PLANAR ALGEBRA OF THE SUBGROUP-SUBFACTOR

By

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As members of the Viva Voce Board, we recommend that the dissertation prepared by Ved Prakash Gupta entitled "Planar Algebra of the Subgroup-Subfactor" may be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

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Declaration

I, hereby declare that the investigation presented in this thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part of a degree/diploma at this or any other Institution/University.

Ved Prakash Gupta

Dedicated to my parents
and my siblings.

To my teachers ....

Gurur Brahma Gurur Vishnu Gurur Devo Maheshwarah
Gurur Sakshat Param Brahma Tasmai Sri Guruve Namaha.

Abstract

We give an identification between the planar algebra of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$ and the G-invariant planar subalgebra of the planar algebra of the bipartite graph $\star_n$ (the graph with 1 odd and n even vertices), where $n = [G : H]$. The crucial step in this identification is the exhibition of a model for the basic construction tower, and thereafter of the standard invariant, of $R \rtimes H \subset R \rtimes G$ in terms of operator matrices.

A brief appendix is devoted to a discussion of the relationship between Jones' planar algebra and Ocneanu's paragroup approaches to the standard invariant.

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Chapter 1

Introduction

The study of von Neumann algebras was initiated by F. Murray and J. von Neumann in a series of seminal papers [MN36, MN37, MN40, MN43, Neu49]. And at quite an early stage of the development of this theory, Murray and von Neumann realized that the building blocks of von Neumann algebras were the ones with trivial centres, usually known as factors. Since then the whole direction of the study of von Neumann algebras shifted towards factors. And among the various types of factors, the study of the II_1-factors saw the evolution of a new branch known as the subfactor theory, which later turned out to have various significant applications.

Murray and von Neumann came up with several constructions of factors. One among them was the method of crossed product $M \rtimes_\alpha G$ of a von Neumann algebra M with a discrete group G acting on it by a group homomorphism $\alpha : G \to Aut(M)$. This was a sort of generalisation of the semi-direct product of a group with another group acting on it.

In particular, when we consider an action α of a finite group G on a II_1-factor P with an additional condition that the automorphisms $\alpha_t, e \neq t \in G$ are not inner, then the crossed product $P \rtimes_\alpha G$ turns out to be yet another II_1-factor. Further, if H is any subgroup of G, then we obtain a pair of II_1-factors

$$P \rtimes_{\alpha_{|_H}} H \subset P \rtimes_\alpha G.$$

Such an inclusion of II_1-factors is sometimes referred to as a *subgroup-subfactor.*

The subfactor theoretic aspects of subgroup-subfactors have been studied quite extensively in literature - see [Kos89, KY92, Ocn91, KS00, Izu02], to mention a few.

Such an action always exists on the hyperfinite II_1-factor R - see [Jon80].

Thus we shall only deal with subgroup-subfactors

$$R \rtimes_{\alpha_{|_H}} H \subset R \rtimes_\alpha G.$$

Further, group actions also yield the fixed algebra subfactor $R^G \subset R^H$.

In order to have a better understanding of subfactors, Jones introduced the notion of planar algebras in [Jon]. Basically, a planar algebra is a collection of vector spaces admitting a compatible action of the operad of coloured tangles.

Given a pair of II_1-factors $N \subset M$, Jones [Jon83] introduced the notion of index $[M : N]$, which can take values in $[1, \infty]$. Further, for such a pair with finite index, by iterating the basic construction, he got hold of a tower of II_1-factors

$$N \subset M \subset^{e_1} M_1 \subset \cdots \subset M_k \subset^{e_{k+1}} M_{k+1} \subset \cdots .$$

Then for the so called extremal pairs of II_1-factors $N \subset M$ with finite index, Jones [Jon] gave a planar algebra structure on the tower of relative commutants

$$\mathbb{C} = N' \cap N \subset N' \cap M \subset \cdots \subset N' \cap M_k \subset \cdots .$$

This planar algebra structure is unique with respect to some conditions to be satisfied by certain tangle maps (corresponding to a 'generating set' of coloured tangles).

Further, to every finite bipartite graph with a spin function, Jones [Jon00a] associated a planar algebra and called it the planar algebra of that bipartite graph.

Each subgroup-subfactor, being irreducible, is extremal, and therefore admits a planar algebra structure on the tower of its relative commutants as above. The main objective of this thesis is to identify this planar algebra with the group-invariant planar subalgebra of the planar algebra of an appropriate bipartite graph.

Overview of the Thesis

Chapter 2 is a brief survey of subfactor theory. Detailed discussions of subfactor theory can be found in many standard texts, for instance [GHJ89, Jon91, JS97, EK98, Bis97, Tak03]. Basically, this chapter is a quick recollection and slight modification of certain terminologies and facts of subfactor theory that we will need in the sequel. The centre of discussion in this chapter is a subfactor with integer index.

Chapter 3 is an important step towards the main theorem. In this chapter, we briefly recall the construction and basic aspects of subgroup-subfactors and set up notations for a proof of the main theorem. The heart of this chapter is the exhibition of a model for the tower of basic constructions of a subgroup-subfactor in terms of matrix algebras of operators. Further, this also gives an explicit description of the standard invariant of the subgroup-subfactor in terms of operator matrices. Apart from this, we show that the fixed algebra subfactor is the dual of the subgroup-subfactor.

Chapter 4 begins with a brief recollection of the basics of planar algebras. We primarily follow [KS04] for these discussions. We then introduce the notion of group action on a planar algebra and discuss the invariant planar subalgebras thus obtained. We then move on to recall the notion of associating a planar algebra to a finite bipartite graph with a spin function as given by Jones [Jon00a]. We do some elementary calculations related to certain tangle maps that we shall use in what follows afterwards. We give an identification between the dual of the planar algebra of a bipartite graph and the planar algebra of the flip of that bipartite graph. Finally, we introduce the notion of group action on a finite bipartite graph with a spin function and analyse its implications on the planar algebra associated to it.

Finally, all preparations at place, in Chapter 5, we prove the main theorem of the thesis, which identifies the planar algebra of the subgroup-subfactor with the group invariant planar subalgebra of the planar algebra of the bipartite graph $\star_n$, the graph with n even vertices and 1 odd vertex, where n is the index of the subgroup in the ambient group. By making use of the fact that the dual of the planar algebra of a subfactor is the planar algebra of the dual subfactor, we also give an identification of the planar algebra of the fixed algebra subfactor $R^G \subset R^H$ with the group invariant planar subalgebra of the planar algebra of the flip of the bipartite graph $\star_n$. And the chapter ends with some natural questions that arise from this work.

Apart from this, we have also made an attempt to understand the rotation maps, that appears in [Jon], in terms of Ocneanu's paragroup, which we discuss in the appendix. The motivation behind this was to obtain a bridge between Jones' planar algebras and Ocneanu's paragroup theory.

Some Remarks

1. There are references in literature giving different models for the basic construction tower and the standard invariant of the subgroup-subfactor - see, for instance, [KY92] and [BL]. Furthermore, Bina Bhattacharyya and Zeph Landau [BL] have given descriptions of planar algebras of an intermediate subfactor and a subgroup-subfactor is

a particular case of it. However, the description of the planar algebra of a subgroup-subfactor that we present here is different from theirs, and is more concrete. One immediate consequence of this description is that, given any pair of finite groups $H \subset G$ with index n, the planar algebra of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$ is sandwiched in between the planar algebras $P(\star_n)^{S_n}$ and $P(\star_n)$ - see Corollary 5.4.9.

2. Whenever we refer to some result, for instance, - see [JS97] -, we do not mean that it is the first text in which the result appeared. It is just to save the time of the reader by letting her/him know of an appropriate reference. The same holds even while citing some reference(s) at the beginning of a Lemma, Proposition or a Theorem.

Chapter 2

Subfactors

The short first section starts with definitions of matrix algebras and matrix maps; and then we fix a convention on indexing the elements of block matrices. §2 is basically a brief recollection of some terminologies and elementary facts from the theory of subfactors related to (Jones) index, basic construction, tower of basic constructions and the standard invariant of a subfactor. Further, we recall the definition of basis for a subfactor and present some related results that we shall need in the sequel. After this brief discussion, we move on to show that $M_\Lambda(\cdot)$ is a functor for basic construction of subfactors, a fact which will be used extensively in Chapter 3. Finally, we give a useful modification of the fact that any finite truncation of the tower of basic constructions of a finite index subfactor can be identified with an appropriate truncation of the amplification of that tower.

2.1 Matrix Algebras and Matrix Maps

We shall only consider algebras over the field $\mathbb{C}$, and they will always be associative.

Let Λ be an index set. Given an algebra P, we set

$$M_\Lambda(P) = \{((a_{\lambda,\lambda'})_{(\lambda,\lambda')\in\Lambda\times\Lambda}) : a_{\lambda,\lambda'} \in P\}.$$

Thus $M_\Lambda(P)$ is also an algebra with usual matrix multiplication, and is a $*$-algebra if P is one, with the obvious involution. Clearly

$$M_\Lambda(P) \cong M_\Lambda(\mathbb{C}) \otimes P.$$

For any two algebras P and Q, and a map $\theta : P \to Q$, we define the matrix map $M_\Lambda(\theta) : M_\Lambda(P) \to M_\Lambda(Q)$ by

$$[M_\Lambda(\theta)(A)]_{\lambda_1,\lambda_2} = \theta(A_{\lambda_1,\lambda_2}),\ \forall\ \lambda_1, \lambda_2 \in \Lambda,\ A \in M_\Lambda(P). \tag{2.1}$$

With these definitions, we note that

1. *$M_\Lambda(\theta)$ is one-one iff θ is so;*
2. *$M_\Lambda(\theta)$ is onto iff θ is so;*
3. *$M_\Lambda(\theta)$ is an algebra map iff θ is so; and*
4. *$M_\Lambda(\theta)$ is a $*$-preserving map iff θ is so, in case P and Q are $*$-algebras.*

Then for any two index sets Λ and Γ, and an algebra P, we view $M_\Lambda(M_\Gamma(P))$ as the set of block matrices, whose blocks are determined by the index set Λ and the matrices in each block are members of $M_\Gamma(P)$. Thus we identify it with the algebra $M_{\Gamma\times\Lambda}(P)$ via the correspondence

$$\begin{aligned} M_\Lambda(M_\Gamma(P)) \ni A \quad &\longmapsto \quad \widetilde{A} \in M_{\Gamma\times\Lambda}(P), \\ \widetilde{A}_{(\gamma_1,\lambda_1),(\gamma_2,\lambda_2)} \quad &:= \quad (A_{\lambda_1,\lambda_2})_{\gamma_1,\gamma_2}, \end{aligned} \tag{2.2}$$

i.e., we make the convention that the last two indices λ_1, λ_2, in a pair of tuples $(\gamma_1, \lambda_1), (\gamma_2, \lambda_2)$, determine the matrix in the λ_1-λ_2 block of A and the first two indices γ_1, γ_2 determine the γ_1-γ_2 entry of that matrix.

With this convention, we shall interchangeably use A and $\widetilde{A}$ during calculations.

Thus, given algebras P and Q, index sets Λ and Γ, and an algebra map $\theta : P \to M_\Gamma(Q)$, the matrix map

$$M_\Lambda(\theta) : M_\Lambda(P) \quad \to \quad M_{\Gamma\times\Lambda}(Q)$$

is given by

$$[M_\Lambda(\theta)(A)]_{(\gamma_1,\lambda_1),(\gamma_2,\lambda_2)} \quad = \quad \theta(A_{\lambda_1,\lambda_2})_{\gamma_1,\gamma_2} =: \theta_{\gamma_1,\gamma_2}(A_{\lambda_1,\lambda_2}), \tag{2.3}$$

$\forall\ (\gamma_i, \lambda_i) \in \Gamma \times \Lambda,\ i = 1, 2,\ \forall\ A \in M_\Lambda(P)$.

We will have many occasions to use this description (2.3) of a matrix map in Chapter 3, where we give a model for the basic construction tower of a subgroup-subfactor.

2.2 Subfactors

We first recall the notion of basic construction, and thereafter the standard invariant, of a subfactor. We then discuss the concept of bases for subfactors.

And finally, we present some facts related to integer index subfactors that we will need in the sequel.

2.2.1 The Standard Invariant

Let $N \subset M$ be a unital inclusion of II_1 factors. Such a pair is usually called a *subfactor*. Jones [Jon83] introduced the notion of index $[M : N]$ for such a pair and showed that it can take values in the set $\{4\,cos^2 \frac{\pi}{n+1} : n \geq 2\} \cup [4, \infty]$. For a finite index subfactor $N \subset M$, he extends M to another II_1-factor M_1 generated by M and a projection e_N ($M_1 = \langle M, e_N \rangle$), where e_N is called the projection which implements the tr_M preserving conditional expectation $E_N : M \to N$, i.e., $e_N x e_N = E_N(x) e_N$ and $tr_N(E_N(x)) = tr_M(x)\ \forall\, x \in M$.

This passage from $N \subset M$ to $N \subset M \subset^{e_N} M_1$ is known as the *basic construction* of the subfactor $N \subset M$, and e_N the *Jones projection* implementing the basic construction. The index $[M_1 : M]$ is the same as the initial index $[M : N]$.

The basic construction for a finite index subfactor is unique in the following sense.

Proposition 2.2.1 *[PP88] Let $N \subset M$ be a finite index subfactor and $N \subset M \subset^{e_N} M_1$ be its basic construction. Suppose $\widetilde{M}$ is another II_1-factor containing M and a projection $\widetilde{e}$. Then the following are equivalent:*

1. *$\exists$ a $*$-isomorphism $\varphi : M_1 \to \widetilde{M}$ such that $\varphi(x) = x$ for all $x \in M$, and $\varphi(e_N) = \tilde{e}$.*
2. *(a) $\tilde{e} x \tilde{e} = E_N(x)\widetilde{e},\ \forall\, x \in M,\ \tilde{e} \neq 0$; and*
   (b) $\widetilde{M} = \langle M, \tilde{e} \rangle$.

Further, given a finite index subfactor $N \subset M$, writing $M_{-1} = N$ and $M_0 = M$, by iterating the above process of basic construction we obtain a tower of II_I-factors

$$N = M_{-1} \subset M_0 = M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \cdots M_k \subset^{e_{k+1}} M_{k+1} \subset \cdots , \quad (2.4)$$

where $M_{k-1} \subset M_k \subset^{e_{k+1}} M_{k+1}$ is the basic construction of the subfactor $M_{k-1} \subset M_k$, $\forall\, k \geq 0$. The tower (2.4) is usually referred to as the (Jones') basic construction tower of the subfactor $N \subset M$.

This tower further gives a grid consisting of two towers of relative commutants

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset \cdots \subset & N' \cap M_k & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & \mathbb{C} = & M' \cap M & \subset & M' \cap M_1 & \subset \cdots \subset & M' \cap M_k & \subset & \cdots . \end{array} \tag{2.5}$$

The above grid is an invariant for the subfactor $N \subset M$, and is known as its *standard invariant*.

Since $[M : N] < \infty$, it is easily seen that the relative commutants $N' \cap M_k$ and $M' \cap M_k$, $k \geq 0$ are all finite dimensional C^*-algebras. If we take the Bratteli diagram of the first tower in the standard invariant (2.5), and remove from it all those parts which are obtained by reflecting the previous stage, the graph that remains is called the *principal graph* of the subfactor $N \subset M$.

Definition 2.2.2 *A finite index subfactor $N \subset M$ is said to be of finite depth if its principal graph is finite.*

A von Neumann algebra M is said to be *approximately finite dimensional (AFD)* or *hyperfinite* if it is the weak closure of the union of an increasing sequence of finite dimensional *-subalgebras.

Murray and von Neumann [MN43] proved that there is only one hyperfinite II_1-factor upto isomorphism.

For some 'good subfactors' of the hyperfinite II_1-factor R, the standard invariant turns out to be a complete invariant.

Remark 2.2.3 *It follows from a much stronger result of Popa [Pop94] that the standard invariant of a finite index subfactor $N \subset M$ with finite depth is a complete invariant if M is isomorphic to the hyperfinite II_1-factor R.*

2.2.2 Bases

Before talking about bases, we first note that given a II_1-factor P, and any finite index set Λ, $M_\Lambda(P)$ is also a II_1-factor, and its faithful normalised trace is given by

$$tr_{M_\Lambda(P)}((p_{\lambda,\lambda'})) = \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} tr_P(p_{\lambda,\lambda}), \, \forall \, (p_{\lambda,\lambda'}) \in M_\Lambda(P).$$

Although elementary, the following is a very useful fact.

Lemma 2.2.4 *[PP86] Let $N \subset M$ be a subfactor with $\tau^{-1} = [M : N] < \infty$. Then, for each $x_1 \in M_1$, there exists a unique $x_0 \in M$ such that $x_1 e_1 = x_0 e_1$; and this element is given by $x_0 = \tau^{-1} E_M(x_1 e_1)$.*

The notion of basis for a subfactor first appeared in [PP86]. We shall, however, recall certain facts related to bases as discussed in [JS97].

Definition 2.2.5 *Let $N \subset M$ be a finite index subfactor and $n \geq [M : N]$ be an integer. A collection $\{\lambda_1, \ldots, \lambda_n\} \subset M$ is said to be a* basis *for M/N if the matrix $q := ((E_N(\lambda_i \lambda_j^*)))$ is a projection in $M_I(N)$ with $tr_{M_I(N)}(q) = \frac{[M:N]}{n}$, where $I = \{1, 2, \ldots, n\}$.*

The following proposition contains every thing about bases that we shall need in the sequel. We refer to [JS97] for the proofs, except for the converse of (3), which we prove here.

Proposition 2.2.6 *[JS97] Let $N \subset M$ be a subfactor with $\tau^{-1} = [M : N] < \infty$. Then*

1. *for any integer $n \geq [M : N]$, and any projection $q \in M_I(N)$ with $tr_{M_I(N)}(q) = \frac{[M:N]}{n}$, there always exists a basis $\{\lambda_1, \ldots, \lambda_n\} \subset M$ for M/N such that $q_{i,j} = E_N(\lambda_i \lambda_j^*)$, $\forall\, i, j \in I = \{1, \ldots, n\}$;*
2. *if a collection $\{\lambda_1, \ldots, \lambda_n\} \subset M$ is a basis for M/N, then*
$$x = \sum_{i=1}^{n} E_N(x \lambda_i^*) \lambda_i, \ \forall\, x \in M;$$
3. *a collection $\{\lambda_1, \ldots, \lambda_n\} \subset M$ is a basis for M/N iff $\sum_{i=1}^{n} \lambda_i^* e_1 \lambda_i = 1$; and*
4. *if $\{\lambda_i : 1 \leq i \leq n\}$ is a basis for M/N, then $\{\tau^{-1/2} e_1 \lambda_i : 1 \leq i \leq n\}$ is a basis for M_1/M.*

Proof: Converse of (3): Let $\{\lambda_1, \ldots, \lambda_n\} \subset M$ be a collection satisfying $\sum_{i=1}^{n} \lambda_i^* e_1 \lambda_i = 1$. Then, taking $q_{i,j} = E_N(\lambda_i \lambda_j^*)$, for all $i, j \in I := \{1, \ldots, n\}$, we note that $q := (q_{i,j}) \in M_I(N)$ is a projection. Indeed, q is clearly self adjoint and for every $i, j \in I$,

$$\sum_{k=1}^{n} q_{i,k} q_{k,j} e_1 \ = \ \sum_{k=1}^{n} E_N(\lambda_i \lambda_k^*) E_N(\lambda_k \lambda_j^*) e_1$$

$$\begin{aligned} &= \sum_k e_1 \lambda_i \lambda_k^* e_1 \lambda_k \lambda_j^* e_1 \\ &= e_1 \lambda_i \lambda_j^* e_1 \\ &= E_N(\lambda_i \lambda_j^*) e_1, \end{aligned}$$

which, by the uniqueness condition in Lemma 2.2.4, implies that $\sum_{k=1}^n q_{i,k} q_{k,j} = q_{i,j}, \ \forall\, i,j \in I$, i.e., q is a projection.

Finally, the hypothesis $\sum_{i=1}^n \lambda_i^* e_1 \lambda_i = 1$, and the fact that $tr_M(e_1 x) = \tau tr_M(x), \forall\, x \in M$ - see [JS97, §3.1], imply that

$$1 = \sum_{i=1}^n tr_M(\lambda_i^* e_1 \lambda_i) = \tau \sum_{i=1}^n tr_M(\lambda_i \lambda_i^*).$$

This gives $tr_{M_I(N)}(q) = \frac{[M:N]}{n}$. Thus $\{\lambda_1, \ldots, \lambda_n\}$ is a basis for M/N. □

Remark 2.2.7 *In particular, we observe that there always exists a basis for M/N, for every finite index subfactor $N \subset M$.*

2.2.3 Some Useful Results

We saw that given a II_1-factor P, and any finite index set Λ, the matrix algebra $M_\Lambda(P)$ is also a II_1-factor. We now show that the matrix functor $M_\Lambda(\cdot)$ is, in fact, a functor for basic construction of subfactors.

Proposition 2.2.8 *Suppose $N \subset M \subset^{e_1} M_1$ is the basic construction for a subfactor $N \subset M$ with $\tau^{-1} = [M:N] < \infty$. Then for any finite index set Λ,*

$$M_\Lambda(N) \subset M_\Lambda(M) \subset M_\Lambda(M_1) \tag{2.6}$$

is the basic construction for the subfactor $M_\Lambda(N) \subset M_\Lambda(M)$, where

1. *the trace preserving conditional expectation $E_{M_\Lambda(N)} : M_\Lambda(M) \to M_\Lambda(N)$ is given by the matrix map $M_\Lambda(E_N)$; and*
2. *the Jones projection which implements the above conditional expectation $E_{M_\Lambda(N)}$ is given by the diagonal matrix $\tilde{e_1} \in M_\Lambda(M_1)$, with diagonal entries given by*

$$(\widetilde{e}_1)_{\lambda,\lambda} = e_1, \ \forall\, \lambda \in \Lambda.$$

Proof: Clearly, (2.6) is a tower of II_1-factors.
(1) Given $(a_{\lambda,\lambda'}) \in M_\Lambda(N)$ and $(x_{\lambda,\lambda'}) \in M_\Lambda(M)$, we have

$$\begin{aligned} M_\Lambda(E_N)((a_{\lambda,\lambda'})(x_{\lambda,\lambda'}))_{\lambda_1,\lambda_2} &= E_N([(a_{\lambda,\lambda'})(x_{\lambda,\lambda'})]_{\lambda_1,\lambda_2}) \\ &= E_N(\sum_{\lambda\in\Lambda} a_{\lambda_1,\lambda}x_{\lambda,\lambda_2}) \\ &= \sum_{\lambda\in\Lambda} a_{\lambda_1,\lambda}E_N(x_{\lambda,\lambda_2}) \\ &= [(a_{\lambda,\lambda'})(E_N(x_{\lambda,\lambda'}))]_{\lambda_1,\lambda_2} \\ &= [(a_{\lambda,\lambda'})M_\Lambda(E_N)((x_{\lambda,\lambda'}))]_{\lambda_1,\lambda_2}, \end{aligned}$$

for all $\lambda_1, \lambda_2 \in \Lambda$, i.e., $M_\Lambda(E_N)$ is left $M_\Lambda(N)$-linear. Likewise, $M_\Lambda(E_N)$ is right $M_\Lambda(N)$-linear as well. Further,

$$\begin{aligned} tr_{M_\Lambda(N)}\left(M_\Lambda(E_N)((x_{\lambda,\lambda'}))\right) &= \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} tr_N(E_N(x_{\lambda,\lambda})) \\ &= \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} tr_M(x_{\lambda,\lambda}) \\ &= tr_{M_\Lambda(M)}((x_{\lambda,\lambda'})), \end{aligned}$$

showing that $M_\Lambda(E_N)$ preserves trace. Thus, by uniqueness of the trace preserving conditional expectation $E_{M_\Lambda(N)}$, we see that $E_{M_\Lambda(N)} = M_\Lambda(E_N)$.
(2) Note that

$$\begin{aligned} \tilde{e_1}(x_{\lambda,\lambda'})\tilde{e_1} &= (e_1 x_{\lambda,\lambda'} e_1) \\ &= (E_N(x_{\lambda,\lambda'})e_1) \\ &= (E_N(x_{\lambda,\lambda'}))\tilde{e_1} \\ &= M_\Lambda(E_N)((x_{\lambda,\lambda'}))\tilde{e_1},\ \forall\,(x_{\lambda,\lambda'}) \in M_\Lambda(M). \end{aligned}$$

Further, we claim that $M_\Lambda(M_1) = \langle M_\Lambda(M), \tilde{e_1}\rangle$. To show this, fix a basis $\{\lambda_1,\ldots,\lambda_n\} \subset M$ for M/N; so, by Proposition 2.2.6, $\{\tau^{-1/2}e_1\lambda_j : 1 \leq j \leq n\}$ is a basis for M_1/M. Thus, $x^{(1)} = \tau^{-1}\sum_{j=1}^n E_M(x^{(1)}\lambda_j^* e_1)e_1\lambda_j$ for all $x^{(1)} \in M_1$. Then, for any $(x^{(1)}_{\lambda,\lambda'}) \in M_\Lambda(M_1)$, we obtain

$$(x^{(1)}_{\lambda,\lambda'}) = \tau^{-1}\sum_{j=1}^n (E_M(x^{(1)}_{\lambda,\lambda'}\lambda_j^* e_1)e_1\lambda_j)$$

$$= \quad \tau^{-1} \sum_{j=1}^{n} \Big(E_M(x^{(1)}_{\lambda,\lambda'}\, \lambda_j^* \, e_1) \Big) \; \tilde{e_1} \, \tilde{\lambda_j} \; \in \langle M_\Lambda(M), \tilde{e_1} \rangle,$$

where $\tilde{\lambda_j} \in M_\Lambda(M),\ 1 \leq j \leq n$, are the diagonal matrices with diagonal entries $(\tilde{\lambda_j})_{\lambda,\lambda} = \lambda_j,\, \forall\, \lambda \in \Lambda,\, 1 \leq j \leq n$.

And we are done in view of the 'uniqueness of basic construction' in the sense of Proposition 2.2.1. □

Remark 2.2.9 *It is readily seen, by Proposition 2.2.6 (3), that $\{\tilde{\lambda_j} : 1 \leq j \leq n\} \subset M_\Lambda(M)$, as in the proof of Proposition 2.2.8, is a basis for $M_\Lambda(M)/M_\Lambda(N)$.*

In Chapter 3, we shall demonstrate a model for the basic construction tower of a subgroup-subfactor. This will be done by using the fact that any finite truncation of the tower of basic constructions of $M_1 \subset M_2$ can be identified with an appropriate amplification of that of $N \subset M$. This has been proved in literature for any finite index subfactor - see [PP86, JS97]; however, since the subgroup-subfactors are of integer index, we present its modified form for integer index subfactors. We refer to [JS97] for a complete proof.

Proposition 2.2.10 *[JS97] Let $N \subset M$ be a subfactor with integer index $n = [M : N]$ and $I = \{1, 2, \ldots, n\}$. Then, for each $k \geq -1$, there exists an isomorphism of towers*

$$(M_I(N) \subset M_I(M) \subset \cdots \subset M_I(M_k)) \cong (M_1 \subset M_2 \subset \cdots \subset M_{k+2}).$$

Proof: We just recall the isomorphisms for the case $k = 0$ and refer to [JS97] for verification of the assertions.

Fix a basis $\{\lambda_1, \ldots, \lambda_n\} \subset M$ for M/N. Define maps $\varphi_1 : M_I(N) \to M_1$ and $\varphi_2 : M_I(M) \to M_2$ by

$$\begin{aligned} \varphi_1((a_{i,j})) &= \textstyle\sum_{i,j=1}^n \lambda_i^* a_{i,j} e_1 \lambda_j,\ \forall\ (a_{i,j}) \in M_I(N);\ \text{and} \\ \varphi_2((m_{i,j})) &= n \textstyle\sum_{i,j=1}^n \lambda_i^* e_1 m_{i,j} e_2 e_1 \lambda_j,\ \forall\ (m_{i,j}) \in M_I(M). \end{aligned}$$

Then $\varphi_i, i = 1, 2$ are both (unital normal) $*$-isomorphisms and φ_2 restricts on $M_I(N)$ to φ_1. □

What we need is a slight extension of the above fact for the case $k = 0$, which we state and prove in following lemma.

Lemma 2.2.11 *Let $N \subset M$ be a subfactor as in Proposition 2.2.10, and $\{\lambda_i : i \in I\}$ be a (n orthonormal) basis for M/N. Then there exists a unital inclusion $\theta : M \hookrightarrow M_I(N)$ and an isomorphism of towers*

$$\left(N \subset M \overset{\theta}{\hookrightarrow} M_I(N) \subset M_I(M)\right) \cong (N \subset M \subset M_1 \subset M_2). \tag{2.7}$$

In particular, the Jones projections $\widetilde{e}_1 \in M_I(N)$ and $\widetilde{e}_2 \in M_I(M)$, corresponding to e_1 and e_2 under the above tower isomorphism, are given by

$$(\widetilde{e}_1)_{i,j} = E_N(\lambda_i)\, E_N(\lambda_j^*) \quad \textit{and}$$
$$(\widetilde{e}_2)_{i,j} = n^{-1}\lambda_i\lambda_j^*,$$

respectively, $\forall\, i, j \in I$. And the trace preserving conditional expectation $E_M : M_I(N) \to M$ is given by

$$E_M((a_{i,j})) = \frac{1}{n}\sum_{i,j\in I} \lambda_i a_{i,j}\lambda_j^*,\ \forall\ (a_{i,j}) \in M_I(N). \tag{2.8}$$

Proof: Consider the projection $1 \in M_I(N)$. By Proposition 2.2.6 (1), we choose a basis $\{\lambda_1, \ldots, \lambda_n\} \subset M$ for M/N such that $E_N(\lambda_i\lambda_j^*) = \delta^i_j$, $\forall\, 1 \leq i, j \leq n$. Define a map $\theta : M \to M_I(N)$ by

$$\theta_{i,j}(x) = E_N(\lambda_i x \lambda_j^*),\ x \in M,\ 1 \leq i, j \leq n.$$

Clearly θ is a normal $*$-homomorphism and, by the choice of the basis $\{\lambda_i : 1 \leq i \leq n\}$, it is unital as well. Further, M being a II_1-factor, θ is injective.

Now consider the isomorphisms φ_i, $i = 1, 2$, as given in the proof of Proposition 2.2.10, with respect to the choice of the (orthonormal) basis made here. We note that the composition $\varphi_1 \circ \theta$ is the natural inclusion $M \hookrightarrow M_1$. Indeed, for each $x \in M$,

$$\varphi_1(\theta(x)) = \sum_{i,j=1}^{n} \lambda_i^* E_N(\lambda_i x \lambda_j^*) e_1 \lambda_j = \sum_{i,j=1}^{n} \lambda_i^* e_1 \lambda_i x \lambda_j^* e_1 \lambda_j = x.$$

Thus we have an isomorphism of the towers in (2.7).

The Jones projections for the tower $N \subset M \overset{\theta}{\hookrightarrow} M_I(N) \subset M_I(M)$ are given by $\widetilde{e}_1 := \varphi_1^{-1}(e_1)$ and $\widetilde{e}_2 := \varphi_2^{-1}(e_2)$. Thus, in order to get its matrix entries, we need to know the inverses of the maps $\varphi_i, i = 1, 2$.

It is easily seen, using (2), (3) and (4) of Proposition 2.2.6, that $\varphi_1((nE_N(E_M(\lambda_i x_1 \lambda_j^* e_1)))) = x_1$ for all $x_1 \in M_1$; so

$$\begin{aligned}\varphi_1^{-1}(x_1)_{i,j} &= nE_N\left(E_M(\lambda_i x_1 \lambda_j^* e_1)\right),\ \forall\, x_1 \in M_1,\ i,j \in I;\ \text{and likewise}\\ \varphi_2^{-1}(x_2)_{i,j} &= n^2 E_M\left(E_{M_1}(e_1 \lambda_i x_2 \lambda_j^* e_1 e_2)\right),\ \forall\, x_2 \in M_2,\ i,j \in I.\end{aligned}$$

In particular, this shows that

$$\begin{aligned}(\widetilde{e}_1)_{i,j} &= nE_N(E_M(\lambda_i e_1 \lambda_j^* e_1)) = nE_N(E_M(\lambda_i E_N(\lambda_j^*) e_1))\\ &= E_N(\lambda_i E_N(\lambda_j^*)) = E_N(\lambda_i) E_N(\lambda_j^*), \quad \text{and} \qquad (2.9)\\ (\widetilde{e}_2)_{i,j} &= n^2 E_M(E_{M_1}(e_1 \lambda_i e_2 \lambda_j^* e_1 e_2)) = n^2 E_M(e_1 \lambda_i E_{M_1}(E_M(\lambda_j^* e_1) e_2))\\ &= nE_M(e_1 \lambda_i) E_M(\lambda_j^* e_1) = n^{-1} \lambda_i \lambda_j^*, \qquad (2.10)\end{aligned}$$

$\forall\, i,j \in I$, where in the above simplifications we have used the fact that $E_M(e_1) = n^{-1} = E_{M_1}(e_2)$ - see [JS97, §3.1].

