Some explicit minimal graded free resolutions

By

Aaloka Kanhere

The Institute of Mathematical Sciences, Chennai.

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DECLARATION

I, hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and the work has not been submitted earlier as a whole or in part for a degree/diploma at this or any other Institution or University.

Aaloka Kanhere

To my parents and all my friends in IMSc and outside.

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Abstract

This thesis has three parts. In the first part we take an irreducible curve $\mathcal{C}$ in $\mathbb{P}^2$. Then we use the Veronese map,(σ) to map it to $\mathbb{P}^5$ and compute the resolution of $\sigma(\mathcal{C})$.In the second part we look at reduced intersection of two distinct curves $\mathcal{C}$ and $\mathcal{C}'$ in $\mathbb{P}^2$. And find the resolution of $\sigma(\mathcal{C} \cap \mathcal{C}')$. In the third part we compute explicit Differential graded algebra for one of the resolutions computed ealier.

Part 1

Let $\mathcal{C}$ be a smooth irreducible homogeneous curve in $\mathbb{P}^2$. Then we know that $\mathcal{C}$ is given by zeros of an irreducible homogeneous polynomial in 3-variables, i.e., $\mathcal{C} = Z(f(x_0, x_1, x_2))$, $f \in K[x_0, x_1, x_2]$ is an irreducible homogeneous polynomial.

Consider the embedding on $\mathbb{P}^2$ in $\mathbb{P}^5$ via the *Veronese embedding* σ, where $\sigma(x, y, z) = (x^2, xy, xz, y^2, yz, z^2)$, this also gives an embedding of $\mathcal{C}$ in $\mathbb{P}^5$.

In this part of the thesis, we look at $S/\mathcal{I}_{\sigma(\mathcal{C})}$, the homogeneous coordinate ring of $\sigma(\mathcal{C})$ in $\mathbb{P}^5$ and explicitly calculate the minimal graded free resolution of $S/\mathcal{I}_{\sigma(\mathcal{C})}$, where S is the homogeneous coordinate ring of $\mathbb{P}^5$.

Let the degree of $\mathcal{C}$ in $\mathbb{P}^2$ be d,i.e, $\mathcal{C}$ be defined by an irreducible homogeneous polynomial, 'f' of degree d in $K[x_0, x_1, x_2]$. Depending on the parity of d, we get the following two results.

Theorem 1: Let $\mathcal{C}$ be an irreducible curve of even degree say $d = 2m$, $m \geq 1$. The homogeneous coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C})}$ of $\sigma(\mathcal{C})$ in $\mathbb{P}^5$ has the following minimal graded free resolution:

$$
\begin{aligned}
0 \rightarrow & S(-m-4)^{\oplus 3} \overset{\alpha_4}{\rightarrow} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \overset{\alpha_3}{\rightarrow} \\
& \overset{\alpha_3}{\rightarrow} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \overset{\alpha_2}{\rightarrow} S(-2)^{\oplus 6} \oplus S(-m) \overset{\alpha_1}{\rightarrow} S \rightarrow S/\mathcal{I}_{\sigma(\mathcal{C})} \rightarrow 0
\end{aligned}
$$

where α_i's are matrices of homogeneous polynomial entries with no non-zero scalars [See Section 2.1]

Theorem 2: Let $\mathcal{C}$ be an irreducible curve of odd degree say $d = 2m - 1$, for $m \geq 2$. The homogeneous coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C})}$ of $\sigma(\mathcal{C})$ in $\mathbb{P}^5$ has the following minimal graded free resolution:

$$
\begin{aligned}
0 \rightarrow & S(-m-4) \overset{\beta_4}{\rightarrow} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \overset{\beta_3}{\rightarrow} \\
& \overset{\beta_3}{\rightarrow} S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \overset{\beta_2}{\rightarrow} S(-2)^{\oplus 6} \oplus S^{\oplus 3}(-m) \overset{\beta_1}{\rightarrow} S \rightarrow S/\mathcal{I}_{\sigma(\mathcal{C})} \rightarrow 0
\end{aligned}
$$

where β_i's are matrices of homogeneous polynomial entries with no non-zero scalars [See Section 2.2]

Corollary 1: Let $\mathcal{C}$ be a smooth, irreducible plane curve of degree d and L be the line bundle $\mathcal{O}_{\mathcal{C}}(2)$.
(a)$S/\mathcal{I}_{\sigma(\mathcal{C})}$ is Gorenstein if 'd' is odd and when 'd' is even $S/\mathcal{I}_{\sigma(\mathcal{C})}$ is Cohen-Maculay but not Gorenstein.
(b)$(\mathcal{C}, L)$ satisfies property N_0 for all $d \geq 2$.
(c)$(\mathcal{C}, L)$ satisfies N_1 iff $d = 3, 4$.

Part 2

Consider two distinct irreducible plane projective curves, $\mathcal{C}$ and $\mathcal{C}'$ of degrees d and d' respectively. Then by Bezout's theorem we know that $\mathcal{C}$ and $\mathcal{C}'$ intersect at $d.d'$ points

counted with multiplicity.

In the second problem, we explicitly write down the minimal graded free resolution of $S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$, where $\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$ is the ideal sheaf of $\sigma(\mathcal{C}\cap\mathcal{C}')$. Depending on the parities of d and d', we get the following three results.

Theorem 3: Let $\mathcal{C}$, $\mathcal{C}'$ be two irreducible curves of even degree say $d = 2m$ and $d' = 2m'$, $m, m' \geq 1$. The homogeneous coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$ of $\sigma(\mathcal{C}\cap\mathcal{C}')$ in $\mathbb{P}^5$ has the following minimal graded free resolution.

$$\begin{aligned} 0 \to & S(-m-m'-4)^{\oplus 3} \overset{\mathcal{P}_5}{\to} S(-m-4)^{\oplus 3} \oplus S(-m'-4)^{\oplus 3} \oplus S(-m-m'-3)^{\oplus 8} \overset{\mathcal{P}_4}{\to} \\ & \overset{\mathcal{P}_4}{\to} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \oplus S(-m'-3)^{\oplus 8} \oplus S(-m-m'-2)^{\oplus 6} \overset{\mathcal{P}_3}{\to} \\ & \overset{\mathcal{P}_3}{\to} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \oplus S(-m'-2)^{\oplus 6} \oplus S(-m-m') \overset{\mathcal{P}_2}{\to} \\ & \overset{\mathcal{P}_2}{\to} S(-2)^{\oplus 6} \oplus S(-m) \oplus S(-m') \overset{\mathcal{P}_1}{\to} S \to S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')} \to 0 \end{aligned}$$

where $\mathcal{P}_i$'s are matrices with homogeneous polynomial entries with no non-zero scalars[See Section 3.1]

Theorem 4: Let $\mathcal{C}$, $\mathcal{C}'$ be two irreducible curves of degrees say $d = 2m$ and $d' = 2m' - 1$, $m, m' \geq 2$. Then the homogeneous coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$ of $\sigma(\mathcal{C}\cap\mathcal{C}')$ in $\mathbb{P}^5$ has the following minimal graded free resolution.

$$\begin{aligned} 0 \to & S(-m-m'-4) \overset{\mathcal{Q}_5}{\to} S(-m-4)^{\oplus 3} \oplus S(-m'-4) \oplus S(-m-m'-2)^{\oplus 6} \overset{\mathcal{Q}_4}{\to} \\ & \overset{\mathcal{Q}_4}{\to} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \oplus S(-m'-2)^{\oplus 6} \oplus S(-m-m'-1)^{\oplus 8} \overset{\mathcal{Q}_3}{\to} \\ & \overset{\mathcal{Q}_3}{\to} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \oplus S(-m'-1)^{\oplus 8} \oplus S(-m-m')^{\oplus 3} \overset{\mathcal{Q}_2}{\to} \\ & \overset{\mathcal{Q}_2}{\to} S(-2)^{\oplus 6} \oplus S(-m) \oplus S(-m')^{\oplus 3} \overset{\mathcal{Q}_1}{\to} S \to S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')} \to 0 \end{aligned}$$

where $\mathcal{Q}_i$'s are matrices with homogeneous polynomial entries with no non-zero scalars[See Section 3.2]

Theorem 5: Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible plane curves of odd degree say $d = 2m-1$ and $d' = 2m' - 1$ for $m, m' \geq 2$. The coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$ of $\sigma(\mathcal{C}\cap\mathcal{C}')$ in $\mathbb{P}^5$ has the following minimal graded free resolution.

$$\begin{aligned} 0 \to & S(-m-m'-3)^{\oplus 3} \overset{\mathcal{R}_5}{\to} S(-m-4) \oplus S(-m'-4) \oplus S(-m-m'-2)^{\oplus 8} \overset{\mathcal{R}_4}{\to} \\ & \overset{\mathcal{R}_4}{\to} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \oplus S(-m'-2)^{\oplus 6} \oplus S(-m-m'-1)^{\oplus 6} \overset{\mathcal{R}_3}{\to} \\ & \overset{\mathcal{R}_3}{\to} S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \oplus S(-m'-1)^{\oplus 8} \oplus S(-m-m'+1) \overset{\mathcal{R}_2}{\to} \\ & \overset{R_2}{\to} S(-2)^{\oplus 6} \oplus S(-m)^{\oplus 3} \oplus S(-m')^{\oplus 3} \overset{\mathcal{R}_1}{\to} S \to S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')} \to 0 \end{aligned}$$

where $\mathcal{R}_i$'s are matrices with homogeneous polynomial entries with no non-zero scalars[See Section 3.3]

Corollary 2: $S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$ is Gorenstein if degrees of $\mathcal{C}$ and $\mathcal{C}'$ are of different parities and is Cohen-Maculay but not Gorenstein otherwise.

Part 3

Consider the resolution in Theorem 2. Namely,

$$0 \rightarrow S(-m-4) \stackrel{\beta_4}{\rightarrow} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \stackrel{\beta_3}{\rightarrow}$$
$$\stackrel{\beta_3}{\rightarrow} S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \stackrel{\beta_2}{\rightarrow} S(-2)^{\oplus 6} \oplus S^{\oplus 3}(-m) \stackrel{\beta_1}{\rightarrow} S \rightarrow S/\mathcal{I}_{\sigma(\mathcal{C})} \rightarrow 0$$

Then

$$\mathbf{P}\bullet .0 \rightarrow S(-m-4) \stackrel{\beta_4}{\rightarrow} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \stackrel{\beta_3}{\rightarrow}$$
$$\stackrel{\beta_3}{\rightarrow} S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \stackrel{\beta_2}{\rightarrow} S(-2)^{\oplus 6} \oplus S^{\oplus 3}(-m) \stackrel{\beta_1}{\rightarrow} S \rightarrow S/\mathcal{I}_{\sigma(\mathcal{C})} \rightarrow 0$$

is a symmetric acyclic complex.
In [KM], the author proves that any length 4, symmetric resolution has a DG Algebra structure. Hence the above resolution has a DG Algebra structure.

Theorem 3.1: We give an explicit DG Algebra structure to the above acyclic complex $\mathbf{P}\bullet$.

Contents

1
Preliminaries

1.1 The d-Uple embedding

Let $\mathbb{P}^n$ be n-dimensional projective space over a field K. Then for $d > 0$, we can define a map $\sigma_d \colon \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$, such that for $\bar{P} \in \mathbb{P}^n$,

$$\sigma_d(\bar{P}) = \big(M_0(\bar{P}), \ldots, M_N(\bar{P})\big)$$

where M_i's are degree d monomials which form a basis of the vector space of all homogeneous polynomials of degree d in $n+1$ variables.

This map which is an embedding, is called the d-Uple embedding.

Now for n and N as above define a map, θ such that

$$\begin{aligned} \theta \quad &: K[y_0, \ldots, y_N] \to K[x_0, \ldots, x_n] \\ &\theta(y_i) = M_i(x_0, \ldots, x_n) \end{aligned}$$

Then $\ker\theta$ is a homogeneous prime ideal of $K[y_0, \ldots, y_N]$ and $Z(\ker(\theta))$ is a projective variety of $\mathbb{P}^N$ and $Z(\ker(\theta)) = \sigma_d(\mathbb{P}^n)$.(See [H] for proof of the statement.)

The 2-uple embedding of $\mathbb{P}^2$ is called the *Veronese Embedding*, and $\sigma_2(\mathbb{P}^2)$ is called the *Veronese Surface*. Now let us look at the map θ with $n = 2$ and $N = 5$. So we have

$$\theta\colon \quad K[y_0, \ldots, y_5] \to K[x_0, x_1, x_2]$$

To see this map more clearly, we will change the notations.

Let us denote, $K[y_0, \ldots, y_5]$ as $K[x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}]$ and

$$\theta(x_{ij}) = x_i.x_j \text{ for } 0 \le i \le j \le 2$$

Then we see that
$\ker(\theta) = \langle \Delta_{ij} \quad : \quad 0 \le i \le j \le 2\rangle$, where

$$\begin{aligned} \Delta_{00} &= x_{11}x_{22} - x_{12}^2 \\ \Delta_{01} &= x_{01}x_{22} - x_{12}x_{02} \\ \Delta_{02} &= x_{01}x_{12} - x_{02}x_{11} \\ \Delta_{11} &= x_{00}x_{22} - x_{02}^2 \\ \Delta_{12} &= x_{00}x_{12} - x_{02}x_{01} \\ \Delta_{22} &= x_{00}x_{11} - x_{01}^2. \end{aligned} \tag{1.1}$$

Hence we get that, $\{\Delta_{ij} = 0 \quad : \quad 0 \leq i \leq j \leq 2\}$ are the 6 defining equations of the Veronese Surface; In fact $Z(ker(\theta)) = \sigma(\mathbb{P}^2)$ as a projective subvariety of $\mathbb{P}^5$

1.2 Syzygies and minimal free resolutions

Note that as we will only look at homogeneous coordinate rings of projective varieties and finitely generated modules over them, our definitions and notations will be adapted accordingly. We know that the homogeneous coordinate ring of the projective space, $\mathbb{P}^n_K$ is the polynomial ring, $S = K[x_0, \dots, x_n]$ in $n+1$ variables, with all the variables of degree one.

Let $M = \oplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded S-module with the d^{th} graded component M_d. Now as M is finitely generated, each M_d is finite dimensional K-vector space.
For any graded module, M, $M(a)$ is the module M shifted(or *'twisted'*) by a, where $a \in \mathbb{Z}$:

$$M(a)_d = M_{a+d}$$

A module M over a graded ring S is called *graded free S-module* if M is decomposable as a direct sum of free S modules: $M = \oplus_i S(a_i)$.

Given homogeneous elements $m_i \in M$ of degree a_i that generate M as an S-module, we define a map from graded free S module $F_0 = \oplus_i S(-a_i)$ onto M, by sending the ${a_i}^{\text{th}}$ degree generators to m_i . Now if N is the kernel of this map, then the elements of N are called *syzygies* of M. We also know that N is finitely generated graded S-module, hence we can define a map onto N from another graded free S-module, F_1 in same way. Continuing this way we can construct a sequence of maps of graded free module. This sequence is called a *graded free resolution* of M.

A complex of graded S-modules

$$\dots \to F_i \xrightarrow{\delta_i} F_{i-1} \to \dots$$

is called *minimal* if for each i, $\delta_i(F_i) \subset \ \boldsymbol{m}F_{i-1}$, where $\boldsymbol{m} = (x_0, \dots, x_n)$, the only homogeneous maximal ideal of S.

Now we are in a position to state a theorem, which we will use extensively in the first two problems.

Theorem 1.1[OP] : The homogenous coordinate ring $S/\mathcal{I}_{\sigma(\mathbb{P}^2)}$ of $\sigma(\mathbb{P}^2)$ in $\mathbb{P}^5$ has the following minimal graded free resolution:

$$0 \to S(-4)^{\oplus 3} \xrightarrow{M_3} S(-3)^{\oplus 8} \xrightarrow{M_2} S(-2)^{\oplus 6} \xrightarrow{M_1} S \to S/\mathcal{I}_{\sigma(\mathbb{P}^2)} \to 0 \tag{1.2}$$

where,

$$M_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02},, & \Delta_{11}, & \Delta_{12},, & \Delta_{22} \end{bmatrix} \tag{1.3}$$

$$M_2 = \begin{bmatrix} x_{02} & 0 & x_{01} & 0 & 0 & x_{00} & 0 & 0 \\ -x_{12} & x_{02} & -x_{11} & x_{01} & 0 & 0 & x_{00} & 0 \\ x_{22} & 0 & x_{12} & x_{02} & x_{01} & x_{02} & 0 & x_{00} \\ 0 & -x_{12} & 0 & -x_{11} & 0 & -x_{11} & -x_{01} & 0 \\ 0 & x_{22} & 0 & 0 & -x_{11} & x_{12} & x_{02} & -x_{01} \\ 0 & 0 & 0 & x_{22} & x_{12} & 0 & 0 & x_{02} \end{bmatrix} \quad (1.4)$$

and let

$$M_2 = \begin{bmatrix} W_1, & W_2, & W_3, & W_4, & W_5, & W_6, & W_7, & W_8 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} x_{01} & -x_{00} & 0 \\ -x_{11} & x_{01} & 0 \\ -x_{02} & 0 & x_{00} \\ x_{12} & -x_{02} & 0 \\ -x_{22} & 0 & x_{02} \\ 0 & x_{02} & -x_{01} \\ 0 & -x_{12} & x_{11} \\ 0 & x_{22} & -x_{12} \end{bmatrix} \quad (1.5)$$

and let

$$M_3 = \begin{bmatrix} G_1, & G_2, & G_3 \end{bmatrix}$$

1.3 N_p-property

Let X be a smooth irreducible projective curve of genus g and L be an very ample line bundle on X generated by global sections. Thus L determines a morphism

$$\Phi_L : \qquad X \longrightarrow \mathbb{P}\left(H^0(L)\right) = \mathbb{P}^r$$

where $r = dim\left(H^0(L)\right) - 1$. If L is very ample then Φ_L is an embedding.

Let S denote the symmetric algebra, $\mathrm{Sym}^{\cdot} H^0(L)$ on $H^0(L)$. So S is a homogeneous coordinate ring of $\mathbb{P}^r$. Consider the graded ring

$$R = R(L) = \oplus_m H^0(X, L^m)$$

associated to L. Then R is in a natural way a finitely generated module over S, and so we can talk about its minimal graded free resolution.$F_\bullet \to R \to 0$ of R; i.e.,

$$0 \to F_{r-1} \overset{f_{r-1}}{\to} \ldots \to F_1 \overset{f_1}{\to} F_0 \to R \to 0 \tag{1.6}$$

is exact where each F_i is a direct sum of twists of S, that is,

$$F_i = \oplus_j S(-a_{i,j}),$$

and hence in particular the maps in equation(1.6) are given by matrices of homogeneous forms. Minimality in this context means that none of the entries in these matrices are non-zero constants.

Definition: [L] For a integer $p \geq 0$, we say that the line bundle L satisfies **Property** (N_p) if

$$F_0(L) = S \text{ and } F_i(L) = \oplus S(-i-1) \text{ for all } 1 \leq i \leq p$$

The above definition means the following:

L satifies N_0	$\Longrightarrow$	Φ_L embeds X as a projectively normal curve;
L satifies N_1	$\Longrightarrow$	N_0 holds for L, and the homogeneous ideal I of X is generated by quadrics;
L satisfies N_2	$\Longrightarrow$	N_0 and N_1 hold for X, and the module of syzygies among the quadrics generators $Q_i \in \mathcal{I}$ is spanned by relations of the form $\sum L_i Q_i = 0$ where the L_i are linear polynomials;
$\vdots$	$\vdots$	
L satifies N_p	$\Longrightarrow$	L satisfies N_{p-1} and the syzygies amongst the generators of F_{p-1} are linear polynomials

1.4 Differential graded(DG) algebras

Let S be a commutative ring.

Let

$$\mathbf{P}_\bullet \qquad \dots \qquad \to P_2 \to P_1 \to P_0 \to 0$$

be an acyclic complex of projective S-modules with $P_0 = S$. We can consider $\mathbf{P}_\bullet$ as a graded module equipped with an endomorphism, $\partial : \mathbf{P}_\bullet \to \mathbf{P}_\bullet$ of degree -1 satisfying $\partial \circ \partial = 0$.

In [BH], the authors give the following definition.

The resolution, $(\mathbf{P}_\bullet, \delta)$ is said to be a *Differential graded(DG) algebra* (or is said to have a DG algebra structure) if we can define an associative multiplication on $\mathbf{P}_\bullet$ satisfying the following conditions,

(i) $P_n.P_m \subset P_{n+m} \quad \forall n, m \geq 0$;

(ii) $1 \in P_0$ acts as the unit element i.e $1.a = a.1 = a \quad \forall a \in \mathbf{P}_\bullet$;

(iii) $a.b = (-1)^{\deg(a).\deg(b)} b.a$, for all homogeneous elements, $a, b \in \mathbf{P}_\bullet$;

(iv) $a.a = 0$ for all odd degree elements, a;

(v) $\partial(a.b) = \partial(a).b + (-1)^{\deg(a)} a.\partial(b)$, for all homogeneous elements $a, b \in \mathbf{P}_\bullet$.

Proposition:[A] If A is a projective resolution of a R-module, M, such that $A_0 = R$ and $A_n = 0$ for $n \geq 4$, then A has a structure of DG algebra.

