi^\pm]) &= T_{s^\pm}[\Pi^\pm]
\end{aligned} \tag{3.39}$$

Representation of the *- Algebra

Denote the kinematic Hilbert space for the $\pm$ embedding sectors by $\mathcal{H}^\pm_E$. $\mathcal{H}^\pm_E$ is the closure of the span of the orthonormal basis of embedding 'charge network states'. Each such state is labelled by a charge network $s^\pm$ and denoted by $T_{s^\pm}$. [1] The inner product is

$$< T_{s^\pm},\; T_{s'^\pm} > = \; \delta_{s^\pm, s'^\pm} \tag{3.40}$$

where $\delta_{s^\pm,s'^\pm}$ is a Kronecker delta function which is unity when the two charge networks are equivalent and vanishes otherwise.

The '$\pm$' sector operators corresponding to the elementary functions of the previous section are denoted by $\hat{X}^\pm(x), \hat{T}_{s^\pm}$. $\hat{T}_{s^\pm}$ acts on the charge network states as:

$$\hat{T}_{s^\pm} T_{s'^\pm} \;:=\; T_{s^\pm + s'^\pm} \tag{3.41}$$

where $s^\pm + s'^\pm$ is the charge network obtained by choosing its underlying graph to be finer than $\gamma(s^\pm)$, $\gamma(s^{\pm\prime})$ dividing $\gamma(s^\pm)$, $\gamma(s'^\pm)$ and assigning charge $k^\pm_{e^\pm} + k^\pm_{e'^\pm}$

[1] More precisely, the labelling is by the equivalence class of $s^\pm$ as in Definition 1, section 3.3.1

to $e^{\pm} \cap e'^{\pm}$ where $e^{\pm} \in \gamma(s^{\pm})$, $e'^{\pm} \in \gamma(s_1^{\pm})$.

The action of $\hat{X}^{\pm}(x)$ is:

$$\hat{X}^{\pm}(x) T_{s^{\pm}} := \lambda_{x,s^{\pm}} T_{s^{\pm}}, \tag{3.42}$$

where, for $\gamma(s^{\pm})$ with $n^{\pm}$ edges,

$$\begin{aligned} \lambda_{x,s^{\pm}} &:= \hbar k^{\pm}_{e^{\pm}_{I^{\pm}}} T_{s^{\pm}} \; if \; x \in \text{Interior}(e^{\pm}_{I^{\pm}}),\; 1 \leq I^{\pm} \leq n^{\pm} \\ &:= \tfrac{\hbar}{2}(k^{\pm}_{e^{\pm}_{I^{\pm}}} + k^{\pm}_{e^{\pm}_{(I+1)^{\pm}}}) T_{s^{\pm}} \text{ if } x \in e^{\pm}_{I^{\pm}} \cap e^{\pm}_{(I^{\pm}+1)} \; 1 \leq I^{\pm} \leq (n^{\pm}-1) \end{aligned} \tag{3.43}$$

$$\begin{aligned} &:= \tfrac{\hbar}{2}(k^{\pm}_{e^{\pm}_{n^{\pm}}} \mp \tfrac{2\pi}{\hbar} + k^{\pm}_{e^{\pm}_{1}}) T_{s^{\pm}} \text{ if } x = 0 \\ &:= \tfrac{\hbar}{2}(k^{\pm}_{e^{\pm}_{1}} \pm \tfrac{2\pi}{\hbar} + k^{\pm}_{e^{\pm}_{n^{\pm}}}) T_{s^{\pm}} \text{ if } x = 2\pi \end{aligned} \tag{3.44}$$

The last two equations, (3.44), implement the boundary condition $X^{\pm}(2\pi) - X^{\pm}(0) = \pm 2\pi$ (see **(i)** of section 3.2.3.)

It is straightforward to check that equations (3.41),(3.42),(3.43),(3.44) provide a representation of the Poisson bracket algebra (3.35) so that quantum commutators equal $i\hbar$ times the Poisson brackets. It is also straightforward to verify that the *- relations (3.36) on $\hat{X}^{\pm}(x), \hat{T}_{s^{\pm}}$ are implemented by the inner product (3.40) so that $\hat{X}^{\pm}(x)$ are self adjoint and $\hat{T}_{s^{\pm}}$ are unitary.

Unitary representation of finite gauge transformations.

Since the Hamiltonian flows of $\alpha_{\phi^{\pm}}$ (3.18) are real, the corresponding quantum operators $\hat{U}(\phi^{\pm})$ must be unitary. Equations (3.18), (3.19) imply that this unitary representation must satisfy

$$\begin{aligned} \hat{U}^{\pm}(\phi_1^{\pm})\hat{U}^{\pm}(\phi_2^{\pm}) &= \hat{U}^{\pm}(\phi_1^{\pm} \circ \phi_2^{\pm}) \\ \hat{U}^{\pm}(\phi^{\pm})\hat{X}^{\pm}(x)\hat{U}^{\pm}(\phi^{\pm})^{-1} &= \hat{X}^{\pm}(y^{\pm}) \pm 2\pi N^{\pm} \\ \hat{U}^{\pm}(\phi^{\pm})\hat{T}_{s^{\pm}}\hat{U}^{\pm}(\phi^{\pm})^{-1} &= \hat{T}_{\phi^{\pm}(s^{\pm})_{ext}|[0,2\pi]}. \end{aligned} \tag{3.45}$$

where $\phi^{\pm}(x) = y^{\pm} + 2\pi N^{\pm}$, with $y^{\pm} \in [0, 2\pi]$ and $N^{\pm} \in \mathbf{Z}$.
We define the action of $\hat{U}(\phi^{\pm})$ to be

$$\begin{aligned} \hat{U}^{\pm}(\phi^{\pm})T_{s^{\pm}} &:= T_{\phi(\overline{s}^{\pm}_{ext})|_{[0,2\pi]}} \\ \hat{U}^{\mp}(\phi^{\mp})T_{s^{\pm}} &:= T_{s^{\pm}}. \end{aligned} \tag{3.46}$$

The appearance of the quasi-periodic extensions $\overline{s}^{\pm}_{ext}$ of the charge networks $s^{\pm}$ (see Definition 3, section 3.3.1) in the first equation above may be anticipated from the quasi- periodic nature of the embedding variables $X^{\pm}(x)$ (3.15). Unitarity of $\hat{U}^{\pm}(\phi^{\pm})$ follows straightforwardly:

$$\begin{aligned} < \hat{U}^{\pm}(\phi^{\pm})T_{s_1^{\pm}} \, , \, \hat{U}^{\pm}(\phi^{\pm})T_{s_2^{\pm}} > &= < T_{\phi(\overline{s}^{\pm}_{1ext})|_{[0,2\pi]}} \, , \, T_{\phi(\overline{s}^{\pm}_{2ext})|_{[0,2\pi]}} > \\ &= \delta_{\phi^{\pm}(\overline{s}^{\pm}_{1ext})|_{[0,2\pi]},\ \phi^{\pm}(\overline{s}^{\pm}_{2ext})|_{[0,2\pi]}} \ \forall \ \phi^{\pm} \\ &= \delta_{s_1^{\pm}, s_2^{\pm}} \end{aligned} \tag{3.47}$$

where we have used the fact that two charge-networks are equal on $[0, 2\pi]$ iff their extensions are equal.

From equation (3.46) and Definitions 3,4 of section 3.3.1, it follows that

$$\begin{aligned} \hat{U}^{\pm}(\phi_1^{\pm})\hat{U}^{\pm}(\phi_2^{\pm}) \, T_{s^{\pm}} &= T_{\phi_1^{\pm}\overline{(\phi_2^{\pm}(\overline{s^{\pm}}_{ext})|_{[0,2\pi]})}_{ext}|_{[0,2\pi]}} \\ &= T_{\phi_1^{\pm}(\phi_2^{\pm}(\overline{s}^{\pm}_{ext}))|_{[0,2\pi]}} \\ &= T_{(\phi_1^{\pm}\circ\phi_2^{\pm})(\overline{s}^{\pm}_{ext})|_{[0,2\pi]}} \\ &= \hat{U}^{\pm}(\phi_1^{\pm} \circ \phi_2^{\pm})T_{s^{\pm}}, \end{aligned} \tag{3.48}$$

thus verifying the first relation in (3.45).

Next, we turn to the second relation of (3.45). We sketch the proof for the '+' sector; the proof for the '-' sector is on similar lines. From (3.46) and (3.42) we have that:

$$\begin{aligned} \hat{U}^{+}(\phi^{+})\hat{X}^{+}(x)\hat{U}^{+}(\phi^{+})^{-1} \, T_{s^{+}} &= \hat{U}^{+}(\phi^{+})\hat{X}^{+}(x)T_{(\phi^{+})^{-1}(\overline{s}^{+}_{ext})|_{[0,2\pi]}} \\ &= \lambda_{x,(\phi^{+})^{-1}(\overline{s}^{+}_{ext}|_{[0,2\pi]})}T_{s^{+}}. \end{aligned} \tag{3.49}$$

It is straightforward to see that

$$\lambda_{x,(\phi^+)^{-1}(\overline{s}^+_{ext}|_{[0,2\pi]})} \;=\; \lambda_{y^+,s^+} + 2\pi N^+, \tag{3.50}$$

which via equation (3.42) obtains the desired result.

Finally, we turn to the last relation of (3.45). Once again, we sketch the proof for the '+' sector; the '-' sector proof follows analogously. We want to show that

$$\hat{U}^+(\phi^+)\hat{T}_{s^+}\hat{U}^+((\phi^+)^{-1}) \;=\; \hat{T}_{\phi^+(s^+_{ext})|_{[0,2\pi]}}. \tag{3.51}$$

Since charge network states form an orthonormal basis in the Hilbert space, it follows that (3.51) is equivalent to the condition that $\forall s_1^+, s_2^+$

$$\langle T_{(\phi^+)^{-1}(\overline{s^+}_1)_{ext}|_{[0,2\pi]}}|\,\hat{T}_{s^+}|\,T_{(\phi^+)^{-1}(\overline{s^+}_2)_{ext}|_{[0,2\pi]}}\rangle = \langle T_{s_1^+}|\,\hat{T}_{\phi^+(s^+_{ext})|_{[0,2\pi]}}|T_{s_2^+}\rangle, \tag{3.52}$$

which from equation (3.41) is, in turn, equivalent to the equation

$$\delta_{(\phi^+)^{-1}(\overline{s^+}_1)_{ext}|_{[0,2\pi]},s^+ + (\phi^+)^{-1}(\overline{s^+}_2)_{ext}|_{[0,2\pi]}} \;=\; \delta_{s_1^+,\phi^+(s^+_{ext})|_{[0,2\pi]}+s_2^+}. \tag{3.53}$$

However, (suppressing the '+' superscript), we have that

$$\begin{aligned}\delta_{\phi^{-1}(\overline{s}_1)_{ext}|_{[0,2\pi]},s+\phi^{-1}(\overline{s}_2)_{ext}|_{[0,2\pi]}} &= \delta_{\phi^{-1}(\overline{s}_1)_{ext},s_{ext}+\phi^{-1}(\overline{s}_2)_{ext}}\\ &= \delta_{(\overline{s}_1)_{ext},\phi(s_{ext})+(\overline{s}_2)_{ext}}\\ &= \delta_{(s_1)_{ext},\phi(s_{ext})+(s_2)_{ext}}\\ &= \delta_{s_1,\phi(s_{ext})|_{[0,2\pi]}+s_2},\end{aligned} \tag{3.54}$$

thus proving (3.51).

3.3.3 Matter sector.

The *- Algebra.

The *- Algebra is generated by the operators corresponding to the classical holonomies $W_{s^\pm}[Y^\pm]$ which are defined as

$$W_{s^\pm}[Y^\pm] \;=\; \exp[i \sum_{e^\pm \in E(\gamma(s^\pm))} l^\pm_{e^\pm} \int_{e^\pm} Y^\pm]. \tag{3.55}$$

Here $s^{\pm} := \{\gamma(s^{\pm}),\ (l^{\pm}_{e^{\pm}_1}, ..., l^{\pm}_{e^{\pm}_{m}})\ \}$ are charge- networks. The algebra for the holonomy operators is the analog of the Weyl algebra for linear quantum fields. Similar to that case, we need to first evaluate the Poisson brackets, $\{\sum_{e^{\pm}} l^{\pm}_{e^{\pm}} \int_{e^{\pm}} Y^{\pm},\ \sum_{e'^{\pm}} l^{\pm}_{e'^{\pm}} \int_{e'^{\pm}} Y^{\pm}\}$, between the exponents of pairs of classical holonomies and then use the Baker- Campbell- Hausdorff Lemma [38] to define the algebra on the holonomy operators in quantum theory.

Let κ_e be the characteristic function associated with a closed interval e and denote the beginning and final points of e by $b(e)$ and $f(e)$ so that

$$\begin{aligned} \kappa_e(x) &= 1 \text{ if } x \in \text{Interior}(e) \\ &= \tfrac{1}{2} \text{ if } x = b(e) \text{ or } f(e) \end{aligned} \tag{3.56}$$

$$\begin{aligned} &= \tfrac{1}{2} \text{ if } x = 0 \text{ and } f(e) = 2\pi \\ &= \tfrac{1}{2} \text{ if } x = 2\pi \text{ and } b(e) = 0. \end{aligned} \tag{3.57}$$

Here, equations (3.57) follow from the periodicity of the delta function. From equation (3.9) it follows that

$$\{\int_{e^{\pm}} Y^{\pm},\ \int_{e'^{\pm}} Y^{\pm}\} = \pm\alpha(e^{\pm}, e'^{\pm}) := \pm(\kappa_{e'^{\pm}}|_{\partial_{e^{\pm}}} - \kappa_{e^{\pm}}|_{\partial_{e'^{\pm}}}), \tag{3.58}$$

where ∂_e refers to the boundary of e and,

$$\kappa_e|_{\partial_{e'}} := \kappa_e(f(e')) - \kappa_e(b(e')), \tag{3.59}$$

so that

$$\{\sum_{e^{\pm}} l^{\pm}_{e^{\pm}} \int_{e^{\pm}} Y^{\pm},\ \sum_{e'^{\pm}} l^{\pm}_{e'^{\pm}} \int_{e'^{\pm}} Y^{\pm}\} = \pm \sum_{e^{\pm}, e'^{\pm}} l^{\pm}_{e^{\pm}} l^{\pm}_{e'^{\pm}} \alpha(e^{\pm}, e'^{\pm}). \tag{3.60}$$

It follows that the 'Weyl algebra' of holonomy operators is:

$$\begin{aligned} \hat{W}(s^{\pm})\hat{W}(s'^{\pm}) &= \exp[\mp\tfrac{i\hbar}{2}\alpha(s^{\pm}, s'^{\pm})]\hat{W}(s^{\pm} + s'^{\pm}), \\ \hat{W}(s^{\pm})^* &= \hat{W}(-s^{\pm}), \end{aligned} \tag{3.61}$$

where

$$\alpha(s^{\pm}, s'^{\pm}) := \sum_{e^{\pm}\in\gamma(s^{\pm})} \sum_{e'^{\pm}\in\gamma(s'^{\pm})} l_e^{\pm} l_{e'^{\pm}}^{\pm} \alpha(e^{\pm}, e'^{\pm}), \tag{3.62}$$

with $\alpha(e, e')$ defined through equations (3.59) and (3.58). From the second equation of (3.9), it follows that the '+' and '-' holonomy operators commute, so that, once again, these sectors can be treated independently.

Representation of the *- Algebra.

It is convenient to define the quantum theory through the Gelfand- Naimark - Segal (GNS) construction [62]. The explicit operator action on the basis of charge network states is provided after we present the GNS state.

We define the GNS states $\omega_M^{\pm}$ on the $\pm$ holonomy algebras by specifying their action on the holonomy operators as follows:

$$\omega_M^{\pm}(\hat{W}(s^{\pm})) = \delta_{s^{\pm},\circ}. \tag{3.63}$$

Here '$\circ$' is the trivial charge network which may be represented by graph $\gamma(\circ)$ consisting of the single edge $e = [0, 2\pi]$ with vanishing charge $l_e^{\pm} = 0$. The Kronecker delta function $\delta_{s^{\pm},\circ}$ is unity iff $s^{\pm} = \circ$ and vanishes otherwise. It follows from the GNS construction that the corresponding GNS Hilbert spaces $\mathcal{H}_M^{\pm}$ are spanned by charge network states denoted by $W(s^{\pm})$. The inner product is

$$< W(s^{\pm}), W(s'^{\pm}) >_{\pm} = \delta_{s^{\pm},s'^{\pm}} \tag{3.64}$$

and the action of the holonomy operators is

$$\hat{W}(s^{\pm})W(s'^{\pm}) = \exp[\mp\frac{i\hbar\alpha(s^{\pm}, s'^{\pm})}{2}]W(s^{\pm} + s'^{\pm}). \tag{3.65}$$

Here, as for the embedding sector, $s^{\pm} + s'^{\pm}$ is defined as in (3.41).[2]

It is straightforward to check, explicitly, that equation (3.65) provides a representation for the first equation of (3.61). Verification of the second equation of

[2]While our notation uses charge network labels, the operators $\hat{W}(s^{\pm})$ and states $W(s^{\pm})$ only depend on the equivalence classes of labels. See also Footnote 1 in this regard.

(3.61) is equivalent to showing that $\forall s^{\pm}, s'^{\pm}, s''^{\pm}$,

$$< W(s'^{\pm}),\ (\hat{W}(s^{\pm}))^{\dagger} W(s''^{\pm})\ >_{\pm} = <\ W(s'^{\pm}),\ \hat{W}(-s^{\pm}) W(s''^{\pm})\ >_{\pm} . \tag{3.66}$$

Equation (3.66) follows straightforwardly from (3.64),(3.65). One needs to use the identity $\delta_{s^{\pm}, -s'^{\pm}+s''^{\pm}} = \delta_{s^{\pm}+s'^{\pm}, s''^{\pm}}$ and the easily verifiable fact that $\alpha(s^{\pm}, s'^{\pm})$ is bilinear and antisymmetric in its arguments.

Unitary representation of finite gauge transformations.

Since $Y^{\pm}$ are periodic scalar densities, under finite gauge transformations their holonomies transform in a similar manner to those of the embedding momenta. Specifically, equation (3.18) in conjunction with the periodicity of $\phi^{\pm}, Y^{\pm}, s^{\pm}_{ext}$ and the various definitions of section 3.3.1, imply that

$$\alpha_{\phi^{\pm}} W_{s^{\pm}}[Y^{\pm}] \ := \ W_{(\phi^{\pm})(s^{\pm}_{ext})|_{[0,2\pi]}}[Y^{\pm}]. \tag{3.67}$$

It is straightforward to see (either explicitly from equation (3.62) or abstractly using the fact that the periodicity of $\phi^{\pm}, Y^{\pm}, s^{\pm}_{ext}$ implies that one is effectively restricting attention to diffeomorphisms, graphs, charge networks and holonomies on S^1) that

$$\alpha(s^{\pm}, s'^{\pm}) = \alpha(\phi^{\pm}(s^{\pm}_{ext})|_{[0,2\pi]}, \phi^{\pm}(s'^{\pm}_{ext})|_{[0,2\pi]}.) \tag{3.68}$$

Equations (3.65) and (3.68) imply that the Hamiltonian flow of (3.67) induces an automorphism of the Weyl algebra of holonomies. Note also that equation (3.63) is invariant under the action of this automorphism. This directly implies that the group of finite gauge transformations is unitarily represented in the quantum theory. Let these unitary operators be denoted, as in the embedding sector, by $\hat{U}^{\pm}(\phi^{\pm})$. Their explicit action on the charge network basis can be defined from the GNS construction to be

$$\hat{U}^{\pm}(\phi^{\pm}) W(s^{\pm}) \ := \ W((\phi^{\pm})(s^{\pm}_{ext})|_{[0,2\pi]}). \tag{3.69}$$

3.3.4 The kinematic Hilbert space.

The kinematic Hilbert space $\mathcal{H}_{kin}$ is the product of the Hilbert spaces $\mathcal{H}^{\pm}_{kin}$ with

$$\mathcal{H}^{\pm}_{kin} = (\mathcal{H}^{\pm}_{E} \otimes \mathcal{H}^{\pm}_{M}) \tag{3.70}$$

so that

$$\mathcal{H}_{kin} = (\mathcal{H}^{+}_{E} \otimes \mathcal{H}^{+}_{M}) \otimes (\mathcal{H}^{-}_{E} \otimes \mathcal{H}^{-}_{M}). \tag{3.71}$$

$\mathcal{H}^{\pm}_{kin}$ is spanned by an orthonormal basis of equivalence classes of charge network states of the form $T_{s^{\pm}} \otimes W(s'^{\pm})$ with $s^{\pm} = \{\gamma(s^{\pm}), (k^{\pm}_{e^{\pm}_1}, ..., k^{\pm}_{e^{\pm}_{n^{\pm}}})\}$, $s'^{\pm} = \{\gamma(s'^{\pm}), (l^{\pm}_{e'^{\pm}_1}, ..., l^{\pm}_{e'^{\pm}_{m^{\pm}}})\}$.

The results of the previous subsections show that $\mathcal{H}_{kin}$ supports a *- representation of the *- algebras for the matter and embedding degrees of freedom, as well as a unitary representation of finite gauge transformations.

Consider, as above, the state $T_{s^{\pm}} \otimes W(s'^{\pm})$. The equivalence relation between charge networks is defined in Definition 1, section 3.3.1. Using this equivalence, it is straightforward to see that we can always choose $s^{\pm}, s'^{\pm}$ such that $\gamma(s^{\pm}) = \gamma(s'^{\pm})$. Then each edge $e^{\pm}$ of $\gamma(s^{\pm})$ is labelled by a pair of real charges $(k^{\pm}_e, l^{\pm}_e)$. Note that such a choice of graph and charge pairs is not unique. However it is easy to see that a unique choice can be made if we require that the pairs of charges, $(k^{\pm}_{e^{\pm}}, l^{\pm}_{e^{\pm}})$, are such that no two consecutive edges are labelled by the same pair of charges. We shall denote this unique labelling by $\mathbf{s}^{\pm}$ so that

$$\mathbf{s}^{\pm} := \{\gamma(\mathbf{s}^{\pm}), (k^{\pm}_{e^{\pm}_1}, l^{\pm}_{e^{\pm}_1}), ..., (k^{\pm}_{e^{\pm}_{n^{\pm}}}, l^{\pm}_{e^{\pm}_{n^{\pm}}})\}, \tag{3.72}$$

with

$$k_{e^{\pm}_{I^{\pm}}} \neq k_{e^{\pm}_{(I^{\pm}+1)}} \text{ or/and } l_{e^{\pm}_{I^{\pm}}} \neq l_{e^{\pm}_{(I^{\pm}+1)}}. \tag{3.73}$$

The corresponding charge network state is denoted by $|\mathbf{s}^{\pm}\rangle$ so that

$$|\mathbf{s}^{\pm}\rangle = T_{s^{\pm}} \otimes W(s'^{\pm}) \tag{3.74}$$

with $\mathbf{s}^{\pm}$ defined from $s^{\pm}, s'^{\pm}$ in the manner discussed above. It follows from (3.46) and (3.69) that $\hat{U}^{\pm}(\phi^{\pm})$ maps $|\mathbf{s}^{\pm}\rangle$ to a new charge network state. We denote the new (unique) charge network label by $\mathbf{s}^{\pm}_{\phi^{\pm}}$ so that

$$|\mathbf{s}^{\pm}_{\phi^{\pm}}\rangle := \hat{U}^{\pm}(\phi^{\pm})|\mathbf{s}^{\pm}\rangle. \tag{3.75}$$

3.4 Unitary representation of Dirac observables.

3.4.1 Exponentials of mode functions.

Whereas $a_{(\pm)n}$ (3.23) depend on $Y^{\pm}(x)$, the basic operators of quantum theory are the holonomies $\hat{W}(s^{\pm})$. As in LQG, the representation of the holonomy operators on $\mathcal{H}_{kin}$ is not regular enough to allow a definition of $\hat{Y}^{\pm}(x)$ via a "shrinking of edges" procedure [34]. For example, let $s^{\pm}(t)$ be a 1 parameter family of charge networks such that $\gamma(s^{\pm}(t))$ has non- vanishing unit charge on only one of its edges. Let this edge contain x and let its coordinate length be t. Whereas, classically, $Y^{\pm}(x) = \lim_{t\to 0}\frac{W(s^{\pm}(t))-1}{it}$, it is easy to check that, as in LQG, the corresponding operators are not weakly continuous in t and the limit cannot be defined on the charge network basis. This leads to a regularization dependence in the definition of $\hat{a}_{(\pm)n}$ [34]. However, as we show below, suitably defined exponential functions of $a_{(\pm)n}, a^{*}_{(\pm)n}$ *can* be promoted to quantum operators in a regularization/triangulation independent manner. Let q_n, p_n be the real and imaginary parts of $a_{(\pm)n}$ so that

$$\begin{aligned} q_{(\pm)n} &= \int_{S^1} Y^{\pm}(x)\cos(nX^{\pm}(x)), \\ p_{(\pm)n} &= \int_{S^1} Y^{\pm}(x)\sin(nX^{\pm}(x)), \end{aligned} \tag{3.76}$$

and consider the functions

$$\begin{aligned} e^{i\alpha q_{(\pm)n}} &= e^{i\alpha\int_{S^1} Y^{\pm}(x)\cos(nX^{\pm}(x))} \\ e^{i\beta p_{(\pm)n}} &= e^{i\beta\int_{S^1} Y^{\pm}(x)\sin(nX^{\pm}(x))} \end{aligned} \tag{3.77}$$

where $\alpha,\ \beta\ \in \mathbf{R}$. These functions can be promoted to quantum operators as follows.[3]

[3] As an aside let us note that physically the exponentiatals of mode functions look rather artificial. However exponential of a Klein-Gordon scalar field on $S^1 \times \mathbf{R}$ has a beautiful geometric interpretation as a representative of certain differential cohomology classes of a cylinder [18]. Such cohomologies are known as Cheeger-Simons cohomologies.

Let $f(X^{\pm})$ be a smooth periodic *real* function of $X^{\pm}$. Then $O^{\pm}_f := \int_{S^1} Y^{\pm}(x) f(X^{\pm}(x))$ are functions on the phase space of PFT. Next, restrict *attention to the embedding sector Hilbert space* $\mathcal{H}^{\pm}_E$ *and consider the operator* valued (on $\mathcal{H}^{\pm}_E$) function on the matter phase space, $O^{\pm}_{\hat{f}} := \int_{S^1} Y^{\pm}(x) f(\hat{X}^{\pm}(x))$. Since charge network states are eigen states of the embedding operator, we have that

$$O^{\pm}_{\hat{f}} T_{s^{\pm}} = (\sum_{i=1}^{n^{\pm}} f(\hbar k^{\pm}_{e^{\pm}_i}) \int_{e^{\pm}_i} Y^{\pm}(x)) T_{s^{\pm}}, \tag{3.78}$$

where $s^{\pm} = \{\gamma(s^{\pm}), (k^{\pm}_{e^{\pm}_1}, .., k^{\pm}_{e^{\pm}_{n^{\pm}}})\}$ and that,

$$\begin{aligned} e^{iO^{\pm}_{\hat{f}}} T_{s^{\pm}} &= e^{i\sum_{i=1}^{n^{\pm}} f(\hbar k^{\pm}_{e^{\pm}_i}) \int_{e^{\pm}_i} Y^{\pm}(x)} T_{s^{\pm}}, \\ &= W(s^{\pm}_f)[Y^{\pm}] T_{s^{\pm}}, \end{aligned} \tag{3.79}$$

where $s^{\pm}_f := \{\gamma(s^{\pm}), (f(\hbar k^{\pm}_{e^{\pm}_1}), ..., f(\hbar k^{\pm}_{e^{\pm}_{n^{\pm}}}))\}$. Equation (3.79) implies that we can define the operators $\widehat{\exp iO^{\pm}_f}$ corresponding to the functions $\exp iO^{\pm}_f$ via their action on the charge network states $T_{s^{\pm}} \otimes W(s'^{\pm}) \in \mathcal{H}^{\pm}$:

$$(\widehat{\exp iO^{\pm}_f}) T_{s^{\pm}} \otimes W(s'^{\pm}) := T_{s^{\pm}} \otimes \hat{W}(s^{\pm}_f) W(s'^{\pm}). \tag{3.80}$$

Clearly, this is a manifestly regularization/triangulation independent definition. Moreover, since $s^{\pm}_f$ is constructed from the embedding part of the charge network, and since f is periodic, it is straightforward to check that $\widehat{e^{iO^{\pm}_f}}$ commute with the unitary operators corresponding to finite gauge transformations. Hence $\widehat{e^{iO^{\pm}_f}}$ are Dirac observables in quantum theory. It is also easy to check that

$$(\widehat{\exp iO^{\pm}_f})^{\dagger} = (\widehat{\exp iO^{\pm}_f})^{-1} = (\widehat{\exp iO^{\pm}_{-f}}) \tag{3.81}$$

so that the classical reality conditions are implemented.

By setting f to be the appropriate cosine (sine) function times α (β), we obtain the operators corresponding to the functions in equation (3.77). Clearly, these operators ($\forall \alpha, \beta \in \mathbf{R}, n > 0$) form an over- complete set of Dirac observables.

3.4.2 Conformal Isometries.

Regularization dependence also manifests in attempts to promote the generators of conformal isometries, $\Pi^{\pm}[U^{\pm}]$ (see equation (3.26), to operators on $\mathcal{H}_{kin}$. Choosing exponentials of these observables only partially alleviates this problem since (unlike the case of $a_{(\pm)n}$) the resulting operator suffers from operator ordering problems stemming from the fact that $\{\Pi_{\pm}(x), U^{\pm}(X^{\pm}(x))\} \neq 0$. Therefore, we focus on the Hamiltonian flows corresponding to finite conformal isometries.

The action of the Hamiltonian flows (corresponding to conformal isometries), $\alpha_{\phi_c^{\pm}}$, on $(X^{\pm}(x), Y^{\pm}(x))$ has been detailed in section 3.2.4. It remains to specify their action on the embedding momenta, $\Pi_{\pm}(x)$. The information in this specification can equally well be seeded in the action of $\alpha_{\phi_c^{\pm}}$ on the Hamiltonian flows $\alpha_{\phi^{\pm}}$ corresponding to finite gauge transformations by virtue of the facts that (a) the constraints (3.10) are linear in the embedding momenta and (b) this linear dependence is invertible by virtue of the non- degeneracy condition **(iv)** of section 3.2.3. Thus $\alpha_{\phi_c^{\pm}}$ are completely specified through equations (3.31),(3.32),(3.33). Accordingly, we seek a unitary representation of $\alpha_{\phi_c^{\pm}}$ by operators $\hat{V}(\phi_c^{\pm})$ such that $\hat{V}^{\pm}(\phi_c^{\pm})$ act trivially on the matter sector, commute with the operators $\hat{U}^{+}(\phi^{+})$ and $\hat{U}^{-}(\phi^{-})$ which implement gauge transformations, and transform $\hat{X}^{\pm}(x)$ through

$$\hat{V}^{\pm}(\phi_c^{\pm})\hat{X}^{\pm}(x)(\hat{V}^{\pm})^{\dagger}(\phi_c^{\pm}) = \phi_c^{\pm}(\hat{X}^{\pm}(x)), \tag{3.82}$$

while leaving $\hat{X}^{\mp}(x)$ invariant.

We define $\hat{V}^{\pm}(\phi_c^{\pm})$ to act trivially on the matter Hilbert spaces $\mathcal{H}_M^{+}, \mathcal{H}_M^{-}$ and on the $\mp$ embedding Hilbert space $\mathcal{H}_E^{\mp}$. The action of $\hat{V}^{\pm}(\phi_c^{\pm})$ on $\mathcal{H}_E^{\pm}$ is defined as follows. Let $s = \{\gamma(s)(k^{\pm}_{e_1^{\pm}}, ..., k^{\pm}_{e_n^{\pm}})\}$ be a charge network. Define the charge networks $\phi_c^{+}(s^{+}), \phi_c^{-}(s^{-})$ by

$$\phi_c^{\pm}(s^{\pm}) := \{\gamma(s^{\pm}), (\phi_c^{\pm}(k^{\pm}_{e_1^{\pm}}), ..., \phi_c^{\pm}(k^{\pm}_{e_n^{\pm}}))\}. \tag{3.83}$$

Then the action of $\hat{V}(\phi_c^{\pm})$ on the charge network state $T_{s^{\pm}} \in \mathcal{H}_E^{\pm}$ is defined to be

$$\hat{V}^{\pm}[\phi_c^{\pm}]T_{s^{\pm}} \;=\; T_{(\phi_c^{\pm})^{-1}(s^{\pm})}. \tag{3.84}$$

To reiterate, in the notation (3.83) we have that

$(\phi_c^{\pm})^{-1}(s^{\pm}) = \{\gamma(s^{\pm}), ((\phi_c^{\pm})^{-1}(k^{\pm}_{e_1^{\pm}}), ..., (\phi_c^{\pm})^{-1}(k_{e_n^{\pm}}))\}$.

From equation (3.84), the invertibility of the functions $\phi_c^{\pm}$ (which follows from equation (3.29)) and the inner product (3.40), it follows that $\langle \hat{V}^{\pm}[\phi_c^{\pm}]T_{s^{\pm}} | \hat{V}^{\pm}[\phi_c^{\pm}]T_{s'^{\pm}} \rangle = \langle T_{s^{\pm}} | T_{s'^{\pm}} \rangle \; \forall s^{\pm}, s'^{\pm}$, thus showing unitarity. It is also straightforward to check, using the quasiperiodicity of the functions $\phi_c^{\pm}$ (3.30), that $\hat{V}^{\pm}[\phi_c^{\pm}]$ commutes with $\hat{U}(\phi^{\pm})$. By definition $\hat{V}^{\pm}[\phi_c^{\pm}]$ commutes with $\hat{U}(\phi^{\mp})$ and with the matter holonomies. Finally, it is easy to check that equation (3.82) holds when applied on any charge network state. Thus, our definition of $\hat{V}^{\pm}[\phi_c^{\pm}]$ provides a satisfactory definition of conformal isometries in quantum theory.

Note also that equation (3.84) implies that

$$\hat{V}^{\pm}[\phi_{1c}^{\pm}]\hat{V}^{\pm}[\phi_{2c}^{\pm}] = \hat{V}^{\pm}[\phi_{2c}^{\pm} \circ \phi_{1c}^{\pm}], \tag{3.85}$$

so that our definition of $\hat{V}^{\pm}[\phi_c^{\pm}]$ implies an anomaly free representation (by right multiplication) of the group of conformal isometries.

3.5 Physical state space by Group Averaging.

Only gauge invariant states are physical so that physical states Ψ must satisfy the condition $\hat{U}^{\pm}(\phi^{\pm})\Psi = \Psi, \; \forall \phi^{\pm}$. A formal solution to this condition is to fix some $|\psi\rangle \in \mathcal{H}_{kin}$ and set $\Psi = \sum |\psi'\rangle$ where the sum is over all distinct $|\psi'\rangle$ which are gauge related to ψ. A mathematically precise implementation of this idea places the gauge invariant states in the dual representation (corresponding to a formal sum over bras rather than kets) and goes by the name of Group Averaging. The "Group" is that of gauge transformations and the "Averaging" corresponds to the construction of a gauge invariant state from a kinematical one by giving meaning to the formal sum over gauge related states. Specifically (for details see Reference [6]), the physical Hilbert space can be constructed if there exists an anti-linear map η from a dense subspace $\mathcal{D}$ of the kinematical Hilbert space $\mathcal{H}_{kin}$, to its algebraic dual $\mathcal{D}^*$, subject to certain requirements. The algebraic dual of $\mathcal{D}$ is defined to be the space of linear mappings from $\mathcal{D}$ to the complex numbers. The requirements which η needs to satisfy are as follows. Let $\psi_1, \psi_2 \in \mathcal{D}$, let $\hat{A}$ be a Dirac observable of interest and let $\phi^{\pm}$ be a gauge transformation with $\hat{U}^{\pm}(\phi^{\pm})$ being its unitary implementation on $\mathcal{H}_{kin}$. Let $\eta(\psi_1) \in \mathcal{D}^*$ denote the image of ψ_1 by η and let

$\eta(\psi_1)[\psi_2]$ denote the complex number obtained by the action of $\eta(\psi_1)$ on ψ_2. Then for all $\psi_1, \psi_2, \hat{A}, \phi$ we require that
(1) $\eta(\psi_1)[\psi_2] = \eta(\psi_1)[\hat{U}(\phi)\psi_2]$
(2) $\eta(\psi_1)[\psi_2] = (\eta(\psi_2)[\psi_1])^*$, $\eta(\psi_1)[\psi_1] \geq 0$.
(3) $\eta(\psi_1)[\hat{A}\psi_2] = \eta(\hat{A}^\dagger\psi_1)[\psi_2]$.