Finally, equation (2.8) readily shows that E_M is M-M linear, i.e.,

$$E_M(\theta(x)(a_{i,j})\theta(y)) = xE_M((a_{i,j}))y,\ \forall\ (a_{i,j}) \in M_I(N),\ x,y \in M,$$

and, by the choice of above basis, it is also $tr_{M_I(N)}$-preserving. Thus, by uniqueness of such a map, it is indeed the trace preserving conditional expectation from $M_I(N)$ onto M. $\square$

In accordance with [JS97], for each $k \geq 1$, we define a map $\theta^{(k)} : M \to M_{I^k}(N)$ and show that these are unital $*$-morphisms.

Lemma 2.2.12 *With $N \subset M$, $\{\lambda_i : i \in I\}$ and θ as in Lemma 2.2.11, for each $k \geq 1$, define $\theta^{(k)} : M \to M_{I^k}(N)$ by*

$$\begin{aligned}\theta^{(k)}_{\underline{i},\underline{j}}(x) &= \theta_{i_1,j_1}(\theta_{i_2,j_2}(\cdots \theta_{i_k,j_k}(x)\cdots))\\ &= E_N(\lambda_{i_1} E_N(\lambda_{i_2} E_N(\cdots E_N(\lambda_{i_k} x \lambda_{j_k}^*)\cdots)\lambda_{j_2}^*)\lambda_{j_1}^*),\end{aligned}$$

for all $\underline{i} = (i_1,\ldots,i_k)$, $\underline{j} = (j_1,\ldots,j_k) \in I^k$. Then $\theta^{(k)}$ is a unital $*$-homomorphism for each $k \geq 1$.

Proof: The choice of the basis $\{\lambda_i : i \in I\}$ for M/N readily implies that $\theta^{(k)}$ is unital for each $k \geq 1$. The other properties of $\theta^{(k)}$ have been proved in [JS97, § A.4]. $\square$

Chapter 3

The Subgroup-Subfactor

The crux of this chapter is the exhibition of an explicit model for the basic construction tower, and thereafter of the standard invariant, of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$ in terms of certain operator matrices. Apart from it, this chapter also serves as a preparation for a proof of the main theorem, which identifies the planar algebra of the subgroup-subfactor with the group invariant planar subalgebra of the planar algebra of a certain bipartite graph, that we shall demonstrate in Chapter 5.

We briefly recall the notion of crossed product of a von Neumann algebra with a discrete group acting on it; and then we restrict ourselves to the crossed product of the hyperfinite II_1-factor R with a finite group acting outerly on it. Thus we arrive at the definition of a subgroup-subfactor. We recall certain results associated to subgroup-subfactors and then discuss another subfactor obtained from the crossed product consideration, namely the fixed algebra subfactor. It turns out that this new subfactor is dual to the subgroup-subfactor. In §2, we obtain a model for the basic construction tower of a subgroup-subfactor and discuss certain useful aspects of this model. Finally in §3, we describe the standard invariant of the subgroup-subfactor via this model. We fix up certain notations for operator matrices and identify certain bases (indexed by some orbits) for the relative commutants appearing in the grid of the standard invariant.

3.1 Basics

Let $M \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and $\alpha : G \rightarrow Aut(M)$ be a group action of a (discrete) group G. Taking $\widetilde{H} = \mathcal{H} \otimes \ell^2(G)$, one obtains a faithful normal $*$-homomorphism $\pi : M \rightarrow \mathcal{L}(\widetilde{\mathcal{H}})$ and a faithful unitary representation $\lambda : G \to \mathcal{L}(\widetilde{\mathcal{H}})$ satisfying the commutation relation

$$\lambda(g)\pi(x)\lambda(g)^* = \pi(\alpha_g(x)), \ \forall\, x \in M,\, g \in G. \tag{3.1}$$

The *crossed product* $M \rtimes_\alpha G$ is defined to be the von Neumann algebra $(\pi(M) \cup \lambda(G))'' \subset \mathcal{L}(\widetilde{\mathcal{H}})$.

It is a fact - see, for instance, [Sak71, Sun87a] - that the isomorphism class of the von Neumann algebra $M \rtimes_\alpha G$ is independent of the Hilbert space $\mathcal{H}$ that we chose M to be realized upon.

Given a von Neumann algebra $M \subset \mathcal{L}(\mathcal{H})$, an automorphism θ of M is said to be *free* if for a given x in M, $xy = \theta(y)x$ for all y in M implies $x = 0$. It is easily seen - see [JS97, § A.4]- that if M is a factor, then $\theta \in Aut(M)$ is free if and only if it is *outer*, i.e., it is not an inner automorphism.

Let R be the hyperfinite II_1-factor viewed as sitting in $\mathcal{L}(L^2(R))$ as left multiplication operators, where $L^2(R)$ is the GNS-construction of R with respect to its faithful tracial state tr_R. Let G be a finite group and $\alpha : G \to Aut(R)$ be an outer action of G on R, i.e., α_g is outer for all $e \neq g \in G$. Such an action always exists on the hyperfinite II_1-factor R - see [Jon80]. Then the crossed product $R_1 := R \rtimes_\alpha G$ is also a II_1-factor, and the subfactor $R \subset R \rtimes_\alpha G$ is *irreducible*, i.e., $\pi(R)' \cap R \rtimes_\alpha G = \mathbb{C}$.

We have another useful realization of the crossed product $R \rtimes G$. By uniqueness of the trace on R, it turns out that R_1 acts on $L^2(R)$ as well:

Given $g \in G$, $u_g(\hat{x}) := \widehat{\alpha_g(x)}$, $\forall\, x \in R$ gives rise to a unitary representation $g \mapsto u_g$ of G on $L^2(R)$, where $\hat{y} := \pi_{tr_R}(y)$, $y \in R$ via the GNS-construction. And the analog of the commutation relation (3.1) holds, i.e.,

$$u_g x u_g^* = \alpha_g(x),\, \forall\, x \in R,\, g \in G.$$

Thus there is a natural homomorphism of R_1 onto the von Neumann algebra $(R \cup \{u_g : g \in G\})'' \subset \mathcal{L}(L^2(R))$. Further, R_1 being a factor, this is a $*$-isomorphism.

Throughout the sequel, we shall stick to this realization of the crossed product $R \rtimes_\alpha G$ in $\mathcal{L}(L^2(R))$.

Let H be a subgroup of G. Then $(R \cup \{u_h : h \in H\})'' \subset \mathcal{L}(L^2(R))$ is isomorphic to the subfactor $R \rtimes_{\alpha/_H} H$, and thus we obtain a subfactor

$$R \rtimes_{\alpha_{|_H}} H \subset R \rtimes_\alpha G.$$

Such a subfactor is usually known as a *subgroup-subfactor*. We usually suppress α and simply write it as $R \rtimes H \subset R \rtimes G$. Once and for all, we fix a set of representatives $\{g_1, \ldots, g_n\}$ for the right H-coset decomposition of G with $g_1 = e$, i.e., $G = \sqcup_{i=1}^{n} Hg_i$, where $n = [G : H]$.

We now list some standard facts of subgroup-subfactors, whose proofs can be found, for instance, in [JS97] or [EK98].

Proposition 3.1.1 *With notations as in the preceding paragraph, we have the following:*

1. *$R \rtimes H \subset R \rtimes G$ is an irreducible subfactor with index $[G : H]$ and is of finite depth.*
2. *Each element of $R \rtimes G$ (resp., $R \rtimes H$) can be expressed as a finite sum $\sum_{g\in G} x_g u_g$, $x_g \in R$ (resp., $\sum_{h\in H} x_h u_h, x_h \in R$).*
3. *The trace on $R \rtimes G$ is given by $tr_{R\rtimes G}(\sum_{g\in G} x_g u_g) = tr_R(x_e)$, and the $tr_{R\rtimes G}$-preserving conditional expectation $E_{R\rtimes H} : R \rtimes G \to R \rtimes H$ is given by*
$$E_{R\rtimes H}(\sum_{g\in G} x_g u_g) = \sum_{h\in H} x_h u_h. \tag{3.2}$$
4. *$\{u_{g_1}, \ldots, u_{g_n}\}$ is a basis for $R\rtimes G/R\rtimes H$ and the corresponding projection $(E_{R\rtimes H}(u_{g_i} u_{g_j}^*))$ is the identity matrix, i.e., it is an orthonormal basis.*

The crossed product construction $R \rtimes G$ yields one more subfactor corresponding to the fixed-point subalgebras of R, namely $R^G \subset R^H$, where $R^K = \{x \in R : \alpha_t(x) = x,\ \forall\, t \in K\}, K = G, H$. And, it turns out that forming the crossed product is dual to taking the fixed-point algebra.

Proposition 3.1.2 *[JS97] With notations as in the previous paragraph, we have*

1. *$R^G \subset R$ is an irreducible subfactor with index $|G|$ and is of finite depth.*
2. *$R^G \subset R \subset R \rtimes G$ is the basic construction for the subfactor $R^G \subset R$.*

Even more is true; the subgroup-subfactor is also the dual of the fixed subfactor $R^G \subset R^H$.

Proposition 3.1.3
1. *$R^G \subset R^H$ is an irreducible subfactor with index $[G : H]$ and has finite depth.*
2. *$(R^G \subset R^H) \cong (M \subset M_1)$, where M_1 is the II_1-factor obtained by the basic construction of the subgroup-subfactor $N := R \rtimes H \subset R \rtimes G =: M$.*

Proof: (1) is a well known fact, and although (2) is probably a folk-lore, for the sake of completeness, we demonstrate its proof here.

Basically, the idea is to amplify the outer action α to an outer action $\widetilde{\alpha}$ of G on the amplification $\widetilde{R} := M_G(R)$ and then to obtain subfactor isomorphisms

$$(R^G \subset R^H) \cong (\widetilde{R}^G \subset \widetilde{R}^H) \cong (M \subset M_1).$$

It will be easier to work with the identification $M_G(R) \cong M_G(\mathbb{C}) \otimes R$ given by the map $(r_{s,t}) \mapsto \sum_{s,t \in G} e_{s,t} \otimes r_{s,t}$, where $\{e_{s,t} : s, t \in G\}$ is the canonical system of matrix units of the matrix algebra $M_G(\mathbb{C})$.

We define $\widetilde{\alpha}_g(e_{s,t} \otimes r) = e_{gs,gt} \otimes \alpha_g(r)$ for all $g, s, t \in G$ and $r \in R$. It is readily seen that the outerness of α on R implies the same for $\widetilde{\alpha}$ on $\widetilde{R}$. Further, it is well known that $\widetilde{R}$ is isomorphic to the hyperfinite II_1-factor R. Let $\widetilde{R} \xrightarrow{\psi} R$ give an isomorphism. Since any two outer actions of a finite group on the hyperfinite II_1-factor R are conjugate - see [Jon80], there exists an automorphism φ of R which intertwines the actions α and $\psi \circ \widetilde{\alpha} \circ \psi^{-1}$ of G, i.e., $\varphi^{-1} \circ \alpha_g \circ \varphi = \psi \circ \widetilde{\alpha}_g \circ \psi^{-1}$, $\forall\, g \in G$. This shows that the isomorphism $\widetilde{R} \xrightarrow{\varphi \circ \psi} R$ intertwines the actions $\widetilde{\alpha}$ and α. Thus

$$(R^G \subset R^H) \cong (\widetilde{R}^G \subset \widetilde{R}^H).$$

In order to obtain the second isomorphism, we make repeated use of Lemma 2.2.11. Note that $\{u_h : h \in H\}$, $\{u_g : g \in G\}$ and $\{u_{g_i} : i \in I\}$ form bases for $R \rtimes H/R$, $R \rtimes G/R$ and $R \rtimes G/R \rtimes H$, respectively. Thus, with these choices of bases, by Lemma 2.2.11, the towers

$$R \subset R \rtimes H \overset{\theta^H}{\hookrightarrow} M_H(R), \quad R \subset R \rtimes G \overset{\theta^G}{\hookrightarrow} M_G(R) \text{ and}$$
$$R \rtimes H \subset R \rtimes G \overset{\theta}{\hookrightarrow} M_I(R \rtimes H)$$

are instances of basic constructions, where the maps θ^H, θ^G and θ are defined as in Lemma 2.2.11 with respect to the above bases. For each $r \in R$ and $g \in G$, we have $\theta^G_{s,t}(r) = \delta^s_t\, \alpha_s(r)$ and $\theta^G_{s,t}(u_g) = \delta^e_{sgt^{-1}}$, for all s, $t \in G$; thus

$$\theta^G(r) = \sum_{s \in G} e_{s,s} \otimes \alpha_s(r) \text{ and } \theta^G(u_g) = \sum_{s \in G} e_{s,sg} \otimes 1.$$

Likewise, for each $r \in R$ and each $h \in H$, we have $\theta^H(r) = \sum_{x \in H} f_{x,x} \otimes \alpha_x(r)$ and $\theta^H(u_h) = \sum_{x \in H} f_{x,xh} \otimes 1$, where $\{f_{x,y} : x, y \in H\}$ is the canonical system of matrix units for the matrix algebra $M_H(\mathbb{C})$, and we have identified

$M_H(R)$ canonically with $M_H(\mathbb{C}) \otimes R$.

We define a map $\Phi : M_I(R \rtimes H) \to M_G(R)$ by

$$\Phi(A)_{xg_i,\, yg_j} \;=\; \theta^H_{x,\,y}(A_{i,\,j}),\ \forall\, A \in M_I(R \rtimes H),\, x, y \in H,\, i, j \in I. \tag{3.3}$$

Note that under the canonical identifications $M_I(M_H(R)) \cong M_{H\times I}(R) \cong M_G(R)$ (the second one being given by $(r_{(x,i),\,(y,j)}) \mapsto (r_{xg_i,\,yg_j})$), the map Φ is precisely the matrix map $M_I(\theta^H)$. In particular, this says that Φ is an injective $*$-homomorphism. Further, it is easily seen that the diagram

$$\begin{tikzcd} R \rtimes G \arrow[d, hook, "\theta"] \arrow[dr, hook, "\theta^G"] & \\ & M_G(R) \\ M_I(R \rtimes H) \arrow[ur, hook, "\Phi"] & \end{tikzcd}$$

commutes. We claim that $\theta^G(R \rtimes G) = M_G(R)^G$ and $\Phi(M_I(R \rtimes H)) = M_G(R)^H$. This would then imply the second isomorphism of the subfactors $(\widetilde{R}^G \subset \widetilde{R}^H) \cong (M \subset M_1)$.

Note that $M_G(R)^G$ is generated as a vector space by the elements of the type $\sum_{g\,\in\, G} e_{gs,\,gt} \otimes \alpha_g(r)$, where $r \in R$ and $(s,\,t)$ varies over a set of representatives of G-orbits of $G \times G$ under the diagonal left multiplication action. Thus from the above description of the map θ^G, we see that $\theta^G(R \rtimes G) \subset \widetilde{R}^G$.

The G-orbits of $G \times G$ are given by $\{s(e,g) : s \in G\}$ with g varying over the group G. Then, for each $r \in R$ and the representative (e,g) of an orbit of $G \times G$, we have $\theta(ru_g) = \sum_{s\,\in\, G} e_{s,\,sg} \otimes \alpha_s(r)$. Thus $\theta^G(R \rtimes G) = \widetilde{R}^G$.

Exactly as for θ^G, we have $\theta^H(R \rtimes H) \subset M_H(R)^H$, where the outer H-action on the hyperfinite II_1-factor $M_H(R)$ is the amplification, as above, of the restricted action $\alpha/_H$. This shows that

$$\alpha_h(\theta^H_{h_1,\,h_2}(\omega)) = \theta^H_{hh_1,\,hh_2}(\omega),\ \forall\, h, h_1, h_2 \in H,\, \omega \in R \rtimes H.$$

Thus, for any $A \in M_I(R \rtimes H)$, we have

$$\begin{aligned} \widetilde{\alpha}_h(\Phi(A)) &= \sum_{x,y\,\in\, H,\, i,j\,\in\, I} e_{hxg_i,\, hyg_j} \otimes \alpha_h(\theta^H_{x,\,y}(A_{i,\,j})) \\ &= \sum_{x,y\,\in\, H,\, i,j\,\in\, I} e_{hxg_i,\, hyg_j} \otimes \theta^H_{hx,\,hy}(A_{i,\,j}) \\ &= \Phi(A),\ \forall\, h \in H, \end{aligned}$$

showing that $\Phi(M_I(R \rtimes H)) \subset M_G(R)^H$. On the other hand, $M_G(R)^H$ is generated as a vector space by elements of the form $\sum_{h \in H} e_{hx,\,hy} \otimes \alpha_h(r)$, where $r \in R$ and $(x,\,y)$ varies over a set of representatives of H-orbits of $G \times G$. Note that, for each $g \in G$, the G-orbit $\{s(e,g) : s \in G\}$ is invariant under H and decomposes as a disjoint union of n H-orbits, i.e.,

$$\{s(e,g) : s \in G\} = \sqcup_{i \in I}\{xg_i(e,g) : x \in H\}.$$

Thus, for each representative (x,y) of an H-orbit of $G \times G$, we have $(x,y) = hg_i(e,g)$, for a unique $(h,i) \in H \times I$. Let $Hg_ig = Hg_j$, for some $j \in I$; and $h_o := g_igg_j^{-1} \in H$. Then

$$\begin{aligned}
\sum_{z \in H} e_{zx,\,zy} \otimes \alpha_z(r) &= \sum_{z \in H} e_{zhg_i,\,zhg_ig} \otimes \alpha_z(r) \\
&= \sum_{z \in H} e_{zg_i,\,zg_ig} \otimes \alpha_{zh^{-1}}(r) \\
&= \sum_{z \in H} e_{zg_i,\,zh_og_j} \otimes \alpha_{zh^{-1}}(r).
\end{aligned}$$

Consider $A \in M_I(R \rtimes H)$ given by $A_{k,\,l} = \delta_i^k\,\delta_j^l \alpha_{h^{-1}}(r)h_o,\ \forall\, k,l \in I$. Then

$$\Phi(A)_{wg_k,\,zg_l} = \theta^H_{w,\,z}(A_{k,\,l}) = \delta_i^k\,\delta_j^l\,\alpha_w(\alpha_{h^{-1}}(r))\,\delta_z^{wh_o},\ \forall\, w,z \in H,\ k,l \in I,$$

i.e., $\Phi(A) = \sum_{z \in H} e_{zg_i,\,zh_og_j} \otimes \alpha_{zh^{-1}}(r)$. This shows that $\Phi(M_I(R \rtimes H)) = M_G(R)^H$ and that completes the proof. □

3.2 The Basic Construction Tower of $R \rtimes H \subset R \rtimes G$

Taking $N = R \rtimes H \subset R \rtimes G = M$, by Lemma 2.2.11, with respect to the orthonormal basis $\{u_{g_i} : i \in I\}$ for M/N, we have

$$(N \subset M \subset M_1 \subset M_2) \cong (N \subset M \overset{\theta}{\hookrightarrow} M_I(N) \subset M_I(M)).$$

Then, by repeated applications of Proposition 2.2.8, we see that

$$\begin{array}{c}
M_{2k-1} \subset M_{2k} \subset M_{2k+1} \subset M_{2k+2} \\
\cong \\
M_{I^k}(N) \subset M_{I^k}(M) \overset{\Theta_{k+1}}{\hookrightarrow} M_{I^{k+1}}(N) \subset M_{I^{k+1}}(M), \quad \forall\, k \geq 0,
\end{array} \tag{3.4}$$

where, as in §2.1, for each $k \geq 1$, we have inductively identified $M_{I^k}(X)$ with $M_I(M_{I^{k-1}}(X))$ for $X \in \{N, M\}$, and $\Theta_{k+1} := M_I(\Theta_k)$ with $\Theta_1 = \theta$. Thus, we have the following:

Theorem 3.2.1 *With notations as in the preceding paragraph, the tower*

$$N \subset M \overset{\Theta_1}{\hookrightarrow} M_I(N) \subset M_I(M) \overset{\Theta_2}{\hookrightarrow} \cdots \subset M_{I^k}(M) \overset{\Theta_{k+1}}{\hookrightarrow} M_{I^{k+1}}(N) \subset \cdots \tag{3.5}$$

is a model for the basic construction tower of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$.

Henceforth, we shall work with the above model as the basic construction tower of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$.

Remark 3.2.2 *There is nothing particular about the subgroup-subfactor in the tower (3.5). In fact, (3.5) gives a model for the basic construction tower of any subfactor $N \subset M$ with integer index.*

Given a k-tuple $\underline{i} \in I^k$, we write $\underline{i}_{[r,\,s]}$ for the $(s-r+1)$-tuple $(i_r, i_{r+1}, \ldots, i_s)$ for all $1 \leq r < s \leq k$. In fact, if $r = 1$ (resp., $s = k$), we simply write $\underline{i}_{\ s]}$ (resp., $\underline{i}_{[r}$) for $\underline{i}_{[r,\,s]}$. And, for any $j \in I$, we write $(j, \underline{i}_{[r,\,s]})$ (resp., $(\underline{i}_{[r,\,s]}, j)$) for the tuple $(j, i_r, \ldots, i_s)$ (resp., $(i_r, \ldots, i_s, j)$).

We point out certain simple yet useful facts related to the model (3.5).

Corollary 3.2.3 *For each $k \geq 1$, let the Jones projections e_{2k-1} and e_{2k} be mapped to the operator matrices $\widetilde{e}_{2k-1} \in M_{I^k}(N)$ and $\widetilde{e}_{2k} \in M_{I^k}(M)$, respectively, under the identifications (3.4). Then*

$$(\widetilde{e}_{2k-1})_{\underline{i},\,\underline{j}} = \delta^{\underline{i}}_{\underline{j}}\, \delta^{i_1}_{1},\ \forall\ \underline{i},\ \underline{j} \in I^k;\ \textit{and} \tag{3.6}$$

$$(\widetilde{e}_{2k})_{\underline{i},\,\underline{j}} = \frac{1}{n}\, \delta^{\underline{i}_{[2}}_{\underline{j}_{[2}}\ u_{g_{i_1} g_{j_1}^{-1}},\ \forall\ \underline{i},\ \underline{j} \in I^k. \tag{3.7}$$

And, for each $k \geq 0$, the trace preserving conditional expectations $E_{M_{2k}} : M_{I^{k+1}}(N) \to M_{I^k}(M)$ and $E_{M_{2k-1}} : M_{I^k}(M) \to M_{I^k}(N)$ are given by the matrix maps

$$E_{M_{2k}} = M_{I^k}(E_M)\ \textit{and}\ E_{M_{2k-1}} = M_{I^k}(E_N), \tag{3.8}$$

respectively, where E_M is given as in Lemma 2.2.11, and as usual $M_{I^k}(T) := M_I(M_{I^{k-1}}(T))$ for $T \in \{E_M, E_N\}$.

Proof: All these assertions follow readily from Proposition 2.2.8 and Lemma 2.2.11. $\square$

In order to obtain a better understanding of the standard invariant of the subgroup-subfactor $R \rtimes H \subset R \rtimes G$, we need to know how N sits in the higher II_1-factors of this model. For that, we first need to know how the maps Θ_k, $k \geq 1$ look like.

Lemma 3.2.4 *For each $k \geq 0$, $\Theta_{k+1} : M_{I^k}(M) \to M_{I^{k+1}}(N)$ is given by*

$$\Theta_{k+1}((a_{\underline{i},\underline{j}}))_{\underline{u},\underline{v}} = \theta_{u_1,v_1}(a_{\underline{u}_{[2},\underline{v}_{[2}}),\ \forall\,(a_{\underline{i},\underline{j}}) \in M_{I^k}(M),\ \underline{u},\ \underline{v} \in I^{k+1}. \tag{3.9}$$

Proof: We prove this fact by induction on k. For $k = 0$, it is the mere definition of Θ_1. Suppose (3.9) holds for some $k > 0$. Then for $((a_{\underline{i},\underline{j}})) \in M_{I^{k+1}}(M)$ and $\underline{u}$, $\underline{v} \in I^{k+2}$, we have

$$\begin{aligned}
\Theta_{k+2}((a_{\underline{i},\underline{j}}))_{\underline{u},\underline{v}} &= [M_I(\Theta_{k+1})((a_{\underline{i},\underline{j}}))]_{\underline{u},\underline{v}} \\
&= \Theta_{k+1}(u_{k+2}\text{-}v_{k+2} \text{ block of } (a_{\underline{i},\underline{j}}))_{\underline{u}_{k+1]},\underline{v}_{k+1]}} \quad \text{(by (2.3))} \\
&= \Theta_{k+1}((a_{(\underline{i'},u_{k+2}),(\underline{j'},v_{k+2})})_{\{\underline{i'},\underline{j'}\in I^k\}})_{\underline{u}_{k+1]},\underline{v}_{k+1]}} \\
&= \theta_{u_1,v_1}(a_{(\underline{u}_{[2,\,k+1]},u_{k+2}),(\underline{v}_{[2,\,k+1]},v_{k+2})}) \\
&= \theta_{u_1,v_1}(a_{\underline{u}_{[2},\underline{v}_{[2}}).
\end{aligned}$$

This completes the proof. $\square$

3.3 The Standard Invariant of $R \rtimes H \subset R \rtimes G$

It turns out that M, and in particular N, sits in the higher II_1-factors in the basic construction tower of $R \rtimes H \subset R \rtimes G$ by the maps $\theta^{(k)}$, as defined in Lemma 2.2.12.

Proposition 3.3.1 *For each $k \geq 1$, we have*

$$\Theta_k \circ \cdots \circ \Theta_1 = \theta^{(k)}. \tag{3.10}$$

Proof: For $k = 1$, by definition, $\Theta_1 = \theta = \theta^{(1)}$. Suppose (3.10) holds for some $k > 1$. Then for $x \in M$ and $\underline{i}, \underline{j} \in I^{k+1}$, by Lemma 3.9, we have

$$\begin{aligned}
[\Theta_{k+1} \circ \Theta_k \circ \cdots \circ \Theta_1(x)]_{\underline{i},\underline{j}} &= \theta_{i_1,j_1}([\Theta_k \circ \cdots \circ \Theta_1(x)]_{\underline{i}_{[2},\underline{j}_{[2}}) \\
&= \theta_{i_1,j_1}(\theta^{(k)}(x)_{\underline{i}_{[2},\underline{j}_{[2}}) \\
&= \theta^{(k+1)}(x)_{\underline{i},\underline{j}}.
\end{aligned}$$

Thus, by induction on k, we see that (3.10) holds for all $k \geq 1$. $\square$

Thus we have explicit maps $\theta^{(k)}$ giving the inclusions of N and M in all the II_1-factors of the basic construction tower of $R \rtimes H \subset R \rtimes G$.

We now recall, from [JS97], certain group actions on the sets $I^k, k \geq 1$, which arise quite naturally while analysing the relative commutants of the subgroup-subfactor; and later they will turn out to be extremely useful for the main theorem.

Let us first see how the map θ acts separately on R and G. It is easily seen from definitions that

$$\begin{aligned} \theta_{i,j}(r) &= \delta^i_j \; \alpha_{g_i}(r), \forall\, r \in R,\, i,j \in I; \text{ and} \qquad (3.11)\\ \theta_{i,j}(u_g) &= \left\{ \begin{array}{ll} u_{g_i g g_j^{-1}}, & \text{if } Hg_jg^{-1} = Hg_i \\ 0\,, & \text{otherwise} \end{array} \right\}, \ \forall\, g \in G,\ i,j \in I. \qquad (3.12) \end{aligned}$$

Given a k-tuple $\underline{i} = (i_1, \ldots, i_k) \in I^k$, we write $\sqcap g_{\underline{i}}$ for the product $g_{i_1} g_{i_2} \cdots g_{i_k}$. For each $k \geq 1$, we have an action β^k of G on the set I^k:

For $\underline{i}, \underline{j} \in I^k$,

$$\beta^k_g(\underline{j}) = \underline{i} \Longleftrightarrow Hg_{j_l}g_{j_{l+1}} \cdots g_{j_k}g^{-1} = Hg_{i_l}g_{i_{l+1}} \cdots g_{i_k}, \ \forall\, 1 \leq l \leq k. \qquad (3.13)$$

As above, from the definitions of $\theta^{(k)}$, for each $k \geq 1$, we have

$$\begin{aligned} \theta^{(k)}_{\underline{i},\underline{j}}(r) &= \delta^{\underline{i}}_{\underline{j}} \; \alpha_{\sqcap g_{\underline{i}}}(r), \forall\, r \in R,\ \underline{i}, \underline{j} \in I^k; \text{ and} \qquad (3.14)\\ \theta^{(k)}_{\underline{i},\underline{j}}(u_g) &= \left\{ \begin{array}{ll} u_{\sqcap g_{\underline{i}} g (\sqcap g_{\underline{j}})^{-1}}, & \text{if } \underline{i} = \beta^k_g(\underline{j}) \\ 0\,, & \text{otherwise} \end{array} \right\}, \ \forall\, g \in G,\ \underline{i}, \underline{j} \in I^k. \qquad (3.15) \end{aligned}$$

We are now well equipped to describe the standard invariant in terms of operator matrices. From the identifications (3.4) and Proposition 3.3.1, for each $k \geq 1$, the grid of relative commutants can be identified as

$$\left(\begin{array}{ccccc} N' \cap M_{2k-1} & \subset & N' \cap M_{2k} & \subset & N' \cap M_{2k+1} \\ \cup & & \cup & & \cup \\ M' \cap M_{2k-1} & \subset & M' \cap M_{2k} & \subset & M' \cap M_{2k+1} \end{array} \right) \cong$$

$$\left(\begin{array}{ccccc} \theta^{(k)}(N)' \cap M_{I^k}(N) & \subset & \theta^{(k)}(N)' \cap M_{I^k}(M) & \overset{\Theta_{k+1}}{\hookrightarrow} & \theta^{(k+1)}(N)' \cap M_{I^{k+1}}(N) \\ \cup & & \cup & & \cup \\ \theta^{(k)}(M)' \cap M_{I^k}(N) & \subset & \theta^{(k)}(M)' \cap M_{I^k}(M) & \overset{\Theta_{k+1}}{\hookrightarrow} & \theta^{(k+1)}(M)' \cap M_{I^{k+1}}(N) \end{array}\right). \tag{3.16}$$

In terms of operator matrices, for each $k \geq 1$, the relative commutants of R are given - see [JS97, § A.4] - by

$$\begin{aligned} R' \cap M_{2k-1} \cong \theta^{(k)}(R)' \cap M_{I^k}(N) = \{X = (x_{\underline{i},\underline{j}}) \in M_{I^k}(N) : \\ \exists\ (C_{\underline{i},\underline{j}}) \in M_{I^k}(\mathbb{C}) \text{ such that } x_{\underline{i},\underline{j}} = C_{\underline{i},\underline{j}} u_{(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1}} \\ \text{and } C_{\underline{i},\underline{j}} = 0 \text{ unless } H \sqcap g_{\underline{i}} = H \sqcap g_{\underline{j}},\ \forall\ \underline{i},\ \underline{j} \in I^k\}; \text{ and} \quad (3.17) \\ R' \cap M_{2k} \cong \theta^{(k)}(R)' \cap M_{I^k}(M) = \{X = (x_{\underline{i},\underline{j}}) \in M_{I^k}(M) : \\ \exists\ (C_{\underline{i},\underline{j}}) \in M_{I^k}(\mathbb{C}) \text{ such that } x_{\underline{i},\underline{j}} = C_{\underline{i},\underline{j}} u_{(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1}}, \forall\ \underline{i},\ \underline{j} \in I^k\}. \quad (3.18) \end{aligned}$$

From these descriptions, it immediately follows that the relative commutants in the grid of the standard invariant are given by

$$\begin{aligned} N' \cap M_{2k-1} \cong \theta^{(k)}(N)' \cap M_{I^k}(N) = \{X = (x_{\underline{i},\underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(N) : \\ C_{\underline{i},\underline{j}} = C_{\beta_h^k(\underline{i}),\beta_h^k(\underline{j})},\ \forall\, h \in H,\ \underline{i},\ \underline{j} \in I^k\}; \quad (3.19) \\ N' \cap M_{2k} \cong \theta^{(k)}(N)' \cap M_{I^k}(M) = \{X = (x_{\underline{i},\underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(M) : \\ C_{\underline{i},\underline{j}} = C_{\beta_h^k(\underline{i}),\beta_h^k(\underline{j})},\ \forall\, h \in H,\ \underline{i},\ \underline{j} \in I^k\}; \quad (3.20) \\ M' \cap M_{2k-1} \cong \theta^{(k)}(M)' \cap M_{I^k}(N) = \{X = (x_{\underline{i},\underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(N) : \\ C_{\underline{i},\underline{j}} = C_{\beta_g^k(\underline{i}),\beta_g^k(\underline{j})},\ \forall\, g \in G,\ \underline{i},\ \underline{j} \in I^k\}; \text{ and} \quad (3.21) \\ M' \cap M_{2k} \cong \theta^{(k)}(M)' \cap M_{I^k}(M) = \{X = (x_{\underline{i},\underline{j}}) \in \theta^{(k)}(R)' \cap M_{I^k}(M) : \\ C_{\underline{i},\underline{j}} = C_{\beta_g^k(\underline{i}),\beta_g^k(\underline{j})},\ \forall\, g \in G,\ \underline{i},\ \underline{j} \in I^k\}, \quad (3.22) \end{aligned}$$

for all $k \geq 1$, where in each of the above equations the scalar matrix $(C_{\underline{i},\underline{j}})$ corresponds to the operator matrix $X = (x_{\underline{i},\underline{j}})$ as in (3.17) and (3.18).