Recall the resolution used in the previous section.

$$0 \to S(-4)^{\oplus 3} \xrightarrow{M_3} S(-3)^{\oplus 8} \xrightarrow{M_2} S(-2)^{\oplus 6} \xrightarrow{M_1} S \to S/\mathcal{I}_{\sigma(\mathbb{P}^2)} \to 0$$

Let us call the above resolution $\mathbf{P}_\bullet$. Notice that this resolution is of length 3, and hence by the earlier proposition this can be given a DG-algebra structure.

So we have $\mathbf{P}_\bullet : 0 \to P_3 \to P_2 \to P_1 \to P_0 = S \to 0$ where,

rank(P_1)= 6, with $\{e_i \quad : \quad i = 1, \dots, 6\}$ as the basis of P_1

rank(P_2)= 8, with $\{e_{w_s} \quad : \quad s = 1, \dots, 8\}$ as the basis of P_2

rank(P_3)= 3, with $\{e_{g_t} \quad : \quad t = 1, 2, 3\}$ as the basis of P_3.

Now with the following conditions,

$$(i) \qquad e_i.e_j = \sum_{s=1,\dots,8} A_{i,j_s} e_{w_s}$$

$$(ii) \qquad e_i.e_{w_s} = \sum_{t=1,2,3} B_{i,s_t}.e_{g_t}$$

$$(iii) \qquad e_i.e_{g_t} = 0 \qquad \forall \quad i = 1, \dots, 6 \quad \text{and} \quad t = 1, 2, 3$$

$$(iv) \qquad e_{w_s}.e_{w_t} = 0 \qquad \forall \quad s, t = 1, \dots, 8$$

$$(v) \qquad \begin{array}{ll} \partial(e_{2i+j+1}) = & \Delta_{ij} \quad i \neq 2,\ 0 \leq i \leq j \leq 2 \\ \partial(e_6) = & \Delta_{22} \end{array}$$

$$(vi) \qquad \partial(e_{w_s}) = \sum_{i=1,\dots,6} W_{s_i}.e_i$$

$$(vii) \qquad \partial(e_{g_t}) = \sum_{s=1,\dots,8} (-1)^{t+1} G_{t_s}.e_{w_s},$$

and with $[A_{i,j}]$,$[B_{i,j}]$ matrices from Chapter 4, we can check that $\mathbf{P}_\bullet$ is a DG-algebra. These structure will be used extensively in the third part of this thesis.

2

Resolutions of plane curves in the Veronese embedding.

Recall from chapter 1, that the map
$x_{ij} \mapsto x_i.x_j$ for $0 \leq i \leq j \leq 2$ induces a homomorphism

$$\theta : \quad K[x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}] \rightarrow K[x_0, x_1, x_2] \text{ of graded rings}$$

From this we get the following lemma.

Lemma 2.1: If $g \in K[x_0, x_1, x_2]$ is a homogeneous polynomial of even degree(say $2n$). Then $g \in Im(\theta)$, which means that the subalgebra $Im(\theta)$ of $K[x_0, x_1, x_2]$ is generated by even polynomials.
Proof: Let

$$g = \sum_{i+j+k=2n} b_{ijk} x_0^i x_1^j x_2^k \tag{2.1}$$

Depending on the parities of i, j, k, we define some homogeneous polynomials in S using the coefficients b_{ijk} appearing in (2.1) i.e., $g = g^I + g^{II} + g^{III} + g^{IV}$ with;

$$g^I = \sum_{\substack{i+j+k=2n,\\ i,j,k \text{ all even}}} b_{ijk} x_0^i x_1^j x_2^k$$

$$g^{II} = \sum_{\substack{i+j+k=2n,\\ i \text{ even, } j,k \text{ odd}}} b_{ijk} x_0^i x_1^j x_2^k$$

$$g^{III} = \sum_{\substack{i+j+k=2n,\\ j \text{ even, } i,k \text{ odd}}} b_{ijk} x_0^i x_1^j x_2^k$$

$$g^{IV} = \sum_{\substack{i+j+k=2n,\\ k \text{ even, } i,j \text{ odd}}} b_{ijk} x_0^i x_1^j x_2^k$$

Case I : When i, j, k are all even, consider

$$G^I = \sum_{\substack{i+j+k=d\\ i,j,k \text{ even}}} b_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}}$$

Notice that $\theta(G^I) = g^I$

Case II: When i is even, j and k odd, consider

$$G^{II} = \sum_{\substack{i+j+k=2n \\ i \text{ even} \\ j,k \text{ odd}}} b_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k-1}{2}} x_{12}$$

Similarly as in *Case I*, $\theta(G^{II}) = g^{II}$

Case III: When i,k are odd and j is even, consider

$$G^{III} = \sum_{\substack{i+j+k=2n \\ j \text{ even} \\ i,k \text{ odd}}} b_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}} x_{02}$$

$\theta(G^{III}) = g^{III}$

Case IV: When i,j are odd and k is even consider,

$$G^{IV} = \sum_{\substack{i+j+k=2n \\ k \text{ even} \\ i,j \text{ odd}}} b_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}} x_{01}$$

$\theta(G^{IV}) = g^{IV}$

Now let

$$G = G^I + G^{II} + G^{III} + G^{IV}$$

Then $\theta(G) = g$.
Hence $g \in Im(\theta)$.

From Section (1.1), we also know that for the embedding, $\mathbb{P}^2 \overset{\sigma}{\hookrightarrow} \mathbb{P}^5$, $\mathbb{Z}(ker(\theta)) = \sigma(\mathbb{P}^2)$.
Let $\mathcal{C}$ be a smooth(or irreducible) plane curve. Hence $\mathcal{C}$ is given by a irreducible polynomial in three variables. The Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ gives an embedding $\mathcal{C} \overset{\sigma}{\hookrightarrow} \mathbb{P}^5$. We will compute the syzygies of the homogeneous ideal $\mathcal{I}_{\sigma(\mathcal{C})}$ of this embedding of $\mathcal{C}$ in $\mathbb{P}^5$ using the resolution of the Veronese embedding talked about in Chapter 1 . Let $\mathcal{C}$ be defined by the polynomial f of degree d in three variables. Let

$$\mathcal{C} = Z\left(f(x_0, x_1, x_2)\right) \text{ where, } f = \sum_{i+j+k=d} a_{ijk} x_0^i x_1^j x_2^k$$

2.1 Degree of $\mathcal{C}$ is even

We have d is even(say $2m$) and

$$f = \sum_{i+j+k=2m} a_{ijk} x_0^i x_1^j x_2^k$$

From Lemma 2.1, we get that $f \in Im(\theta)$. Let F be a homogeneous polynomial in S such that $\theta(F) = f$.

Lemma 2.2: Let $G \in S$ such that, G homogeneous and $Z(\theta(F)) \subset Z(\theta(G)) \subset \mathbb{P}^2$. Then $G \in < F, \Delta_{i,j} : 0 \leq i \leq j \leq 2 >$, where $\langle F, \Delta_{ij} : 0 \leq i \leq j \leq 2 \rangle$ is the homogeneous ideal generated by F and Δ_{ij} in S. i.e is
Proof: Let $\theta(G) = g$, then g is a homogeneous polynomial of even degree and,

$$Z(f) \subset Z(g)$$

Hence $g \in (f)$. As $\mathcal{C}$ is an irreducible curve f is irreducible, hence,

$$g = f.h \text{ for some } h \text{ homogeneous in } K[x_0, x_1, x_2]$$

Now f and g are even degree implies that h is of even degree hence, by Lemma(2.1) we can find a homogeneous $H \in S$, such that $\theta(H) = h$.
Thus $\theta(G) = \theta(F).\theta(H) = \theta(F.H)$,

$$\Rightarrow \theta(G - F.H) = 0$$

$$\Rightarrow G - F.H \in ker(\theta)$$

$$\Rightarrow G - F.H = \sum_{0 \leq i \leq j \leq 2} \Delta_{ij} S_{ij} \text{ for some } S_{ij} \in S, S_{ij} \text{ homogeneous}$$

$$\Rightarrow G \in < F, \Delta_{ij} : 0 \leq i \leq j \leq 2 >$$

This completes the proof of the lemma.

From now on we will denote M_1, M_2 and M_3 from equations (1.4), (1.5) of section(1.2) as below: The i^{th} row of M_2 will be W_i and the j^{th} of M_3 will be G_j, for $1 \leq i \leq 8$ and $j = 1, 2, 3$. So we have,

$$M_2 = \begin{bmatrix} W_1, & W_2, & W_3, & W_4, & W_5, & W_6, & W_7, & W_8 \end{bmatrix} \tag{2.2}$$

$$M_3 = \begin{bmatrix} G_1, & G_2, & G_3 \end{bmatrix} \tag{2.3}$$

Theorem 2.1: Let $\mathcal{C}$ be an irreducible curve of even degree say $d = 2m$, $m \geq 1$. The homogeneous co-ordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C})}$ of $\sigma(\mathcal{C})$ in $\mathbb{P}^5$ has the following minimal free resolution.

$$\begin{aligned} 0 \rightarrow & S(-m-4)^{\oplus 3} \overset{\alpha_4}{\rightarrow} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \overset{\alpha_3}{\rightarrow} \\ & \overset{\alpha_3}{\rightarrow} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \overset{\alpha_2}{\rightarrow} S(-2)^{\oplus 6} \oplus S(-m) \overset{\alpha_1}{\rightarrow} S \rightarrow S/\mathcal{I}_{\sigma(\mathcal{C})} \rightarrow 0 \end{aligned} \tag{2.4}$$

where α_i's are as follows,

$$\alpha_1 = \begin{bmatrix} [M_1], & F \end{bmatrix} \tag{2.5}$$

If

$$\alpha_2' = \begin{bmatrix} -F & 0 & 0 & 0 & 0 & 0 & \Delta_{00} \\ 0 & -F & 0 & 0 & 0 & 0 & \Delta_{01} \\ 0 & 0 & -F & 0 & 0 & 0 & \Delta_{02} \\ 0 & 0 & 0 & -F & 0 & 0 & \Delta_{11} \\ 0 & 0 & 0 & 0 & -F & 0 & \Delta_{12} \\ 0 & 0 & 0 & 0 & 0 & -F & \Delta_{22} \end{bmatrix}$$

write α_2' as

$$\alpha_2' = \begin{bmatrix} U_{00}, & U_{01}, & U_{02}, & U_{11}, & U_{12}, & U_{22} \end{bmatrix}^T$$

Then

$$\alpha_2 = \begin{bmatrix} W_1', & W_2', & W_3', & W_4', & W_5', & W_6', & W_7', & W_8', & U_{00}, & U_{01}, & U_{02}, & U_{11}, & U_{12}, & U_{22} \end{bmatrix} \tag{2.6}$$

with

$$W_i' = \begin{bmatrix} W_i \\ 0 \end{bmatrix} \qquad \forall i = 1, \dots, 8$$

with W_i as in (2.1)
That is,

$$\alpha_2 = \begin{bmatrix} [M_2] & -FI_6 \\ 0 & [M_1] \end{bmatrix}$$

If

$$H_i = \begin{bmatrix} [F.I_i^8] \\ [W_i] \end{bmatrix}$$

with

$$I_i^k = \begin{bmatrix} 0, & 0, & \dots, & \overset{i^{th} \text{ position}}{1}, & 0, & \dots, & 0 \end{bmatrix}^T \text{ is a } k \times 1 \text{ vector}$$

$$\text{Then, } \alpha_3 = \begin{bmatrix} G_1', & G_2', & G_3', & H_1, & \dots, & H_8 \end{bmatrix} \tag{2.7}$$

where

$$G_i' = \begin{bmatrix} G_i \\ [\bar{0}] \end{bmatrix} \qquad \text{for } i = 1, 2, 3$$

where G_i as in (2.2) and $[\bar{0}]$ is a 0 matrix of appropriate dimension.
That is,

$$\alpha_3 = \begin{bmatrix} [M_3] & FI_8 \\ 0 & [M_2] \end{bmatrix}$$

Finally,

$$\alpha_4 = \begin{bmatrix} \begin{pmatrix} [-F.I_1^3] \\ [G_1] \end{pmatrix}, & \begin{pmatrix} [-F.I_2^3] \\ [G_2] \end{pmatrix}, & \begin{pmatrix} [-F.I_3^3] \\ [G_3] \end{pmatrix} \end{bmatrix} \tag{2.8}$$

That is,

$$\alpha_4 = \begin{bmatrix} -FI_3 \\ [M_3] \end{bmatrix}$$

Proof: From Lemma 2.2, it is clear that

$$\alpha_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02}, & \Delta_{11}, & \Delta_{12}, & \Delta_{22}, & F \end{bmatrix}$$

Now consider $B \in S$ homogenous and

$$A = \begin{bmatrix} a_{00}, & a_{01}, & a_{02}, & a_{11}, & a_{12}, & a_{22} \end{bmatrix}$$

where $a_{ij} \in S$ homogeneous such that

$$\sum_{i,j} a_{ij}.\Delta_{ij} + B.F = 0$$

$$\Rightarrow \theta(B.F) = 0$$

$$\Rightarrow \theta(B).f = 0$$

$$\Rightarrow B \in < \Delta_{ij} : 0 \leq i \leq j \leq 2 >$$

Hence, $B = \sum(b_{ij}\Delta_{ij})$ for some homogeneous polynomials $b_{ij} \in S$.

$$\Rightarrow \sum(a_{ij} + b_{ij}.F).\Delta_{ij} = 0$$

Now if $a_{ij}\Delta_{ij} + b_{ij}.F = 0$ for all (a_{ij}, b_{ij}) then such a $[A, B]$ is generated by U_{ij}. If not then,
$\Rightarrow \sum(a_{ij} + b_{ij}F) \in \mathrm{Syz}^1(< \Delta_{ij} : 0 \le i \le j \le 2 >)$
Hence, the relations between Δ_{ij} and F are generated by $U_{ij} : 0 \le i \le j \le 2$ and $W'_k : k = 1, \ldots, 8$.
Hence we get,

$$\alpha_2 = \begin{bmatrix} W'_1, & W'_2, & W'_3, & W'_4, & W'_5, & W'_6, & W'_7, & W'_8, & U_{00}, & U_{01}, & U_{02}, & U_{11}, & U_{12}, & U_{22} \end{bmatrix}$$

Now consider
$A = \begin{bmatrix} a_{00}, & a_{01}, & a_{02}, & a_{11}, & a_{12}, & a_{22} \end{bmatrix}^T, a_{ij} \in S$, a_{ij} homogeneous $\forall 0 \le i \le j \le 2$ and,

$$B = \begin{bmatrix} (b_k) \end{bmatrix}, b_k \in S, \text{ homogeneous}$$

such that

$$\sum_{0 \le i \le j \le 2} a_{ij}.U_{ij} + \sum_{1 \le k \le 8} b_k.W'_k = 0$$

$$\Rightarrow \sum_{i,j} a_{ij}\Delta_{ij} = 0$$

as the last column of each W'_k, $k = 1, \ldots, 8$ is zero and the last column of U_{ij} is Δ_{ij} for $0 \le i \le j \le 2$

$$\Rightarrow A \in < W_k : k = 1, \ldots, 8 >$$

Let $A = \sum_k (c_k W_k)$, for some homogeneous polynomial, $c_k \in S$

$$\Rightarrow -\sum_k c_k W_k F.Id_6 + \sum_k b_k W_k = 0$$

where Id_n is a $n \times n$ identity matrix.

$$\Rightarrow \sum_{i,k} W_k(-c_k F + b_k) = 0$$

Hence if $-c_k.F + b_k = 0$ for all k, this implies $b_k = c_k.F$ for all k then such (b_k, a_{ij}) are generated by $< \left[[F.[I_i^8]], [W_i]\right] >$ for $i = 1, \ldots, 8$. If not then, $[(-c_k F + b_k)I_k]_{k=1,\ldots,6} \in \mathrm{Syz}^1(< W_j : j = 1, \ldots, 8 >)$.

Hence the relations between W'_k and U_{ij} are generated by G'_i and H_k. Hence we get

$$\alpha_3 = \begin{bmatrix} G'_1, & G'_2, & G'_3, & H_1, & \ldots, & H_8 \end{bmatrix}$$

Now consider

$A = \begin{bmatrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & a_7, & a_8 \end{bmatrix}^T, a_i \in S$, homogeneous for $i = 1, \ldots, 8$

$B = \begin{bmatrix} (b_k) \end{bmatrix}$, $b_k \in S$, homogeneous for $k = 1, 2, 3$ such that

$$\sum_i a_i.H_i + \sum_k b_k.G'_k = 0$$

$$\Rightarrow \sum_i a_i W_i = 0,$$

as the last six columns of each G'_k, $k = 1, 2, 3$ are zero.

$$\Rightarrow A \in < G_k : k = 1, 2, 3 >$$

Let $A = \sum_k (c_k G_k)$, for some homogeneous polynomial, $c_k \in S$.
Then we have, $\sum_k (c_k G_k).(F.Id_8) + \sum_k b_k.G_k = 0$

$$\Rightarrow \sum_k (c_k.F.Id_8 + b_k)\, G_k = 0$$

Now if $c_k.F + b_k = 0$ for every k, then $b_k = -c_k.F$ for all p, then we can say that $([b_p], [c_p])$ is generated by $< \left(\left[-F.I_i^3\right], \left[I_i^3\right]\right) : i = 1, 2, 3 >$, hence $([b_p], [a_k])$ is generated by $< \left(\left[-F.I_i^3\right], [G_i]\right) i = 1, 2, 3 >$

Also from Theorem 1.1, we have that $G'_k : k = 0, 1, 2$ are independent. Hence $\mathrm{Syz}^1(< G'_i, H_j : i = 1, 2, 3$ and $j = 1, \ldots 8 >) =< \left(\left[-F.I_i^3\right], [G_i]\right) : i = 1, 2, 3 >$
Hence,

$$\alpha_4 = \left[\begin{pmatrix} \left[-F.I_i^3\right] \\ [G_i] \end{pmatrix}_{1 \le i \le 3} \right]$$

2.2 Degree of $\mathcal{C}$ is odd

Recall

$$f = \sum_{i+j+k=d} a_{ijk} x_0^i x_1^j x_2^k$$

Now let $f_0 = x_0.f$, $f_1 = x_1.f$, $f_2 = x_2.f$. Then f_n is of even degree and hence according to Lemma 2.1, $f_n \in Im(\theta)$ for $n = 0, 1, 2$. We have the following lemma.
Lemma 2.3: $Z(f) = Z(f_0) \cap Z(f_1) \cap Z(f_2)$.
Proof: Clearly, $Z(f) \subset Z(f_0) \cap Z(f_1) \cap Z(f_2)$
Also if $\exists\ \bar{p} = (p_0, p_1, p_2) \in Z(f_0) \cap Z(f_1) \cap Z(f_2)$ and $\bar{p} \notin Z(f)$. Then $\bar{p} \in Z(x_i) \quad \forall i = 0, 1, 2$. This implies $p_i = 0 \forall i = 0, 1, 2$. But this contradicts the fact that $\bar{p} \in \mathbb{P}^2$. Hence $Z(f) = Z(f_0) \cap Z(f_1) \cap Z(f_2)$.
In the same way as proof of Lemma 2.1, we split f in four parts depending on the parities of i,j,k.
Case I: i, j, k are all odd. Let

$$\text{Let } h_I = \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k-1}{2}}$$

$$\begin{aligned} {F_0}^I &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i+1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k-1}{2}} x_{12} \\ {F_1}^I &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j+1}{2}} x_{22}^{\frac{k-1}{2}} x_{02} \\ {F_2}^I &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k+1}{2}} x_{01} \end{aligned}$$

Then,

$$\begin{aligned} {F_0}^I &= x_{00} x_{12} h_I \\ {F_1}^I &= x_{11} x_{02} h_I \\ {F_2}^I &= x_{22} x_{01} h_I \end{aligned}$$

Case II: i odd, j even, k even. Now

$$\text{Let } h_{II} = \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}}$$

$$\begin{aligned} {F_0}^{II} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i+1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}} \\ {F_1}^{II} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}} x_{01} \\ {F_2}^{II} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i-1}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k}{2}} x_{02} \end{aligned}$$

Then,

$$\begin{aligned} {F_0}^{II} &= x_{00} h_{II} \\ {F_1}^{II} &= x_{01} h_{II} \\ {F_2}^{II} &= x_{02} h_{II} \end{aligned}$$

Case III: i even, j odd, k even. Now

$$\text{Let } h_{III} = \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}}$$

$$\begin{aligned} {F_0}^{III} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}} x_{01} \\ {F_1}^{III} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j+1}{2}} x_{22}^{\frac{k}{2}} \\ {F_2}^{III} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k}{2}} x_{12} \end{aligned}$$

Then,

$$\begin{aligned} {F_0}^{III} &= x_{01} h_{III} \\ {F_1}^{III} &= x_{11} h_{III} \\ {F_2}^{III} &= x_{12} h_{III} \end{aligned}$$

Case IV: i even, j even, k odd. Now

$$\text{Let } h_{IV} = \sum_{i,j,k} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}}$$

$$\begin{aligned} {F_0}^{IV} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}} x_{02} \\ {F_1}^{IV} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j}{2}} x_{22}^{\frac{k-1}{2}} x_{12} \\ {F_2}^{IV} &= \sum_{i+j+k=d} a_{ijk} x_{00}^{\frac{i}{2}} x_{11}^{\frac{j-1}{2}} x_{22}^{\frac{k+1}{2}} \end{aligned}$$