Here, we choose $\mathcal{D}$ to be the finite span of charge network states. Clearly due to the split of '+' and '−' structures, we may consider averaging maps $\eta^\pm$ on the dense sets $\mathcal{D}^\pm \subset \mathcal{H}^\pm_{kin}$ separately. Here $\mathcal{D}^\pm$ is the finite span of states of the form $|\mathbf{s}^\pm\rangle$ (see section 3.3.4 for the notation used here and below). Define the action of $\eta^\pm$ on $|\mathbf{s}^\pm\rangle$ as

$$\begin{aligned} \eta^\pm(|\mathbf{s}^\pm\rangle) &= \eta_{[\mathbf{s}^\pm]} \textstyle\sum_{\mathbf{s}'^\pm \in [\mathbf{s}^\pm]} < \mathbf{s}'^\pm| \\ &= \eta_{[\mathbf{s}^\pm]} \textstyle\sum_{\phi^\pm \in Diff^P_{[\mathbf{s}^\pm]}\mathbf{R}} < \mathbf{s}^\pm_{\phi^\pm}|, \end{aligned} \tag{3.86}$$

where $[\mathbf{s}^\pm] = \{\mathbf{s}'^\pm | \mathbf{s}'^\pm = \mathbf{s}^\pm_{\phi^\pm} \text{ for some } \phi^\pm\}$, $Diff^P_{[\mathbf{s}^\pm]}\mathbf{R}$ is a set of gauge transformations such that for each $\mathbf{s}'^\pm \in [\mathbf{s}^\pm]$ there is precisely one gauge transformation in the set which maps $\mathbf{s}^\pm$ to $\mathbf{s}'^\pm$ and $\eta_{[\mathbf{s}^\pm]}$ is a positive real number depending only on the gauge orbit $[\mathbf{s}^\pm]$. The right hand side of equation (3.86) inherits an action on states in $\mathcal{D}$ from that of each of its summands. Due to the inner product (3.40), (3.64), only a finite number of terms in the sum contribute so that $\eta^\pm(|\mathbf{s}^\pm\rangle)$ is indeed in $\mathcal{D}^*$. It is straightforward to see that $\eta^\pm$ satisfies the requirements **(1)**, **(2)** and that a positive definite inner product $<,>_{phys}$ on the space $\eta^\pm(\mathcal{D}^\pm)$ can be defined through

$$< \eta^\pm(|\mathbf{s}^\pm_1\rangle), \eta^\pm(|\mathbf{s}^\pm_2\rangle) >_{phys} = \eta^\pm(|\mathbf{s}^\pm_1\rangle)[|\mathbf{s}^\pm_2\rangle]. \tag{3.87}$$

If in addition, **(3)** is also satisfied by $\eta^\pm$, the group averaging technique guarantees that the above inner product automatically implements the adjointness conditions on the Dirac observables (which act by dual action on $\mathcal{D}^{\pm *}$) [4] of section 3.4, by virtue of the fact that these conditions are implemented on $\mathcal{H}_{kin}$.

In section 3.5.2 we use the requirement **(3)** to constrain the positive real numbers $\eta_{[\mathbf{s}^\pm]}$ and thus bring down the enormous ambiguity in the inner product (3.87).

[4] Given $\Psi^\pm \in \mathcal{D}^{\pm *}$, $\psi^\pm \in \mathcal{D}^\pm$ and $\hat{A}_\pm$ such that $\hat{A}^\dagger_\pm\psi^\pm \in \mathcal{D}^\pm$, define $\hat{A}_\pm\Psi^\pm$ through $\hat{A}_\pm\Psi^\pm[\psi^\pm] := \Psi^\pm[\hat{A}^\dagger_\pm\psi^\pm]$. This is the dual action.

While the analysis can be done, in principle, for all of $\eta^{\pm}[\mathcal{D}^{\pm}]$, we shall, for simplicity, restrict attention to a certain subspace of $\mathcal{D}^{\pm}$ which is left invariant by finite gauge transformations as well as the Dirac observables of section 3.4. In section 3.5.1 we define this 'superselected' subspace. Finally, in section 3.5.3 we display a cyclic representation of the operator algebra generated by the Dirac observables in conjunction with the gauge transformations.

3.5.1 The chosen subspace of $\mathcal{D}$.

Consider the charge network state $T_{s^{\pm}} \otimes W_{s'^{\pm}}$. Let $\gamma(s^{\pm})$ have $n^{\pm}$ edges and let the embedding charges on these edges be such that:
(a) $\pm k^{\pm}_{e^{\pm}_{I^{\pm}}} \geq \pm k^{\pm}_{e^{\pm}_{(I^{\pm}-1)}} \quad I^{\pm} = 2, .., n^{\pm}$.
(b) $\pm(k^{\pm}_{e^{\pm}_{n^{\pm}}} - k^{\pm}_{e^{\pm}_{1}}) \leq \frac{2\pi}{\hbar}$.

These conditions are physically motivated. Conditions **(a)**, **(b)** are the quantum analogs of the classical non-degeneracy condition **(iv)** of section 3.2.3. when $x \in (0, 2\pi)$, and when $x \in \{0, 2\pi\}$ respectively.

Henceforth we shall restrict attention to charge network states subject to **(a)** and **(b)**. Note that these conditions define a superselection sector of $\mathcal{D}$ with respect to gauge transformations as well as the observables of section 3.4. We will refer to this subspace as $\mathcal{D}_{(\mathbf{a})(\mathbf{b})}$.

3.5.2 Commutativity of $\eta^{\pm}$ with Dirac observables.

We focus on the '+' case and supress the '+' superscripts wherever possible. The '−' case follows analogously. We aim to restrict $\eta_{[\mathbf{s}]}$ by subjecting it to condition **(3)** above. We choose $\hat{A} := \widehat{e^{iO^{+}_{f}}}$ (recall, from section 3.4.1, that $O^{+}_{f} := \int_{S^1} Y^{+}(x) f(X^{+}(x)))$. Thus we require that $\forall$ **s**,

$$e^{i\int \widehat{Y^{+}f(X^{+})}}\eta(|\mathbf{s}\rangle) \;=\; \eta(e^{i\int \widehat{Y^{+}f(X^{+})}}|\mathbf{s}\rangle) \; . \tag{3.88}$$

As in equation (3.74) we set $|\mathbf{s}^{\pm}\rangle = T_{s^{\pm}} \otimes W(s'^{\pm})$. The equivalence relation between charge network labels allows us, without loss of generality, to choose $\gamma(\mathbf{s}) = \gamma(s) =$

$\gamma(s')$. Equations (3.80), (3.65), (3.62) imply that

$$e^{i\widehat{\int Y^+ f(X^+)}}|\mathbf{s}\rangle = \hat{W}_{s_f}|\mathbf{s}\rangle := e^{-\frac{i\hbar\alpha(s_f,s')}{2}}|\mathbf{s}(f)\rangle \tag{3.89}$$

where

$$\mathbf{s} = \{\gamma(\mathbf{s}), ((k_{e_1}, l_{e_1}), .., (k_{e_n}, l_{e_n}))\} \tag{3.90}$$
$$s' = \{\gamma(\mathbf{s}), (l_{e_1}, .., l_{e_n})\} \tag{3.91}$$
$$s_f = \{\gamma(\mathbf{s}), (f(\hbar k_{e_1}), .., f(\hbar k_{e_n}))\} \tag{3.92}$$
$$\mathbf{s}(f) = \{\gamma(\mathbf{s}), ((k_{e_1}, l_{e_1} + f(\hbar k_{e_1}), .., (k_{e_n}, l_{e_n} + f(\hbar k_{e_n})\} \tag{3.93}$$
$$\alpha(s_f, s') = \sum_{I=1}^{n} f(\hbar k_{e_I})[l_{e_{I+1}} - l_{e_{I-1}}],\ e_0 := e_n,\ e_{n+1} := e_1 \tag{3.94}$$

Recall (see section 3.3.4) that $\mathbf{s}$ denotes the unique labelling such that no two consecutive edges of $\gamma(\mathbf{s})$ have the same pair of charges. It is straightforward to see from equation (3.94) that for $I = 1, .., n-1$,

$$\begin{aligned} & k_{e_I} \neq k_{e_{I+1}} \quad \text{or/and} \quad l_{e_I} \neq l_{e_{I+1}} \\ \Rightarrow\ & k_{e_I} \neq k_{e_{I+1}} \quad \text{or/and} \quad l_{e_I} + f(\hbar k_{e_I}) \neq l_{e_{I+1}} + f(\hbar k_{e_{I+1}}). \end{aligned} \tag{3.95}$$

Thus, consistent with the use of bold face notation (see section 3.3.4), $\mathbf{s}(f)$ is also the unique labelling such that no two consecutive edges of its underlying graph (also chosen to be $\gamma(\mathbf{s})$) have the same pair of charges.

From footnote 4 (3.89), (3.68), the fact that $e^{i\widehat{\int Y^+ f(X^+)}}$ commutes with gauge transformations, and(3.86), it follows that the left hand side of (3.88) is

$$e^{i\widehat{\int Y^+ f(X^+)}}\eta(|\mathbf{s}\rangle) = \eta_{[\mathbf{s}]}e^{\frac{i\hbar\alpha(s_f,s')}{2}} \sum_{\phi\in Diff^P_{[\mathbf{s}]}\mathbf{R}} < \mathbf{s}(f)_\phi|. \tag{3.96}$$

and that the right hand side of (3.88) is

$$\eta(e^{i\widehat{\int Y^+ f(X^+)}}|\mathbf{s}\rangle) = \eta_{[\mathbf{s}(f)]}e^{\frac{i\hbar\alpha(s_f,s')}{2}} \sum_{\phi\in Diff^P_{[\mathbf{s}(f)]}\mathbf{R}} < \mathbf{s}(f)_\phi| \tag{3.97}$$

where $|\mathrm{s}(f)_\phi\rangle := \hat{U}(\phi)|\mathrm{s}(f)\rangle$. Thus we need to impose

$$\eta_{[\mathrm{s}]} \sum_{\phi \in Diff^P_{[\mathrm{s}]}\mathbf{R}} < \mathrm{s}(f)_\phi| = \eta_{[\mathrm{s}(f)]} \sum_{\phi \in Diff^P_{[\mathrm{s}(f)]}\mathbf{R}} < \mathrm{s}(f)_\phi| \tag{3.98}$$

We will now show that,

$$Diff^P_{[\mathrm{s}]}\mathbf{R} = Diff^P_{[\mathrm{s}(f)]}\mathbf{R}. \tag{3.99}$$

Claim 3.5.1 *Given a* $\mathrm{s} = \{\gamma(\mathrm{s}), ((k_{e_1}, l_{e_1}), .., (k_{e_n}, l_{e_n}))\}$, *let* $\tilde{\mathrm{s}} = \{\gamma(\mathrm{s}), ((k_{e_1}, l_{e_1})\}$, *then* $Diff^P_{[\mathrm{s}]}\mathbf{R} \cong Diff^P_{[\tilde{\mathrm{s}}]}\mathbf{R}$.

Proof

Injection : Let ϕ_1 *in* $Diff^P_{[\mathrm{s}]}\mathbf{R}$, $\phi_1 \cdot \mathrm{s} \neq \mathrm{s}$.

We want to show that $\phi_1 \cdot \tilde{\mathrm{s}} \neq \tilde{\mathrm{s}}$.

As $\phi_1 \cdot \mathrm{s} \neq \mathrm{s}$, one of the following must be true.

1. $\phi_1 \cdot \gamma(\mathrm{s}) \neq \gamma(\mathrm{s})$ or,

2. $\phi_1 \cdot \gamma(\mathrm{s}) = \gamma(|bfs)$, $(l_{e_1}, ..., l_{e_n}) \rightarrow (l_{e_1}, ..., l_{e_n})$ and $(k_{e_1}, ...k_{e_n}) \rightarrow (k_{e_1} + 2\pi M, ..., k_{e_n} + 2\pi M)$

3. $\phi_1 \cdot \gamma(\mathrm{s}) = \gamma(\mathrm{s})$, $(l_{e_1}, ..., l_{e_n}) \rightarrow (l_{e_I}, ..., l_{e_{I-1}})$, and $(k_{e_1}, ...k_{e_n}) \rightarrow (k_{e_I} + 2\pi M, ..., k_{e_{I-1}} + 2\pi(M+1))$.

If s is such that $(k_{e_n}, l_{e_n}) = (k_{e_1} + 2\pi, l_{e_1})$ then there is another possibility.

4. $\phi_1 \cdot \gamma(\mathrm{s}) = \gamma(\mathrm{s})$, $(l_{e_1}, ..., l_{e_n}) \rightarrow (l_{e_I}, ..., l_{e_I})$, and $(k_{e_1}, ...k_{e_n}) \rightarrow (k_{e_I} + 2\pi M, ...k_{e_n} + 2\pi M, k_{e_2} + 2\pi(M+1), ..., k_{e_I} + 2\pi(M+1))$.

Possibility 1. and 2. clearly imply that $\phi \cdot \tilde{\mathrm{s}} \neq \tilde{\mathrm{s}}$. $\tilde{\mathrm{s}}$ will remain invariant under 3. iff $\exists$ $(k_{e_1}, ..., k_{e_n})$ such that $(k_{e_1}, ..., k_{e_n}) = (k_{e_I} + 2\pi M, ..., k_{e_{I-1}} + 2\pi(M+1))$ for some I and M.

Now we show that this is not possible.

Suppose it were possible. i.e. $\exists\ (k_{e_1}, ..., k_{e_n})$ such that,

$$\begin{aligned}
k_{e_1} &= k_{e_I} + 2\pi M \\
k_{e_2} &= k_{e_{I+1}} + 2\pi M \\
&. \\
&. \\
&. \\
k_{e_{n-I+2}} &= k_{e_1} + 2\pi(M+1) \\
k_{e_{n-I+3}} &= k_{e_2} + 2\pi(M+1) \\
&. \\
&. \\
&. \\
k_{e_n} &= k_{e_{I-1}} + 2\pi(M+1)
\end{aligned} \tag{3.100}$$

for some $I \neq 1$[5] and M. Summing both sides, we get

$$\begin{aligned}
&\sum_{i=1}^{n} k_{e_i} = \sum_{i=1}^{n} k_{e_i} + (n-I+1)2\pi M + (I-1)2\pi(M+1) \\
&\Rightarrow n(2\pi M) + (I-1)2\pi = 0 \\
&\Rightarrow nM = 1-I
\end{aligned} \tag{3.101}$$

The above equation has no solution for $I > 1$ as $n \geq I$ and $M \in \mathbf{Z}$.

[5] I=1 is possibility 2

A similar proof shows that even in the case of 4. Let $\exists$ $(k_{e_1}, ..., k_{e_n})$ such that

$$\begin{aligned}
k_{e_1} &= k_{e_I} + 2\pi M \\
k_{e_2} &= k_{e_{I+1}} + 2\pi M \\
&. \\
&. \\
&. \\
k_{e_{n-I+1}} &= k_{e_n} + 2\pi M \\
k_{e_{n-I+2}} &= k_{e_2} + 2\pi(M+1) \\
k_{e_{n-I+3}} &= k_{e_3} + 2\pi(M+1) \\
&. \\
&. \\
&. \\
k_{e_{n-1}} &= k_{e_{I-1}} + 2\pi(M+1)
\end{aligned} \tag{3.102}$$

for some $2 \leq I \leq n$ and M. Summing both sides, we get,

$$\begin{aligned}
(n-I-1)2\pi M + (I-2)2\pi(M+1) &= 0 \\
\Rightarrow (n+1)M &= 2-I
\end{aligned} \tag{3.103}$$

The above equation has no solution for $I > 1$ as $n \geq I$ and $M \in \mathbf{Z}$.

Whence, $\phi \cdot \tilde{\mathrm{s}}$ cannot equal $\tilde{\mathrm{s}}$. This in turn proves injection.

Surjection between $Diff^P_{[s]}\mathbf{R}$ and $Diff^P_{[\tilde{s}]}\mathbf{R}$ is obvious as any ϕ that changes $\tilde{s}$ will change s as well.

Now let us go back to (??). As $\tilde{\mathbf{s}_f} = \tilde{\mathrm{s}}$, we have,

$$Diff^P_{[\mathbf{s}_f]}\mathbf{R} \;=\; Diff^P_{[\tilde{\mathbf{s}}_f]}\mathbf{R} \;=\; Diff^P_{[\ \tilde{bfs}]}\mathbf{R} \;=\; Diff^P_{[\mathrm{s}]}\mathbf{R} \tag{3.104}$$

Equations (3.99), (3.98) imply that

$$\eta_{[\mathbf{s}]} = \eta_{[\mathbf{s}(f)]}. \tag{3.105}$$

Next, we analyse the consequences of the restriction (3.105). There are 2 cases:
Case 1: [s] is such that there exists some $\mathbf{s} \in [\mathrm{s}]$, $\mathbf{s} = \{\gamma(\mathbf{s}), ((k_{e_1}, l_{e_1}), .., (k_{e_n}, l_{e_n}))\}$

with

$$k_{e_1} < k_{e_2} < .... < k_{e_n}, \quad (k_{e_n} - k_{e_1}) < 2\pi. \tag{3.106}$$

Case 2: The complement of Case 1.

We have analysed both cases. The analysis for Case 2 is quite involved and, in the interests of pedagogy, we present it in Appendix A. Here we only focus on case 1. Accordingly, consider $\mathbf{s}$ as in Case 1. We define $\tilde{\mathbf{s}}$ to be the *embedding* charge network label which is obtained by dropping the matter charge labels from $\mathbf{s}$ so that $\gamma(\tilde{\mathbf{s}}) = \gamma(\mathbf{s})$ with the edges of $\gamma(\tilde{\mathbf{s}})$ carrying the same embedding charges as in $\mathbf{s}$. Since $\mathbf{s}, \mathbf{s}(f)$ have the same embedding charges and the same underlying graph, we could equally well have obtained $\tilde{\mathbf{s}}$ by dropping the matter charge labels from $\mathbf{s}(f)$. Thus, using the '$\tilde{}$' notation, we have that

$$\tilde{\mathbf{s}} = \tilde{\mathbf{s}}(f) = (\gamma(\mathbf{s}), (k_{e_1}, .., k_{e_n})). \tag{3.107}$$

Next, note that we can always choose f such that $f(\hbar k_{e_I}) = -l_{e_I}$, $I = 1, .., n$ so that $\mathbf{s}(f)$ has vanishing matter charges. Clearly the property that all matter charges vanish is a gauge invariant statement. This fact together with equation (3.107) implies that the set $[\mathbf{s}(f)]$ (with f chosen as above) is isomorphic to the set of *embedding* charge networks which are gauge equivalent to $\tilde{\mathbf{s}}$. Denoting the latter set by $[\tilde{\mathbf{s}}]$ we have, from equation (3.105) that $\eta_{[\mathbf{s}]}$ can only depend on the set $[\tilde{\mathbf{s}}]$. We denote this dependence through the notation

$$\eta_{[\tilde{\mathbf{s}}]} := \eta_{[\mathbf{s}]}. \tag{3.108}$$

An identical analysis holds for the conformal isometry operators $\hat{V}(\phi_c)$. Equation (3.84) implies that

$$\hat{V}(\phi_c)|\mathbf{s}\rangle =: |\phi_c^{-1}(\mathbf{s})\rangle. \tag{3.109}$$

$\mathbf{s}$ is given by equations (3.90), (3.106) and

$$\phi_c^{-1}(\mathbf{s}) = \{\gamma(\mathbf{s}), ((\phi_c^{-1}(k_{e_1}), l_{e_1}), .., (\phi_c^{-1}(k_{e_n}), l_{e_n}))\}. \tag{3.110}$$

The invertibility of ϕ_c and its quasi-periodicity imply that $\phi_c^{-1}(\mathbf{s})$ is the unique labelling such that no 2 consecutive edges have the same pairs of charges, and that the condition (3.106) is preserved by the action of $\hat{V}(\phi_c)$.

Condition (**3**) implies that, in obvious notation,

$$\eta_{[\mathbf{s}]} \sum_{\phi \in Diff^P_{[\mathbf{s}]}\mathbf{R}} < \phi_c^{-1}(\mathbf{s})|\hat{U}^\dagger(\phi) = \eta_{[\phi_c^{-1}(\mathbf{s})]} \sum_{\phi \in Diff^P_{[\phi_c^{-1}(\mathbf{s})]}\mathbf{R}} < \phi_c^{-1}(\mathbf{s})|\hat{U}^\dagger(\phi). \quad (3.111)$$

An argument identical to that in (??) implies that $Diff^P_{[\mathbf{s}]}\mathbf{R} = Diff^P_{[\phi_c^{-1}(\mathbf{s})]}\mathbf{R}$ so that

$$\eta_{[\mathbf{s}]} = \eta_{[\phi_c^{-1}(\mathbf{s})]}. \quad (3.112)$$

Once again case 2, as defined on the previous page is quite involved and is analyzed in appendix B. Here we only analyze case 1.

Clearly, given any pair of charge networks $\mathbf{s}_1, \mathbf{s}_2$ as in Case 1, with $\gamma(\mathbf{s}_1) = \gamma(\mathbf{s}_2)$ and with identical matter charges, there exists some ϕ_c such that $|\mathbf{s}_2\rangle = \hat{V}(\phi_c)|\mathbf{s}_1\rangle$. This, in conjunction with equations (3.112), (3.108) implies that $\eta_{[\mathbf{s}]}$ can only depend on the set of graphs $[\gamma(\mathbf{s})]$ which are obtained by the action of gauge transformations on $\gamma(\mathbf{s})$. Specifically,

$$\begin{aligned} [\gamma(\mathbf{s})] &= \{\gamma' \text{ s.t. } \exists\phi \text{ s.t. } \gamma'_{ext} = \phi(\gamma_{ext})\} \\ \gamma &:= \gamma(\mathbf{s}), \end{aligned} \quad (3.113)$$

where we have used the notation defined in section 3.3.1. We denote this dependence of $\eta_{[\mathbf{s}]}$ through the notation

$$\eta_{[\mathbf{s}]} = \eta_{[\gamma(\mathbf{s})]} \quad (3.114)$$

This completes our analysis of the rigging map.

3.5.3 Cyclic representation

We focus on the '+' sector of the algebra of operators and the '+' sector of the state space. As in section 3.5.2 we suppress '+' superscripts. The analysis for the '−' case follows analogously. Cyclicity is defined with respect to an algebra of operators. Here the putative generators of the algebra are the Dirac observables of section 3.4 and the finite gauge transformations. As we shall see in section 3.6, neither does the commutator of two of the observables of section 3.4.1 yield a representation of the corresponding Poisson brackets nor does their product yield a representation of the appropriate Weyl algebra. As shown in section 3.6, the connection with classical theory is state dependent and only holds for semiclassical states (this is roughly similar to what happens for area operators in LQG [3]). Given this situation, we define the operator algebra in terms of the concrete representation on $\mathcal{H}_{kin}$ (or $\mathcal{H}_{phys}$) of the relevant operators rather than in terms of abstract representations of classical structures.

Since the operators of section 3.4 as well as those for finite gauge transformations are unitary (and hence bounded), the finite span of their products is well defined on $\mathcal{H}_{kin}$ so that it is possible to define the algebra of operators generated by these elementary ones in terms of the action of elements of this algebra on $\mathcal{H}_{kin}$. We denote this algebra of operators as $\mathcal{A}^{kin}_{D,G}$. In a similar manner, consider the algebra of operators generated by the action of the Dirac observables of section 3.4 on $\mathcal{H}_{phys}$. Denote this algebra by $\mathcal{A}^{phys}_{D}$.

Fix a graph γ. Let $\mathbf{s}_\gamma$ be the set of charge networks such that $\forall \mathbf{s} \in \mathbf{s}_\gamma, \gamma(\mathbf{s}) = \gamma$ and $\mathbf{s}$ satisfies condition (3.106) on its embedding charges. Let $[\mathbf{s}_\gamma]$ be the set of charge networks which are gauge related to elements of $\mathbf{s}_\gamma$ i.e. $\forall \mathbf{s}' \in [\mathbf{s}_\gamma] \exists$ some gauge transformation ϕ and some $\mathbf{s} \in \mathbf{s}_\gamma$ such that $\mathbf{s}' = \mathbf{s}_\phi$. Finally, let $\mathcal{H}_{[\gamma]}$ be the (Cauchy completion of the) finite span $\mathcal{D}_{[\gamma]}$ ($\subset \mathcal{D}_{(\mathbf{a})(\mathbf{b})}$) of charge network states $|\mathbf{s}'\rangle$, $\mathbf{s}' \in [\mathbf{s}_\gamma]$.

The analysis of the preceding section shows that:
(1) $\mathcal{H}_{[\gamma]} \subset \mathcal{H}_{kin}$ provides a cyclic representation of the algebra $\mathcal{A}^{kin}_{D,G}$. Any charge network state in $\mathcal{H}_{[\gamma]}$ is a cyclic state.
(2) Group averaging of states in $\mathcal{D}_{[\gamma]}$ yields a cyclic representation of the algebra $\mathcal{A}^{phys}_{D}$ i.e. $\mathcal{A}^{phys}_{D}$ is represented cyclically on $\mathcal{H}_{[\gamma],phys} \subset \mathcal{H}_{phys}$ where $\mathcal{H}_{[\gamma],phys}$ is the Cauchy completion (in the physical inner product) of $\eta(\mathcal{D}_{[\gamma]})$. The group average of any charge network state in $\mathcal{D}_{[\gamma]}$ is a cyclic state.

Note that both $\mathcal{H}_{[\gamma]}$ and $\mathcal{H}_{[\gamma],phys}$ are *non- seperable.*

3.6 Semiclassical Issues.

An exhaustive analysis of semiclassical states is outside the scope of this thesis. Instead, we focus on two issues related to semiclassicality. In section 3.6.1 we show that semiclassical states must be based on suitably defined 'weaves'. In section

3.6.2 we show that semiclassicality can be exhibited with respect to, at most, a countable number of the mode function operators of section 3.4.1.

3.6.1 Semiclassicality and Weaves.

Recall that in LQG, states which exhibit semiclassical behaviour for spatial geometry operators are based on graphs called weaves [8]. Here the (flat) spacetime geometry is encoded in the behaviour of the $\hat{X}^{\pm}(x)$ operators. Hence we define the notion of a weave as follows. The embedding charge network $s^{\pm} = \{\gamma(s^{\pm}), (k^{\pm}_{e_1^{\pm}}, .., k^{\pm}_{e^{\pm}_{N^{\pm}}})\}$ will be called a *weave* iff the embedding charges satisfy (a),(b) of section 3.5.1 together with $k^{\pm}_{e_N^{\pm}} - k^{\pm}_{e_1^{\pm}} \approx \pm 2\pi$ and iff $N >> 1$. This is, of course, not a precise definition since $k^{\pm}_{e_N^{\pm}} - k^{\pm}_{e_1^{\pm}} \approx 2\pi$ and $N >> 1$. are not precise statements. Nevertheless this 'working' definition will suffice for our purposes.

Let $\psi^{\pm} \in \mathcal{H}^{\pm}_{kin}$ exhibit semiclassicality with respect to the $\pm$ sector observables of section 3.4.1. Further, let $\psi^{\pm}$ be an eigen state of $\hat{X}^{\pm}(x)$ (we shall relax this assumption later) so that $\psi^{\pm} = T_{s^{\pm}} \otimes \psi^{\pm}_M$, $\psi^{\pm}_M \in \mathcal{H}^{\pm}_M$. The analysis below is for the + sector and can be trivially extended to the $-$ sector. In what follows we suppress the + superscript. From equation (3.80) it follows straightforwardly that

$$\langle [\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}] \rangle = -2i \sin(\frac{\alpha\beta\hbar}{2} f_{s,m}) \langle \widehat{e^{i\alpha q_m + i\beta p_m}} \rangle, \tag{3.115}$$

where

$$f_{s,m} := \sum_{I=1}^{N} \cos(\hbar m k_{e_I})(\sin(\hbar m k_{e_{I+1}}) - \sin(\hbar m k_{e_I})). \tag{3.116}$$

where $k_{e_{N+1}} := k_{e_1}$. In order to write (3.116) in a more useful form, we define the following:

$$\Delta k_{e_I} := k_{e_{I+1}} - k_{e_I}, \quad I = 1, .., N-1 \tag{3.117}$$

$$\Delta k_{e_N} := k_{e_1} - k_{e_N} + \frac{2\pi}{\hbar}. \tag{3.118}$$

Rearranging terms in (3.116) and using standard trigonometric identities we obtain that

$$f_{s,m} = \sum_{I=1}^{N} \sin(\hbar m \Delta k_{e_I}). \tag{3.119}$$

Since ψ is semiclassical we assume that, for some classical data (q_m, p_m),

$$\langle \widehat{e^{i\alpha q_m + i\beta p_m}} \rangle \approx e^{i\alpha q_m + i\beta p_m}, \tag{3.120}$$

and we require that as $\hbar \to 0$

$$\langle [\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}] \rangle \to i\hbar \{e^{i\alpha q_m}, e^{i\beta p_m}\} \tag{3.121}$$

where the Poisson bracket evaluates to

$$\{e^{i\alpha q_m}, e^{i\beta p_m}\} = -\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m}. \tag{3.122}$$

Equations (3.115)- (3.122) imply that to leading order in $\hbar$

$$f_{s,m=1} \approx 2\pi. \tag{3.123}$$

Note that the eigen values of the embedding operators are in terms of

$$k_I := \hbar k_{e_I} \tag{3.124}$$

so that in the $\hbar \to 0$ (classical) limit, k_I does not vanish (except when $k_{e_I} = 0$). Hence, we investigate the conditions imposed on s by the requirement

$$|2\pi - \sum_{I=1}^{N} \sin(\Delta k_I)| < \epsilon, \quad \epsilon << 1. \tag{3.125}$$

where, similar to (3.117) we have defined

$$\Delta k_I := k_{I+1} - k_I, \;\; I = 1, .., N-1 \tag{3.126}$$

$$\Delta k_N := k_1 - k_N + 2\pi. \tag{3.127}$$

Note that conditions (**a**), (**b**) of section 3.5.1 imply that

$$\Delta k_I \geq 0, \quad \sum_{I=1}^{N} \Delta k_I = 2\pi. \tag{3.128}$$

Intuitively, since $|\frac{\sin x}{x}| \leq 1$ and $= 1$ at $x = 0$, equations (3.125), (3.128) lead us to expect that $\Delta k_I, I = 1, .., N$ should be small. That this is indeed the case is shown in Lemmas 1- 3 in the Appendix. Clearly, the fact that $\Delta k_I \rightarrow 0$ as $\epsilon \rightarrow 0$ (see Appendix) implies that s is a weave. Thus, we have shown that any kinematic semiclassical state which is an eigen state of the embedding operators must be based on a weave.

Next, consider an arbitrary kinematic state $|\psi\rangle = \sum a_i |s_i > \otimes |\psi_{iM}\rangle$ where a_i are complex coefficients, $|s_i\rangle$ are an orthonormal set of embedding charge network states and $|\psi_{iM}\rangle \in \mathcal{H}_M$. In order that this state satisfies equation (3.121), it turns out that $|\psi\rangle$ must be peaked around s_i such that s_i are weaves. This is shown in Lemma 4 of the Appendix. Finally, consider an arbitrary physical state. Such a state is a linear combination of averages over embedding eigen states. Lemma 5 shows that such a state is peaked around averages of embedding eigen states which are based on weaves.

3.6.2 Semiclassicality and mode function operators: a no-go result.

We show that no states exist which are semiclassical with respect to the uncountable set of operators $\{\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}, |\alpha - \alpha_0| < \epsilon, |\beta - \beta_0| < \delta\}$ for any fixed m, α_0, β_0 and any $\epsilon, \delta > 0$. First, consider states $|\psi\rangle$ which are embedding eigen states so that $|\psi\rangle = |s\rangle \otimes |\psi_M\rangle$. Here s is an embedding charge network and $|\psi_M\rangle \in \mathcal{H}_M$ can expanded as $|\psi_M\rangle = \sum_r b_r |s'_r\rangle$ where $\{|s'_r\rangle\}$ is a countable set of orthonormal matter charge networks.

The operators $\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}$ act by changing the matter charge labels by sines and cosines of (m times) the embedding charges (see (3.80)). Consider the set L of all matter charges on $s_r \forall r$ and construct the set ΔL of differences between all pairs of elements of L i.e. $\Delta L := \{l - l' \forall l, l' \in L\}$. Let $k_e, e \subset \gamma(s)$ be such that $\cos m\hbar k_e \neq 0$. Then, in any neighbourhood of α_0 we can choose uncountably many

α such that $\alpha \cos m\hbar k_e \notin \Delta L$. Clearly for such α we have that $\langle \widehat{e^{i\alpha q_m}} \rangle = 0$. If $\cos m\hbar k_e = 0$ we can repeat the same argument with $\sin m\hbar k_e$ and conclude that $\langle \widehat{e^{i\beta p_m}} \rangle = 0$ for uncountable many β near β_0. Clearly, such behaviour is far from semiclassical. This argument can be suitably generalised for arbitrary states in $\mathcal{H}_{kin}$ as well as in $\mathcal{H}_{phys}$. The relevant material is in Lemma 6 and Lemma 7 of Appendix D.

3.7 Two open issues and their resolution.

Before we conclude this chapter, a couple of points remain which we have not addressed as yet. First, it still remains to enforce **(ii)**, section 3.2.3 in order to ensure that the spatial topology is a circle. Second, we need to take care of the zero modes by imposing equation (3.25) in quantum theory and show that the results of section 3.6 continue to hold after this is done. We address these points in sections 3.7.1 and 3.7.2 below.

3.7.1 Identifying 2π shifted embeddings

Although the spatial inertial co-ordinate X ranges over $(-\infty, \infty)$, we need to identify $X \sim X + 2\pi$ in accordance with the discussion in section 3.2.3. Condition **(ii)**, section 3.2.3 states that two embeddings (X_1, T_1), (X_2, T_2) are equivalent if the following conditions are satisfied:

$$\begin{aligned} X_1^+(x) &= X_2^+(x) + 2m\pi \ \forall \ x \in \ [0, 2\pi], \\ X_1^-(x) &= X_2^-(x) - 2m\pi \ \forall \ x \in \ [0, 2\pi]. \end{aligned} \tag{3.129}$$

We now show that this equivalence has already been taken care of at the physical state-space level. Let

$$\begin{aligned} \mathbf{s}^+ &= \{ \ \gamma(\mathbf{s}^+), (k^+_{e^+_1}, ..., k^+_{e^+_N}), (l^+_{e^+_1}, ..., l^+_{e^+_N}) \ \} \\ \mathbf{s}^- &= \{ \ \gamma(\mathbf{s}^-), (k^-_{e^-_1}, ..., k^-_{e^-_M}), (l^-_{e^-_1}, ..., l^-_{e^-_M}) \} \end{aligned} \tag{3.130}$$

The identification (3.130) in the classical theory implies the following equivalence condition in quantum theory:

$$|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle \sim |\mathbf{s}^+_{2\pi m}\rangle \otimes |\mathbf{s}^-_{-2\pi m}\rangle \tag{3.131}$$

where,

$$\begin{aligned} \mathbf{s}^+_{2\pi m} &= \{\ \gamma(\mathbf{s}^+), (k^+_{e^+_1} + 2m\pi, ..., k^+_{e^+_N} + 2m\pi), (l^+_{e^+_1}, ..., l^+_{e^+_N})\ \}, \\ \mathbf{s}^-_{-2\pi m} &= \{\ \gamma(\mathbf{s}^-), (k^-_{e^-_1} - 2m\pi, ..., k^-_{e^-_M} - 2m\pi), (l^-_{e^-_1}, ..., l^-_{e^-_M})\ \}. \end{aligned} \tag{3.132}$$

Next, note that for any integer m, there exist gauge transformations $\phi^{\pm}_{(m)}$ such that $\phi^{\pm}_{(m)} \cdot \mathbf{s}^{\pm} = \{\gamma(\mathbf{s}^{\pm}),\ (k^{\pm}_{e^{\pm}_1} \pm 2m\pi, ..., k^{\pm}_{e^{\pm}_N} \pm 2m\pi),\ (l^{\pm}_{e^{\pm}_1}, ..., l^{\pm}_{e^{\pm}_N})\}$. Thus $|\mathbf{s}^{\pm}\rangle$ and $|\mathbf{s}^{\pm}_{\pm 2\pi m}\rangle$ are gauge related so that

$$\eta^{\pm}(|\mathbf{s}^{\pm}\rangle) = \eta^{\pm}(|\mathbf{s}^{\pm}_{\pm 2\pi m}\rangle), \tag{3.133}$$

$$\Rightarrow \eta^+(|\mathbf{s}^+\rangle) \otimes \eta^-(|\mathbf{s}^-\rangle) = \eta^+(|\mathbf{s}^+_{2\pi m}\rangle) \otimes \eta^-(|\mathbf{s}^-_{-2\pi m}\rangle). \tag{3.134}$$

Equation (3.134) shows that the identification of 2π-shifted embeddings is *subsumed* by the identification of embeddings related by gauge transformations.