From now onwards, we identify the relative commutants $X' \cap M_k$, $X \in \{R, N, M\}$, $k \geq 1$, *with their matrix counterparts as given in equations (3.17 – 3.22).*

We need some more notations to simplify the things coming ahead. For each $k \geq 1$, and given k-tuples $\underline{i},\ \underline{j} \in I^k$, we define operator matrices

$[\underline{i},\, \underline{j}]^{ev} \in M_{I^k}(M)$ and $[\underline{i},\, \underline{j}]^{od} \in M_{I^k}(N)$ by

$$[\underline{i},\, \underline{j}]^{ev}_{\underline{u},\, \underline{v}} = \delta^{\underline{i}}_{\underline{u}}\, \delta^{\underline{j}}_{\underline{v}}\, u_{(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1}} \text{ and} \tag{3.23}$$

$$[\underline{i},\, \underline{j}]^{od}_{\underline{u},\, \underline{v}} = \delta^{\underline{i}}_{\underline{u}}\, \delta^{\underline{j}}_{\underline{v}}\; 1_H((\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1})\, u_{(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1}}, \tag{3.24}$$

respectively, $\forall\ \underline{u},\ \underline{v} \in I^k$. (The superscripts *ev* and *od* just signify the even and odd indexes $2k$ and $2k-1$ of the respective relative commutants.)

Further, for each $k \geq 1$, we set $Y^k = \{(\underline{i},\, \underline{j}) \in I^k \times I^k : H \sqcap g_{\underline{i}} = H \sqcap g_{\underline{j}}\}$. With these simplified notations, it is not hard to show that, for each $k \geq 1$,

1. G acts diagonally by β^k on $I^k \times I^k$ and Y^k is invariant under this action, i.e., $(\beta^k_g(\underline{i}),\, \beta^k_g(\underline{j})) \in Y^k,\ \forall\ g \in G,\ (\underline{i},\, \underline{j}) \in Y^k$;

2. $\{[\underline{i},\, \underline{j}]^{od} : (\underline{i},\, \underline{j}) \in Y^k\}$ forms a basis for $R' \cap M_{2k-1}$; and

3. $\{[\underline{i},\, \underline{j}]^{ev} : (\underline{i},\, \underline{j}) \in I^k \times I^k\}$ forms a basis for $R' \cap M_{2k}$.

Thus, it makes sense to write $g[\underline{i},\, \underline{j}]^x$ for $[\beta^k_g(\underline{i}),\, \beta^k_g(\underline{j})]^x$ for all $g \in G,\ (\underline{i},\, \underline{j}) \in I^k \times I^k,\ k \geq 1,\ x \in \{ev,\ od\}$.

Given a G-action υ on a set X, we write $G\backslash X(\upsilon)$ for a set of representatives of G-orbits of X under the action υ, and when the group action is clear from the context we simply denote it by $G\backslash X$. For instance, we write $G\backslash Y_k$ for a set of orbit representatives for the diagonal G-action on Y_k by β^k.

The descriptions $(3.19 - 3.22)$ give the following list of bases for the relative commutants:

Lemma 3.3.2 *For each $k \geq 1$ and for every choice of sets of orbit representatives $H\backslash X$ and $G\backslash X$ $(X = Y_k,\ I^k \times I^k)$,*

1. *$\{\sum_{g\in G} g[\underline{i},\, \underline{j}]^{od} : (\underline{i},\, \underline{j}) \in G\backslash Y_k\}$ (resp., $\{\sum_{h\in H} h[\underline{i},\, \underline{j}]^{od} : (\underline{i},\, \underline{j}) \in H\backslash Y_k\}$) forms a basis for $M' \cap M_{2k-1}$ (resp., $N' \cap M_{2k-1}$).*
2. *$\{\sum_{g\in G} g[\underline{i},\, \underline{j}]^{ev} : (\underline{i},\, \underline{j}) \in G\backslash(I^k \times I^k)\}$ (resp., $\{\sum_{h\in H} h[\underline{i},\, \underline{j}]^{ev} : (\underline{i},\, \underline{j}) \in H\backslash(I^k \times I^k)\}$) forms a basis for $M' \cap M_{2k}$ (resp., $N' \cap M_{2k}$).*

Since the above bases do not vary with the choice of orbit representatives, we shall no more trouble ourselves by repeating the phrase "given a set of representatives of K-orbits of X" and simply write $K\backslash X$ for such a choice.

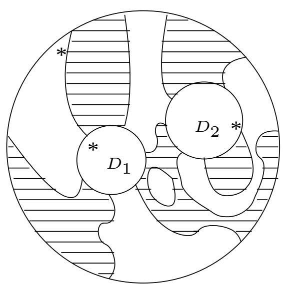
Chapter 4

Planar Algebras

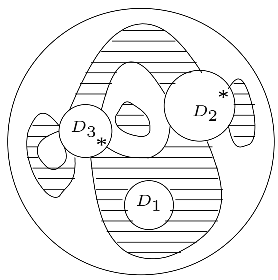
The notion of planar algebras was first introduced by Jones [Jon] in order to obtain a better understanding of the standard invariant of certain subfactors. However, since its introduction, the definition of planar algebras has undergone some modifications (see [Jon00a, Jon00b, Jon03b, Jon03a, KLS03, KS04]), and for completeness we shall present the formalism that appears in [KS04].

We first recall the definition of the operad of coloured planar tangles and list certain sets of generating tangles that we shall use in the sequel. In §2, we present a brief overview of planar algebras and their importance to subfactors; and introduce the notion of action of a (finite) group on a planar algebra. Then, in §3, we recall (with slight modification) the description of the planar algebra associated to a bipartite graph with a spin function as given in [Jon00a]. Finally, we analyse the effect a finite group acting on a bipartite graph has on the planar algebra associated to it.

4.1 Tangles



$S:$ 4-tangle

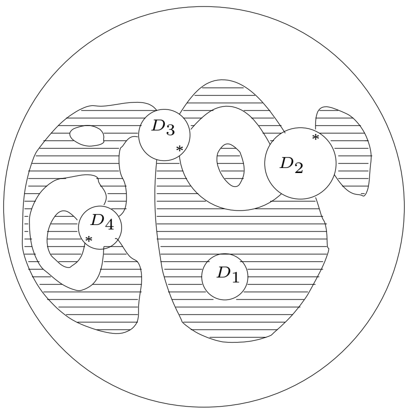


$T:$ 0_+-tangle

Loosely speaking, a *coloured tangle* is the equivalence class of a picture in the plane consisting of an outer disc containing finitely many ordered internal discs and finitely many non-intersecting smooth strings either joining these discs on their boundaries or floating around as simple closed curves.

Each disc can either have an even number (half of which is defined to be its colour) of marked points on it where the strings meet it or have no marked point and thus is not connected to any string (in this case, its colour is defined to be $0_\pm$ according as the shading (described below) around the box is white or black). One among the marked points, if any, on a disc is distinguished as a $*$-vertex and the regions in the picture outside the internal boxes are endowed with a chequerboard shading such that the region to the right while moving away from (resp., towards) a $*$-vertex on an internal (resp., external) disc is shaded black and that no two adjacent regions are both black or both white. Two such pictures are in the same equivalence class if there is an isotopy between the pictures which preserves the order of the discs, the $*$-vertices on the discs and the shading of the regions. And the colour of the tangle is defined to be the colour of the outer disc. Note that the possible set of colours is $Col := \{0_\pm, 1, 2, \ldots\}$.

By a slight abuse of language, for a k-tangle T, we pick up any element in that isotopy class and call that picture to be a k-tangle and denote it again by T. This does no harm to our requirements as our primary concern regarding a tangle is the "tangle map" that it gives on a given "planar algebra", as we shall see below.



$T \circ_3 S$

Figure 4.1: Composition of two tangles

Composition of tangles: If T and S are two coloured tangles such that the colour of S is same as the colour of some internal disc of T, then we can compose T and S by plugging S into that disc of T, so that the $*$-vertex of S meets the $*$-vertex of that disc; and the ordering of the internal discs follows the convention as described in § 4.2.1.

For instance, if T and S are the coloured tangles as above, then, up to isotopy of coloured tangles, the 'composite tangle' $T \circ_3 S$ is the 0_+-tangle as shown in Figure 4.1.

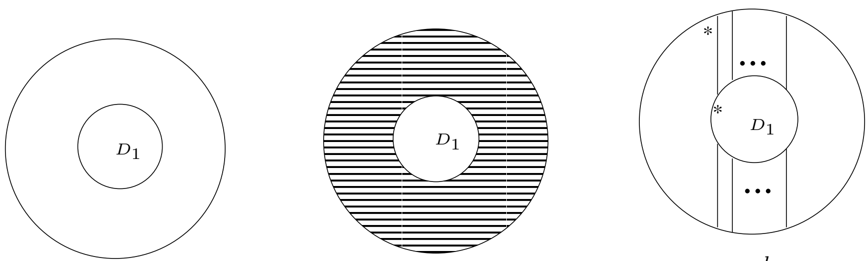
Note that, if a coloured tangle has a disc with marked points (irrespective of outer or inner), then once the $*$-vertex for the disc is fixed, the shading of the regions of the tangle is uniquely determined and hence becomes redundant. Taking this observation into account, from now onwards, we shall ignore the shading whenever we describe a coloured tangle with at least one disc with marked points.

We list some important coloured tangles in Figures 4.2–4.7. With respect to composition of tangles, [KS04] contains the following generating sets:

Theorem 4.1.1 *[KS04] Let $\mathcal{T}$ be a collection of coloured tangles containing*

- $\mathcal{G}_0 := \{1^{0_\pm}\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in Col\} \cup \{\mathcal{E}^{k+1}, (E')^k_k : k \geq 1\}$, *or*
- $\mathcal{G}_1 := \{1^{0_\pm}\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in Col\} \cup \{R_k : k \geq 2\}$,

and suppose $\mathcal{T}$ is closed under composition of coloured tangles, whenever it makes sense. Then $\mathcal{T}$ contains all coloured tangles.

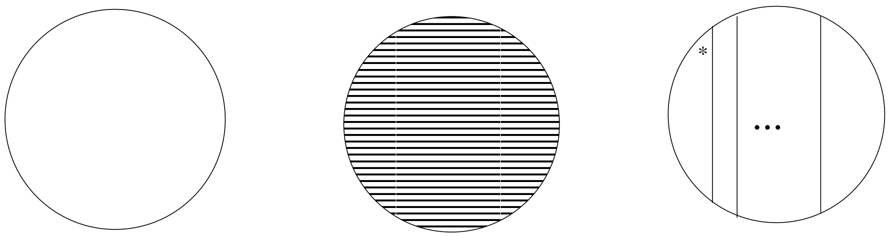


$I^{0_+}_{0_+}$ $\qquad$ $I^{0_-}_{0_-}$ $\qquad$ I^k_k

Figure 4.2: The Identity Tangles

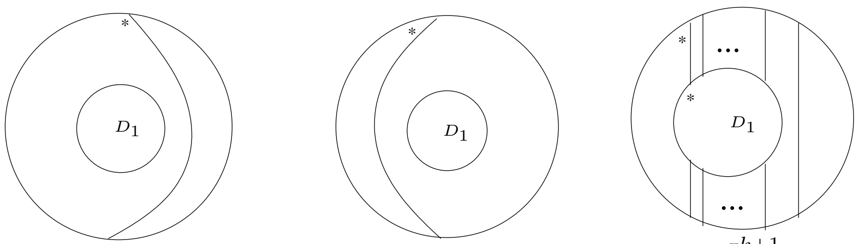
4.2 Planar Algebras

As mentioned before, we follow [KS04] for this overview of planar algebras. This section is almost self-contained as it includes all those definitions that one needs to know in order to understand subfactor planar algebras.



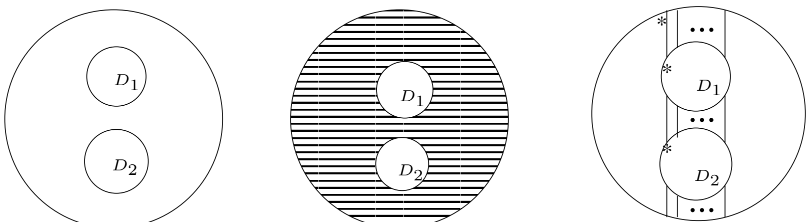
1^{0_+} 1^{0_-} 1^k (k-strings)

Figure 4.3: The Unit Tangles



$I^1_{0_+}$ $I^1_{0_-}$ I^{k+1}_k

Figure 4.4: The Inclusion Tangles



M_{0_+} M_{0_-} M_k : all three are k-discs

Figure 4.5: The Multiplication Tangles

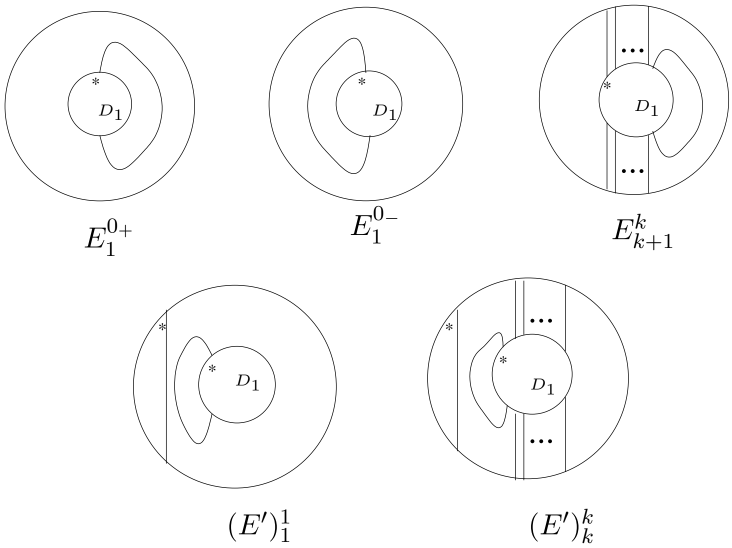


Figure 4.6: The Conditional Expectation Tangles

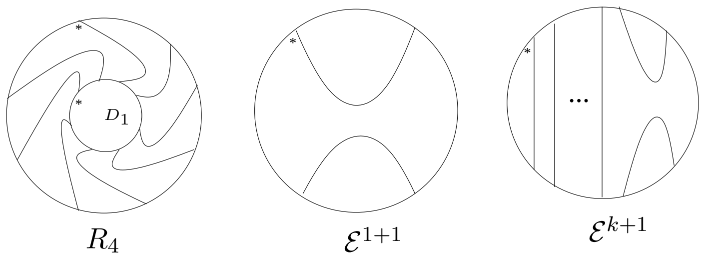


Figure 4.7: The Rotation and Jones' Projections Tangles

4.2.1 Definitions

A *planar algebra* is a collection of vector spaces $P = \{P_k : k \in Col\}$ which admits an action of the operad of coloured tangles, i.e., if T is a k_0-tangle with b internal discs of colours $k_1, k_2, \ldots, k_b$, respectively, then there is an associated linear map (called tangle map)

$$Z_T^P : \begin{cases} P_{k_1} \otimes P_{k_2} \otimes \cdots \otimes P_{k_b} \to P_{k_0} & \text{, if } b > 0; \\ \mathbb{C} \to P_{k_0} & \text{, if } b = 0. \end{cases}$$

These tangle maps are compatible with the composition of coloured tangles in the following sense:

If S is an l_0-tangle with $d(d > 0)$ internal discs of colours $l_1, \ldots l_d$, respectively, such that the i-th disc of T is of the same colour as that of S, i.e., $k_i = l_0$, then the composite tangle $T \circ_i S$ is a k_0-tangle with $b + d - 1$ internal discs in the following order:

For each $1 \le j \le b + d - 1$, the j-th disc of $T \circ_i S$ is the

$$\begin{cases} j\text{-th disc of } T & ,\text{if } 1 \le j \le i-1; \\ j-i+1\text{-th disc of } S & ,\text{if } i \le j \le i+b-1; \text{ and the} \\ j-b+1\text{-th disc of } T & ,\text{if } i+b \le j \le b+d-1. \end{cases}$$

Then the compatibility requirement for the tangle maps is that the diagram

$$\begin{tikzcd}
(\otimes_{j=1}^{i-1} P_{k_j}) \otimes (\otimes_{r=1}^{d} P_{l_r}) \otimes (\otimes_{j=i+1}^{b} P_{k_j}) \arrow[dd, "id \otimes Z_S^P \otimes id"'] \arrow[dr, "Z^P_{T \circ_i S}"] & \\
& P_{k_0} \\
\otimes_{j=1}^{b} P_{k_j} \arrow[ur, "Z_T^P"'] &
\end{tikzcd} \tag{4.1}$$

must commute. And if the l_0-tangle S has no internal disc, i.e., if $d = 0$, then compatibility requirement is that the following modified diagram must commute.

$$\begin{tikzcd}
\otimes_{j \ne i} P_{k_j} \arrow[d, "\cong"'] \arrow[dr, "Z^P_{T \circ_i S}"] & \\
(\otimes_{j=1}^{i-1} P_{k_j}) \otimes \mathbb{C} \otimes (\otimes_{j=i+1}^{b} P_{k_j}) \arrow[d, "id \otimes Z_S^P \otimes id"'] & P_{k_0} \\
\otimes_{j=1}^{b} P_{k_j} \arrow[ur, "Z_T^P"'] &
\end{tikzcd} \tag{4.2}$$

In order to avoid unnecessary complications, the assignment $T \to Z_T^P$ must be independent of the ordering of the internal boxes in the following sense:

If T is a k_0-tangle as in (4.1) with $b > 0$, and if σ is permutation on b symbols $\{1, 2, \ldots, b\}$, then the diagram

$$\begin{tikzcd}
P_{k_{\sigma(1)}} \otimes P_{k_{\sigma(2)}} \otimes \cdots \otimes P_{k_{\sigma(b)}} \arrow[rr, "Z^P_{\sigma^{-1}(T)}"] \arrow[dr, "U_\sigma"'] & & P_{k_0} \\
& P_{k_1} \otimes P_{k_2} \otimes \cdots \otimes P_{k_b} \arrow[ur, "Z_T^P"'] &
\end{tikzcd} \tag{4.3}$$

must commute, where U_σ is the map given by

$$\otimes_{j=1}^{b} P_{k_{\sigma(j)}} \ \ni \ \otimes_{j=1}^{b} x_{\sigma(j)} \stackrel{U_\sigma}{\longmapsto} \otimes_{j=1}^{b} x_j \ \in \ \otimes_{j=1}^{b} P_{k_j},$$

and $\sigma^{-1}(T)$ is the coloured tangle which differs from T only in the ordering of its internal discs; for each $1 \leq i \leq b$, the i-th disc of $\sigma^{-1}(T)$ is the $\sigma(i)$-th disc of T.

Furthermore, in order to avoid certain degeneracies, P must satisfy

$$Z^P_{I^k_k} = id_{P_k}, \forall\, k \in \ Col. \tag{4.4}$$

Definition 4.2.1 *A planar algebra is a collection $P = \{P_k : k \in Col\}$ of vector spaces, equipped with an assignment $T \mapsto Z^P_T$ of linear maps to coloured tangles, in such a manner that equations (4.1–4.4) are satisfied.*

Note that, for each $k \in \ Col$, P_k is a unital associative algebra with respect to multiplication given by $x_1 x_2 := Z^P_{M_k}(x_1 \otimes x_2)$, for all $x_1, x_2 \in P_k$, and the multiplicative identity being the element $1_k := Z^P_{1^k}(1) \in P_k$.

Definition 4.2.2 *Let P be a planar algebra, and for each $k \in Col$, let Q_k be a subspace of P_k. Then $Q := \{Q_k : k \in Col\}$ is said to be a planar subalgebra of P if Q is invariant under the tangle maps, i.e., for every k_0-tangle T as in (4.1), $Z^P_T(x_1 \otimes \cdots \otimes x_b) \in Q_{k_0}$, for all $x_i \in Q_{k_i}$, $1 \leq i \leq b$, and the subset $\{Z^P_S(1) : S$ is a k-tangle without internal discs $\} \subset Q_k$, for all $k \in Col$. The tangle maps Z^Q_T are simply the restrictions of Z^P_T to Q.*

Definition 4.2.3 *Let P and Q be two planar algebras. A* planar algebra morphism *from P to Q is a collection $\varphi = \{\varphi_k : k \in Col\}$ of linear maps $\varphi_k : P_k \to Q_k$ which commutes with all the tangle maps, i.e., if T is a k_0-tangle with b internal boxes of colours $k_1, \ldots, k_b$, then*

$$\varphi_{k_0} \circ Z^P_T = \begin{cases} Z^Q_T \circ (\otimes_{i=1}^{b} \varphi_k) & , \text{ if } b > 0; \text{ and} \\ Z^Q_T & , \text{ if } b = 0. \end{cases} \tag{4.5}$$

φ is said to be a planar algebra isomorphism *if the maps φ_k are all linear isomorphisms.*

On more than one occasion, we will find ourselves with a bunch of maps and will have to show that they collectively form a planar algebra morphism. In order to do that, we shall invariably invoke to Theorem 4.1.1; for which it would be quite handy to note the following:

Lemma 4.2.4 *Let P and Q be planar algebras, and let $\varphi_k : P_k \to Q_k$, $k \in$ Col be a collection of linear maps. If $\mathcal{T}$ is the set of those tangles T for which equation (4.5) holds, then $\mathcal{T}$ is closed under composition of tangles.*

Proof: Let T and S be any two coloured tangles as in (4.1), i.e., S has positive number of internal boxes. Then a few seconds of gaze at the diagram 4.6 readily shows that the composition tangle $T \circ_i S$ also belongs to the collection $\mathcal{T}$. And, if S has no internal box, then the appropriate modification of this diagram shows that the composite tangle $T \circ_i S$ is also in $\mathcal{T}$.

$$\begin{tikzcd}
(\otimes_{j=1}^{i-1} P_{k_j}) \otimes (\otimes_{r=1}^{d} P_{l_r}) \otimes (\otimes_{j=i+1}^{b} P_{k_j}) \arrow[rr, "Z^P_{T\circ_i S}"] \arrow[dr, "id \otimes Z^P_S \otimes id"] \arrow[dd, "(\otimes_{j=1}^{i-1}\varphi_{k_j})\otimes(\otimes_{r=1}^{d}\varphi_{l_r})\otimes(\otimes_{j=i+1}^{b}\varphi_{k_j})"'] & & P_{k_0} \arrow[dd, "\varphi_{k_0}"] \\
& \otimes_{j=1}^{b} P_{k_j} \arrow[ur, "Z^P_T"'] \arrow[d, "(\otimes_{j=1}^{b}\varphi_{k_j})"] & \\
& \otimes_{j=1}^{b} Q_{k_j} \arrow[dr, "Z^Q_T"] & \\
(\otimes_{j=1}^{i-1} Q_{k_j}) \otimes (\otimes_{r=1}^{d} Q_{l_r}) \otimes (\otimes_{j=i+1}^{b} Q_{k_j}) \arrow[rr, "Z^Q_{T\circ_i S}"'] \arrow[ur, "id \otimes Z^Q_S \otimes id"] & & Q_{k_0}
\end{tikzcd} \tag{4.6}$$

□

4.2.2 Subfactor Planar Algebras

The importance of planar algebras to subfactors lies in Theorem 4.2.7. Thus we first need to understand what we mean by a subfactor planar algebra. This involves the following definitions.

A planar algebra P is said to be *connected* (resp., *irreducible*) if $dim\, P_{0_\pm} = 1$ (resp., $dim\, P_1 = 1$). The next type of planar algebra involves certain behaviour of the loops in the coloured tangles. A connected planar algebra P is said to have *modulus* δ if there is a scalar δ such that

$$Z^P_{T^{\pm}_{\mp}}(1_{0_\pm}) = \delta 1_{0_\mp},$$

where the annular coloured tangles T^+_- and T^-_+ are illustrated in Figure 4.8.

This basically amounts to saying that a loop comes out as the constant

δ. It can be easily seen that if a planar algebra has positive modulus, then inclusion tangles give injective maps.

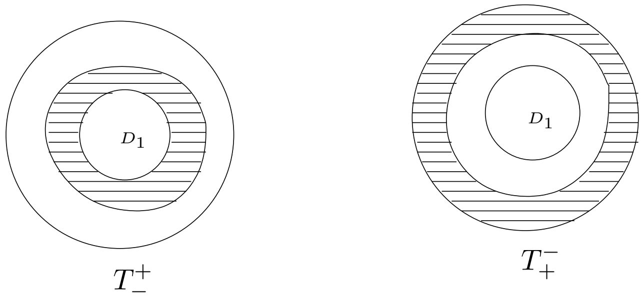


Figure 4.8: The Tangles T_-^+ and T_+^-

For a connected planar algebra P, we identify $P_{0_\pm}$ with $\mathbb{C}$. Then for any 0_+- or 0_--tangle T with b internal discs of colours $k_1, \ldots, k_b$, respectively, and for $x_i \in P_{k_i}$, we get scalars $Z_T^P(\otimes_{i=1}^b x_{k_i})$. Thus considering the discs of T to be labelled by the vectors $x_i \in P_{k_i}$, we have an assignment of scalars from the set of labelled $0_\pm$-tangles. This assignment of scalars to the labelled $0_\pm$-tangles is known as the *partition function associated to the planar algebra* P.

Further, for a $0_\pm$-tangle T, by its *network*, we mean the system of strings and discs of T excluding its outer disc, along with the shadings of the regions. The unbounded region of the network of T gets a shading of white or black according as T is a 0_+- or 0_--tangle.

A connected planar algebra is said to be *spherical* if its partition function assigns same value to any two $0_\pm$-tangles whose associated networks are isotopic on the 2-sphere such that the $*$-vertices on the discs, if any, are preserved and so are the shaded regions..

This is the right place to recall a useful fact from [KS06].

Lemma 4.2.5 *[KS06] If P is any connected and irreducible planar algebra with modulus $\delta > 0$, then P is spherical.*

One last definition before we finally formalise what we mean by a subfactor planar algebra.

Loosely speaking, the *adjoint* T^* of a coloured tangle T is the tangle obtained by reflecting the tangle in the horizontal diameter, keeping the order of the internal discs same; and the $*$-vertex of a disc with marked points now being the one which was the immediate left marked point on it before reflection. This forces the shadings of the regions to remain same as in the original tangle. We show it by an example in Figure 4.9.

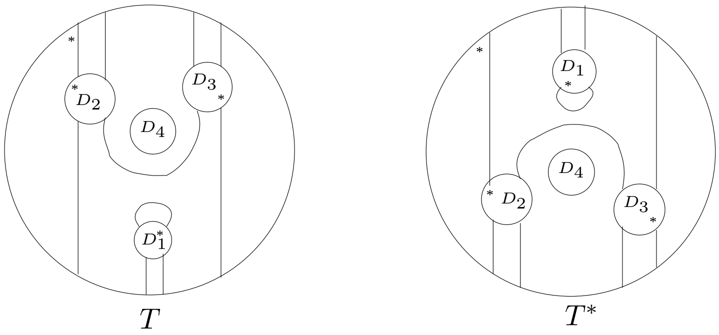


Figure 4.9: The adjoint of a tangle

Subfactor Planar Algebra: A planar algebra P is said to be a *subfactor planar algebra* if:
1. P is connected, spherical, has a positive modulus;
2. each P_k is a finite dimensional C^*-algebra with the property that, for any k_0-tangle T with b internal boxes of colours $k_1, \ldots, k_b$,

$$Z_T^P(x_1 \otimes \cdots \otimes x_b)^* = Z_{T^*}^P(x_1^* \otimes \cdots \otimes x_b^*), \tag{4.7}$$

for all $x_i \in P_{k_i}, 1 \leq i \leq b$; and
3. if we define the pictorial trace on P by

$$tr_{k+1}(x)1_{0_+} = \frac{1}{\delta^{k+1}} Z^P_{E_1^{0_+}} \circ Z^P_{E_2^1} \circ \cdots \circ Z^P_{E_{k+1}^k}(x), \forall\, x \in P_{k+1}, \tag{4.8}$$

then tr_m is a faithful positive trace on P_m, for all $m \geq 1$.

Remark 4.2.6 *For a connected, spherical planar algebra P with modulus $\delta > 0$, and each P_k being a finite dimensional C^*-algebra,*

1. *the pictorial traces tr_m, $m \in Col$ are consistent with respect to inclusions and thus define a global trace on P, where for $0_\pm$ the trace for $P_{0_\pm} \equiv \mathbb{C}$ is the obvious identity map; and*
2. *if tr_m is faithful for all $m \geq 1$, then for each $k \in Col$, $\frac{1}{\delta} Z^P_{E^k_{k+1}}$ (resp., $\frac{1}{\delta} Z^P_{(E')^{k+1}_{k+1}}$) is the unique tr_{k+1} preserving conditional expectation from P_{k+1} onto P_k (resp., $P_{1,\,k+1}$), where $P_{1,\,k+1} := Image(Z^P_{(E')^{k+1}_{k+1}})$.*

The importance of planar algebras in subfactor theory lies in the following theorem of Jones:

Theorem 4.2.7 *[Jon] Let $N \subset M$ be a finite index, extremal subfactor with $[M : N] = \delta^2$ and let*

$$N \subset M(=: M_0) \subset^{e_1} M_1 \subset \cdots \subset^{e_k} M_k \subset^{e_{k+1}} M_{k+1} \subset \cdots$$

be its associated tower of basic construction. Then with $P_{0_\pm} := \mathbb{C}$, and $P_k := N' \cap M_{k-1}$, $k \geq 1$, there exists a unique subfactor planar algebra structure on the collection $P^{N \subset M} := P = \{P_k : k \in Col\}$ satisfying:

1. $Z^P_{\mathcal{E}^{k+1}}(1) = \delta\, e_k$, $\forall\, k \geq 1$;
2. $Z^P_{(E')^k_k}(x) = \delta\, E_{M' \cap M_{k-1}}(x)$, $\forall\, x \in N' \cap M_{k-1}$, $\forall\, k \geq 1$;
3. $Z^P_{E^k_{k+1}}(x) = \delta\, E_{N' \cap M_{k-1}}(x)$, $\forall\, x \in N' \cap M_k, \forall\, k \in Col$, *where for $k = 0_\pm$, the equation is read as*

$$Z^P_{E_1^{0_\pm}}(x) = \delta\, tr_M(x),\ \forall\, x \in N' \cap M.$$

Conversely, any subfactor planar algebra P with modulus δ arises from an extremal subfactor of index δ^2 in the above fashion.

We aim to give a description of the subfactor planar algebra $P^{R \rtimes H \subset R \rtimes G}$, as the 'invariant planar subalgebra of the planar algebra of an appropriate bipartite graph'.

Remark 4.2.8 *The planar algebra $P = P^{N \subset M}$ of an extremal, finite index subfactor $N \subset M$ also contains the data of its standard invariant. Define $P_{1,k} = E_{M' \cap M_k}(N' \cap M_{k-1}) = M' \cap M_{k-1}$ for all $k \geq 1$. Then the grid*

$$\begin{array}{ccccccccc} \mathbb{C} = P_{0_\pm} & \subset & P_1 & \subset & \cdots & \subset & P_k & \subset & \cdots \\ & & \cup & & & & \cup & & \\ & & P_{1,1} & \subset & \cdots & \subset & P_{1,k} & \subset & \cdots \end{array}$$

is precisely the standard invariant of the subfactor $N \subset M$. Thus if the planar algebras of two such subfactors are isomorphic, then it follows, in particular, that their standard invariants are isomorphic.

4.2.3 Dual of a Planar Algebra

Given a planar algebra P, there is a notion of its dual planar algebra ${}^-P$, which we describe here.

We define $(0_\pm)^- = 0_\mp$ and $k^- = k$ for all $k > 0$. Then, given an l-tangle T, we write T^- for the l^--tangle which as a picture without shading is same as T, whose $*$-vertex for any disc is that marked point on the disc which was immediate left to $*$-vertex on it in T, and thus the shadings of the regions are opposite to that of T. T^- is said to be the dual of the tangle T. Examples of duals of coloured tangles can be seen in the proof of Theorem 4.3.2.

With this definition, given a planar algebra P, we set $({}^-P)_k = P_{k^-}$ for all $k \in Col$. Then for any k-tangle T, taking $Z_T^{-P} = Z_{T^-}^P$, we get a planar algebra structure on ${}^-P$, where the tangle map for any coloured tangle T is the map Z_T^{-P} - see [KS04]. ${}^-P$ is said to be the *dual planar algebra* of P. We have the following important facts:

Proposition 4.2.9 *[KS04]* 1. *Given a planar algebra P, we have ${}^{--}P \cong P$.* 2. *Suppose $N \subset M$ is a finite index, extremal subfactor and $P = P^{N \subset M}$. Then*

$$ {}^-P \cong P^{M \subset M_1}. $$

4.2.4 Group actions on Planar Algebras

We shall be only interested in actions of finite groups.