Then,

$$\begin{aligned} {F_0}^{IV} &= x_{02} h_{IV} \\ {F_1}^{IV} &= x_{12} h_{IV} \\ {F_2}^{IV} &= x_{22} h_{IV} \end{aligned}$$

Write $F_n = {F_n}^I + {F_n}^{II} + {F_n}^{III} + {F_n}^{IV}$ $\forall n = 0, 1, 2$
Also notice $\theta(F_n) = f_n$ for $n = 0, 1, 2$

Lemma 2.4: Let $G \in k[x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}]$ be homogeneous and

$$Z(\theta(F_0)) \cap Z(\theta(F_1)) \cap Z(\theta(F_2)) \subset Z(\theta(G)) \subset \mathbb{P}^2$$

Then $G \in < F_k, \Delta_{i,j} : 0 \leq k \leq 2, 0 \leq i \leq j \leq 2 >$.
Proof: Now let $\theta(G) = g$, then degree(g) is even.

$$Z(f_0) \cap Z(f_1) \cap Z(f_2) \subset Z(g)$$
$$\Rightarrow Z(f) \subset Z(g)$$
$$\Rightarrow g \in (f) \text{ as } \mathcal{C} \text{ is an irreducible curve and } f \text{ is irreducible}$$

$$\Rightarrow g = f.h \text{ for some } h \in k[x_0, x_1, x_2]$$

$\Rightarrow h \neq 1$ as degree of f is odd while degree of g is even. Moreover h is an odd-degree polynomial

$$\Rightarrow g = \sum_{i=0,1,2} f_i h_i \text{ for some homogeneous polynomial } h_i \in k[x_0, x_1, x_2],$$

where degree of h_i is even and $h = x_0h_0 + x_1h_1 + x_2h_2$

$$\text{Hence } \Rightarrow G = \sum_{i=0,1,2} F_i H_i, \text{ where } \theta(H_i) = h_i \forall i = 0,1,2.$$

Such a H_i exists as the degree of h_i is even.

$$\Rightarrow \theta(G - \sum_{i=0,1,2} F_i H_i) = 0$$

$$\Rightarrow G - \sum_{i=0,1,2} F_i H_i \in ker(\theta)$$

$$\Rightarrow G = \sum_{i=0,1,2} F_i H_i + \sum_{i,j=0,1,2} \Delta_{ij} S_{ij} \quad \text{for some} \quad S_{ij} \in k[x_{00}, \ldots, x_{22}]$$

$$\text{Hence} \Rightarrow G \in < F_k, \Delta_{ij} : i,j,k = 0,1,2 >$$

Theorem 2.2: Let $\mathcal{C}$ be an irreducible curve of odd degree say $d = 2m-1$, for $m \geq 2$. The homogeneous coordinate ring $S/\mathcal{I}_\mathcal{C}$ of $\sigma(\mathcal{C})$ in $\mathbb{P}^5$ has the following resolution.

$$\begin{aligned} 0 \rightarrow & S(-m-4) \xrightarrow{\beta_4} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \xrightarrow{\beta_3} \\ & \xrightarrow{\beta_3} S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \xrightarrow{\beta_2} S(-2)^{\oplus 6} \oplus S(-m)^{\oplus 3} \xrightarrow{\beta_1} S \rightarrow S/\mathcal{I}_\mathcal{C} \rightarrow 0 \end{aligned} \tag{2.9}$$

Proof:
From Lemma 2.3 and Lemma 2.4, it is clear that

$$\beta_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02}, & \Delta_{11}, & \Delta_{12}, & \Delta_{22}, & F_0, & F_1, & F_2 \end{bmatrix}$$

Now consider $A = \begin{bmatrix} a_{00}, & a_{01}, & a_{02}, & a_{11}, & a_{12}, & a_{22} \end{bmatrix}$, $a_{ij} \in S$, homogeneous $\forall 0 \leq i \leq æ \leq 2$ and $b = \begin{bmatrix} b_0, & b_1, & b_2 \end{bmatrix}$ where $b_l \in S$, homogeneous, for $k = 0,1,2$ such that,

$$\sum_{i,j} a_{ij}.\Delta_{ij} + \sum_k b_k.F_k = 0 \tag{2.10}$$

$$\Rightarrow \theta(\sum_k (b_k.F_k)) = 0$$

$$\Rightarrow \sum_k (\theta(b_k).f_k) = 0$$

$$\Rightarrow \sum_k (\theta(b_k).f.x_k) = 0$$

$$\Rightarrow \sum_k (\theta(b_k).x_k) = 0$$

Let $\theta(b_k) = B_k$, then degree of B_k is even. Then

$$B = (B_0, B_1, B_2)^T \in \text{Syz}^1(x_0, x_1, x_2)$$

Now by simple computation we get

$$\text{Syz}^1(x_0, x_1, x_2) = < \begin{pmatrix} x_1 \\ -x_0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \\ -x_0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ -x_1 \end{pmatrix} >$$

hence $B \in < Y_0, Y_1, Y_2 >$ where

$$Y_0 = \begin{pmatrix} x_1, & -x_0, & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} x_2, & 0, & -x_0 \end{pmatrix}$$

$$Y_2 = \begin{pmatrix} 0, & x_2, & -x_1 \end{pmatrix}$$

But degree of B_k is even, hence, $B \in < x_k Y_l : k, l = 0, 1, 2 >$.
Hence, $(b_0, b_1, b_2) \in < Y_{lk} : k, l = 0, 1, 2 >$

where

$$Y_{00} = \begin{pmatrix} x_{01}, & -x_{00}, & 0 \end{pmatrix}$$
$$Y_{01} = \begin{pmatrix} x_{11}, & -x_{01}, & 0 \end{pmatrix}$$
$$Y_{02} = \begin{pmatrix} x_{12}, & -x_{02}, & 0 \end{pmatrix}$$

$$Y_{10} = \begin{pmatrix} x_{02}, & 0, & -x_{00} \end{pmatrix}$$
$$Y_{11} = \begin{pmatrix} x_{12}, & 0, & -x_{01} \end{pmatrix}$$
$$Y_{12} = \begin{pmatrix} x_{22}, & 0, & -x_{02} \end{pmatrix}$$

$$Y_{20} = \begin{pmatrix} 0, & x_{02}, & -x_{01} \end{pmatrix}$$
$$Y_{21} = \begin{pmatrix} 0, & x_{12}, & -x_{11} \end{pmatrix}$$
$$Y_{22} = \begin{pmatrix} 0, & x_{22}, & -x_{12} \end{pmatrix}$$

Also note that,

$$Y_{02} = Y_{11} - Y_{20}$$

Now substituting all Y_{ij} for $i, j = 0, 1, 2$ except for Y_{02} for b in equation(2.8) we get, the following 8 vectors,
Note that if A is a nXm matrix then by A^T we denote the transpose of A.

$$V_1 = \begin{bmatrix} 0, & 0, & -x_{00}h_I, & 0, & h_{IV}, & h_{III}, & [Y_{00}] \end{bmatrix}^T$$

$$V_2 = \begin{bmatrix} 0, & 0, & h_{IV}, & 0, & -x_{11}h_I, & -h_{II}, & [Y_{01}] \end{bmatrix}^T$$

$$V_3 = \begin{bmatrix} 0, & x_{00}h_I, & 0, & h_{IV}, & h_{III}, & 0, & [Y_{10}] \end{bmatrix}^T$$

$$V_4 = \begin{bmatrix} x_{00}h_I, & h_{IV}, & 0, & 0, & -h_{II}, & -x_{22}h_I, & [Y_{11}] \end{bmatrix}^T$$

$$V_5 = \begin{bmatrix} 0, & -h_{III}, & 0, & -h_{II}, & -x_{22}h_I, & 0, & [Y_{12}] \end{bmatrix}^T$$

$$V_6 = \begin{bmatrix} 0, & h_{IV}, & h_{III}, & x_{11}h_I, & 0, & -x_{22}h_I, & [Y_{20}] \end{bmatrix}^T$$

$$V_7 = \begin{bmatrix} h_{IV}, & x_{11}h_I, & -h_{II}, & 0, & 0, & 0, & [Y_{21}] \end{bmatrix}^T$$

$$V_8 = \begin{bmatrix} -h_{III}, & -h_{II}, & x_{22}h_I, & 0, & 0, & 0, & [Y_{22}] \end{bmatrix}^T$$

Let

$$\beta_2' = \begin{bmatrix} [V_1], & [V_2], & [V_3], & [V_4], & [V_5], & [V_6], & [V_7], & [V_8] \end{bmatrix}$$

Now all the relations between F_n's and Δ_{ij}'s are generated by V_k's and W_l''s and all the relations between only Δ_{ij}'s are generated by W_l's. Hence all relations between F_n, Δ_{jk} are generated by V_k, W_l'.
Hence $\mathrm{Syz}^1(< F_n, \Delta_{ij} >) =< V_k, W_l' : \quad 1 \leq k, l \leq 8 >$ and

$$\beta_2 = \left([W_1'] \, [W_2'] \ldots [W_8'] \, [V_1] \ldots [V_8] \right)$$

where $W_k' = [[W_k] \, [\bar{0}]]$ with $[\bar{0}]$ a 1×3 zero vector

Now consider, $\bar{A} = (a_i)$ with $a_i \in S$ homogeneous and $\bar{B} = (b_k)$ with $b_k \in S$, homogeneous

$$\sum_i a_i V_i + \sum_k b_k W_k' = 0 \tag{2.11}$$

Let $\bar{A} = [a_1, \ldots, a_8]$ and $\mathbf{V} = [V_1, \ldots, V_8]^T$ then equation(2.11) can be written as

$$\bar{A}. \ \mathbf{V} \ + \sum_k b_k . W_k' = 0 \tag{2.12}$$

Now as all the entries in the last 3 columns in each of W_i' are zero we have,

$$\sum_i a_i Y_{ij} = 0$$

.
Now it can be computed that $\mathrm{Syz}^1(Y_{ij}) =< K_l' : 1 \leq l \leq 6 >$ where

$$K_1' = \left[\ x_{02}, \quad 0, \quad -x_{01}, \quad 0, \quad 0, \quad x_{00}, \quad 0, \quad 0 \ \right]$$

$$K_2' = \left[\ x_{12}, \quad x_{02}, \quad -x_{11}, \quad -x_{01}, \quad 0, \quad x_{01}, \quad x_{00}, \quad 0 \ \right]$$

$$K_3' = \left[\ x_{22}, \quad 0, \quad -x_{12}, \quad x_{02}, \quad -x_{01}, \quad 0, \quad 0, \quad x_{00} \ \right]$$

$$K_4' = \left[\ 0, \quad x_{12}, \quad 0, \quad -x_{11}, \quad 0, \quad 0, \quad x_{01}, \quad 0 \ \right]$$

$$K_5' = \left[\ 0, \quad x_{22}, \quad 0, \quad 0, \quad -x_{11}, \quad -x_{12}, \quad x_{02}, \quad x_{01} \ \right]$$

$$K_6' = \left[\ 0, \quad 0, \quad 0, \quad x_{22}, \quad -x_{12}, \quad -x_{22}, \quad 0, \quad x_{02} \ \right]$$

Hence $\bar{A} = \sum_l d_l . K_l'$ in equation (2.12), we get

$$\sum_{l=1}^{8} d_l K_l' . \ \mathbf{V} + \sum_k b_k W_k' = 0$$

where d_l are homogenous polynomials in S for all $l = 1, \ldots, 6$. Simple calculation gives us that, $\left[\bar{B}, \sum_l d_l K_l'\right] \in \langle K_l : \quad l = 1, \ldots, 6 \rangle$, where K_l's for $1 \leq l \leq 6$ are as follows

$$K_1 = \left[\ 0, \quad 0, \quad 0, \quad x_{00} h_I, \quad 0, \quad 0, \quad -h_{IV}, \quad h_{III}, \quad [K_1'] \ \right]^T$$

$$K_2 = \left[\ 0, \quad 0, \quad x_{00} h_I, \quad 0, \quad -h_{III}, \quad -h_{IV}, \quad x_{11} h_I, \quad h_{II}, \quad [K_2'] \ \right]^T$$

$$K_3 = \left[\ -x_{00} h_I, \quad -h_{IV}, \quad 0, \quad -h_{III}, \quad 0, \quad h_{III}, \quad h_{II}, \quad x_{22} h_I, \quad [K_3'] \ \right]^T$$

$$K_4 = \left[\ 0, \quad 0, \quad -h_{IV}, \quad -x_{11} h_I, \quad h_{II}, \quad x_{11} h_I, \quad 0, \quad 0, \quad [K_4'] \ \right]^T$$

$$K_5 = \left[\ -h_{IV}, \quad -x_{11} h_I, \quad h_{III}, \quad h_{II}, -x_{22} h_I, \quad 0, \quad 0, \quad 0 \quad [K_5'] \ \right]^T$$

$$K_6 = \left[\ h_{III}, \quad h_{II}, \quad 0, \quad 0, \quad 0, \quad -x_{22} h_I, \quad 0, \quad 0, \quad [K_6'] \ \right]^T$$

Now all the relations between V_i's and W'_j's are generated by $\{K_l, G'_k, 1 \leq l \leq 6, k = 1,2,3\}$ and all the relations between only W'_j's (which are actually W_j) are generated by G'_k's. Hence we have that all relations between $\{\{V_i\},\{W'_j\}\}$ are generated by $\{K_l, G'_k\ 1 \leq l \leq 6, k = 1,2,3\ \}$. So, $\text{Syz}^1(< V_i, W'_j >) =< K_l, G'_k >$. So we get that,

$$\beta_3 = \begin{bmatrix} [G'_0] & [G'_1] & [G'_2] & [K_1] & \dots & [K_6] \end{bmatrix}$$

where, $G'_i = \begin{bmatrix} [G_i] & [\bar{0}] \end{bmatrix}$ where $[\bar{0}]$ is an appropriate dimensional zero matrix.
Now consider $\bar{A} = (A_i)$, such that $A_i \in S$, homogeneous and $\bar{B} = (B_k)$, such that $B_k \in S$, homogeneous such that,

$$\sum_l A_l K_l + \sum_k B_k G'_k = 0 \tag{2.13}$$

Hence we have,

$$\sum_l A_l K_l'^T = 0$$

(as the last eight columns of G'_i's are zero entries)
Now it can be computed that $\text{Syz}^1(K'_l) =< J' >$ where,

$$J' = \begin{bmatrix} J'_1 \\ J'_2 \\ J'_3 \\ J'_4 \\ J'_5 \\ J'_6 \end{bmatrix} = \begin{bmatrix} {x_{12}}^2 - x_{11}x_{22} \\ -x_{02}x_{12} + x_{01}x_{22} \\ x_{11}x_{02} - x_{01}x_{12} \\ {x_{02}}^2 - x_{00}x_{22} \\ -x_{01}x_{02} + x_{00}x_{12} \\ {x_{01}}^2 - x_{00}x_{11} \end{bmatrix}$$

Like in the calculation of K_l's, substitute J' in equation(2.13). Then we get,

$$J = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ J_5 \\ J_6 \\ J_7 \\ J_8 \\ J_9 \end{bmatrix} = \begin{bmatrix} -x_{00}x_{12}h_I - x_{00}h_{II} - x_{01}h_{III} - x_{02}h_{IV} \\ \\ -x_{11}x_{02}h_I + x_{01}h_{II} + x_{11}h_{III} + x_{12}h_{IV} \\ \\ -x_{01}x_{22}h_I - x_{02}h_{II} - x_{12}h_{III} - x_{22}h_{IV} \\ \\ \begin{bmatrix} [J'] \end{bmatrix} \end{bmatrix}$$

Now all the relations between K_l's and G'_k's are generated by J and there are no relations between only G'_k's as there are no non-trivial relations between G_k's. Hence all relations between K_l, G'_k are generated by J. Hence $\text{Syz}^1(< K_l, G'_k >) = \langle J_i : 1 \leq i \leq 9 \rangle$. Hence

$$\beta_4 = [J]$$

This completes the proof of the theorem.

2.3 Some remarks on N_p properties on plane curves under Veronese embedding

Due to the explicit computation of resolutions done in earlier sections we get some results about Property N_p of the line bundle, $\mathcal{O}_{\mathcal{C}}(2)$ of a plane curves, $\mathcal{C}$ with degree $d \geq 2$.

Consider $L = \mathcal{O}_{\mathcal{C}}(2)$. Now $\mathcal{O}_{\mathcal{C}}(1)$ is very ample by definition. So L is very ample and hence globally generated and so determines an embedding, Φ_L such that:

$$\Phi_L : \mathcal{C} \to \mathbb{P}^5$$

Also we have $\sigma_{|\mathcal{C}} : \mathcal{C} \hookrightarrow \mathbb{P}^5$. Hence we get the following diagram.

$$\begin{array}{ccc} & & \mathbb{P}^5 \\ & \overset{\Phi_L}{\nearrow} & \uparrow \sigma \\ \mathcal{C} & \longrightarrow & \mathbb{P}^2 \end{array}$$

We claim that the above diagram is commutative.

Notice that ${\Phi_L}^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathcal{C}}(2)$ and also $\sigma^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_{\mathbb{P}^2}(2)$. We have by definition of $\mathcal{C} \hookrightarrow \mathbb{P}^2$, that $\mathcal{O}_{\mathcal{C}}(1) = \mathcal{O}_{{\mathbb{P}^2}_{|\mathcal{C}}}(1)$. Hence we get that ${\Phi_L}^*(\mathcal{O}_{\mathbb{P}^5}(1)) = {\sigma_{|\mathcal{C}}}^*(\mathcal{O}_{\mathbb{P}^5}(1))$ and so, we get that the above diagram is commutative.

Remark-1: $\mathcal{C}$ is as above with degree d, then $(\mathcal{C}, L)$ satisfies Property N_0 for every $d \geq 2$.

Proof: Now $\mathcal{I}_{\mathcal{C}}$ is the ideal sheaf of $\mathcal{C}$ in $\mathbb{P}^5$ in $\mathbb{P}(H^0(L)))$. Then we have the following short exact sequence:

$$0 \to \mathcal{I}_{\mathcal{C}} \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_{\mathcal{C}} \to 0$$

So for every $n \in \mathbb{Z}$, we have

$$0 \to \mathcal{I}_{\mathcal{C}}(n) \to \mathcal{O}_{\mathbb{P}^5}(n) \to \mathcal{O}_{\mathcal{C}}(2n) \to 0$$

Also as $\mathcal{C} \hookrightarrow \mathbb{P}^2$, we get a map from $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \to H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n))$ for all $n \in \mathbb{Z}$. Let this map be γ_n.

To prove that $(\mathcal{C}, L)$ satisfies N_0. We have to prove that the map, $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(n)) \xrightarrow{\Phi_{Ln}} H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(2n))$ is surjective for all $n \in \mathbb{Z}$. Now we have the following commutative diagram.

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}^5}(n)) & & \\ \downarrow \tilde{\sigma}_n & \overset{\tilde{\Phi}_{Ln}}{\searrow} & \\ H^0(\mathcal{O}_{\mathbb{P}^2}(2n)) & \xrightarrow{\gamma_{2n}} & H^0(\mathcal{O}_{\mathcal{C}}(2n)) \end{array}$$

Claim 1: $H^0(\mathcal{O}_{\mathbb{P}^5}(n)) \xrightarrow{\tilde{\sigma}_n} H^0(\mathcal{O}_{\mathbb{P}^2}(2n))$ surjects for all n.
From [OP], we know that $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies Property N_p, $\forall d \geq p$ and $\forall n$. Hence we have that $\mathcal{O}_{\mathbb{P}^2}(2)$ satisfies N_0, and so we have that $\tilde{\sigma}_n$ surjects for all n.

Claim 2: γ_n surjects for all n.
As $\mathcal{C} \hookrightarrow \mathbb{P}^2$, we get the following short exact sequence.

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathcal{C}} \to 0$$

for every $n \in \mathbb{Z}$, we get the following long exact sequence:

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^2}(n-d) \to H^0(\mathcal{O}_{\mathbb{P}^2}(n)) \xrightarrow{\gamma_n} H^0(\mathcal{O}_{\mathcal{C}}(n)) \to$$

$$\to H^1(\mathcal{O}_{\mathbb{P}^2}(n-d)) \to H^1(\mathcal{O}_{\mathbb{P}^2}(n)) \to H^1(\mathcal{O}_{\mathcal{C}}(n) \to 0$$

But,

$$H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-d)) = 0 \qquad \forall n, d \in \mathbb{Z}$$

Hence γ_n surjects for all $n \in \mathbb{Z}$.

So we get that $\tilde{\Phi}_{Ln}$ surjects for all n. This implies that $(\mathcal{C}, L)$ satisfies Property N_0 for all plane curves, $\mathcal{C}$ with degree, $d \geq 2$.

Remark-2: If $(\mathcal{C}, L)$ as above, then L satisfies N_1 iff degree of $\mathcal{C} = 3$ or 4.

Proof: A very ample line bundle L is said to satisfy property N_1 if α_1 from (A) has degree 2 entries, implying that $\mathcal{I_C}$ is generated by quadrics.
Now if the curve $\mathcal{C}$ has even degree, $d = 2m$, then the degree of f is $2m$. And from lemma 1 we know that, $\mathcal{I_C} =< F, \Delta_{ij} >$ where, degree (F) is m and degree$(\Delta_{ij}) = 2$. Hence for $d = 4$, $\mathcal{I_C}$ is generated by quadrics, moreover for any even d, $d \neq 4$, $\mathcal{I_C}$ is cannot be generated by quadrics.