3.7.2 Taking care of the zero mode in quantum theory.

In section 3.7.2.1 we impose the condition $p = 0$ (see equation (3.25)) by appropriate group averaging. In section 3.7.2.2 we show that this does not alter the conclusions of section 3.6.

Imposition of $p = 0$ by averaging.

The conditions $\int_{S^1} Y^{\pm} = 0$ of equation (3.25) are equivalent to the conditions $e^{i\lambda^{\pm}\int_{S^1} Y^{\pm}} = 1,\ \forall \lambda^{\pm}$. The latter can be imposed by group averaging with respect to the operators $e^{i\widehat{\lambda^{\pm}\int_{S^1} Y^{\pm}}}$. Let $s^{\pm}_{\lambda^{\pm}}$ be matter charge networks with a single edge $e^{\pm} = [0, 2\pi]$ labelled by the charge $\lambda^{\pm}$ i.e. $s^{\pm}_{\lambda^{\pm}} = \{\gamma(s^{\pm}_{\lambda^{\pm}}) = [0, 2\pi], l^{\pm}_{e^{\pm}} = \lambda^{\pm}\}$. Clearly, we have that $e^{i\widehat{\lambda^{\pm}\int_{S^1} Y^{\pm}}} = \hat{W}(s^{\pm}_{\lambda^{\pm}})$. Note that $\hat{W}(s^{\pm}_{\lambda^{\pm}})$ commutes with all the gauge transformations as well as observables of section 3.4. Since we have already averaged over the group of gauge transformations, the map $\overline{\eta}^{\pm}$

which implements (3.25) is defined from the space $\eta^{\pm}(\mathcal{D}^{\pm}_{(a)(b)})$ to its algebraic dual $\eta^{\pm}(\mathcal{D}^{\pm}_{(a)(b)})*$. Recall that $\mathcal{D}^{\pm}_{(a)(b)}$ (defined in section 3.5.1) is the finite span of charge networks subject to the conditions (**a**), (**b**) of section 3.5.1. Before defining $\overline{\eta}^{\pm}$, note that,

$$\hat{W}(s^{\pm}_{\lambda^{\pm}})|\mathbf{s}^{\pm}\rangle \;=:\; |\mathbf{s}^{\pm}_{\lambda^{\pm}}\rangle \tag{3.135}$$

where $\mathbf{s}^{\pm}_{\lambda^{\pm}}$ is obtained from $\mathbf{s}^{\pm} = \{\gamma(\mathbf{s})^{\pm}, (k^{\pm}_{e^{\pm}_1}, ..., k^{\pm}_{e^{\pm}_N}), (l^{\pm}_{e^{\pm}_1}, ..., l^{\pm}_{e^{\pm}_N})\}$ by adding $\lambda^{\pm}$ to all the matter charges. We now define,

$$\overline{\eta}^{\pm}(\eta^{\pm}(|\mathbf{s}^{\pm}\rangle)) = \overline{\eta}_{[[\mathbf{s}^{\pm}]]_0}\eta_{[\mathbf{s}^{\pm}]}(\bigoplus\nolimits_{\lambda^{\pm}\in\mathbf{R}}\sum\nolimits_{\phi^{\pm}\in Diff^P_{[\gamma(\mathbf{s}^{\pm})]}\mathbf{R}}\langle(\mathbf{s}^{\pm}_{\phi^{\pm}})_{\lambda^{\pm}}| \tag{3.136}$$

The equivalence class $[[\mathbf{s}^{\pm}]]_0$ is defined via following relation.
$[\mathbf{s}^{\pm}] \sim [\mathbf{s}^{\pm}_1]$ iff for any $\{\gamma(\mathbf{s}^{\pm}), (k^{\pm}_{e^{\pm}_1}, ..., k^{\pm}_{e^{\pm}_N}), (l^{\pm}_{e^{\pm}_1}, ..., l^{\pm}_{e^{\pm}_N})\} \in [\mathbf{s}^{\pm}]$,
$\exists$ $(\{\gamma(\mathbf{s}^{\pm}), (k^{\pm}_{e^{\pm}_1}, ..., k^{\pm}_{e^{\pm}_N}), (l^{\pm}_{e^{\pm}_1} + \lambda^{\pm}, ..., l^{\pm}_{e^{\pm}_N} + \lambda^{\pm})\} \in [\mathbf{s}^{\pm}_1]$ for some $\lambda_{\pm} \in \mathbf{R}$.

Once again the ambiguity in the rigging map contained in $\overline{\eta}_{[[\mathbf{s}^{\pm}]]_0}$ can be reduced by demanding that $\overline{\eta}^{\pm}$ commutes with the observables. It can be checked that for the super-selected sector of $\mathcal{H}_{phys}$ defined in section 3.5.2, we have $\overline{\eta}_{[[\mathbf{s}]]_0} = \overline{\eta}_{[\gamma]}$, where as in section 3.5.2, 3.5.3 we have once again suppressed $\pm$ superscripts and where $[\gamma]$ is defined as in section 3.5.3. Setting $\tilde{\eta}_{[\gamma(\mathbf{s})]} := \overline{\eta}_{[\gamma]}\eta_{[\gamma]}$, we have that the inner product on $\overline{\eta}^{\pm}(\mathcal{D}^{\pm\,ss}_{phy})$ is given by,

$$\langle\overline{\eta}(\eta(|\mathbf{s}\rangle)|\overline{\eta}(\eta(|\mathbf{s}_1\rangle))\rangle \;=\; \tilde{\eta}_{[\gamma(\mathbf{s})]}\bigoplus_{\lambda}\Big(\;\eta(|\mathbf{s}\rangle)[|\mathbf{s}_{1,\lambda}\rangle]\Big), \tag{3.137}$$

Semiclassical Issues.

Since the zero mode operator $\hat{W}(s^{\pm}_{\lambda^{\pm}})$ leaves the embedding part of the states in $\mathcal{H}_{kin}$ and $\mathcal{H}_{phys}$ untouched, it is easy to see that the proofs of section 3.6.1 and appendix C still apply after the zero mode averaging is done. Thus, semiclassical states which satisfy the $p = 0$ constraint are necessarily based on weaves.

It is also straightforward to see that the results of section 3.6.2 apply after zero mode group averaging. While the line of argument is roughly similar to that in section 3.6.2 and appendix D, there are some differences. In the interests of brevity, we provide only a skeleton of the argument below. As usual we shall suppress the $\pm$ superscripts.

The averaging with respect to $\overline{\eta}$ slightly complicates matters because there is an additional sum over matter charge networks wherein matter charges associated with charge network states are all incremented by the same amount. As a result, it is necessary to consider pairs of edges subject to conditions on their embedding charges. This is in contrast to the role of single edges (with cosines or sines of ($\hbar$ times) their embedding charges being non- vanishing) in the arguments of section 3.6.1 and appendix D. Specifically, consider a state decomposition defined in terms of embedding charge networks s_j as in equations (3.151) and (3.164). Separate the values taken by the index j into a set C_1 and its complement, C_2, where $j \in C_1$ iff for fixed m, there exist a pair of edges $e_I(j), e_J(j) \in \gamma(s_j)$ such that $\cos m\hbar k_{e_I(j)} \neq \cos m\hbar k_{e_J(j)}$.

Next, with a slight abuse of notation, for each $j \in C_1$ fix a pair of edges $e_I(j), e_J(j) \in \gamma(s_j)$ such that $\cos m\hbar k_{e_I(j)} \neq \cos m\hbar k_{e_J(j)}$. As in appendix D, define ΔL to be the set of differences of all matter charges which occur in the expansions (3.151), (3.164), (3.170). Also define $\Delta^2 L$ to be the set of all differences between pairs of elements of ΔL. For each $j \in C_1$ define $\Delta^2 L_j$ to be the set of elements obtained by dividing each element of $\Delta^2 L$ by $\cos m\hbar k_{e_I(j)} - \cos m\hbar k_{e_J(j)}$. Let $\Delta^2 L_{C_1} := \cup_{j \in C_1} \Delta^2 L_j$. The set $\Delta^2 L_{C_1}$ is countable so that there are uncountably many α in any neighbourhood of α_0 such that $\alpha \notin \Delta^2 L_{C_1}$. It can then be checked that $\langle \widehat{e^{i\alpha q_m}} \rangle$ obtains contributions only from terms labelled by $j \in C_2$.

Finally, we show that such terms are of negligible measure. Note that for $j \in C_2$ we have that $\cos m\hbar k_{e_I(j)} = \cos m\hbar k_{e_J(j)}$ for any pair of edges $e_I(j), e_J(j) \in \gamma(s_j)$. It is then straightforward to see that for such j, the function $f_{s_j,m}$ (defined by equations (3.157), (3.116)) vanishes identically. Then the arguments of section 3.6.1 and appendix C imply that the contribution from $j \in C_2$ must be negligible for semiclassicality to hold.

Similar arguments can be made for $\langle \widehat{e^{i\beta p_m}} \rangle$ by replacing cosines with sines in the above argument.

3.8 Discussion of results and open issues.

In this chapter, we constructed a quantization of PFT similar to that used in LQG. Quantum states are in correspondence with graphs (i.e. collections of edges) in the spatial manifold. The edges of these graphs are labelled by a set of real valued embedding and matter charges. These charge network states are analogs of the spin network states in LQG. There, however, the labels are integer valued. Such a labelling is also, in principle, possible here. Had the holonomies of section 3.3 been based on charge networks with embedding charges which were integer multiples of $\frac{2\pi}{L}$ for some fixed integer L and matter charges which were also integer multiples of some appropriate dimensionful unit, such holonomies would still separate points in phase space by virtue of the fact that they were based on arbitrary graphs (this is similar to what happens in LQG). Such a choice would lead to states with integer valued charges. However it is not clear if a large enough subset of the Dirac observables of section 3.4 preserve the space spanned by these integer-charge network states. It would be useful to investigate this issue in detail.

The polymer quantization of the embedding variables replaces the classical (flat) spacetime continuum with a discrete structure consisting of a countable set of points. This can be seen as follows. The canonical data $X^{\pm}(x)$ is a map from S^1 into the flat spacetime $(S^1 \times R, \eta)$ and embeds the former into the latter as a spatial Cauchy slice. Any gauge transformation generated by the constraints maps this data to new embedding data which, in turn, define a new Cauchy slice in the flat spacetime. In particular, the action of the one parameter family of gauge transformations generated by smearing the constraints with some choice of "lapse-shift" type functions N^A (see section 3.2) generates a foliation of $(S^1 \times R, \eta)$. Consider the image set in $(S^1 \times R, \eta)$ of the set of all embeddings which are gauge related to a given one. From the above discussion it follows that this image set is exactly the flat spacetime $(S^1 \times R, \eta)$ itself. Next, consider the corresponding quantum structures. Any charge network state is an eigen state of $\hat{X}^{\pm}(x)$. Consider a charge network state, $|\mathbf{s}^+\rangle \otimes |\mathbf{s}^-\rangle$ with $|\mathbf{s}^{\pm}\rangle = T_{s^{\pm}} \otimes W_{s'^{\pm}}$, where $s^{\pm}$ satisfy the conditions $(\mathbf{a}), (\mathbf{b})$ of section 3.5.1. From equation (3.42)- (3.44) it follows that the set of eigen values $\lambda_{x,s^{\pm}}$ for all $x \in [0, 2\pi]$ describes a finite set of points on a spacelike Cauchy surface in $(S^1 \times R, \eta)$. These points have light cone coordinates $(X^+, X^-) = (\lambda_{x,s^+}, \lambda_{x,s^-})$. The action of any gauge transformation on such a charge network state yields another charge network state whose eigen values lie, once again, on a Cauchy slice in $(S^1 \times R, \eta)$. From equation (3.46) it follows that the set of eigen values for all possible gauge related charge network states is countable and defines a corresponding set of points in $(S^1 \times R, \eta)$. The gauge invariant state obtained by group averaging a charge network state is a sum over all distinct gauge equivalent states and hence contains the elements of this discrete structure. This answers the first question raised in the introduction. The discrete structure is a good approximant of the continuum spacetime $(S^1 \times R, \eta)$ for charge networks with a large number of embedding charges i.e. for weave states. Thus, it is not surprising that semiclassicality requires states to be based on weaves as in section 3.6.1 and appendix A.

In contrast to the embedding charges, the matter charges do not have a direct physical interpretation because charge network states are not eigen states of the matter holonomies. As a tentative, provisional interpretation we choose to think of them, rather imprecisely, as measuring excitations of the matter. Since, on the constraint surface, the classical data $(X^{\pm}(x), Y^{\pm}(x))$ correspond to free scalar field data $Y^{\pm}(x)$ on the slice $(X^{+}(x), X^{-}(x))$ in flat spacetime, we interpret a charge network state $|\mathbf{s}^{+}\rangle \otimes |\mathbf{s}^{-}\rangle \in \mathcal{H}_{kin}$ as specifying excitations of matter on the discretized "quantum" slice specified by the embedding charges. The action of a gauge transformation on a charge network state can then be interpreted as evolving the matter excitations on the 'initial' quantum slice specified by this state to the new one specified by the gauge related charge network state. Since the physical state obtained as the group average of a charge network state contains all distinct gauge related states, it follows that such a physical state may be interpreted, roughly, as a "history". It may be useful to attempt an interpretation of physical states in LQG along these lines.

An over- complete set of Dirac observables corresponding to exponential functions of the standard annihilation- creation modes of free scalar field theory are represented as (unitary) operators in the polymer representation. Note that in contrast to the assumption of Reference [6], here the commutator between two such operators does not close as in the case of Weyl algebras. Indeed, as shown in section 3.6.1, the commutator only approximates the corresponding Poisson bracket for semiclassical states based on weaves. This underlines the fact that in a general covariant theory involving spacetime geometry, classical structures are typically not approximated in the $\hbar \rightarrow 0$ limit unless it is possible to coarse grain/smoothen away the underlying discreteness of the quantum spacetime. Nevertheless the action of the basic Dirac observables is well defined and there is no obstruction to the quantization procedure.

The results of section 3.6.2 imply that semiclassical analysis requires a choice of a countable subset of these observables. One possibility is to choose, for each n, a pair $\alpha, \beta << \frac{1}{\sqrt{\hbar}}$ and define the approximants to $\hat{q}_n, \hat{p}_n$ by $\frac{\widehat{e^{i\alpha q_n}} - \widehat{e^{-i\alpha q_n}}}{2i\alpha}$, $\frac{\widehat{e^{i\beta p_n}} - \widehat{e^{-i\beta p_n}}}{2i\beta}$. However, there is no natural choice of α, β and so, while the quantization constructed in this paper is free of the "triangularization" choices which occur in the definition of the quantum dynamics of LQG, an element of choice does appear when semiclassical issues are confronted. Note, however, that the results of section 3.6.1 indicate that any physical semiclassical state necessarily has an associated (gauge invariant) structure, namely that of a weave. [6] The "spacing" of the weave (i.e. $\hbar\Delta k_I$ of section 3.6.1 and the Appendix C) provides a natural scale for α, β. Thus, our viewpoint is that since choices of Dirac observables can be tied however tenuously to structures already present in the semiclassical states, ambiguities, if present in definitions of the quantum dynamics are more worrying because quantum dynamics is defined for all states, not only semiclassical ones.

While the general covariance of PFT is encoded in the gauge transformations generated by the constraints, the conformal invariance of the underlying free scalar field theory is reflected in the canonical transformations which correspond to the Dirac observables of section 3.2.4.3. The results of sections 3.3 and 3.4.2 show that the group of gauge transformations as well as that of conformal isometries are represented in an anomaly free manner. While the anomaly free nature of the former is necessary for the consistency of the quantum theory, it is possible, in principle, for the latter to admit anomalies. Indeed this is exactly what happens in the representation of PFT constructed in Reference [29, 58]. While the algebra of gauge transformations is anomaly free, the physical Hilbert space representation is equivalent to the standard free field Fock representation and the algebra of the generators of conformal isometries displays the standard Virasoro central extension. Motivated by the results of References [29, 58] and [31], we believe that the anomaly manifests as result of the Poincare invariance of the Fock representation i.e. as a result of the existence of the Poincare invariant vacuum. From this point of view the absence of anomalies in the group of gauge transformations as well as the group of conformal isometries in the polymer quantization is related to the absence of a Poincare invariant state (Poincare transformations are a subset of the

[6] Note that in contract to the weaves of Reference [8] which approximate a spatial geometry, here it is the (flat) *spacetime* geometry which is being approximated by virtue of the discussion in the second paragraph of this section.

conformal isometry group and it is easy to see that no kinematic or physical state is Poincare invariant). We shall return to the issue of Poincare invariance towards the end of this section.

Next we turn to the discussion of the efficacy of polymer PFT as a toy model for LQG. We believe that the quantization provided here is a useful testing ground for proposed definitions of quantum dynamics in canonical LQG. It would be of interest to construct the quantum dynamics of the model along the lines of Reference [51] and compare the resulting physical Hilbert space with the one considered here. Proposals for examining semiclassical issues [43] may also be tested here. One of the outstanding problems in LQG [61, 11] is the relation between states in LQG and the Fock states of perturbative gravity. Since PFT admits a Fock quantization [29, 58] equivalent to the standard flat spacetime free scalar field Fock representation, one may enquire as to how Fock states arise from the polymer Hilbert space. Since the results of section 3.6.2 suggest that the operators corresponding to exponentials of mode functions do not possess the recquisite continuity for the annihilation- creation modes themselves to be defined as operators, it is difficult to identify Fock states in terms of their properties with respect to the action of the annihilation- creation operators. However, as a first step, it may be possible to identify candidate states corresponding to the Fock vacuum by using the Poincare invariance of the latter. Specifically, since the operators corresponding to finite Poincare transformations are available (as a subset of the conformal isometry operators of section 3.4), one could try and group average with respect to these operators.

Another open issue pertains to the representation appropriate to the case of non- compact spatial topology. We analyze this issue in the next two chapters.

Appendix

A. Reducing the ambiguities in the rigging map

In this appendix we would like analyze the consequence of case 2 for (3.105) as noted in the main text. The constraints on the embedding charges in case 2 are as follows.

Case 2 : Let $J > I$ be such that $k_{e_J} = k_{e_I} + 2\pi$. Thus $k_{e_1} = ... = k_{e_I}$ and $k_{e_J} = k_{e_{J+1}} = ... = k_{e_n}$. Also let $\exists$ a family of tuples $(M_1, M_1+1, ...M_1+L_1), (M_2, ..., M_2+L_2), ..., (M_m, ..., M_m+L_m)$ such that $I < M_1 < M_1+L_1 < M_2 < M_2+L_2 < ... < M_m < M_m+L_m < J$
and $k_{e_{M_1}} = ... = k_{e_{M_1+L_1}}$, $k_{e_{M_2}} = ... = k_{e_{M_{L_2}}}$,...,$k_{e_{M_m}} = ... = k_{e_{M_m+L_m}}$.

As f is periodic ($f(k+2\pi) = f(k)$), under $l_{e_I} \to l_{e_I} + f(k_{e_I})$, the elements in each of the following set will change by the same amount.

$$\{(l_{e_1}, ..., l_{e_I}; l_{e_J}, ...l_{e_n}), (l_{e_{M_1}}, ..., l_{e_{M_1+L_1}}), ..., (l_{e_{M_m}}, ..., L_{e_{M_m+L_m}})\}$$

Thus the invariant data is given by,

$$\begin{aligned} L_s \;=\; \{(\; (\amalg_{i\in(1,...,I,J,...,n), i<j} l_{e_i} - l_{e_{i+1}}), (\amalg_{i\in(M_1,...,M_1+L_1)} l_{e_i} - l_{e_{i+1}}), ..., \\ (\amalg_{i\in(M_m,...,M_m+L_m)} l_{e_i} \;-\; l_{e_{i+1}})\} \end{aligned} \quad (3.138)$$

where $\amalg$ stands for disjoint union. $I+1 \equiv J$, $n+1 \equiv 1$ and $M_i + L_i + 1 \equiv M_i$.

- If $k_{e_n} = k_{e_1} + 2\pi$ and $|l_{e_1} - l_{e_n}| = 0$ then we exclude this zero from the first (sub)set in L_s. Reason for this exclusion will become clear below.

Now we define an equivalence relation on $(\gamma(\mathbf{s}), \vec{k}(\mathbf{s}), L_{\mathbf{s}})$ as follows.
$(\gamma(\mathbf{s}), \vec{k}(\mathbf{s}), L_{\mathbf{s}}) \sim (\gamma(\mathbf{s}'), \vec{k}(\mathbf{s}'), L_{\mathbf{s}'})$ if $\exists$ a $\phi \in Diff^{P}_{\tilde{\mathbf{s}}}\mathbf{R}$ such that $(\gamma(\mathbf{s}'), \vec{k}(\mathbf{s}'), L_{\mathbf{s}'}) = (\gamma(\phi\cdot\mathbf{s}), \vec{k}(\phi\cdot\mathbf{s}), L_{\phi\cdot\mathbf{s}})$.

Given a $L_{\mathbf{s}}$, It is a straightforward exercise to show that (with the exclusion of a certain zero specified above) $L_{\phi\cdot\mathbf{s}}$ is one of the following types for any ϕ.
Denote the subsets of $L_{\mathbf{s}}$ by A, A_1,...,A_k respectively.

(e.g. $A = (\Pi_{i\in(1,...,I,J,...,n),i<j} l_{e_i} - l_{e_{i+1}})$.)

$$\begin{aligned} L_{\phi\cdot \mathbf{s}} &= \{\sigma(A), A_1, ..., A_k\} \\ &\text{or} \\ &= \{\sigma(A_i), A_{i+1}, ..., A_k, ((l_{e_J} - l_{e_{J+1}}), ..., (l_{e_n} - l_{e_1}), ..., (l_{e_I} - l_{e_J})), A_1, ..., A_{i-1}\} \\ &\text{or} \\ &= \{A_i, ..., A_k, ((l_{e_J} - l_{e_{J+1}}), ..., (l_{e_n} - l_{e_1}), ..., (l_{e_I} - l_{e_J})), A_1, ..., A_{i-1}\} \end{aligned} \tag{3.139}$$

where σ is any cyclic permutation.

Claim 3.8.1

$$\eta_{[\mathbf{s}]} = \eta_{[\mathbf{s}_f]} \ \forall f \iff \eta_{[\mathbf{s}]} = \eta_{[\gamma(\mathbf{s}),\vec{k},L_\mathbf{s}]}.$$

Proof: $\Leftarrow$ is obvious as $L_{\mathbf{s}_f} = L_\mathbf{s} \ \forall f$. (By definition)
So it remains to show that $\eta_{[\mathbf{s}_f]} = \eta_{[\mathbf{s}]} \ \forall f \Rightarrow \eta_{[\mathbf{s}]} = \eta_{[\gamma,\vec{k},L_\mathbf{s}]}$.
But, $[\mathbf{s}]\bigcap(\cap_{f\in C^0(S^1)}[\mathbf{s}_f]) = [\mathbf{s}\bigcap(\cap_{f\in C^0(S^1)}\mathbf{s}_f)] = [\gamma(\mathbf{s}), \vec{k}, L_\mathbf{s}]$.
(Where the first equality uses the fact that $\hat{W}(s_f)$ is an observable.) This proves the if-side of the claim.

Thus the condition that the rigging map commutes with $\hat{W}^+(s_f)$ has reduced the ambiguity in the definition of η.

B. Making the rigging map commute with Conformal isometries

In this appendix we derive conditions on $\eta_{[s]}$ such that the Rigging map commutes with the action of $\hat{V}[\phi_c]$, when $\mathbf{s} \in [\mathbf{s}]$ is such that the embedding charges on s satisfy the following condition.
$\exists\ 1 \leq I < M_1 < M_L < J \leq n$ such that $k_J - k_I = 2\pi$ and $k_{M_1-1} \neq k_{M_1} = ... = k_{M_L} \neq k_{M_L+1}$.
As before, we only analyze the right-moving sector and supress the +-indices.
We proceed in two steps. First we list all the possible orbits that can be linked to each other via action of $\hat{V}[\phi_c]$ for some ϕ_c. That is, starting with a $\mathbf{s} = \{\gamma(\mathbf{s}), (k_{e_1}, ..., k_{e_n})\}$ (with $k_{e_J} - k_{e_I} = 2\pi$, $k_{e_{M_1}} = ... = k_{e_{M_L}}$), what are the possible s′s that can be obtained from s via an action of $\hat{V}[\phi_c]$, $\hat{U}(\phi)$ on $|\mathbf{s}\rangle$.

We then define a Virasoro-invariant set $[\mathbf{s}^I, L_\mathbf{s}]$ which is the data common to all the orbits mentioned above. This will imply that $\eta_{[\mathbf{s}]} = \eta_{[\mathbf{s}^I, L_\mathbf{s}]}$.

lemma 1: Let $s = \{\gamma, (k_{e_1}, ..., k_{e_n})\}$ such that $k_{e_J} - k_{e_I} = 2\pi.(\Rightarrow k_{e_1} = ... = k_{e_I}, k_{e_J} = ... = k_{e_n})$ And let $k_{e_{M_1}} = ... = k_{e_{M_L}}$, with $I < M_1 < M_L < J$. $\hat{U}(\phi)\hat{V}[\phi_c]|\mathbf{s}\rangle = |\mathbf{s}_1\rangle$ for any ϕ and ϕ_c, is equivalent to $\mathbf{s}_1 = \{\gamma(\mathbf{s}_1, (k'_{e'_1}, ..., k'_{e'_m})\}$ being one of the following type.

1. $m = n$, $\exists\ 1 \leq I_1 < J_1 \leq n$ such that $k'_{e'_{J_1}} - k'_{e'_{I_1}} = 2\pi$ with $J_1 - I_1 = J - I$ and $\exists M_2$ such that $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(M_L-M_1)}}$ with $M_2 - I_1 = M_1 - I$.
2. $m = n$, $\exists\ 1 \leq I_1 < J_1 \leq n$ such that $k'_{e'_{J_1}} - k'_{e'_{I_1}} = 2\pi$ with $(N - J_1 + 1) - I_1 = M_L - M_1$ and $\exists M_2$ such that $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(N-I+1)+J}}$ with $M_2 - I_1 = J - M_L$.
3. $m = n$, $\exists\ M_2, M_3$ such that $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(M_L-M_1)}}$, $k'_{e'_{M_3}} = ... = k'_{e'_{M_3+(N-J+1)+I}}$ with $M_3 - (M_2 + (M_L - M_1)) = J - M_L$, $M_2 - 1 \leq (M_1 - (I-1))$.
4. $m = n$, $\exists\ M_2, M_3$ such that $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(N-J+1)+I}}$, $k'_{e'_{M_3}} = ... = k'_{e'_{M_3+(M_L-M_1)}}$ with $M_3 - (M_2 + (N - J + 1) + I) = M_1 - I$, $M_2 - 1 \leq (J - (M_L + 1))$.
5. $m = n+1$, $\exists\ 1 \leq I_1 < J_1 \leq n+1$ such that $k'_{e'_{J_1}} - k'_{e'_{I_1}} = 2\pi$ with $J_1 - I_1 = J - I$, $I_1 \leq I$ and $\exists M_2$ such that $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(M_L-M_1)}}$ with $M_2 - I_1 = M_1 - I$.
6. $m = n + 1$, $\exists\ 1 \leq I_1 < J_1 \leq n + 1$ such that $k'_{e'_{J_1}} - k'_{e'_{I_1}} = 2\pi$ with $J_1 - I_1 = J - I$, $I_1 \leq (N - J + 1)$ and $\exists\ M_2$ such that $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(M_L-M_1)}}$ with $M_2 - I_1 = M_1 - I$.
7. $m = n + 1$, $\exists\ I_1 < J_1$ such that $k'_{e'_{J_1}} - k'_{e'_{I_1}} = 2\pi$ with $I_1 \leq (M_L - M_1) + 1$, $J_1 - I_1 = N - (M_L - M_1) + 1$ and $\exists\ M_2$ such that $k'_{e'_{M_2}} = ... = k'_{e'_{(N-J+1)+I}}$ with $M_2 - I_1 = J - M_L$.
8. $m = n + 1$, $k'_{e'_1} + 2\pi = k'_{e'_{n+1}}$ and $\exists\ M_2, M_3$ such that $M_2 < M_3$, $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(M_L-M_1)}}$, $k'_{e'_{M_3}} = ... = k'_{e'_{M_3+(N-J+1)+I}}$ and $M_2 - 1 \leq M_2 - (I - 1)$, $M_3 - (M_2 + (M_L - M_1)) = J - M_L$.
9. $m = n + 1$, $k'_{e'_1} + 2\pi = k'_{e'_{n+1}}$ and $\exists\ M_2, M_3$ such that $M_2 < M_3$, $k'_{e'_{M_2}} = ... = k'_{e'_{M_2+(N-J+1)+I}}$, $k'_{e'_{M_3}} = ... = k_{e'_{M_3+(M_L-M_1)}}$ and $M_2 - 1 \leq J - (M_L + 1)$, $M_3 - (M_2 + (N - J + 1) + I) = M_1 - I$.

Note Instead of considering $\tilde{\tilde{\mathbf{s}}} = \{\gamma(\mathbf{s}), \vec{k}, L_\mathbf{s}\}$ we are only considering the embedding charge networks in this claim. This is because neither $\hat{U}(\phi)$ nor $\hat{V}[\phi_c]$ can change $L_\mathbf{s}$.

proof : We will first show that given any one of the above $\mathbf{s}'$, it can be "linked" to s via action of $\hat{U}(\phi)$ and $\hat{V}[\phi_c]$ for some ϕ and ϕ_c. Consider $\mathbf{s}'$ which is of type 1. Recall that

$$\overline{\mathrm{s}}_{ext} = \{\cup_{N\in\mathbf{Z}}T_N(\gamma(\mathbf{s})), (..., (k_{e_1}^{-N}, ..., k_{e_n}^{-N}), ..., (k_{e_1}, .., k_{e_n}), ..., (k_{e_1}^{N}, ..., k_{e_n}^{N}), ...\} \tag{3.140}$$

where $k_{e_I}^N = k_{e_I} + 2\pi N$.

case-1. $I > I_1$. Let ϕ be a diffeo which maps $e_{I-I_1+1}^{-N} \rightarrow e_1'$,..., $e_{I-I_1}^{-N+1} \rightarrow e_n'$. (All the in between edges should be mapped onto corresponding edges of $\gamma(\mathrm{s}'$.) The corresponding charges on $\phi^{-1}(\overline{\mathrm{s}}_{ext})|_{[0,2\pi]}$ are given by,

$$(k_{e_{I-I_1+1}}^{-N}, ..., k_{e_{M_1}}^{-N}, ..., k_{e_{M_L}}^{-N}, ..., k_{e_J}^{-N}, ..., k_{e_{I-I_1}}^{-N+1}) \tag{3.141}$$

where the charges $k_{e_{M_1}}^{-N}, ..., k_{e_{M_L}}^{-N}$ are sitting on the edges $\phi(e_{M_1}^N) = e'_{M_2}$ etc. Now choose a ϕ_c such that $\phi_c^{-1}(k_{e_{I-I_1+1}}^{-N}) = k'_{e_1'}, ..., \phi_c^{-1}(k_{e_{M_1}}^{-N}) = k'_{e'_{M_2}}, ..., \phi_c^{-1}(k_{e_J}^{-N}) = k'_{e'_{J_1}}$. (Such a ϕ_c always exists). Thus in this case one can map $|\mathrm{s}\rangle$ to $T_{|\mathrm{s}'\rangle}$ using some ϕ and some ϕ_c.

case-2. $I < I_1$. In this case, choose a diffeo which maps $(e_{n-(I_1-(I+1))}^{-N+1}, ..., e_{n-(I_1-(I+1))-1}^{-N})$ onto $(e_1', ..., e_n')$ and then choose a ϕ_c which maps the charges on $(e_{n-(I_1-(I+1))}^{-N+1}, ..., e_{n-(I_1-(I+1))-1}^{-N})$ to $(k'_{e_1'}, ..., k'_{e_n'})$.

When s$'$ is of type-2. One has to choose a diffeo which will map $(e_i^{-N}, ..., e_{i-1}^{-N+1})$ with $M_1 \leq i \leq M_{L_1}$ onto $(e_1', ..., e_n')$ such that $M_{L_1} - i = I_1$. Then choose a ϕ_c which will map the charges on $(e_i^{-N}, ..., e_{i-1}^{-N+1})$ onto $(k'_{e_1'}, ..., k'_{e_n'})$. (Such a ϕ_c will exist due to relations between (I, J, M_1, M_L) and (I_1, J_1, M_2, M_{L_2}).

The analysis for the remaining cases is similar. It merely involves lot of book-keeping so we do not reproduce it here.

Now we need to show that, if $T_{\mathrm{s}'} = \hat{U}(\phi)\hat{V}[\phi_c]T_{\mathrm{s}}$ then s$'$ has to be one of the types in the list.

Let ϕ be a diffeo which maps initial vertex of e_i^N (with $1 \leq i \leq I$) to x=0. Then,

$$\begin{aligned}\phi^{-1}(\overline{s}_{ext})|_{[0,2\pi]} = \{ & (\phi^{-1}(e_i^N), ..., \phi^{-1}(e_I^N), ..., \phi^{-}(e_J^N), ..., \phi^{-1}(e_{i-1}^{N+1})),\\ & (k_{e_i} + 2N\pi = k_{e_1} + 2N\pi, ..., k_{e_I} + 2N\pi = k_{e_1} + 2N\pi, ...,\\ & k_{e_j} + 2N\pi = k_{e_1} + 4N\pi, ..., k_{e_{i-1}} + 4N\pi = k_{e_1} + 4N\pi)\}\end{aligned}$$

where we have used $k_{e_1} = ... = k_{e_I} = k_{e_J} - 2\pi = ... = k_{e_n} - 2\pi$.

Now, applying $\hat{V}[\phi_c]$ will change first I-i charges and final n-J+i charges by equal amount and the intermediate $M_L - M_1$ charges by equal amount. It is easy to see that the resulting charge-network is of type 1.
Depending on which vertex or interior point of γ is mapped onto x=0 under a diffeo, one will obtain $\mathbf{s}'$ which will be in one of the 9 classes listed in the claim. This finishes the sketch of the proof.

The goal of the above claim is to help us isolate the data that is common to $([\mathbf{s}], [\phi_c(\mathbf{s})]) \ \forall \ \phi_c$ in the case when $\exists \ \mathbf{s} \in \ [\mathbf{s}]$ such that $\mathbf{s} = \{\gamma(\mathbf{s}), (k_{e_1}, ..., k_{e_n})\}$ (with $k_{e_J} - k_{e_I} = 2\pi$, $k_{e_{M_1}} = ... = k_{e_{M_L}}$). But before doing that, we will need to introduce some more notations.
Set-A : Disjoint union of two sets (set-a, set-b) of equal charges such that charges in set-a differ by charges in set-b by 2π.
e.g. For the $\mathbf{s}$ used in claim above, set-a $= (k_{e_1}, ..., k_{e_I})$, set-b $= (k_{e_J}, ..., k_{e_n})$.
Set-B : Set of charges which are equal to each other and are neither equal to, nor differ by 2π from any other charges. Note that in a given s, there can be more then one set of this type.
e.g. For the $\mathbf{s}$ used in claim above, set-B $= (k_{e_{M_1}}, ..., k_{e_{M_L}})$.
Definition : Given an embedding set, $(k_{e_1}, ..., k_{e_n})$ and a pair of subsets, $(k_{e_{i_1}}, ..., k_{e_{i_m}})$, $(k_{e_{j_1}}, ..., k_{e_{j_l}})$, such that $i_1 < i_2... < i_m < j_1 < ... < j_l$, the distance between two subsets is defined to be $(j_1 - i_m)$.
Definition : Given a $\mathbf{s} = \{\gamma(\mathbf{s}), (k_{e_1}, ..., k_{e_n})\}$, define

$$\mathbf{s}_I := \{\gamma(\mathbf{s}), (\text{Distance between set-b and set-a; cardinality of set-B, } \text{Distance betwen set-b and set-B.})\} \tag{3.142}$$

e.g. For $\mathbf{s}$ in Claim.B.1, $\mathbf{s}_I \ = \ \{\gamma(\mathbf{s}), (J - I, M_L - M_1, J - M_L)\ \}$.[7]
Note : If there is more then one set of type-B, then $\mathbf{s}_I$ should contain cardinality of all such sets, and also contain distance between various sets of this type.