Definition 4.2.10 *Let G be a finite group and $P = \{P_k : k \in Col\}$ be a planar algebra. We say that G acts on P if for each $k \in Col$, we have group homomorphisms $\alpha_k : G \to GL(P_k)$ such that, for each $g \in G$, $\alpha(g) := \{\alpha_k(g) : k \in Col\}$ is a planar algebra automorphism of P.*

For convenience, we write gx for the element $\alpha_k(g)(x)$, for $g \in G, x \in P_k$ and $k \in Col$. Under such an action of G on a planar algebra P, taking

$$ P_k^G := \{x \in P_k : gx = x, \forall g \in G\},\ k \in Col, $$

we note that the collection $P^G := \{P_k^G : k \in Col\}$ is a planar subalgebra of P. Furthermore, the above action induces a natural G-action on the dual planar algebra ${}^-P$ as well. Indeed, for each $k > 0$, since ${}^-P_k = P_k$, G acts on ${}^-P_k$ as it does on P_k; and as ${}^-P_{0_\pm} = P_{0_\mp}$, again G acts on these as it does on $P_{0_\mp}$.

And what follows immediately is that the dual of the invariant planar subalgebra is isomorphic to the invariant planar subalgebra of the dual planar algebra.

Proposition 4.2.11 *If a finite group G acts on a planar algebra P, then the planar algebras ${}^-(P^G)$ and $({}^-P)^G$ are isomorphic.*

Proof: The proof of this fact relies on the fact that the constituent vector spaces of these two planar algebras are same, and the identity morphism is readily seen to form a planar morphism between them. Indeed, for ${}^{-}(P^G)$, we have

$$\begin{array}{rclcl} {}^{-}(P^G)_{0_\pm} & = & (P^G)_{0_\mp} & = & P^G_{0_\mp}, \\ {}^{-}(P^G)_k & = & P^G_k & & , \text{ if } k > 0; \end{array}$$

and for $({}^{-}P)^G$, we have

$$\begin{array}{rclcl} ({}^{-}P)^G_{0_\pm} & = & ({}^{-}P_{0_\pm})^G & = & P^G_{0_\mp}, \\ ({}^{-}P)^G_k & = & ({}^{-}P_k)^G & = & P^G_k, \text{ if } k > 0. \end{array}$$

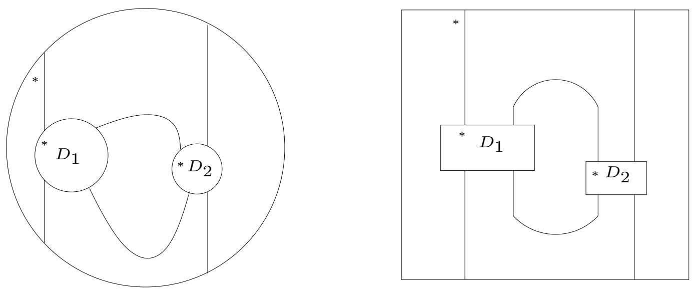
□

4.3 Planar Algebra of a Bipartite Graph

Given a finite (possibly with multiple edges), connected bipartite graph with a 'spin function', Jones demonstrates a way to associate a planar algebra to it in [Jon00a]. However, in [Jon00a], there is a slight conflict between the notion of 'state' and its 'compatibility' with 'loops'. For completeness, we present the appropriate modification of that description; so that the results obtained there hold verbatim.

A graph (possibly with multiple edges) Γ with edge set $\mathcal{E}$ is said to be bipartite if its vertex set is of the form $\mathcal{U} = \mathcal{U}^+ \sqcup \mathcal{U}^-$ such that a vertex in $\mathcal{U}^+$ is connected only to a vertex in $\mathcal{U}^-$ and not to any vertex in $\mathcal{U}^+$ and vice versa. We usually denote such a graph by the data $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$, and call $\mathcal{U}^+$ and $\mathcal{U}^-$ to be the set of even and odd vertices of Γ, respectively.

For convenience, from now onwards, we shall replace discs by boxes in the description of a coloured tangle.



4.3.1 The Description

Let $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ be a finite, connected bipartite (i.e., Γ has finitely many vertices and edges); and suppose there is a function $\mu : \mathcal{U} := \mathcal{U}^+ \sqcup \mathcal{U}^- \to (0, \infty)$. The function μ is usually referred to as a *spin function* and in most of the 'interesting' cases $(\mu_u^2)_{u\in\mathcal{U}}$ is an eigen vector for the adjacency matrix of the graph Γ.

For each $k > 0$, we denote a loop on Γ of length $2k$, based at a vertex $\pi_0 \in \mathcal{U}^+$ by a pair (π, ϵ) of maps $\pi : \{0, 1, \ldots, 2k-1\} \to \mathcal{U}$ and $\epsilon : \{0, 1, \ldots, 2k-1\} \to \mathcal{E}$ such that $\pi_{2i} \in \mathcal{U}^+$ and $\pi_{2i+1} \in \mathcal{U}^-$ for all $0 \le i \le k-1$; and for $0 \le i \le 2k-1$, ϵ_i is the edge joining the vertices π_i and π_{i+1}, where we follow, here and elsewhere, the convention of labelling the vertices of $2k$-loops *modulo* $2k$; thus $\pi_{2k} = \pi_0$. Pictorially, the pair (π, ϵ) denotes the loop

$$\begin{array}{ccccccc} & & \pi_1 \xrightarrow{\epsilon_1} \pi_2 \longrightarrow \cdots \longrightarrow \pi_{k-1} & & \\ & {}^{\epsilon_0}\nearrow & & \searrow^{\epsilon_{k-1}} & \\ \pi_0 & & & & \pi_k\ . \\ & {}_{\epsilon_{2k-1}}\nwarrow & & \swarrow_{\epsilon_k} & \\ & & \pi_{2k-1} \xleftarrow[\epsilon_{2k-2}]{} \pi_{2k-2} \longleftarrow \cdots \longleftarrow \pi_{k+1} & & \end{array}$$

For each $k > 0$ and a $2k$-loop (π, ϵ), for every $1 \le i < j \le 2k$, we define the truncation

$$\pi_{[i,j]} = \pi_i \xrightarrow{\epsilon_i} \pi_{i+1} \longrightarrow \cdots \xrightarrow{\epsilon_{j-1}} \pi_j,$$

and its reverse $\widetilde{\pi_{[i,j]}}$ to be the path

$$\widetilde{\pi_{[i,j]}} = \pi_j \xrightarrow{\epsilon_{j-1}} \pi_{j-1} \longrightarrow \cdots \xrightarrow{\epsilon_i} \pi_i\,;$$

and if $0 \le r < s \le 2l$ are such that $\lambda_r = \pi_j$ for some $2l$-loop (λ, η), then we define the concatenation $\pi_{[i,j]} \circ \lambda_{[r,s]}$ to be the path

$$\pi_{[i,j]} \circ \lambda_{[r,s]} = \pi_i \xrightarrow{\epsilon_i} \pi_{i+1} \longrightarrow \cdots \xrightarrow{\epsilon_{j-1}} \pi_j (= \lambda_r) \xrightarrow{\eta_r} \cdots \xrightarrow{\eta_{s-1}} \lambda_s.$$

Further, for any two vertices $u^+ \in \mathcal{U}^+$ and $u^- \in \mathcal{U}^-$, we set

$$\mathcal{E}(u^+, u^-) = \{\epsilon \in \mathcal{E} : \epsilon \text{ joins } u^+ \text{ and } u^-\}.$$

Note that the graph Γ is not directed, and the arrows for the edges in the above loops or their truncations just give the order in which the vertices appear. We shall interchangeably use the pictorial form and the pair of maps form for a loop according to our convenience.

We can now present Jones' description of the planar algebra that he associates to the bipartite graph Γ.

For each $k > 0$, let $P_k(\Gamma)$ be the $\mathbb{C}$-vector space whose basis consists of loops on Γ of length $2k$ based at vertices in $\mathcal{U}^+$; and as for $k = 0_\pm$, a loop of length 0 is just a vertex on Γ, we set $P_{0_\pm}(\Gamma)$ to be the $\mathbb{C}$-vector space with

$\mathcal{U}^\pm$ as basis. We wish to give a planar algebra structure on the collection

$$P(\Gamma) := \{P_k(\Gamma) : k \in Col\}.$$

The main ingredient in the definition of the tangle maps for this collection of vector spaces is the notion of labelling of the regions and strings of coloured tangles in a consistent manner by vertices and edges, respectively, of the graph Γ.

Definition 4.3.1 *A* state σ *of a coloured tangle* T *is a function*

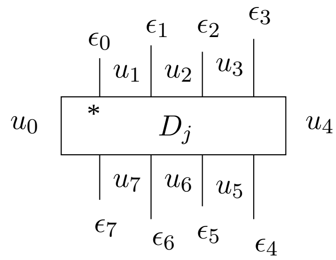
$$\sigma : \{\textit{regions of } T\} \sqcup \{\textit{strings of } T\} \to \mathcal{U} \sqcup \mathcal{E},$$

such that

1. $\sigma(\{\textit{unshaded regions}\}) \subset \mathcal{U}^+$, $\sigma(\{\textit{shaded regions}\}) \subset \mathcal{U}^-$*;*
2. $\sigma(\{\textit{strings}\}) \subset \mathcal{E}$*; and*
3. *if a string* s *lies in the closure of two regions* r_1 *and* r_2*, then* $\sigma(s)$ *is the edge joining the vertices* $\sigma(r_1)$ *and* $\sigma(r_2)$*.*

The first thing we observe is that the state σ induces unique loops at the internal as well as the external boxes of the tangle T:

We show this by an example. Suppose around the j-th internal box D_j of T, say of colour $k_j = 4$, the state σ looks like



,

then, by condition (3) in Definition 4.3.1, we obtain the loop

$$\begin{tikzcd} & u_1 \arrow[r, "\epsilon_1"] & u_2 \arrow[r, "\epsilon_2"] & u_3 \arrow[dr, "\epsilon_3"] & \\ u_0 \arrow[ur, "\epsilon_0"] & & & & u_4 \arrow[dl, "\epsilon_4"] \\ & u_7 \arrow[ul, "\epsilon_7"] & u_6 \arrow[l, "\epsilon_6"] & u_5 \arrow[l, "\epsilon_5"] & \end{tikzcd} .$$

And if $k_j \in \{0_\pm\}$, then the state σ simply induces a vertex in $\mathcal{U}^+$ or $\mathcal{U}^-$ according as D_j is a 0_+- or 0_--box. We take this vertex as the $2k_j$-loop induced by σ at D_j. Similarly, σ also induces a $2k_0$-loop at the external box D_0. For each $0 \leq j \leq b$, we denote this unique $2k_j$-loop by the pair $(\pi^\sigma_{D_j}, \epsilon^\sigma_{D_j})$. We say that the state σ is compatible with a $2k_j$-loop (π, ϵ) at the j-th box D_j if $(\pi^\sigma_{D_j}, \epsilon^\sigma_{D_j}) = (\pi, \epsilon)$. We can now define the tangle maps on the collection $P(\Gamma)$.

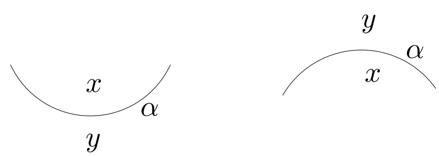
Given a k_0-tangle T as above, we first isotope it to a "standard form", i.e.,

- first rotate all the internal boxes of T so that their $*$-vertices are on top left-corner; and
- then isotope all the strings, if necessary, so that any singularity of the y-coordinate function for strings is either a local maximum or a local minimum.

Then, given a set of $2k_j$-loops $\{(\pi^{(j)}, \epsilon^{(j)}) : 0 \leq j \leq b\}$ (with the convention that $2k_j = 0_\pm$, if $k_j = 0_\pm$, and $(\pi^{(j)}, \epsilon^{(j)})$ is an appropriate element of $\mathcal{U}$), we define the coefficient of $(\pi^{(0)}, \epsilon^{(0)})$ in the vector $Z_T^{P(\Gamma)}((\pi^{(1)}, \epsilon^{(1)}) \otimes \cdots \otimes (\pi^{(b)}, \epsilon^{(b)}))$[1] by

$$\begin{aligned} & Z_T^{P(\Gamma)}((\pi^{(1)}, \epsilon^{(1)}) \otimes \cdots \otimes (\pi^{(b)}, \epsilon^{(b)}))_{(\pi^{(0)},\, \epsilon^{(0)})} \\ &= \sum_{\sigma \in \left\{ \begin{array}{c} \text{states of } T \\ \text{compatible with} \\ (\pi^{(j)},\, \epsilon^{(j)}) \text{ at } D_j, \\ \forall\, 0 \leq j \leq b \end{array} \right\}} \quad \prod_{\begin{array}{c} \text{singularities } \alpha \\ \text{of } y\text{-coordinate} \\ \text{on strings} \end{array}} \mu_\alpha, \end{aligned} \tag{4.9}$$

where, taking $x = \sigma(\text{inner region at } \alpha)$ and $y = \sigma(\text{outer region at } \alpha)$, we define $\mu_\alpha = \mu_x / \mu_y$.



This gives the required tangle map $Z_T^{P(\Gamma)} : P_{k_1}(\Gamma) \otimes P_{k_2}(\Gamma) \otimes \cdots \otimes P_{k_b}(\Gamma) \to P_{k_0}(\Gamma)$. The fact that this definition does not depend upon the way we isotope the tangle T to a standard form and that these tangle maps satisfy the compatibility conditions for composition of tangles is precisely the crux of Theorem 3.1 in [Jon00a].

[1] We follow the convention that empty tensor product denotes the scalars.

***-structure on $P(\Gamma)$:** There is a natural $*$-algebra structure on each $P_k(\Gamma)$, such that $*$-compatibility condition (4.7) holds:

On $P_{0_\pm}(\Gamma)$, we define $(u^\pm)^* = u^\pm$. And for each $k > 0$, given a $2k$-loop (π, ϵ), we define $(\pi, \epsilon)^*$ to be the reverse of the loop (π, ϵ), i.e.,

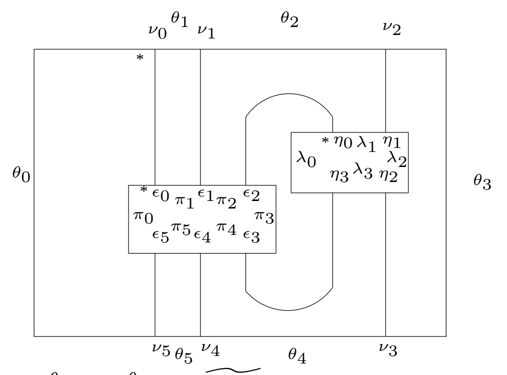
$$(\pi,\epsilon)^* = \begin{tikzcd} & \pi_{2k-1} \arrow[r, "\epsilon_{2k-2}"] & \pi_{2k-2} \arrow[r] & \cdots \arrow[r] & \pi_{k+1} \arrow[dr, "\epsilon_k"] & \\ \pi_0 \arrow[ur, "\epsilon_{2k-1}"] & & & & & \pi_k . \arrow[dl, "\epsilon_{k-1}"] \\ & \pi_1 \arrow[ul, "\epsilon_0"] & \pi_2 \arrow[l, "\epsilon_1"] & \cdots \arrow[l] & \pi_{k-1} \arrow[l] & \end{tikzcd}$$

This gives an involution on $P_k(\Gamma)$, with respect to the multiplication induced by $Z^{P(\Gamma)}_{M_k}$ such that the $*$-compatibility condition (4.7) holds.

4.3.2 Some Calculations

For a coloured tangle T with b internal boxes, it is quite illustrative to write the coefficient $Z^{P(\Gamma)}_T((\pi^{(1)}, \epsilon^{(1)}) \otimes \cdots \otimes (\pi^{(b)}, \epsilon^{(b)}))_{(\pi^{(0)},\, \epsilon^{(0)})}$ as a picture. Basically, the regions in between the marked points on the i-th box of T are labelled by the vertices of the loop $(\pi^{(i)}, \epsilon^{(i)})$ in a clockwise order with $\pi^{(i)}_0$ appearing at the region immediate left to the $*$-vertex, and the marked points are labelled by the edges of the loop joining the corresponding vertices. And a box without any marked point is labelled by the corresponding vertex. We illustrate this by the self explanatory example in equation (4.10), where T is the coloured tangle obtained by removing the labels from the picture.

$$Z^{P(\Gamma)}_T((\pi, \epsilon) \otimes (\lambda, \eta))_{(\theta, \nu)} =$$



$$= \delta^{\theta_{[0,\,2]}}_{\pi_{[0,\,2]}} \, \delta^{\theta_{[4,\,6]}}_{\pi_{[4,\,6]}} \, \delta^{\theta_{[2,\,4]}}_{\lambda_{[1,\,3]}} \, \delta^{\widetilde{\pi_{[2,\,4]}}}_{\lambda_{[3,\,4]} \circ \lambda_{[0,\,1]}} . \qquad (4.10)$$

Note that, the labellings of the marked points of the boxes is redundant. Thus we shall only label the regions on the boxes during calculations.

Given a finite, connected bipartite graph $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ with a spin function μ, making use of the above diagrammatic approach, we describe the expressions for some tangle maps on $P(\Gamma)$, which we shall need in the sequel.

$\boxed{1^{0_\pm} :}$

From the picture of 1^{0_+}, for every $u^+ \in \mathcal{U}^+$, we have $Z^{P(\Gamma)}_{1^{0_+}}(1)_{u^+} = 1$. Thus

$$Z^{P(\Gamma)}_{1^{0_+}}(1) = \sum_{u^+ \in \mathcal{U}^+} u^+. \tag{4.11}$$

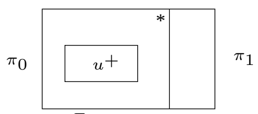
Likewise, we have

$$Z^{P(\Gamma)}_{1^{0_-}}(1) = \sum_{u^- \in \mathcal{U}^-} u^-. \tag{4.12}$$

$\boxed{I_k^{k+1},\ k \in Col :}$

Case I: $k = 0_\pm$: For each $u^+ \in \mathcal{U}^+$, and a 2-loop (π, ϵ), we have

$$Z^{P(\Gamma)}_{I^1_{0_+}}(u^+)_{(\pi,\epsilon)} =$$



$$= \delta^{\epsilon_0}_{\epsilon_1}\, \delta^{\pi_0}_{u^+}.$$

Thus

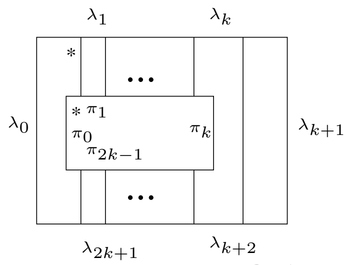
$$Z^{P(\Gamma)}_{I^1_{0_+}}(u^+) = \sum_{\left\{ \begin{array}{c} \text{2-loops}\,(\pi,\epsilon)\ : \\ \epsilon_0 = \epsilon_1, \pi_0 = u^+ \end{array} \right\}} (\pi, \epsilon). \tag{4.13}$$

Likewise, for each $u^- \in \mathcal{U}^-$,

$$Z^{P(\Gamma)}_{I^1_{0_-}}(u^-) = \sum_{\left\{ \begin{array}{c} \text{2-loops}\,(\pi,\epsilon)\ : \\ \epsilon_0 = \epsilon_1, \pi_1 = u^- \end{array} \right\}} (\pi, \epsilon). \tag{4.14}$$

Case II: $k > 0$: For a $2k$-loop (π, ϵ) and a $2(k+1)$-loop (λ, η), we have

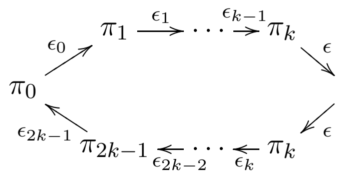
$$Z^{P(\Gamma)}_{I_k^{k+1}}((\pi,\epsilon))_{(\lambda,\eta)} \quad =$$



$$= \quad \delta^{\lambda_{[0,\,k]}}_{\pi_{[0,\,k]}}\ \delta^{\lambda_{[k+2,\,0]}}_{\pi_{[k,\,0]}}\ \delta^{\widetilde{\lambda_{[k+1,\,k+2]}}}_{\lambda_{[k,\,k+1]}}.$$

Thus

$$Z^{P(\Gamma)}_{I_k^{k+1}}((\pi,\epsilon)) = \sum_{\left\{\begin{array}{c}\text{possible}\, v \in \mathcal{U},\\ \epsilon \in \mathcal{E}(\pi_k, v)\end{array}\right\}} \tag{4.15}$$



$\boxed{M_k,\ k \in \ Col:}$

Case I: $k = 0_\pm$: For any three even vertices $u^+, v^+, w^+ \in \mathcal{U}^+$, from the picture of M_{0_+}, we see that $Z^{P(\Gamma)}_{M_{0_+}}(u^+ \otimes v^+)_{w^+} = \delta^{u^+}_{v^+}\,\delta^{v^+}_{w^+}$. Thus

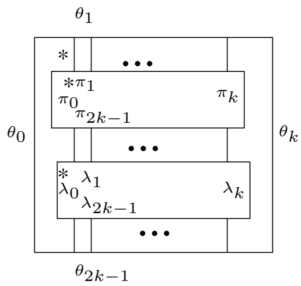
$$Z^{P(\Gamma)}_{M_{0_+}}(u^+ \otimes v^+) = \delta^{u^+}_{v^+}\, u^+. \tag{4.16}$$

Likewise, for any two odd vertices $u^-, v^- \in \mathcal{U}^-$, we get

$$Z^{P(\Gamma)}_{M_{0_-}}(u^- \otimes v^-) = \delta^{u^-}_{v^-}\, u^-. \tag{4.17}$$

Case II: $k > 0$: Let $(\pi,\epsilon), (\lambda,\eta)$ and (θ,ζ) be any three $2k$-loops. Then

$$Z^{P(\Gamma)}_{M_k}((\pi,\epsilon)\otimes(\lambda,\eta))_{(\theta,\zeta)} \quad =$$



$$= \quad \delta^{\pi[0,\,k]}_{\theta_{[0,\,k]}}\ \delta^{\lambda[k,\,0]}_{\theta_{[k,\,0]}}\ \delta^{\widetilde{\pi[k,\,0]}}_{\lambda_{[0,k]}}.$$

Thus

$$Z^{P(\Gamma)}_{M_k}((\pi,\epsilon)\otimes(\lambda,\eta)) = \delta^{\widetilde{\pi[k,\,0]}}_{\lambda_{[0,k]}} \quad \begin{tikzcd} & \pi_1 \arrow[r, "\epsilon_1"] & \cdots \arrow[r] & \pi_{k-1} \arrow[dr, "\epsilon_{k-1}"] & \\ \pi_0 \arrow[ur, "\epsilon_0"] & & & & \lambda_k \arrow[dl, "\eta_k"] \\ & \lambda_{2k-1} \arrow[ul, "\eta_{2k-1}"] & \cdots \arrow[l, "\eta_{2k-2}"] & \lambda_{k+1} \arrow[l] & \end{tikzcd} . \tag{4.18}$$

$\boxed{E^k_{k+1},\ k\in Col:}$

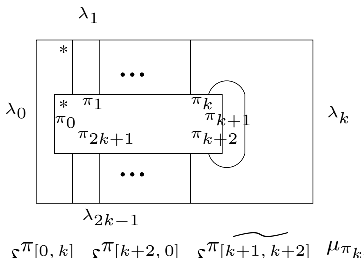
Case I: $k=0_\pm$: Again, looking at the figure of E^{0+}_1, for any 2-loop (π,ϵ) and a vertex $u^+\in\mathcal{U}^+$, we have $Z^{P(\Gamma)}_{E^{0+}_1}((\pi,\epsilon))_{u^+} = \delta^{\pi_0}_{u^+}\,\delta^{\epsilon_0}_{\epsilon_1}\,\frac{\mu^2_{\pi_1}}{\mu^2_{\pi_0}}$. Thus

$$Z^{P(\Gamma)}_{E^{0+}_1}((\pi,\epsilon)) \quad = \quad \delta^{\epsilon_0}_{\epsilon_1}\,\frac{\mu^2_{\pi_1}}{\mu^2_{\pi_0}}\ \pi_0;\ \text{and likewise,} \tag{4.19}$$

$$Z^{P(\Gamma)}_{E^{0-}_1}((\pi,\epsilon)) \quad = \quad \delta^{\epsilon_0}_{\epsilon_1}\,\frac{\mu^2_{\pi_0}}{\mu^2_{\pi_1}}\ \pi_1. \tag{4.20}$$

Case II: $k>0$: Given a $2(k+1)$-loop (π,ϵ) and a $2k$-loop (λ,η), we have

$$Z^{P(\Gamma)}_{E^k_{k+1}}((\pi,\epsilon))_{(\lambda,\eta)} \quad =$$



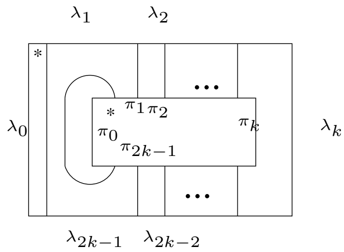
$$= \quad \delta^{\pi[0,\,k]}_{\lambda_{[0,\,k]}}\ \delta^{\pi[k+2,\,0]}_{\lambda_{[k,\,0]}}\ \delta^{\pi\widetilde{[k+1,\,k+2]}}_{\pi_{[k,\,k+1]}}\ \frac{\mu_{\pi_{k+1}}}{\mu_{\pi_k}}\ \frac{\mu_{\pi_{k+1}}}{\mu_{\pi_{k+2}}}.$$

Thus

$$Z^{P(\Gamma)}_{E^k_{k+1}}((\pi,\epsilon)) = \delta^{\pi\widetilde{[k+1,\,k+2]}}_{\pi_{[k,\,k+1]}}\ \frac{\mu^2_{\pi_{k+1}}}{\mu^2_{\pi_k}} \quad \begin{tikzcd} & \pi_1 \arrow[r, "\epsilon_1"] & \cdots \arrow[r] & \pi_{k-1} \arrow[dr, "\epsilon_{k-1}"] & \\ \pi_0 \arrow[ur, "\epsilon_0"] & & & & \pi_k \arrow[dl, "\epsilon_{k+2}"] \\ & \pi_{2k+1} \arrow[ul, "\epsilon_{2k+1}"] & \cdots \arrow[l, "\epsilon_{2k}"] & \pi_{k+3} \arrow[l] & \end{tikzcd} . \tag{4.21}$$

$\boxed{(E')^k_k,\ k \geq 1 :}$ For any two $2k$-loops (π, ϵ) and (λ, η), we have

$$Z^{P(\Gamma)}_{(E')^k_k}((\pi,\epsilon))_{(\lambda,\eta)} =$$



$$= \delta^{\lambda_{[1,\,2k-1]}}_{\pi_{[1\,2k-1]}}\ \delta^{\widetilde{\pi_{[2k-1,\,0]}}}_{\pi_{[0,\,1]}}\ \delta^{\eta_0}_{\eta_{2k-1}}\ \frac{\mu_{\pi_0}}{\pi_1}\ \frac{\mu_{\pi_0}}{\pi_{2k-1}}.$$

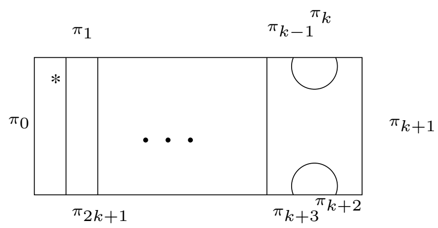
Thus

$$Z^{P(\Gamma)}_{(E')^k_k}((\pi,\epsilon)) = \tag{4.22}$$

$$\delta^{\widetilde{\pi_{[2k-1,\,0]}}}_{\pi_{[0,\,1]}}\ \frac{\mu^2_{\pi_0}}{\mu^2_{\pi_1}} \sum_{\left\{\begin{array}{c}\text{possible}\, u \in \mathcal{U}^+,\\ \epsilon \in \mathcal{E}(u,\pi_1)\end{array}\right\}} \begin{tikzcd} & \pi_1 \arrow[r, "\epsilon_1"] & \cdots \arrow[r] & \pi_{k-1} \arrow[dr, "\epsilon_{k-1}"] & \\ u \arrow[ur, "\epsilon"] & & & & \pi_k \arrow[dl, "\epsilon_k"] \\ & \pi_{2k-1} \arrow[ul, "\epsilon"] & \cdots \arrow[l, "\epsilon_{2k-2}"'] & \pi_{k+1} \arrow[l] & \end{tikzcd}.$$

$\boxed{\mathcal{E}^{k+1},\ k \geq 1 :}$ For any $2(k+1)$-loop (π, ϵ), we have

$$Z^{P(\Gamma)}_{\mathcal{E}^{k+1}}(1)_{(\pi,\epsilon)} =$$



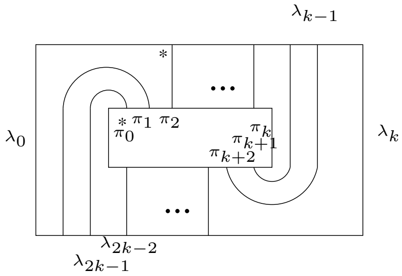
$$= \delta^{\widetilde{\pi_{[k+3,\,0]}}}_{\pi_{[0,\,k-1]}}\ \delta^{\widetilde{\pi_{[k,\,k+1]}}}_{\pi_{[k-1,\,k]}}\ \delta^{\widetilde{\pi_{[k+2,\,k+3]}}}_{\pi_{[k+1,\,k+2]}}\ \frac{\mu_{\pi_k}}{\mu_{\pi_{k-1}}}\ \frac{\mu_{\pi_{k+2}}}{\mu_{\pi_{k+1}}}.$$

Thus

$$Z^{P(\Gamma)}_{\mathcal{E}^{k+1}}(1) = \sum_{\left\{\begin{array}{c}\text{2(k+1)-loops}\ \widetilde{(\pi,\epsilon)}\ :\\ \pi_{[0,\,k-1]} = \widetilde{\pi_{[k+3,\,0]}},\\ \pi_{[k-1,\,k]} = \widetilde{\pi_{[k,\,k+1]}},\\ \pi_{[k+1,\,k+2]} = \pi_{[k+2,\,k+3]}\end{array}\right\}} \frac{\mu_{\pi_k}\,\mu_{\pi_{k+2}}}{\mu^2_{\pi_{k-1}}}\,(\pi,\epsilon). \tag{4.23}$$

$\boxed{R_k,\ k \geq 2:}$ For any two $2k$-loops (π, ϵ) and (λ, η), we have

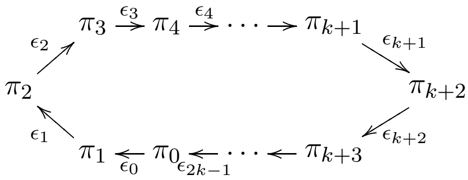
$$Z_{R_k}^{P(\Gamma)}((\pi,\epsilon))_{(\lambda,\eta)} =$$



$$= \delta^{\lambda_{[0,\,k]}}_{\pi_{[2,\,k+2]}}\ \delta^{\lambda_{[k,\,0]}}_{\pi_{[k+2,\,0]}\circ\widetilde{\pi_{[0,\,2]}}}\ \frac{\mu_{\pi_0}}{\mu_{\pi_2}}\ \frac{\mu_{\pi_k}}{\mu_{\pi_{k+2}}}.$$

Thus

$$Z_{R_k}^{P(\Gamma)}((\pi,\epsilon)) = \frac{\mu_{\pi_0}\,\mu_{\pi_k}}{\mu_{\pi_2}\,\mu_{\pi_{k+2}}}$$



(4.24)

4.3.3 Flip and Dual

Given a bipartite graph $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$, we define another bipartite graph $\overline{\Gamma} = (\mathcal{U}^-, \mathcal{U}^+, \mathcal{E})$ such that if ϵ is an edge in Γ joining an even vertex u_+ with an odd vertex u_-, then in $\overline{\Gamma}$, ϵ is an edge joining the even vertex u_- with the odd vertex u_+. We call $\overline{\Gamma}$ to be the *flip* of the graph Γ.

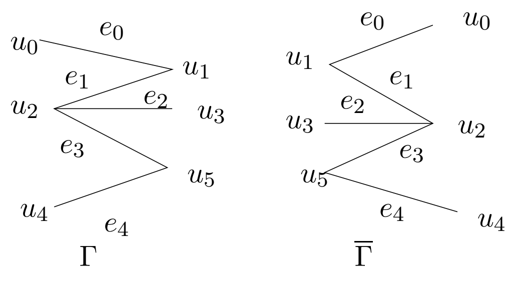


Figure 4.10: Flip of a bipartite graph

Given a finite, connected bipartite graph $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ with a spin function μ, we note that its flip $\overline{\Gamma}$ is also finite and connected, and μ is a spin function for $\overline{\Gamma}$ as well.