Now if the curve $\mathcal{C}$ has odd degree, $d = 2m - 1$, then the degree of f_i is $2m$ for $i = 0, 1, 2$. Now from lemma 3, we know that $\mathcal{I_C} =< F_0, F_1, F_2, \Delta_{ij} >$. where degree $(F_i) = m$ and degree$(\Delta_{ij}) = 2$. Hence for $d = 3$, $\mathcal{I_C}$ is generated by quadrics,and moreover for any odd d, $d \neq 3$, $\mathcal{I_C}$ is cannot be generated by quadrics.

Remark-3: Let $(\mathcal{C}, L)$ be as above, with degree$(\mathcal{C}) = 2m$, $m \geq 1$. Then $(\mathcal{C}, L)$ fails to satisfy Property N_2, and hence Property N_p, $p \geq 2$.

Proof: Let $d = 4$, then the matrix α_2 has degree 2 entries, hence the resolution is not linear. Hence such a $\mathcal{C}$ fails to satisfy Property N_2. And for $d \neq 4$ we know from result 2, that such a $\mathcal{C}$ fails to satisfy Property N_p for $p \geq 1$. Hence we have the above result.

Remark-4: Let $(\mathcal{C}, L)$ be as above, with degree$(\mathcal{C}) = 3$, then such a $\mathcal{C}$ satisfies Property N_3 but fails to satisfy Property N_4.

Proof: Notice that, if the degree of $\mathcal{C} = 2m - 1$, then degree$(h_i) = m - 1$, for $\forall i = II, III, IV$ and degree$(h_I) = m - 2.h_i$ for $i = I, II, III, IV$ as defined in Chapter 2 Now when the degree$(\mathcal{C}) = 3$, degree$(h_I) = 0$ and degree$(h_i) = 1$, for $\forall i = II, III, IV$, hence β_i has linear entries, for $i = 2, 3$. So we have that the resolution is linear till the third step while β_4 has quadratic entries, implies that the resolution is not linear in the fourth step, which implies that L satisfies N_3 but fails to satisfy N_4.

Remark-5 For all plane curve, $\mathcal{C}$, (degree ≥ 2), $\mathcal{O}_{\mathcal{C}}(2)$ fails to satisfy N_p for $p \geq 4$.

Proof: With all but degree 3 and 4 curves failing to satisfy N_1, degree 4 curve failing to satisfy N_2 and degree 3 curve failing to satify N_4, we get the above result.

3

Resolutions of Veronese embedding of complete intersections of curves in the plane.

If $\mathcal{C}$ and $\mathcal{C}'$ are two distinct plane curves, then consider, $\mathcal{C}\cap\mathcal{C}' \hookrightarrow \mathbb{P}^2 \overset{\sigma}{\hookrightarrow} \mathbb{P}^5$, where $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ is the Veronese embedding. We will compute the syzygies of the homogeneous ideal, $\mathcal{I}_{\sigma(\mathcal{C}\mathcal{C}')}$ of $\sigma\,(\mathcal{C}\cap\mathcal{C}')$ in $\mathbb{P}^5$.
Throughout we assume $(\mathcal{C}\cap\mathcal{C}')$ is reduced.

Now let $\mathcal{C}$ be defined by the polynomial f of degree d in three variables, and $\mathcal{C}'$ be defined by $\tilde{f}$ of degree d' in three variables. Hence

$$\mathcal{C} = Z(f(x_0,x_1,x_2)), \qquad \mathcal{C}' = Z(\tilde{f}(x_0,x_1,x_2))$$

Let us recall Theorem 2.1. and Theorem 2.2

Theorem Let $\mathcal{C}$ be an irreducible curve of even degree say $d=2m$, $m\geq 1$. The homogeneous ideal $\mathcal{I}_{\sigma(\mathcal{C})}$ of $\sigma(\mathcal{C})$ in $\mathbb{P}^5$ has the following minimal free graded resolution.

$$\begin{aligned} 0 \to & S(-m-4)^{\oplus 3} \overset{\alpha_4}{\to} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \overset{\alpha_3}{\to} \\ & \overset{\alpha_3}{\to} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \overset{\alpha_2}{\to} S(-2)^{\oplus 6} \oplus S(-m) \overset{\alpha_1}{\to} S \to S/\mathcal{I}_{\sigma(\mathcal{C})} \to 0 \end{aligned}$$

where

$$\alpha_1 = \left[\ [M_1], \quad F\ \right]$$

$$\alpha_2 = \left[\ [W_i, 0], \quad [U_{jk}]\ \right]$$

where $i=1,\ldots,8$ and $0\leq j\leq k\leq 2$

$$\alpha_3 = \left[\ [G_i', \bar{0}], \quad [H_j]\ \right]$$

where $i=1,2,3$ and $j=1,\ldots,8$.

$$\alpha_4 = \left[\ \begin{pmatrix} [-F.I_1^3] \\ [G_1] \end{pmatrix}, \ \begin{pmatrix} [-F.I_2^3] \\ [G_2] \end{pmatrix}, \ \begin{pmatrix} [-F.I_3^3] \\ [G_3] \end{pmatrix}\ \right]$$

Also when we consider the above resolution for the curve, $\mathcal{C}'$, we will denote the matrices in the resolution with '$\sim$'.
Before recalling Theorem 2.2, we introduce a change in the notations for V_i, K_i and J

appearing in Theorem 2.2 for the sake of convinence, so from now on we will denote $V_1 = [\ [V_{00}],\ \ [Y_{00}]\ ]$, $V_2 = [\ [V_{01}],\ \ [Y_{01}]\ ]$, $V_3 = [\ [V_{10}],\ \ [Y_{10}]\ ]$, $V_4 = [\ [V_{11}],\ \ [Y_{11}]\ ]$,

$V_5 = [\ [V_{12}],\ \ [Y_{12}]\ ]$, $V_6 = [\ [V_{20}],\ \ [Y_{20}]\ ]$, $V_7 = [\ [V_{21}],\ \ [Y_{21}]\ ]$, $V_8 = [\ [V_{22}],\ \ [Y_{22}]\ ]$.

$K_i = [\ [K_i''],\ \ [K_i']\ ]$, and, $J = [\ [J''],\ \ [J']\ ]$

Hence we have
Theorem: Let $\mathcal{C}$ be an irreducible curve of odd degree say $d = 2m - 1$, for $m \geq 2$. The ideal $\mathcal{I}_{\sigma(\mathcal{C})}$ of $\sigma(\mathcal{C})$ in $\mathbb{P}^5$ has the following minimal free graded resolution.

$$\begin{aligned} 0 \to & S(-m-4) \xrightarrow{\beta_4} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \xrightarrow{\beta_3} \\ & \xrightarrow{\beta_3} S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \xrightarrow{\beta_2} S(-2)^{\oplus 6} \oplus S^{\oplus 3}(-m) \xrightarrow{\beta_1} S \to S/\mathcal{I}_{\sigma(\mathcal{C})} \to 0 \end{aligned}$$

where

$$\beta_1 = [\ \Delta_{00},\ \ \Delta_{01},\ \ \Delta_{02},\ \ \Delta_{11},\ \ \Delta_{12},\ \ \Delta_{22},\ \ F_0,\ \ F_1,\ \ F_2,\ ]$$

$$\beta_2 = [\ [\ W_i,\ \ \bar{0}\ ],\ \ [\ [V_{jk}],\ \ [\mathbf{Y_{jk}}]\ ]\ ]$$

where $i = 1, \ldots, 8$ and $0 \leq j, k \leq 2$ with $(jk) \neq (02)$

$$\beta_3 = [\ [\ G_i,\ \ \bar{0}\ ],\ \ [\ [K_1,\ \ \mathbf{K'}_j\ ]\ ]$$

where $i = 1, 2, 3$, $j = 1, \ldots, 6$ and $\bar{0}$ is an appropriate dimensional zero matrix.

$$\beta_4 = [J'', \mathbf{J'}]$$

Note that all the matrices in bold print are independent of the curve considered. Also like in the case of theorem 2.2, we will denote the matrices occuring in the resolution of $\mathcal{C}'$ with a '$\sim$'

3.1 Degrees of $\mathcal{C}$ and $\mathcal{C}'$ are even.

In this case d is even(say $2m$), and $d' = 2m'$.

$$\begin{aligned} f &= \sum_{i+j+k=2m} a_{ijk} x_0^i x_1^j x_2^k \quad \text{and} \\ \tilde{f} &= \sum_{i+j+k=2m'} \tilde{a}_{ijk} x_0^i x_1^j x_2^k \end{aligned}$$

As the degrees of f and $\tilde{f}$ are even, from Lemma 2.1, we have that f , $\tilde{f} \in Im(\theta)$. Let $F, \tilde{F} \in S$ be homogeneous polynomials such that $\theta(F) = f$ and $\theta(\tilde{F}) = \tilde{f}$

Lemma 3.1: Let $G \in S$ such that G homogeneous and $Z(\theta(F)) \cap Z(\theta(\tilde{F})) \subset Z(\theta(G)) \subset \mathbb{P}^2$. Then $G \in < F, \tilde{F}, \Delta_{i,j} : 0 \leq i \leq j \leq 2 >$.

Proof: Let $\theta(G) = g$, then g is a homogeneous even degree polynomial and

$$Z(f) \cap Z(\tilde{f}) \subset Z(g)$$

$\Rightarrow g \in (f, \tilde{f})$ as $\mathcal{C}$ and $\mathcal{C}'$ are irreducible curves and hence f and $\tilde{f}$ are irreducible.

$\Rightarrow g = f.h + \tilde{f}.\tilde{h}$ for some h and $\tilde{h}$ homogeneous, in $K[x_0, x_1, x_2]$

Now as f, $\tilde{f}$ and g are even degree homogeneous polynomials we get that h and $\tilde{h}$ are both even degree polynomials hence $\exists$ H and $\tilde{H} \in S$, homogeneous such that $\theta(H) = h$ and $\theta(\tilde{H}) = \tilde{h}$.
Thus we have

$$\theta\Big(G - (F.H + \tilde{F}.\tilde{H})\Big) = 0$$

Hence $G - F.H - \tilde{F}.\tilde{H} \in ker(\theta)$

So we get $G - F.H - \tilde{F}.\tilde{H} = \sum_{0 \leq i \leq j \leq 2} \Delta_{ij} S_{ij}$ for some $S_{ij} \in S, S_{ij}$ homogeneous

$$\Rightarrow G \in < F, \tilde{F}, \Delta_{ij} : 0 \leq i \leq j \leq 2 >$$

This completes the proof of the lemma.

Theorem 3.1: Let $\mathcal{C}$, $\mathcal{C}'$ be two irreducible curves of even degree say $d = 2m$ and $d' = 2m'$, $m, m' \geq 1$. The homogeneous coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C} \cap \mathcal{C}')}$ of $\sigma(\mathcal{C} \cap \mathcal{C}')$ in $\mathbb{P}^5$ has the following minimal free graded resolution.

$$\begin{aligned} 0 \rightarrow & S(-m-m'-4)^{\oplus 3} \xrightarrow{\mathcal{P}_5} S(-m-4)^{\oplus 3} \oplus S(-m'-4)^{\oplus 3} \oplus S(-m-m'-3)^{\oplus 8} \xrightarrow{\mathcal{P}_4} \\ & \xrightarrow{\mathcal{P}_4} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \oplus S(-m'-3)^{\oplus 8} \oplus S(-m-m'-2)^{\oplus 6} \xrightarrow{\mathcal{P}_3} \\ & \xrightarrow{\mathcal{P}_3} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \oplus S(-m'-2)^{\oplus 6} \oplus S(-m-m') \xrightarrow{\mathcal{P}_2} \\ & \xrightarrow{\mathcal{P}_2} S(-2)^{\oplus 6} \oplus S(-m) \oplus S(-m') \xrightarrow{\mathcal{P}_1} S \rightarrow S/\mathcal{I}_{\sigma(\mathcal{C} \cap \mathcal{C}')} \rightarrow 0 \end{aligned} \tag{3.1}$$

where the matrices $\mathcal{P}_i$ are given as follows:

$$\mathcal{P}_1 = \begin{bmatrix} [M_1], & F, & \tilde{F} \end{bmatrix} \tag{3.2}$$

Let

$$P_2 = \begin{bmatrix} -F & 0 & 0 & 0 & 0 & 0 & \Delta_{00} & 0 \\ 0 & -F & 0 & 0 & 0 & 0 & \Delta_{01} & 0 \\ 0 & 0 & -F & 0 & 0 & 0 & \Delta_{02} & 0 \\ 0 & 0 & 0 & -F & 0 & 0 & \Delta_{11} & 0 \\ 0 & 0 & 0 & 0 & -F & 0 & \Delta_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & -F & \Delta_{22} & 0 \end{bmatrix}$$

Similarly we get

$$\tilde{P}_2 = \begin{bmatrix} -\tilde{F} & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{00} \\ 0 & -\tilde{F} & 0 & 0 & 0 & 0 & 0 & \Delta_{01} \\ 0 & 0 & -\tilde{F} & 0 & 0 & 0 & 0 & \Delta_{02} \\ 0 & 0 & 0 & -\tilde{F} & 0 & 0 & 0 & \Delta_{11} \\ 0 & 0 & 0 & 0 & -\tilde{F} & 0 & 0 & \Delta_{12} \\ 0 & 0 & 0 & 0 & 0 & -\tilde{F} & 0 & \Delta_{22} \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} U_{00}, & U_{01}, & U_{02}, & U_{11}, & U_{12}, & U_{22} \end{bmatrix}^T$$

We have $\tilde{U}_{ij}$ for $0 \le i \le j \le 2$ and hence $\tilde{P}_2$

Also let

$$\mathcal{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \tilde{F} & -F \end{bmatrix}^T$$

$$\mathcal{P}_2 = \begin{bmatrix} W_1', & W_2', & W_3', & W_4', & W_5', & W_6', & W_7', & W_8', & [P_2], & \left[\tilde{P}_2\right], & [\mathcal{S}] \end{bmatrix} \tag{3.3}$$

where

$$W_i' = \begin{bmatrix} W_i \\ 0 \\ 0 \end{bmatrix} \qquad \forall i = 1, \ldots, 8$$

with W_i as in equation (2.2) of Chapter 2.

Let

$$H_i = \begin{bmatrix} [F.I_i^8] \\ [W_i] \\ [\bar{0}] \\ 0 \end{bmatrix}$$

$$\tilde{H}_i = \begin{bmatrix} \left[\tilde{F}.I_i^8\right] \\ [\bar{0}] \\ [W_i] \\ 0 \end{bmatrix}$$

where $i = 1, \ldots 8$, $[\bar{0}]$ is a zero-matrix of appropriate dimension and

$$I_j^k = \begin{bmatrix} 0, & 0, & \ldots, & \overset{j^{th} \text{ position}}{1}, & 0, & \ldots, & 0 \end{bmatrix}^T \text{ is a } k \times 1 \text{ vector}$$

Let

$$L_{ij} = \begin{bmatrix} [\bar{0}] \\ \left[-\tilde{F} I_{2i+j+1}^6\right] \\ \left[F I_{2i+j+1}^6\right] \\ \Delta_{ij} \end{bmatrix} \qquad \forall \ \ 0 \le i \le j \le 2$$

And let

$$L = \begin{bmatrix} [L_{00}], & \ldots, & [L_{22}] \end{bmatrix}$$

$$\mathcal{P}_3 = \begin{bmatrix} [G'_i]_{1\leq i\leq 3}, & [H_j]_{1\leq j\leq 8}, & [\tilde{H}_j]_{1\leq j\leq 8}, & [L] \end{bmatrix} \tag{3.4}$$

where

$$G'_i = \begin{bmatrix} G_i \\ [\bar{0}] \end{bmatrix} \qquad \text{for } i = 1,2,3$$

where G_i as in equation(2.3) of Chapter 2 and $[\bar{0}]$ is a 0 matrix of appropriate dimension, and $j = 1,\ldots,8$.

We define

$$P_4 = \left[\begin{pmatrix} [-F.I_1^3] \\ [G_1] \\ [\bar{0}] \\ [\bar{0}] \end{pmatrix}, \begin{pmatrix} [-F.I_2^3] \\ [G_2] \\ [\bar{0}] \\ [\bar{0}] \end{pmatrix}, \begin{pmatrix} [-F.I_3^3] \\ [G_3] \\ [\bar{0}] \\ [\bar{0}] \end{pmatrix} \right]$$

$$\tilde{P}_4 = \left[\begin{pmatrix} \left[-\tilde{F}.I_1^3\right] \\ [\bar{0}] \\ [G_1] \\ [\bar{0}] \end{pmatrix}, \begin{pmatrix} \left[-\tilde{F}.I_2^3\right] \\ [\bar{0}] \\ [G_2] \\ [\bar{0}] \end{pmatrix}, \begin{pmatrix} \left[-\tilde{F}.I_3^3\right] \\ [\bar{0}] \\ [G_3] \\ [\bar{0}] \end{pmatrix} \right]$$

And let

$$\mathcal{W}_i = \begin{bmatrix} [\bar{0}] \\ \left[-\tilde{F}.I_i^8\right] \\ \left[F.I_i^8\right] \\ [W_i] \end{bmatrix} \qquad i = 1,\ldots,8.$$

Let

$$\mathcal{P}_4 = \begin{bmatrix} [P_4], & [P'_4], & [\mathcal{W}_1], & \ldots, & [\mathcal{W}_8] \end{bmatrix} \tag{3.5}$$

And

$$\mathcal{G}_i = \begin{bmatrix} \left[\tilde{F}I_i^3\right] \\ [-FI_i^3] \\ [G_i] \end{bmatrix}$$

$$\mathcal{P}_5 = \begin{bmatrix} [\mathcal{G}_1], & [\mathcal{G}_2], & [\mathcal{G}_3] \end{bmatrix} \tag{3.6}$$

Proof:
From Lemma 3.1, it is clear that

$$\mathcal{P}_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02}, & \Delta_{11}, & \Delta_{12}, & \Delta_{22}, & F, & \tilde{F} \end{bmatrix}$$

Now consider

$$A = \begin{bmatrix} A_{00}, & A_{01}, & A_{02}, & A_{11}, & A_{12}, & A_{22} \end{bmatrix}$$

where $a_{ij} \in S$, homogeneous. And B, $B' \in S$, homogeneous such that

$$\sum_{i,j} A_{ij}.\Delta_{ij} + B.F + \tilde{B}.\tilde{F} = 0$$

$$\Rightarrow \theta(B.F + \tilde{B}.\tilde{F}) = 0$$
$$\Rightarrow \theta(B).f = -\theta(\tilde{B})\tilde{f}$$

Now if $B = 0$, then we get $\theta(\tilde{B}) = 0$, hence $\tilde{B} \in <\Delta_{ij} : 0 \le i \le j \le 2>$
So we get, $\tilde{P}_2$. Similar reasoning for $\tilde{B} = 0$, gives us, P_2.
Now if B and $\tilde{B}$ both non-zero, we get,

$$\theta(B) \in <\tilde{f}> \text{ and } \theta(\tilde{B}) \in <f>$$

So let $\theta(B) = p.\tilde{f}$, then $\theta(\tilde{B}) = -p.f$, where $p \in k[x_0, x_1, x_2]$. Degree of p is even, therefore $\exists P \in S$, such that $\theta(P) = p$. Hence $[B, \tilde{B}] \in <\mathcal{S}>$. So we get that the relations between Δ_{ij}, F and $\tilde{F}$ are generated by $U_{ij} : 0 \le i \le j \le 2$, $\tilde{U}_{ij} : 0 \le i \le j \le 2$, $W'_k : k = 1, \ldots, 8$ and $\mathcal{S}$

Now we get

$$\mathcal{P}_2 = \left[\ [W'_i]_{1 \le i \le 8}, \quad [P_2], \quad \left[\tilde{P}_2\right], \quad [\mathcal{S}]\ \right]$$

for $i = 1, \ldots, 8$
Now consider

$$A = [A_k]_{1 \le k \le 8}, A_k \in S, \quad A_k \text{ homogeneous } \forall 1 \le k \le 8 \quad \text{and}$$

$$B = [\ (B_{ij})\ ], B_{ij} \in S, \text{ homogeneous}$$
$$\tilde{B} = [\ (\tilde{B}_{ij})\ ], \tilde{B}_{ij} \in S, \text{ homogeneous}$$

for $0 \le i \le j \le 2$, and $D \in S$ homogeneous, such that

$$\sum_{1 \le k \le 8} A_k.W'_k + \sum_{0 \le i \le j \le 2} B_{ij}.U_{ij} + \sum_{0 \le i \le j \le 2} \tilde{B}_{ij}.\tilde{U}_{ij} + D.\mathcal{S} = 0 \tag{3.7}$$

Now let $\tilde{B}_{ij} = 0$ for all i, j, so D=0. Then we have

$$B \in <W_k : k = 1, \ldots, 8>$$

Hence from Theorem 2.1, we get that the relations between W'_k and U_{ij} are generated by G'_i and H_k. Similarly, when $B_{ij} = 0$, for all i, j, we get that all relations between W'_k and $\tilde{U}_{ij}$ are generated by G'_i and $\tilde{H}_k$