[7] Instead of specifying $J - M_L$, one can also specify $M_1 - I$.

Defining $[\mathbf{s}_I]$

Given a $\mathbf{s}_I = \{\gamma(\mathbf{s}), (J-I, M_L - M_1, M_1 - I)\}$ (with $|\gamma(\mathbf{s})| = n$), define an equivalence relation on the set of all $\mathbf{s}_I$'s as follows.[8]

$$\mathbf{s}_I \sim \tag{3.143}$$

$$\begin{aligned}
&(1)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\mathbf{s})| = n,\ (J-I,\ M_L - M_1,\ M_1 - I)\} \\
&(2)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\gamma(\mathbf{s}))| = n, (0, (\ M_L - M_1, n - J + 1 + I),\ (J - M_L))\} \\
&(3)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\gamma(\mathbf{s}))| = n, (0, (n - J + 1 + I, M_L - M_1), (M_1 - I))\} \\
&(4)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\gamma(\mathbf{s}))| = n, (n - M_L + 1 + M_1, n - J + 1 + I, J - M_L)\} \\
&(5)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\gamma(\mathbf{s}))| = n+1, (J-I,\ M_L - M_1,\ M_1 - I)\} \\
&(6)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\gamma(\mathbf{s}))| = n+1, (n, (M_L - M_1, n - J + 1 + I), J - M_L)\} \\
&(7)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\gamma(\mathbf{s}))| = n+1, (n, (n - J + 1 + I, M_L - M_1), M_1 - I)\} \\
&(8)\{\phi^{-1}(\gamma(\mathbf{s})), |\phi^{-1}(\gamma(\mathbf{s}))| = n+1, (n - M_L + M_1 + 1, n - J + 1 + I, J - M_L)\}
\end{aligned}$$

For the $\mathbf{s}_I$'s in (2), set-a and set-b are empty. There are two sets of type B with cardinalities $M_L - M_1$, $n - J + 1 + I$ respectively. The distance between these two sets is $J - M_L$. Other classes can be understood in a similar way.

We refer to this equivalence class as $[\mathbf{s}_I]$. This set is Virasoro-invariant by construction.

Claim 3.8.2

$$[\mathbf{s}] \cap (\cap_{\phi_c \in \Phi}[\phi_c(\mathbf{s})]) = [\mathbf{s}_I] \tag{3.144}$$

Note : Here $\cap$ is not to be understood as set-theoretic intersection, but rather as a (formal) symbol which indicates the invariants of $[\mathbf{s}]$ under action of $\hat{V}[\phi_c]$.
proof : From claim.A.1 it can be seen that any $\mathbf{s}'$ that is the "image" of $\mathbf{s}$ under the action of $\hat{U}(\phi)$ and $\hat{V}[\phi_c]$ has $\mathbf{s}'_I$ that is equivalent to $\mathbf{s}_I$. This implies that

$$[\mathbf{s}_I] \subset [\mathbf{s}] \cap (\cap_{\phi_c \in \Phi}[\phi_c(\mathbf{s})]).$$

[8]More precisely an equivalence relation should be defined on the pair $(\mathbf{s}_I, L_{\mathbf{s}})$. The equivalence relation defined below can be combined with relation defined on $L_{\mathbf{s}}$ to define relation on the pair. However we spare reader of all the gory details.

Now given s, choose a ϕ_c which satisfies the following.

$$\begin{aligned}
\phi_c(k_{e_1}) &\neq k_{e_i} + 2N\pi \text{ for any N and i} \\
\phi_c(k_{e_j}) &\neq k_{e_i} + 2N\pi\forall\ I+1 \leq j < M_1 - 1 \text{ for any N and i} \\
\phi_c(k_{e_{M_1}}) &\neq k_{e_i} + 2N\pi \text{ for any N and i} \\
\phi_c(k_{e_j}) &\neq k_{e_i} + 2N\pi\forall\ M_1 + L_1 + 1 \leq j < J \text{ for any N and i}
\end{aligned} \tag{3.145}$$

(Obviously such a ϕ_c always exists).

In this case, $[\mathbf{s}] \cap [\phi_c(\mathbf{s})] = [\mathbf{s}_I]$. Which implies $[\mathbf{s}] \cap (\cap_{\phi_c \in \Phi}[\phi_c(\mathbf{s})]) \subset [\mathbf{s}_I]$. Whence $[\mathbf{s}] \cap (\cap_{\phi_c \in \Phi}[\phi_c(\mathbf{s})]) = [\mathbf{s}_I]$.

Thus we finally have (re-inserting L_s),

$$[\tilde{\mathbf{s}}] \cap (\cap_{\phi_c \in \Phi}[\phi_c(\tilde{\mathbf{s}})]) = ([\mathbf{s}_I, L_{\mathbf{s}}]) \tag{3.146}$$

C. Lemmas concerning Semiclassicality and Weaves.

Lemma 1: If $\Delta k_J \geq \pi$ (see (3.124),(3.126)) for some J, $1 \leq J \leq N$ then $-1 \leq f_{s,m=1} \leq \pi$.

Proof: Let $\Delta k_J \geq \pi$. Equations (3.128) imply that

$$\sum_{I \neq J} \Delta k_I \leq \pi, \tag{3.147}$$

and, hence, that

$$\Delta k_I|_{I \neq J} \leq \pi. \tag{3.148}$$

This in conjunction with the fact that $|\frac{\sin x}{x}| \leq 1$ implies that

$$\sum_{I=1}^{N} \sin \Delta k_I \leq \sum_{I \neq J} \Delta k_I + \sin \Delta k_J \leq \pi. \tag{3.149}$$

From equation (3.148) and $\Delta k_J \geq \pi$, we have that

$$\sum_{I=1}^{N} \sin \Delta k_I \geq -1. \tag{3.150}$$

The Lemma follows immediately from equations (3.149), (3.150) and the definition (3.119) of $f_{s,m=1}$

Lemma 2: If $\Delta k_I \leq \pi,\ I = 1,..,N$ (see (3.124),(3.126)) then $0 \leq f_{s,m=1} \leq 2\pi$.

Proof: This follows immediately from the fact that $|\frac{\sin x}{x}| \leq 1$ in conjunction with equations (3.128) and the definition (3.119) of $f_{s,m=1}$.

Lemma 3: Equation (3.125) implies that as $\epsilon \to 0,\ \Delta k_I \to 0,\ I = 1,..,N$ and $N \to \infty$.

Proof: From Lemma 1 and equation (3.125) it follows that for sufficiently small ϵ, it must be the case that $\Delta k_I \leq \pi, I = 1,..,N$.

Next, let α be the minimum value of the bounded, continuous function $\frac{\sin\theta}{\theta}$ in the interval $[0, \frac{\pi}{2}]$ (here $\frac{\sin\theta}{\theta}|_{\theta=0} := 1$). Define the function $f(x) := x - \sin x - \frac{\alpha}{6}x^3$. It is easy to check that $\frac{df}{dx} \geq 0,\ x \in [0,\pi]$ and that $f(x=0) = 0$. This implies that $x - \sin x \geq \frac{\alpha}{6}x^3,\ x \in [0,\pi]$. This in conjunction with equations (3.128), (3.125) implies that $\sum_{I=1}^{N}(\Delta k_I)^3 < \frac{6\epsilon}{\alpha}$ so that $\Delta k_I \to 0,\ I = 1,..,N$ as $\epsilon \to 0$. This in turn, together with (3.128), implies that $N \to \infty$ as $\epsilon \to 0$.

Lemma 4: Any normalised $|\psi\rangle \in \mathcal{H}_{kin}$ admits the expansion:

$$|\psi\rangle = \sum_j a_j|s_j, \psi_{jM}\rangle,\ \ |s_j, \psi_{jM}\rangle := |s_j\rangle \otimes |\psi_{jM}\rangle, \tag{3.151}$$

$$\langle s_i|s_j\rangle = \delta ij,\ s_j = \{\gamma(s_j), (k_{e_1^j}, .., k_{e_{n_j}^j})\} \tag{3.152}$$

$$\langle \psi_{jM}|\psi_{jM}\rangle = 1, \tag{3.153}$$

$$\sum_j |a_j|^2 = 1. \tag{3.154}$$

Here s_j are embedding charge labels, $e_I^j, I = 1,..,n_j$ are the edges of the graph underlying s_j, a_j are complex coefficients and $|\psi_{jM}\rangle \in \mathcal{H}_M$.

If $|\psi\rangle$ is semiclassical then the coefficients a_j are such that $|\psi\rangle$ is peaked around s_j such that s_j are weaves.

Proof: The proof closely mirrors the arguments of section 3.6.1. Semiclassicality implies that to leading order in $\hbar$,

$$\langle\psi|[\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}]|\psi\rangle \approx i\hbar\{e^{i\alpha q_m}, e^{i\beta p_m}\} = -i\hbar\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m} \tag{3.155}$$

Using equations (3.151), (3.80), (3.155), we have that

$$\sum_j |a_j|^2 2\sin(\frac{\alpha\beta\hbar}{2} f_{s_j,m})\langle s_j, \psi_{jM}|\widehat{e^{i\alpha q_m+i\beta p_m}}|s_j, \psi_{jM}\rangle \approx \hbar\alpha\beta 2\pi m e^{i\alpha q_m+i\beta p_m} \tag{3.156}$$

where

$$f_{s_j,m} = \sum_{I=1}^{n_j} \sin m\Delta k_I^j, \tag{3.157}$$

and $\Delta k_I^j := k_{I+1}^j - k_I^j$ for $1 \leq I \leq n_j - 1$, $\Delta k_{n_j}^j := k_1^j - k_{n_j}^j + 2\pi$ and we have set $k_I^j := \hbar k_{e_I^j}$. From Lemmas 1 and 2 it follows that

$$-1 \leq f_{s_j,m=1} \leq 2\pi \ \forall j. \tag{3.158}$$

Since $f_{s_j,m=1}$ is bounded, equation (3.156) implies that to leading order in $\hbar$, we have that

$$\sum_j |a_j|^2 f_{s_j,m=1}\langle s_j, \psi_{jM}|\widehat{e^{i\alpha q_1+i\beta p_1}}|s_j, \psi_{jM}\rangle e^{-i\alpha q_1-i\beta p_1} \approx 2\pi \tag{3.159}$$

Denote the left handside of equation (3.159) by LHS. Equation (3.159) implies that

$$|LHS - 2\pi| \leq \delta, \quad \delta << 1. \tag{3.160}$$

Taking absolute values of both sides of equation (3.159) and using (3.158), (3.154) and the fact that $\widehat{e^{i\alpha q_m+i\beta p_m}}$ is a bounded operator of norm 1, we have that

$$2\pi \geq \sum_j |a_j|^2 |f_{s_j,m=1}| \geq |LHS|. \tag{3.161}$$

From (3.161), (3.160) we have that $\delta \geq |2\pi - LHS| \geq 2\pi - |LHS| \geq 2\pi - \sum_j |a_j|^2 |f_{s_j,m=1}|$, so that

$$\sum_j |a_j|^2 |f_{s_j,m=1}| \geq 2\pi - \delta. \tag{3.162}$$

Let $J_<$ be the set of all j such that $|f_{s_j,m=1}| \leq 2\pi - \delta^{\frac{1}{2}}$ and let $\sum_{j\in J_<} |a_j|^2 = P_<$. Then (3.158),(3.162) imply that $P_<(2\pi - \delta^{\frac{1}{2}}) + (1-P_<)2\pi \geq 2\pi - \delta$ so that $P_< \leq \delta^{\frac{1}{2}}$.

Thus as $\delta \rightarrow 0$, almost all j are such that $|f_{s_j,m=1}| \geq 2\pi - \epsilon$, where we have set $\epsilon := \delta^{\frac{1}{2}}$. Using (3.158), this, in turn, implies that for small enough ϵ,

$$f_{s_j,m=1} \geq 2\pi - \epsilon. \tag{3.163}$$

This brings us back to equation (3.123) with $s = s_j, m = 1$. The analysis subsequent to that equation implies that such s_j must be a weave.

Lemma 5: Let $|\psi\rangle \in \mathcal{H}_{phys}$ be semiclassical. Then $|\psi\rangle$ is peaked at group averages of embedding eigen states which are based on weaves.

Proof: Recall that $|\psi\rangle$ is in the completion of $\eta(\mathcal{D})$ where $\mathcal{D}$ is the finite span of charge network states. It is then straightforward to see that any such $|\psi\rangle$ admits the expansion:

$$|\psi\rangle = \sum_j a_j \eta(|s_j\rangle \otimes |\psi_{jM}\rangle), \tag{3.164}$$

such that

$$\eta(|s_i\rangle \otimes |\psi_{iM}\rangle)[|s_j\rangle \otimes |\psi_{jM}\rangle] = \delta_{ij}, \tag{3.165}$$

and $|s_i\rangle$, $|s_j\rangle$ are not gauge related if $i \neq j$ i.e. for $i \neq j$ and $\forall \phi$,

$$|s_i\rangle \neq \hat{U}(\phi)|s_j\rangle. \tag{3.166}$$

Here s_j is an embedding charge network label, ϕ is a gauge tranformation and $|\psi_{jM}\rangle \in \mathcal{H}_M$. We shall use the notation of Lemma 4 for the edges and charge labels of s_j. Note that $|\psi_{jM}\rangle$ is such that $\eta(|s_j\rangle \otimes |\psi_{jM}\rangle) \in \mathcal{H}_{phys}$ as implied by (3.165). Using (3.87), the normalization $< \psi|\psi >_{phys} = 1$ implies that

$$\sum_j |a_j|^2 = 1 \tag{3.167}$$

Semiclassicality implies that, to leading order in $\hbar$,

$$\langle\psi|[\widehat{e^{i\alpha q_m}}, \widehat{e^{i\alpha p_m}}]|\psi\rangle_{phys} \approx +i\hbar\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m}, \tag{3.168}$$

where the '+' sign in the right hand side is due to the fact that operators act on $\mathcal{H}_{phys}$ by dual action (see Footnote 4). Using equations (3.80) and (3.166), we have that

$$\sum_j |a_j|^2 2i\sin(\frac{\alpha\beta\hbar}{2} f_{s_j,m})\langle\eta(|s_j\rangle \otimes |\psi_{jM}\rangle),\ \widehat{e^{i\alpha q_m + i\beta p_m}}\eta(|s_j\rangle \otimes |\psi_{jM}\rangle)\rangle_{phys}$$
$$\approx\ i\hbar\alpha\beta 2\pi m e^{i\alpha q_m + i\beta p_m}. \qquad (3.169)$$

Here $f_{s_j,m}$ is defined as in Lemma 4. [9] This is the analog of equation (3.156) of Lemma 4. The analysis of Lemma 4 subsequent to that equation applies here identically thus proving Lemma 5.

D. Lemmas concerning the no go result of section 3.6.2.

Lemma 6: No states $|\psi\rangle \in \mathcal{H}_{kin}$ exist which are semiclassical with respect to the uncountable set of operators $\{\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}, |\alpha - \alpha_0| < \epsilon, |\beta - \beta_0| < \delta\}$ for any fixed m, α_0, β_0 and any $\epsilon, \delta > 0$.

Proof: As in Lemma 4 of Appendix A, any $|\psi\rangle \in \mathcal{H}_{kin}$ admits the expansion (3.151)- (3.154). Additionally we may expand $|\psi_{jM}\rangle$ in terms of matter charge networks so that for any fixed j,

$$|\psi_{jM}\rangle \ = \ \sum_{r^j} b_{r^j}|s'_{r^j}\rangle \qquad (3.170)$$

$$\langle s'_{r_1^j}|s'_{r_2^j}\rangle = \delta_{r_1^j, r_2^j} \qquad (3.171)$$

where r^j varies over a countable set (as, of course, does j), b_{r^j} are complex coefficients and and s'_{r^j} are matter charge networks.

Let C be the set of all j such that $\gamma(s_j)$ has at least one edge $e(j)$ with embedding charge $k_{e(j)}$ such that $\cos m\hbar k_{e(j)} \neq 0$. For every $j \in C$ choose an edge $e^j \subset \gamma(s_j)$ with embedding charge k_{e^j} such that

$$c_j := \cos m\hbar k_{e(j)} \neq 0. \qquad (3.172)$$

Let S be the set of all j such that $j \notin C$. Clearly, for each $j \in S$ we can fix an

[9] It is straightforward to check that $f_{s_j,m}$ in (3.157) is a *gauge invariant* function of s_j i.e. $f_{s_j,m} = f_{s'_j,m} \forall s'_j$ such that $\exists$ a gauge transformation ϕ such that $|s'_j\rangle = \hat{U}(\phi)|s_j\rangle$.

edge $e^j \in \gamma(s_j)$ such that its charge label k_{e^j} satisfies

$$s_j := \sin m\hbar k_{e(j)} \neq 0. \tag{3.173}$$

Next, let L be the set of all matter charges which occur in $s'_{rj} \forall j, r$. Let ΔL be the set of differences between all pairs of elements of L i.e. $\Delta L = \{l - l' \forall l, l' \in L\}$. For every $j_C \in C$, $j_S \in S$, define the sets $\Delta L_{j_C}, \Delta L_{j_S}$ whose elements are obtained by dividing elements of ΔL by c_{j_C}, s_{j_S} (see (3.172),(3.173)) i.e. $\Delta L_{j_C} := \{\frac{x}{c_{j_C}} \forall x \in \Delta L\}$, $\Delta L_{j_S} := \{\frac{x}{s_{j_S}} \forall x \in \Delta L\}$. Finally, let $\Delta L_C := \cup_{j_C \in C} L_{j_C}, \Delta L_S := \cup_{j_S \in S} L_{j_S}$.

Note that $\Delta L_C, \Delta L_S$ are both countable sets. It follows that in any neighbourhood of α_0, β_0 there exist uncountably many α, β such that $\alpha \notin \Delta L_C, \beta \notin \Delta L_S$. Then from (3.80) and the fact that $\widehat{e^{i\beta p_m}}$ is an operator of unit norm, it follows that for such α, β,

$$|\langle\psi|\widehat{e^{i\alpha q_m}}|\psi\rangle| = \sum_{j \in S} |a_j|^2, \tag{3.174}$$

$$|\langle\psi|\widehat{e^{i\beta p_m}}|\psi\rangle| \leq \sum_{j \in C} |a_j|^2 = 1 - \sum_{j \in S} |a_j|^2. \tag{3.175}$$

Semiclassicality requires that both (3.174) and (3.175) be close to unity. Clearly, this is not possible.

Lemma 7: No states $|\psi\rangle \in \mathcal{H}_{phys}$ exist which are semiclassical with respect to the uncountable set of operators $\{\widehat{e^{i\alpha q_m}}, \widehat{e^{i\beta p_m}}, |\alpha - \alpha_0| < \epsilon, |\beta - \beta_0| < \delta\}$ for any fixed m, α_0, β_0 and any $\epsilon, \delta > 0$.

Proof: As in Lemma 5, Appendix A, any $|\psi\rangle \in \mathcal{H}_{phys}$ admits the expansion (3.164)- (3.166). Further $|\psi_{jM}\rangle$ can be expanded as in equation (3.170)- (3.171) of Lemma 6. Note that the antilinearity of η implies that we may rewrite equation (3.164) as

$$|\psi\rangle = \eta(\sum_j a_j^* |s_j\rangle \otimes |\psi_{jM}\rangle). \tag{3.176}$$

Next, let us construct the sets $\Delta L_C, \Delta L_S$ (as defined in Lemma 6) for the state $\sum_j a_j^* |s_j\rangle \otimes |\psi_{jM}\rangle \in \mathcal{H}_{kin}$. It follows straightforwardly from the periodicity of the cosine and sine functions in conjunction with the action of gauge transformations (3.75) that we may choose the sets $\Delta L_C, \Delta L_S$ in such a way that they are identical for any (kinematic) state which is gauge related to the state $\sum_j a_j^* |s_j\rangle \otimes |\psi_{jM}\rangle$. Thus the sets ΔL_C, ΔL_S can be chosen so as to depend only on the physical state $|\psi\rangle$, and it is straightforward to see that, as in Lemma 6, if we choose $\alpha \notin \Delta L_C, \beta \notin \Delta L_S$, we obtain equations (3.174), (3.175) with $|\psi\rangle$ as in (3.176). This proves the Lemma.

E. Choice of units.

In this appendix we summarize dimensions of various operators and parameters of the theory. We have set the speed of light c to be unity.

$$\begin{aligned}
&[S_0] = ML = [\hbar] \\
&[f] = M^{\frac{1}{2}} L^{\frac{1}{2}}, \ [\pi_f] = M^{\frac{1}{2}} L^{-\frac{1}{2}} \\
&[X^{\pm}] = L, [\Pi_{\pm}] = ML^{-1} \\
&[q_{(\pm)n}] = M^{\frac{1}{2}} L^{\frac{1}{2}} = [p_{(\pm)n}]
\end{aligned} \tag{3.177}$$

where $[n] = L^{-1}$.

The dimensions of the above fields naturally imply the dimensions of the various charges and parameters involved in the theory.

$$\begin{aligned}
&[k_e] = M^{-1}, \ [l_e] = M^{-\frac{1}{2}} L^{-\frac{1}{2}} \\
&[\alpha] = M^{-\frac{1}{2}} L^{-\frac{1}{2}}
\end{aligned} \tag{3.178}$$

where the parameter α occurs in the exponentiated observables defined in (3.77).

Throughout this chapter, we have fixed the units such that length of the $T=$ constant circle is 2π. Thus the only arbitrary scale in the theory is the mass scale.

4

polymer quantization of PFT on $\mathbf{R}^2$: CGHS model

4.1 Introduction

In this chapter we turn to the Polymer quantization of the CGHS model [14]. Having briefly mentioned the model in the introduction, let us first see why it is an interesting field theory.

In the past two decades, two dimensional theories of gravity have received quite a bit of attention [21] as toy models to address questions arising in (four dimensional) quantum gravity. In particular the CGHS model whose action is inspired from the effective target space action of 2-d non-critical string theory constitutes a highly desirable choice, due to its various features like classical integrability, existence of Black-hole space-times in its solution space and the presence of Hawking radiation and evaporation at 1-loop level.

Semi-classical analysis of this model has been carried out by number of authors ([14], [42], [46], [20] and ref.therein). By incorporating a large number of conformal scalar fields, Hawking radiation (arising from trace anomaly) and the back reaction take place at the 1-loop level. However during the final stages of the collapse, the semi-classical approximation breaks down signaling a need to incorporate higher order quantum corrections and non-perturbative effects. It is always believed that a non-perturbative quantum theory is required in order to answer questions regarding final fate of singularity, information loss etc. (see however [42]).

In the canonical formulation the non-perturbative quantization of CGHS model has been carried out in detail in various papers ([33], [10], [15], [39] and ref.therein). After a rescaling of the metric, the model becomes amenable to Dirac constraint quantization as well as BRST methods. Although the complete spectrum is known in the BRST-approch as well as in the Dirac method (in the so called Heisenberg picture), so far it has not been possible to ask the questions regarding quantum geometry using this spectrum.

In this paper we begin the analysis of the rescaled-CGHS (KRV) model using the methods of loop quantum gravity (LQG) ([54], [5]) more generally known as polymer quantization ([7]). More in detail, we derive a quantum theory of dilaton gravity (starting from classical CGHS model) which can be used to understand the near-plankian physics of CGHS model. The aim of this work is two-fold. First, we would eventually like to understand if the methods of loop quantization sheds new light on the structure of quantum geometry close to singularity of the CGHS Black holes. Although we do not answer this question in this paper, we setup a framework where this question can be asked. Secondly as the model offers a greater degree of analytic control than its higher-dimensional avatars, we can study in detail various structures which arise in LQG but have so far remained rather formal. (physical Hilbert space, Dirac observables, relational dynamics)

We begin by reviewing the classical CGHS model and its canonical formulation in section 4.2. We recast it as a free parametrized scalar field theory on a fiducial flat space-time [33]. As we have already seen in the previous chapter, PFTs on fixed background have a very rich mathematical and conceptual structure and are ideal arenas to test methods of LQG. In this chapter we aim to show that by combining the ideas from parametrized field theories and LQG, one obtains a potentially interesting quantum theory of dilaton gravity.

In section 4.3 we kinematically quantize the phase space (i.e. prior to solving the constraints). By choosing an appropriate sub-algebra of full Poisson algebra and performing the so called GNS quantization using a positive linear functional (analogous to the Ashtekar-Lewandowski functional used in LQG), we obtain a Hilbert space which carries a unitary representation of the space-time diffeomorphism group of the theory. We use the group averaging method in section 4.4 to solve the constraints and obtain the complete spectrum (physical Hilbert space) of the theory. This section is analogous to section 3 of chapter 3. However as the topology of background space-time is trivial, the (unitary) action of space-time

diffeomorphisms of spin-network states in quite different. To make the section self-contained we present all the relevant details, although reader familiar with chapter 2 could skim through it.
In section 4.5 we show how to quantize the algebra of Dirac observables on the physical Hilbert space and show how the physical Hilbert space is not a representation space for this algebra (in other words the algebra gets deformed in the quantum theory).

At this point we could argue that we have a complete non-perturbative quantum theory of CGHS model. However as there is no true Hamiltonian in the system, there is no dynamics. Thus we are faced with the so-called problem of time which as observed in chapter 2 is a generic feature of all parametrized systems ([23], [24]). We tackle this problem in our model using the idea of certain relational observables. In principle this gives us a framework for asking dynamical questions related to the evaporation of Black-holes, fate of the Black-hole singularities in a non-perturbative framework.

The rest of the chapter is organized as follows.

In section 4.6, using key ideas due to Dittrich and Hajicek ([26], [24], [16]) we define an elementary set of dynamical observables (referred to as complete observables) in the classical theory. These observables are elementary in the sense that more complicated observables (e.g. the observable corresponding to the dilaton) can be built from them. Along with the diffeomorphisms of the background spacetime, complete observables are then used to define time evolution in the system.

In the section 4.7 we perform canonical quantization of the complete observables and also define (non-unitary) time evolution in quantum theory in the Heisenberg picture. The canonically quantized observables can in turn be used to define the physical dilaton operator on $\mathcal{H}_{phy}$. This operator (which turns out to be a distribution) contains complete information about the quantum geometry and thus is a crucial ingredient for any future investigations that one might wish to carry out in this framework. With an eye toward future application (e.g. semi-classical analysis) we show how to calculate its expectation value in a generic basis-state in $\mathcal{H}_{phy}$.

This finishes the basic construction of a non-perturbative dilatonic theory of gravity quantized via methods used in LQG. There is however a conceptual problem with the canonically quantized complete observables. Unlike their classical counterparts, they do not admit relational interpretation. Whence in the penultimate section we propose an alternate definition of complete observables *directly* in the quantum theory which admit the same interpretation as the classical observables. Moreover the time evolution of these observables is naturally discrete. This is interesting as discrete (internal) time evolution has been the feature of other toy models quantized via LQG methods.

Although the Dirac observables defined in the first part of the chapter are densely defined on $\mathcal{H}_{phy}$, the complete observables built from them diverge for certain values of the internal time (the embeddings) and are not densely defined. In the last section we take a closer look at these divergences and argue that they can be resolved by changing measure in the Fourier space which is used in the definition of complete observables.

4.2 Classical theory

In this section we briefly recall the (rescaled) action of the CGHS model along with the solution of the field equations and the structure of the canonical theory.
The original CGHS action[1] describing a two dimensional theory of dilatonic gravity is given by,

$$S_{CGHS} = \frac{1}{4}\int d^2X \sqrt{-g}[\, e^{-2\phi}(\, R[g] + 4(\nabla\phi)^2 + 4\lambda^2\,) - (\nabla f)^2\,]. \quad (4.1)$$

Here ϕ is the dilaton field, g is the space-time metric (signature (-,+)) and f is a conformally coupled scalar field.
Rescaling the metric $g_{\mu\nu} = e^{2\phi}\,\gamma_{\mu\nu}$ one obtains the KRV action [33],

$$S_{KRV} = \frac{1}{2}\int d^2X \sqrt{-\gamma}[\,(\, yR[\gamma] + 4\lambda^2\,) - \gamma^{\alpha\beta}\nabla_\alpha f\nabla_\beta f\,)]. \quad (4.2)$$

where $y = e^{2\phi}$.
The field equations obtained by varying S_{KRV} can be analyzed in the conformal gauge. The solution is as follows. $\gamma_{\alpha\beta}$ is flat. The remaining fields can be described most elegantly in terms of null-coordinates $X^{\pm} = Z \pm T$ on the flat space-time. The scalar field f is simply free field propagating on the flat space-time

$$f(X) = f_+(X^+) + f_-(X^-) \quad (4.3)$$

and the dilaton is

$$y(X) = \lambda^2 X^+X^- - \tfrac{1}{2}\int^{X^+} d\overline{X}^+ \int^{\overline{X}^+} d\overline{\overline{X}}^+ \partial_+ f\partial_+ f - \tfrac{1}{2}\int^{X^-} d\overline{X}^- \int^{\overline{X}^-} d\overline{\overline{X}}^+ \partial_- f\partial_- f,$$

where $(\overline{X}^+, \overline{X}^-)$, $(\overline{\overline{X}}^+, \overline{\overline{X}}^-)$ are null-coordinates on Minkowski space.
Thus the solution space of the original CGHS model, namely $(g_{\mu\nu}, f)$ is completely determined in terms of the matter field f. This space contains black hole space-times as well. Easiest way to see this is to look at vacuum solutions. Taking

[1] We choose c=G=1. Thus only basic dimension in the theory is L and [M] $= L^{-1}$. In these units $\hbar$ becomes a dimensionless number.

$f(X) = 0$, one can show that the dilaton is given by,

$$y(X) = \lambda^2 X^+ X^- - \frac{M}{\lambda} \tag{4.4}$$

and the associated physical metric is,

$$g_{\mu\nu} = \frac{1}{\lambda^2 X^+ X^- - \frac{M}{\lambda}} \gamma_{\mu\nu} \tag{4.5}$$

which correspond to black holes of mass M in 2 dimensions.(M=0 is the linear dilaton vacuum). The singularity occurs where $y(X) = 0$. One can obtain more generic black hole space-times by sending in left-moving matter pulses from past null infinity. In all these cases locus of singularity is defined by $y(X) = 0$.[2][19]

4.2.1 Canonical description

The reason for using the rescaled-KRV action rather than the original (and perhaps more interesting) CGHS action is the following. One can perform a canonical transformation on the canonical co-ordinates of the KRV phasespace and obtain a parametrized free field theory on flat background. This will be our starting point for quantization. The details of this canonical transformation (also known as Kuchar decomposition [27]) are given in [33], here we only summarize the main results.[3]

The KRV spacetime action can be cast into canonical form by using an arbitrary foliation $X^\alpha = X^\alpha(x,t)$ of space-time by (t=const) space-like hypersurfaces.

$$S_{KRV} = \int dt \int_{-\infty}^{\infty} dx \, (\pi_y \dot{y} + \pi_\sigma \dot{\sigma} + \pi_f \dot{f} - NH - N^1 H_1) \tag{4.6}$$

[2]There is an important difference between the CGHS and KRV action at the semi-classical level. In the path integral quantization, Hawking radiation is encoded in a one loop term obtained by integrating out the matter field. This term is known as the Polyakov-Liouville term and is zero if one uses the flat metric γ (naturally appearing in the KRV action) to define the measure for the matter field. It is however non-zero if one uses the physical metric g (which appears in the CGHS action). Whence it is often claimed that the theory defined by KRV action does not contain Hawking radiation.[46]

[3]It is interesting to note that even the phasespace of the CGHS action can be mapped onto a parametrized scalar field theory on Kruskal spacetime. However the canonical transformation are singular in a portion of phasespace. [59]

where $(y(x), \sigma(x), f(x))$ are the pullback of the dilaton, space-time metric $\gamma_{\mu\nu}$ and the scalar field onto the hypersurface Σ respectively and π_y, π_σ, π_f are their conjugate momenta. (N, N^1) are the usual lapse and shift functions and H, H_1 are Hamiltonian and momentum constraints respectively and are constrained to vanish.

At this point it is important to note that by choosing appropriate gauge fixing conditions ($\dot{\rho} = \pi_\phi = 0$), one obtains a reduced phase space co-ordinatized by (f, π_f) with a true Hamiltonian given by,

$$H = \frac{1}{2}\int dx(\pi_f^2 + (f')^2). \tag{4.7}$$

One can than quantize this free field theory on a Fock-space and obtain a non-perturbative quantum theory. However as our primary motivation is to gain insights into structure of LQG, where constraints are solved directly in quantum theory, we do not solve the constraints classically.

A series of non-local canonical transformations maps the above action into that of a parametrized free field theory of flat background [33],

$$\begin{aligned} S[X^{\pm}, \Pi^{\pm}, f, \pi_f, N, N^1, p, m_R] = \int dt \int_{-\infty}^{\infty} dx(\Pi_+\dot{X}^+ + \Pi_-\dot{X}^- + \pi_f\dot{f} \\ -N\tilde{\tilde{H}} - N^1\tilde{H}_1 + \int dtp(t)\dot{m}_R(t)) \end{aligned} \tag{4.8}$$

where $X^{\pm}(x)$ are the embedding variables [4] and correspond to the light-cone coordinates on the Minkowski spacetime, $\Pi_\pm$ are conjugate momenta and the Hamiltonian constraint has been rescaled so as to have the same density weight as the momentum constraint.[5] The boundary term $\int p\dot{m}_R$ arises due to asymptotic conditions(Note that there are 2 boundaries in the problem, left and right infinity but only 1 boundary term in the action) on the initial data. m_R is the right mass of spacetime and it is conjugate momentum p is the difference between the parametrized time and proper time at right infinity when the parametrized time at left infinity is chosen to agree with the proper time.[6]. This action is the canonical action for a parametrized massless scalar field theory on flat spacetime.

[4] Here $X^{\pm}(x)$ means a phase-space function evaluated at x.

[5] Here our notation is $\tilde{f}$ means f is a density of weight 1.

[6] As the physical metric $g_{\mu\nu}$ is asymptotically flat, it has a asymptotic stationary killing field. proper time is the time measured by clock along the orbit is of this killing field. Parametrized time is the time defined by asymptotic value of the lapse function. For more details see [32]

The 2 constraints can be combined to form two Virasoro constraints $\tilde{\tilde{H}}^{\pm} = \frac{1}{2}(\tilde{\tilde{H}} \pm \tilde{\tilde{H}}_1)$. These two Virasoro constraints mutually commute with each other. Thus the constraint algebra can be written as a direct sum of two Lie algebras each of which generates Diff($\mathbf{R}$).

4.2.2 Boundary conditions

The boundary conditions on various fields are chosen so that the action is functionally differentiable are such that all the fields except the Embedding fields tend to zero as $x \rightarrow \pm\infty$. The boundary conditions on the embedding fields are

$$X^{\pm}(x) \rightarrow \pm x \; as \; \rightarrow \pm\infty \tag{4.9}$$

We now proceed to the quantum theory.

4.3 Quantum theory

In this section we quantize the classical theory using the techniques of polymer quantization. Although the steps involved are quite analogous to the kinematical quantization of PFT presented in the last chapter; we present all the details here for the sake of self-containedness. The analysis presented here is also closer in spirit to the kinematical quantization of LQG as summarized in chapter 2.
Recall that in the last chapter, the embedding variables were quasi-periodic. This introduced additional subtleties in their quantization which are absent from the quantization of embedding variables in this model.[7] However the quantization of matter sector is exactly analogous to the one presented in the last chapter (the only difference being that graphs are embedded in $\mathbf{R}$ instead of in $[0, 2\pi]$) and the reader familiar with chapter 2 can safely skip the section on matter sector.

[7] Those subtleties are replaced by the as yet unresolved issue of asymptotic conditions.

4.3.1 Embedding sector

The first step toward canonical quantization is a suitable choice of quantum algebra. Let us first describe our choice of quantum algebra for the embedding sector. Recall that $\Pi^{\pm}$ are scalar densities of weight $+1$ (equivalently 1-forms in 1-dim.) and $X^{\pm}$ are scalars (equivalently densitized vector fields).

Definition 1 : Consider a graph γ in the spatial slice Σ as a collection of finite number of edges and vertices. (By an edge we mean a closed interval in $\mathbf{R}$.) A cylindrical function for both the right moving(+) and left-moving(-) embedding sectors is defined as

$$f^{\pm}_{\gamma^{\pm}} \;=\; \Pi_{e^{\pm}\in E(\gamma^{\pm})}\, exp(ik^{\pm}_{e^{\pm}} \int_{e^{\pm}} \Pi^{\pm}) \tag{4.10}$$

where $k^{\pm}_{e} \in \mathbf{R}$.