The flip and the dual 'commute' with each other at the level of planar algebras.

Theorem 4.3.2 *The dual of the planar algebra of a finite, connected bipartite graph* $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E}, \mu)$ *is isomorphic to the planar algebra of its flip* $\overline{\Gamma} = (\mathcal{U}^-, \mathcal{U}^+, \mathcal{E}, \mu)$*, i.e.,*
$$^-P(\Gamma) \cong P(\overline{\Gamma}).$$

Proof: We have $^-P(\Gamma)_{0_\pm} = P_{\mp 0}(\Gamma) = \mathbb{C}[\mathcal{U}^\mp] = P_{\pm 0}(\overline{\Gamma})$; and for each $k > 0$,

$$\begin{array}{rcl} ^-P(\Gamma)_k & = & P_k(\Gamma) = \mathbb{C}[\,2k\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^+\,], \text{ and} \\ P(\overline{\Gamma})_k & = & \mathbb{C}[\,2k\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^-\,]. \end{array}$$

Let $\varphi_{0_\pm} : {}^-P(\Gamma)_{0_\pm} \to P_{0_\pm}(\overline{\Gamma})$ be the identity morphism of the underlying vector space; and for each $k > 0$, define $\varphi_k : {}^-P(\Gamma)_k \to P_k(\overline{\Gamma})$ to be the map whose action on basis vectors is given by

$$\varphi_k((\pi,\epsilon)) = \frac{\mu_{\pi_0}\,\mu_{\pi_k}}{\mu_{\pi_1}\,\mu_{\pi_{k+1}}} \begin{array}{ccccc} & \pi_2 \xrightarrow{\epsilon_2} \cdots \longrightarrow \pi_k & \\ {}^{\epsilon_1}\nearrow & & \searrow^{\epsilon_k} \\ \pi_1 & & \pi_{k+1} \\ {}_{\epsilon_0}\nwarrow & & \swarrow_{\epsilon_{k+1}} \\ & \pi_0 \xleftarrow[\epsilon_{2k-1}]{} \cdots \longleftarrow \pi_{k+2} & \end{array}, \tag{4.25}$$

for all $2k$-loops (π,ϵ) on Γ based at vertices in $\mathcal{U}^+$. Clearly, $\varphi_k((\pi,\epsilon))$ is a $2k$-loop on $\overline{\Gamma}$ based at a vertex in $\mathcal{U}^-$.

And, as the above correspondence is a bijection between $2k$-loops on Γ and those on its flip, each φ_k is a linear isomorphism. We claim that $\varphi := \{\varphi_k;\, k \in \mathit{Col}\}$ is a planar algebra isomorphism from $^-P(\Gamma)$ onto $P(\overline{\Gamma})$.

Let $\mathcal{T}$ be the collection of tangles T for which equation (**??**) holds. By Lemma 4.2.4, $\mathcal{T}$ is closed under composition of tangles. Thus, in order to show that $\mathcal{T}$ contains all the coloured tangles, it is enough to show, by Theorem 4.1.1, that it contains the collection

$$\mathcal{G}_1 = \{1^{0_\pm}\} \cup \{I_k^{k+1}, M_k, E_{k+1}^k : k \in \mathit{Col}\} \cup \{R_k : k \geq 2\}.$$

We show this by explicitly checking the commutativity of φ with all tangle maps for tangles in $\mathcal{G}_1$.

$\boxed{1^{0_\pm} :}$

Note that the dual $(1^{0_\pm})^-$ is $1^{0_\mp}$. Clearly,

$$Z_{1^{0_\pm}}^{^-P(\Gamma)}(1)_{u^\mp} = Z_{1^{0_\mp}}^{P(\Gamma)}(1)_{u^\mp} = 1; \text{ and also } Z_{1^{0_\pm}}^{P(\overline{\Gamma})}(1)_{u^\mp} = 1,$$

for all $u^{\mp} \in \mathcal{U}^{\mp}$. Thus $\varphi_{0_\pm} \circ Z^{-P(\Gamma)}_{1^{0_\pm}} = Z^{P(\overline{\Gamma})}_{1^{0_\pm}}$, and $1^{0_\pm} \in \mathcal{T}$.

$\boxed{I_k^{k+1},\ k \in Col:}$

Case I : $k = 0_\pm$*:* The dual $(I^1_{0_\pm})^- = I^1_{0_\mp}$. Thus for any odd vertex $u^- \in \mathcal{U}^-$, by (4.14), we have

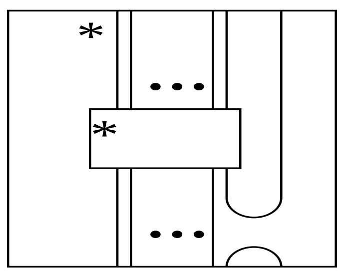
$$Z^{-P(\Gamma)}_{I^1_{0_+}}(u^-) = Z^{P(\Gamma)}_{I^1_{0_-}}(u^-) = \sum_{\left\{\substack{\text{2-loops } (\pi,\epsilon) \text{ on } \Gamma: \\ \pi_1 = u_-,\ \epsilon_0 = \epsilon_1}\right\}} (\pi, \epsilon)$$
$$\xmapsto{\varphi_1} \sum_{\left\{\substack{\text{2-loops } (\lambda,\eta) \text{ on } \overline{\Gamma}: \\ \lambda_0 = u_-,\ \eta_0 = \eta_1}\right\}} (\lambda, \eta).$$

By (4.13) for $P(\overline{\Gamma})$, we have

$$Z^{P(\overline{\Gamma})}_{I^1_{0_+}}(\varphi_{0_+}(u^-)) = Z^{P(\overline{\Gamma})}_{I^1_{0_+}}(u^-) = \sum_{\left\{\substack{\text{2-loops } (\lambda,\eta) \text{ on } \overline{\Gamma}: \\ \lambda_0 = u_-,\ \eta_0 = \eta_1}\right\}} (\lambda, \eta).$$

This shows that $\varphi_1 \circ Z^{-P(\Gamma)}_{I^1_{0_+}} = Z^{P(\overline{\Gamma})}_{I^1_{0_+}} \circ \varphi_{0_+}$, and $I^1_{0_+} \in \mathcal{T}$. Exactly on similar lines we see that $I^1_{0_-} \in \mathcal{T}$.

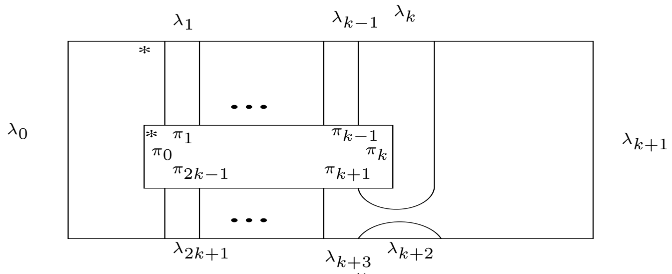
Case II: $k > 0$*:* We have

$(I_k^{k+1})^- =$ .

Thus, for each $2k$-loop (π, ϵ) and a $2(k+1)$-loop (λ, η) on Γ based at a vertex in $\mathcal{U}^+$, we have

$$Z^{-P(\Gamma)}_{I_k^{k+1}}((\pi,\epsilon))_{(\lambda,\eta)} = Z^{P(\Gamma)}_{(I_k^{k+1})^-}((\pi,\epsilon))_{(\lambda,\eta)}$$

$$=$$



$$= \quad \delta^{\pi_{[0,\,k+1]}}_{\lambda_{[0,\,k+1]}}\, \delta^{\pi_{[k+1,\,0]}}_{\lambda_{[k+3,\,0]}}\, \delta^{\eta_{k+1}}_{\eta_{k+2}}\, \frac{\mu_{\pi_k}}{\mu_{\pi_{k+1}}}\, \frac{\mu_{\lambda_{k+2}}}{\mu_{\pi_{k+1}}}.$$

Thus

$$Z^{-P(\Gamma)}_{I_k^{k+1}}((\pi,\epsilon)) \quad = \quad Z^{P(\Gamma)}_{(I_k^{k+1})^-}((\pi,\epsilon))$$

$$= \sum_{\left\{\begin{array}{l}\text{possible } v \in \mathcal{U},\\ \epsilon \in \mathcal{E}(\pi_{k+1}, v)\end{array}\right\}} \frac{\mu_{\pi_k}\mu_v}{\mu^2_{\pi_{k+1}}} \begin{tikzcd} & \pi_1 \arrow[r, "\epsilon_1"] & \cdots \arrow[r, "\epsilon_{k-1}"] & \pi_k \arrow[dr, "\epsilon_k"] & \\ \pi_0 \arrow[ur, "\epsilon_0"] & & & & \pi_{k+1} \arrow[dl, "\epsilon"] \\ & \pi_{2k-1} \arrow[ul, "\epsilon_{2k-1}"] & \cdots \arrow[l, "\epsilon_{2k-2}"] & v \arrow[l, "\epsilon"] & \end{tikzcd}$$

$$\overset{\varphi_{k+1}}{\longmapsto} \quad \sum_{\left\{\begin{array}{l}\text{possible } v \in \mathcal{U},\\ \epsilon \in \mathcal{E}(\pi_{k+1}, v)\end{array}\right\}} \frac{\mu_{\pi_k}\mu_{\pi_0}}{\mu_{\pi_{k+1}}\mu_{\pi_1}} \begin{tikzcd} & \pi_2 \arrow[r, "\epsilon_2"] & \cdots \arrow[r, "\epsilon_k"] & \pi_{k+1} \arrow[dr, "\epsilon"] & \\ \pi_1 \arrow[ur, "\epsilon_1"] & & & & v \arrow[dl, "\epsilon"] \\ & \pi_0 \arrow[ul, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_{2k-1}"] & \pi_{k+1} \arrow[l, "\epsilon_{k+1}"] & \end{tikzcd} .$$

On the other hand, by (4.15) for $P(\overline{\Gamma})$, we have

$$Z^{P(\overline{\Gamma})}_{I_k^{k+1}}(\varphi_k(\pi,\epsilon)) = \sum_{\left\{\begin{array}{l}\text{possible } v \in \mathcal{U},\\ \epsilon \in \mathcal{E}(\pi_{k+1}, v)\end{array}\right\}} \frac{\mu_{\pi_0}\,\mu_{\pi_k}}{\mu_{\pi_1}\,\mu_{\pi_{k+1}}} \begin{tikzcd} & \pi_2 \arrow[r, "\epsilon_2"] & \cdots \arrow[r, "\epsilon_k"] & \pi_{k+1} \arrow[dr, "\epsilon"] & \\ \pi_1 \arrow[ur, "\epsilon_1"] & & & & v \arrow[dl, "\epsilon"] \\ & \pi_0 \arrow[ul, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_{2k-1}"] & \pi_{k+1} \arrow[l, "\epsilon_{k+1}"] & \end{tikzcd} ,$$

which shows that $\varphi_{k+1} \circ Z^{-P(\Gamma)}_{I_k^{k+1}} = Z^{P(\overline{\Gamma})}_{I_k^{k+1}} \circ \varphi_k$, and thus $I_k^{k+1} \in \mathcal{T}$.

$\boxed{M_k,\ k \in Col:}$

Case I: $k = 0_\pm$: We have $(M_{0_\pm})^- = M_{0_\mp}$. For any two even vertices

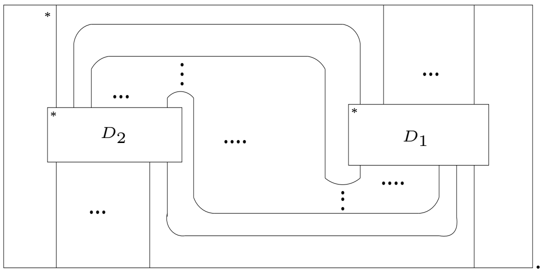
$u_1, u_2 \in \mathcal{U}^+$, by (4.17), we get

$$Z^{-P(\Gamma)}_{M_{0_-}}(u_1 \otimes u_2) = Z^{P(\Gamma)}_{M_{0_+}}(u_1 \otimes u_2) = \delta^{u_1}_{u_2} u_1.$$

By (4.16) for $P(\overline{\Gamma})$, we have $Z^{P(\overline{\Gamma})}_{M_{0_-}}(u_1 \otimes u_2) = \delta^{u_1}_{u_2} u_1$. This shows that $\varphi_{0_-} \circ Z^{-P(\Gamma)}_{M_{0_-}} = Z^{P(\overline{\Gamma})}_{M_{0_-}} \circ \varphi_{0_-}$, and thus $M_{0_-} \in \mathcal{T}$. Similarly $M_{0_+} \in \mathcal{T}$.

Case II: $k > 0$: The dual of M_k is given by

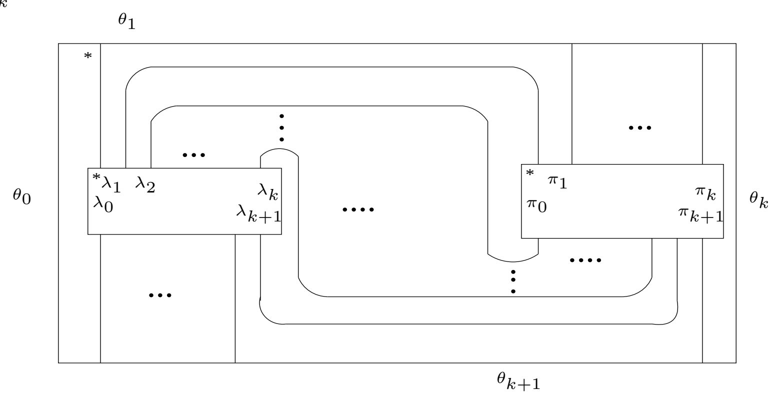
$$M_k^- =$$



Thus, for any triplet $(\pi, \epsilon), (\lambda, \eta)$ and (θ, ζ) of $2k$-loops, we have

$$\begin{aligned}
&Z^{-P(\Gamma)}_{M_k}((\pi, \epsilon) \otimes (\lambda, \eta))_{(\theta,\zeta)} \\
&= \; Z^{P(\Gamma)}_{M_k^-}((\pi, \epsilon) \otimes (\lambda, \eta))_{(\theta,\zeta)}
\end{aligned}$$

$$=$$



$$= \; \delta^{\lambda_{[1,\,k+1]}}_{\widetilde{\pi_{[0,\,1]}} \circ \widetilde{\pi_{[k+1,\,0]}}} \; \delta^{\theta_{[0,\,k+1]}}_{\lambda_{[0,\,1]} \circ \pi_{[1,\,k+1]}} \; \delta^{\theta_{[k+1,\,0]}}_{\lambda_{[k+1,\,0]}} \; \times$$

$$\frac{\mu_{\lambda_2}}{\mu_{\lambda_1}} \frac{\mu_{\lambda_3}}{\mu_{\lambda_2}} \cdots \frac{\mu_{\lambda_k}}{\mu_{\lambda_{k-1}}} \frac{\mu_{\pi_0}}{\mu_{\pi_{2k-1}}} \frac{\mu_{\pi_{2k-1}}}{\mu_{\pi_{2k-2}}} \cdots \frac{\mu_{\pi_{k+2}}}{\mu_{\pi_{k+1}}}.$$

This gives

$$Z^{-P(\Gamma)}_{M_k}((\pi,\epsilon)\otimes(\lambda,\eta))$$

$$= \delta^{\widetilde{\lambda_{[1,\,k+1]}}}_{\widetilde{\pi_{[0,\,1]}\circ\pi_{[k+1,\,0]}}} \frac{\mu_{\lambda_k}\mu_{\pi_0}}{\mu_{\lambda_1}\mu_{\pi_{k+1}}} \begin{tikzcd} & \pi_1 \arrow[r, "\epsilon_1"] & \cdots \arrow[r] & \pi_{k-1} \arrow[dr, "\epsilon_{k-1}"] & \\ \lambda_0 \arrow[ur, "\epsilon_0"] & & & & \pi_k \arrow[dl, "\epsilon_k"] \\ & \lambda_{2k-1} \arrow[ul, "\eta_{2k-1}"] & \cdots \arrow[l, "\eta_{2k-2}"] & \lambda_{k+1} \arrow[l] & \end{tikzcd}$$

$$\overset{\varphi_k}{\longmapsto} \delta^{\widetilde{\lambda_{[1,\,k+1]}}}_{\widetilde{\pi_{[0,\,1]}\circ\pi_{[k+1,\,0]}}} \frac{\mu_{\lambda_k}\,\mu_{\pi_0}}{\mu_{\lambda_1}\,\mu_{\pi_{k+1}}} \frac{\mu_{\lambda_0}\,\mu_{\pi_k}}{\mu_{\pi_1}\,\mu_{\lambda_{k+1}}} \begin{tikzcd} & \pi_2 \arrow[r, "\epsilon_2"] & \cdots \arrow[r] & \pi_k \arrow[dr, "\epsilon_k"] & \\ \pi_1 \arrow[ur, "\epsilon_{\eta_1}"] & & & & \pi_{k+1} \arrow[dl, "\eta_{k+1}"] \\ & \lambda_0 \arrow[ul, "\eta_1"] & \cdots \arrow[l, "\eta_{2k-1}"] & \lambda_{k+2} \arrow[l] & \end{tikzcd} .$$

By (4.18) for $P(\overline{\Gamma})$, we have

$$Z^{P(\overline{\Gamma})}_{M_k}(\varphi_k((\pi,\epsilon))\otimes\varphi_k((\lambda,\eta)))$$

$$= \delta^{\widetilde{\lambda_{[1,\,k+1]}}}_{\widetilde{\pi_{[0,\,1]}\circ\pi_{[k+1,\,0]}}} \frac{\mu_{\lambda_k}\,\mu_{\pi_0}}{\mu_{\lambda_1}\,\mu_{\pi_{k+1}}} \frac{\mu_{\lambda_0}\,\mu_{\pi_k}}{\mu_{\pi_1}\,\mu_{\lambda_{k+1}}} \begin{tikzcd} & \pi_2 \arrow[r, "\epsilon_2"] & \cdots \arrow[r] & \pi_k \arrow[dr, "\epsilon_k"] & \\ \pi_1 \arrow[ur, "\epsilon_1"] & & & & \pi_{k+1} \arrow[dl, "\eta_{k+1}"] \\ & \lambda_0 \arrow[ul, "\eta_1"] & \cdots \arrow[l, "\eta_{2k-1}"] & \lambda_{k+2} \arrow[l] & \end{tikzcd} .$$

Thus M_k commutes with φ and belongs to $\mathcal{T}$.

$\boxed{E^k_{k+1},\, k \in Col:}$

Case I: $k = \pm$: We have $(E^{0\pm}_1)^- = E^{0\mp}_1$. Thus for any 2-loop (π,ϵ), by (4.20), we get

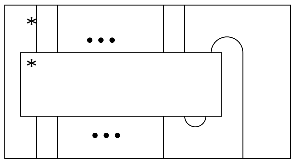
$$\begin{aligned} Z^{-P(\Gamma)}_{E^{0+}_1}((\pi,\epsilon)) = Z^{P(\Gamma)}_{E^{0-}_1}((\pi,\epsilon)) &= \delta^{\epsilon_0}_{\epsilon_1}\frac{\mu^2_{\pi_0}}{\mu^2_{\pi_1}}\pi_1 \\ &\overset{\varphi_{0_+}}{\longmapsto} \delta^{\epsilon_0}_{\epsilon_1}\frac{\mu^2_{\pi_0}}{\mu^2_{\pi_1}}\pi_1. \end{aligned}$$

By (4.19) for $P(\overline{\Gamma})$, we have

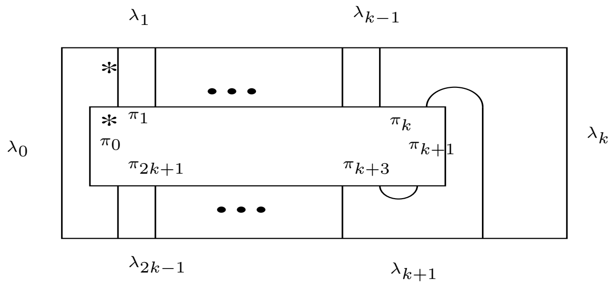
$$Z^{P(\overline{\Gamma})}_{E^{0+}_1}(\varphi_1((\pi,\epsilon))) = \frac{\mu^2_{\pi_0}}{\mu^2_{\pi_1}}\delta^{\epsilon_0}_{\epsilon_1}\pi_1.$$

Thus $\varphi_{0_+}\circ Z^{-P(\Gamma)}_{E^{0+}_1} = Z^{P(\overline{\Gamma})}_{E^{0+}_1}\circ\varphi_1$, and $E^{0+}_1 \in \mathcal{T}$.

Case II: $k > 0$: A candidate for the dual tangle $(E^k_{k+1})^-$ is given by

$(E^k_{k+1})^- \quad =$ 

Thus for any two loops (π, ϵ) and (λ, η) of lengths $2(k+1)$ and $2k$, respectively, we have

$Z^{-P(\Gamma)}_{E^k_{k+1}}((\pi,\epsilon))_{(\lambda,\eta)} \quad =$ 

$$= \quad \delta^{\pi_{[0,\,k]}}_{\lambda_{[0,\,k]}}\, \delta^{\pi_{[k+3,\,0]}}_{\lambda_{[k+1,\,0]}}\, \delta^{\pi_{\widetilde{[k+1,\,k+2]}}}_{\pi_{[k+2,\,k+3]}}\, \frac{\mu_{\pi_{k+1}}}{\mu_{\pi_k}}\, \frac{\mu_{\pi_{k+2}}}{\mu_{\pi_{k+1}}}.$$

This gives

$$Z^{-P(\Gamma)}_{E^k_{k+1}}((\pi,\epsilon)) \quad = \quad \delta^{\pi_{\widetilde{[k+1,\,k+2]}}}_{\pi_{[k+2,\,k+3]}}\, \frac{\mu_{\pi_{k+2}}}{\mu_{\pi_k}} \begin{tikzcd} & \pi_1 \arrow[r, "\epsilon_1"] & \cdots \arrow[r, "\epsilon_{k-2}"] & \pi_{k-1} \arrow[dr, "\epsilon_{k-1}"] & \\ \pi_0 \arrow[ur, "\epsilon_0"] & & & & \pi_k \arrow[dl, "\epsilon_k"] \\ & \pi_{2k+1} \arrow[ul, "\epsilon_{2k+1}"] & \cdots \arrow[l, "\epsilon_{2k}"] & \pi_{k+3} \arrow[l, "\epsilon_{k+3}"] & \end{tikzcd}$$

$$\overset{\varphi_k}{\longmapsto} \quad \delta^{\pi_{\widetilde{[k+1,\,k+2]}}}_{\pi_{[k+2,\,k+3]}}\, \frac{\mu_{\pi_0}\,\mu_{\pi_{k+2}}}{\mu_{\pi_1}\,\mu_{\pi_{k+1}}} \begin{tikzcd} & \pi_2 \arrow[r, "\epsilon_2"] & \cdots \arrow[r] & \pi_k \arrow[dr, "\epsilon_k"] & \\ \pi_1 \arrow[ur, "\epsilon_1"] & & & & \pi_{k+3} \arrow[dl, "\epsilon_{k+3}"] \\ & \pi_0 \arrow[ul, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_{2k+1}"] & \pi_{k+4} \arrow[l] & \end{tikzcd} .$$

By (4.21) for $P(\overline{\Gamma})$, we see that

$$Z^{P(\overline{\Gamma})}_{E^k_{k+1}}(\varphi_{k+1}((\pi,\epsilon))) = \delta^{\pi_{\widetilde{[k+1,\,k+2]}}}_{\pi_{[k+2,\,k+3]}}\, \frac{\mu_{\pi_0}\,\mu_{\pi_{k+2}}}{\mu_{\pi_1}\,\mu_{\pi_{k+1}}} \begin{tikzcd} & \pi_2 \arrow[r, "\epsilon_2"] & \cdots \arrow[r, "\epsilon_{k-1}"] & \pi_k \arrow[dr, "\epsilon_k"] & \\ \pi_1 \arrow[ur, "\epsilon_1"] & & & & \pi_{k+3} \arrow[dl, "\epsilon_{k+3}"] \\ & \pi_0 \arrow[ul, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_{2k+1}"] & \pi_{k+4} \arrow[l, "\epsilon_{k+4}"] & \end{tikzcd} .$$

Thus $\varphi_k \circ Z^{-P(\Gamma)}_{E^k_{k+1}} = Z^{P(\overline{\Gamma})}_{E^k_{k+1}} \circ \varphi_{k+1}$, and $E^k_{k+1} \in \mathcal{T}$.

$\boxed{R_k,\ k \geq 2}$

We note that $(R_k)^- = R_k$, $\forall\, k \geq 2$. Thus for any pair of $2k$-loops (π, ϵ) and (λ, η), by (4.24), we have

$$Z^{-P(\Gamma)}_{R_k}((\pi,\epsilon)) = Z^{P(\Gamma)}_{R_k}((\pi,\epsilon))$$

$$= \frac{\mu_{\pi_0}\,\mu_{\pi_k}}{\mu_{\pi_2}\,\mu_{\pi_{k+2}}} \begin{tikzcd}[column sep=small] & \pi_3 \arrow[r, "\epsilon_3"] & \pi_4 \arrow[r, "\epsilon_4"] & \cdots \arrow[r] & \pi_{k+1} \arrow[dr, "\epsilon_{k+1}"] & \\ \pi_2 \arrow[ur, "\epsilon_2"] & & & & & \pi_{k+2} \arrow[dl, "\epsilon_{k+2}"] \\ & \pi_1 \arrow[ul, "\epsilon_1"] & \pi_0 \arrow[l, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_{2k-1}"] & \pi_{k+3} \arrow[l] & \end{tikzcd}$$

$$\stackrel{\varphi_k}{\longmapsto} \frac{\mu_{\pi_0}\,\mu_{\pi_k}}{\mu_{\pi_3}\,\mu_{\pi_{k+3}}} \begin{tikzcd}[column sep=small] & \pi_4 \arrow[r, "\epsilon_4"] & \pi_5 \arrow[r, "\epsilon_5"] & \cdots \arrow[r] & \pi_{k+2} \arrow[dr, "\epsilon_{k+2}"] & \\ \pi_3 \arrow[ur, "\epsilon_3"] & & & & & \pi_{k+3} \arrow[dl, "\epsilon_{k+3}"] \\ & \pi_2 \arrow[ul, "\epsilon_2"] & \pi_1 \arrow[l, "\epsilon_1"] & \cdots \arrow[l, "\epsilon_0"] & \pi_{k+4} \arrow[l] & \end{tikzcd} .$$

And, by (4.24) for $P(\overline{\Gamma})$, we have

$$Z^{P(\overline{\Gamma})}_{R_k}((\pi,\epsilon)) = \frac{\mu_{\pi_0}\,\mu_{\pi_k}}{\mu_{\pi_3}\,\mu_{\pi_{k+3}}} \begin{tikzcd}[column sep=small] & \pi_4 \arrow[r, "\epsilon_4"] & \pi_5 \arrow[r, "\epsilon_5"] & \cdots \arrow[r] & \pi_{k+2} \arrow[dr, "\epsilon_{k+2}"] & \\ \pi_3 \arrow[ur, "\epsilon_3"] & & & & & \pi_{k+3} \arrow[dl, "\epsilon_{k+3}"] \\ & \pi_2 \arrow[ul, "\epsilon_2"] & \pi_1 \arrow[l, "\epsilon_1"] & \cdots \arrow[l, "\epsilon_0"] & \pi_{k+4} \arrow[l] & \end{tikzcd} .$$

Thus φ commutes with the tangle maps for the rotation tangles, and so $R_k \in \mathcal{T}$. This completes the proof. $\square$

4.3.4 Group Action on a Bipartite Graph with Spin Function

Definition 4.3.3 *Let $\Gamma = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$ be a bipartite graph with a spin function μ. We say that a finite group G acts on Γ if*

- *G acts on each of the sets $\mathcal{U}^+$, $\mathcal{U}^-$ and $\mathcal{E}$; such that*
- *if ϵ is an edge joining the vertices u_+ and u_-, then $g\epsilon$ is an edge between the vertices gu_+ and gu_-.*

Further, we say that G acts on (Γ, μ) if G acts on Γ and the spin function μ is constant on the G-orbits of $\mathcal{U}^\pm$.

As in the case of flip and duality, group invariance and duality 'commute' with each other at the level of planar algebras.

Theorem 4.3.4 *Let $\Gamma = (\mathcal{U}^+,\mathcal{U}^-,\mathcal{E})$ be a finite, connected bipartite graph with a spin function μ, and suppose that a finite group G acts on (Γ,μ). Then there is a canonical action of G on the planar algebra $P(\overline{\Gamma})$, and its G-invariant planar subalgebra $P(\overline{\Gamma})^G$ is isomorphic to the dual planar algebra $^-(P(\Gamma)^G)$, i.e.,*

$$P(\overline{\Gamma})^G \cong {}^-(P(\Gamma)^G).$$

Proof: For each $k > 0$, the action of G on Γ induces an action on the set $\{2k\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^+\}$ by

$$g(\pi,\epsilon) = \begin{tikzcd} & g\pi_1 \arrow[r, "g\,\epsilon_1"] & g\pi_2 \arrow[r] & \cdots \arrow[r] & g\pi_{k-1} \arrow[dr, "g\,\epsilon_{k-1}"] & \\ g\pi_0 \arrow[ur, "g\,\epsilon_0"] & & & & & g\pi_k, \arrow[dl, "g\,\epsilon_k"] \\ & g\pi_{2k-1} \arrow[ul, "g\,\epsilon_{2k-1}"] & g\pi_{2k-2} \arrow[l, "g\,\epsilon_{2k-2}"] & \cdots \arrow[l] & g\pi_{k+1} \arrow[l] & \end{tikzcd}$$

for all $2k$-loops (π,ϵ) and $g \in G$.

Already, G acts on $\mathcal{U}^{\pm}$, thus we get a group action on $P_k(\Gamma)$, for all $k \in \mathit{Col}$. Thanks to the invariance of the spin function on the G-orbits of $\mathcal{U}^{\pm}$, Lemma 4.2.4 and Theorem 4.1.1, a bit of straight forward checking readily shows that it is in fact a G-action on $P(\Gamma)$.

And the second assertion that $P(\overline{\Gamma})^G \cong {}^-(P(\Gamma)^G)$ is a mere consequence of Proposition 4.2.11 and the fact that the morphism $\varphi = \{\varphi_k;\, k \in \mathit{Col}\}$ giving the planar isomorphism $P(\overline{\Gamma}) \cong {}^-P(\Gamma)$ in Theorem 4.3.2 is in fact a G-map, i.e., $\varphi_k(gx) = g\varphi_k(x)$, $\forall g \in G, x \in P_k, k \in \mathit{Col}$. $\square$

We now list some simple yet useful observations, which were also suggested in [Jon00a].

Lemma 4.3.5 *Let Γ and G be as in Theorem 4.3.4. Then the G-invariant planar subalgebra $P(\Gamma)^G$*

1. *is connected iff G acts transitively on $\mathcal{U}^{\pm}$.*
2. *has modulus $||\Gamma||$ if in addition to (1), $(\mu_u^2)_{u\in\mathcal{U}}$ is the Perron-Frobenius eigen-vector of the adjacency matrix A_Γ of Γ, where $||\Gamma||$ is the norm of A_Γ.*
3. *is irreducible iff G acts transitively on the set $\{2\text{-loops on } \Gamma \text{ based at vertices in } \mathcal{U}^+\}$.*

Proof: (1) follows from the easily verifiable fact that $P_{0_\pm}(\Gamma)$ has a basis given by $\{\sum_{g\in G} gu^\pm : u^\pm \in$ a set of representatives of G-orbits of $\mathcal{U}^\pm\}$.

For (2), we first note that in $P(\Gamma)$, given any odd vertex u^- and an even vertex u^+,

$$Z^{P(\Gamma)}_{T^+_-}(u^-)_{u^+} = \begin{cases} \frac{\mu^2_{u^-}}{\mu^2_{u^+}} & \text{, if } A_\Gamma(u^+, u^-) \neq 0, \\ 0 & \text{, otherwise.} \end{cases}$$

Thus $Z^{P(\Gamma)}_{T^+_-}(u^-) = \sum_{u^+\in\,\mathcal{U}^+} \frac{\mu^2_{u^-}}{\mu^2_{u^+}} A_\Gamma(u^+, u^-)\, u^+$.

For $P(\Gamma)^G$, when it is connected, we have $1_{0_\pm} = \sum_{u^\pm\in\mathcal{U}^\pm} u^\pm$ (recall that $1_k := Z^P_{1^k}(1)$). Thus

$$\begin{aligned} Z^{P(\Gamma)^G}_{T^+_-}(1_{0_-}) &= \sum_{u^-\in\mathcal{U}^-} \sum_{u^+\in\mathcal{U}^+} \frac{\mu^2_{u^-}}{\mu^2_{u^+}} A_\Gamma(u^+, u^-)\, u^+ \\ &= \sum_{u^+\in\mathcal{U}^+} \left(\sum_{u^-\in\mathcal{U}^-} A_\Gamma(u^+, u^-)\, \mu^2_{u^-} \right) \frac{u^+}{\mu^2_{u^+}} \\ &= \sum_{u^+\in\mathcal{U}^+} ||\Gamma||\, \mu^2_{u^+} \frac{u^+}{\mu^2_{u^+}} \\ &= ||\Gamma||\, 1_{0_+}. \end{aligned}$$

Likewise, $Z^{P(\Gamma)^G}_{T^-_+}(1_{0_+}) = ||\Gamma||\, 1_{0_-}$. Thus we conclude that $P(\Gamma)^G$ has (positive) modulus $||\Gamma||$.