Now if $B_{ij}, \tilde{B}_{kl} \ne 0$ for some i, j, k, l, then it is clear that $D \ne 0$ in (3.7) from the definitions of W'_i, U_{ij},and $\tilde{U}_{ij}$.
So we have

$$\sum_{ij} B_{ij}\Delta_{ij} + D.\tilde{F} = 0\ , \qquad \sum_{ij} \tilde{B}_{ij}\Delta_{ij} - D.F = 0$$

This implies that $D \in <\Delta_{ij} : 0 \le i \le j \le 2>$. So for some $C_{ij} \in S$, homogeneous we have $D = \sum_{ij} C_{ij}\Delta_{ij}$ and hence

$$\sum_{ij} \left(B_{ij} + C_{ij}.\tilde{F}\right)\Delta_{ij} = 0 \text{ and } \sum_{ij} \left(\tilde{B}_{ij} - C_{ij}.F\right)\Delta_{ij} = 0$$

If $B_{ij} - C_{ij}.\tilde{F} = 0$ for all i, j and $\tilde{B}_{ij} + C_{ij}.F = 0$ for all i, j, then $C_{ij}.\tilde{F} = B_{ij}$ and $C_{ij}.F = -\tilde{B}_{ij}$ for all i, j then such $(B_{ij}, \tilde{B}_{ij}, C_{ij})$ are generated by $<H_k, \tilde{H}_k, L_{ij}>$ for $0 \le i \le j \le 2$ and $k = 1, \ldots, 8$.
And if not then $\sum(\tilde{B}_{ij} + C_{ij}.F) \in \mathrm{Syz}^1(<W'_j : 1 \le j \le 8>)$.
Similarly $\sum(B_{ij} - C_{ij}.\tilde{F}) \in \mathrm{Syz}^1(<W'_j : 1 \le j \le 8>)$.
Hence the relations between $\{W'_k,\ U_{ij},\ \tilde{U}_{ij},\ J\ \}$are generated by $G'_k : k = 1, 2, 3$, H_i, $\tilde{H}_i : 1 \le i \le 8$ and $L_{jk} : 0 \le j \le k \le 2$.
Hence

$$\mathcal{P}_3 = [\ [G'_1], \quad [G'_2], \quad [G'_3], \quad [H_i], \quad [\tilde{H}_i] \quad [L]\ ]$$

for $i = 1, \ldots, 8$

Now consider

$A = \left[\ (A_i)\ \right]_{1 \le i \le 3}$ $A_i \in S$ homogeneous for $i = 1, 2, 3$

$B = \left[\ (B_j)\ \right]_{1 \le j \le 8}$ and
$\tilde{B} = \left[\ (\tilde{B}_j)\ \right]_{1 \le j \le 8}$, where B_i and $\tilde{B}_i$ homogeneous in S for $i = 1, \ldots, 8$,

$C = \left[\ (C_{kl})\ \right] 1 \le k \le l \le 2$,where $C_{ij} \in S$ homogeneous for $0 \le k \le l \le 2$ such that

$$\sum_i A_i.G'_i + \sum_i B_i.H_i + \sum_i \tilde{B}_i.\tilde{H}_i + \sum_{i,j} C_{ij}.L_{ij} = 0$$

Now if $\left[\tilde{B}\right] = [\bar{0}]$, then $C = [0]$, hence we have
$\sum_i B_i W_i = 0$ then, $B \in < G_p : p = 1, 2, 3 >$ Now theorem 2.2, we get P_4. Similarly we get $\tilde{P}_4$, when $B_{ij} = 0$ for all (i, j)
If $B \neq [0]$ and $\tilde{B} \neq [0]$, then we have that $C_{ij} \neq 0$ for some i, j. So we get that $\sum_{i,j}(C_{ij}.\Delta_{ij}) = 0$. This implies that $C \in < W_k >$, then with similar arguments as in the proof of theorem 2.2, we get $\mathcal{W}_i$ for $i = 1, \ldots, 8$.

Hence we get

$$\mathcal{P}_4 = \left[\ [P_4],\ \ \left[\tilde{P}_4\right],\ \ [\mathcal{W}_1],\ \ \ldots,\ \ [\mathcal{W}_8]\ \right]$$

Now let

$$B = \left[\ (B_i)\ \right]$$
$$\tilde{B} = \left[\ (\tilde{B}_i)\ \right]$$
$$A = \left[\ (A_j)\ \right]$$

such that for $i = 1, 2, 3$, B_i,$\tilde{B}_i$ are homogeneous in S, and for $j = 1, \ldots, 8$, A_j are homogeneous in S

$$B.P_4 + \tilde{B}.\tilde{P}_4 + \sum_i A_i.\mathcal{W}_i = 0$$

Then we get that $\sum_i A_i.W_i = 0$, this implies that $A \in < G_k : k = 1, 2, 3 >$, with the same arguments as earlier we get, $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$. Hence

$$\mathcal{P}_5 = \left[\ [\mathcal{G}_1],\ \ [\mathcal{G}_2],\ \ [\mathcal{G}_3]\ \right]$$

3.2 Degree of $\mathcal{C}$ is even and degree of $\mathcal{C}'$ is odd

let $d = 2m$, and $d' = 2m' - 1$

$$\begin{aligned} f &= \sum_{i+j+k=2m} a_{ijk} x_0^i x_1^j x_2^k \quad \text{and ,} \\ \tilde{f} &= \sum_{i+j+k=2m'-1} \tilde{a}_{ijk} x_0^i x_1^j x_2^k \end{aligned}$$

As the degree of f is even from Lemma 2.1, we have that $f \in Im(\theta)$. Also from Lemmas 2.2 and 2.3, we also know that for $\tilde{f}$ with odd degree, we have $\tilde{f}_i = x_i.\tilde{f}$ for $0 \le i \le 2$ such that $Z(\tilde{f}) = \cap_{i=0}^{2} Z(\tilde{f}_i)$ and that each $\tilde{f}_i \in Im(\theta)$ for $i = 0, 1, 2$.
Like in Chapter 2 we also have, $\tilde{h}_I$, $\tilde{h}_{II}$, $\tilde{h}_{III}$, and $\tilde{h}_{IV}$ such that

$$\tilde{F}_0 = x_{00} x_{12} \tilde{h}_I + x_{00} \tilde{h}_{II} + x_{01} \tilde{h}_{III} + x_{02} \tilde{h}_{IV}$$

$$\begin{aligned} \tilde{F}_1 &= x_{11} x_{02} \tilde{h}_I + x_{01} \tilde{h}_{II} + x_{11} \tilde{h}_{III} + x_{12} \tilde{h}_{IV} \\ \tilde{F}_2 &= x_{22} x_{01} \tilde{h}_I + x_{02} \tilde{h}_{II} + x_{12} \tilde{h}_{III} + x_{22} \tilde{h}_{IV} \end{aligned}$$

Lemma 3.2: Let $G \in S$ such that G homogeneous and

$$Z(\theta(F)) \cap Z(\theta(\tilde{F}_0)) \cap Z(\theta(\tilde{F}_1)) \cap Z(\theta(\tilde{F}_2)) \subset Z(\theta(G)) \subset \mathbb{P}^2$$

. Then $G \in < F, \tilde{F}_k, \Delta_{i,j} : 0 \le i \le j \le 2 \ , \ k = 0, 1, 2 >$.
Proof:Let $\theta(G) = g$, then g is a homogeneous even degree polynomial and,

$$Z(f) \cap Z(\tilde{f}_0) \cap Z(\tilde{f}_1) \cap Z(\tilde{f}_2) \subset Z(g)$$

$$Z(f) \cap Z(\tilde{f}) \subset Z(g)$$

$\Rightarrow g \in (f, \tilde{f})$ as $\mathcal{C}$ and $\mathcal{C}'$ are irreducible curves and by assumption(i.e. $\mathcal{C} \cap \mathcal{C}'$ is reduced. So

$$g = f.h + \tilde{f}.\tilde{h} \text{ for some } h \text{ and } \tilde{h} \text{ homogeneous in } K[x_0, x_1, x_2]$$

Now as f and g are even degree homogeneous polynomials and $\tilde{f}$ is homogeneous of odd degree, we get that degree of h is even and $\tilde{h}$ is a odd degree polynomial hence, there exists $H \in S$, homogeneous such that $\theta(H) = h$ and $\tilde{h} = \sum_i \tilde{h}_i x_i$, where $\tilde{h}_i \in K[x_0, x_1, x_2]$, $\tilde{h}_i$ homogeneous of even degree.
So there exists $\tilde{H}_i$s such that $\theta(\tilde{H}_i) = \tilde{h}_i$ for $i = 0, 1, 2$
Thus we have

$$\theta\left(G - \left(F.H + \sum_i (\tilde{F}_i.\tilde{H}_i)\right)\right) = 0$$

$$\text{hence, } G - \left(F.H + \sum_i (\tilde{F}_i.\tilde{H}_i)\right) \in ker(\theta)$$

So we get

$$G - \left(F.H + \sum_i (\tilde{F}_i.H'_i)\right) = \sum_{0 \le i \le j \le 2} \Delta_{ij} S_{ij} \text{ for some } S_{ij} \in S, S_{ij} \text{ homogeneous}$$

$$\Rightarrow G \in < F, \tilde{F}_k, \Delta_{ij} : 0 \le i \le j \le 2 \text{ and } k = 0, 1, 2 >$$

This completes the proof of the lemma.

Theorem 3.2: Let $\mathcal{C}$, $\mathcal{C}'$ be two irreducible curves of degrees say $d = 2m$ and

$d' = 2m' - 1$, $m, m' \geq 2$. Then the homogeneous coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$ of $(\sigma(\mathcal{C}) \cap \sigma(\mathcal{C}'))$ in $\mathbb{P}^5$ has the following minimal free graded resolution.

$$\begin{aligned} 0 \to & S(-m-m'-4) \overset{Q_5}{\to} S(-m-4)^{\oplus 3} \oplus S(-m'-4) \oplus S(-m-m'-2)^{\oplus 6} \overset{Q_4}{\to} \\ & \overset{Q_4}{\to} S(-4)^{\oplus 3} \oplus S(-m-3)^{\oplus 8} \oplus S(-m'-2)^{\oplus 6} \oplus S(-m-m'-1)^{\oplus 8} \overset{Q_3}{\to} \\ & \overset{Q_3}{\to} S(-3)^{\oplus 8} \oplus S(-m-2)^{\oplus 6} \oplus S(-m'-1)^{\oplus 8} \oplus S(-m-m')^{\oplus 3} \overset{Q_2}{\to} \\ & \overset{Q_2}{\to} S(-2)^{\oplus 6} \oplus S(-m) \oplus S(-m')^{\oplus 3} \overset{Q_1}{\to} S \to S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')} \to 0 \end{aligned} \tag{3.8}$$

Proof:
From Lemma 3.2, we get that

$$Q_1 = \begin{bmatrix} [M_1], & F, & \tilde{F}_0, & \tilde{F}_1, & \tilde{F}_2 \end{bmatrix} \tag{3.9}$$

Now let

$A = \begin{bmatrix} (A_{ij}) \end{bmatrix}$, $A_{ij} \in S$, homogeneous for $0 \leq i \leq j \leq 2$ and,
$B \in S$, homogeneous, $\tilde{B} = \begin{bmatrix} \tilde{B}_0, & \tilde{B}_1, & \tilde{B}_2 \end{bmatrix}$, $\tilde{B}_i \in S$, homogeneous for $i = 0, 1, 2$, such that

$$\sum_{ij}(A_{ij}.\Delta_{ij}) + B.F + \sum_i(\tilde{B}_i.\tilde{F}_i) = 0$$

Now if $\tilde{B}_i = 0$, for all $i = 0, 1, 2$, then we have,

$$\sum_{ij}(A_{ij}.\Delta_{ij}) + B.F = 0$$

By theorem 2.1 we get that
$[\ [A_{ij}], B\ ] \in <[\ [W_i], 0], [U_{jk}]\ \ : i = 1, \ldots, 8,\ 0 \leq j \leq k \leq 2>$.
Hence we get that
$[[A_{ij}], B, [\bar{0}]] \in < [[W_i], 0, [\bar{0}]]\,, [[U_{jk}], [\bar{0}]] : i = 1, \ldots, 8,\ 0 \leq j \leq k \leq 2 >$.

Similarly if $B = 0$, by theorem 2.2 we get that
$[[A_{ij}], 0, [\tilde{B}_k]] \in < [[W_i], [\bar{0}]]\,, [\tilde{V}_j] : i, j = 1, \ldots, 8 >$.

Now we have,

$$\Big[[A_{ij}], \quad 0, \quad [\tilde{B}_k]\Big] \in \Big\langle \begin{bmatrix} [W_i], & [\bar{0}] \end{bmatrix}, \quad \begin{bmatrix} [V_{jk}], & 0, & [\mathbf{Y_{jk}}] \end{bmatrix} \Big\rangle$$

for $i = 1, \ldots, 8$ and $0 \leq j, k \leq 2, (j, k) \neq (0, 2)$.

Now let $B \neq 0$ and $\tilde{B}_i \neq 0$ for some i.
Then we have,

$$b.f + \sum_i(\tilde{b}_i.\tilde{f}_i) = 0$$

where $b = \theta(B)$ and $\tilde{b}_i = \theta(\tilde{B}_i)$ for $i = 0, 1, 2$
Hence we have, $b \in < \tilde{f} >$ and $\sum_i(b'_i.x_i) \in < f >$,
but the degree of b is even, so $b \in < x_i.\tilde{f} : i = 0, 1, 2 >$. So we get,
$B = \sum_i C_i.\tilde{F}_i$.
This gives us that

$$\begin{bmatrix} B, & [\tilde{B}_0, & \tilde{B}_1, & \tilde{B}_2] \end{bmatrix} \in \Big\langle \begin{bmatrix} \tilde{F}_i, & [-F.I_i^3] \end{bmatrix} : \quad i = 0, 1, 2 \Big\rangle$$

Let $L_i = \begin{bmatrix} [\bar{0}]_6, & \tilde{F}_i, & [-F.I_i^3] \end{bmatrix}$, where $[\bar{0}]_i$ is a $1 \times i$ zero-vector. Hence we get that

$$\Big[[A_{ij}], B, [\tilde{B}_k]\Big] \in \Big\langle \begin{bmatrix} [W_i], & 0, & [\bar{0}] \end{bmatrix}, \quad \begin{bmatrix} [U_{jk}], & [\bar{0}] \end{bmatrix}, \quad \begin{bmatrix} [\tilde{V}_{ln}], [\bar{0}], [Y_{ln}] \end{bmatrix}, \quad [L_s] \Big\rangle$$

for $i = 1, \ldots, 8$, $0 \leq j \leq k \leq 2$, $0 \leq l, n \leq 2$ ($(l, n) \neq (0, 2)$) and $s = 0, 1, 2$

Hence we get Q_2.

Now let
$A = \begin{bmatrix} (A_i) & _{1 \leq i \leq 8} \end{bmatrix}^T$, $A_i \in S$, homogeneous for $i = 1, \ldots, 8$
$B = \begin{bmatrix} (B_{ij}) & _{0 \leq i \leq j \leq 2} \end{bmatrix}^T$, $B_{ij} \in S$, homogeneous for $0 \leq i \leq j \leq 2$
$\tilde{B} = \begin{bmatrix} (\tilde{B}_{ij}) & _{0 \leq i \leq j \leq 2} \end{bmatrix}^T$, $\tilde{B}_{ij} \in S$, homogeneous for $0 \leq i, j \leq 2$,
$C = \begin{bmatrix} C_0, & C_1, & C_2 \end{bmatrix}^T$, $C_i \in S$, homogeneous for $i = 0, 1, 2$.
such that

$$\begin{aligned}\sum_{i=1,\ldots,8} A_i \begin{bmatrix} [W_i], & 0, & [\bar{0}]_3 \end{bmatrix} + \sum_{0 \leq i \leq j \leq 2} B_{ij} \begin{bmatrix} [U_{ij}], & [\bar{0}]_3 \end{bmatrix} \\ + \sum_{ij} \tilde{B}_{ij} \begin{bmatrix} [\tilde{V}_{ij}], & 0, & [\mathbf{Y_{ij}}] \end{bmatrix} + \sum_{i=0,1,2} C_i . L_i = 0\end{aligned} \tag{3.10}$$

Consider the following cases:

(1)Let $B = [\bar{0}]$, $\tilde{B} = [\bar{0}]$ and $C = [\bar{0}]$, then $A \in < G_i : i = 1, 2, 3 >$. Hence

$$\begin{bmatrix} [A_{ij}], & 0, & [\bar{0}], & [\bar{0}] \end{bmatrix} \in < \begin{bmatrix} [G_i], & 0, & [\bar{0}], & [\bar{0}] \end{bmatrix} >$$

for $i = 1, 2, 3$

(2)Let $\tilde{B} = [\bar{0}]$, $C = [\bar{0}]$, but $B \neq [\bar{0}]$, then theorem 2.1 we get, $[[A], [B]] \in < [G_i, 0], [H_j] : i = 1, 2, 3, j = 1, \ldots, 8 >$. Hence,

$$\begin{bmatrix} [A_{ij}], & B, & [\bar{0}], & [\bar{0}] \end{bmatrix} \in \left\langle \begin{bmatrix} [G_i], & [\bar{0}], & [\bar{0}], & [\bar{0}] \end{bmatrix}, \begin{bmatrix} [H_j], & [\bar{0}], & [\bar{0}] \end{bmatrix} \right\rangle$$

for $i = 1, 2, 3$ and $j = 1, \ldots, 8$

(3)Let $B = 0$, $C = [\bar{0}]$, but $\tilde{B} \neq [\bar{0}]$, then like the previous case we get, $[[A], [\tilde{B}]] \in < [G_i, 0], [\tilde{K}_j] : i = 1, 2, 3\ j = 1, \ldots, 6 >$. Hence

$$\begin{bmatrix} [A], & [\bar{0}], & [\tilde{B}], & [\bar{0}] \end{bmatrix} \in \left\langle \begin{bmatrix} [G_i], & [\bar{0}], & [\bar{0}] \end{bmatrix}, \begin{bmatrix} [\tilde{K''}_j], & [\bar{0}], & [\mathbf{K'}_j], & [\bar{0}] \end{bmatrix} \right\rangle$$

for $i = 1, 2, 3$ and $j = 1 \ldots 6$

(4)Let $B \neq [\bar{0}]$ and $\tilde{B} \neq [\bar{0}]$, then we have ,

$$\sum_{0 \leq i \leq j \leq 2} (B_{ij} \Delta_{ij}) + \sum_{i=0,1,2} (C_i . \tilde{F}_i) = 0$$

Hence

$$\sum_{i=0,1,2} (c_i . \tilde{f}_i) = 0$$

where $c_i = \theta(C_i)$ for all $i = 0, 1, 2$
So we get that $\begin{bmatrix} C_0, & C_1, & C_2 \end{bmatrix} \in \langle \mathbf{Y_{ij}} : 0 \leq i, j \leq 2, (i, j) \neq (0, 2) \rangle$. Hence,

$$[C] = \sum_{k,l} D_{kl}[Y_{kl}] \text{ where } D_{kl} \in S, \text{ homogeneous for, } 0 \leq i, j \leq 2, \quad (l, k) \neq (0, 2)$$

So

$$\sum_{ij} (B_{ij}).[\mathbf{Y_{ij}}] = \sum_{ij} F(D_{ij})[\mathbf{Y_{ij}}]$$

Now if, $B_{ij} - F.D_{ij} = 0$ for all i, j, then $([B_{ij}], [C]) \in < [F.I]^8_k, [Y_{kl}] >$,
where $k = 2i + j + 1$ if $i = 0, 1$ and $k = 6 + j$ for $i = 2$.
Hence we get that $([A], [B], [\tilde{B}], [C]) \in \langle [\ [\bar{0}], \quad [\tilde{V}_{ij}], \quad [F.I]^8_k, \quad [\mathbf{Y_{ij}}]\ ]\rangle$. Define,

$$\mathcal{V}_{ij} = [\ [\bar{0}], \quad -V_{ij}, \quad [F.I^8_k], \quad [\mathbf{Y_{ij}}]\ ],$$

for $i, j = 0, 1, 2$, and for $k = 2i + j + 1$, if $i \neq 2$ and $k = 6 + j$ if $i = 2$.
If not then, $[(B_{ij} - F.D_{ij})] \in < K'_l : 1 \le l \le 6 > (\ \mathrm{Syz}^1(< \mathbf{Y_{ij}} >))$.
Hence we get Q_3.
Let
$A = \Big[\ (A_i)\ _{1\le i\le 3}\Big]^T, A_i \in S$, homogeneous for $i = 1, 2, 3$
$B = \Big[\ (B_i)\ _{1\le i\le 8}\Big]^T, B_i \in S$, homogeneous for $i = 1, \dots 8$
$\tilde{B} = \Big[\ (\tilde{B}_i)\ _{1\le i\le 3}\Big]^T, \tilde{B}_i \in S$, homogeneous for $i = 1, \dots, 6$,
$C = \Big[\ C_{ij}\ _{0\le i,j\le 2}\Big]^T, C_{ij} \in S$, homogeneous for $i, j = 0, 1, 2$.
such that

$$\sum_{i=1,2,3} A_i [\ [G_i], \quad [\bar{0}]_6, \quad [\bar{0}]_8, \quad [\bar{0}]_3\ ] + \sum_{0\le i\le j\le 2} B_{ij} [\ [H_i], \quad [\bar{0}]_8, \quad [\bar{0}]_3\ ] + \sum_i \tilde{B}_i [\ [\tilde{K''}_i], \quad [\bar{0}]_8, \quad [\mathbf{K'_i}], \quad [\bar{0}]_3\ ] + \sum_{i,j} C_{ij}.[\mathcal{V}_{ij}] = 0 \tag{3.11}$$

Consider the following cases,
(1)$\tilde{B} = [\bar{0}]$, hence $B \neq [\bar{0}]$ and $C = [\bar{0}]$, then from Theorem 2.1, we get that

$$[[A], [B]] \in \big\langle ([-F.I^3_i], [G_i]) \qquad : i = 1, 2, 3 \big\rangle$$

Hence $\Big[[A], [B], [\tilde{B}], [C]\Big] \in \big\langle [\ [-F.I^3_i], \quad [G_i], \quad [\bar{0}], \quad [\bar{0}]\ ] : i = 1, 2, 3 \big\rangle$
Denote $[\ [-F.I^3_i], \quad [G_i], \quad [\bar{0}], \quad [\bar{0}]\ ]$ as $[\Im_i]$ for $i = 1, 2, 3$

(2)$B, C = [\bar{0}]$ and $\tilde{B} \neq [\bar{0}]$, then from theorem 2.1, we get that $\Big[[A], [\tilde{B}]\Big] \in \Big\langle [\tilde{J}] \Big\rangle$

Hence $\Big[[A], [B], [\tilde{B}], [C]\Big] \in \Big\langle \Big[[\tilde{J''}], [\bar{0}], [\mathbf{J'}], [\bar{0}]\Big] \Big\rangle$

(3)$C \neq [\bar{0}]$, then we have, $\sum_{ij} C_{ij}.[\mathbf{Y_{ij}}] = 0$, hence $C \in < [\mathbf{K'_i}] : i = 1, \dots, 6 >$.