Define Abelian *-algebras $Cyl^{\pm} \;=\; \cup_{\gamma\in\Gamma}\, Cyl^{\pm}_{\gamma}$. Let Vec denote the complexified Lie algebra of vector fields $X^{\pm}(x)$ which are maps $Cyl^{\pm} \longrightarrow Cyl^{\pm}$, (via Poisson brackets) that satisfy Leibniz rule and annihilate constants.

Consider the Lie-* algebra V defined by[8]

$$[(f, X(x)), (f', X'(x'))] \;=\; (\{X(x), f'\} \;-\; \{X'(x'), f\}\;,\; 0) \tag{4.11}$$

where (f,f') are in Cyl and (X,X') are vector fields. *-operation is just complex conjugation. (Conjugation of vector fields is defined by $X(x)^{*}f \;:=\; (X(x)f)^{*}$.) We now define the quantum algebras for the embedding sector. Our derivation mimics the derivation of quantum algebra for LQG given in [36].
Let us denote the (abstract) pair $(\hat{f}^{\pm}, \hat{X}^{\pm}(x))$ by a symbol $a^{\pm}$. Consider the *-algebra of finite linear combinations of finite sequences of the form $(a^{\pm}_1, ..., a^{\pm}_n)$ with an associative product,

$$(a^{\pm}_1, ..., a^{\pm}_n) \cdot (a^{\pm}_{n+1}, ..., a^{\pm}_m) \;=\; (a^{\pm}_1, ..., a^{\pm}_m) \tag{4.12}$$

[8]we have suppressed $\pm$ symbol so as to not clutter the formulae.

and an involution,

$$(a_1^{\pm}, ..., a_n^{\pm})^* = (a_n^{\pm *}, ..., a_1^{\pm *}) \tag{4.13}$$

Divide this algebra by a two sided ideal defined by elements of the form:

$$\begin{aligned} &(ka^{\pm}) - k(a^{\pm}) \\ &(a_1^{\pm} + a_2^{\pm}) - (a_1^{\pm}) - (a_2^{\pm}) \end{aligned} \tag{4.14}$$

The resulting algebras (for both $\pm$ sectors) are nothing but the free tensor algebras generated by $a^{\pm}$. The algebras $\mathcal{U}_E^{\pm}$ that we will quantize are defined as the free tensor algebras defined above modulo the 2-sided ideal generated by elements of the form $a_1^{\pm} \otimes a_2^{\pm} - a_2^{\pm} \otimes a_1^{\pm} - [a_1^{\pm}, a_2^{\pm}]$.[9]

So finally the algebra that we choose for quantization is $\mathcal{U}_E = \mathcal{U}_E^{+} \otimes \mathcal{U}_E^{-}$. The group generated by the two Virasoro constraints which is a direct product of two copies of Diff($\mathbf{R}$) has a natural representation as a group of outer automorphisms on $\mathcal{U}_E$. Abusing the standard nomenclature we refer to this group as Virasoro group.

$$\alpha_{\phi}^{\pm}(\hat{f}_{\gamma^{\pm}}^{\pm}) = \hat{f}_{(\phi^{\pm})^{-1}(\gamma^{\pm})}^{\pm} \tag{4.15}$$

$$\alpha_{\phi}^{\pm}(\hat{X}^{\pm}(x)) = \hat{X}^{\pm}((\phi^{\pm})^{-1}(x)) \tag{4.16}$$

The representation of $\mathcal{U}_E$ should be such that the outer automorphisms of $\mathcal{U}_E$ are represented via unitary operators as inner automorphisms. i.e.

$$U_{\pi}(\phi^{\pm})\pi(a^{\pm})U_{\pi}(\phi^{\pm})^{-1} = \pi(\alpha_{\phi^{\pm}}(a^{\pm})) \ \forall a \in \mathcal{U}_E \tag{4.17}$$

The GNS-quantization of the C* sub-algebra generated by $Cyl^{\pm}$ [36] proceeds via a positive linear functional $\omega_0^{\pm}$ which is motivated by the Ashtekar-Lewandowski positive linear functional of LQG.

[9] $\mathcal{U}_E^{\pm}$ is nothing but the universal enveloping algebra of the Lie algebra $V^{\pm}$.

$$\omega_0^{\pm}(f_\gamma^{\pm}) = \delta_{\gamma,0} \tag{4.18}$$

The Hilbert spaces $\mathcal{H}_E^{\pm}$ are the closure of the finite linear span of the cylindrical functions $f_{\gamma^\pm}^{\pm}$ w.r.t the inner-product defined by ω_0.
On $\mathcal{H}_E^{\pm}$ cylindrical functions act as multiplication operators and one can show that the embedding variables act as derivations,

$$\begin{aligned} \hat{X}^{\pm}(x) f_\gamma^{\pm} &= (-i\hbar)\, ik_e\, f_\gamma^{\pm} \;\; if\ x \in e \\ &= (-i\hbar)\, i\frac{(k_e + k_{e'})}{2}\, f_\gamma^{\pm}(\Pi^{\pm}) \;\; if\ x \in e \cap e' \\ &= 0 \text{ otherwise} \end{aligned} \tag{4.19}$$

The Virasoro group acts unitarily on $\mathcal{H}_E^{\pm}$ as

$$\begin{aligned} \hat{U}^{\pm}(\phi^{\pm})\, f_\gamma^{\pm}(\pi^{\pm}) &= f^{\pm}_{(\phi^{\pm})^{-1}\gamma}(\pi^{\pm}) \\ \hat{U}^{\pm}(\phi^{\pm})\, \hat{f}_\gamma^{\pm}\, \hat{U}^{\pm}(\phi^{\pm}) &= \hat{f}^{\pm}_{(\phi^{\pm})^{-1}\gamma} \\ \hat{U}^{\pm}(\phi^{\pm})\, \hat{X}^{\pm}(x)\, \hat{U}^{\pm}(\phi^{\pm}) &= \hat{X}^{\pm}(\phi^{\pm}(x)) \end{aligned} \tag{4.20}$$

The complete embedding Hilbert space is of course given by $\mathcal{H}_E = \mathcal{H}_E^{+} \otimes \mathcal{H}_E^{-}$.

4.3.2 Matter sector

Now we consider the kinametical quantization of the matter sector. The quantization given here is unitarily inequivalent to the Bohr quantization of scalar field but it is the same quantization that is used by Thiemann to quantize the Bosonic string. For more details we refer the reader to [55].

Once again the choice of quantum algebra will be motivated by the fact that we want the Virasoro group to act as group of outer automorphisms on this algebra. Following observations help us make such a choice.

Consider the canonical transformation $(\pi_f, f) \rightarrow (Y^{\pm} = \pi_f \pm f')$. $Y^{\pm}$

satisfy the Poisson bracket relations,

$$\begin{aligned} \{\, Y^{\pm}(x)\,,\ Y^{\pm}(x')\,\} \;&=\; \mp(\ \partial_{x'}\delta(x',x) \;-\; \partial_x\delta(x,x')\ ) \\ \{\, Y^{\pm}(x)\,,\ Y^{\mp}(x')\,\} \;&=\; 0. \end{aligned} \tag{4.21}$$

In terms of these variables the Virasoro constraints are given by,

$$\begin{aligned} H^+(x) \;&=\; \Pi_+ X^{+'} \;+\; \tfrac{1}{4}(\pi_f \;+\; f')^2 \\ H^-(x) \;&=\; \Pi_- X^{-'} \;-\; \tfrac{1}{4}(\pi_f \;-\; f')^2 \end{aligned} \tag{4.22}$$

Whence one can see that under the Lie-derivative along the Hamiltonian vector field of the constraints,

$$\begin{aligned} \mathcal{L}_{H^{\pm}[N_{\pm}]}\, Y^{\pm}(x) \;&=\; (N_{\pm}\, Y^{\pm})'(x) \\ \mathcal{L}_{H^{\pm}[N_{\pm}]}\, Y^{\mp}(x) \;&=\; 0. \end{aligned} \tag{4.23}$$

Thus it is clear that the 2 generators of the Virasoro algebra $H^{\pm}$ act as generators of spatial diffeomorphisms on $Y^{\pm}$. These considerations motivate the following.

Once again let Γ be the set of all graphs γ embedded in Σ consisting of finite number of edges and vertices. We start by defining momentum network (similar to spin-network in LQG)s_m as a pair $(\gamma,\ \vec{l}(\gamma)\ :=\ (l_{e_1},...,l_{e_N}))$ where l_e are real numbers. A momentum network operator for both the right and left moving sectors is defined as,

$$W^{\pm}(s_m^{\pm}) \;:=\; exp(i\ [\ \sum_{e\in E(\gamma(s_m^{\pm})}\ l_e^{\pm} \int_e Y^{\pm}]\ ) \tag{4.24}$$

The Weyl relations obeyed by $W^{\pm}(s_m^{\pm})$ can be easily derived from (4.21) (using BHC formula [38]),

$$\begin{aligned} W^{\pm}(s_m^{\pm})W^{\pm}(s_m^{\pm'}) \;&=\; e^{\mp\frac{i\hbar}{2}[\alpha(s_m^{\pm},s_m^{\pm'})]}W^{\pm}(s_m^{\pm} \;+\; s_m^{\pm'}) \\ W^{\pm}(s_m^{\pm})^* \;&=\; W^{\pm}(-s_m^{\pm}) \end{aligned} \tag{4.25}$$

where

$$\alpha(s_m^{\pm}, s_m^{\pm\prime}) = \sum_{e_1 \in \gamma(s_m^{\pm})} \sum_{e_2 \in \gamma(s_m^{\pm\prime})} l^{e_1}(s_1) l^{e_2}(s_2) \alpha(e_1, e_2) \tag{4.26}$$

with $\alpha(e_1, e_2) = [\kappa_{e_1}]_{\partial e_2} - [\kappa_{e_2}]_{\partial e_1}$. Here κ_e is the characteristic function of e. $\kappa_e(x) = 1$ for $x \in Int(e)$, $\kappa_e(x) = \frac{1}{2}$ for $x \in boundary(e)$ and 0 otherwise. In (4.25) notation $(s_1 + s_2)$ means we decompose all edges e_1 and e_2 in their maximal mutually non-overlapping segments and assign $l^{e_1} + l^{e_2}$ to $e_1 \cap e_2$, l^{e_1} to $e_1 - \gamma(s_2)$ and l^{e_2} to $e_2 - \gamma(s_1)$ respectively.

Now we define the algebra that we will be interested in quantizing. Consider an associative algebra generated by formal finite linear combinations of formal sequences of the form $(W_{s_1}^{\pm}, ..., W_{s_n}^{\pm})$ with associative multiplication given by,

$$(W_{s_1}^{\pm}, ..., W_{s_n}^{\pm}) \cdot (W_{s_{n+1}}^{\pm}, ..., W_{s_m}^{\pm}) := (W_{s_1}^{\pm}, ..., W_{s_m}^{\pm}) \tag{4.27}$$

We give this algebra tensor product structure by moding out 2-sided ideals generated by elements of the form,

$$\begin{aligned} &(\alpha\, W^{\pm}(s)) - \alpha(W^{\pm}(s))\ \alpha \in \mathbf{C} \\ &(W^{\pm}(s_1) + W^{\pm}(s_2)) - (W^{\pm}(s_1)) + (W^{\pm}(s_2)) \end{aligned} \tag{4.28}$$

We refer to this tensor algebra as $Cyl_M^{\pm}$. The *-algebra that we will quantize is $Cyl_M^{\pm}$ modulo the 2 sided ideal implied by (4.25).We denote this algebra as $\mathcal{U}_M^{\pm}$. Finally the full algebra for both sectors is given by $\mathcal{U}_M = \mathcal{U}_M^{+} \otimes \mathcal{U}_M^{-}$.

Action of the Virasoro group on this algebra is given by,

$$\alpha_{\phi^{\pm}}^{\pm}(W^{\pm}(s_m^{\pm})) = W_{(\phi^{\pm})(s_m^{\pm})}^{\pm} \tag{4.29}$$

where $\phi(s) := (\phi^{-1}(\gamma)\ ,\ \vec{l}(\gamma))$.
Now just like for the embedding sector we perform a GNS quantization of $\mathcal{U}_M$ using a Virasoro-invariant positive linear functional,

$$\omega_{\pm}(W^{\pm}(s)) = \delta_{s,0} \tag{4.30}$$

where 0 in $\delta_{s,0}$ stands for a graph with zero edges and an empty label set.

This functional is clearly motivated by the Ashtekar-Lewandowski functional used in LQG. It can be easily shown to be virasoro-invariant. The resulting Hilbert space for both $(\pm)$ sectors is given by Cauchy completion of $\mathcal{U}_M^{\pm}$. and the representation is,

$$\hat{W}^{\pm}(s_1)\,W^{\pm}(s_2)(Y^{\pm}) \;=\; W^{\pm}(s_1)(Y^{\pm})W^{\pm}(s_2)(Y^{\pm}) \;=\; e^{\mp\frac{i\hbar}{2}[\alpha(s_1,s_2)]}W^{\pm}(s_1+s_2)(Y^{\pm}) \tag{4.31}$$

As $\omega_{\pm}$ is Virasoro invariant, it implies that the Virasoro group acts unitarily and anomaly-freely on $\mathcal{H}_M^{\pm}$

$$\begin{aligned}\hat{U}^{\pm}(\phi^{\pm})\,W^{\pm}(s)(Y^{\pm}) &= W^{\pm}((\phi^{\pm})s)(Y^{\pm})\\ \hat{U}^{\pm}(\phi^{\pm})\,\hat{W}^{\pm}(s)\,\hat{U}^{\pm^{-1}}(\phi^{\pm}) &= \hat{W}^{\pm}(\phi^{\pm}s)\end{aligned} \tag{4.32}$$

Finally the Matter Hilbert space is $\mathcal{H}_M \;=\; \mathcal{H}_M^{+}\otimes\mathcal{H}_M^{-}$.[10]

4.3.3 Quantizing the asymptotic degrees of freedom

The final component of kinematical Hilbert space is the Schroedinger representation of the boundary data $(m\ ,\ p)$. As the Virasoro group has trivial action on the corresponding Heisenberg algebra, we do not need to loop quantize these asymptotic degrees of freedom. Thus we choose for the boundary Hilbert space $L^2(\mathbf{R}, dm)$[11] with,

$$\begin{aligned}\hat{m}\,\Psi(m) &= m\,\Psi(m)\\ \hat{p}\,\Psi(m) &= -i\tfrac{\partial}{\partial m}\,\Psi(m)\end{aligned} \tag{4.33}$$

This finishes the construction of kinematical Hilbert space of the quantum theory. We rewrite the final Hilbert space as tensor product of 3 Hilbert spaces,

[10] Note that this Hilbert space is not of the form $L^2(\overline{Y^{\pm}}, d\mu)$. It is (the completion of) algebra itself considered as a vector space with inner product defined by the GNS state.

[11] More precisely we need to perform quantization on a half-line in order to restrict ourselves to $m \geq 0$ configurations. dm is the suitable measure on the half line.

$$\begin{aligned}\mathcal{H}_{kin} &= \mathcal{H}_M \otimes \mathcal{H}_E \otimes \mathcal{H}_m = (\mathcal{H}_M^+ \otimes \mathcal{H}_M^-) \otimes (\mathcal{H}_E^+ \otimes \mathcal{H}_E^-) \otimes \mathcal{H}_m \\ &= (\mathcal{H}_M^+ \otimes \mathcal{H}_E^+) \otimes (\mathcal{H}_M^- \otimes \mathcal{H}_E^-) \otimes \mathcal{H}_m.\end{aligned} \tag{4.34}$$

one for the left moving(-) sector ($\mathcal{H}^+$), one for the right moving(+) sector ($\mathcal{H}^-$) and one for the asymptotic sector. The Virasoro group acts unitarily on the Hilbert space as 2 mutually commuting copies of spatial diffeomorphisms.
We define the basis for $\mathcal{H}$ as follows.
Definition 2 : Consider a graph γ with a set of pair of real numbers $((k_1^\pm, l_1^\pm), ..., (k_N^\pm, l_N^\pm))$ where in outermost pairs $((k_1^\pm, l_1^\pm)$ and $(k_N^\pm, l_N^\pm)))$ either k_i or l_i can be zero but not both. in the interior edges $(e_2, ..., e_{N-1})$ we even allow both the charges (k,l) to be zero.
We call the pair $(\gamma, ((k_1^\pm, l_1^\pm), ..., (k_N^\pm, l_N^\pm)))$ charge-network (in analogy with spin-networks in LQG) and denote it by s. The state associated with s will be denoted by $f_{s^\pm}^\pm$.

4.4 Physical Hilbert space

The motivation behind choosing a particular quantum algebra and a peculiar GNS functional has been the unitary and anomaly-free representation of the Virasoro group on the Hilbert space. One again we can use the framework of Refined algebraic quantization to solve the Virasoro constraints. The ideas behind the Rigging map were already summarized in section 3.5 of chapter 3. We refer the reader to that section for details.
Recall that we are seeking for distributions in Φ_{kin}^* which satisfy,

$$\Psi(\hat{U}(\phi^\pm) f_{s^\pm}^\pm) = \Psi(f_{s^\pm}^\pm) \;\; \forall \phi \in Diff(\mathbf{R}) \tag{4.35}$$

The rigging map is defined as follows. Given a charge network s, define $\{[s] = \phi \cdot s | \phi \in Diff(\mathbf{R})\}$.
Then the rigging map (which is tied to the charge-network basis) is given by,

$$\eta(f_s^+ \otimes f_{s'}^-) := (\eta^+ \otimes \eta^-)(f_s^+ \otimes f_{s'}^-) = \\ \eta_{[s]} \sum_{\phi \in Diff_{[s]}(\mathbf{R})} < f_s^+ \hat{U}^+(\phi) \, , > \otimes \, \eta_{[s']} \sum_{\phi' \in Diff_{[s']}(\mathbf{R})} < f_{s'}^- \hat{U}^-(\phi') \, , > \tag{4.36}$$

where the sum is over all the diffeomorphisms ϕ which take a charge-network $s = (\gamma, (\vec{k^{\pm}}, \vec{l^{\pm}}))$ to a different charge network $s' = (\phi^{-1}(\gamma), (\vec{k^{\pm}}, \vec{l^{\pm}}))$ As can be clearly seen from the definition of rigging map, the solution space is a tensor product of 2 vector spaces. Inner product on both of them can be defined as,

$$< \eta^{\pm}(f_s^{\pm}) | \eta^{\pm}(f_{s'}^{\pm}) > = [\eta^{\pm}(f_{s'}^{\pm})](f_s^{\pm}) \tag{4.37}$$

The physical Hilbert space is thus characterized by diffeomorphism-equivalence class of charge networks [s] which in 1 dimensions can be classified by the following data.

1. Number of edges $|E(\gamma)| = N$
2.The ordered set $((k_1^{\pm}, l_1^{\pm}), ..., (k_N^{\pm}, l_N^{\pm}))$

Thus we can write a ket in $\mathcal{H}_{phy}^{\pm}$ as

$$|\Psi>_{\pm} = \eta_{[s]} |N, \, (k_1^{\pm}, l_1^{\pm}), ..., (k_N^{\pm}, l_N^{\pm}) > \tag{4.38}$$

So far $\eta_{[s]}$ are completely arbitrary positive real numbers. As in the previous chapters, by demanding that the Rigging map commute with Quantum observables (to be defined in the next section), we will derive conditions on these numbers.
Finally as the Virasoro group acts trivially on $\mathcal{H}_m$ it remains unchanged under group averaging, whence complete $\mathcal{H}_{phy} = \mathcal{H}_{phy}^+ \otimes \mathcal{H}_{phy}^- \otimes \mathcal{H}_m$.
At this point we would like to comment on the anomaly-freeness of our representation. In the Fock space quantization of the model ([33], [15]), one obtains a Virasoro anomaly in the constraint algebra due to the Schwinger term in the commutator of the energy momentum tensor for the matter field. In [33], the anomaly is removed by modifying the Embedding sector of the theory where as when one uses BRST methods to quantize the model, anomaly is removed by adding background charges(enhancing the central charge) and ghost fields(which define so called bc-CFT). Our choice of Poisson sub-algebra coupled with a unusual choice of GNS functional results in a discontinuous(but anomaly free) representation of the Virasoro group. Here it is important to note that even in Fock space

one can normal order the constraints with respect to so called squeezed vacuum state [15] such that the central charge is zero. However these states have peculiar properties like the action of finite gauge transformations is ill-defined on them.[12]

4.5 Complete set of Dirac observables

By group averaging the Virasoro constraints we have obtained a physical Hilbert space $\mathcal{H}_{phy} = \mathcal{H}^{+}_{phy} \otimes \mathcal{H}^{-}_{phy}$. Now we encounter (what one always encounters at some stage in canonical quantization of diffeo-invariant theories) problem of time. There is apriori no dynamics on the physical Hilbert space. In order to ask the dynamical questions about for e.g. singularity resulting from the collapse of scalar field in quantum theory, some notion of dynamics should be defined on $\mathcal{H}_{phy}$. We do this by employing ideas due to [16], [25] which goes back to the old idea of evolving constants of motion by [40].

However first we show how to define a complete set of Dirac observables (Perennials) for our model and how to represent them as well-defined operators on $\mathcal{H}_{phy}$. For the classical theory, these perennials have been known for a long time ([33] , [56]) and are analogous to the DDF observables of bosonic string theory [45].

The basic idea behind constructing Dirac observables in parametrized field theory is fairly simple.(This is a general algorithm for defining Dirac observables in parametrized field theories and is also known as Kuchar decomposition [27].) Given the phase-space of the theory co-ordinatized by $(X^{\pm},\ \Pi^{\pm},\ f,\ \pi_f)$, one can perform a canonical transformation to the so-called Heisenberg chart $(X^{\pm},\ \mathbf{\Pi}^{\pm},\ \mathrm{f},\ \pi_{\mathrm{f}})$ [33] where $\mathbf{\Pi}^{\pm}$ are the two Virasoro constraints and $(\mathbf{f},\ \pi_{\mathbf{f}})$ are the scalar field data on an initial(fixed) slice. $(X^{\pm},\ \mathbf{\Pi}^{\pm})$ and $(\mathbf{f},\ \pi_{\mathbf{f}})$ form a mutually commuting canonically conjugate pair whence it is clear that $(\mathbf{f},\ \pi_{\mathbf{f}})$ are Dirac observables.

Choosing the initial slice as $(X_0^{\pm}(x) = x)$ we can expand these observables in terms of an orthonormal set of mode functions e^{ikx},

$$\mathrm{f}(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dk}{|k|} e^{ikx} a_k + c.c.$$

[12] contrast this with our representation where infinitesimal gauge transformations are ill-defined.

It is clear that (a_k, a_k^*) are also Dirac observables. It is also clear in the Heisenberg chart that they form a complete set(describe true degrees of freedom of the theory). Now by expressing a_k in terms of the original(Schrodinger) canonical chart we will obtain Dirac observables that will be promoted to operators on $\mathcal{H}_{phy}$.
In order to write $a_k \;=\; a_k[X^{\pm}, \Pi^{\pm}, f, \pi_f]$ one has to appeal to the space-time picture of a parametrized field theory propagating on flat background. We just summarize the main results and refer the reader to [56] for details.
The scalar field f(X) in space-time satisfies,

$$\mathrm{f}(X) \;=\; 0$$

The solution can be expanded as,

$$f(X) \;=\; \int \frac{dk}{|k|} e^{ik\cdot X} a_k \;+\; c.c.$$

a_k's can be projected out of the solutions f(X)'s on any hyper-surface using the Klein-Gordon inner product.

$$a_k \;=\; i \int_{\Sigma} \sqrt{g} [\, e^{-ik\cdot X(x)} n^{\alpha} \nabla_{\alpha} f(x) \;-\; f(x) n^{\alpha} \nabla_{\alpha} e^{-ik\cdot X(x)}\,] \tag{4.39}$$

where f(x) = f(X(x)), n^{α} is a unit normal to the embedding $(X^+(x), X^-(x))$, and in fact $\sqrt{g} n^{\alpha} \nabla_{\alpha} f(x) \;=\; \pi_f(x)$. Thus given (f(x),$\pi_f(x)$) on a spatial slice, we can obtain (a_k, a_k^*).

$$\begin{aligned} a_k \;&=\; \int \sqrt{g}\, [\, u_k^* n^{\alpha} \nabla_{\alpha} f \;-\; f n^{\alpha} \nabla_{\alpha} u_k^*\,] \\ &=\; \int [\, u_k^* \pi_f \;-\; \sqrt{g} f\, (\, -ik^- \sqrt{\tfrac{X^{+\prime}}{X^{-\prime}}} \;+\; ik^+ \sqrt{\tfrac{X^{-\prime}}{X^{+\prime}}}\,)] \\ &=\; \int [\pi_f \;+\; ik^- X^{+\prime} f \;-\; ik^+ X^{-\prime} f] \end{aligned} \tag{4.40}$$

Where we have used $n^+ \;=\; \sqrt{\frac{X^{+\prime}}{X^{-\prime}}}$, $n^- \;=\; -\sqrt{\frac{X^{-\prime}}{X^{+\prime}}}$ and $\sqrt{g} \;=\; \sqrt{X^{+\prime} X^{-\prime}}$.
Using $k^{\pm} \;=\; \frac{1}{2}(k \;\pm |k|)$ we can show that,

$$\begin{aligned} a_k \;&=\; \int e^{-ikX^-} Y^- \qquad k > 0 \\ a_k \;&=\; \int e^{-ikX^+} Y^+ \qquad k < 0 \\ a_0 \;&=\; \int \pi_f \end{aligned} \tag{4.41}$$

a_k^* are defined by complex conjugating the a_k.
By explicit calculations one can check that these functions are Dirac observables. Their Poisson algebra is given by,

$$\begin{aligned}\{a_k, a_l\} &= 0. \\ \{a_k^*, a_l^*\} &= 0. \\ \{a_k, a_l^*\} &= |k|\delta(k,l)\end{aligned} \tag{4.42}$$

4.5.1 Quantization of a_k

In this section we show how to promote (a_k, a_k^*) to densely defined operators on $\mathcal{H}_{phy}$. This prescription can at best be viewed as an ad-hoc way of trying to promote regulated expressions from $\mathcal{H}_{kin}$ to $\mathcal{H}_{phy}$. We hope that a better scheme for doing this emerges in future or that the one given here is more justified.
Given a (strong) Dirac observable (one that strongly commutes with the Virasoro constraints), ideal way to promote it to an operator on $\mathcal{H}_{phy}$ is as follows. One first defines an operator on $\mathcal{H}_{kin}$ and if this operator is G-equivariant(where G here is the direct product of 2 copies of diffeomorphisms acting on $\mathbf{R}$), then one can define an operator on $\mathcal{H}_{phy}$ simply by dual action. We will show how this procedure fails here [55]. (This is analogous to a generic problem in LQG of defining connection dependent operators on $\mathcal{H}_{diff}$.)
for $k > 0$,

$$a_k = \int e^{ikX^-}Y^- \tag{4.43}$$

In order to represent a_k on $\mathcal{H}_{kin}$ we have to triangulate our spatial slice Σ by 1-simplices (closed intervals). Let T be a triangulation of σ. Given a state $f_{s^-}^-$ for the left moving sector, we choose a triangulation $T(\gamma(s^-))$ adapted to $\gamma(s)$ i.e. the triangulation is such that all the vertices of $\gamma(s)$ are vertices of $T(\gamma(s^-))$. Classically we know that,

$$\frac{h_{\triangle_m}(Y^-) - h_{\triangle_m^{-1}}(Y^-)}{|\triangle_m|} = Y^-(\frac{v_m + v_{m+1}}{2}) + O((|\triangle_m|)^2). \tag{4.44}$$

Where $\triangle_m \in T(\gamma(s^-))$ (It is a closed interval in say Cartesian co-ordinate system), and (v_m, v_{m+1}) are beginning and terminating vertices of $\triangle_m$ respectively.
Now we can write a_k as the limit of a Riemann sum,

$$a_k \;=\; lim_{T\rightarrow\Sigma}\, a_{k,T(\gamma(s^-))} \tag{4.45}$$

where

$$a_{k,T(\gamma(s^-))} \;=\; \sum_{\triangle_m\in T(\gamma(s^-))} e^{ik\hat{X}^-(v_m)}[h_{\triangle_m}Y^- \;-\; h_{\triangle_m^{-1}}(Y^-)]. \tag{4.46}$$

$a_{k,T(\gamma(s))}$ can be represented on H_{kin} as follows.

$$\hat{a}_{k,T(\gamma(s))}f^-_{s^-} \;=\; \sum_{\triangle_m\in T(\gamma(s^-))} e^{ik\hat{X}^-(v_m)}[h_{\triangle_m} \;-\; h_{\triangle_m^{-1}}]\, f^-_{s^-} \tag{4.47}$$

Similar expression holds for $k < 0$ with (X^-, Y^-) replaced by (X^+, Y^+) and the resulting operator acting on $f^+_{s^+}$. Also one can (densely) define $\hat{a}^\dagger_{k,T(\gamma(s))}$ using the inner product on $\mathcal{H}_{kin}$.

$$\hat{a}^\dagger_{k,T(\gamma(s^-))}f^-_{s^-} \;:=\; \sum_{\triangle_m\in T(\gamma(s^-))} e^{-ik\hat{X}^-(v_m)}[h_{\triangle_m} \;-\; h_{\triangle_m^{-1}}]\, f^-_{s^-} \tag{4.48}$$

At finite triangulation (i.e. when number of simplices in T are finite) $\hat{a}_{k,T(\gamma(s^-))}$ is not Virasoro-equivariant,

$$\begin{aligned}
&\hat{U}(\phi^-)a_{k,T(\gamma(s^-))}\hat{U}^{-1}(\phi^-) = \hat{U}(\phi^-)\textstyle\sum_{\triangle_m\in T(\gamma(s))} e^{ik\hat{X}^-(v_m)}[h_{\triangle_m} - h_{\triangle_m^{-1}}]\,\hat{U}^{-1}(\phi^-)\\
&= \textstyle\sum_{\triangle_m} \hat{U}(\phi^-)e^{ik\hat{X}^-(v_m)}\hat{U}^{-1}(\phi^-)\,\hat{U}(\phi^-)[h_{\triangle_m} - h_{\triangle_m^{-1}}]\hat{U}^{-1}(\phi^-)\\
&= \textstyle\sum_{\triangle_m} e^{ikX^-(\phi^{-1}(v_m))}\,[h_{\phi^{-1}(\triangle_m)} - h_{\phi^{-1}(\triangle_m^{-1})}]\\
&= \textstyle\sum_{\overline{\triangle_m}\in\phi(T(\gamma(s^-)))} e^{ik\hat{X}^-(\overline{v}_m)}[\,h_{\overline{\triangle}_m} - h_{\overline{\triangle}_m^{-1}}\,]\\
&= a_{k,\phi(T(\gamma(s^-)))}.
\end{aligned} \tag{4.49}$$

Thus, $\hat{a}_{k,T(\gamma(s^\pm))}$ cannot be promoted to an operator on $\mathcal{H}_{phy}$ simply by dual action. This problem was also encountered by Thiemann in [55]. As argued by him, if we try to remove the triangulation by taking the continuum limit then we either get zero(in weak operator topology) or infinity(in strong operator topology).

There are two ways to get around this problem. First way is due to Thiemann. There the idea was to use the graph(underlying a state) itself as a triangulation and define a strongly Virasoro-invariant operator on $\mathcal{H}_{kin}$. Here we propose a different way.[13] Essentially we use a sort of gauge-fixing in the space of (diff) equivalence class of charge-networks (defined as the triple $(\gamma,\ \vec{l}(\gamma),\ \vec{k}(\gamma))$) to define an operator corresponding to a_k on $\mathcal{H}_{phy}$. As will be argued later, Thiemann's proposal can be considered as a special case of ours.

Given an orbit of diffeomorphism equivalence class of charge-networks, (we supress the $\pm$ indices on charge-networks in rest of this section) $[s] = \{\phi\cdot s|\ \phi\in Diff\Sigma\}$ we fix <u>once and for all</u> a network $s_0 = (\gamma_0,\ \vec{l}(\gamma),\ \vec{k}(\gamma))$ and a triangulation $T(\gamma_0(s_0))$ adapted to it. Now for any s in the orbit, such that $s = \phi\cdot s_0 = (\phi^{-1}(\gamma_0),\ \vec{l}(\gamma),\ \vec{k}(\gamma))$ we choose the corresponding triangulation $T(\gamma(s))$ such that $\hat{a}_{T(\gamma(s))} = \hat{U}(\phi)\hat{a}_{T(\gamma_0(s_0))}\hat{U}^{-1}(\phi)$. Now let $\Psi\in\mathcal{H}_{phy}$. One can show that this family of operators are cylindrically consistent and define a operator on $\mathcal{H}_{kin}$. The resulting operator on $\mathcal{H}_{phy}$ defined by the dual action turns out to be densely defined.

$$(\hat{a}_{k'}\Psi)[f_s] = \Psi[\,\hat{a}^{\dagger}_{k,T(\gamma_0)}\,f_{s_0}\,] \tag{4.50}$$

Here as defined earlier $\gamma_0(s_0)$ is the graph which is fixed in the orbit of $\gamma(s)$, and $T(\gamma_0(s_0))$ is a fixed triangulation adapted to it. This proposal (of defining $\hat{a_{k'}}$ on $\mathcal{H}_{phy}$) is as we emphasized earlier rather ad-hoc as it involves an arbitrary choice s_0 and triangulation $T(\gamma_0(s_0))$. It nonetheless results in a "regulated" and densely defined operator on $\mathcal{H}_{phy}$.

We will now argue that Thiemann's proposal of defining a Virasoro-invariant operator directly on $\mathcal{H}_{kin}$ ([55] pg.28) can be subsumed by the prescription given above. (Note that in [55] spatial topology is compact (S^1), whence we have to modify the proposal given there accordingly as in our case the spatial manifold is $\mathbf{R}$.) Let us first note how Thiemann's prescription applies to our perennials.

1. Choose the graph underlying a state itself as a triangulation (by adding fiducial edges if necessary).

2.Then the operator (in our case $\hat{a}_{k,\gamma(s)}$) acting on a basis-state f_s results in a linear

[13] The idea of defining regulated operators on $\mathcal{H}_{phy}$ in this way was suggested to us by Madhavan Varadarajan.

combination of states $f_{s'}$ such that $\gamma(s') \subset \gamma(s)$. Thus,

$$\hat{U}(\phi)\ \hat{a}_{k,\gamma(s)} f_s \ = \ \sum b_I \hat{U}(\phi) f_{s_I} \ = \ \sum b_I f_{\phi(s_I)}. \tag{4.51}$$

Using (4.46) one can write this more explicitly as,

$$\begin{aligned} &\hat{U}(\phi)\ (\textstyle\sum_{e\in E(\gamma)}\ e^{ik\hat{X}^-(b(e))}\ [h_e(Y^-)\ -\ h_{e^{-1}}(Y^-)])\ f_s \\ &= \hat{U}(\phi)\ (\textstyle\sum_{e\in E(\gamma)}\ e^{ik\frac{1}{2}(k_e^- + k_{e-1}^-)}\ e^{i\hbar\alpha(e,\gamma)}[f_{s'}\ -\ f_{s''}]) \\ &= \textstyle\sum_{e\in E(\gamma)}\ e^{ik\frac{1}{2}(k_e^- + k_{e-1}^-)}\ e^{i\hbar\alpha(e,\gamma)}[f_{\phi\cdot s'}\ -\ f_{\phi\cdot s''}] \\ &= \textstyle\sum_{e\in E(\phi^{-1}(\gamma))}\ e^{ik\frac{1}{2}(k_e^- + k_{e-1}^-)}\ e^{i\hbar\alpha(e,\phi^{-1}(\gamma))}[f_{\phi\cdot s'}\ -\ f_{\phi\cdot s''}] \end{aligned} \tag{4.52}$$

where $s' = (\gamma, ((k_1, l_1), ..., (k_e, l_e{+}1), ..., (k_N, l_N)))$ and $s'' = (\gamma, ((k_1, l_1), ..., (k_e, l_e - 1), ..., (k_N, l_N)))$. and in the last line we have made use of the fact that $(k_{\phi^{-1}(e)}, l_{\phi^{-1}(e)}) = (k_{(e)}, l_{(e)})$.
However it is easy to convince oneself that the last line in (4.52) equals,

$$\hat{a}_{k,\gamma(\phi\cdot s)}\ f_{\phi\cdot s} \ = \ \hat{a}_{k,T}\ \hat{U}(\phi)\ f_s. \tag{4.53}$$

This shows Virasoro invariance of $\hat{a}_{k,\gamma(s)}$.
The above proof crucially relies on the fact that the triangulation used to regulate the operator is same as the graph (underlying the state on which the operator acts) itself. We now show how to achieve this by adding fiducial edges to the graph (This is where Thiemann's prescription has to be slightly modified as in [55] spatial topology is that of S^1.)
Given any basis-state f_s we can always write it as a state $\tilde{f}_{\tilde{s}} \in \mathcal{H}_{kin}$ such that $\tilde{\gamma}(\tilde{s}) = e_L \cup \gamma(s) \cup e_R$, where e_L and e_R are edges from $-\infty$ to initial vertex of $\gamma(s)$ and from final vertex of $\gamma(s)$ to ∞ respectively. (See figure below) In fact we can define a new basis for $\mathcal{H}_{kin}$ as follows. Any element of the basis $f_{\tilde{s}}$ is defined to be based on a graph which is of the form $\tilde{\gamma}(\tilde{s}) = e_L \cup \gamma \cup e_R$ where γ is a subgraph of $\tilde{\gamma}(\tilde{s})$ such that e_L and e_R are as defined above. The charge-pairs $(k_{e_{L/R}}, l_{e_{L/R}})$ are allowed to be (0,0) but (k_{e_1}, l_{e_1}) and (k_{e_N}, l_{e_N}) are not allowed to be (0,0). (Here e_1 and e_N are initial and final edges of γ respectively.)[14] Thus

[14] The introduction of new basis is only to show how Thiemann's prescription is consistent with ours and will not be used in the rest of the paper anywhere.