Finally, (3) follows on similar lines as (1).

□

CHAPTER 5

PLANAR ALGEBRA OF THE SUBGROUP-SUBFACTOR

One among the many equivalent conditions of extremality of finite index subfactors - see [PP86]- shows that every finite index irreducible subfactor is extremal. Thus the subgroup-subfactor $R \rtimes H \subset R \rtimes G$ is also extremal. In particular, by Theorem 4.2.7, we have a subfactor planar algebra $P^{R \rtimes H \subset R \rtimes G}$. We shall show that this planar algebra can be identified with the G-invariant planar subalgebra of the planar algebra of the bipartite graph $\star_n$.

In §1, we give the description of the bipartite graph $\star_n$, the spin function and the G-action on it with which we shall obtain the G-invariant planar subalgebra $P(\star_n)^G$. Then in §2, after simplifying the notations of loops, we identify a basis of $P(\star_n)^G$, indexed by G-orbits of $(H\backslash G)^k$; and define the map from $P^{R \rtimes H \subset R \rtimes G}$ to $P(\star_n)^G$, which would give the required identification. §3 is basically a collection of certain properties of $P^{R \rtimes H \subset R \rtimes G}$ and $P(\star_n)^G$, and some general facts, which will be needed to verify the equivariance of the above map with the tangle maps. In §4, we prove all the necessary details of the main theorem. Finally, in §5, we conclude the chapter with some questions arising from the work undertaken in this thesis.

5.1 THE BIPARTITE GRAPH $\star_n$

Let R be the hyperfinite II_1-factor. Let G be a finite group with a subgroup H of index n, with a fixed set $\{g_1, \ldots, g_n\}$ of right H-coset representatives, and α be an outer action of G on R as fixed in Chapter 3.

Consider the bipartite graph $\star_n = (\mathcal{U}^+, \mathcal{U}^-, \mathcal{E})$, as in Figure 5.1, where $\mathcal{U}^+ = \{Hg_1, \ldots, Hg_n\}$, $\mathcal{U}^- = \{*\}$ and the edge set $\mathcal{E} := \{\epsilon_1, \ldots, \epsilon_n\}$, where,

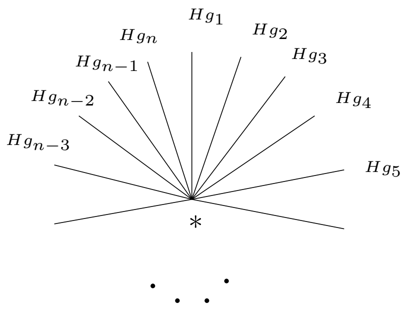


Figure 5.1: The bipartite graph $\star_n$

for each $1 \leq i \leq n$, ϵ_i is the edge joining the vertices Hg_i and $*$. Let $\mu : \mathcal{U}^+ \sqcup \mathcal{U}^- \to (0, \infty)$ be the spin function given by $\mu(*) = n^{1/4}$ and $\mu(Hg_i) = 1$ for all $1 \leq i \leq n$.

Note that $(\mu_u^2)_{u \in \mathcal{U}^+ \sqcup \mathcal{U}^-}$ is the Perron-Frobenius eigen vector of the adjacency matrix of the bipartite graph $\star_n$, where the norm of the adjacency matrix of this bipartite graph is $|| \star_n || = \sqrt{n}$. Further, with this set up, there is a natural G-action on $(\star_n, \mu)$:

G acts trivially on $\mathcal{U}^-$; on $\mathcal{U}^+$ it acts by natural left action, i.e., $g \cdot Hg_i = Hg_j \Leftrightarrow Hg_i g^{-1} = Hg_j$; and on $\mathcal{E}$ its action is induced by that on $\mathcal{U}^+$, i.e., $g\epsilon_i = \epsilon_j \Leftrightarrow g \cdot Hg_i = Hg_j$, $g \in G, i, j \in I$.

Remark 5.1.1 *G acts transitively on $\mathcal{U}^{\pm}$. Thus, in view of Lemma 4.3.5, the planar algebra $P(\star_n)^G$ is connected and has a positive modulus $\sqrt{n}$.*

5.2 Towards the Isomorphism

There being only one odd vertex and no multiple edges in $\star_n$, we can ignore the odd vertex $*$ and the edge joining it to an even vertex in the notation of a loop on $\star_n$. Thus, if k is even, say $k = 2r$, we simply write

$$\left(Hg_{i_0} \begin{array}{c} Hg_{i_1}, \;\; \cdots, Hg_{i_{r-1}} \\ Hg_{i_{2r-1}}, \cdots, Hg_{i_{r+1}} \end{array} Hg_{i_r} \right) \text{ for the } 2k\text{-loop}$$

$$\begin{tikzcd}
 & * \arrow[r, "\epsilon_{i_1}"] & Hg_{i_1} \arrow[r] & \cdots \arrow[r] & Hg_{i_{r-1}} \arrow[r, "\epsilon_{i_{r-1}}"] & * \arrow[dr, "\epsilon_{i_r}"] & \\
Hg_{i_0} \arrow[ur, "\epsilon_{i_0}"] & & & & & & Hg_{i_r}, \arrow[dl, "\epsilon_{i_r}"] \\
 & * \arrow[ul, "\epsilon_{i_0}"] & Hg_{i_{2r-1}} \arrow[l, "\epsilon_{i_{2r-1}}"] & \cdots \arrow[l] & Hg_{i_{r+1}} \arrow[l] & * \arrow[l, "\epsilon_{i_{r+1}}"] &
\end{tikzcd}$$

and if k is odd, say $k = 2s+1$, we denote by $\begin{pmatrix} & Hg_{i_1}, \cdots, \ Hg_{i_s} \\ Hg_{i_0} & \\ & Hg_{i_{2s}}, \cdots, Hg_{i_{s+1}} \end{pmatrix}$ the $2k$-loop

$$\begin{tikzcd}
 & * \arrow[r, "\epsilon_{i_1}"] & Hg_{i_1} \arrow[r] & \cdots \arrow[r] & * \arrow[r, "\epsilon_{i_s}"] & Hg_{i_s} \arrow[dr, "\epsilon_{i_s}"] & \\
Hg_{i_0} \arrow[ur, "\epsilon_{i_0}"] & & & & & & * \arrow[dl, "\epsilon_{is+1}"] \\
 & * \arrow[ul, "\epsilon_{i_0}"] & Hg_{i_{2s}} \arrow[l, "\epsilon_{i_{2s}}"] & \cdots \arrow[l] & * \arrow[l] & Hg_{i_{s+1}} \arrow[l, "\epsilon_{i_{s+1}}"] &
\end{tikzcd},$$

for all $\underline{i} = (i_0, i_1, \ldots, i_{k-1}) \in I^k$.

We shall also view these new descriptions of $2k$-loops as elements of $(H\backslash G)^k$. Also note that the action β^1 of G on I is basically the natural left action of G on the set $H\backslash G = \{Hg_1, \ldots, Hg_n\}^1$.

With these simplified notations we have the following set of (orbit) bases.

Lemma 5.2.1 *$\{\sum_{i\in I} Hg_i\}$ and $\{*\}$ form bases for $P_{0_+}(\star_n)^G$ and $P_{0_-}(\star_n)^G$, respectively. For every choice of orbit representatives $G\backslash(H\backslash G)^k$ under the diagonal β^1-action,*

$$\left\{ \sum_{g\in G} \begin{pmatrix} & Hg_{i_1}, \ \cdots, Hg_{i_{r-1}} & \\ Hg_{i_0} & & Hg_{i_r} \\ & Hg_{i_{2r-1}}, \cdots, Hg_{i_{r+1}} & \end{pmatrix} : \begin{pmatrix} & Hg_{i_1}, \ \cdots, Hg_{i_{r-1}} & \\ Hg_{i_0} & & Hg_{i_r} \\ & Hg_{i_{2r-1}}, \cdots, Hg_{i_{r+1}} & \end{pmatrix} \in G\backslash(H\backslash G)^{2r} \right\}$$

forms a basis for $P_{2r}(\star_n)^G$ for all $r \geq 1$; and

$$\left\{ \sum_{g\in G} \begin{pmatrix} & Hg_{i_1}, \cdots, \ Hg_{i_r} \\ Hg_{i_0} & \\ & Hg_{i_{2r}}, \cdots, \ Hg_{i_{r+1}} \end{pmatrix} : \begin{pmatrix} & Hg_{i_1}, \cdots, \ Hg_{i_r} \\ Hg_{i_0} & \\ & Hg_{i_{2r}}, \cdots, \ Hg_{i_{r+1}} \end{pmatrix} \in G\backslash(H\backslash G)^{2r+1} \right\}$$

forms a basis for $P_{2r+1}(\star_n)^G$ for all $r \geq 0$.

Proof: The first assertion follows from the fact that G acts transitively on $\mathcal{U}^+$ and trivially on the singleton $\mathcal{U}^-$.

For the other two, it is easily seen that any G-invariant element in $P_k(\star_n)^G$ is a linear sum of the members of above linearly independent sets, with corresponding k, and that the dimension of $P_k(\star_n)^G$ is precisely the

[1] Our running notation for orbit representatives (§3.3) is not to be confused with $H\backslash G$.

number of G orbits of $(H\backslash G)^k$ under the diagonal β^1-action. $\square$

We point out the following simple yet useful observation:

Lemma 5.2.2 *With notations as in §5.1, the planar algebra $P(\star_n)^G$ is irreducible, spherical and admits a global trace.*

Proof: By Remark 5.1.1, $P(\star_n)^G$ is connected and has modulus $\sqrt{n}$. Further, the action of G on $\mathcal{U}^+ = H\backslash G$ being transitive, by Lemma 5.2.1, the vector space $P_1(\star_n)^G$ is one dimensional, i.e., $P(\star_n)^G$ is irreducible. Therefore, by Lemma 4.2.5, we conclude that $P(\star_n)^G$ is a spherical planar algebra.

Thus the pictorial trace, as defined in Remark 4.2.6, gives a global trace on $P(\star_n)^G$. $\square$

We shall later show that this trace on $P(\star_n)^G$ is in fact faithful.

The stage is now set for giving the identification between the subgroup-subfactor planar algebra $P^{R\rtimes H\subset R\rtimes G}$ and the G-invariant planar subalgebra $P(\star_n)^G$.

We define $\varphi_{0_\pm} : P_{0_\pm}^{R\rtimes H\subset R\rtimes G} \to P_{0_\pm}(\star_n)^G$ by

$$P_{0_\pm}^{R\rtimes H\subset R\rtimes G} := \mathbb{C} \ni \lambda \overset{\varphi_{0_\pm}}{\longmapsto} \lambda 1_{0_\pm} \in P_{0_\pm}(\star_n)^G, \tag{5.1}$$

where $1_{0_+} = \sum_{i\in I} Hg_i$ and $1_{0_-} = *$ are the respective multiplicative units of $P_{0_\pm}(\star_n)^G$; and $\varphi_1 : P_1^{R\rtimes H\subset R\rtimes G} = \mathbb{C} \to P_1(\star_n)^G$ by

$$P_1^{R\rtimes H\subset R\rtimes G} \cong \mathbb{C} \ni \lambda \overset{\varphi_1}{\longmapsto} \lambda 1_1 \in P_1(\star_n)^G, \tag{5.2}$$

where 1_1 is the multiplicative unit $\sum_{i\in I}(Hg_i)$ of $P_1(\star_n)^G$.

For each $k \geq 2$, we define $\varphi_k : P_k^{R\rtimes H\subset R\rtimes G} \to P_k(\star_n)^G$ on the basis vectors described in Lemma 3.3.2 by

$$\begin{aligned} P_{2k+1}^{R\rtimes H\subset R\rtimes G} &= N' \cap M_{2k} \ni \sum_{h\in H} h[\underline{i},\, \underline{j}]^{ev} \\ &\overset{\varphi_{2k+1}}{\longmapsto} \textstyle\sum_{g\in G} g \left(H \begin{matrix} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \cdots, H \sqcap g_{\underline{j}} \end{matrix} \right) \in P_{2k+1}(\star_n)^G, \end{aligned} \tag{5.3}$$

for all $(\underline{i},\, \underline{j}) \in H\backslash(I^k \times I^k)$, $k \geq 1$; and

$$P_{2k}^{R\rtimes H\subset R\rtimes G} = N' \cap M_{2k-1} \ni \sum_{h\in H} h[\underline{i},\, \underline{j}]^{od}$$

$$\stackrel{\varphi_{2k}}{\longmapsto} \sum_{g\in G} g \left(H \begin{matrix} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \cdots, Hg_{i_2}\cdots g_{i_k} \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \cdots, Hg_{j_2}\cdots g_{j_k} \end{matrix} H \sqcap g_{\underline{i}} \right) \in P_{2k}(\star_n)^G, \tag{5.4}$$

for all $(\underline{i}, \underline{j}) \in H\backslash Y_k$, $k \geq 1$.

Main Theorem: *The theme of this thesis is to show that the map*

$$\{\varphi_k : k \in Col\} =: \varphi : P^{R\rtimes H \subset R\rtimes G} \to P(\star_n)^G$$

is a planar algebra isomorphism.

5.3 Ingredients of the Proof

The first observation we make is that the dimensions of the constituent vector spaces of the two planar algebras in the theorem are same.

Lemma 5.3.1 *With running notations, we have*

$$dim\, P_k^{R\rtimes H\subset R\rtimes G} = dim\, P_k(\star_n)^G, \;\forall\; k \in Col.$$

Proof: From Lemma 5.2.1 (1), we note that $P_{0_+}(\star_n)^G$, $P_{0_-}(\star_n)^G$ and $P_1(\star_n)^G$ are all one dimensional and so are $P_{0_\pm}^{R\rtimes H\subset R\rtimes G} := \mathbb{C}$ and $P_1^{R\rtimes H\subset R\rtimes G} \cong \mathbb{C}$.

For each $k \geq 1$, by Lemma 3.3.2, $dim\, N' \cap M_{2k}$ (resp., $dim\, N' \cap M_{2k-1}$) is same as the number of H-orbits of $I^k \times I^k$ (resp., Y_k) under the diagonal β^k-action. On the other hand, by Lemma 5.2.1, $dim\, P_{2k+1}(\star_n)^G$ (resp., $dim\, P_{2k}(\star_n)^G$) is equal to the number of G-orbits of $(H\backslash G)^{2k+1}$ (resp., $(H\backslash G)^{2k}$) under the diagonal β^1-action.

The corresponding dimensions are same because the number of H-orbits of $I^k \times I^k$ (resp., Y_k) under the diagonal β^k-action is equal to the number of G-orbits of $(H\backslash G)^{2k+1}$ (resp., $(H\backslash G)^{2k}$) under the diagonal β^1-action. We include a proof of the assertion in the parentheses, and the other follows similarly.

Note that the number of G-orbits of $(H\backslash G)^{2k}$ is same as the number of H-orbits of $(H\backslash G)^{2k-1}$ under the diagonal actions mentioned in the previous paragraph. This is true because any element of $(H\backslash G)^{2k}$ lies in the G-orbit of an element of the type $(H, Hg_{i_1}, \ldots, Hg_{i_{2k-1}})$; and

$$g(H, Hg_{i_1}, \ldots, Hg_{i_{2k-1}}) = (H, Hg_{j_1}, \ldots, Hg_{j_{2k-1}})$$

if and only if $g \in H$ and $g(Hg_{i_1}, \ldots, Hg_{i_{2k-1}}) = (Hg_{j_1}, \ldots, Hg_{j_{2k-1}})$.

Thus the assertion in the parentheses follows from the facts that the correspondence

$$Y_k \ni (\underline{i},\, \underline{j}) \mapsto \begin{pmatrix} Hg_{i_k},\, Hg_{i_{k-1}}g_{i_k},\, \ldots,\, Hg_{i_2}\cdots g_{i_k} & \\ & H \sqcap g_{\underline{i}} \\ Hg_{j_k},\, Hg_{j_{k-1}}g_{j_k},\, \ldots,\, Hg_{j_2}\cdots g_{j_k} & \end{pmatrix} \in (H\backslash G)^{2k-1},$$

by the very definition of the action β^k, is an H-injection, and that $|Y_k| = |(H\backslash G)^{2k-1}|$. □

The above counts make it clear that the maps φ_k, $k \in Col$ are linear isomorphisms.

Lemma 5.3.2 *With notations as in equations (5.1–5.4), for each* $k \in Col$*, the map* $\varphi_k : P_k^{R\rtimes H \subset R\rtimes G} \to P_k(\star_n)^G$ *is a linear isomorphism.*

Proof: Let $k \in Col$. In view of Lemma 5.3.1, it is enough to show that φ_k is injective and that is clear from its definition. □

We conclude this section with two lemmas that we shall need for verifying that the φ_k's are equivariant with respect to the inclusion and conditional expectation tangle maps.

In order to understand how the inclusion tangles act on the subgroup-subfactor planar algebra, and the way we defined the maps φ_k, we need to know the inclusions in terms of the bases that we obtained in Lemma 3.3.2. For k odd, say $k = 2r-1$, it is simply the usual matrix algebra inclusion $\theta^{(r)}(N)' \cap M_{I^r}(N) \subset \theta^{(r)}(N)' \cap M_{I^r}(M)$, whereas for k even, the inclusion is given by an appropriate Θ_r.

Lemma 5.3.3 *For each* $k \geq 1$*, the inclusion* $N' \cap M_{2k} \subset N' \cap M_{2k+1}$ *is given by the map* Θ_{k+1}*:*

$$\begin{aligned} &\theta^{(k)}(N)' \cap M_{I^k}(M) \ni \textstyle\sum_{h\in H} h[\underline{i},\, \underline{j}]^{ev} \\ &\quad \stackrel{\Theta_{k+1}}{\longmapsto} \textstyle\sum_{\left\{\substack{h\in H,\, r,\, s\,\in I\,:\\ ((r,\underline{i}),\,(s,\underline{j}))\in Y_{k+1}}\right\}} h[(r, \underline{i}),\, (s, \underline{j})]^{od} \in \theta^{(k+1)}(N)' \cap M_{I^{k+1}}(N), \end{aligned} \tag{5.5}$$

for all $(\underline{i},\, \underline{j}) \in H\backslash(I^k \times I^k)$.

Proof: First note that, for each $k \geq 1$ and $(\underline{i},\, \underline{j}) \in I^k \times I^k$,

$$\Theta_{k+1}([\underline{i},\, \underline{j}]^{ev}) = \sum_{\substack{r,s\,\in I:\\ ((r,\,\underline{i}),\,(s,\,\underline{j}))\in Y_{k+1}}} [(r,\, \underline{i}),\, (s,\, \underline{j})]^{od}.$$

Indeed, by the description of Θ_{k+1} as in Lemma 3.2.4, we have

$$\begin{aligned}\Theta_{k+1}([\underline{i},\, \underline{j}]^{ev})_{\underline{u},\,\underline{v}} &= \theta_{u_1,\,v_1}([\underline{i},\, \underline{j}]^{ev}_{\underline{u}_{[2},\,\underline{v}_{[2}}) \\ &= \theta_{u_1,\,v_1}(\delta^{\underline{i}}_{\underline{u}_{[2}}\, \delta^{\underline{j}}_{\underline{v}_{[2}}\, u_{(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1}}) \\ &= \delta^{\underline{i}}_{\underline{u}_{[2}}\, \delta^{\underline{j}}_{\underline{v}_{[2}}\, 1_H(g_{u_1}(\sqcap g_{\underline{i}})(\sqcap g_{\underline{j}})^{-1} g_{v_1}^{-1})\; u_{(\sqcap g_{(u_1,\,\underline{i})})(\sqcap g_{(v_1,\,\underline{j})})^{-1}},\end{aligned}$$

$\forall\, \underline{u},\, \underline{v} \in I^{k+1}$. Thus we just need to show that

$$\begin{aligned}\Theta_{k+1}(h[\underline{i},\, \underline{j}]^{ev}) &= \sum_{\substack{r,\, s \in I\,:\\ ((r,\,\underline{i}),\,(s,\,\underline{j}))\in Y_{k+1}}} h[(r,\, \underline{i}),\, (s,\, \underline{j})]^{od},\ \forall\, h \in H,\\ \text{i.e.,}\quad & \sum_{\substack{x,\, y \in I\,:\\ ((x,\,\beta^k_h(\underline{i})),\,(y,\,\beta^k_h(\underline{j})))\in Y_{k+1}}} [(x, \beta^k_h(\underline{i})),\, (y, \beta^k_h(\underline{j}))]^{od} \qquad (5.6)\\ &= \sum_{\substack{r,\, s \in I\,:\\ ((r,\,\underline{i}),\,(s,\,\underline{j}))\in Y_{k+1}}} [\beta^{k+1}_h(r,\, \underline{i}),\, \beta^{k+1}_h(s,\, \underline{j})]^{od},\ \forall\, h \in H.\end{aligned}$$

For each $h \in H$ and $(\underline{u},\, \underline{v}) \in Y_{k+1}$, the coefficient of $[\underline{u},\, \underline{v}]^{od}$ on *L.H.S.* (resp., *R.H.S.*) of equation (5.6) is the number of elements in the set

$$\begin{aligned}L_{\underline{u},\,\underline{v}} := \{(x, y) \in I \times I : ((x, \beta^k_h(\underline{i})),\, (y, \beta^k_h(\underline{j}))) \in Y_{k+1},\\ [(x, \beta^k_h(\underline{i})),\, (y, \beta^k_h(\underline{j}))]^{od} = [\underline{u},\, \underline{v}]^{od}\}\end{aligned}$$

$$\begin{aligned}(\text{resp., } R_{\underline{u},\,\underline{v}} := \{(r, s) \in I \times I : ((r, \underline{i}),\, (s, \underline{j})) \in Y_{k+1},\\ [\beta^{k+1}_h((r, \underline{i})),\, \beta^{k+1}_h((s, \underline{j}))]^{od} = [\underline{u},\, \underline{v}]^{od}\}).\end{aligned}$$

Viewing the two sides of equation (5.6) as elements of $R' \cap M_{2k+1}$, we shall be done once we show that these coefficients are same.

Note that if $(\beta^{k+1}_h((r, \underline{i})),\, \beta^{k+1}_h((s, \underline{j}))) = (\underline{u},\, \underline{v})$, for some $h \in H$ and $((r, \underline{i}),\, (s, \underline{j})) \in Y_{k+1}$, then $\underline{u}_{[2} = \beta^k_h(\underline{i})$, $\underline{v}_{[2} = \beta^k_h(\underline{j})$ and

$$((u_1, \beta^k_h(\underline{i})),\, (v_1, \beta^k_h(\underline{j}))) \in Y_{k+1}.$$

Consider the maps $L_{\underline{u},\,\underline{v}} \underset{\psi}{\overset{\phi}{\rightleftarrows}} R_{\underline{u},\,\underline{v}}$ given as under:

For each $(x, y) \in L_{\underline{u},\,\underline{v}}$ (resp., $(r, s) \in R_{\underline{u},\,\underline{v}}$), $\phi((x, y)) = (p, q) \in R_{\underline{u},\,\underline{v}}$ (resp., $\psi((r, s)) = (w, z)$), where (p, q) (resp., (w, z)) is given by the equations

$$\begin{cases} Hg_p = Hg_x(\sqcap g_{\beta^k_h(\underline{i})})h(\sqcap g_{\underline{i}})^{-1} \text{ and}\\ Hg_q = Hg_y(\sqcap g_{\beta^k_h(\underline{j})})h(\sqcap g_{\underline{j}})^{-1}\end{cases}$$

$$(\text{resp., } \beta_h^{k+1}(r, \underline{i}) = (w, \beta_h^k(\underline{i})) \text{ and } \beta_h^{k+1}(s, \underline{j}) = (z, \beta_h^k(\underline{j}))).$$

It can be easily seen that the maps ϕ and ψ are inverses of each other. Thus the coefficient of $[\underline{u},\, \underline{v}]^{od}$ is same on both sides of equation (5.6), and we are done. $\square$

Lemma 5.3.4 *Let $A_0 \subset A_1$ and $B_0 \subset B_1$ be inclusions of finite dimensional C^*-algebras. Suppose that tr_{A_1} and tr_{B_1} are faithful traces on A_1 and B_1, respectively. Let $\phi_i : A_i \to B_i,\ i = 0, 1$ be C^*-isomorphisms preserving the traces such that $\phi_1/A_0 = \phi_0$. Then $E^{B_1}_{B_0} \circ \phi_1 = \phi_0 \circ E^{A_1}_{A_0}$, where $E^{X_1}_{X_0} : X_1 \to X_0$ is the unique tr_{X_1}-trace preserving conditional expectation for $X_i \in \{A_i, B_i\}, i = 0, 1$.*

Proof: By uniqueness of the trace preserving conditional expectation, we get $E^{B_1}_{B_0} = \phi_0 \circ E^{A_1}_{B_1} \circ \phi_1^{-1}$. $\square$

5.4 The Main Theorem

We now give a detailed proof of our main theorem. We need to show that $\varphi := \{\varphi_k : k \in Col\}$ is a planar algebra isomorphism, i.e., φ is equivariant with respect to all the tangle maps. Again, as seen before, it is enough to show the equivariance of φ with respect to tangle maps for a generating set of tangles.

For notational convenience, in what follows, we simply write P (resp., Q) for the subgroup-subfactor planar algebra $P^{R \rtimes H \subset R \rtimes G}$ (resp., the invariant planar subalgebra $P(\star_n)^G$) .

Lemma 5.4.1 $\varphi_{k+1} \circ Z^P_{I^{k+1}_k} = Z^Q_{I^{k+1}_k} \circ \varphi_k,\ \forall\, k \in Col.$

Proof: By equations (5.1) and (5.2), we clearly have $\varphi_1 \circ Z^P_{I^1_{0_\pm}} = Z^Q_{I^1_{0_\pm}} \circ \varphi_{0_\pm}$.

Now for $k \geq 1$, we give a proof for odd k, the even case is similar. Suppose $k = 2r+1$ for some $r \geq 0$. Then, by Lemma 5.3.3, the tangle map $Z^P_{I^{2r+2}_{2r+1}} : N' \cap M_{2r} \hookrightarrow N' \cap M_{2r+1}$ is given by the inclusion map Θ_{r+1}. On the other hand, by (4.15) for the bipartite graph $\star_n$, we have

$$Q_{2r+1} \ni \textstyle\sum_{g \in G} g \left(Hg_{i_0} \begin{array}{c} Hg_{i_1}, \cdots, \ Hg_{i_r} \\ Hg_{i_{2r}}, \cdots, Hg_{i_{r+1}} \end{array} \right)$$

$$\overset{Z^Q_{I^{2r+2}_{2r+1}}}{\longmapsto} \textstyle\sum_{j \in J,\, g \in G} g \left(Hg_{i_0} \begin{array}{c} Hg_{i_1}, \ \cdots, \ Hg_{i_r} \\ Hg_{i_{2r-1}}, \cdots, Hg_{i_{r+1}} \end{array} Hg_j \right) \ \in \ Q_{2r+2}.$$

If $r = 0$, then for each $\lambda \in \mathbb{C} = P_1$, by the defining equations (5.2) and (5.4), we have

$$\begin{aligned}
(Z^Q_{I_1^2} \circ \varphi_1)(\lambda) &= Z^Q_{I_1^2}(\lambda \sum_{i\in I}(Hg_i)) \\
&= Z^Q_{I_1^2}\left(\lambda \frac{1}{|\mathrm{Iso}_H(H)|}\sum_{g\in G} g(H)\right) \\
&= \frac{\lambda}{|H|}\sum_{i\in I,\, g\in G} g(H,\, Hg_i)
\end{aligned}$$

and

$$\begin{aligned}
(\varphi_2 \circ Z^P_{I_1^2})(\lambda) &= \varphi_2\left(\frac{1}{|\mathrm{Iso}_H((1,1))|}\sum_{g\in G} g[1,1]^{od}\right) \\
&= \frac{\lambda}{|H|}\varphi_2\left(\sum_{i\in I,\, h\in H} hg_i[1,1]^{od}\right) \\
&= \frac{\lambda}{|H|}\sum_{i\in I,\, g\in G} g(H,\, Hg_i),
\end{aligned}$$

showing that $\varphi_2 \circ Z^P_{I_1^2} = Z^Q_{I_1^1} \circ \varphi_1$, where for an H-set X and $x \in X$, $\mathrm{Iso}_H(x) := \{h \in H : hx = x\}$.

And for $r \geq 1$, for each $(\underline{i},\, \underline{j}) \in H\backslash(I^r \times I^r)$, by Lemma 5.3.3, we have

$$\begin{aligned}
&\left(\varphi_{2r+2} \circ Z^P_{I_{2r+1}^{2r+2}}\right)\left(\sum_{h\in H} h[\underline{i},\, \underline{j}]^{ev}\right) \\
&= \varphi_{2r+2}\left(\sum_{\substack{h\in H,\ x,y\in I: \\ ((x,\underline{i}),\,(y,\underline{j}))\in Y_{r+1}}} h[(x,\underline{i}),\, (y,\underline{j})]^{od}\right) \\
&= \sum_{g\in G,\ x\in I} g\left(H \begin{matrix} Hg_{i_r},\, Hg_{i_{r-1}}g_{i_r},\, \cdots,\, Hg_{i_1}\cdots g_{i_r} \\ Hg_{j_r},\, Hg_{j_{r-1}}g_{j_r},\, \cdots,\, Hg_{j_1}\cdots g_{j_r} \end{matrix} H \sqcap g_{(x,\,\underline{i})}\right)
\end{aligned}$$

and

$$\left(Z^Q_{I_{2r+1}^{2r+2}} \circ \varphi_{2r+1}\right)\left(\sum_{h\in H} h[\underline{i},\, \underline{j}]^{ev}\right)$$

$$
\begin{aligned}
&= Z^Q_{I^{2r+2}_{2r+1}} \left(\sum_{g\in G} g \left(H \begin{array}{l} Hg_{i_k},\ Hg_{i_{k-1}}g_{i_k},\ \cdots,\ H\sqcap g_{\underline{i}} \\ Hg_{j_k},\ Hg_{j_{k-1}}g_{j_k},\ \cdots,\ H\sqcap g_{\underline{j}} \end{array} \right) \right) \\
&= \sum_{x\in I,\ g\in G} g \left(H \begin{array}{l} Hg_{i_k},\ Hg_{i_{k-1}}g_{i_k},\ \cdots,\ H\sqcap g_{\underline{i}} \\ Hg_{j_k},\ Hg_{j_{k-1}}g_{j_k},\ \cdots,\ H\sqcap g_{\underline{j}} \end{array} Hg_x \right) \\
&= \sum_{j\in I,\ g\in G} g \left(H \begin{array}{l} Hg_{i_k},\ Hg_{i_{k-1}}g_{i_k},\ \cdots,\ Hg_{i_1}\cdots g_{i_k} \\ Hg_{j_k},\ Hg_{j_{k-1}}g_{j_k},\ \cdots,\ Hg_{j_1}\cdots g_{j_k} \end{array} H\sqcap g_{(j,\underline{i})} \right).
\end{aligned}
$$

Thus $\varphi_{k+1}\circ Z^P_{I^{k+1}_k} = Z^Q_{I^{k+1}_k}\circ\varphi_k$. □

Lemma 5.4.2 $\varphi_k \circ Z^P_{M_k} = Z^Q_{M_k}\circ(\varphi_k\otimes\varphi_k),\ \forall\, k\in Col.$

Proof: There is nothing to prove for $k=0_\pm$. Both P and Q being irreducible, there is nothing to be proved for $k=1$ either. For $k\geq 2$, we give a proof for the case when k is odd, say $k=2r+1$ for some $r\geq 1$, and the other case can be proved on exactly similar lines. Let

$$
X_{2r+1} = \left\{ \left(Hg_x \begin{array}{l} Hg_{i_1},\cdots,Hg_{i_r} \\ Hg_{j_1},\cdots,Hg_{j_r} \end{array} \right) \in (H/G)^{2r+1} : x=1,\ \underline{i},\underline{j}\in I^r \right\}.
$$

Then X_{2r+1} is invariant under the diagonal β^1-action of H on $(H\backslash G)^{2r+1}$ and it can also be identified with $(H\backslash G)^{2r}$ as an H-set. Thus, as in the proof of Lemma 5.3.1, the correspondence

$$
I^r\times I^r \ni (\underline{i},\ \underline{j}) \longmapsto \left(H \begin{array}{l} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r},\cdots,H\sqcap g_{\underline{i}} \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r},\cdots,H\sqcap g_{\underline{j}} \end{array} \right) \in X_{2r+1}
$$

is an H-bijection; and a set of representatives of H-orbits of X_{2r+1} is also a set of representatives of G-orbits of $(H\backslash G)^{2r+1}$. This shows that

$$
\left\{ \left(H \begin{array}{l} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r},\cdots,H\sqcap g_{\underline{i}} \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r},\cdots,H\sqcap g_{\underline{j}} \end{array} \right) : (\underline{i},\ \underline{j})\in H\backslash(I^r\times I^r) \right\}
$$

is a set of representatives of G-orbits of $(H\backslash G)^{2r+1}$. In particular, by Lemma 5.2.1,

$$\left\{ \sum_{g \in G} g \left(H \begin{matrix} Hg_{i_r}, Hg_{i_{r-1}}g_{i_r}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_r}, Hg_{j_{r-1}}g_{j_r}, \cdots, H \sqcap g_{\underline{j}} \end{matrix} \right) : (\underline{i}, \underline{j}) \in H \backslash (I^r \times I^r) \right\}$$

forms a basis for Q_{2r+1}.