Hence we have

$$\Big[[A], [B], [\tilde{B}], [C]\Big] \in < \Big[[\bar{0}], [\tilde{K''}_i], [-F.I^6_i], [\mathbf{K'}_i]\Big] : i = i, \dots, 6 >$$

Lets denote the above set of vectors as $\tilde{\mathcal{K}}_i$, $i = 1, \dots, 6$
Hence we have,

$$[\ [A], \quad [B], \quad [\tilde{B}], \quad [C]\ ] \in < \Big(\ [\Im_i], \quad [\ \tilde{J''}, \quad [\bar{0}], \quad [J'], \quad [\bar{0}]\ ], \quad \Big[\tilde{\mathcal{K}}_j\Big]\ \Big) >$$

$i = 1, 2, 3$, and $j = 1, \dots, 6$

Hence we get Q_4.

Let
$A = [\ (A_i)\ ], A_i \in S$, homogeneous for $i = 1, 2, 3$
$B \in S$, homogeneous. $C = [\ (C_i)\ ], C_i \in S$, homogeneous for $i = 1, \dots, 6$, such that

$$\sum_{i=1,2,3} A_i [\Im_i] + B.J + \sum_i C_i [\tilde{\mathcal{K}}_i] = 0 \tag{3.12}$$

From theorem 2 and 3 in [A], we get that if $[C] = [\bar{0}]$ then $[A]$ and $[B]$ are also equal to $[\bar{0}]$. So $[C] \neq [\bar{0}]$, then we have
$\sum_i C_i.\mathbf{K'}_i = 0$, hence $[C] \in < \mathbf{J'} >$.
Hence we have $\left[\; [A], \quad B, \quad [C] \; \right] \in \left\langle \left[\; [-\tilde{J}''], \quad -F, \quad [\mathbf{J}'] \; \right] \right\rangle$.

Hence we get Q_5.

3.3 Degrees of $\mathcal{C}$ and $\mathcal{C}'$ are odd

Let the degrees of f,$\tilde{f}$ be $2m-1$ and $2m'-1$ respectively. Then we have,

$$f = \sum_{i+j+k=2m-1} a_{ijk}x_0^i x_1^j x_2^k \quad \text{and,}$$
$$\tilde{f} = \sum_{i+j+k=2m'-1} \tilde{a}_{ijk}x_0^i x_1^j x_2^k$$

Now let $f_0 = x_0.f$, $f_1 = x_1.f$, $f_2 = x_2.f$. Similary define $\tilde{f}_i$ for $i = 0, 1, 2$
Then f_i and $\tilde{f}_i$ are of even degree and hence according to Lemma 2.1, f_i , $\tilde{f}_i \in Im(\theta)$ for $i = 0, 1, 2$. Also like in Section 3.2, we have,

$$F_0 = x_{00}x_{12}h_I + x_{00}h_{II} + x_{01}h_{III} + x_{02}h_{IV}$$

$$F_1 = x_{11}x_{02}h_I + x_{01}h_{II} + x_{11}h_{III} + x_{12}h_{IV}$$
$$F_2 = x_{22}x_{01}h_I + x_{02}h_{II} + x_{12}h_{III} + x_{22}h_{IV}$$

Lemma 3.4: Let $G \in k[x_{00}, x_{01}, x_{02}, x_{11}, x_{12}, x_{22}]$ such that G homogeneous and $(\cap_i Z(\theta(F_i)))\cap(\cap_i Z(\theta(\tilde{F}_i))) \subset Z(\theta(G)) \subset \mathbb{P}^2$. Then $G \in< F_k, \tilde{F}_k, \Delta_{i,j} : 0 \le k \le 2, 0 \le i \le j \le 2 >$.
Proof: Now let $\theta(G) = g$, then degree(g) is even.

$$(\cap_i Z(f_i)) \cap (\cap_i Z(\tilde{f}_i)) \subset Z(g)$$

$$\Rightarrow Z(f) \cap Z(\tilde{f}) \subset Z(g)$$

$\Rightarrow g \in< f, \tilde{f} >$ as $\mathcal{C}$ and $\mathcal{C}'$ are irreducible curves and the assumption about the intersection of $\mathcal{C}$ and $\mathcal{C}'$.

$$\Rightarrow g = f.h + \tilde{f}.\tilde{h} \text{ for some } h, \tilde{h} \text{ homogeneous } \in k[x_0, x_1, x_2]$$
$$\Rightarrow h \neq 1 \text{ as degree } f \text{ is odd while degree } g \text{ is even}$$

Similarly $\tilde{h} \neq 1$, hence, $g = \sum_{i=0,1,2} f_i h_i + \sum_{i=0,1,2} \tilde{f}_i \tilde{h}_i$,
for some homogeneous polynomials $h_i, \tilde{h}_i \in k[x_0, x_1, x_2]$ with even degrees.

$$\Rightarrow G = \sum_{i=0,1,2} F_i H_i + \sum_{i=0,1,2} \tilde{F}_i \tilde{H}_i, \text{ where } \theta(H_i) = h_i \text{ and } \theta(\tilde{H}_i) = \tilde{h}_i,$$

for all $i = 0, 1, 2$ and such H_is, and $\tilde{H}_i$s exists as the degrees of both h_i and $\tilde{h}_i$ are even from Lemma 2.1.

$$\Rightarrow \theta\left(G - \sum_{i=0,1,2} F_i H_i + \sum_{i=0,1,2} \tilde{F}_i \tilde{H}_i\right) = 0$$

$$\Rightarrow G - \left(\sum_{i=0,1,2} F_i H_i + \sum_{i=0,1,2} \tilde{F}_i \tilde{H}_i\right) \in ker(\theta)$$
$$\Rightarrow G = \sum_{i=0,1,2} F_i H_i + \sum_{i=0,1,2} \tilde{F}_i \tilde{H}_i + \sum_{i,j=0,1,2} \Delta_{ij} S_{ij}$$

for some S_{ij} homogeneous $\in k[x_{00}, \ldots, x_{22}]$

$$\Rightarrow G \in< F_k, \tilde{F}_k, \Delta_{ij} : i, j, k = 0, 1, 2 >$$

Theorem 3.3: Let $\mathcal{C}$ and $\mathcal{C}'$ be two irreducible plane curves of odd degree say $d =$

$2m-1$ and $d' = 2m'-1$ for $m, m' \geq 2$. The homogenous coordinate ring $S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')}$ of the intersection of $\sigma(\mathcal{C})$ and $\sigma(\mathcal{C}')$ in $\mathbb{P}^5$ has the following minimal free graded resolution.

$$\begin{aligned} 0 \to & S(-m-m'-3)^{\oplus 3} \overset{R_5}{\to} S(-m-4) \oplus S(-m'-4) \oplus S(-m-m'-2)^{\oplus 8} \overset{R_4}{\to} \\ \overset{R_4}{\to} & S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \oplus S(-m'-2)^{\oplus 6} \oplus S(-m-m'-1)^{\oplus 6} \overset{R_3}{\to} \\ \overset{R_3}{\to} & S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \oplus S(-m'-1)^{\oplus 8} \oplus S(-m-m'+1) \overset{R_2}{\to} \\ \overset{R_2}{\to} & S(-2)^{\oplus 6} \oplus S(-m)^{\oplus 3} \oplus S(-m')^{\oplus 3} \overset{R_1}{\to} S \to S/\mathcal{I}_{\sigma(\mathcal{C}\cap\mathcal{C}')} \to 0 \end{aligned} \tag{3.13}$$

Proof:
From lemma 3.2 and 3.4, it is clear that

$$R_1 = \begin{bmatrix} \Delta_{00}, & \Delta_{01}, & \Delta_{02}, & \Delta_{11}, & \Delta_{12}, & \Delta_{22}, & F_0, & F_1, & F_2, & \tilde{F}_0, & \tilde{F}_1, & \tilde{F}_2 \end{bmatrix}$$

Hence we get R_1
Now consider
$A = \begin{bmatrix} A_{00}, & A_{01}, & A_{02}, & A_{11}, & A_{12}, & A_{22} \end{bmatrix}$, $A_{ij} \in S$, homogeneous $\forall 0 \leq i \leq æ \leq 2$,
$B = \begin{bmatrix} B_0, & B_1, & B_2 \end{bmatrix}$ where $B_k \in S$, homogeneous, for $k = 0,1,2$ and
$\tilde{B} = \begin{bmatrix} \tilde{B}_0, & \tilde{B}_1, & \tilde{B}_2 \end{bmatrix}$ where $\tilde{B}_l \in S$, homogeneous, for $l = 0,1,2$
such that

$$\sum_{i,j} A_{ij}.\Delta_{ij} + \sum_k B_k.F_k + \sum_k \tilde{B}_k.\tilde{F}_k = 0 \tag{3.14}$$

Consider the following cases:

(1)$[B] = [\tilde{B}] = [\bar{0}]$, then we get that $[A] \in < W_i : i = 1, \ldots, 8 >$. Hence we get

$$\begin{bmatrix} [A], & [\bar{0}], & [\bar{0}] \end{bmatrix} \in \left\langle \begin{bmatrix} W_1, & \bar{0}, & \bar{0} \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} W_8, & \bar{0}, & \bar{0} \end{bmatrix} \right\rangle$$

(2)$[B] \neq [\bar{0}]$, but $[\tilde{B}] = [\bar{0}]$, then from [A] we get that

$$\begin{bmatrix} [A], & [B], & [\bar{0}] \end{bmatrix} \in \left\langle \begin{bmatrix} V_{00}, & Y_{00}, & \bar{0} \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} V_{22}, & Y_{22}, & \bar{0} \end{bmatrix}, \right\rangle$$

(3)Similarly for $[B] = [\bar{0}]$, but $[\tilde{B}] \neq [\bar{0}]$, we get that

$$\begin{bmatrix} [A], & [B], & [\bar{0}] \end{bmatrix} \in \left\langle \begin{bmatrix} \tilde{V}_{00}, & \bar{0}, & \mathbf{Y_{00}} \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} \tilde{V}_{22}, & \bar{0}, & \mathbf{Y_{22}} \end{bmatrix}, \right\rangle$$

(4)$B, \tilde{B} \neq [\bar{0}]$, hence we get that

$$\theta\left(\sum_k \left((B_k.F_k) + (\tilde{B}_k.\tilde{F}_k)\right)\right) = 0$$

Let $b_k = \theta(B_k)$ and $\tilde{b}_k = \theta(\tilde{B}_k)$. Now note that the degrees of b_k and $\tilde{b}_k$ are even, for $k = 0,1,2$
Hence we have that

$$\sum_k (b_k.x_k).f + \sum_k (\tilde{b}_k.x_k).\tilde{f} = 0$$

Now as f and $\tilde{f}$ are irreducible polynomials and by assumption that $\mathcal{C} \cap \mathcal{C}'$ is reduced we get that

$$\sum_k (b_k.x_k) \in < \tilde{f} > \text{ and } \sum_k (\tilde{b}_k.x_k) \in < f > \tag{3.15}$$

To get b_k and $\tilde{b}_k$ satisfying the above equation, consider the two vectors ,

$$(h_0, h_1, h_2) = \big(\; x_1.x_2.\theta(\tilde{h}_I) + \theta(\tilde{h}_{II}), \quad \theta(\tilde{h}_{III}), \quad \theta(\tilde{h}_{IV}) \; \big)$$

$$\left(\tilde{h}_0, \tilde{h}_1, \tilde{h}_2\right) = \big(\; -x_1.x_2.\theta(h_I) - \theta(h_{II}), \quad -\theta(h_{III}), \quad -\theta(h_{IV}) \; \big)$$

And let [H] $= \big[\; x_{12}.\tilde{h}_I, \quad \tilde{h}_{II}, \quad \tilde{h}_{III}, \quad \tilde{h}_{IV} \; \big]$ and [H̃] $= \big[\; -x_{12}.h_I - h_{II}, \quad -h_{III}, \quad -h_{IV} \; \big]$

Now substituting h_i as b_i and $\tilde{h}_i$ as $\tilde{b}_i$, we get that $\sum_i (h_i.x_i) = \tilde{f}$ and $\sum_i (\tilde{h}_i.x_i) = -f$. So $\sum_i (h_i.x_i) f + \sum_i (\tilde{h}_i.x_i).\tilde{f} = 0$

Now for any vectors satisfying (3.15) the following holds

$$\sum_i b_i.x_i = p.\tilde{f} \text{ and } \sum_i \tilde{b}_i.x_i = -p.f, \text{ for some homogeneous } p \in S$$

Notice that degree of p is even, as degree of b_i is even and degree of f is odd.

Hence we get that

$$\sum_i b_i.x_i = p.(\sum_i (h_i.x_i)) \text{ and } \sum_i \tilde{b}_i.x_i = p.(\sum_i (\tilde{h}_i.x_i)),$$

So

$$\sum_i (b_i - p.h_i).x_i = 0 \text{ and } \sum_i (\tilde{b}_i - p.\tilde{h}_i) x_i = 0$$

Hence

$$\big(\; (b_0 - p.h_0, b_1 - p.h_1, b_2 - p.h_2) \, , \quad (\tilde{b}_0 - p.\tilde{h}_0, \tilde{b}_1 - p.\tilde{h}_1, \tilde{b}_2 - p.\tilde{h}_2) \; \big) \in \mathrm{Syz}^1(x_0, x_1, x_2)$$

Now using the same arguments as Theorem 2.2, we get that

$$[B - P.H], [\tilde{B} - P.H'] \in < \mathbf{Y_{ij}} : 0 \leq i, j \leq 2 >,$$

where P such that $\theta(P) = p$.
Hence $[B] \in \langle \mathbf{Y_{ij}}, H : 0 \leq i, j \leq 2 \rangle$ and $[\tilde{B}] \in < \mathbf{Y_{ij}}, \tilde{H} : 0 \leq i, j \leq 2 >$. Let $\mathcal{H} = [\; (\; 0, \quad \tau_1, \quad \tau_2, \quad 0, \quad 0, \quad , 0 \;), \quad H, \quad \tilde{H} \;]$,
where $\tau_1 = \tilde{h_I}.h_{IV} - \tilde{h_{IV}}.h_I$ and $\tau_2 = \tilde{h_{III}}.h_I - \tilde{h_I}.h_{III}$, then we get

$$[\; A, \quad B, \quad \tilde{B} \;] \in \left\langle \; [W_i, \bar{0}, \bar{0}] \, , \quad [V_{jk}, \mathbf{Y_{jk}}, \bar{0}] \, , \quad \left[\tilde{V}_{jk}, \bar{0}, \mathbf{Y_{jk}}\right] , \quad [\mathcal{H}] \; \right\rangle$$

for $i = 1, \ldots, 8$,and $0 \leq j, k \leq 2$, $(j, k) \neq (0, 2)$
Hence we get R_2

Consider
$A = [\; (A_i) \;]$, $B = [\; (B_{jk}) \;]$, $\tilde{B} = [\; (\tilde{B}_{jk}) \;]$ and C,
where A_i,B_{jk},$\tilde{B}_{jk}$,$C \in S$, homogeneous, for $i = 1, \ldots, 8$, $j, k = 0, 1, 2$ and $(j, k) \neq (0, 2)$ such that

$$\sum_i A_i.[W_i, \bar{0}, \bar{0}, 0] + \sum_{jk} B_{jk}.[V_{jk}, \mathbf{Y_{jk}}, \bar{0}, 0] + \sum_{jk} \tilde{B}_{jk}.[\tilde{V}_{jk}, \bar{0}, \mathbf{Y_{jk}}, 0] + C.[\mathcal{H}] = 0 \tag{3.16}$$

Like in the earlier part of this proof, we consider four cases
(1) $B = [\bar{0}]$ and $\tilde{B} = [\bar{0}]$, hence $C = 0$, then we get that $[A] \in \langle G_1, G_2, G_3 \rangle$
Hence $([A], [\bar{0}]) \in \Big\langle \ [G_1, \bar{0}], \quad [G_2, \bar{0}], \quad [G_3, \bar{0}] \ \Big\rangle$
(2)$B \neq [\bar{0}]$ but $\tilde{B} = [\bar{0}]$, then we get that $C = 0$.
Then $[A, B] \in \Big\langle \ [G_i, \bar{0}], \quad [K_j, \mathbf{K_j'}] \ : i = 1, 2, 3, j = 1, \ldots, 6 \Big\rangle$.
Hence $[A, B, \bar{0}, 0] \in \Big\langle \ [G_i, \bar{0}], \quad [K_j, \mathbf{K'}_j, \bar{0}, 0] \ : i = 1, 2, 3, j, k = 0, 1, 2 \Big\rangle$.
(3)$\tilde{B} \neq [\bar{0}]$ but $B = [\bar{0}]$, hence $C = 0$. Similarly to case(2) we get,
$[A, \bar{0}, \tilde{B}, 0] \in \Big\langle [G_i, \bar{0}], [\tilde{K}_j, \bar{0}, \mathbf{K'}_j, 0] : i = 1, 2, 3, 1 \leq j \leq 6 \Big\rangle$.
(4))$B, \tilde{B} \neq [\bar{0}]$, Then we have

$$\sum_{jk} B_{jk}.\mathbf{Y_{jk}} + C.H = 0 \text{ and } \sum_{jk} \tilde{B}_{jk}.\mathbf{Y_{jk}} + C.H' = 0 \tag{3.17}$$

$$\text{So }, \sum_{jk} x_j b_{jk}.\mathbf{Y_k} + c.h = 0 \text{ and } \sum_{jk} x_j b'_{jk}.\mathbf{Y_k} + c.h' = 0 \tag{3.18}$$

where $\langle Y_0, Y_1, Y_2 \rangle = \mathrm{Syz}^1(x_0, x_1, x_2)$
(see Theorem 2.2)
Now multiplying (3.18) by $[x_0, x_1, x_2]^T$, we get

$$c.f = c.\tilde{f} = 0 \Rightarrow c = 0 \Rightarrow C \in < \Delta_{ij} >$$

Now substituting $C = \Delta_{ij}$, $\forall 0 \leq i \leq j \leq 2$ in (3.17), we get a set of six vectors, lets call them $\mathcal{D}_{ij}$. So we have

$$\mathcal{D}_{ij} = \begin{bmatrix} \delta_{ij}, & \Delta_{ij} \end{bmatrix}$$

Hence $[A, B, \tilde{B}, C] \in \Big\langle \ [W_i, \bar{0}, \bar{0}], \quad [K_j, \mathbf{K'}_j, \bar{0}, 0], \quad \Big[\tilde{K}_j, \bar{0}, \mathbf{K'}_j, \Big], \quad [\mathcal{D}_{kl}] \ \Big\rangle$
for $1 \leq i \leq 8$, $1 \leq j \leq 6$, $0 \leq k \leq l \leq 2$.
Hence we get R_3
Consider,

$$A = \begin{bmatrix} (A_i) \end{bmatrix}, \quad B = \begin{bmatrix} (B_j) \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} (\tilde{B}_j) \end{bmatrix}, \quad C = [(C_{kl})]$$

,
where A_i,B_j,$\tilde{B}_j$,$C_{kl} \in S$, homogeneous, for $i = 1, 2, 3$, $j = 1, \ldots, 6$
and $0 \leq k \leq l \leq 2$ with $(k, l) \neq (0, 2)$ such that

$$\sum_i A_i.[G_i, \bar{0}, \bar{0}, \bar{0}] + \sum_j B_j.[K''_j, \mathbf{K'_j}, \bar{0}, \bar{0}] + \sum_j \tilde{B}_j.[\tilde{K}''_j, \bar{0}, \mathbf{K'_j}, \bar{0}] + \sum_{k,l} C_{kl}.[\mathcal{D}_{kl}] = 0 \tag{3.19}$$

If we take similar cases as in the earlier part of the proof, we get

(1)If $C = \bar{0}$, then $[A, B, \tilde{B}, \bar{0}] \in \Big\langle \ [J'', \mathbf{J'}, \bar{0}, \bar{0}], \quad [\tilde{J''}, \bar{0}, \mathbf{J'}, \bar{0}] \ \Big\rangle$.