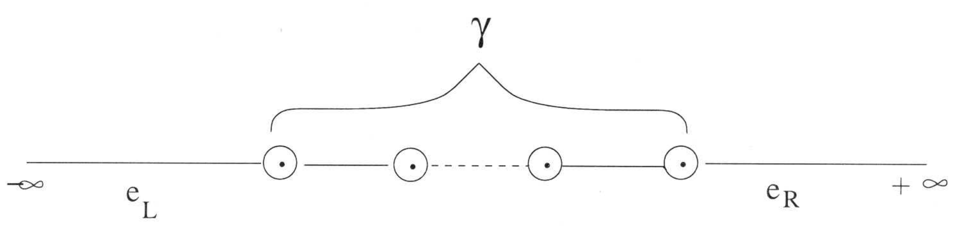


Figure 4.1: Triangulation based on a state

the graphs on which the new basis is defined itself becomes triangulation of Σ and Thiemann's prescription follows.

How does our definition of $\hat{a'}_k$ subsume Thiemann's definition as a special case? The answer is as follows. Once we choose an s_0 in the orbit of s choose $T(\gamma_0(s_0)) = e_L \cup \gamma_0 \cup e_R$ (As shown in the figure).

The resulting operator $\hat{a}^{\dagger}_{T(\gamma_0(s_0))}$ is Virasoro invariant on $\mathcal{H}_{kin}$. Whence $\hat{a'}_k$ is the dual of a linear operator $\hat{a}^{\dagger}_T$ obtained on $\mathcal{H}_{kin}$ via cylindrical consistency.

Now we can analyze the conditions under which $\hat{a}_{T(\gamma_0(s_0))}$ commutes with the rigging map defined above. An analysis similar to the one performed in chapter 3 leads to the following condition on $\eta_{[s]}$.

Given any charge network s, we denote by $s_{\triangle_m}$ one of the following.

- $s_{\triangle_m} = \{\gamma(s) \cup \triangle_m, (k_{e_1}, ...k_{e_{I-1}}, k_{e_I}, k_{e_I}, k_{e_{I+1}}, ..., k_{e_n}), (l_{e_1}, ..., l_{e_I}, l_{e_I} \pm 1, ..., l_{e_n})\}$ when $\triangle_m \cap e_I = \triangle_m$ with $f(e_I) = f(\triangle_m)$
- $s_{\triangle_m} = \{\gamma(s) \cup \triangle_m, (k_{e_1}, ..., k_{e_{I-1}}, k_{e_I}, k_{e_I}, k_{e_I}, k_{e_{I+1}}, ..., k_{e_n}), (l_{e_1}, ..., l_{e_I}, l_{e_I} \pm 1, l_{e_I}, ..., l_{e_n})\}$ when $\triangle_m \subset Int(e_I)$
- $s_{\triangle_m} = \{\gamma(s) \cup \triangle_m, (0, k_{e_1}, ..., k_{e_n}), (\pm 1, l_{e_1}, ..., l_{e_n})\}$ when $\triangle_m \cap e_1 = b(e_1)$
- $s_{\triangle_m} = \{\gamma(s) \cup \triangle_m, (k_{e_1}, ..., k_{e_n}, 0), (l_{e_1}, ..., l_{e_n}, \pm 1)\}$ when $\triangle_m \cap e_n = f(e_n)$

One can show that the Rigging map commutes with $\hat{a}_{T(\gamma_0(s_0))}$ iff $\eta_{[s]} = \eta_{[s_{\triangle_m}]}$ when $s_{\triangle_m}$ is any of the above. Note that, the exponentiated observables of the previous chapter only commute with the Rigging map under much stronger conditions on $\eta_{[s]}$ and also result in super-selection sectors on $\mathcal{H}_{phy}$. Such sectors do not seem to exist for the observables defined in this chapter.

4.5.2 Commutation relations

Next we study the commutator algebra generated by the Dirac observables (a_k, a_k^*) in quantum theory. Contrary to the classical Poisson algebra which closes, we show that in the quantum theory even $(a_k\, ,\, a_l)$ do not in general commute with each other. It is plausible that this will have serious implications on causal structure of the quantum theory and the issue is far from being resolved. Recall that the physical content of parametrized free field theory (at least classically) is same as that of ordinary free field theory on flat space-time. Whence we could have started with the reduced phase space co-ordinatized by (a_k, a_k^*), and its representation on Fock space will result in a quantum theory in which fields separated by space-like interval will commute. Also the two point functions will decay exponentially outside the light cone. However If the commutator algebra of (a_k, a_k^*) gets deformed in the quantum theory then it is not clear in what sense the causal structure defined by the background space-time is preserved. In fact as we are not aware of a state (or a class of states) in $\mathcal{H}_{phy}$ which correspond to the Fock vacuum, it is not even known how to define two point functions. (using which we can study causal relations.)

The commutator

When $k < 0$ and $l > 0$ it is clear that $[\hat{a}_{k'}\, ,\, \hat{a}_{l'}]$ will be trivially zero as $\hat{a}_{k'}$ acts on right-moving sector $(\mathcal{H}^{+}_{phy})$ and $\hat{a}_{l'}$ acts on left-moving sector $(\mathcal{H}^{-}_{phy})$ whence they commute.

Let us consider the case when k,l < 0. Remaining case (k,l > 0) can be handled similarly.

As $a_k = \int Y^{+}(x)e^{ik\,X^{+}(x)}$, we only look at the right-moving(+) sector of $\mathcal{H}_{phy}$.

$$
\begin{aligned}
(\,[\hat{a}_{k'}\, ,\, \hat{a}_{l'}]\Psi^{+}\,)f^{+}_{s} &= ((\hat{a}_{k'}\hat{a}_{l'}\; - \;\hat{a}_{k'}\hat{a}_{l'})\Psi^{+}\,)f^{+}_{s} \\
&= (\hat{a}_{k'}\hat{a}_{l'}\Psi^{+}\,)f^{+}_{s} \; - \; (\hat{a}_{l'}\hat{a}_{k'}\Psi^{+}\,)f^{+}_{s} \\
&= (\hat{a}_{l'}\Psi^{+}\,)(\hat{a}^{\dagger}_{k,T(\gamma_0(s_0))}f^{+}_{\gamma_0}) \; - \; (\hat{a}_{k'}\Psi^{+}\,)(\hat{a}^{\dagger}_{l,T(\gamma_0(s_0))}f^{+}_{s_0})
\end{aligned}
\tag{4.54}
$$

Where f_s^+ is an arbitrary state in the kinametical Hilbert space of the right moving sector $\mathcal{H}_{kin}^+$ and as before s_0 is a fixed charge-network in the orbit of s. Let us look at both the terms separately.

Term 1 - $(\hat{a}_{l'}\Psi^+)(\hat{a}^\dagger_{k,T(\gamma_0(s_0))}f^+_{s_0})$

Now we employ a specific choice of triangulation $T(\gamma_0(s_0))$. This choice is motivated by the requirement of simplicity. We will argue shortly that the result(at least qualitatively) does not depend on this particular choice.

So let us choose $T(\gamma_0(s_0)) = \gamma_0 \cup e_L \cup e_R$. where e_L and e_R are as shown in the figure 1.

(Remark : With this choice of the triangulation the continuum limit is approached only when $|E(\gamma_0)|$ tends to ∞.)

so,

$$\hat{a}^\dagger_{k,T(\gamma_0(s_0))} = \frac{1}{2i}\sum_{e_I\in(\gamma_0\cup e_L\cup e_R)} e^{-ik\hat{X}^+(v_I)}\,[\,h_{e_I} - h_{e_I^{-1}}\,] \tag{4.55}$$

Here $v_I = b(e_I)$.

Similarly we choose $T(\gamma_0(s_0)) = \gamma_0 \cup e_L \cup e_R$, which implies,

$$\begin{aligned}\Psi(\hat{a}^\dagger_{l,T(\gamma_0(s_0))}\,\hat{a}^\dagger_{k,T(\gamma_0(s_0))}\,f^\dagger_{\gamma_0}) = -\tfrac{1}{4}\,\Psi\,(\textstyle\sum_{e_J} e^{-il\hat{X}^+(v_J)}[\,h_{e_J} - h_{e_J^{-1}}\,]\\ \textstyle\sum_{e_I} e^{-ik\hat{X}^+(v_I)}[\,h_{e_I} - h_{e_I^{-1}}\,])\end{aligned} \tag{4.56}$$

Second term $(\hat{a}_{k'}\Psi^+)(\hat{a}^\dagger_{l,T(\gamma_0(s_0))}f^+_{s_0})$ can be evaluated similarly and we get,

$$([\hat{a}_{k'}\,,\,\hat{a}_{l'}]\Psi\,)(f_s^+) = \Psi\,(\sum_{e_J,e_I} e^{-il\hat{X}^+(v_J)}e^{-ik\hat{X}^+(V_I)}[(h_{e_J} - h_{e_J^{-1}}\,)\,,\,(\,h_{e_I} - h_{e_I^{-1}}\,)]\,) \tag{4.57}$$

In the above double sum only those edges $(e_I\ e_J)$ contribute for which $e_I \cap e_J \neq 0$. Consider the following 2 pairs (I= M,J=M+1) and (I+M+1,J=M) for some fixed M.

The contribution to the commutator coming from the above pairs is,

$$
\begin{aligned}
-\tfrac{1}{4}\Psi(e^{-il\hat{X}^+(v_{M+1})}e^{-ik\hat{X}^+(v_M)}[\,(h_{e_{M+1}} - h_{e^{-1}_{M+1}})\,(h_{e_M} - h_{e^{-1}_M})\,] \\
+e^{-ik\hat{X}^+(v_{M+1})}e^{-il\hat{X}^+(v_M)}[(h_{e_M} - h_{e^{-1}_M})\,(h_{e_{M+1}} - h_{e^{-1}_{M+1}})]\, f^+_{s0}) \\
= -\tfrac{1}{4}\Psi\,(\,[\,(h_{e_{M+1}} - h_{e^{-1}_{M+1}})\,,\,(h_{e_M} - h_{e^{-1}_M})\,] \\
(\,e^{-il\hat{X}^+(v_{M+1})}e^{-ik\hat{X}^+(v_M)} - e^{-ik\hat{X}^+(v_{M+1})}e^{-il\hat{X}^+(v_M)}\,)\,)f^+_{s0}
\end{aligned}
\tag{4.58}
$$

Now using (4.25) one can show that,

$$
[h_{e_I},\, h_{e_J}] \;=\; \frac{1}{2i}\, sin(\hbar\alpha(e_I, e_J))\, h_{e_I+e_J} \tag{4.59}
$$

It is now straight-forward to evaluate the commutators in the (4.58). Whence given a pair of successive edges which lie within the graph $(e_I\,,\, e_{I+1})$ (I = 1,...N-1) their contribution to $(\,[\hat{a}_{k'}\,,\,\hat{a}_{l'}]\Psi\,)(f^+_s)$ is,

$$
\begin{aligned}
-\tfrac{1}{4}\Psi\,[\,\tfrac{1}{2i}\, sin(\tfrac{1}{2}\hbar i)\textstyle\sum_{I=1}^{N-1}(h_{e_I+e_{I+1}} + h_{e_{I-1}+e_{I+1-1}} + h_{e_{I-1}+e_{I+1}} + h_{e_I+e_{I+1-1}}) \\
[\,e^{-il\hat{X}^+(v_{I+1})}e^{-ik\hat{X}^+(v_I)} - e^{-ik\hat{X}^+(v_{I+1})}e^{-il\hat{X}^+(v_I)}\,]f^+_{s0} \\
= -\tfrac{1}{4}\Psi\,[\,\tfrac{1}{2i}\, sin(\tfrac{1}{2}\hbar)\textstyle\sum_{I=1}^{N-1}(h_{e_I+e_{I+1}} + h_{e_{I-1}+e_{I+1-1}} + h_{e_{I-1}+e_{I+1}} + h_{e_I+e_{I+1-1}}) \\
[e^{-\frac{1}{2}i\hbar l(k_{e_I}+k_{e_{I+1}})}e^{-\frac{1}{2}i\hbar k(k_{e_{I-1}}+k_{e_I})} - e^{-\frac{1}{2}i\hbar k(k_{e_I}+k_{e_{I+1}})}e^{-\frac{1}{2}i\hbar l(k_{e_{I-1}}+k_{e_I})}]f^+_{s0}
\end{aligned}
\tag{4.60}
$$

Where we have used $\hat{X}^+(v_I)f^+_{s0} = \frac{1}{2}\hbar(k^+_{e_I} + k^+_{e_{I+1}})f^+_{\gamma 0}$ and defined $k_{e_0} = 0$. Finally there are contributions from the pair (e_L, e_1) and (e_N, e_R),

$$
\begin{aligned}
-\tfrac{1}{4}\Psi\,((e^{-ilk_{e_1}} - e^{-ikk_{e_1}})\,\tfrac{1}{2i}sin(\tfrac{1}{2}\hbar)[h_{e_1+e_L} + h_{e_1^{-1}+e_L^{-1}} + h_{e_1^{-1}+e_L} + h_{e_1+e_L^{-1}}]f^+_{s0}) \\
-\tfrac{1}{4}\Psi((e^{-il\hbar k_{e_N}}e^{-i\frac{1}{2}k\hbar(k_{e_{N-1}}+k_{e_N})} - e^{-ik\hbar k_{e_N}}e^{-i\frac{1}{2}l\hbar(k_{e_{N-1}}+k_{e_N})}) \\
\tfrac{1}{2i}\, sin(\tfrac{1}{2}\hbar)[h_{e_R+e_N} + h_{e_{R-1}+e_{N-1}} + h_{e_{R-1}+e_N} + h_{e_R+e_{N-1}}]f^+_{s0})
\end{aligned}
$$

Let $e_L = e_0$ and $e_R = e_{N+1}$, $k_{e_0} = k_{e_{N+1}} = 0$ we finally get,

$$
\begin{aligned}
([\hat{a}_{k'}\ \hat{a}_{l'}]\Psi)(f_s^+) \ =& \\
-\tfrac{1}{4}\Psi\Bigg(&\tfrac{1}{2i} sin(\tfrac{1}{2}\hbar) \textstyle\sum_{I=0}^{N} [h_{e_I + e_{I+1}} + h_{e_{I-1}+e_{I+1-1}} + h_{e_{I-1}+e_{I+1}} + h_{e_I + e_{I+1-1}}] \\
&(e^{-i\frac{1}{2}\hbar l(k_{e_I}+k_{e_{I+1}})} e^{-i\frac{1}{2}\hbar k(k_{e_{I-1}}+k_{e_I})} - e^{-i\frac{1}{2}\hbar k(k_{e_I}+k_{e_{I+1}})} e^{-i\frac{1}{2}\hbar l(k_{e_{I-1}}+k_{e_I})})f_{s0}^+ \Bigg)
\end{aligned} \tag{4.61}
$$

Thus it is clear that in general the commutator $[\hat{a}_{k'}\ \hat{a}_{l'}]$ (k , l < 0)does not vanish on $\mathcal{H}_{phy}$. The commutator for $[\hat{a}_{k'}\ \hat{a}_{l'}]$ with (k , l > 0) is exactly similar with all operators acting on the left moving sector.

Now we give a heuristic proposal showing existence of (a class of) states on which the commutator $[\hat{a}_{k'}\ \hat{a}_{l'}]$ vanishes. Ideally one would like to do a semi-classical analysis of the expectation value of the commutators to see if the non-zero contributions are sub-leading. This is an open question that we have not addressed in the present paper. In what follows we argue for the existence of states (possibly in ITP(infinite tensor product extension [43]) of $\mathcal{H}_{phy}$) on which the commutator vanishes.

Notice that given a $\Psi \in \mathcal{H}_{phy}$, $([\hat{a}_{k'}\ \hat{a}_{l'}]\Psi)(f_s^+)$ is non-zero iff the "embedding-component" of Ψ is group averaged distribution obtained from the "embedding-component" of f_s^+. In other words if $\Psi = |2N{+}1, 2N{+}2, ([-N, l_1], ..., [N, l_{2N}]) >$ where the matter-charges $(l_1, ...l_{2N})$ are arbitrary but non-zero then $([\hat{a}_{k'}\ \hat{a}_{l'}]\Psi)(f_s^+)$ is non-zero iff $|E(\gamma)| =$ 2N+1 , the embedding charges on the edges of γ constitute the set (-N,...,N) and the matter charges form a set $(l_1, ..., l_I \pm 1, l_{I+1} \pm 1, ...l_{2N})$ for some I.

$$
\begin{aligned}
([\hat{a}_{k'}\ \hat{a}_{l'}]\Psi)(f_s^+) \ =& \\
-\tfrac{1}{4}\Psi sin(\tfrac{1}{2}\hbar i)\Bigg(&\textstyle\sum_{I=0}^{N} [h_{e_I + e_{I+1}} + h_{e_{I-1}+e_{I+1-1}} + h_{e_{I-1}+e_{I+1}} + h_{e_I + e_{I+1-1}}] \\
&\textstyle\sum_{n=-N}^{N} [e^{-i\hbar(l+k)n} e^{-\frac{1}{2}i\hbar(l-k)} - e^{-i\hbar(l+k)n} e^{-\frac{1}{2}i\hbar(k-l)}] f_{s0}^+ \Bigg)
\end{aligned} \tag{4.62}
$$

Now as $N \to \infty$ and each e_I shrinks to it is vertex v_I, and if we assume that to leading order in $\frac{1}{N}$, $h_{e_I} \to 1$ then one gets,

$$
\begin{aligned}
([\hat{a}_{k'}, \hat{a}_{l'}]\Psi)(f_s^+) &= \\
-\tfrac{1}{4}\Psi &\left(\sum_{n\in\mathbf{Z}} [e^{-i\hbar(l+k)n}e^{-\frac{1}{2}i\hbar(l-k)} - e^{-i\hbar(l+k)n}e^{-\frac{1}{2}i\hbar(k-l)}] f_{s_0}^+ \right) \\
= -\tfrac{1}{4}\Psi &\left(\delta(l+k) sin(\tfrac{1}{2}\hbar(l-k))\ f_{s_0}^+ \right)
\end{aligned}
\tag{4.63}
$$

which equals 0 for l,k < 0.

Couple of comments are in order :

1. We have not displayed semi-classicality in the sense that we have not shown that the non-vanishing terms in $[\hat{a}_{k'}\ \hat{a}_{l'}]$ are sub-leading corrections on a class of states in $\mathcal{H}_{phy}$.

2. The above result does not depend on our choice of triangulation $T(\gamma_0(s_0)) = \gamma_0 \cup e_L \cup e_R$. Consider any triangulation, T which is adapted to γ_0 in the sense that the vertex set of γ_0 is a subset of the vertex-set of T. Then it can be shown that only those edges which intersect the vertices of the graph contribute. Contributions from all other edges cancel out pairwise.

The calculation of $[\hat{a}_{k'}\ \hat{a}_l^{\dagger'}]$ proceeds similarly.

$$
\begin{aligned}
&([\hat{a}_{k'}, \hat{a}_l*']\Psi)(f_s^+) = \\
&\tfrac{1}{4}\Psi sin(\tfrac{1}{2}\hbar i)\Bigg(\sum_{I=0}^{N}[h_{e_I+e_{I+1}} + h_{e_{I-1}+e_{I+1-1}} + h_{e_{I-1}+e_{I+1}} + h_{e_I+e_{I+1-1}}] \\
&(e^{i\frac{1}{2}\hbar l(k_{e_I}+k_{e_{I+1}})}e^{-i\frac{1}{2}\hbar k(k_{e_{I-1}}+k_{e_I})} - e^{-i\frac{1}{2}\hbar k(k_{e_I}+k_{e_{I+1}})}e^{i\frac{1}{2}\hbar l(k_{e_{I-1}}+k_{e_I})}) f_{s_0}^+ \Bigg)
\end{aligned}
\tag{4.64}
$$

which as $N \to \infty$ and each e_I shrinks to it is vertex v_I gives,

$$
([\hat{a}_{k'}\ \hat{a}_l^{*'}]\Psi)(f_s^+) = \frac{1}{4\hbar} sin(\frac{\hbar i}{2})\Psi(\ \delta(l-k) sin(\frac{1}{2}\hbar(l+k))\ f_{\gamma_0}^+\ ) \tag{4.65}
$$

which is a specific quantum deformation of the classical Poisson bracket.

Our heuristic calculations show that it is plausible that on a specific class of states with countably infinite edges the commutator algebra generated by ($a^{k'}$, $a^{k^{*'}}$ and 1) closes and is a specific deformation of the Poisson algebra. Such states cannot lie in $\mathcal{H}_{phy}$ but in infinite tensor product extension thereof [43].

This suggests : In [55] semi-classical states have been defined by using graphs with large but finite number of edges. However based on the heuristic calculations displayed above we believe that when spatial slice is non-compact, ideal home for semi-classical states is the ITP extension of $\mathcal{H}_{phy}$.

We now turn to the construction of Evolving observables in the polymer quantized CGHS model. But before we begin, let us briefly summarize the analysis done so far in this chapter. We started with the canonical action of rescaled-CGHS model defining a parametrized scalar field theory on flat space-time. By performing a background-independent GNS quantization of a suitable Poisson algebra, we obtained a kinematical Hilbert space $\mathcal{H}_{kin}$ which admitted a unitary representation of the space-time diffeomorphism group. By group averaging the diffeomorphisms we obtained a physical Hilbert space $\mathcal{H}_{phy}$. Given any solution of Klein-Gordon equation on flat spacetime, its Fourier coefficients $(a_k,\ a{*}_k)$ on any arbitrary space-like slice became the Dirac observables of the theory. We defined corresponding (regulated) operators on $\mathcal{H}_{phy}$ and showed that their commutators are deformed away from their classical Poisson algebra.

Let us very briefly summarize the results obtained in this chapter so far.
Basis for $\mathcal{H}_{kin}$.

Consider a graph γ with a set of pair of real numbers $((k_1^{\pm}, l_1^{\pm}), ..., (k_N^{\pm}, l_N^{\pm}))$ where in outermost pairs ($(k_1^{\pm}, l_1^{\pm})$ and $(k_N^{\pm}, l_N^{\pm})$)) either k_i or l_i can be zero but not both. in the interior edges $(e_2, ..., e_{N-1})$ we even allow both the charges (k,l) to be zero. We call the pair $(\gamma, ((k_1^{\pm}, l_1^{\pm}), ..., (k_N^{\pm}, l_N^{\pm})))$ charge-network (in analogy with spin-networks in LQG) and denote it by $s^{\pm}$. The state associated with s is denoted by $f_{s^{\pm}}^{\pm}$.
Representation of embedding variables.

$$\begin{aligned} \hat{X}^{\pm}(x) f_{s^{\pm}}^{\pm} &= (-i\hbar)\ ik_e\ f_s^{\pm}\ \ if\ x \in e \\ &= (-i\hbar)\ i\frac{(k_e + k_{e'})}{2}\ f_{\gamma}^{\pm}(\Pi^{\pm})\ \ if\ x \in e \cap e' \\ &= 0 \text{ otherwise} \end{aligned} \tag{4.66}$$

A complete set dirac observables in classical theory.

$$\begin{aligned} a_k &= \int e^{-ikX^-}Y^- \qquad k>0 \\ a_k &= \int e^{-ikX^+}Y^+ \qquad k<0 \\ a_0 &= \int \pi_f \end{aligned} \tag{4.67}$$

a_k^* are defined by complex conjugating the a_k.
Representation of observables on $\mathcal{H}_{kin}$.

$$\hat{a}_{k,T(\gamma(s))}f_{s^-}^- = \sum_{\triangle_m \in T(\gamma(s^-))} e^{ik\hat{X}^-(v_m)}[h_{\triangle_m} - h_{\triangle_m^{-1}}]\, f_{s^-}^- \tag{4.68}$$

Similar expression holds for $k < 0$ with (X^-, Y^-) replaced by (X^+, Y^+) and the resulting operator acting on f_s^+.
Also recall that we denoted the physical state obtained by group averaging $f_{s^+}^+ \otimes f_{s^-}^- \otimes |m_R\rangle$ (with $s^\pm = \{\gamma(s^\pm), \vec{k}^\pm, \vec{l}^\pm\}$) as $\Psi = |N, \vec{k}^+, \vec{l}^+\rangle \otimes |M, \vec{k}^-, \vec{l}^-\rangle \otimes |m_R\rangle$ is a physical basis state with N(M) being the number of edges, and $\vec{k}^\pm$, $\vec{l}^\pm$ the embedding and matter charges respectively.

4.6 Classical complete observables

The canonical co-ordinates on the phase-space are $(f(x),\ \pi_f(x),\ X^\pm(x),\ \Pi_\pm(x))$. In what follows, we will treat $x \in \sigma$ as a label set and think of the canonical fields as functionals on the phase-space labeled by x i.e. $f(x) : \mathcal{M} \to \mathbf{R}$ Now choose the following gauge fixing conditions for the two Virasoro constraints,

$$X^\pm(x) = X_p^\pm(x)\ \forall\ x \in \sigma \tag{4.69}$$

where $X_p^\pm : \sigma \to M^2$ is a prescribed embedding of σ in Minkowski space M^2. As $\{H^\pm(x)\ ,\ X^\pm(x')\ \} = -X^\pm(x)'\delta(x,x') \neq 0$, these are good gauge fixing conditions in the sense that they define global gauge slices, i.e. one can draw a gauge orbit passing through any point on the constraint surface which intersects this slice transversally.
Using the functional on phase-space f(x), for a given x, we can now construct a complete observable $f(x)[X_p^+, X_p^-; m]$ as follows.

Given a point $m = (f,\ \pi_f,\ X^+,\ X^-)$ on the constraint surface, and a gauge orbit $\mathcal{G}_m$ passing through it, we ask for the value of f(x) at that point m' on the gauge orbit which intersect the gauge slice defined by the above gauge fixing conditions. i.e. we define,

$$f(x)[X_p^+, X_p^-; m] \;=\; f(x)[m'] \tag{4.70}$$

It is immediately clear from the definition that $f(x)[X_p^+, X_p^-; m]$ is invariant under gauge transformations, and one can show that [16] it has a (weakly) unique extension off the constraint surface.

A complete observable $\pi_f(x)[X_p^+, X_p^-; m]$ corresponding to π_f can be defined analogously.

There is an alternate characterization of complete observables (as given in section 2.2.4 of chapter 2) in parametrized field theory which immediately yield the explicit expression for these quantities.[15] Given a maximal classical solution to $\Box f(X) = 0$ which lies in the gauge orbit $\mathcal{G}$ passing through $(f, \pi_f, X^\pm)$, what is the Cauchy data corresponding to it on the slice given by $X_p^\pm : \sigma \rightarrow M^2$. The answer is immediate,

$$\begin{aligned} f(x)[X_p^+, X_p^-; m] &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tfrac{dk}{|k|} [\, a_k(m) e^{ik\cdot X_p(x)} \;+\; a_k^*(m) e^{-ik\cdot X_p(x)}] \\ \pi_f(x)[X_p^+, X_p^-; m] &= \tfrac{i}{\sqrt{2\pi}} [\, \int_0^\infty dk \; X_p^{-'}(x) a_k(m) e^{ik\cdot X_p} \;+\; c.c. \\ &\quad - \int_{-\infty}^0 dk \; X_p^{+'}(x) a_k(m) e^{ik\cdot X_p} \;-\; c.c. \,] \end{aligned} \tag{4.71}$$

Thus given a free scalar field on flat space-time, its Cauchy data on a prescribed slice gives complete observables of the corresponding parametrized field theory.

As shown in [16], Poisson bracket of two complete observables is a complete observable. Thus the space of all complete observables form an Poisson *-algebra.

[15] The complete observables also satisfy a functional differential equation see [16] which in our case can be explicitly solved to get the same expression for the complete observables that are given here.

4.6.1 Time evolution or Dynamical interpretation

One can define non-trivial gauge transformations $\hat{\alpha}_\tau$ on the space of observables which generalizes Rovelli's idea of evolving constant of motion to arbitrary number of constraints. [[16], [40]] The basic idea is to see how $f(x)[X_p^+, X_p^-; m]$ changes when one changes the gauge-fixing slices $X^\pm = X_p^\pm$ under gauge transformations. In our case these transformations simply amount to changing the parameters $X_p^\pm(x)$ additively.

$$X_p^\pm(x) \to X_p^\pm(x) + \tau^\pm(x) \tag{4.72}$$

where $\tau^\pm(x) \in [-\infty, \infty]$. Here $\tau^\pm(x) = N^\pm(x)$ where $N^\pm(x)$ are linear combinations of Lapse and Shift functions. Whence,

$$\hat{\alpha}_\tau \, f(x)[X_p^+, X_p^-; m] = f(x)[X_p^+ + \tau^+, X_p^- + \tau^-; m] \tag{4.73}$$

It can be shown that $\hat{\alpha}_\tau$ act as automorphisms on the algebra of observables [16],

$$\begin{aligned} &\hat{\alpha}_\tau\{f(x)[X_p^+, X_p^-; m], f(x')[X_p^+, X_p^-; m]\} = \\ &\qquad \{\hat{\alpha}_\tau f(x)[X_p^+, X_p^-; m], \hat{\alpha}_\tau f(x')[X_p^+, X_p^-; m]\} \end{aligned} \tag{4.74}$$

So far it is not clear in what sense these transformations define temporal evolution (dynamics) of the complete observables. The easiest way to see this is by using an alternate characterization of the above automorphisms [26].

Given a 1 parameter group of time-like diffeomorphisms $\theta(t) : M \to M$ of the auxiliary Minkowski background, one can associate to it a 1 parameter group of symplectic diffeomorphisms [[25]] $\overline{\theta}(t)$ which are defined as follows.

Given a point $m = (\phi, \pi_\phi, X^+, X^-)$ in the constraint surface Γ_c define $\overline{\theta}(t) : \Gamma_c \to \Gamma_c$ as, $\overline{\theta}(t)(\phi, \pi_\phi, X^+, X^-) = (\phi, \pi_\phi, X^+ \circ \theta(t), X^- \circ \theta(t))$. These symplectomorphisms shift the gauge-fixing slice $X = X_p$ to $X \circ \theta(t) = X_p$. This can also be understood as changing the prescribed embedding X_P to some new embedding $\theta(-t)X_p$ [16]. Time evolution is the evolution of complete observables under above change.

Consider for example two parameter family of timelike killing fields $V =$

[16] $\theta(-t)X_p$ is merely a notation. It is not to be understood in the same sense as $\theta(-t)X$

$A^+\partial_+ + A^-\partial_-$ where $A^\pm$ are real numbers. using $\theta(t) = exp(tV)$

$$\begin{aligned} X^+ &\rightarrow X^+ + A^+t \\ X^- &\rightarrow X^- + A^-t \end{aligned} \tag{4.75}$$

The corresponding change in the gauge fixing slice $X = X_p$ is the same as that would be obtained by transforming $X_p^\pm$ as,

$$\begin{aligned} X_p^+ &\rightarrow X_p^+ - A^+t \\ X_p^- &\rightarrow X_p^- - A^-t \end{aligned} \tag{4.76}$$

As V is timelike, $|A^+ + A^-| < |A^+ - A^-|$. Comparing (4.76) with (4.72) we can rewrite (4.72) with $\tau^+ = A^+t$ and $\tau^- = A^-t$. V is timelike implies,

$$|\tau^+ + \tau^-| < |\tau^+ - \tau^-| \tag{4.77}$$

Thus change in complete observables under (4.72) which satisfy the inequalities (4.77) define a class of time evolution for the system.

Whence by combining the complete set of Dirac observables obtained in the previous section with the notion of gauge-fixed slices in constraint surface we have defined dynamical observables of our theory.

Note that this method of defining time evolution is rather generic in parametrized field theory, as explained beautifully in ([26],[23],[24]).

4.7 Canonical quantization of complete observables

Now we consider quantization of the complete observables $f(x)[X_p^+, X_p^-]$ and $\pi_f(x)[X_p^+, X_p^-]$ on $\mathcal{H}_{phy}$. As $X_p^\pm(x)$ are mere c-numbers, the canonical quantization of these observables follow directly from the quantization of the Dirac observables $(a_k,\ a_k*)$ on $\mathcal{H}_{phy}$[34],

$$\begin{aligned}(f(x)\widehat{[X_p^+, X_p^-]}_{can}\ \Psi)(f_{s^+}^+ \otimes f_{s^-}^-) = \\ \Psi(\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tfrac{dk}{|k|}[\ \hat{a}_{k,T(\gamma(s))}e^{ik\cdot X_p(x)} + \hat{a}^\dagger_{k,T(\gamma(s))}e^{-ik\cdot X_p(x)}]\ f_{s^+}^+ \otimes f_{s^-}^-)\end{aligned} \tag{4.78}$$

where $\Psi \in \mathcal{H}_{phy}$.

Thus the action of $f(x)\widehat{[X_p^+, X_p^-]}_{can}$ on Ψ is obtained via its dual action on a charge-network state $f_{s^+}^+ \otimes f_{s^-}^-$. The dual action is in turn defined in terms of $\hat{a}_{k,T(\gamma(s))}$ and $\hat{a}^\dagger_{k,T(\gamma(s))}$.

Note that although it is not possible to define an operator corresponding to f(x) on $\mathcal{H}_{kin}$, we have been able to define an operator for the corresponding complete observable on $\mathcal{H}_{phy}$. However as we will see in 4.8.2 depending on the value of the parameters ($X_p^\pm(x)$), this operator is not well defined on all charge networks.

One can similarly define an operator valued distribution for $\pi_f(x)\widehat{[X_p^+, X_p^-; m]}_{can}$ on $\mathcal{H}_{phy}$.

As before let Ψ be a physical state given above.

Recall that in order to define $\hat{a}_k$, $\hat{a}_k^\dagger$ on $\mathcal{H}_{phy}$ we have to fix a pair of charge-networks (s_0^+, s_0^-) in the orbit of (s^+, s^-) and a pair of triangulations $(T(\gamma_0(s_0^+)), T(\gamma_0(s_0^-)))$.

$$
\begin{aligned}
(\pi_f(x)\widehat{[X_p^+, X_p^-]}_{can}\Psi)(f_{s^+}^+ \otimes f_{s^-}^-) \; = \\
\Psi\, [\, \frac{i}{2\sqrt{\pi}} \int_0^\infty dk e^{ikX_p^-} \frac{(X_p^-(x) - X_p^-(v_m))}{|\triangle_m|} a_{k,T(\gamma_0(s_0^+))} \\
- \frac{i}{2\sqrt{\pi}} \int_0^\infty dk e^{-ikX_p^-} \frac{(X_p^-(x) - X_p^-(v_m))}{|\triangle_m|} a^\dagger_{k,T(\gamma_0(s_0^+))} \\
- \frac{i}{2\sqrt{\pi}} \int_0^\infty dk e^{ikX_p^+} \frac{(X_p^+(x) - X_p^+(v_m))}{|\triangle_m|} a_{k,T(\gamma_0(s_0^-))} \\
+ \frac{i}{2\sqrt{\pi}} \int_0^\infty dk e^{-ikX_p^+} \frac{(X_p^+(x) - X_p^+(v_m))}{|\triangle_m|} a^\dagger_{k,T(\gamma_0(s_0^-))}\,] \\
(f_{s_0^+}^+ \otimes f_{s_0^-}^-)
\end{aligned}
\tag{4.79}
$$

$\triangle_m$ is a simplex which begins at v_m and terminates at x. The derivative of prescribed embedding at x has been replaced by finite difference.[17] Here we have chosen $T(\gamma_0(s_0^+))$ and $T(\gamma_0(s_0^-))$ such that $\triangle_m$ is a simplex in both the triangula-

[17] It isn't essential to do this here as X_p are classical functions but it will be required when we define new quantum observables in the next section.

tions and v_m, x are its initial and final vertices respectively.