We now prove that φ commutes with the tangle map for the multiplication tangle M_{2r+1}. For each pair of representatives $(\underline{i}, \underline{j}), (\underline{\tilde{i}}, \underline{\tilde{j}}) \in H \backslash (I^r \times I^r)$, we have

$$\begin{aligned} \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) \left(\sum_{\tilde{h} \in H} \tilde{h}[\underline{\tilde{i}}, \underline{\tilde{j}}]^{ev} \right) &= \sum_{h, \tilde{h} \in H} \delta^{\beta^r_h(\underline{j})}_{\beta^r_{\tilde{h}}(\underline{\tilde{i}})} [\beta^r_h(\underline{i}), \beta^r_{\tilde{h}}(\underline{\tilde{j}})]^{ev} \\ &= \sum_{\hat{h} \in H, (\underline{\hat{i}}, \underline{\hat{j}}) \in H \backslash (I^r \times I^r)} C_{\underline{\hat{i}}, \underline{\hat{j}}} \hat{h}[\underline{\hat{i}}, \underline{\hat{j}}]^{ev} \end{aligned}$$

$$\xmapsto{\varphi_{2r+1}} \sum_{\hat{g} \in G, (\underline{\hat{i}}, \underline{\hat{j}}) \in H \backslash (I^r \times I^r)} C_{\underline{\hat{i}}, \underline{\hat{j}}} \, \hat{g} \left(H \begin{matrix} Hg_{\hat{i}_r}, Hg_{\hat{i}_{r-1}}g_{\hat{i}_r}, \cdots, H \sqcap g_{\underline{\hat{i}}} \\ Hg_{\hat{j}_r}, Hg_{\hat{j}_{r-1}}g_{\hat{j}_r}, \cdots, H \sqcap g_{\underline{\hat{j}}} \end{matrix} \right),$$

where $C_{\underline{\hat{i}}, \underline{\hat{j}}}$ is the number of elements in the set

$$\widetilde{C}_{\underline{\hat{i}}, \underline{\hat{j}}} := \{(h, \tilde{h}) \in H \times H : \beta^r_h(\underline{j}) = \beta^r_{\tilde{h}}(\underline{\tilde{i}}), \ \beta^r_h(\underline{i}) = \underline{\hat{i}} \text{ and } \beta^r_{\tilde{h}}(\underline{\tilde{j}}) = \underline{\hat{j}}\}.$$

We have used the fact that since the above product is in $N' \cap M_{2r}$, the coefficient of $\hat{h}[\underline{\hat{i}}, \underline{\hat{j}}]^{ev}$ in it is the same as that of $[\underline{\hat{i}}, \underline{\hat{j}}]^{ev}$ for all $\hat{h} \in H$.

On the other hand, by (4.18) for the bipartite graph $\star_n$, we have

$$\begin{aligned} &\varphi_{2r+1} \left(\sum_{h \in H} h[\underline{i}, \underline{j}]^{ev} \right) \ \varphi_{2r+1} \left(\sum_{\tilde{h} \in H} \tilde{h}[\underline{\tilde{i}}, \underline{\tilde{j}}]^{ev} \right) \\ &= \sum_{g \in G} g \left(H \begin{matrix} Hg_{i_r}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_r}, \cdots, H \sqcap g_{\underline{j}} \end{matrix} \right) \sum_{\tilde{g} \in G} \tilde{g} \left(H \begin{matrix} Hg_{\tilde{i}_r}, \cdots, H \sqcap g_{\underline{\tilde{i}}} \\ Hg_{\tilde{j}_r}, \cdots, H \sqcap g_{\underline{\tilde{j}}} \end{matrix} \right) \\ &= \sum_{g, \tilde{g} \in G} \delta^{g(H, Hg_{j_r}, \cdots, H \sqcap g_{\underline{j}})}_{\tilde{g}(H, Hg_{\tilde{i}_r}, \cdots, H \sqcap g_{\underline{\tilde{i}}})} \left(g \cdot H \begin{matrix} g \cdot Hg_{i_r}, \cdots, g \cdot H \sqcap g_{\underline{i}} \\ \tilde{g} \cdot Hg_{\tilde{j}_r}, \cdots, \tilde{g} \cdot H \sqcap g_{\underline{\tilde{j}}} \end{matrix} \right) \end{aligned}$$

$$= \sum_{\hat{g} \in G,\ (\hat{\underline{i}}, \hat{\underline{j}}) \in H \backslash (I^r \times I^r)} D_{\hat{\underline{i}}, \hat{\underline{j}}}\, \hat{g} \left(H \begin{matrix} Hg_{\hat{i}_r}, \cdots, H \sqcap g_{\hat{\underline{i}}} \\ Hg_{\hat{j}_r}, \cdots, H \sqcap g_{\hat{\underline{j}}} \end{matrix} \right),$$

where $D_{\hat{\underline{i}}, \hat{\underline{j}}}$ is the number of elements in the set

$$\widetilde{D}_{\hat{\underline{i}}, \hat{\underline{j}}} := \left\{ \begin{array}{l} (g, \tilde{g}) \in G \times G : \\ \quad \left\{ \begin{array}{l} g(H, Hg_{j_r}, \cdots, H \sqcap g_{\underline{j}}) = \tilde{g}(H, Hg_{\tilde{i}_r}, \cdots, H \sqcap g_{\tilde{\underline{i}}}), \\ g(H, Hg_{i_r}, \cdots, H \sqcap g_{\underline{i}}) = (H, Hg_{\hat{i}_r}, \cdots, H \sqcap g_{\hat{\underline{i}}}) \text{ and} \\ \tilde{g}(H, Hg_{\tilde{j}_r}, \cdots, H \sqcap g_{\tilde{\underline{j}}}) = (H, Hg_{\hat{j}_r}, \cdots, H \sqcap g_{\hat{\underline{j}}}) \end{array} \right. \end{array} \right\},$$

and the coefficients are constant on each orbit as in the former case.

It can be easily seen that, for each $(\hat{\underline{i}}, \hat{\underline{j}}) \in H \backslash (I^r \times I^r)$, the sets $\widetilde{C}_{\hat{\underline{i}}, \hat{\underline{j}}}$ and $\widetilde{D}_{\hat{\underline{i}}, \hat{\underline{j}}}$ are same.

Thus we conclude that $\varphi_{2r+1} \circ Z^P_{M_{2r+1}} = Z^Q_{M_{2r+1}} \circ (\varphi_{2r+1} \otimes \varphi_{2r+1})$. $\square$

Note that $Q_{1,k} = [P(\star_n)^G]_{1,k} = P_{1,k}(\star_n)^G,\ \forall\, k \geq 1$.

Lemma 5.4.3 *With running notations, we have*

$$\varphi_k \circ Z^P_{E^{k+1}_k} = Z^Q_{E^k_{k+1}} \circ \varphi_{k+1},\ \textit{and} \tag{5.7}$$

$$\varphi_{k+1} \circ Z^P_{(E')^{k+1}_{k+1}} = Z^Q_{(E')^{k+1}_{k+1}} \circ \varphi_{k+1},\ \forall\, k \in Col. \tag{5.8}$$

Proof: First note that the maps $\varphi_k, k \in Col$ are all $*$-preserving, where the $*$-structure on $P(\star_n)^G$ is given as in the last paragraph of §4.3.1; and, by Lemma 5.4.2, they are algebra homomorphisms as well.

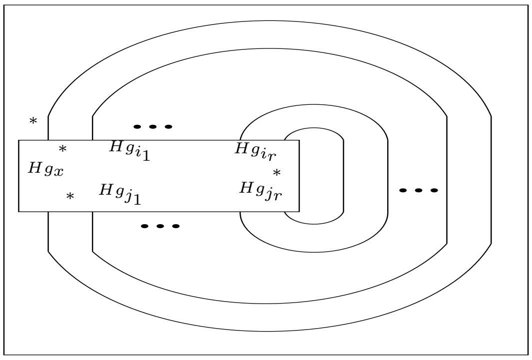
We next show that these maps preserve the traces as well, where Q is equipped with the global pictorial trace as given in Lemma 5.2.2. Then by the fact that they are $*$-preserving algebra isomorphisms, it follows that the global pictorial trace on Q is in fact faithful. Thus (5.7) holds by Lemma 5.3.4, and also by the same result, (5.8) will hold once we show that $\varphi_k(P_{1,k}) = Q_{1,k} := Image\,(Z^Q_{(E')^k_k}),\ \forall\, k \geq 1$.

We calculate the trace on Q_k for odd k, say $k = 2r + 1$ for some $r \geq 1$. In the planar algebra $P(\star_n)$, given $\underline{i}, \underline{j} \in I^r$ and $x, y \in I$, we have

$$Z^{P(\star_n)}_{E_1^{0+}\circ\cdots\circ E_{2r+1}^{2r}}\left(\left(Hg_x \begin{matrix} Hg_{i_1},\cdots,Hg_{i_r} \\ Hg_{j_1},\cdots,Hg_{j_r} \end{matrix}\right)\right)_{Hg_y}$$

Hg_y

$$=$$



$$= \delta^{\underline{i}}_{\underline{j}}\,\delta^{x}_{y}\,\mu_*^2.$$

Thus the pictorial trace on Q_{2r+1} is given by

$$\begin{aligned} tr_{2r+1}\left(\sum_{g\in G}\left(Hg_x \begin{matrix} Hg_{i_1},\cdots,Hg_{i_r} \\ Hg_{j_1},\cdots,Hg_{j_r} \end{matrix}\right)\right) &= \frac{1}{(\sqrt{n})^{2r+1}}\,\mu_*^2\,\delta^{\underline{i}}_{\underline{j}}\,|\mathrm{Iso}_G(Hg_x)| \\ &= \frac{|H|}{n^r}\,\delta^{\underline{i}}_{\underline{j}}, \end{aligned}$$

for all $x \in I$ and $\underline{i}$, $\underline{j} \in I^r$, where we have used the fact that, since G acts transitively on $H\backslash G$, $\mathrm{Iso}_G(Hg_x) \cong \mathrm{Iso}_G(H) = H$, $\forall x \in I$.

On the other hand, for each $(\underline{i}, \underline{j}) \in H\backslash(I^r \times I^r)$, we have $tr_{M_{I^r}(M)}\left([\underline{i}, \underline{j}]^{ev}\right) = \frac{1}{n^r}\,\delta^{\underline{i}}_{\underline{j}}$; so that

$$tr_{M_{I^r}(M)}\left(\sum_{h\in H} h[\underline{i}, \underline{j}]^{ev}\right) = \sum_{h\in H}\delta^{\beta_h^r(\underline{i})}_{\beta_h^r(\underline{j})}\,\frac{1}{n^r} = \frac{|H|}{n^r}\,\delta^{\underline{i}}_{\underline{j}},$$

and thus

$$\begin{aligned} tr_{2r+1}\left(\varphi_{2r+1}\left(\sum_{h\in H} h[\underline{i}, \underline{j}]^{ev}\right)\right) &= tr_{2r+1}\left(\sum_{g\in G}\left(H \begin{matrix} Hg_{i_1},\cdots,Hg_{i_r} \\ Hg_{j_1},\cdots,Hg_{j_r} \end{matrix}\right)\right) \\ &= \frac{|H|}{n^r}\,\delta^{\underline{i}}_{\underline{j}}. \end{aligned}$$

This shows that φ_{2r+1} preserves trace. With exactly similar calculations one can show that φ_k preserves trace for even k as well.

It only remains to show that $\varphi_k(P_{1,k}) = Q_{1,k}$ for all $k \geq 1$. Again, there is nothing to prove for $k = 1$. We prove the assertion for odd k, say $k = 2r+1$ for some $r \geq 1$. As usual, the other case follows on similar lines. Note that

$$\left\{ \sum_{g \in G} g \begin{pmatrix} & Hg_{i_r}, Hg_{i_{r-1}}g_{i_r}, \cdots, H \sqcap g_{\underline{i}} \\ H & \\ & Hg_{j_r}, Hg_{j_{r-1}}g_{j_r}, \cdots, H \sqcap g_{\underline{j}} \end{pmatrix} : (\underline{i}, \underline{j}) \in H \backslash (I^r \times I^r) \right\}$$

forms a basis for Q_{2r+1}. By (4.22) for the bipartite graph $\star_n$, for each $(\underline{i}, \underline{j}) \in H\backslash(I^r \times I^r)$, we have

$$\begin{aligned} & Z^Q_{(E')^k_k} \left(\sum_{g \in G} g \begin{pmatrix} & Hg_{i_r},\, Hg_{i_{r-1}}g_{i_r},\, \cdots,\, H \sqcap g_{\underline{i}} \\ H & \\ & Hg_{j_r},\, Hg_{j_{r-1}}g_{j_r},\, \cdots,\, H \sqcap g_{\underline{j}} \end{pmatrix} \right) \\ & = \frac{1}{\sqrt{n}} \sum_{z \in I,\, g \in G} \begin{pmatrix} & g \cdot Hg_{i_r},\, g \cdot Hg_{i_{r-1}}g_{i_r},\, \cdots,\, g \cdot H \sqcap g_{\underline{i}} \\ Hg_z & \\ & g \cdot Hg_{j_r},\, g \cdot Hg_{j_{r-1}}g_{j_r},\, \cdots,\, g \cdot H \sqcap g_{\underline{j}} \end{pmatrix} \\ & = \frac{1}{\sqrt{n}} \sum_{x \in I,\, g \in G} g \begin{pmatrix} & Hg_{i_r},\, Hg_{i_{r-1}}g_{i_r},\, \cdots,\, H \sqcap g_{\underline{i}} \\ Hg_x & \\ & Hg_{j_r},\, Hg_{j_{r-1}}g_{j_r},\, \cdots,\, H \sqcap g_{\underline{j}} \end{pmatrix}; \end{aligned}$$

and these elements generate $Q_{1,\,2r+1}$ as a vector space.

Further, for each $(\underline{i}, \underline{j}) \in H\backslash(I^r \times I^r)$, $\sum_{g \in G} g[\underline{i}, \underline{j}]^{ev} \in M' \cap M_{2r}$ and

$$\begin{aligned} \sum_{g \in G} g[\underline{i}, \underline{j}]^{ev} &= \sum_{x \in I,\, h \in H} hg_x[\underline{i}, \underline{j}]^{ev} \\ &\overset{\varphi_{2r+1}}{\longmapsto} \sum_{x \in I,\, g \in G} g \begin{pmatrix} & Hg_{i_r}g_x^{-1},\, Hg_{i_{r-1}}g_{i_r}g_x^{-1},\, \cdots,\, H(\sqcap g_{\underline{i}})g_x^{-1} \\ H & \\ & Hg_{j_r}g_x^{-1},\, Hg_{j_{r-1}}g_{j_r}g_x^{-1},\, \cdots,\, H(\sqcap g_{\underline{j}})g_x^{-1} \end{pmatrix} \\ &= \sum_{x \in I,\, g \in G} gg_x \begin{pmatrix} & Hg_{i_r},\, Hg_{i_{r-1}}g_{i_r},\, \cdots,\, H \sqcap g_{\underline{i}} \\ Hg_x & \\ & Hg_{j_r},\, Hg_{j_{r-1}}g_{j_r},\, \cdots,\, H \sqcap g_{\underline{j}} \end{pmatrix} \\ &= \sum_{x \in I,\, \tilde{g} \in G} \tilde{g} \begin{pmatrix} & Hg_{i_r},\, Hg_{i_{r-1}}g_{i_r},\, \cdots,\, H \sqcap g_{\underline{i}} \\ Hg_x & \\ & Hg_{j_r},\, Hg_{j_{r-1}}g_{j_r},\, \cdots,\, H \sqcap g_{\underline{j}} \end{pmatrix}. \end{aligned}$$

Thus the elements of the above generating set for $Q_{1,2r+1}$ are in the space $\varphi_{2r+1}(M' \cap M_{2r})$, and we know that $P_{1,2r+1} = E_{M' \cap M_{2r}}(N' \cap M_{2r}) = M' \cap$

M_{2r}. This proves our second claim. $\square$

Lemma 5.4.4 $(\varphi_{k+1} \circ Z^P_{\mathcal{E}^{k+1}}) = Z^Q_{\mathcal{E}^{k+1}}, \ \forall\, k \geq 1.$

Proof: We first prove the assertion for even k.

We need to consider some H- and G-invariant subsets of $I^k \times I^k$ and $(H\backslash G)^{2k+1}$. For each $k \geq 1$, we set $W_k = \{(\underline{i},\, \underline{j}) \in I^k \times I^k : \underline{i}_{[2} = \underline{j}_{[2}\}$,

$$F_{2k+1} \quad = \quad \left\{ \begin{pmatrix} Hg_x \begin{matrix} Hg_{i_1}, \cdots, Hg_{i_k} \\ Hg_{j_1}, \cdots, Hg_{j_k} \end{matrix} \end{pmatrix} : x \in I,\, \underline{i}, \underline{j} \in I^k \text{ with } \underline{i}_{\,k-1]} = \underline{j}_{\,k-1]} \right\}$$

and

$$G_{2k+1} \quad = \quad \left\{ \begin{pmatrix} H \begin{matrix} Hg_{i_1}, \cdots, Hg_{i_k} \\ Hg_{j_1}, \cdots, Hg_{j_k} \end{matrix} \end{pmatrix} : \underline{i}, \underline{j} \in I^k \text{ with } \underline{i}_{\,k-1]} = \underline{j}_{\,k-1]} \right\}.$$

Then W_k (resp., F_{2k+1}) is G-invariant under the diagonal β^k(resp., β^1)-action, and $G_{2k+1} \subset F_{2k+1}$ is H-invariant under the restricted action.

Further, the correspondence

$$W_k \ni (\underline{i},\, \underline{j}) \mapsto \begin{pmatrix} H \begin{matrix} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \cdots, H \sqcap g_{\underline{j}} \end{matrix} \end{pmatrix} \in G_{2k+1}$$

is an H-bijection. So $\left\{ \begin{pmatrix} H \begin{matrix} Hg_{i_k}, Hg_{i_{k-1}}g_{i_k}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, Hg_{j_{k-1}}g_{j_k}, \cdots, H \sqcap g_{\underline{j}} \end{matrix} \end{pmatrix} : (\underline{i},\, \underline{j}) \in H\backslash W_k \right\}$ is a set of representatives of H-orbits of G_{2k+1}. Clearly this is also a set of representatives of G-orbits of F_{2k+1}.

Note that, by (4.23) for the graph $\star_n$, we have

$$Z^Q_{\mathcal{E}^{2k+1}}(1) = \frac{1}{\sqrt{n}} \sum_{x,\, y,\, z \in I,\ \underline{i} \in I^{k-1}} \begin{pmatrix} Hg_x \begin{matrix} Hg_{i_1}, \cdots, Hg_{i_{k-1}}, Hg_y \\ Hg_{i_1}, \cdots, Hg_{j_{k-1}}, Hg_z \end{matrix} \end{pmatrix}.$$

Recall, from Theorem 4.2.7, that $Z^P_{\mathcal{E}^{2k+1}}(1) = \sqrt{n}\, \widetilde{e}_{2k}$, where the Jones projection $\widetilde{e}_{2k} \in M_{I^k}(M)$ is given, as in Corollary 3.2.3, by

$$(\widetilde{e}_{2k})_{\underline{i}, \underline{j}} = \frac{1}{n}\, \delta^{\underline{i}_{[2}}_{\underline{j}_{[2}}\ u_{g_{i_1} g_{j_1}^{-1}},\ \forall\, \underline{i}, \underline{j} \in I^k.$$

Thus

$$
\begin{aligned}
\widetilde{e}_{2k} &= \frac{1}{n} \sum_{(\underline{i},\, \underline{j}) \in W_k} [\underline{i},\, \underline{j}]^{ev} \\
&= \frac{1}{n} \sum_{h \in H,\ (\underline{i},\underline{j}) \in H \backslash W_k} \frac{1}{|\mathrm{Iso}_H((\underline{i},\, \underline{j}))|}\ h[\underline{i},\, \underline{j}]^{ev} \\
\xmapsto{\varphi_{2k+1}} & \quad \frac{1}{n} \sum_{g \in G,\ (\underline{i},\underline{j}) \in H \backslash W_k} \frac{1}{|\mathrm{Iso}_H((\underline{i},\, \underline{j}))|}\ g \left(H \begin{array}{l} Hg_{i_k}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, \cdots, H \sqcap g_{\underline{j}} \end{array} \right) \\
&= \frac{1}{n} \sum_{g \in G,\ (\underline{i},\underline{j}) \in H \backslash W_k} \frac{1}{\left| \mathrm{Iso}_G \left(\left(H \begin{array}{l} Hg_{i_k}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, \cdots, H \sqcap g_{\underline{j}} \end{array} \right) \right) \right|} \times \\
&\qquad\qquad g \left(H \begin{array}{l} Hg_{i_k}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, \cdots, H \sqcap g_{\underline{j}} \end{array} \right) \\
&= \frac{1}{n} \sum_{x, y, z \in I,\ \underline{i} \in I^{k-1}} \left(Hg_x \begin{array}{l} Hg_{i_1}, \cdots, Hg_{i_{k-1}}, Hg_y \\ Hg_{i_1}, \cdots, Hg_{j_{k-1}}, Hg_z \end{array} \right),
\end{aligned}
$$

where we have used the fact that

$$
\mathrm{Iso}_H((\underline{i},\, \underline{j})) = \mathrm{Iso}_G \left(\left(H \begin{array}{l} Hg_{i_k}, \cdots, H \sqcap g_{\underline{i}} \\ Hg_{j_k}, \cdots, H \sqcap g_{\underline{j}} \end{array} \right) \right).
$$

This shows that $Z^Q_{\mathcal{E}^{2k+1}} = \varphi_{2k+1} \circ Z^P_{\mathcal{E}^{2k+1}}$.

The case for odd k follows similarly by taking analogues of W_k, F_{2k+1} and G_{2k+1} to be the sets Z_k, F_{2k} and G_{2k}, respectively, which are given by $Z_k = \{(\underline{i},\, \underline{j}) \in I^k \times I^k : \underline{i} = \underline{j}, i_1 = 1\}$,

$$
F_{2k} = \left\{ \left(Hg_x \begin{array}{l} Hg_{i_1}, \cdots, Hg_{i_{k-1}} \\ Hg_{i_1}, \cdots, Hg_{i_{k-1}} \end{array} Hg_y \right) : x, y \in I,\, \underline{i} \in I^{k-1} \text{ and } y = i_{k-1} \right\}
$$

and

$$
G_{2k} = \left\{ \left(H \begin{array}{l} Hg_{i_1}, \cdots, Hg_{i_{k-1}} \\ Hg_{i_1}, \cdots, Hg_{i_{k-1}} \end{array} Hg_y \right) : y \in I,\, \underline{i} \in I^{k-1} \text{ and } y = i_{k-1} \right\}.
$$

□

Finally, all the calculations at place, we collect some of the lemmas proved above to give a complete proof of the main theorem.

Theorem 5.4.5 *Given a finite group G, a subgroup H of index, say n, and an outer action α of G on the hyperfinite II_1-factor R, the planar algebra of the subgroup-subfactor $R \rtimes_{\alpha_{|_H}} H \subset R \rtimes_\alpha G$ is isomorphic to the G-invariant planar subalgebra of $P(\star_n)$, i.e.,*

$$P^{R\rtimes_{\alpha_{|_H}} H \,\subset\, R\rtimes_\alpha G} \cong P(\star_n)^G.$$

Proof: Let φ_k, $k \in Col$ be the maps defined as in equations (5.1−5.4). We claim that

$$\{\varphi_k : k \in Col\} =: \varphi : P^{R\rtimes_{\alpha_{|_H}} H \,\subset\, R\rtimes_\alpha G} \to P(\star_n)^G$$

is a planar algebra isomorphism.

We already know, by Lemma 5.3.2, that the maps $\varphi_k, k \in Col$ are all linear isomorphisms. Thus what remains to be shown is that φ is a planar algebra morphism.

Let $\mathcal{T}$ be the collection of coloured tangles T which commute with φ. Then, by Theorem 4.1.1, it is enough to show that $\mathcal{T}$ is closed under composition of tangles, whenever it makes sense, and that it contains the generating set of tangles $\mathcal{G}_0 = \{1^{0_\pm}\} \cup \{E^k_{k+1}, M_k, I^{k+1}_k : k \in Col\} \cup \{\mathcal{E}^{k+1}, (E')^k_k : k \geq 1\}$.

The first assertion follows from Lemma 4.2.4. Then, it readily follows from definitions that $\varphi_{0_\pm} \circ Z^P_{1^{0_\pm}} = Z^{P(\star_n)^G}_{1^{0_\pm}}$. Thus $\{1^{0_+}, 1^{0_-}\} \subset \mathcal{T}$. Finally, Lemmas 5.4.1−5.4.4 show that the collection $\{I^{k+1}_k, M_k : k \in Col\} \cup \{E^k_{k+1}, (E')^{k+1}_{k+1} : k \in Col\} \cup \{\mathcal{E}^{k+1} : k \geq 1\}$ is also contained in $\mathcal{T}$. Thus $\mathcal{G}_0 \subset \mathcal{T}$. This completes the proof. $\square$

What follows immediately is the following fact, which, however, already exists in literature.

Corollary 5.4.6 *Given a finite group G and a subgroup H, the subgroup subfactor $R \rtimes_{\alpha_{|_H}} H \subset R \rtimes_\alpha G$ does not depend upon the outer action α of G on the hyperfinite II_1-factor R.*

Proof: It follows from Theorem 5.4.5 that the planar algebra of the subgroup-subfactor $R \rtimes_{\alpha_{|_H}} H \subset R \rtimes_\alpha G$ is independent of the outer action α. In particular, by Remark 4.2.8, it follows that the standard invariant of the subfactor is independent of α. Further, since every II_1-factor containing the hyperfinite II_1-factor R as a subfactor of finite index is itself hyperfinite - see [Jon83], it follows that the II_1-factor $R \rtimes_\alpha G$ is also hyperfinite. Thus, by a consequence to Popa's result [Pop94] (as stated in Remark 2.2.3), the standard invariant of the subgroup-subfactor is a complete invariant and therefore $R \rtimes_{\alpha_{|_H}} H \subset R \rtimes_\alpha G$ is independent of the outer action α. $\square$

We can now easily calculate the dimensions of the constituent vector spaces of the planar algebra $P^{R\rtimes H\subset R\rtimes G}$. For this we need the following counting lemma, which is also known as *'Not Burnside's Lemma'*:

Lemma 5.4.7 *let G be a finite group acting on a set X. For each g in G, let $X^g = \{x \in X : gx = x\}$. Then the number of G-orbits of X is given by the formula*[2]

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

Corollary 5.4.8 *For each $k \geq 1$,*

$$dim\ P_k^{R\rtimes H\subset R\rtimes G} = \frac{1}{|G|}\sum_{C\in\mathcal{C}_G}|C|\left(\frac{|C\cap H||G|}{|C||H|}\right)^k, \tag{5.9}$$

where $\mathcal{C}_G$ is the set of conjugacy classes of G.

Proof: Let $X = H\backslash G = \{Hg_1, \ldots, Hg_n\}$, and G act on it as usual by β^1. Let $k \geq 1$ and $n_k := dim\ P_k^{R\rtimes H\subset R\rtimes G}$. Then, by Theorem 5.4.5 (basically by Lemma 5.2.1 and Lemma 5.3.1), $n_k = |G\backslash X^k|$, with respect to the diagonal β^1-action of G on X^k. Note that, for each $g \in G$, $(X^k)^g = (X^g)^k$, and that $|X^g| = |X^{xgx^{-1}}|$, $\forall x \in G$. Thus, by Lemma 5.4.7, we have

$$n_k = \frac{1}{|G|}\sum_{g\in G}|X^g|^k = \frac{1}{|G|}\sum_{C\in\mathcal{C}_G}|C|\ |X^{g_C}|^k, \tag{5.10}$$

where we have fixed a representative g_C for each conjugacy class $C \in \mathcal{C}_G$.

Consider the set $Z_C := \{g \in G : gg_Cg^{-1} \in H\}$. Then $Z_C \subset G$ is invariant under left multiplication action of H. By Lemma 5.4.7 and the fact that $hg = g$ if and only if $h = e$, we have

$$|H\backslash Z_C| = \frac{1}{|H|}\sum_{h\in H}|Z_C^h| = \frac{|Z_C|}{|H|}.$$

[2] The left expression in this formula does not conflict with our previous notation for a set of orbit representatives.

Further, the collection of H-orbits of Z_C is given by $\{Hg : gg_C g^{-1} \in H\}$ (without repetitions), which is precisely the set $X^{g_C} = \{Hg_i : g_i g_C g_i^{-1} \in H\}$. Thus $|X^{g_C}| = |Z_C|/|H|$ for all $C \in \mathcal{C}_G$.

Now, let $C_G(g_C) = \{x \in G : xg_C = g_C x\}$ be the centraliser of g_C in G. Then $|C_G(g_C)| = |G|/|C|$, and $C_G(g_C)$ acts naturally on Z_C by the map $C_G(g_C) \times Z_C \to Z_C$, $(x, g) \mapsto gx^{-1}$. Note that the map

$$Z_C \ni g \mapsto gg_C g^{-1} \in C \cap H$$

induces a bijection between the $C_G(g_C)$-orbits of Z_C and $C \cap H$. Moreover, as seen for the number of H-orbits of Z_C, we obtain

$$|C_G(g_C)\backslash Z_C| = \frac{1}{|C_G(g_C)|} \sum_{g \in C_G(g_C)} |Z_C^g| = \frac{|Z_C|}{|C_G(g_C)|}.$$

Thus we see that $|Z_C| = |C \cap H|\,|C_G(g_C)| = |C \cap H|\,|G|/|C|$, and so $|X^{g_C}| = \frac{|C\cap H|\,|G|}{|H|\,|C|}$ for all $C \in \mathcal{C}_G$. Substituting this value in (5.10) we obtain the expression for $dim\ P_k^{R^G \subset R^H}$ as asserted in (5.9).

□

As promised in Chapter 1, we note that since the automorphism group of the bipartite graph $\star_n$ is S_n, the symmetric group of n elements, we have the following:

Corollary 5.4.9 *Given any pair of finite groups $H \subset G$ with index n, the planar algebra $P^{R\rtimes H \subset R \rtimes G}$ is a planar subalgebra of the planar algebra of the bipartite graph $\star_n$ and contains the planar algebra $P(\star_n)^{S_n} \cong P^{R\rtimes S_{n-1} \subset R\rtimes S_n}$, where $\star_n$ is equipped with the Perron-Frobenius spin function as in Theorem 5.4.5. Thus*

$$P(\star_n)^{S_n} \subseteq P^{R\rtimes H \subset R\rtimes G} \subset P(\star_n).$$

We can also identify the planar algebra of the fixed algebra subfactor with the invariant planar algebra of the flip of $\star_n$.

Corollary 5.4.10 *Given a finite group G, a subgroup H of finite index, say n, and an outer action α of G on the hyperfinite II_1-factor R, the planar algebra $P^{R^G \subset R^H}$ is isomorphic to the G-invariant planar subalgebra of $P(\overline{\star_n})$, i.e.,*

$$P^{R^G \subset R^H} \cong P(\overline{\star_n})^G,$$

where $\overline{\star_n}$ is the flip of the bipartite graph $\star_n$. In particular, the subfactor $R^G \subset R^H$ is independent of the outer action α.

Proof: Recall, from Proposition 3.1.3, that if $N := R \rtimes_{\alpha/H} H$ and $M := R \rtimes_\alpha G$, then $P^{R^G \subset R^H} \cong P^{M \subset M_1}$, where M_1 is the II_1-factor obtained by the basic construction of the subfactor $N \subset M$. Further, by Proposition 4.2.9, $P^{M \subset M_1}$ is isomorphic to the dual planar algebra ${}^-P^{N \subset M}$. Thus, by Theorems 4.3.4 and 5.4.5, we conclude that

$$P^{R^G \subset R^H} \cong P(\overline{\star_n})^G.$$

This shows that the planar algebra $P^{R^G \subset R^H}$ is independent of the outer action α. Further, R^H, being a II_1-factor sitting in R, is itself hyperfinite - see [Con76]. Thus, as in the preceding corollary, the subfactor $R^G \subset R^H$ is independent of the action α. □

Thus, as in the case of subgroup-subfactors, the planar algebra of the fixed algebra subfactor $P^{R^G \subset R^H}$ also gets sandwiched between the two planar algebras $P(\overline{\star_n})^{S_n}$ and $P(\overline{\star_n})$.