(2)$C \neq \bar{0}$, then $[C] \in \langle [W_i] : i = 1, \ldots, 8 \rangle$
Substituting $[C] = [W_i]$ for some i in (3.19), we get a set of 8 vectors

$$W_i = [[\omega_i], [W_i]] .$$

Hence we have,

$$[A, B, \tilde{B}, C] \in \Big\langle \begin{bmatrix} J'', & \mathbf{J'}, & \bar{0}, & \bar{0} \end{bmatrix}, \begin{bmatrix} \tilde{J''}, & \bar{0}, & \mathbf{J'}, & \bar{0} \end{bmatrix}, [W_i] \Big\rangle$$

for $1 \leq i \leq 8$.

Hence we get R_4

Let $A = \Big[\ (A_i)\ _{1\leq i\leq 8}\Big]$, B, $\tilde{B}$
where A_i, B, $\tilde{B} \in S$, homogeneous for $i = 1, \ldots, 8$ such that

$$\sum_i A_i.[\omega_i, W_i] + B.[J'', \mathbf{J}', 0, \bar{0}] + \tilde{B}.[\tilde{J''}, 0, \mathbf{J}', \bar{0}] = 0 \tag{3.20}$$

As the last rows of the last two vectors are zero we have $\sum_i A_i.W_i = [\bar{0}]$

This implies that, $[A] \in \langle G_k : k = 1, 2, 3\rangle$. Substituting this in (3.20), we get 3 vectors, let us call them Γ_k.

$$\Gamma_k = [\mathsf{G}_k, G_k] \text{ for } k = 1, 2, 3$$

So $[B, \tilde{B}, A] \in \langle \Gamma_k : k = 1, 2, 3\rangle$

Hence we get R_5.

4

DG -Algebra

Recall from Chapter 2, that for $\mathcal{C}$ a smooth(or irreducible) plane curve, the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ gives an embedding $\mathcal{C} \overset{\sigma}{\hookrightarrow} \mathbb{P}^5$. In Chapter 2 we computed the syzygies of the homogeneous ideal $\mathcal{I}_{\sigma(\mathcal{C})}$ of this embedding of $\mathcal{C}$ in $\mathbb{P}^5$. Now if the degree of $\mathcal{C}$ is odd then from theorem 2.2, we have that the minimal graded free resolution of $S/\mathcal{I}_{\sigma(\mathcal{C})}$ is as follows:

$$\begin{aligned} 0 \to & S(-m-4) \overset{\beta_4'}{\to} S(-4)^{\oplus 3} \oplus S(-m-2)^{\oplus 6} \overset{\beta_3'}{\to} \\ & \overset{\beta_3'}{\to} S(-3)^{\oplus 8} \oplus S(-m-1)^{\oplus 8} \overset{\beta_2'}{\to} S(-2)^{\oplus 6} \oplus S^{\oplus 3}(-m) \overset{\beta_1'}{\to} S \to S/\mathcal{I}_{\mathcal{C}} \to 0 \end{aligned}$$

where

$$\begin{aligned} \beta_1' &= \begin{bmatrix} \Delta_{00}, & \ldots, & \Delta_{22}, & F_0, & F_1, & F_2 \end{bmatrix} \\ \beta_2' &= \begin{bmatrix} [W_1, \bar{0}], & ,\ldots, & [W_8, \bar{0}] & , & [\ [Y_1'], & [Y_1]\ ], & \ldots, & [\ [Y_8'], & [Y_8]\ ] \end{bmatrix} \\ \beta_3' &= \begin{bmatrix} [G_1', \bar{0}], & ,[G_2', \bar{0}], & [G_3', \bar{0}] & , & [\ [K_1'], & [K_1]\ ], & \ldots, & [\ [K_6'], & [K_6]\ ] \end{bmatrix} \\ \beta_4' &= [\beta_1']^T \end{aligned}$$

where

$$\left.\begin{array}{l} (1)\ W_i \text{ are matrices from equation (2.2)} \\ (2)\ G_1' = G_1, \quad G_2' = -G_2 \text{ and } G_3' = G_3 \text{ from equation (2.3)} \\ (3)\ Y_{t_s} = G_{s_t}, \text{ for all } s = 1,2,3 \text{ and } t = 1,\ldots,8 \\ (4)\ K_{t_s} = W_{s_t}, \text{ for all } t = 1,\ldots,6 \text{ and } s = 1,\ldots,8 \\ (5) \left\{ \begin{array}{llll} Y_1' = V_{00}, & Y_2' = -V_{01}, & Y_3' = -V_{10}, & Y_4' = \begin{bmatrix} x_{00}h_I, & 0, & ,-h_{III}, & -x_{11}h_I, & ,h_{II}, & 0 \end{bmatrix}^T \\ Y_5' = -V_{12}, & Y_6' = V_{20}, & Y_7' = -V_{21}, & Y_8' = V_{22} \end{array} \right. \\ (6)\ K_{t_s}' = Y_{s_t}' \text{ for all } t = 1,\ldots,6 \text{ and } s = 1,\ldots,8 \end{array}\right\} \tag{4.1}$$

where V_{ij} are matrices from Chapter 3.

Note that the β_i's in the above resolution are not the same as the β_is defined in Theorem 2.2. But because the above resolution is symmetric, columns of W_i's and G_i's are linearly independent and the fact that,

$$\sum_i W_{i_n}.Y_{i_m}' + \sum_j Y_{j_n}'.W_{j_m} = 0 \qquad \forall \qquad n,\ m = 1,\ldots,6$$

gives us that the above β_i's also define a resolution.

Let us call the above exact sequence $\mathcal{P}\bullet$. So we have

$$\mathcal{P}\bullet:\ 0 \to \mathcal{P}_4 \to \mathcal{P}_3 \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 = S \to 0$$

where rank $(\mathcal{P}_0)$ = rank $(S) = 1$, rank $(\mathcal{P}_1) = 9$, rank $(\mathcal{P}_2) = 16$, rank $(\mathcal{P}_3) = 9$, rank$(\mathcal{P}_4) = 1$

Let $\{e_i, e_{F_{n-1}}\}$ be basis of $\mathcal{P}_1$, $\{e_{w_s}, e_{v_s}\}$ be basis of $\mathcal{P}_2$, $\{e_{g_n}, e_{k_i}\}$ be basis of $\mathcal{P}_3$, $\{e_{\mathcal{J}}\}$ be basis of $\mathcal{P}_4$. where $i = 1, \ldots, 6$, $n = 1, 2, 3$, $s = 1, \ldots, 8$.
In [KM] the authors prove that any symmetric resolution of length 4 has a DG algebra structure. Hence we know that the above resolution has a DG-algebra structure. In this chapter we will define a DG-algebra structure for the resolution, $\mathcal{P}\bullet$.

4.1 Defining $(*)$

Let us define the multiplication $(*)$ on the above basis elements

$$\begin{array}{ll}
(i) & e_i * e_j = \sum_{t=1}^{8} A_{i,j_s}.e_{w_t} \\
(ii) & e_i * e_{w_s} = \sum_{t=1,2,3} B_{i,s_t}.e_{g_t} \\
(iii) & e_i * e_{g_s} = 0 \\
(iv) & e_{w_s} * e_{w_t} = 0 \\
(v) & e_i * e_{F_{n-1}} = \sum_{t=1}^{8} B_{i,t_n}.e_{v_t} + \sum_{t=1}^{8} \alpha'_{i,n-1_t}.e_{w_t} \\
(vi) & e_i * e_{v_s} = \sum_{t=1}^{6} A_{i,t_s}.e_{k_t} + \sum_{t=1}^{3} \alpha'_{i,t-1_s}.e_{g_t} \\
(vii) & e_{F_{n-1}} * e_{F_{m-1}} = \sum_{t=1}^{8} \mathcal{A}_{n-1,m-1_t}.e_{v_t} \\
(viii) & e_{F_{n-1}} * e_{v_s} = -\sum_{t=1}^{6} \alpha'_{t,n-1_s}.e_{k_t} \\
(ix) & e_{F_{n-1}} * e_{w_s} = -\sum_{t=1}^{6} B_{t,s_n}.e_{k_t}
\end{array} \tag{4.2}$$

$$\begin{array}{ll}
(x) & e_i * e_{k_s} = \delta_{is}.e_{\mathcal{J}} \\
(xi) & e_{F_{i-1}} * e_{k_s} = 0 \\
(xii) & e_{F_{i-1}} * e_{g_s} = \delta_{is}.e_{\mathcal{J}} \\
(xiii) & e_{w_i} * e_{v_s} = -\delta_{is}.e_{\mathcal{J}} \\
(xiv) & e_{v_s} * e_{v_t} = 0
\end{array} \tag{4.3}$$

where
$A_{i,j}$, $B_{i,s}$, $\mathcal{A}_{n-1,m-1}$, $\alpha'_{i,n-1}$ are matrices given below and

$$\delta_{is} = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}$$

$$A_i = \begin{bmatrix} [A_{i,1}], & \dots & [A_{i,6}] \end{bmatrix} \text{ for } i = 1, \dots, 6$$

$$A_1 = \begin{bmatrix} 0 & x_{12} & x_{11} & x_{02} & 0 & 0 \\ 0 & 0 & 0 & x_{12} & x_{11} & 0 \\ 0 & -x_{22} & -x_{12} & 0 & x_{02} & x_{01} \\ 0 & 0 & 0 & 0 & 0 & x_{11} \\ 0 & 0 & 0 & 0 & 0 & -x_{12} \\ 0 & 0 & 0 & -x_{22} & -x_{12} & -x_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -x_{12} & 0 & x_{01} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{02} & x_{01} & 0 \\ x_{22} & 0 & -x_{02} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{01} \\ 0 & 0 & 0 & 0 & 0 & -x_{02} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_{22} & -x_{12} & -x_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -x_{11} & -x_{01} & 0 & -x_{00} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ x_{12} & x_{02} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{02} & x_{01} \\ 0 & 0 & 0 & x_{02} & 0 & 0 \\ 0 & 0 & 0 & -x_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_{12} & -x_{11} \end{bmatrix}, \quad A_4 = \begin{bmatrix} -x_{02} & 0 & x_{00} & 0 & 0 & 0 \\ -x_{12} & -x_{02} & 0 & 0 & x_{00} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{00} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ x_{22} & 0 & -x_{02} & 0 & 0 & 0 \\ 0 & -x_{22} & -x_{12} & 0 & -x_{02} & -x_{01} \\ 0 & 0 & 0 & 0 & 0 & -x_{02} \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -x_{11} & -x_{01} & 0 & -x_{00} & 0 & 0 \\ -x_{02} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_{02} & 0 & 0 & x_{00} \\ x_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_{12} & 0 & x_{02} & 0 & 0 \\ 0 & 0 & x_{12} & 0 & 0 & -x_{01} \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -x_{01} & 0 & 0 & 0 & 0 & 0 \\ -x_{11} & -x_{01} & 0 & -x_{00} & 0 & 0 \\ x_{12} & x_{02} & -x_{01} & 0 & -x_{00} & 0 \\ x_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{11} & 0 & x_{01} & 0 & 0 \\ 0 & 0 & x_{11} & x_{02} & x_{01} & 0 \end{bmatrix}$$

And

$$B_i = \begin{bmatrix} [B_{i,1}], & \ldots, & [B_{i,8}] \end{bmatrix} \text{ for } i = 1, \ldots, 8$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & -x_{12} & -x_{11} & 0 & x_{02} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{12} & x_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{22} & x_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & -x_{02} & -x_{01} & -x_{02} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_{12} & 0 & x_{01} \\ 0 & 0 & 0 & 0 & 0 & -x_{22} & 0 & x_{02} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & x_{02} & 0 & x_{01} & 0 & 0 & 0 & 0 \\ 0 & x_{12} & 0 & x_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_{12} & -x_{02} & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & x_{02} & 0 & -x_{00} & 0 & 0 & 0 \\ 0 & 0 & x_{12} & x_{02} & 0 & 0 & 0 & x_{00} \\ 0 & 0 & x_{22} & 0 & -x_{02} & 0 & 0 & 0 \end{bmatrix}$$

$$B_5 = \begin{bmatrix} -x_{02} & 0 & 0 & x_{00} & 0 & 0 & 0 & 0 \\ -x_{12} & 0 & 0 & x_{01} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{12} & x_{02} & 0 & x_{02} & 0 & 0 \end{bmatrix}, \quad B_6 = \begin{bmatrix} -x_{01} & -x_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_{11} & -x_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_{02} & x_{11} & 0 & 0 & x_{01} & 0 & 0 \end{bmatrix}$$

$$\mathcal{A}_{0,1} = \begin{bmatrix} -h_{II} \\ h_{III} \\ x_{11}h_I \\ -h_{IV} \\ 0 \\ 0 \\ -x_{00}h_I \\ 0 \end{bmatrix}, \quad \mathcal{A}_{0,2} = \begin{bmatrix} -x_{22}h_I \\ 0 \\ h_{II} \\ -h_{III} \\ h_{IV} \\ -h_{III} \\ 0 \\ -x_{00}h_I \end{bmatrix}, \quad \mathcal{A}_{1,2} = \begin{bmatrix} 0 \\ -x_{22}h_I \\ 0 \\ 0 \\ x_{11}h_I \\ -h_{II} \\ h_{III} \\ -h_{IV} \end{bmatrix}$$

$$\alpha'_i = \begin{bmatrix} [\alpha'_{i,0}] & [\alpha'_{i,1}] & [\alpha'_{i,2}] \end{bmatrix}$$

$$\alpha_1' = \begin{bmatrix} 0 & -x_{11}h_I & -h_{II} \\ x_{11}h_I & 0 & 0 \\ -h_{III} & -h_{II} & -x_{22}h_I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -h_{II} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \alpha_2' = \begin{bmatrix} x_{00}h_I & 0 & h_{III} \\ 0 & -x_{11}h_I & 0 \\ 0 & h_{III} & 0 \\ 0 & 0 & -x_{22}h_I \\ 0 & -x_{22}h_I & 0 \\ 0 & 0 & 0 \\ -h_{II} & 0 & 0 \\ -x_{22}h_I & 0 & 0 \end{bmatrix}$$

$$\alpha_3' = \begin{bmatrix} 0 & 0 & -h_{IV} \\ 0 & 0 & -x_{11}h_I \\ -x_{00}h_I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -h_{II} & -x_{22}h_I \\ 0 & -x_{11}h_I & 0 \\ -x_{11}h_I & 0 & 0 \\ -h_{II} & 0 & 0 \end{bmatrix}, \qquad \alpha_4' = \begin{bmatrix} 0 & -x_{00}h_I & 0 \\ x_{00}h_I & 0 & h_{III} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & h_{III} & 0 \\ h_{III} & h_{II} & 0 \\ 0 & 0 & x_{22}h_I \end{bmatrix}$$

$$\alpha_5' = \begin{bmatrix} 0 & 0 & -x_{00}h_I \\ 0 & 0 & -h_{IV} \\ 0 & -x_{00}h_I & 0 \\ 0 & 0 & 0 \\ 0 & h_{III} & -x_{22}h_I \\ -x_{00}h_I & 0 & 0 \\ 0 & -x_{11}h_I & 0 \\ h_{III} & 0 & x_{22}h_I \end{bmatrix}, \qquad \alpha_6' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -h_{IV} \\ -x_{00}h_I & -h_{IV} & -h_{III} \\ 0 & 0 & 0 \\ 0 & 0 & x_{11}h_I \\ -h_{IV} & -x_{11}h_I & 0 \end{bmatrix}$$

4.2 Associativity

1.Check that

$$\sum_{s=1}^{8} A_{i,j_s}.B_{l,s_t} = \sum_{s=1}^{8} A_{j,l_s}.B_{i,s_t} \text{for all } t=1,2,3 \text{ and for all } 1 \leq i,j,k \leq 6$$

This implies that $(e_i * e_j) * e_l = e_i * (e_j * e_l)$ for all $1 \leq i,j,k \leq 6$

2. Check that

$$\sum_{s=1}^{8} B_{j,s_{l+1}}.\alpha'_{i,t-1_s} + \sum_{s=1}^{8} B_{i,s_t}\alpha'_{i,t-1_s} = \sum_{s=1}^{8} A_{i,j_s}\mathcal{A}_{l,t-1_s}$$

for all $1 \leq i,j \leq 6$ and $l = 0,1,2$.

And, $\sum_{s=1}^{8} A_{i,j_s}.B_{t,s_{l+1}} = \sum_{s=1}^{8} A_{t,i_s}.B_{j,s_{l+1}}$ for all $1 \leq i,j,t \leq 6$ and $l = 0,1,2$, by 1. above

This implies that $(e_i * e_j) * e_{F_l} = e_i * (e_j * e_{F_l})$ for all $1 \leq i,j \leq 6$ and $l = 0,1,2$

3. Check that

$$\sum_{s=1}^{8} \mathcal{A}_{0,1_s}\alpha'_{t,2_s} = \sum_{s=1}^{8} \mathcal{A}_{1,2_s}\alpha'_{t,0_s} \text{for all } 1 \leq t \leq 8$$

This implies that $(e_{F_0} * e_{F_1}) * e_{F_2} = e_{F_0} * (e_{F_1} * e_{F_2})$

4.Check that

$$(e_i * e_j) * e_{v_s} = e_i * (e_j * e_{v_s}) \text{ , by } 4.2.(i), 4.2.(vi), \text{ and } 4.3$$

5.Check that

$$(e_i * e_{F_j}) * e_{w_s} = e_i * (e_{F_j} * e_{w_s}) \text{ , by } 4.2.(v), 4.2.(ix), \text{ and } 4.3$$

6.Check that

$$(e_i * e_{F_j}) * e_{v_s} = e_i * (e_{F_j} * e_{v_s}) \text{ , by } 4.2.(v), 4.2.(viii), \text{ and } 4.3$$

7.Check that

$$(e_{F_i} * e_{F_j}) * e_{w_s} = e_{F_i} * (e_{F_j} * e_{w_s}) \text{ , by } 4.2.(vii), 4.2.(ix), \text{ and } 4.3$$

8.Check that

$$(e_{F_i} * e_{F_j}) * e_{v_s} = e_{F_i} * (e_{F_j} * e_{v_s}) = 0 \text{ , by } 4.2.(vii), 4.2.(viii), \text{ and } 4.3$$

Hence we get that $*$ is associative.