The Heisenberg dynamics is defined by promoting automorphism of the algebra of complete observables (4.73) to an automorphism on the algebra of corresponding operators, but this automorphism is not generated by any unitary operator. Thus the quantum dynamics is not unitary.

Note: There is a potential problem with the above definitions of complete observables in quantum theory. In classical theory although $X_p^{\pm}(x)$ are just parameters they are also the value of prescribed embeddings at spatial point x (recall the gauge fixing conditions $X^{\pm} = X_p^{\pm}$ required to define the complete observables). However in quantum theory there seems to be no relation between $X_p^{\pm}(x)$ and the embedding charges which label a given state. In view of this, we propose an alternative definition of $f(x)\widehat{[X_p^+, X_p^-]}$, $\pi_f(x)\widehat{[X_p^+, X_p^-]}$ in the next section and show how it leads to several interesting consequences in the quantum theory.

In this section we continue to work with the canonically quantized observables.

4.7.1 Complete observable from the dilaton field

We now apply the formalism we have developed so far to quantize the complete observable corresponding to the dilaton field on $\mathcal{H}_{phy}$. This operator is the starting point for the discussions about physical quantum geometry.

Using the above expressions for the complete observables corresponding to the scalar field and its conjugate momenta, we can obtain an observable corresponding to the dilaton as follows.

As shown in [33] the canonical transformation relating the dilaton to the embedding chart on the phase space is given by,

$$\begin{aligned} y(x) = & \lambda^2 X^+(x)X^-(x) - \int_{\infty}^{x} dx_1 X^{-'}(x_1) \int_{\infty}^{x_1} dx_2 \Pi_-(x_2) \\ & + \int_{-\infty}^{x} dx_1 X^{+'}(x_1) \int_{-\infty}^{x_1} dx_2 \Pi_+(x_2) + \int_{-\infty}^{\infty} X^+(x)\Pi_+(x) + \frac{m_R}{\lambda}. \end{aligned} \tag{4.80}$$

This expression can be rearranged using integration by parts as follows.

$$\begin{aligned} y(x) \ = \ & \lambda^2 X^+(x)X^-(x) \ - \ X^-(x)\int_\infty^x dx_1 \Pi_-(x_1) \ + \ X^+(x)\int_{-\infty}^x dx_1 \Pi_+(x_1) \ + \\ & \int_\infty^x dx_1 X^-(x_1)\Pi_-(x_1) \ + \ \int_{-\infty}^x dx_1 X^+(x_1)\Pi_+(x_1) \ + \\ & \int_{-\infty}^\infty X^+(x)\Pi_+(x) + \frac{m_R}{\lambda}. \end{aligned} \tag{4.81}$$

One can go to the constraint surface by solving for embedding momenta in terms of the scalar field and its conjugate momenta and by substituting the complete observable corresponding to the scalar field content, we obtain the observable corresponding to the dilaton. In the spacetime picture one can think of dilaton y(X), as a function of the spacetime scalar field, and the complete observable $y(x)[X_p^+, X_p^-]$ (As mentioned above $x \in \Sigma$ should be thought of as a label set.) corresponds to the pull back of y(X) on a prescribed spatial slice when the free scalar field f(X) is pulled back on it.
Whence,

$$\begin{aligned} y(x)[X_p^+, X_p^-] \ = \ & \lambda^2 X_p^+(x)X_p^-(x) \ - \ \frac{X_p^-(x)}{4} \int_\infty^x dx_1 \ \frac{Y_-(x_1)[X_p^+, X_p^-]^2}{X_p^{-'}(x_1)} \\ - \ \frac{X_p^+(x)}{4} \int_{-\infty}^x dx_1 \ & \frac{Y_+(x_1)[X_p^+, X_p^-]^2}{X_p^{+'}(x_1)} \ + \ \int_\infty^x dx_1 \ \frac{X_p^-(x_1)}{X_{p'}^-(x_1)} \ Y_-(x_1)[X_p^+, X_p^-]^2 \\ + \ \int_\infty^x dx_1 \ & \frac{X_p^+(x_1)}{X_{p'}^+(x_1)} \ Y_+(x_1)[X_p^+, X_p^-]^2 + \ \frac{m_R}{\lambda}. \end{aligned} \tag{4.82}$$

We would like to promote $y(x)[X_p^+, X_p^-]$ to an operator on $\mathcal{H}_{phy}$. This is the operator which contains the information about physical quantum geometry and thus is the most important ingredient in asking non-perturbative questions regarding fate of black-hole singularities, quantum fluctuation of event horizons and even semi-classical issues like Hawking radiation.
Once again it is important to note that as our previous complete observables, $\widehat{y(x)[X_p^+, X_p^-]}_{can}$ is defined via its dual action on $\mathcal{H}_{kin}$. The derivation is given in appendix A. Here we quote the final result.[18]

[18]In order to not clutter the formulae more, we denote $\gamma_0(s_0^{\pm})$ as $\gamma_0^{\pm}$ from here onwards.

$$\left[y(x)\widehat{[X_p^+, X_p^-]}_{can}\Psi \right](f_{s^+}^+ \otimes f_{s^-}^- \otimes |m\rangle) =$$
$$\Psi\Bigg(\Big[\lambda^2 X_p^+(x)X_p^-(x) \; - \; X_p^-(x) \textstyle\sum_{T_x(\gamma_0^+)} \hat{\mathbf{A}} \; - \; X_p^+(x) \textstyle\sum_{\bar{T}_x(\gamma_0^-)} \hat{\mathbf{B}}$$
$$+ \textstyle\sum_{\triangle_m \in T_x(\gamma_0^+)} X_p^+(v_m)\hat{\mathbf{B}} \; + \; \textstyle\sum_{\triangle_m \in T_x(\gamma_0^-)} X_p^-(v_m)\hat{\mathbf{A}} \; + \frac{\hat{m_R}}{\lambda} \Big](f_{s_0^+}^+ \otimes f_{s_0^-}^-) \Bigg) \tag{4.83}$$

where $\Psi = |\, N,\, N{+}1,\, (k_1^+, l_1^+), ..., (k_N^+, l_N^+) >_+ \otimes |\, M,\, M{+}1,\, (k_1^-, l_1^-), ..., (k_N^-, l_N^-) >_- \otimes |m>$ is a physical state and $f_{s^+}^+ \otimes f_{s^-}^- \otimes |m\rangle$ is an arbitrary charge network in $\mathcal{H}_{kin}$.

We note the following.

1.$T_x(\gamma_0^+)$ is the sub-complex of $T(\gamma_0^+)$ from $-\infty$ to x.

2.$\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are defined in 4.106, 4.107 respectively. Their expressions are not very enlightening however and it suffices to note that their origin lies in various composite operators of the form $\hat{a}_k \hat{a}_k{}^\dagger$.

Thus we have a "regulated" expression $y(x)\widehat{[X_p^+, X_p^-]}_{can}$ on $\mathcal{H}_{phy}$. Several ad-hoc choices are involved, notably fixing a pair of graphs (γ_0^+, γ_0^-) in the orbit of (γ^+, γ^-) and a pair of triangulations $(T(\gamma_0^+), T(\gamma_0^-))$ corresponding to them.

In classical theory the singularities in physical spacetimes usually occur when $y(X) = 0$. Hence to understand the singularity structure of quantum geometry , the expectation value of $y(x)\widehat{[X_p^+, X_p^-]}_{can}$ in a given physical state will play a central role. The expectation value of the dilaton is also a primary object in obtaining the semi-classical geometry from the non-perturbative quantum theory[39]. Thus in order to ask the physical questions using the framework setup in this paper, evaluating the $\langle\Psi| y(x)\widehat{[X_p^+, X_p^-]}_{can}|\Psi\rangle_{phy}$ is of crucial importance. Hence we now calculate the expectation value of $y(x)\widehat{[X_p^+, X_p^-]}_{can}$ for a generic basis-state Ψ. It is possible(and of course essential) to extend our results to obtain expectation value in an arbitrary state in $\mathcal{H}_{phy}$, however the computations are quite complicated and are summarized in appendix-B. The result is:

For an arbitrary basis-state Ψ in $\mathcal{H}_{phy}$, $\langle\Psi| y(x)\widehat{[X_p^+, X_p^-]}_{can}|\Psi\rangle_{phy}$ equals,

$$\langle\Psi|y(x)[\widehat{X_p^+, X_p^-}]_{can}|\Psi\rangle_{phy} = \Big[\lambda^2 X_p^+(x)X_p^-(x) + \tfrac{X_p^-(x)}{16\pi}\textstyle\sum_{\triangle_m\in T_x}\mathcal{A}_m - \tfrac{X_p^+(x)}{16\pi}\sum_{\triangle_m\in T_x}\mathcal{B}_m$$
$$-\tfrac{1}{16\pi}\textstyle\sum_{\triangle_m\in T_x} X_p^-(v_m)\mathcal{A}_m - \tfrac{1}{16\pi}\sum_{\overline{T}_x} X_p^+(v_m)\mathcal{B}_m + \tfrac{\hat{m}_R}{\lambda}\Big] \tag{4.84}$$

Where T_x is sub-complex from -∞ to x and $\overline{T}_x$ is the sub-complex from x to ∞. $\mathcal{A}_m$ and $\mathcal{B}_m$ are evaluated in appendix-B.

$$\mathcal{B}_m = [X_p^-(v_m) - X_p^-(v_{m-1})]$$
$$\int_0^\infty dk d\overline{k}\Big\{e^{i(k+\overline{k})X_p^-(v_m)}\big(-2\textstyle\sum_{n=0}^M e^{-\frac{i}{2}(k+\overline{k})(k_n^- + k_{n+1}^-)} - 4\big) + c.c.$$
$$+e^{-i(k-\overline{k})X_p^-(v_m)}\big(-2\textstyle\sum_{n=0}^M e^{-\frac{i}{2}(k-\overline{k})(k_n^- + k_{n+1}^-)} - 4\big) + c.c.\Big\} \tag{4.85}$$

$$\mathcal{A}_m = [X_p^+(v_m) - X_p^+(v_{m-1})]$$
$$\int_{-\infty}^0 dk d\overline{k}\Big\{e^{i(k+\overline{k})X_p^+(v_m)}\big(-2\textstyle\sum_{n=0}^N e^{-\frac{i}{2}(k+\overline{k})(k_n^+ + k_{n+1}^+)} - 4\big) + c.c.$$
$$+e^{-i(k-\overline{k})X_p^+(v_m)}\big(-2\textstyle\sum_{n=0}^N e^{-\frac{i}{2}(k-\overline{k})(k_n^+ + k_{n+1}^+)} - 4\big) + c.c.\Big\} \tag{4.86}$$

Note that we have obtained a closed-form expression for the expectation value of the dilaton in any basis-state Ψ. Referring to appendix-B one notes that this calculation has been performed for a specific choice of charge-networks $(f_{s_0^+}^+, f_{s_0^-}^-)$ and a specific choice of triangulation. The final result clearly depends on this choice, whence this calculation is regularization dependent.

However note that the expectation value will diverge whenever the value of prescribed embedding at any vertex of the triangulation equals the value of embedding charge at any(not necessarily same) vertex. We will address this issue in detail below. Here we merely note that although the complete observable corresponding to dilaton is an operator on the spatial slice, but is not well defined on the entire

Hilbert space. It might seem surprising that the matter charges $(l_1, ...l_M)$ don't figure in these expressions (as they should!), but this is due to the fact that we have evaluated the expectation value in a basis-state. If Ψ was an arbitrary state (linear combination of basis-states) $\langle\Psi|y(x)\widehat{[X_p^+, X_p^-]}_{can}|\Psi\rangle_{phy}$ will depend on the matter charges as well.

4.8 New observables

In this section we *define* a set of observables in the quantum theory which we believe to be more appropriate counter-parts of the classical complete observables than the canonically quantized operators of the previous section.

The basic idea is the following. Recall that the value of $f(x)[X_p^+, X_p^-]$ on a gauge orbit $\mathcal{G}_m$ passing through point m in the constraint surface is the value of the scalar field f(x) at m' where the gauge fixed slice $X^\pm = X_p^\pm$ intersects $\mathcal{G}_m$. Roughly speaking we try to mimic this construction directly at quantum level and obtain a new class of operators well defined on $\mathcal{H}_{phy}$.

The gauge fixing condition (at a point x in the spatial slice) translates on $\mathcal{H}_{kin}$ as,

$$\hat{X}^\pm(x) f_s = X_p^\pm(x) f_s \tag{4.87}$$

As the quantum counter-part of gauge orbit in the classical theory is the orbit of charge network states in the quantum theory (represented by a state in $\mathcal{H}_{phy}$, we define $\widehat{f(x)[X_p^+, X_p^-]}$ as,

$$\begin{aligned} \widehat{f(x)[X_p^+, X_p^-; m]}\ \Psi &:= \widehat{f(x)[X_p^+, X_p^-; m]}_{can}\ \Psi \\ &\quad \text{if } X_p^+(x) \in (k_1^+, ..., k_N^+) \text{ and } X_p^-(x) \in (k_1^-, ..., k_M^-), \\ &:= 0 \text{ otherwise} \end{aligned} \tag{4.88}$$

where in the above,

$\Psi = |N, N+1, \vec{k}^+, \vec{l}^+\rangle \otimes |M, M+1, \vec{k}^-, \vec{l}^-\rangle \otimes |m_R\rangle$ is a physical basis-state.

This means the following. If the quantum gauge fixing condition (4.87) intersects an orbit of equivalence class of states, then the action of $\widehat{f(x)[X_p^+, X_p^-; m]}$ on the orbit-Ψ equals the action $\widehat{f(x)[X_p^+, X_p^-; m]}_{can}$. And those quantum orbits which do not intersect the gauge fixing slice (4.87) are in the kernel of $\widehat{f(x)[X_p^+, X_p^-; m]}$. Its action can be extended to an arbitrary state in $\mathcal{H}_{phy}$ by linearity.

We would like to emphasize that we are defining a two parameter $(X_p^+(x), X_p^-(x))$ family of observables in quantum theory which admit the same relational interpretation as $f(x)[X_p^+, X_p^-; m]$ of classical theory. Eventually one has to do semi-classical analysis to verify if these operators approximate the classical complete observables in the appropriate limit. However the main motivation behind defining these observables has been to tie the prescribed embeddings $X_p^+(x), X_p^-(x)$ to embedding charges $(k_I^{\pm})$ in some suitable manner.

Several features of this definition are worth noting.

1. **Discrete evolution**:
Given a state Ψ, let $X_p^{\pm}(x)$ "evolve" from $-\infty$ to ∞ in accordance with 4.76. Then $f(x)\widehat{[X_p^+, X_p^-]}\ \Psi\ \neq 0$ only when X_p^+ takes the discrete values ($k_1^+, ..., k_N^+$) and X_p^- takes the discrete values ($k_1^-, ..., k_N^-$). In this sense the underlying dynamics in the quantum theory is discrete. It is important to note that we do not have a unitary dynamics so far. Automorphism of the algebra of complete observables is promoted to a automorphism on the algebra of corresponding operators, but this automorphism is not generated by any unitary operator.
2. Now consider an arbitrary (possibly non-local in x!) function of two prescribed embeddings X_p^1, X_p^2, say $F(X_p^1(x), X_p^2(x'))$. This is trivially an observable, and naively one would think that it should be promoted to multiple of identity operator on $\mathcal{H}_{phy}$. However a careful look at our formulation of quantum complete observable implies that this is not quite true. The corresponding quantum observable on $\mathcal{H}_{phy}$ is,

$$(F(X_p^1\widehat{(x), X}_p^2(x'))\ \Psi)(f_s)\ =\ \Psi(F(X_p^1(x), X_p^2(x'))\mathbf{1}f_s) \tag{4.89}$$

if the 4 parameters $(X_p^{1,+}(x), X_p^{2,+}(x'), X_p^{1,-}(x), X_p^{2,-}(x'))$ lie in the set of the embedding charges of Ψ, and

$$(F(X_p^1\widehat{(x), X}_p^2(x'))\ \Psi)(f_s)\ =\ 0 \tag{4.90}$$

otherwise.
Whence consider the classical function $sin[k\cdot(X(x)\ -\ X(x'))]$ whose pull-back on prescribed slice is given by $sin[k\cdot(X_p(x)\ -\ X_p(x'))]$. The corresponding quantum observable is given by,

$$(sin[k\cdot(X_p\widehat{(x)\ -}\ X_p(x'))]\ \Psi)(f_\gamma)\ =\ \Psi(sin[k\cdot(X_p(x)\ -\ X_p(x'))]\mathbf{1}f_\gamma) \tag{4.91}$$

if $X^{\pm}(x) \in (k_1^{\pm}, ..., k_N^{\pm})$ and $X^{\pm}(x') \in (k_1^{\pm}, ..., k_N^{\pm})$ and zero otherwise.
This observation ensures that if as mentioned earlier, there exist class of semi-classical states on which the commutator algebra generated by $(a_k, a_k^{\dagger})$ mirrors the classical Poisson algebra (to leading order in $\hbar$) then on those states one can easily show that the commutator between $f(x)[X_p^+, X_p^-]$, $f(x')[X_p^+, X_p^-]$ vanishes (again to leading order in $\hbar$) as $\vec{X}_p(x)$ and $\vec{X}_p(x')$ are space-like separated.

Definition of $\pi_f(x)\widehat{[X_p^+, X_p^-]}$
As the classical observable $\pi_f(x)[X_p^+, X_p^-; m]$ in (4.71) involves $X_p^{\pm}(x)$ as well as $X_p^{\pm'}(x)$, the operator valued distribution $\pi_f(x)\widehat{[X_p^+, X_p^-]}$ can be defined as follows. As before, let Ψ be a physical state given above. Then,

$$\pi_f(x)\widehat{[X_p^+, X_p^-]}\ \Psi \ := \ \pi_f(x)\widehat{[X_p^+, X_p^-]}_{can}\ \Psi \tag{4.92}$$

if $(X_p^+(x), X_p^+(v_m)) \in (k_I^+, k_{I+1}^+), 1 \leq I \leq N$ and $(X_p^-(x), X_p^-(v_m)) \in (k_J^-, k_{J+1}^-), 1 \leq J \leq M$.
and,

$$\pi_f(x)\widehat{[X_p^+, X_p^-]}\ \Psi \ := \ 0 \tag{4.93}$$

otherwise.
As in (4.79), we have chosen $T(\gamma_0^+)$ and $T(\gamma_0^-)$ such that $\triangle_m$ is a simplex in both the triangulations and v_m, x are its initial and final vertices respectively.

4.8.1 New dilaton operator

Using the quantum observables $\widehat{f(x)[X_p^+, X_p^-]}$, $\widehat{\pi_f(x)[X_p^+, X_p^-]}$ one can define a new complete observable corresponding to the dilaton ($\widehat{y(x)[X_p^+, X_p^-]}$) in the quantum theory. This operator will obviously differ from $\widehat{y(x)[X_p^+, X_p^-]}_{can}$ and if as argued earlier $\widehat{f(x)[X_p^+, X_p^-]}$, $\widehat{\pi_f(x)[X_p^+, X_p^-]}$ are the appropriate dynamical observables of the quantum theory then information about quantum geometry will be encoded in $\widehat{y(x)[X_p^+, X_p^-]}$.
Using defining equations (4.88) , (4.92) for $\widehat{f(x)[X_p^+, X_p^-]}$ and $\widehat{\pi_f(x)[X_p^+, X_p^-]}$ we can show that the formal expression for $\widehat{y(x)[X_p^+, X_p^-]}$ remains same as that for $\widehat{y(x)[X_p^+, X_p^-]}_{can}$ as given in (4.110).

$$\left[\widehat{y(x)[X_p^+, X_p^-]}_{can'} \Psi \right] (f_{\vec{s}}^+ \otimes f_{\vec{s}}^- \otimes |m\rangle) =$$
$$\Psi\Bigg(\Bigg[\lambda^2 X_p^+(x) X_p^-(x) \;-\; X_p^-(x) \; \sum_{T_x(\gamma_0^+)} \hat{\mathbf{A}} \;-\; X_p^+(x) \; \sum_{\overline{T}_x(\gamma_0^-)} \hat{\mathbf{B}}$$
$$+ \sum_{\triangle_m \in T_x(\gamma_0^+)} X_p^+(v_m) \hat{\mathbf{B}} \;+\; \sum_{\triangle_m \in T_x(\gamma_0^-)} X_p^-(v_m) \hat{\mathbf{A}} \;+ \frac{\hat{m_R}}{\lambda} \Bigg] (f_{\gamma_0^+}^+ \otimes f_{\gamma_0^-}^-) \Bigg) \tag{4.94}$$

Where as before, Ψ is a basis-state in $\mathcal{H}_{phy}$. $\Psi = | N, N{+}1, (k_1^+, l_1^+), ..., (k_N^+, l_N^+) >_+ \otimes | M, M+1, (k_1^-, l_1^-), ..., (k_N^-, l_N^-) >_- \otimes |m>$ and $f_{s^+}^+ \otimes f_{s^-}^- \otimes |m\rangle$ is a charge-network state in $\mathcal{H}_{kin}$.

However the action of this operator on physical states will be quite different compared to $\widehat{y(x)[X_p^+, X_p^-]}_{can}$. Let us consider the action of each term in (4.94) separately.

1. As argued in the paragraph preceding (4.91) $\lambda^2 X_p^+(x) X_p^-(x)$ equals a multiple of identity operator on states for which $X_p^+(x) \in (k_1^+, ..., k_N^+)$, $X_p^-(x) \in (k_1^-, ..., k_M^-)$ and equals zero otherwise.

2(a). Firstly the operator $X_p^-(x) \sum_{T_x(\gamma_0^+)} \hat{\mathbf{A}}$ will have a non-zero action iff $X_p^-(x) \in (k_1^-, ..., k_M^-)$.
2(b). Secondly, in the summation $\sum_{\triangle_m}$ only those simplices $\triangle_m$ contribute for

which

$$(X_p^+(v_{m+1}), X_p^+(V_m)) \in [(k_{I+1}^+, k_I^+)\ ]$$

for some $I \in (1, ..., N-1)$.

$$(X_p^-(v_{m+1}), X_p^-(V_m)) \in [(k_{I+1}^-, k_I^-)\ ]$$

for some $I \in (1, ..., M-1)$.
One immediate consequence of this criterion is that, as we require $X_p^{\pm'}(x) \neq 0$, *only* a finite number of simplices will contribute to the outermost summation $\sum_{\triangle_m}$.
3. Similarly $X_p^+(x)\ \sum_{\overline{T}_x(\gamma_0^-)} \hat{\mathbf{B}}$ will have a non-zero action iff $X_p^+(x) \in (k_1^+, ..., k_N^+)$. Also, Only those simplices in the Riemann sum $\sum_{\overline{T}_x(\gamma_0^-)}$ which satisfy condition given in 2-b above.
Similar remarks apply to fourth and fifth terms.
Let us summarize.
Action of $y(x)[\widehat{X_p^+, X_p^-}]$ on $\mathcal{H}_{phy}$ can be defined by its dual action as,

$$\begin{aligned}
&\left[\ y(x)[\widehat{X_p^+, X_p^-}]^{'}\ |\Psi\ \rangle\ \right](\ f_{s^+}^+ \otimes f_{s^-}^- \otimes |m\rangle\ ) = \\
&\langle\ \Psi|\left(\ \left[\lambda^2 X_p^+(x) X_p^-(x)\ -\ X_p^-(x)\ \textstyle\sum_{T_x(\gamma_0^+)} \hat{\mathbf{A}}\ -\ X_p^+(x)\ \sum_{\overline{T}_x(\gamma_0^-)} \hat{\mathbf{B}}\right.\right. \\
&\left.\left.+\ \textstyle\sum_{\triangle_m \in T_x(\gamma_0^+)} X_p^+(v_m)\hat{\mathbf{B}}\ +\ \sum_{\triangle_m \in \overline{T}_x(\gamma_0^-)} X_p^-(v_m)\hat{\mathbf{A}}\ +\ \frac{\hat{m_R}}{\lambda}\ \right](\ f_{\gamma_0^+}^+ \otimes\ f_{\gamma_0^-}^-)\ \right)
\end{aligned} \tag{4.95}$$

1. The first 3 terms in (4.95) are non-zero iff $(X_p^+(x) \in (k_1^+, ..., k_N^+)$ and $(X_p^-(x) \in (k_1^-, ..., k_M^-)$.
2. There are various Riemann sums involved in each term involving $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$. In the summations over T_x in (4.95), only those simplices contribute for which $(X_p^\pm(v_m), X_p^\pm(v_{m+1}))$ lie in a certain finite set as explained above.

4.8.2 Understanding divergences

Now we address the issue of divergences mentioned in passing above. The expectation value of $y(x)[\widehat{X_p^+, X^-}, p]_{can}$ given in (4.114) involve principal values. These values will diverge when $X_p^{\pm}(v_m)$ equals the value of the embedding charge at some vertex of the triangulation.

Let us first try to understand how these divergences arise in the quantum theory.
We consider the action of a (canonically quantized) complete observable corresponding to scalar field momentum π_f on a charge network $f_{s^+}^+ \otimes f_{s^-}^- \in \mathcal{H}_{kin}$. from (4.79) we can show that,

$$\begin{aligned}
&\pi_f(x)\widehat{[X_p^+, X_p^-]}_{can}(f_{s^+}^+ \otimes f_{s^-}^-) = \\
&\frac{-i}{2\sqrt{\pi}}\ [\ \frac{(X_p^-(x)\ -\ X_p^-(v_n))}{|\triangle_n|}\int_{-\infty}^{0} dk \sum_{\triangle_m \in T(\gamma^+)} e^{(ik(X_p^+(x)\ -\ \frac{k_{m-1}^+ + k_m^+}{2})}(h_{\triangle_m}(Y^+)\ -\ h_{\triangle_m^{-1}}(Y^+)\ ) \\
&-\frac{(X_p^+(x)\ -\ X_p^+(v_n))}{|\triangle_n|}\int_0^{\infty} dk \sum_{\triangle_m \in T(\gamma^-)} e^{(ik(X_p^-(x)\ -\ \frac{k_{m-1}^- + k_m^-}{2})}(h_{\triangle_m}(Y^-)\ -\ h_{\triangle_m^{-1}}(Y^-)\ ) \\
&\qquad -c.c.]
\end{aligned} \tag{4.96}$$

Without doing further computation one can see that the integrals diverge when $X_p^{\pm}(x)\ -\ \frac{k_{m-1}^{\pm} + k_m^{\pm}}{2}$. These are precisely the divergences that have crept in the expectation value of the dilaton. A geometric interpretation helps us in locating these divergences in the background(flat) spacetime.
Given an orbit $([s^+], [s^-])$, we have associated to it, a pair (s_0^+, s_0^-) and a (pair of) triangulations $(T(\gamma_0^+), T(\gamma_0^-))$.(Note that these are the ad-hoc inputs in the quantum theory). Whence a state $\Psi^+ \otimes \Psi^- \in \mathcal{H}_{phy}$, defines a lattice $\mathcal{L}$ in the background spacetime. This lattice is defined by the value of the embedding charges(contained in (s_0^+, s_0^-)) at the vertices of the triangulations. More in detail, let $s_0^{\pm}\ =\ (\ \gamma_0^{\pm}, (k_1^{\pm}, ..., k_N^{\pm})\ )$. And let, $T(\gamma_0^+)\ =\ T(\gamma_0^-)\ =\ e_L \cup \gamma_0^+ \cup e_R$. (This is the choice of triangulation we have made throughout this paper). Here, as always e_L is a simplex from $-\infty$ to the left-most vertex of γ_0^+ and e_R is a simplex from the right-most vertex of γ_0^+ to ∞. The embedding charges at the vertices of $T(\gamma_0^+)$ are $(0, \frac{k_1^{\pm}}{2}, \frac{k_1^{\pm}+k_2^{\pm}}{2}, ..., \frac{k_{N-1}^{\pm}+k_N^{\pm}}{2}, \frac{k_N^{\pm}}{2}, 0)$. These charges define a lattice in the background spacetime (spanned by the null-lines $X^{\pm}\ =\ \frac{k_{I-1}^{\pm}+k_I^{\pm}}{2}$ for some I.)

Whenever the spacetime point governed by the prescribed embeddings $(X_p^+(x), X_p^-(x))$ "sits" on $\mathcal{L}$, the above operator $\pi_f(x)\widehat{[X_p^+, X_p^-]}$ diverges. Note

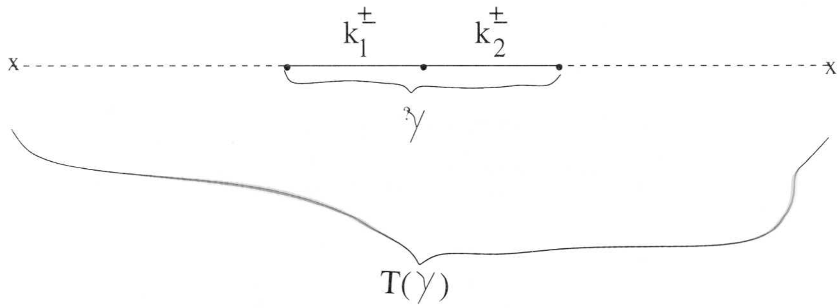
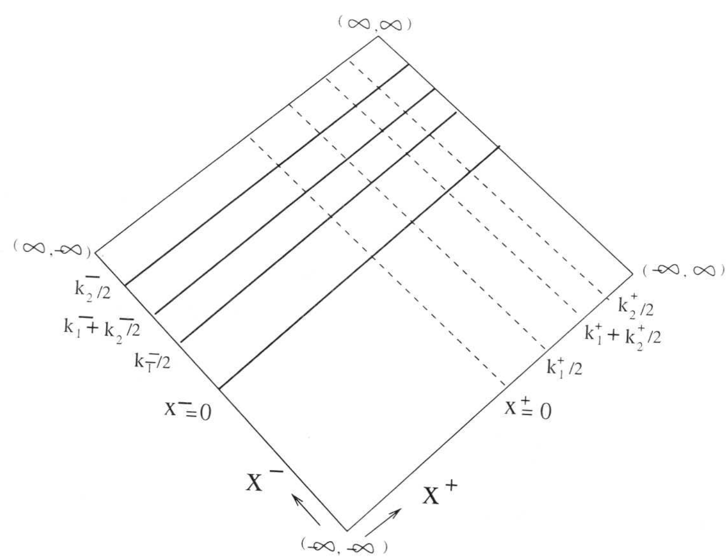


Figure 4.2: A choice of triangulation



Lattice $\mathcal{L}$ in M defined by charges at vertices of T(γ)

Figure 4.3: Understanding divergence

that the reason we are looking at $\pi_f(x)\widehat{[X_p^+, X_p^-]}$ (which is a distribution in x) and not $f(x)\widehat{[X_p^+, X_p^-]}$ is because the scalar field operators also suffer from infra-red divergence as they involve $\frac{1}{k}$ in the integrand. The divergences showing up in the dilaton operator are purely due to the UV limit of the k-integral, and so in order to only focus on the UV-divergences we have considered $\pi_f(x)$ instead of f(x).[19]

Note that the new observables defined in the previous section do not suffer from these divergences. This is because the triangulation we have chosen is such that the embedding charges at its vertex are either zero or $\frac{k_I^{\pm}+k_{I+1}^{\pm}}{2}$ for some edge e_I. Now if we assume that the embedding charges $(k_1^{\pm},...,k_N^{\pm})$ are monotonically increasing (which is a reasonable restriction on the state-space as classically $X^{\pm'} > 0$.) then $k_J \neq \frac{k_I^{\pm}+k_{I+1}^{\pm}}{2} \forall I, J$. Whence $\frac{1}{X_p^{\pm}(v_m)-\frac{k_I^{\pm}+k_{I+1}^{\pm}}{2}}$ can never diverge for $X_p^{\pm} \in (k_1^{\pm},...,k_N^{\pm})$. In fact the requirement that the new observables be well defined in the quantum theory leads to a unique choice of triangulation. If the triangulation was any finer (then the one we have chosen in this paper) then contributions from certain simplices will be of the form $\frac{1}{X_p^{\pm}(v_m)\ -\ k_I^{\pm}}$ which will diverge even for the new observables. However this resolution is far from satisfactory as the (more conventional) canonically quantized observables still diverge and more importantly we do not understand the source of this divergence.

The reason why the divergences occur can be understood by the following heuristic argument. Consider a (spacetime) scalar g(X).(For the sake of simplicity we work in one dimension. i.e. X co-ordinatizes one dimensional flat spacetime) (We assume g(X) to be in the Schwarz space so that its Fourier transform is well defined.)

$$g(X) \ = \ \int dk e^{-ikX} \int d\overline{X} e^{ik\overline{X}} g(\overline{X}) \tag{4.97}$$

Let $f_s \in \mathcal{H}_{kin}$ such that $\hat{X} f_s \ = \ k_0 f_s$. Then in the quantum theory one has,

$$g(k_0) \ = \ \int dk e^{-ikk_0} \sum_{k_e \in (k_1,...,k_N)} g(k_e) e^{ikk_e} \tag{4.98}$$

[19]The issue of how to tackle the IR divergence in the observable corresponding to the scalar field remains open and we do not discuss it in this paper.

Let us understand how we arrived at the above equation. As argued above, a state f_s will pick out a lattice(in one dimension this is a set of discrete points) $\mathcal{L}$ in the spacetime. Integrating with respect to X now means one only picks contributions from those (discrete) spacetime points X, which belong to this lattice. (i.e. those X's which belong to the set of embedding charges $(k_1, ..., k_N)$ of s). Thus the measure in the X-space now is a pure-point measure. Now if we assume that the measure in the k-space to be the ordinary Lebesgue measure, then,

$$g(k_0) = \sum_{k_e \in (k_1,...,k_N)} g(k_e)\delta(k_e - k_0) \tag{4.99}$$

which is non-sensical. (L.H.S is a distribution where as R.H.S is a number.)[20]

Thus it is obvious that the measure in the k-space cannot be the Lebesgue measure but should be such that[21],

$$\int dk e^{-ik(k_0 - k_e)} = \delta_{k_0,k_e} \tag{4.100}$$

Such a measure does exist, and is induced from the Haar measure on the Bohr compactification of the real line [2]!

$$\int_{-\infty}^{\infty} dk e^{-ik(k_0-k_e)} = lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dk e^{-ik(k_0-k_e)} \tag{4.101}$$

Somewhat surprisingly this is the measure on the quantum configuration space of a point particle when one quantizes it via polymer quantization.[22]

The upshot of this argument is that in the polymer quantized parametrized field theory, a given state defines a lattice in the background spacetime and any operator which is defined via some spacetime tensor(i.e. it is a functional of the embedding

[20] In two dimensions the integration over k splits into integration over k^+ which ranges over $[0,\infty]$ and integration over k^- which ranges over $[0,-\infty]$ whence the distribution one gets is not the Dirac-distribution but the principal value distribution.

[21] Or should atleast be such that $\int dk e^{iak}$ is a function and not a distribution. In which case one would get equality between R.H.S. and L.H.S of (4.98) up to certain error terms.

[22] In our case, the measure will be slightly different. As all the integrals are from 0 to $\pm\infty$, we use $\int_0^{\pm\infty} dk e^{-ik(k_0-k_e)} = \pm lim_{T\to\infty}\frac{1}{T}\int_0^{\pm T} dk e^{-ik(k_0-k_e)}$.

variables.) will only "see" this lattice in the quantum theory (when acting on that state). As a result of this the measure $d\vec{X}$, and hence dk are different then the Lebesgue measures associated with the continuum spacetime. This argument is heuristic but it leads us to believe that the appropriate measure to use in the k-space must be different then the measure used in classical theory. (One plausible candidate is the one defined in (4.101))

Let us see how this measure cures the divergence problem of $\pi_f(x)\widehat{[X_p^+, X_p^-]}$. Using (4.96) one gets,

$$\begin{aligned} &\pi_f(x)\widehat{[X_p^+, X_p^-]}_{can}(f_{s^+}^+ \otimes f_{s^-}^-) = \\ &\frac{-i}{2\sqrt{\pi}}\ [\ \frac{(X_p^-(x)\ -\ X_p^-(v_n))}{|\triangle_n|} \sum_{\triangle_m \in T(\gamma^+)} \delta_{X_p^+(x)\ ,\ \frac{k_{m-1}^+ + k_m^+}{2}} (h_{\triangle_m}(Y^+)\ -\ h_{\triangle_m^{-1}}(Y^+)\ ) \\ &-\frac{(X_p^+(x)\ -\ X_p^+(v_n))}{|\triangle_n|} \sum_{\triangle_m \in T(\gamma^-)} \delta_{X_p^-(x)\ ,\ \frac{k_{m-1}^- + k_m^-}{2}} (h_{\triangle_m}(Y^-)\ -\ h_{\triangle_m^{-1}}(Y^-)\ ) \\ &\qquad -c.c.]. \end{aligned} \tag{4.102}$$

Where the new measure for integration over k has been used.