4.3 Defining ∂

$$
\begin{aligned}
&1)\quad \partial(e_{2i+j+1}) = \begin{cases} \Delta_{ij} & \text{when } 0 \le i \le j < 2 \\ \Delta_{22} & i = j = 2 \end{cases}\\
&2)\quad \partial(e_{F_n}) = F_n\\
&3)\quad \partial(e_{w_s}) = \sum_{t=1}^{6} W_{s_t} e_t\\
&4)\quad \partial(e_{v_s}) = \sum_{t=1}^{3} G'_{t_s} e_{F_{t-1}} + \sum_{t'=1}^{6} Y'_{s_t}\\
&5)\quad \partial(e_{g_s}) = \sum_{t=1}^{8} G'_{s_t} e_{w_t}\\
&6)\quad \partial(e_{k_s}) = \sum_{t=1}^{8} W_{s_t} e_{v_t} + \sum_{t=1}^{8} Y'_{s_t} e_{w_t}\\
&7)\quad \partial(e_{\mathcal{J}}) = \sum_{t=1}^{6} (\partial(e_t))\, e_{k_t} + \sum_{t=1}^{3} F_{t-1} e_{g_t}
\end{aligned}
$$

where W_i as in equation (2.2), G'_j as in equation (2.3), Y'_k as inequation (4.1). To prove that ∂ is well-defined,

(1)To check that

$$\partial(e_i * e_j) = \partial(\sum_{t,n} A_{i,j_t} e_{w_t})$$

Now $\{A_{i,j_t}\}$ is computed such that $[A_{i,j}]$ satisfies the following conditins,

$$
\begin{aligned}
\sum_t A_{i,j_t} W_{t_i} &= -\Delta_j \\
\sum_t A_{i,j_t} W_{t_j} &= \Delta_i \\
\sum_t A_{i,j_t} W_{t_n} &= 0 \text{ for } n \neq i, j \text{ and } n = 1, \ldots, 6
\end{aligned}
\tag{4.4}
$$

(2)To check that

$$\partial(e_i * e_{w_s}) = \partial(\sum_{t,n} B_{i,s_t} e_{w_t})$$

Now $\{B_{i,s_t}\}$ is computed such that $[B_{i,s}]$ satisfies the following conditions,

$$
\begin{aligned}
\sum_t B_{i,s_t} G'_{t_s} &= \Delta_i - \sum_t W_{s_t} A_{i,t_s} \\
\sum_t B_{i,s_t} G'_{t_n} &= -\sum_t W_{s_t} A_{i,t_n} \text{ for } n \neq i, j \text{ and } n = 1, \ldots, 8
\end{aligned}
\tag{4.5}
$$

So you get that $\partial(e_i * e_{w_s}) = \partial(\sum_{t,n} B_{i,s_t} e_{w_t})$

(3)To check that

$$\partial(e_i * e_{g_s}) = \partial(0), \text{ check that}$$

$$\begin{aligned}\sum_t B_{i,t_s} G'_{s_t} &= \Delta_i \\ \sum_t B_{i,t_s} G'_{s_t} &= 0 \text{ for } n \neq s \text{ and } n = 1,2,3\end{aligned} \tag{4.6}$$

(4)To check that

$$\partial(e_{w_i} * e_{w_j}) = \partial(0), \text{ check that}$$

$$\sum_t W_{i_t} B_{t,j_n} + \sum_t W_{j_t} B_{t,i_n} = 0 \text{ for all } n = 1,2,3 \tag{4.7}$$

(5)To check that

$$\partial(e_i * e_{F_j}) = \partial\left(\sum_t B_{i,t_{j+1}} e_{v_t}\right) + \partial\left(\sum_t \alpha'_{i,j_t} e_{w_t}\right)$$

Now from equation(4.6) we get that the coefficients of e_{v_n} of both the sides are equal and further, $\{\alpha'_{i,j}\}$ have computed such that $[\alpha'_{i,j}]$ satisfy the following conditions,

$$\begin{aligned}\sum_t \alpha'_{i,j_t} W_{t_i} &= -F_j - \sum_t B_{i,t_{j+1}} Y'_{t_i} \\ \sum_t \alpha'_{i,j_t} W_{t_n} &= -\sum_t B_{i,t_{j+1}} Y'_{t_n} \text{ such that } n \neq i \text{ and } n = 1,\ldots,6\end{aligned} \tag{4.8}$$

(6)To check that

$$\partial(e_{F_i} * e_{F_j}) = \partial(\sum_t \mathcal{A}_{i,j_t} e_{v_t})$$

Now $\{\mathcal{A}_{i,j_t}\}$ have been computed such that $[\mathcal{A}_{i,j}]$ satisfy the following conditions

$$\begin{aligned}\sum_t \mathcal{A}_{i,j_t} G'_{i+1_t} &= -F_j \\ \sum_t \mathcal{A}_{i,j_t} G'_{j+1_t} &= F_i \\ \sum_t \mathcal{A}_{i,j_t} G'_{n_t} &= 0 \text{ for } n \neq i+1, j+1 \text{ and } n = 1,2,3\end{aligned} \tag{4.9}$$

$$\sum_t \mathcal{A}_{i,j_t} Y'_{t_{n'}} = 0 \text{ for } n' = 1,\ldots,6 \tag{4.10}$$

(7)To check

$$\partial(e_{F_i} * e_{v_s}) = \partial(-\sum_t \alpha'_{t,i_s} e_{k_t}), \text{ check that}$$

$$\begin{aligned}\sum_t \mathcal{A}_{i,t-1_s} G'_{t_s} - \sum_t Y'_{s_t} B_{t,s_{i+1}} - \sum_t \alpha'_{t,i_s} W_{s_t} &= F_i \\ \sum_t \mathcal{A}_{i,t-1_n} G'_{t_s} - \sum_t Y'_{s_t} B_{t,n_{i+1}} - \sum_t \alpha'_{t,i_s} W_{n_t} &= 0 \text{ for } n \neq s \text{ and } n = 1,\ldots,8\end{aligned} \tag{4.11}$$

$$\sum_t \alpha'_{t,i_n} Y'_{s_t} + \sum_t \alpha'_{t,i_s} Y'_{n_t} = 0 \text{ for all } n = 1,\ldots,8, \tag{4.12}$$

(8)To check

$$\partial(e_{F_i} * e_{w_s}) = \partial\left(-\sum_t B_{t,s_{i+1}} e_{k_t}\right) + \partial(\sum_t \mathcal{A}_{i,t-1_s} e_{g_t}) \text{ check that}$$

from equation(4.11) we get that the coefficients of e_{w_n} are equal for both the sides, and from equation(4.7) we get that the coefficients of e_{v_n} on both the sides are equal.

(9)To check

$$\partial(e_i * e_{k_j}) = \delta_{ij}.e_{\mathcal{J}} \text{ notice that}$$

from equation(4.4) we get that the coefficients of e_{k_n} are equal for both the sides,and similarly equation(4.8) gives us that the same holds for the coefficients of e_{g_n}.

(10)To check

$$\partial(e_{F_{i-1}} * e_{g_j}) = \delta_{ij}.e_{\mathcal{J}} \text{ notice that}$$

from equation(4.9) we get that the coefficients of e_{k_n} are equal for both the sides,and similarly equation(4.6) gives us that the same holds for the coefficients of e_{g_n}.

(11)To check

$$\partial(e_{F_{i-1}} * e_{k_j}) = 0 \text{ notice that}$$

from equation(4.8) we get that the coefficients of e_{k_n} are zero for the LHS, and similarly equation(4.10) gives us that the coefficients of e_{g_n} of the LHS are zero.

(12)To check

$$\partial(e_{w_i} * e_{v_j}) = -\delta_{ij}.e_{\mathcal{J}} \text{ notice that}$$

from equation(4.5) we get that the coefficients of e_{k_n} are equal for both the sides,and similarly equation(4.11) gives us that the same holds for the coefficients of e_{g_n}.

(13)To check

$$\partial(e_{v_i} * e_{v_j}) = 0 \text{ notice that}$$

from equation(4.12) we get that the coefficients of e_{g_n} are zero for the LHS, further check that

$$\sum_t G'_{t_i} \alpha'_{n,t-1_j} + \sum_t G'_{t_j} \alpha'_{n,t-1_i} = \sum_t Y'_{it} A_{t,n_j} + \sum_t Y'_{jt} A_{t,n_i} \text{ for all } n = 1, \ldots, 6$$

This gives us that the coefficients of e_{k_n} are zero.

5
Appendix

Here we record some observations by one of the referees about the calculations in the thesis.

1. If we have a short exact sequence sequence of finitely generated modules M_1, M_2, M_3 over a polynomial ring,

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

   and if we know the minimal free resolution of M_1 and M_2 we can build a free resolution of M_3 which may not be minimal.

   Therefore as a consequence, the matrices (or the maps) in the free resolution of M_3 is built up from the free resolutions of on M_1 and M_2 will naturally be built up from the up in the free resolution of M_3. This is called the *mapping cone*. The free resolution built this way naturally turns out to be a complex, but also an exact sequence.

2. In our case in Theorem 2.1 we have the short exact sequence of ideals:

$$0 \longrightarrow (\Delta \cap F) = F\Delta \longrightarrow \Delta \oplus (F) \longrightarrow (\Delta, F) \longrightarrow 0$$

   and the corresponding free resolution of $(\Delta \cap F)$ and $\Delta \oplus F$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F_3 & \longrightarrow & G_3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F_2 & \longrightarrow & G_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F_1 & \longrightarrow & G_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (\Delta \cap F) = F\Delta & \longrightarrow & \Delta \oplus (F) & \longrightarrow & (\Delta, F) & \longrightarrow & 0
\end{array}$$

where F_i and G_i are free modules.

The mapping cone gives the following free resolution for (Δ, F):

$$0 \longrightarrow F_3 \longrightarrow F_2 \oplus G_3 \longrightarrow F_1 \oplus G_2 \longrightarrow G_1 \longrightarrow (\Delta, F) \longrightarrow 0.$$

3. Similarly, once we know the free resolution in Theorem 2.1 and Theorem 2.2, the free resolution in Theorem 3.1, Theorem 3.2 and Theorem 3.3 can be built up from them.
4. Hence the maps (or matrices) in minimal free resolution of $\Delta = (\Delta_{00}, \ldots, \Delta_{22})$ does appear in the Theorem 2.1, Theorem 2.2, Theorem 3.1, Theorem 3.2 and Theorem 3.3.
5. The interesting thing here is that all the free resolutions in Theorem 2.1, Theorem 2.2, Theorem 3.1, Theorem 3.2 and Theorem 3.3 are indeed minimal free resolutions which can be seen from the maps.

Theorem 2.1: Since the minimal free resolution for Δ is:

$$0 \longrightarrow S(-4)^3 \xrightarrow{[M_3]_{8\times 3}} S(-3)^8 \xrightarrow{[M_2]_{6\times 8}} S(-2)^6 \xrightarrow{[M_1]_{1\times 6}} \Delta \longrightarrow 0,$$

the minimal free resolution for $\Delta \cap (F) = F\Delta$ is:

$$0 \longrightarrow S(-m-4)^3 \xrightarrow{[M_3]_{8\times 3}} S(-m-3)^8 \xrightarrow{[M_2]_{6\times 8}} S(-m-2)^6 \xrightarrow{[FM_1]_{1\times 6}} \Delta \cap F \longrightarrow 0$$

where $[FM_1]_{1\times 6} = [F\Delta_{00}, \ldots, F\Delta_{22}]$.

The minimal free resolution of (F) is

$$0 \longrightarrow S(-m) \longrightarrow (F) \longrightarrow 0$$

Therefore the minimal free resolution for $\Delta \oplus (F)$ is

$$0 \longrightarrow S(-4)^3 \xrightarrow{[M_3]_{8\times 3}} S(-3)^8 \xrightarrow{\begin{bmatrix} [M_2]_{6\times 8} \\ [0]_{1\times 8} \end{bmatrix}} S(-2)^6 \oplus S(-m) \xrightarrow{[F,[M_1]_{1\times 6}]} \Delta \oplus (F) \longrightarrow 0,$$

Hence we get the commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & S(-m-4)^3 & \xrightarrow{[M_3]_{8\times 3}} & S(-m-3)^8 & \xrightarrow{[M_2]_{6\times 8}} & S(-m-2)^6 & \xrightarrow{[FM_1]_{1\times 6}} & \Delta \cap (F) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle FI_{3\times 3}} & & \downarrow{\scriptstyle FI_{8\times 8}} & & \downarrow{\scriptstyle \begin{bmatrix} -FI_{6\times 6} \\ [M_1]_{1\times 6} \end{bmatrix}} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & S(-4)^3 & \xrightarrow{[M_3]_{8\times 3}} & S(-3)^8 & \xrightarrow{\begin{bmatrix} [M_2]_{6\times 8} \\ [0]_{1\times 8} \end{bmatrix}} & S(-2)^6 \oplus S(-m) & \xrightarrow{[[M_1]_{1\times 6}, F]} & \Delta \oplus (F) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

$$
0 \longrightarrow S(-m-4)^3 \xrightarrow{\begin{bmatrix} [M_3]_{8\times 3} \\ [FI]_{3\times 3} \end{bmatrix}} S(-m-3)^8 \oplus S(-4)^3 \xrightarrow{\begin{bmatrix} [M_2]_{6\times 8} & [0]_{6\times 3} \\ [FI]_{8\times 8} & [M_3]_{8\times 3} \end{bmatrix}} S(-m-2)^6 \oplus S(-3)^8 \xrightarrow{\begin{bmatrix} [-FI]_{6\times 6} & [M_2]_{6\times 8} \\ [M_1]_{1\times 6} & [0]_{1\times 8} \end{bmatrix}} S(-2)^6 \oplus S(-m) \xrightarrow{[[M_1]_{1\times 6}, F]} (\Delta, F) \longrightarrow 0,
$$

Hence the maps can be given as block matrices as follows:

$$\alpha_1 = [[M_1]_{1\times 6}\ F] \quad \alpha_2 = \begin{bmatrix} -FI_{6\times 6} & [M_2]_{6\times 8} \\ [M_1]_{1\times 6} & 0_{1\times 8} \end{bmatrix} \alpha_3 = \begin{bmatrix} [M_2]_{6\times 8} & 0_{6\times 3} \\ -FI_{8\times 8} & [M_3]_{8\times 3} \end{bmatrix} \quad \alpha_4 = \begin{bmatrix} [M_2]_{8\times 3} \\ '-FI_{3\times 3} \end{bmatrix}$$

Remark 2: The above argument can be used for Theorem 2.2, Theorem 3.1, Theorem 3.2 and Theorem 3.3.

Theorem 2.2:

$$\beta_1 = [[M_1]_{1\times 6}\ F_1\ F_2\ F_3] \quad \beta_2 = \begin{bmatrix} [M_2]_{6\times 8} & [V]_{6\times 8} \\ 0_{3\times 8} & [Y]_{3\times 6} \end{bmatrix}$$

$$\beta_3 = \begin{bmatrix} [M_3]_{8\times 3} & -[V^T]_{8\times 6} \\ 0_{8\times 3} & -[M_2]^T_{8\times 6} \end{bmatrix} \qquad \beta_4 = \begin{bmatrix} -[J"]_{8\times 1} \\ [J']_{6\times 1} \end{bmatrix}$$

Remark 3 The complex in Theorem 3.1, can be built up from Theorem 2.1 as follows

Theorem 3.1:

We have the following commutative diagrams:

$$0 \longrightarrow S(-m-m'-4)^3 \xrightarrow{\begin{bmatrix}[M_3]_{8\times3}\\ [FI]_{3\times3}\end{bmatrix}} S(-m-m'-3)^8 \oplus S(-m'-4)^3 \xrightarrow{\begin{bmatrix}[M_2]_{6\times8} & [0]_{6\times3}\\ [FI]_{8\times8} & [M_3]_{8\times3}\end{bmatrix}} S(-m-m'-2)^6 \oplus S(-m'-3)^8 \xrightarrow{\begin{bmatrix}[-FI]_{6\times6} & [M_2]_{6\times8}\\ [M_1]_{1\times6} & [0]_{1\times8}\end{bmatrix}} S(-m'-2)^6 \oplus S(-m-m') \xrightarrow{[[F_2M_1]_{1\times6}, FF_2]} (\Delta F_2, FF_2) \longrightarrow 0$$

Vertical maps (top row to middle row):

$$[\tilde{F}I]_{3\times3}, \quad \begin{bmatrix}[\tilde{F}I]_{8\times8} & [0]_{8\times3}\\ [0]_{8\times3} & [\tilde{F}I]_{3\times3}\end{bmatrix}, \quad \begin{bmatrix}[\tilde{F}I]_{6\times6} & [0]_{6\times8}\\ [0]_{8\times6} & [\tilde{F}I]_{8\times8}\end{bmatrix}, \quad \begin{bmatrix}[\tilde{F}I]_{6\times6} & [0]_{6\times1}\\ [0]_{1\times6} & [\tilde{F}I]_{8\times1}\\ [0]_{1\times6} & [FI]_{1\times1}\end{bmatrix}$$

$$0 \longrightarrow S(-m-4)^3 \xrightarrow{\begin{bmatrix}[M_3]_{8\times3}\\ [FI]_{3\times3}\end{bmatrix}} S(-m-3)^8 \oplus S(-4)^3 \xrightarrow{\begin{bmatrix}[M_2]_{6\times8} & [0]_{6\times3}\\ [FI]_{8\times8} & [M_3]_{8\times3}\end{bmatrix}} S(-m-2)^6 \oplus S(-3)^8 \xrightarrow{\begin{bmatrix}[-FI]_{6\times6} & [M_2]_{6\times8}\\ [M_1]_{1\times6} & [0]_{1\times8}\end{bmatrix}} S(-2)^6 \oplus S(-m) \oplus S(-m') \xrightarrow{[[M_1]_{1\times6}, F, F_2]} (\Delta, F) \oplus F_2 \longrightarrow 0$$

$$0 \to S(-m-m'-4) \xrightarrow{\mathcal{P}_5} \begin{matrix}S(-m-m'-3)^8\\ \oplus S(-m'-4)^3\\ \oplus S(-m-4)^3\end{matrix} \xrightarrow{\mathcal{P}_4} \begin{matrix}S(-m-m'-2)^6 \oplus S(-m'-3)^8\\ \oplus S(-m-3)^8 \oplus S(-4)^3\end{matrix} \xrightarrow{\mathcal{P}_3} \begin{matrix}S(-m'-2) \oplus S(-m-m')\\ \oplus S(-m-2)^6 \oplus S(-3)^8\end{matrix} \xrightarrow{\mathcal{P}_2} \begin{matrix}S(-2)^6 \oplus S(-m)\\ \oplus S(-m')\end{matrix} \xrightarrow{\mathcal{P}_1} (\Delta, F, F_2) \to 0,$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S(-m-m'-4)^3 & \xrightarrow{[M_3]_{8\times 3}} & S(-m-m'-3)^8 & \xrightarrow{[M_2]_{6\times 8}} & S(-m-m'-2)^6 & \xrightarrow{[FF_2M_1]_{1\times 6}} & \Delta FF_2 \longrightarrow 0 \\
 & & \Big\downarrow {\scriptstyle FI_{3\times 3}} & & \Big\downarrow {\scriptstyle FI_{8\times 8}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} -FI_{6\times 6} \\ [M_1]_{1\times 6} \end{bmatrix}} & & \Big\downarrow {\scriptstyle i} \\
0 & \longrightarrow & S(-m'-4)^3 & \xrightarrow{[M_3]_{8\times 3}} & S(-m'-3)^8 & \xrightarrow{\begin{bmatrix} [M_2]_{6\times 8} \\ [0]_{1\times 8} \end{bmatrix}} & S(-m'-2)^6 \oplus S(-m-m') & \xrightarrow{[[F_2M_1]_{1\times 6}, FF_2]} & \Delta F_2 \oplus FF_2 \longrightarrow 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow
\end{array}
$$

$$
0 \longrightarrow S(-m-m'-4)^3 \xrightarrow{\begin{bmatrix} [M_3]_{8\times 3} \\ [FI]_{3\times 3} \end{bmatrix}} S(-m-m'-3)^8 \oplus S(-m'-4)^3 \xrightarrow{\begin{bmatrix} [M_2]_{6\times 8} & [0]_{6\times 3} \\ [FI]_{8\times 8} & [M_3]_{8\times 3} \end{bmatrix}} S(-m-m'-2)^6 \oplus S(-m'-3)^8 \xrightarrow{\begin{bmatrix} [-FI]_{6\times 6} & [M_2]_{6\times 8} \\ [M_1]_{1\times 6} & [0]_{1\times 8} \end{bmatrix}} S(-m'-2)^6 \oplus S(-m-m') \xrightarrow{[[F_2M_1]_{1\times 6}, FF_2]} (\Delta F_2, FF_2) \longrightarrow 0,
$$

$$\mathcal{P}_1 = \Big[[M_1]_{1\times 6}\ F\ \tilde{F}\Big] \qquad \mathcal{P}_2 = \begin{bmatrix} -\tilde{F}I_{6\times 6} & [0]_{6\times 1} & [-FI]_{6\times 6} & [M_2]_{6\times 8} \\ [0]_{1\times 6} & [FI]_{1\times 1} & [M_1]_{1\times 6} & [0]_{1\times 8} \\ [M_1]_{1\times 6} & [0]_{1\times 6} & [\tilde{F}]_{1\times 1} & [0]_{1\times 8} \end{bmatrix}$$

$$\mathcal{P}_3 = \begin{bmatrix} -FI_{8\times 6} & [M_2]_{6\times 8} & [0]_{6\times 8} & [0]_{6\times 3} \\ [M_1]_{1\times 6} & [0]_{1\times 8} & [0]_{1\times 8} & [0]_{1\times 3} \\ [-\tilde{F}I]_{6\times 6} & [0]_{6\times 8} & [M_2]_{6\times 8} & [0]_{6\times 3} \\ [0]_{8\times 6} & -\tilde{F}I_{8\times 8} & [-FI]_{8\times 8} & [M_3]_{8\times 3} \end{bmatrix} \quad \mathcal{P}_4 = \begin{bmatrix} [M_2]_{6\times 8} & [0]_{6\times 3} & [0]_{6\times 3} \\ FI_{8\times 8} & [M_3]_{8\times 3} & [0]_{6\times 3} \\ [-\tilde{F}I]_{8\times 8} & [0]_{8\times 3} & [0]_{6\times 3} \\ [0]_{3\times 8} & [-\tilde{F}I]_{3\times 3} & [0]_{6\times 3} \end{bmatrix}$$

$$\mathcal{P}_5 = \begin{bmatrix} [M_3]_{8\times 3} \\ [FI]_{3\times 3} \\ [-\tilde{F}I]_{3\times 3} \end{bmatrix}$$

Similarly, we can write the maps for Theorem 3.2 and Theorem 3.2.

Bibliography

[ACGH] E.Arbarello, M.Cornalba, P.A.Griffiths, J.Harris, *Geometry of Algebraic Curves*, no.267, Springer-Verlag New York, (1987)

[A] Luchezar L.Avramov, *Infinite Free Resolutions*, Six Lectures on Commutative Algebra, no.166,Birkhauser Verlag,1-118

[BH] W.Bruns, J.Herzog, *Cohen Macaulay rings*, no.39, Cambridge University Press,(1993)

[H] R.Hartshorne, *Algebraic Geometry*, no.52, Springer-Verlag New York,(1977)

[KM] A.R.Kustin, M.Miller, *Algebra structures on minimal resolutions of Gorenstein rings of embedding codimension four*, Math.Z., 173(1980), 171-184

[L] R.Lazarsfeld, *A Sampling of Vector Bundle Techniques in the Study of Linear Series*, Lectures on Riemann Surfaces, ICTP, World Scientific New York(1987) 500-559.

[OP] G.Ottaviani,R.Paoletti, *Syzygies of Veronese embeddings*, Compositio Math., 125(2001), 31-37