One can immediately see that not only is $\pi_f(x)\widehat{[X_p^+, X_p^-]}_{can}$ a well defined operator (valued distribution) but also, only those simplices $\triangle_m$ contribute to the sum over triangulation for which $X_p^{\pm}(x) = \frac{k_{m-1}^{\pm} + k_m^{\pm}}{2}$. This is qualitatively similar to the definition of quantum complete observables defined in the previous section. We believe that this result in itself makes the new measure interesting and the idea(of changing the measure due to the discrete nature of embeddings) worth investigating further.

Now let us see what happens when we apply the above idea to the dilaton operator. The action of $y(x)\widehat{[X_p^+, X_p^-]}_{can}$ on a physical state Ψ is given by 4.110. For the convenience of the reader we rewrite the equation here.

$$\left[y(x)[\widehat{X^+_p, X^-_p}]_{can'} \Psi \right](f^+_{s^+} \otimes f^-_{s^-} \otimes |m\rangle) =$$
$$\Psi\Bigg(\Bigg[\lambda^2 X^+_p(x) X^-_p(x) \; - \; X^-_p(x) \; \textstyle\sum_{T_x(\gamma^-_0)} \hat{\mathbf{B}} \; - \; X^+_p(x) \; \sum_{\overline{T}_x(\gamma^+_0)} \hat{\mathbf{A}}$$

$$+ \textstyle\sum_{\triangle_m \in T_x(\gamma^-_0)} X^+_p(v_m) \hat{\mathbf{A}} \; + \; \sum_{\triangle_m \in T_x(\gamma^+_0)} X^-_p(v_m) \hat{\mathbf{B}} \; + \frac{\hat{m_R}}{\lambda} \Bigg](f^+_{\gamma^+_0} \otimes \; f^-_{\gamma^-_0}) \Bigg) \tag{4.103}$$

where
$\Psi \;=\; |\; N, \; N+1, \; (k^+_1, l^+_1), ..., (k^+_N, l^+_N) >_+ \otimes|\; M, \; M+1, \; (k^-_1, l^-_1), ..., (k^-_N, l^-_N) >_-$ $\otimes|m>$ is a physical state and $f^+_{s^+} \otimes f^-_{s^-} \otimes |m\rangle$ is an arbitrary charge network in $\mathcal{H}_{kin}$.

The operators $\hat{\mathbf{B}}$, $\hat{\mathbf{A}}$ which earlier were principal valued distributions on the embedding sector are now Kronecker delta-functions. A straightforward computation reveals,

$$\begin{aligned}
\hat{\mathbf{B}} \; = \; & \tfrac{1}{16\pi} \textstyle\sum_{\triangle_m \in T_x(\gamma^-_0)} \left(X^-_p(v_{m+1}) \; - \; X^-_p(v_m) \right) \\
& \Big[\textstyle\sum_{\triangle_n} \delta_{X^-_p(v_m), \frac{k^-_{n-1}+k^-_n}{2}} [h_{\triangle_n}(Y^-) \; - \; h_{\triangle^{-1}_n}(Y^-)] \times \\
& \textstyle\sum_{\triangle_l} \delta_{X^-_p(v_m), \frac{k^-_{l-1}+k^-_l}{2}} [h_{\triangle_l}(Y^-) \; - \; h_{\triangle^{-1}_l}(Y^-)] \\
& - \textstyle\sum_{\triangle_n} \delta_{X^-_p(v_m), \frac{k^-_{n-1}+k^-_n}{2}} [h_{\triangle_n}(Y^-) \; - \; h_{\triangle^{-1}_n}(Y^-)] \times \\
& \textstyle\sum_{\triangle_l} \delta_{X^-_p(v_m), \frac{k^-_{l-1}+k^-_l}{2}} [h_{\triangle_l}(Y^-) \; - \; h_{\triangle^{-1}_l}(Y^-)] \\
& - \textstyle\sum_{\triangle_n} \delta_{X^-_p(v_m), \frac{k^-_{n-1}+k^-_n}{2}} [h_{\triangle_n}(Y^-) \; - \; h_{\triangle^{-1}_n}(Y^-)] \times \\
& \textstyle\sum_{\triangle_l} \delta_{X^-_p(v_m), \frac{k^-_{l-1}+k^-_l}{2}} [h_{\triangle_l}(Y^-) \; - \; h_{\triangle^{-1}_l}(Y^-)] \\
& + \textstyle\sum_{\triangle_n} \delta_{X^-_p(v_m), \frac{k^-_{n-1}+k^-_n}{2}} [h_{\triangle_n}(Y^-) \; - \; h_{\triangle^{-1}_n}(Y^-)] \times \\
& \textstyle\sum_{\triangle_l} \delta_{X^-_p(v_m), \frac{k^-_{l-1}+k^-_l}{2}} [h_{\triangle_l}(Y^-) \; - \; h_{\triangle^{-1}_l}(Y^-)] \; \Big]
\end{aligned} \tag{4.104}$$

which vanishes! One can similarly show that $\hat{\mathbf{A}}$ is identically zero as well. Thus the simplest choice of k-measure which we were led to, due the discrete nature of embeddings (and hence the background spacetime), and which removed the divergence seems to completely remove the matter degrees of freedom from the quantum theory and reduces the theory to a pure dilaton gravity model. This is clearly incorrect. We thus conclude that the issue of divergence is far from being resolved. However we do believe that the resolution should come via a new measure in the Fourier space. We plan to investigate this further in near future.

4.9 Discussion

The primary aim of this chapter is to obtain a quantum theory of dilaton gravity by combining the ideas of parametrized field theory and polymer (loop) quantization. We started with a parametrized field theory which is canonically equivalent to the KRV action. By choosing appropriate quantum algebras for the embedding and matter sectors, we obtained a Hilbert space which carries a unitary (and anomaly-free) representation of the space-time diffeomorphism group. Using the so called group averaging method, we were able to get rid of the quantum gauge degrees of freedom and obtain the physical spectrum of the theory in a rather straightforward manner. The parametrized field theory framework gave us a complete set of Dirac observables which we could promote to well defined operators on $\mathcal{H}_{phy}$. This required rather ad hoc choices of triangulations and the final operators are dependent on choice of triangulation. This ad-hocness permeates all the consequent constructions and calculations performed later in the chapter. However this is the construction which is traditionally followed in loop quantized field theories (atleast at the kinematical level). We have used it here and shown how even at finite triangulation one can promote the kinematical operators to physical observables. We encourage the reader to contrast quantum observables defined in this chapter to those defined in chapter 3 and the differing effects they have on underlying graphs and matter charges.

Unlike the Fock space which by definition is an irreducible representation of the Poisson algebra of mode oscillators $(a_k\ ,\ a_k^*)$, $\mathcal{H}_{phy}$ carries a representation of a deformed algebra. It is a faithful deformation of the classical algebra in the sense that all the corrections are $O(\hbar)$. It is an interesting open question to hunt for the full quantum algebra and try to find physical interpretation of its elements which do not have a well defined classical limit (the commutator $[\hat{a}_k, \hat{a}_l]$ defines one such element).

Time evolution could be defined in polymer quantized CGHS model by using the complete (dynamical) observables and seeing how they change under symplectomorphisms which arise from certain diffeomorphisms of the background spacetime. We gave two inequivalent definitions of complete observables in quantum theory and the corresponding Heisenberg dynamics. The first definition was through the canonical quantization of classical observables. However as argued in the chapter, we believe that canonically quantizing the complete observables does not preserve its physical interpretation. This led us to the second definition which captures the relational nature of classical complete observables in a more transparent manner than the canonically quantized counterparts. Adapting the second definition to define dynamical observables in the quantum theory implies discrete temporal evolution. Finally using either definition of quantum complete observables, we defined physical dilaton operator which is well defined without smearing on $\mathcal{H}_{phy}$. For certain values of the prescribed embeddings the canonically quantized complete observables were ill defined. We argued that the source of the divergence was in the incorrect measure in the Fourier space. A possible candidate for the correct measure is induced from the Haar measure on Bohr compactification of the real line. Although this measure did remove the divergence from the observable corresponding to the scalar field-momentum, it reduced the dilaton operator to that of pure gravity without matter. It is however tempting to speculate that a suitable variant of this measure will cure the divergence problem without killing all the physical (matter) degrees of freedom.

We also calculated its expectation value on an arbitrary basis-state in $\mathcal{H}_{phy}$ (for a specific choice of triangulation). This, we believe, gives us a framework to address the semi-classical and non-perturbative issues arising in the CGHS model.

Appendix

A. Definition of the physical dilaton operator

In this appendix we canonically quantize the complete observable corresponding to the dilaton. We start with an arbitrary basis state Ψ and derive the (dual) action of $y(x)\widehat{[X_p^+, X_p^-]}_{can}$ on it. The operator can be extended to an arbitrary physical state by linearity.

Let $\Psi = \mid N,\ N{+}1,\ (k_1^+, l_1^+), ..., (k_N^+, l_N^+) >_+ \otimes \mid M,\ M{+}1,\ (k_1^-, l_1^-), ..., (k_N^-, l_N^-) >_- \otimes |m>$. Without loss of generality, let us assume that N > M. Also let $f_{s^+}^+ \otimes f_{s^-}^- \otimes |m>$ be any state in $\mathcal{H}_{kin}$. In the orbit of $(s^+,\ s^-)$ fix a pair $(s_0^+,\ s_0^-)$, with the corresponding graphs $(\gamma_0^+,\ \gamma_0^-)$. Using our scheme of how to define operator corresponding to complete observable in quantum theory $y(x)\widehat{[X_p^+, X_p^-]}'$ is as follows.[23]

$$
\begin{aligned}
&[y(x)\widehat{[X_p^+, X_p^-]}'_{can}\Psi\,](f_{s^+}^+ \otimes\ f_{s^-}^- \otimes |m>) := \\
&\Bigg[\ [\lambda^2 X_p^+(x) X_p^-(x)\mathbf{1}\ -\ \tfrac{X_p^-(x)}{4}\int_\infty^x d\overline{x}\ \tfrac{Y_-(\overline{x})\widehat{[X_p^+, X_p^-]}^2}{X_P^{-'}(\overline{x})}\ -\ \tfrac{X_p^+(x)}{4}\int_{-\infty}^x d\overline{x}\ \tfrac{Y_+(\overline{x})\widehat{[X_p^+, X_p^-]}^2}{X_P^{+'}(\overline{x})} \\
&+\ \int_\infty^x d\acute{\overline{x}}\ \tfrac{X_p^-(\overline{x})}{X_p^{-'}(\overline{x})}\ Y_-(\overline{x})\widehat{[X_p^+, X_p^-]}^2\ +\ \int_\infty^x d\overline{x}\ \tfrac{X_p^+(\overline{x})}{X_{p'}^+(\overline{x})}\ Y_+(\overline{x})\widehat{[X_p^+, X_p^-]}^2\ +\ \tfrac{\hat{m_R}}{\lambda}\ ]\Psi\Bigg] \\
&\qquad\qquad\qquad\qquad\qquad\qquad (f_{s_o^+}^+ \otimes\ f_{s_0^-}^-\ \otimes |m>)
\end{aligned} \tag{4.105}
$$

Where the first term is a multiple of identity operator. Note that as x is a fixed point in the spatial slice, $X_p^{\pm}(x)$ are just parameters.

Remaining terms can be calculated using expressions for $f(x)[X_p^+, X_p^-]$, $\pi_f(x)[X_p^+, X_p^-]$ on $\mathcal{H}_{kin}$.

[23] Here prime over $y(x)[X_p^+, X_p^-]$ denotes the dual action.

Term 2

Second term in $y(x)[\widehat{X_p^+, X_p^-}]'$ is given by,

$$
\begin{aligned}
&\Psi(\tfrac{X_p^-(x)}{4} \int_\infty^x d\overline{x}\ \widehat{Y_-(\overline{x})^\dagger [X_p^+, X_p^-]^2}\ f_{s^+}^+ \otimes\ f_{s^-}^-)\\
&= \Psi \tfrac{1}{16\pi} \textstyle\sum_{\triangle_m \in T_x(\gamma_0^-)} (X_p^-(v_{m+1})\ -\ X_p^-(v_m))\\
&\Bigg[\int_0^\infty dk e^{ikX_p^-(v_m)} \textstyle\sum_{\triangle_n} e^{-ik\hat{X}^-(v_n)}[h_{\triangle_n}(Y^-)\ -\ h_{\triangle_n^{-1}}(Y^-)]\ \times\\
&\int_0^\infty d\overline{k} e^{i\overline{k}X_p^-(v_m)} \textstyle\sum_{\triangle_l} e^{-i\overline{k}\hat{X}^-(v_l)}[h_{\triangle_l}(Y^-)\ -\ h_{\triangle_l^{-1}}(Y^-)]\\
&-\int_0^\infty dk e^{-ikX_p^-(v_m)} \textstyle\sum_{\triangle_n} e^{ik\hat{X}^-(v_n)}[h_{\triangle_n}(Y^-)\ -\ h_{\triangle_n^{-1}}(Y^-)]\ \times\\
&\int_0^\infty d\overline{k} e^{i\overline{k}X_p^-(v_m)} \textstyle\sum_{\triangle_l} e^{-i\overline{k}\hat{X}^-(v_l)}[h_{\triangle_l}(Y^-)\ -\ h_{\triangle_l^{-1}}(Y^-)]\\
&-\int_0^\infty dk e^{ikX_p^-(v_m)}\ \textstyle\sum_{\triangle_n} e^{-ik\hat{X}^-(v_n)}[h_{\triangle_n}(Y^-)\ -\ h_{\triangle_n^{-1}}(Y^-)]\times\\
&\int_0^\infty dk e^{-ikX_p^-(v_m)}\ \textstyle\sum_{\triangle_n} e^{-ik\hat{X}^-(v_n)}[h_{\triangle_n}(Y^-)\ -\ h_{\triangle_n^{-1}}(Y^-)]\\
&+\int_0^\infty dk e^{-ikX_p^-(v_m)}\ \textstyle\sum_{\triangle_n} e^{ik\hat{X}^-(v_n)}[h_{\triangle_n}(Y^-)\ -\ h_{\triangle_n^{-1}}(Y^-)]\times\\
&\int_0^\infty d\overline{k} e^{-i\overline{k}X_p^-(v_m)}\ \textstyle\sum_{\triangle_l} e^{i\overline{k}\hat{X}^-(v_l)}[h_{\triangle_l}(Y^-)\ -\ h_{\triangle_l^{-1}}(Y^-)] \Bigg]\ (f_{s_0^+}^+ \otimes\ f_{s_0^-}^-)
\end{aligned}
\tag{4.106}
$$

where $T(\gamma_0^-)$ is a fixed triangulation adapted to γ_0^-. Here adapted means in the image of the graph all the vertices of the graph are vertices of triangulation and "outside" the graph $T(\gamma_0^-)$ is arbitrary(However we will make a more specific choice of triangulation when calculating expectation value. It is the same choice that we made when evaluating the commutators between Dirac observables in [34].) As it should be clear from the classical expression for $y(x)[X_p^+, X_p^-]$, $T_x(\gamma_0^-)$ is the triangulation of the spatial manifold from x to ∞.

In the above expression we will denote everything inside [..] as $\hat{\mathbf{B}}$.

Term 3

Here we denote the triangulation adapted to γ_0^+ $T(\gamma_0^+)$ and the sub-complex ranging from $-\infty$ to x as $\overline{T}_x(\gamma_0^+)$.

$$
\begin{aligned}
&[\ \tfrac{X_p^+(x)}{4}\int_{-\infty}^{x} d\overline{x}\ \tfrac{\widehat{Y_+(\overline{x})[X_p^+,X_p^-]^2}}{X_p^{+'}}(\overline{x})'\Psi\ ](f_{s^+}^+\otimes\ f_{s^-}^-\ )\\
&=\ \tfrac{1}{16\pi}\Psi\ \textstyle\sum_{\triangle_m\in\overline{T}_x(\gamma_0^-)}\ (X_p^+(v_{m+1})\ -\ X_p^+(v_m))\\
&\quad\Big[\int_{-\infty}^{0} dk e^{ikX_p^+(v_m)}\textstyle\sum_{\triangle_n\in T(\gamma_0^+)} e^{-ik\hat{X}^+(v_n)}[h_{\triangle_n}(Y^+)\ -\ h_{\triangle_n^{-1}}(Y^+)]\ \times\\
&\quad\int_{-\infty}^{0} d\overline{k} e^{i\overline{k}X_p^+(v_m)}\textstyle\sum_{\triangle_l\in T(\gamma_0^+)} e^{-i\overline{k}\hat{X}^+(v_l)}[h_{\triangle_l}(Y^+)\ -\ h_{\triangle_l^{-1}}(Y^+)]\\
&\quad-\int_{-\infty}^{0} dk e^{ikX_p^+(v_m)}\textstyle\sum_{\triangle_n\in T(\gamma_0^+)} e^{-ik\hat{X}^+(v_n)}[h_{\triangle_n}(Y^+)\ -\ h_{\triangle_n^{-1}}(Y^+)]\ \times\\
&\quad\int_{-\infty}^{0} d\overline{k} e^{i\overline{k}X_p^+(v_m)}\textstyle\sum_{\triangle_l} e^{-i\overline{k}\hat{X}^+(v_l)}[h_{\triangle_l}(Y^+)\ -\ h_{\triangle_l^{-1}}(Y^+)]\\
&\quad-\int_{-\infty}^{0} dk e^{-ikX_p^+(v_m)}\ \textstyle\sum_{\triangle_n} e^{ik\hat{X}^+(v_n)}[h_{\triangle_n}(Y^+)\ -\ h_{\triangle_n^{-1}}(Y^+)]\times\\
&\quad\int_{-\infty}^{0} d\overline{k} e^{i\overline{k}X_p^+(v_m)}\ \textstyle\sum_{\triangle_l} e^{-i\overline{k}\hat{X}^+(v_l)}[h_{\triangle_l}(Y^+)\ -\ h_{\triangle_l^{-1}}(Y^+)]\\
&\quad+\int_{-\infty}^{0} dk e^{-ikX_p^+(v_m)}\ \textstyle\sum_{\triangle_n} e^{-ik\hat{X}^+(v_n)}[h_{\triangle_n}(Y^+)\ -\ h_{\triangle_n^{-1}}(Y^+)]\\
&\quad\int_{-\infty}^{0} d\overline{k} e^{-i\overline{k}X_p^+(v_m)}\ \textstyle\sum_{\triangle_l} e^{-i\overline{k}\hat{X}^+(v_l)}[h_{\triangle_l}(Y^+)\ -\ h_{\triangle_l^{-1}}(Y^+)]\Big] f_{s_0^+}^+\otimes\ f_{s_0^-}^-
\end{aligned}
\tag{4.107}
$$

We denote everything inside $\sum_{\overline{T}_x(\gamma_0^+)}$ $as\hat{\mathbf{A}}$.

Term 4

This term can be directly derived from term-2.

$$\begin{aligned}&\Psi(\tfrac{1}{4}\int_{\infty}^{x} d\overline{x}\; X_p^-(\overline{x})\frac{Y_-(\overline{x})^{\dagger}\widehat{[X_p^+,X_p^-]}^2}{X_p^{-'}(\overline{x})}\;(f_{s^+}^+ \otimes\; f_{s^-}^-)\;)\\ &= \;\Psi(\;\textstyle\sum_{\triangle_m\in T_x(\gamma_0^-)} X_p^-(v_m)\hat{\mathbf{B}}(f_{s^+}^+ \otimes\; f_{s^-}^-)\;)\end{aligned} \tag{4.108}$$

Term 5

Finally using term-3 we see that,

$$\begin{aligned}&\Psi(\tfrac{1}{4}\int_{\infty}^{x} d\overline{x}\; X_p^+(\overline{x})\frac{Y_+(\overline{x})^{\dagger}\widehat{[X_p^+,X_p^-]}^2}{X_p^{+'}(\overline{x})}\;(f_{s^+}^+ \otimes\; f_{s^-}^-)\;)\\ &= \;\Psi(\;\textstyle\sum_{\triangle_m\in T_x(\gamma_0^+)} X_p^+(v_m)\hat{\mathbf{A}}(f_{s^+}^+ \otimes\; f_{s^-}^-)\;)\end{aligned} \tag{4.109}$$

Whence the final expression for the dilaton operator $y(x)\widehat{[X_p^+, X_p^-]}$ at a given point x is,

$$\begin{aligned}&\left[y(x)\widehat{[X_p^+, X_p^-]'}_{can}\Psi\right](f_{s^+}^+ \otimes f_{s^-}^- \otimes |m\rangle) =\\ &\Psi\Bigg(\;\Bigg[\lambda^2 X_p^+(x)X_p^-(x) \;-\; X_p^-(x)\;\textstyle\sum_{T_x(\gamma_0^-)}\hat{\mathbf{B}} \;-\; X_p^+(x)\sum_{\overline{T}_x(\gamma_0^+)}\hat{\mathbf{A}}\\ &+\;\textstyle\sum_{\triangle_m\in T_x(\gamma_0^-)} X_p^+(v_m)\hat{\mathbf{A}} \;+\; \sum_{\triangle_m\in T_x(\gamma_0^+)} X_p^-(v_m)\hat{\mathbf{B}} \;+\frac{\hat{m_R}}{\lambda}\;\Bigg](\;f_{\gamma_0^+}^+ \otimes\; f_{\gamma_0^-}^-)\Bigg)\end{aligned} \tag{4.110}$$

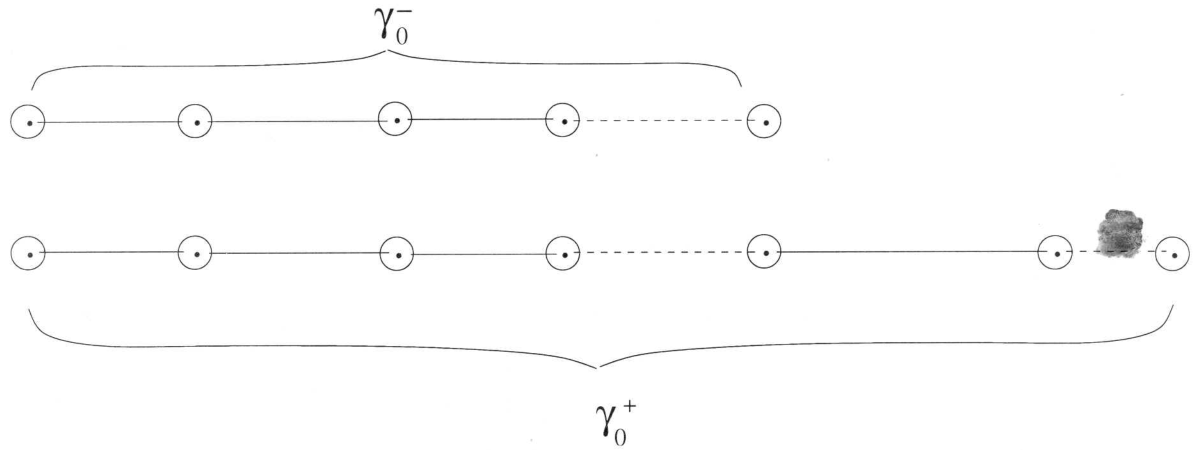


Figure 4.4: Quantum gauge fixing

B. Expectation value of the physical dilaton operator

Let $f^+_{s^+} \otimes\ f^-_{s^-} \otimes |m>$ be such that $\eta(\ f^+_{s^+} \otimes\ f^-_{s^-}\ \otimes |m>\ )\ =\ \Psi$.
Whence,

$$\langle\ \Psi|\ y(x)\widehat{[X_p^+, X_p^-]'}_{can}\ |\Psi\ \rangle\ =\ \Psi(\ y(x)\widehat{[X_p^+, X_p^-]}^{\dagger}\ f^+_{s_0^+} \otimes\ f^-_{s_0^-}\ ). \tag{4.111}$$

We make the following choice for triangulation.
Given an orbit of diffeomorphism-equivalence class of charge-networks $(s^+\ ,\ s^-)$ we choose $(s_0^-\ ,\ s_0^-)$ such that $\mathrm{r}(\gamma_0^-) \subset \mathrm{r}(\gamma_0^+)$ and over the range $\mathrm{r}(\gamma_0^-)$, the subgraph of γ_0^+ coincides with γ_0^-. (see figure below) We choose the triangulations adapted to $\gamma_0^+\ ,\ \gamma_0^-$ as $T(\gamma_0^+)\ =\ T(\gamma_0^-)\ =\ \gamma_0^+\ \cup e_L\ \cup e_R$ where e_L and e_R are arbitrary 1-simplices from $-\infty$ to initial(left-most) vertex of γ_0^+ and from final(right-most) vertex of γ_0^+ to ∞ respectively.
Now let us see how each term simplifies.

Term 2

It is easy to show that with the above choice of $(\gamma_0^+\ ,\ \gamma_0^-)\ ,\ (T(\gamma_0^+)\ ,\ T(\gamma_0^-))$ term-2 can be written as,

$$\langle\ \Psi|\ \frac{X_p^-(x)}{4}\ \int_\infty^x d\overline{x}\frac{Y_-(\overline{x})[\widehat{X_p^+,X_p^-]^{2'}}}{X_p^{-'}(\overline{x})}|\Psi\ \rangle = \langle\ \Psi\ [\frac{X_p^-(x)}{4}\ \int_\infty^x d\overline{x}\frac{Y_-(\overline{x})[\widehat{X_p^+,X_p^-]^{2}}^\dagger}{X_p^{-'}(\overline{x})}]\ (f_{s_0^+}^+ \otimes f_{s_0^-}^-)$$

$$= \langle\ \Psi[\frac{X_p^-(x)}{16\pi}\sum_{\triangle_m\in T_x}\Bigg[\ (X_p^-(v_m)\ -\ X_p^+(v_m))$$

$$\int_0^\infty dkd\overline{k}\Bigg(e^{i(k+\overline{k})X_p^-(v_m)}(-2\sum_{n=0}^M e^{-\frac{i}{2}(k+\overline{k})(k_n^-+k_{n+1}^-)} - 4)$$

$$-e^{-i(k-\overline{k})X_p^-(v_m)}(\ -2\sum_{n=0}^M e^{-\frac{i}{2}(k-\overline{k})(k_n^-+k_{n+1}^-)} - 4)$$

$$-e^{i(k-\overline{k})X_p^-(v_m)}(\ -2\sum_{n=0}^M e^{\frac{i}{2}(k-\overline{k})(k_n^-+k_{n+1}^-)} - 4)$$

$$+e^{-i(k+\overline{k})X_p^-(v_m)}(\ -2\sum_{n=0}^M e^{\frac{i}{2}(k+\overline{k})(k_n^-+k_{n+1}^-)} - 4)\Bigg)\Bigg]$$

$$(f_{s_0^+}^+ \otimes f_{s_0^-}^-) \tag{4.112}$$

where we have set $k_0^- = k_{M+1}^- = 0$.

Similarly **Term-3** can be written as,

$$\langle\ \Psi|\ \frac{X_p^+(x)}{4}\int_{-\infty}^{x}d\overline{x}\frac{Y^+(\overline{x})\widehat{[X_p^+,X_p^-]^{2'}}}{X_p^{+'}(\overline{x})}|\Psi\ \rangle = \langle\ \Psi\ [\frac{X_p^+(x)}{4}\int_{-\infty}^{x}d\overline{x}\frac{Y^+(\overline{x})\widehat{[X_p^+,X_p^-]^{2}}^{\dagger}}{X_p^{+'}(\overline{x})}]\ (f^+_{s_0^+}\otimes f^-_{s_0^-})$$
$$=\langle\ \Psi[\frac{X_p^+(x)}{16\pi}\sum_{\triangle_m\in\overline{T}_x}\Bigg[(X_p^+(v_m)-X_p^+(v_m)$$

$$\int_{-\infty}^{0}dkd\overline{k}\Bigg(e^{i(k+\overline{k})X_p^+(v_m)}(-2\sum_{n=0}^{N}e^{-\frac{i}{2}(k+\overline{k})(k_n^++k_{n+1}^+)}-4)$$

$$-e^{-i(k-\overline{k})X_p^+(v_m)}(-2\sum_{n=0}^{N}e^{-\frac{i}{2}(k-\overline{k})(k_n^++k_{n+1}^+)}-4)$$

$$-e^{i(k-\overline{k})X_p^+(v_m)}(-2\sum_{n=0}^{N}e^{\frac{i}{2}(k-\overline{k})(k_n^++k_{n+1}^+)}-4)$$

$$+e^{-i(k+\overline{k})X_p^+(v_m)}(-2\sum_{n=0}^{N}e^{\frac{i}{2}(k+\overline{k})(k_n^++k_{n+1}^+)}-4)\Bigg)\Bigg]$$
$$(f^+_{s_0^+}\otimes f^-_{s_0^-})$$

(4.113)

where we have set $k_0^+ = k_{N+1}^+ = 0$.

Rest of the terms can be written in a similar fashion and we can write the final expression for $<y(x)\widehat{[X_p^+,X_p^-]}>$ as follows,

$$\langle\Psi|y(x)\widehat{[X_p^+,X_p^-]}_{can}|\Psi\rangle_{phy}\ =$$
$$\Psi\Bigg[\ \lambda^2X_p^+(x)X_p^-(x)\ +\ \frac{X_p^-(x)}{16\pi}\sum_{\triangle_m\in T_x}\mathcal{B}_m\ -\ \frac{X_p^+(x)}{16\pi}\sum_{\triangle_m\in T_x}\mathcal{A}_m$$
$$-\frac{1}{16\pi}\sum_{\triangle_m\in T_x}X_p^-(v_m)\mathcal{B}_m-\ \frac{1}{16\pi}\sum_{\overline{T}_x}X_p^+(v_m)\mathcal{A}_m\ +\ \frac{\hat{m}_R}{\lambda}\ \Bigg](f^+_{s_0^+}\otimes f^-_{s_0^-}\otimes|m>)$$

(4.114)

Recall that T_x is subcomplex from -∞ to x and $\overline{T}_x$ is the subcomplex from x to ∞.

$\mathcal{B}_m$ and $\mathcal{A}_m$ are respectively given by,

$$\begin{aligned}\mathcal{B}_m &= [X_p^-(v_m) - X_p^-(v_{m-1})] \\ & \int_0^\infty dk d\overline{k}\Big\{ e^{i(k+\overline{k})X_p^-(v_m)}\big(-2\sum_{n=0}^M e^{-\frac{i}{2}(k+\overline{k})(k_n^- + k_{n+1}^-)} - 4 \big) + c.c. \\ & + e^{-i(k-\overline{k})X_p^-(v_m)}\big(-2\sum_{n=0}^M e^{-\frac{i}{2}(k-\overline{k})(k_n^- + k_{n+1}^-)} - 4 \big) + c.c.\Big\}\end{aligned} \tag{4.115}$$

$$\begin{aligned}\mathcal{A}_m &= [X_p^+(v_m) - X_p^+(v_{m-1})] \\ & \int_{-\infty}^0 dk d\overline{k}\Big\{ e^{i(k+\overline{k})X_p^+(v_m)}\big(-2\sum_{n=0}^N e^{-\frac{i}{2}(k+\overline{k})(k_n^+ + k_{n+1}^+)} - 4 \big) + c.c. \\ & + e^{-i(k-\overline{k})X_p^+(v_m)}\big(-2\sum_{n=0}^N e^{-\frac{i}{2}(k-\overline{k})(k_n^+ + k_{n+1}^+)} - 4 \big) + c.c.\Big\}\end{aligned} \tag{4.116}$$

One can further simplify this expression as follows. Using,

$$\int_{-\infty}^0 e^{ikX_p^+(v_m)} := \int_{-\infty}^0 e^{ik[X_p^+(v_m) - i\epsilon]} = \frac{1}{X_p^+(v_m) - i\epsilon} = P(\frac{1}{X_p^+(v_m)}) - i\epsilon\delta(X_p^+(v_m)) \tag{4.117}$$

one can show that $\mathcal{B}_m$, $\mathcal{A}_m$ simplify to,

$$\begin{aligned}\mathcal{B}_m &= [X_p^-(v_m) - X_p^-(v_{m-1})] \\ & \Big\{ -4\sum_{n=0}^{M+1} P(\frac{1}{X_p^-(v_m) - \frac{1}{2}(k_n^- + k_{n+1}^-)})^2 - 8\, P(\frac{1}{X_p^-(v_m)})^2 \Big\}\end{aligned} \tag{4.118}$$

$$\begin{aligned}\mathcal{A}_m \;&=\; [X_p^+(v_m) \;-\; X_p^+(v_{m-1})] \\ &\left\{-4\sum_{n=0}^{M+1} P(\frac{1}{X_p^+(v_m) \,-\, \frac{1}{2}(k_n^+ + k_{n+1}^+)})^2 \;-\; 8\; P(\frac{1}{X_p^+(v_m)})^2\right\}\end{aligned} \tag{4.119}$$

5

Conclusions and open issues

In this thesis we presented a case for two dimensional Parametrized field theories being perfect toy models for studying various aspects of loop quantization. Let us very briefly summarise the main results and insights obtained in this work.

- To quantize two dimensional PFTs, we constructed the representation of the analogs of the Holonomy-flux algebra of LQG which moreover admit a unitary action of the (canonical) Lie group generated by constraints.
- Even though the classical PFTs are defined on a background spacetime, In the quantum theory this continuum was "replaced" by a discrete structure consisting of countable number of points. This, we believe provides a glimpse into how a quantum spacetime "constructed" out of a putative physical state of LQG might be discrete.
- In chapter 3, we showed that although the (countable) set of Dirac observables cannot be quantized in a regularization independent manner, an (uncountable) set of certain functionals of these observables could be represented without any ambiguities on the Polymer Hilbert space.
- Out of the uncountable number of Dirac observables at most of a countable number would admit a classical limit.
- As shown in chapter 4, the problem of time can be explicitly solved in two dimensional PFTs using beautiful ideas of Rovelli, Hajicek, Dittrich. However

naive quantization of these observables lead to many technical and conceptual problems in the quantum theory and we suggested a new definition of Relational observables in quantum theory which however would reduce to the classical definitions in the limit of a weave state.

Ironically and interestingly more questions (even in the context of these simple models) have been raised by our work than those that have been answered.

- The matter charges (denoted by l_e when associated to an edge e) have no physical interpretation so far.
- For PFT on $\mathbf{R}^2$, we do not know how to impose the classical asymptotic conditions for the embedding data in quantum theory. Generically in LQG, one way to define Hilbert spaces on spatially non-compact slices is via Infinite tensor product extension. However it is not clear to us, how if at all even this construction would ensure that the embedding eigenstates which lie in the ITP would obey the non-trivial asymptotic conditions.
- Apart from certain no-go results obtained in chapter 3, the issues related to semi-classical limit of polymer quantized PFTs remain completely open. More specifically, we havent constructed any semi-classical state with respect to either the triangulated observables of chapter 4, or the exponentiated observables of chapter 3. It is also completely unclear how a continuum background spacetime can emerge from the discrete structure associated to any state in quantum theory. Note that whereas the classical embedding fields are smooth, the quantum embedding data are piecewise constant. Whence it is not clear, how if at all the smooth foliation of classical theory can be recovered from a discrete foliation.
- How does one polymer quantize four dimensional PFTs.

Temporarily forgetting the open issues mentioned above, what implication does this work have on quantization of four dimensional canonical gravity? As it is well known pure canonical gravity cannot be cast into parametrized form [57]. However constraints of gravity coupled to an incoherent dust [12] can certainly be written in a form of a Parametrized field theory where the embedding fields describe an embedding of a three dimensional manifold into what is known as a dust spacetime. Although the form of the constraints is rather different (most notably, the Hamiltonian constraint doesnot depend on the embedding variables at all), it is tempting to hope that some of the qualitative features of the two dimensional polymer quantized PFTs might carry over into their higher dimensional avatars. We end on this optimistic note.

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