5.5 Some Questions

We conclude the main part of the thesis with the following natural questions that arise from the work done above.

1. We saw that, given a pair of finite groups $H \subset G$ with index n, the planar algebra $P^{R \rtimes H \subset R \rtimes G}$ is sandwiched between the two planar algebras $P(\star_n)^{S_n}$ and $P(\star_n)$. The converse of this statement remains unanswered, i.e., given any subfactor planar algebra P such that $P(\star_n)^{S_n} \subset P \subset P(\star_n)$, is it isomorphic to $P^{R \rtimes H \subset R \rtimes G}$ for some pair of finite groups $H \subset G$ with index n?

2. One can look for an identification of the type obtained in Theorem 5.4.5 in the context of Kac algebras, i.e., does there exist an identification between the planar algebra of a subfactor obtained by an outer action of a finite dimensional Kac algebra on the hyperfinite II_1-factor and a(n invariant) planar subalgebra of the planar algebra of an appropriate bipartite graph.

3. The notion of group action on a planar algebra came up quite naturally in the above work. However, in the context of Kac algebras, it is not

clear how to formulate a definition of an action of a Kac algebra on a planar algebra. And subsequently, one can analyse the invariant planar subalgebra under such a action.

Appendix A

Rotation via Paragroups

A.1 Introduction

The aim of this appendix is to show how to express rotation maps of Jones' planar algebras in terms of Ocneanu's paragroup structure.

We shall only consider inclusions of II_1-factors $N \subset M$ (usually called a subfactor) with finite index ($[M : N] = \delta^2$) and finite depth, unless otherwise specified.

Recall that for a subfactor $N \subset M$ with finite index, its standard invariant is the grid of relative commutants

$$\begin{array}{ccccccccc} & & \mathbb{C} & = & M' \cap M & \subset & M' \cap M_1 & \subset & \cdots \quad (D) \\ & & & & \cap & & \cap & & \\ \mathbb{C} & = & N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \cdots \quad (P). \end{array} \tag{A.1}$$

Further, Jones - see [Jon] - gave a planar algebra structure on the tower (P) of relative commutants for every extremal finite index subfactor $N \subset M$. The rotation map (Definition (A.2.1)) is an integral part of that planar algebra structure.

For finite index subfactors with finite depth, Ocneanu - see [Ocn88, Ocn91] - came up with another invariant, a combinatorial structure, known as a *paragroup*. We discuss this structure, in some detail (without proofs) in §4 .

Before this, we briefly recall some aspects of Frobenius reciprocity in §3.

Then we move on to see the path algebra models for the towers (P) and (D) occurring in the standard invariant in §5, and do some calculations for the conditional expectation $E_{M' \cap N_k}$.

Finally, in §6, we derive the equation for the coefficients of the rotation maps in terms of connection values for the so called 'macro-cells'.

A.2 Rotation

Definition A.2.1 (Jones) *Let $N \subset M$ be an extremal, finite index subfactor with $[M:N] = \delta^2$ $(\delta > 0)$. For each $k \geq 2$, the rotation map on the relative commutant $N' \cap M_{k-1}$ is given by*

$$R_k(x) = \delta^2 E_{N' \cap M_{k-1}}(V_k E_{M' \cap M_k}(x V_k)), \; x \in N' \cap M_{k-1}, \tag{A.2}$$

where $V_k := E_k E_{k-1} \cdots E_2 E_1$, and $E_i := \delta e_i$.

This definition first appeared in [Jon], where it was also shown that R_k is of order k, for all $k \geq 2$.

A.3 Frobenius Reciprocity

Before we move on to describe the paragroup of a subfactor, we fix our notations and recall certain facts related to Frobenius reciprocity for bimodules.

In general, for any three II_1-factors P, Q and R, and irreducible bimodules ${}_P\gamma^1_Q$, ${}_P\gamma^2_R$ and ${}_Q\gamma^3_R$, we have - see [KS01b], [EK98] - anti-unitary involutions

$$F_{(12)} : \mathcal{H}^{{}_P\gamma^2_R}_{{}_P\gamma^1_Q,\, {}_Q\gamma^3_R} \to \mathcal{H}^{{}_P\gamma^1_Q}_{{}_P\gamma^2_R,\, {}_R\overline{\gamma^3}_Q}, \quad F_{(23)} : \mathcal{H}^{{}_P\gamma^2_R}_{{}_P\gamma^1_Q,\, {}_Q\gamma^3_R} \to \mathcal{H}^{{}_Q\gamma^3_R}_{{}_Q\overline{\gamma^1}_P,\, {}_P\gamma^2_R}, \tag{A.3}$$

where $\mathcal{H}^{{}_P\pi_R}_{{}_P\gamma_Q,\, {}_Q\lambda_R}$ is the finite dimensional intertwiner space $Hom({}_P\gamma \otimes_Q \lambda_R,\; {}_P\pi_R)$, equipped with the inner product $\langle \cdot, \cdot \rangle$ given by the equation $\langle \sigma', \sigma \rangle id_\pi = \sigma' \circ \sigma^*$.

A.4 Paragroup of a Subfactor

We shall freely borrow notations from [KS01a].

For a finite depth finite index subfactor $N \subset M$, we set $B_{i,j}$ to be the set of irreducible R_i-R_j $(i, j \in \{0, 1\})$ bimodules arising as submodules of

$$L^2(M) \otimes_N L^2(M) \otimes_N \cdots \otimes_N L^2(M) \; (k \text{ factors}), \; k \geq 1,$$

where $R_0 := N$ and $R_1 := M$. We consider two bipartite (multi-) graphs $V_0 = H_0$ and $V_1 = H_1$, the principal and the dual graphs of $N \subset M$, respectively; so that the vertex sets for V_0 and V_1 (resp., H_0 and H_1) are $B_{0,0} \sqcup B_{1,0}$ and $B_{0,1} \sqcup B_{1,1}$ (resp., $B_{0,0} \sqcup B_{0,1}$ and $B_{1,0} \sqcup B_{1,1}$) respectively:

$$\begin{tikzcd} B_{0,0} \arrow[r, dash, "H_0"] \arrow[d, dash, "V_0"] & B_{0,1} \arrow[d, dash, "V_1"] \\ B_{1,0} \arrow[r, dash, "H_1"] & B_{1,1} \end{tikzcd} .$$

In the vertex sets $B_{i,j},\ i,j \in \{0,1\}$, we have some specific vertices, namely $*_i := L^2(R_i), i = 0,1;\ \alpha :=_N L^2(M)_M,\ \alpha_{0,1} := \alpha$ and $\alpha_{1,0} := \bar{\alpha}$.

Note that any two vertices $\gamma_{i,j}$ and $\gamma_{i,1-j}$ in the horizontal graph H_i are connected by $dim\ \mathcal{H}^{\gamma_{i,1-j}}_{\gamma_{i,j},\ \alpha_{j,1-j}}$ edges. Likewise, two vertices $\gamma_{i,j}$ and $\gamma_{1-i,j}$ in the vertical graph V_j are connected by $dim\ \mathcal{H}^{\gamma_{1-i,j}}_{\alpha_{1-i,i},\ \gamma_{i,j}}$ edges. We can, therefore, label the edges in the graphs V_i and H_j, $i,j \in \{0,1\}$ by some choice of intertwiners. For each pair of vertices $\gamma_{i,j}$ and $\gamma_{i,1-j}$ in the horizontal graphs $H_i,\ i \in \{0,1\}$, we fix an orthonormal basis in $\mathcal{H}^{\gamma_{i,1-j}}_{\gamma_{i,j},\ \alpha_{j,1-j}}$; and as a gift, the Frobenius map $F_{(12)}$ gives an orthonormal basis for the intertwiner space $\mathcal{H}^{\gamma_{i,j}}_{\gamma_{i,1-j},\ \alpha_{1-j,j}}$. We label an edge in the direction $\gamma_{i,j} \to \gamma_{i,1-j}$ by an element of the fixed orthonormal basis of $\mathcal{H}^{\gamma_{i,1-j}}_{\gamma_{i,j},\ \alpha_{j,1-j}}$ and an edge in the reverse direction $\gamma_{i,1-j} \to \gamma_{i,j}$ by that of $\mathcal{H}^{\gamma_{i,j}}_{\gamma_{i,1-j},\ \alpha_{1-j,j}}$, as obtained above. Likewise, after making choices of orthonormal intertwiners and using the Frobenius maps $F_{(23)}$, we label the edges of the vertical graphs $V_j,\ j \in \{0,1\}$, by those intertwiners. Note that the involutive nature of the Frobenius maps $F_{(12)}$ and $F_{(23)}$ makes our labelling convention consistent.

With these conventions, for vertices $\gamma_{i,j} \in B_{i,j},\ \gamma_{1-i,j} \in B_{1-i,j},\ \gamma_{i,1-j} \in B_{i,1-j}$ and $\gamma_{1-i,1-j} \in B_{1-i,1-j}$, $i,j \in \{0,1\}$, a *cell* is a diagram of the form

$$\begin{array}{ccc} \gamma_{i,j} & \tau & \gamma_{i,1-j} \\ \sigma & & \sigma' \\ \gamma_{1-i,j} & \tau' & \gamma_{1-i,1-j} \end{array}\ , \tag{A.4}$$

where the edges σ, σ', τ and τ' are intertwiners between the corresponding vertices, as described above. The above cell also gives intertwiners

$$\tau' \circ (\sigma \otimes id_{\alpha_{j,1-j}}),\ \sigma' \circ (id_{\alpha_{1-i,i}} \otimes \tau) \in\ Hom\,(\alpha_{1-i,i} \otimes_{R_i} \gamma_{i,j} \otimes_{R_j} \alpha_{j,1-j}, \gamma_{1-i,1-j}),$$

and thus we have a function, called a *connection*, that associates to each cell, as above, a complex number $W^{\sigma \cdot \tau'}_{\tau \cdot \sigma'}$ given by

$$W^{\sigma \cdot \tau'}_{\tau \cdot \sigma'} \cdot id_{\gamma_{1-i,1-j}} = \sigma' \circ (id_{\alpha_{1-i,i}} \otimes \tau) \circ (\sigma \otimes id_{\alpha_{j,1-j}})^* \circ \tau'^*.$$

We shall also use the following symbols for connection:

$$W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'} = W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'}(\gamma_{i,\,j}, \gamma_{1-i,\,1-j}) = \begin{array}{ccc} \gamma_{i,\,j} & \xrightarrow{\tau} & \gamma_{i,\,1-j} \\ \sigma\downarrow & & \downarrow\sigma' \\ \gamma_{1-i,\,j} & \xrightarrow{\tau'} & \gamma_{1-i,\,1-j} \end{array} . \tag{A.5}$$

For an edge ω on any of the above graphs, by $s(\omega)$ and $r(\omega)$, we mean the source vertex and range vertex of ω, respectively. A path ξ of length k, on any of the above graphs, is a k-tuple of edges $\xi = (\xi_1, \xi_2, \ldots, \xi_k)$ with $s(\xi_i) = r(\xi_{i-1})$, $2 \leq i \leq k$; and its source and range are given by $s(\xi) := s(\xi_1)$ and $r(\xi) := r(\xi_k)$, respectively. We write $l(\xi)$ or $|\xi|$ for the length of the path ξ. For $1 \leq r \leq s \leq k$, we use the symbol $\xi_{[r,\,s]}$ to denote the truncated path $(\xi_r, \xi_{r+1}, \ldots, \xi_s)$, with the understanding that $\xi_{[r,\,r]}$ denotes the edge ξ_r. As in [KS01a], for any two vertices γ and λ in the same graph, we set $P^H_k(\gamma, \lambda)$ (resp., $P^V_k(\gamma, \lambda)$) to be the collection of horizontal (resp., vertical) paths ξ of length k with $s(\xi) = \gamma$ and $r(\xi) = \lambda$.

We also need to consider paths which are not necessarily straight; thus for any two vertices $\gamma_{i,\,j} \in B_{i,\,j}$ and $\gamma_{r,\,s} \in B_{r,\,s}$, for north-west paths we set $\mathcal{P}^{HV}_{k\,l}(\gamma_{i,\,j}, \gamma_{r,\,s}) = \{\xi \cdot \eta : \xi \in P^H_k(\gamma_{i,\,j}, -), \eta \in P^V_l(-, \lambda_{r,\,s}) \text{ with } r(\xi) = s(\eta)\}$, and likewise for south-east paths we set $P^{VH}_{k\,l}(\gamma_{i,\,j}, \lambda_{r,\,s}) = \{\xi \cdot \eta : \xi \in P^V_k(\gamma_{i,\,j}, -), \eta \in P^H_l(-, \lambda_{r,\,s}) \text{ with } r(\xi) = s(\eta)\}$, where the concatenation of two paths has the obvious meaning. With the above notations, given vertices $\gamma_{i,\,j} \in B_{i,\,j}$, $\gamma_{i,\,r} \in B_{i,\,r}$, $\gamma_{p,\,j} \in B_{p,\,j}$ and $\gamma_{p,\,r} \in B_{p,\,r}$, by a *macro cell*, as in [Ocn88], we mean a diagram of the form

$$\begin{array}{ccc} \gamma_{i,\,j} & \tau & \gamma_{i,\,r} \\ \sigma & & \sigma' \\ \gamma_{p,\,j} & \tau' & \gamma_{p,\,r} \end{array} , \tag{A.6}$$

where τ, τ' are horizontal paths of same lengths and σ, σ' are vertical paths of same lengths, with appropriate source and range. We fill up such a macro cell with a possible family of cells, take the product of the connection values of those cells, sum these values over all possible ways of filling up the macro cell and denote the sum by the symbols:

$$W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'} = W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'}(\gamma_{i,\,j}, \gamma_{p,\,r})_{(|\sigma|,\,|\tau|)} = \begin{array}{ccc} \gamma_{i,\,j} & \xrightarrow{\tau} & \gamma_{i,\,r} \\ \sigma\downarrow & & \downarrow\sigma' \\ \gamma_{p,\,j} & \xrightarrow{\tau'} & \gamma_{p,\,r} \end{array} , \tag{A.7}$$

where the suffix $(|\sigma|,\,|\tau|)$ gives the size of the macro-cell. We shall ignore such suffix for connection values of cells, i.e., for macro cells of size $(1,1)$.

We have one more description of the connection value for the above macro cell. For $\sigma \in P_l^V(\gamma_{i,j}, \gamma_{p,j})$ and $\tau \in P_k^H(\gamma_{i,j}, \gamma_{i,r})$; $i,j,p,r \in \{0,1\}$, we set

$$\begin{aligned} X_\sigma &= \alpha_{p,1-p} \otimes_{R_{1-p}} \alpha_{1-p,p} \otimes_{R_p} \cdots \otimes_{R_{1-i}} \alpha_{1-i,i} \text{ (l factors)} \quad \text{and} \\ Y_\tau &= \alpha_{i,1-i} \otimes_{R_{1-i}} \alpha_{1-i,i} \otimes_{R_i} \cdots \otimes_{R_{1-r}} \alpha_{1-r,r} \text{ (k factors).} \end{aligned}$$

Note that X_σ (resp., Y_τ) is same for all $\sigma \in P_l^V(\gamma_{i,j}, \gamma_{p,j})$ (resp., $\tau \in P_k^H(\gamma_{i,j}, \gamma_{i,r})$).

Suppose, in the above macro-cell (A.6), σ and τ are of lengths l and k, respectively. Then we have intertwiners

$$\sigma := \sigma_l \circ (id_{X_{\sigma_{[l,\,l]}}} \otimes \sigma_{l-1}) \circ \cdots \circ (id_{X_{\sigma_{[2,\,l]}}} \otimes \sigma_1) : X_\sigma \otimes_{R_i} \gamma_{i,j} \to \gamma_{p,j} \quad \text{(A.8)}$$

and

$$\tau := \tau_k \circ (\tau_{k-1} \otimes id_{Y_{\tau_{[k,\,k]}}}) \circ \cdots \circ (\tau_1 \otimes id_{Y_{\tau_{[2,\,k]}}}) : \gamma_{i,j} \otimes_{R_j} Y_\tau \to \gamma_{i,r}. \quad \text{(A.9)}$$

Likewise, we get intertwiners for σ' and τ'. And by iterated inductions on k and l, it is readily seen that

$$W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'} \cdot id_{\gamma_{p,r}} = \sigma' \circ (id_{X_{\sigma'}} \otimes \tau) \circ (\sigma \otimes id_{Y_{\tau'}})^* \circ \tau'^* \,. \quad \text{(A.10)}$$

We also note that the above prescriptions (A.8) and (A.9) give us two orthonormal bases $\{\tau' \circ (\sigma \otimes id_{Y_{\tau'}}) : \sigma \in P_l^V(\gamma_{i,j}, -),\ \tau' \in P_k^H(r(\sigma), \gamma_{p,r})\}$ and $\{\sigma' \circ (id_{X_{\sigma'}} \otimes \tau) : \tau \in P_k^H(\gamma_{i,j}, -),\ \sigma' \in P_l^V(r(\tau), \gamma_{p,r})\}$ of the intertwiner space

$$Hom(X_\sigma \otimes_{R_i} \gamma_{i,j} \otimes_{R_j} Y_\tau,\ \gamma_{p,r}),$$

along the south-west and the north-east paths, respectively.

We have - see [EK98, Ocn91, KS01a] - the following facts:

(V) Vertex Requirements : Each of the four graphs (V_0, V_1, H_0, H_1) is a finite, connected bipartite graph, and the decomposition of its even and odd vertices are as above. And we have two marked vertices in $B_{0,0}$ and $B_{1,1}$, namely $*_0$ and $*_1$, respectively.

(H) Harmonicity : The map μ given by

$$\sqcup_{i,j=0,1} B_{i,j} \ni \gamma_{r,s} \xrightarrow{\mu} \sqrt{dim_{R_r}(\gamma_{r,s})\, dim(\gamma_{r,s})_{R_s}} \in (0,\infty),$$

satisfies:
(i) $\mu(*_i) = 1$, for $i = 0, 1$; and
(ii) for any graph $G \in \{V_0, V_1, H_0, H_1\}$, its adjacency matrix A_G satisfies

$$\sum_{\lambda \in V(G)} A_G(\pi, \lambda)\mu(\lambda) = \delta\, \mu(\pi), \ \forall\, \pi \in V(G),$$

where $V(G)$ is the vertex set for the graph G and $\delta^2 = [M : N]$.

(U) Unitarity : For any two vertices $\gamma_{i,j}$ and $\gamma_{1-i,1-j}$, the (connection) matrix

$$W(\gamma_{i,j}, \gamma_{1-i,1-j}) = \begin{array}{ccc} \gamma_{i,j} & \to & \cdot \\ \downarrow & & \downarrow \\ \cdot & \to & \gamma_{1-i,1-j} \end{array} ,$$

whose entries are given by

$$W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'}(\gamma_{i,j}, \gamma_{1-i,1-j}) = W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'} = \begin{array}{ccc} \gamma_{i,j} & \xrightarrow{\tau} & \gamma_{i,1-j} \\ \sigma\downarrow & & \downarrow\sigma' \\ \gamma_{1-i,j} & \xrightarrow{\tau'} & \gamma_{1-i,1-j} \end{array} ,$$

with $|\sigma| = |\sigma'| = |\tau| = |\tau'| = 1$, is a unitary matrix.

(I) Invariance under flips: Given a cell, as in (A.4), we have:

$$\begin{aligned} &\begin{array}{ccc} \gamma_{i,j} & \xrightarrow{\tau} & \gamma_{i,1-j} \\ \sigma\downarrow & & \downarrow\sigma' \\ \gamma_{1-i,j} & \xrightarrow{\tau'} & \gamma_{1-i,1-j} \end{array} \\ &= \sqrt{\frac{\mu(\gamma_{1-i,j})\mu(\gamma_{i,1-j})}{\mu(\gamma_{i,j})\mu(\gamma_{1-i,1-j})}} \left(\begin{array}{ccc} \gamma_{1-i,j} & \xrightarrow{\tau'} & \gamma_{1-i,1-j} \\ \widetilde{\sigma}\downarrow & & \downarrow\widetilde{\sigma'} \\ \gamma_{i,j} & \xrightarrow{\tau'} & \gamma_{i,1-j} \end{array} \right)^{-} \\ &= \sqrt{\frac{\mu(\gamma_{1-i,j})\mu(\gamma_{i,1-j})}{\mu(\gamma_{i,j})\mu(\gamma_{1-i,1-j})}} \left(\begin{array}{ccc} \gamma_{i,1-j} & \xrightarrow{\widetilde{\tau}} & \gamma_{i,j} \\ \sigma'\downarrow & & \downarrow\sigma \\ \gamma_{1-i,1-j} & \xrightarrow{\widetilde{\tau'}} & \gamma_{1-i,j} \end{array} \right)^{-} \end{aligned}$$

$$= \begin{array}{ccc} \gamma_{1-i,\,1-j} & \xrightarrow{\widetilde{\tau'}} & \gamma_{1-i,\,j} \\ \widetilde{\sigma'} \downarrow & & \downarrow \widetilde{\sigma} \\ \gamma_{i,\,1-j} & \xrightarrow{\widetilde{\tau}} & \gamma_{i,\,j} \end{array} ,$$

where $\widetilde{\omega}$ denotes the opposite of the edge (or path) ω.

(F) Flatness : For $* = *_0, *_1$; $\sigma, \sigma' \in P^V_{2l}(*, *)$; and $\tau, \tau' \in P^H_{2k}(*, *)$, we have:

$$\begin{array}{ccc} * & \xrightarrow{\tau} & * \\ \sigma \downarrow & & \downarrow \sigma' \\ * & \xrightarrow{\tau'} & * \end{array} = \delta^{\sigma}_{\sigma'}\ \delta^{\tau}_{\tau'},\ \forall\, k, l \geq 1. \tag{A.11}$$

The properties **(U)** and **(I)** generalise to macro-cells as well:

For any two vertices $\gamma_{i,\,j} \in B_{i,\,j}$ and $\gamma_{p,\,r} \in B_{p,\,r}$, and $l \geq 1$, $k \geq 1$, where l (resp., k) being even or odd according as $i = p$ or $i \neq p$ (resp., $j = r$ or $j \neq r$), the connection matrix

$$W(\gamma_{i,\,j}, \gamma_{p,\,r})_{(l,\,k)} = \begin{array}{ccc} \gamma_{i,\,j} & \rightarrow & \cdot \\ \downarrow_l & & \downarrow \\ \cdot & \xrightarrow{k} & \gamma_{p,\,r} \end{array}$$

is unitary, whose entries are given by

$$W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'}(\gamma_{i,\,j}, \gamma_{p,\,r})_{(l,\,k)} := W^{\sigma\cdot\tau'}_{\tau\cdot\sigma'} = \begin{array}{ccc} \gamma_{i,\,j} & \xrightarrow{\tau} & \gamma_{i,\,r} \\ \sigma \downarrow & & \downarrow \sigma' \\ \gamma_{p,\,j} & \xrightarrow{\tau'} & \gamma_{p,\,r} \end{array} ,$$

where $\tau, \tau' \in P^H_k(-, -)$ and $\sigma, \sigma' \in P^V_l(-, -)$ with appropriate source and range.

In fact, this (and, in particular **(U)**) also follows from the fact that $W(\gamma_{i,\,j}, \gamma_{p,\,r})_{(l,\,k)}$ is the matrix of the identity map on the intertwiner space

$$Hom(X_\sigma \otimes_{R_i} \gamma_{i,\,j} \otimes_{R_j} Y_\tau,\ \gamma_{p,\,r}),$$

(for any choice of $\sigma \in P^V_l(\gamma_{i,\,j}, -)$ and $\tau \in P^H_k(\gamma_{i,\,j}, -)$) w.r.t. the orthonormal bases given by the intertwiners, as obtained above, along the north-east paths and the south-west paths. Apart from this, the connection for macro-cells follows invariance under flips as well - see [KS01a]. For convenience, when clear from the context, we shall ignore the suffix (l, k) of connection matrices for macro-cells.

Further, we have - see [Ocn91, EK98] - an equivalent version of flatness involving only two $*$ vertices. For $* = *_0, *_1$; $\sigma, \sigma' \in P^V_{2l}(-,-)$; and $\tau, \tau' \in P^H_{2k}(-,-)$, with appropriate source and range, we have:

$$\begin{array}{ccc} * & \xrightarrow{\tau} & \cdot \\ \sigma\downarrow & & \downarrow\sigma' \\ * & \xrightarrow{\tau'} & \cdot \end{array} = \delta^{\tau}_{\tau'}\, C_{\sigma,\sigma'}, \quad \begin{array}{ccc} * & \xrightarrow{\tau} & * \\ \sigma\downarrow & & \downarrow\sigma' \\ \cdot & \xrightarrow{\tau'} & \cdot \end{array} = \delta^{\sigma}_{\sigma'}\, C_{\tau,\tau'}\ , \tag{A.12}$$

where the scalars $C_{\sigma,\sigma'}$ and $C_{\tau,\tau'}$ do not depend upon τ, τ' and σ, σ', respectively. This is often referred to as 2^*-flatness.

A.5 The Standard Invariant

In order to understand the rotation map in terms of the paragroup, we need to consider the path algebra models (also referred as string algebras) - see [JS97, EK98] - of the two towers of relative commutants occurring in the standard invariant of a subfactor.

We identify $N' \cap N$ with $M' \cap M$ and consider the standard invariant in the form:

$$\begin{array}{ccccccccc} \mathbb{C} & = & M' \cap M & \subset & M' \cap M_1 & \subset & \cdots & \subset M' \cap M_k & \cdots \\ & & \cap & & \cap & & & \cap & \\ & & N' \cap M & \subset & N' \cap M_1 & \subset & \cdots & \subset N' \cap M_k & \cdots . \end{array}$$

Since the Bratteli diagrams of the towers in the standard invariant for the subfactor $N \subset M$ are obtainable from its principal and dual graphs, we label their edges as done in above section. Thus, with respect to its two path algebra models, along south-west and north-east paths, respectively, $N' \cap M_k$ has - see [JS97, EK98] - two systems of 'generalised matrix units' $\mathcal{B}_{1\,k}$ and $\mathcal{B}_{k\,1}$ given by:

$$\begin{array}{rcl} \mathcal{B}_{1\,k} & = & \{(\xi^+, \xi^-) : \xi^{\pm} \in P^{VH}_{1\,k}(*_1, -), \text{with } r(\xi^+) = r(\xi^-)\}, \text{ and} \\ \mathcal{B}_{k\,1} & = & \{(\eta^+, \eta^-) : \eta^{\pm} \in P^{HV}_{k\,1}(*_1, -), \text{with } r(\eta^+) = r(\eta^-)\}, \end{array}$$

which are compatible with respect to the corresponding towers.

We now look at the change of bases induced by the connection matrices W. Recall that for $(\xi^+, \xi^-) \in \mathcal{B}_{1\,k}$, we have intertwiners $\xi^{\pm} \in Hom(\alpha_{0,1} \otimes_{R_1} *_1 \otimes_{R_1} Y_{\xi_{[2,\,k]}}, r(\xi^+))$ and thus the intertwiners

$$\xi^{+^*} \circ \xi^- \in End\,(\alpha_{0,1} \otimes_{R_1} *_1 \otimes_{R_1} Y_{\xi_{[2,\,k]}}) \cong N' \cap M_k$$

(expand $Y_{\xi_{[2,\,k]}}$ and see [JS97, §4.4]).

The identification $\xi^{+^*} \circ \xi^- \leftrightarrow (\xi^+, \xi^-)$, $(\xi^+, \xi^-) \in \mathcal{B}_{1\,k}$, gives a $*$-isomorphism between $N' \cap M_k$ and its south-west path algebra model.

Likewise, we have another identification $\eta^{+^*} \circ \eta \leftrightarrow (\eta^+, \eta^-)$, $(\eta^+, \eta^-) \in \mathcal{B}_{k\,1}$, between $N' \cap M_k$ and its north-east path algebra model.

As seen above, for each $k \geq 1$ and $\gamma \in B_{0,\,j}$ (with j being 0 or 1 according as k is even or odd), $P^{VH}_{1\,k}(*_1, \gamma)$ and $P^{HV}_{k\,1}(*_1, \gamma)$ give two orthonormal bases for $Hom\,(\alpha_{0,1} \otimes_{R_1} *_1 \otimes_{R_1} Y_\tau,\, \gamma)$, for any choice of τ in $P^H_k(\alpha_{1,0}, \gamma)$. And the connection matrix W being unitary with respect to these bases, we have:

$$\begin{aligned}
\xi &= \sum_{\eta \in P^{HV}_{k\,1}(*_1, r(\xi))} [W(*_1, r(\xi))^*]^\eta_\xi \cdot \eta \\
&= \sum_{\eta \in P^{HV}_{k\,1}(*_1,\, r(\xi))} \overline{W^\xi_\eta} \cdot \eta, \quad \forall\, \xi \in P^{VH}_{1\,k}(*_1, -), \text{ and} \\
\eta &= \sum_{\xi \in P^{VH}_{1\,k}(*_1,\, r(\eta))} W^\xi_\eta \cdot \xi, \quad \forall \eta \in P^{HV}_{k\,1}(*_1, -).
\end{aligned}$$

Thus, for $N' \cap M_k$, the change of bases induced by the connection are given by:

$$(\xi^+, \xi^-) = \sum_{\substack{(\eta^+,\, \eta^-) \,\in\, \mathcal{B}_{k\,1}: \\ r(\eta^\pm) = r(\xi^\pm)}} W^{\xi^+}_{\eta^+} \overline{W^{\xi^-}_{\eta^-}} (\eta^+, \eta^-), \ \forall\, (\xi^+, \xi^-) \in \mathcal{B}_{1\,k}, \quad \text{(A.13)}$$

$$(\eta^+, \eta^-) = \sum_{\substack{(\xi^+,\, \xi^-) \,\in\, \mathcal{B}_{1\,k}: \\ r(\xi^\pm) = r(\eta^\pm)}} \overline{W^{\xi^+}_{\eta^+}} W^{\xi^-}_{\eta^-} (\xi^+, \xi^-), \ \forall\, (\eta^+, \eta^-) \in \mathcal{B}_{k\,1}. \quad \text{(A.14)}$$

With these relations, we are ready to calculate the conditional expectation $E_{M' \cap M_k}$ in terms of the path algebra models.

Lemma A.5.1 *The conditional expectation* $E_{M' \cap M_k} : N' \cap M_k \to M' \cap M_k$ *is given by*

$$\begin{aligned}
&E_{M' \cap M_k}((\xi^+, \xi^-)) \\
&= \sum_{\zeta^\pm \in P^{VH}_{1\,k}(*_1,\ r(\xi^+)):\ \zeta^+_1 = \zeta^-_1} \frac{\sqrt{\mu(r(\xi^-_1))\mu(r(\zeta^+_1))}}{\delta}\ \delta^{\xi^+_1}_{\xi^-_1} \ \times
\end{aligned}$$

$$\begin{array}{ccccc} \cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & r(\zeta^+) & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \downarrow & & & & \downarrow \\ *_1 & & & & *_1 \\ \downarrow & & & & \downarrow \\ \cdot & \xrightarrow{\widetilde{\xi^+_{[2,\ k+1]}}} & r(\xi^+) & \xrightarrow{\widetilde{\xi^-_{[2,\ k+1]}}} & \cdot \end{array} \quad (\zeta^+, \zeta^-), \qquad \text{(A.15)}$$

for all $(\xi^+, \xi^-) \in \mathcal{B}_{1\,k}$.

Proof: For $(\xi^+, \xi^-) \in \mathcal{B}_{1\,k}$, the above path algebra models of $N' \cap M_k$ give (c.f. [JS97, EK98]):

$$\begin{aligned}
& E_{M'\cap M_k}(\xi^+, \xi^-) \\
&= \sum_{(\eta^+,\eta^-)\in\mathcal{B}_{k\,1}:\ r(\eta^\pm)=r(\xi^\pm)} W^{\xi^+}_{\eta^+}\, \overline{W^{\xi^-}_{\eta^-}}\, E_{M'\cap M_k}(\eta^+, \eta^-) \qquad \text{(by (A.14))} \\
&= \sum_{(\eta^+,\eta^-)\in\mathcal{B}_{k\,1}:\ r(\eta^\pm)=r(\xi^\pm)} W^{\xi^+}_{\eta^+}\, \overline{W^{\xi^-}_{\eta^-}}\, \delta^{\eta^+_{k+1}}_{\eta^-_{k+1}} \frac{\mu(r(\eta^+))}{\delta\,\mu(r(\eta^+_k))} (\eta^+_{[1,\ k]}, \eta^-_{[1,\ k]}) \\
&= \sum_{\substack{\eta^\pm\,\in\, P^H_k(*_1,\,-)\,:\, r(\eta^+) = r(\eta^-), \\ \theta\,\in\, P^V_1(r(\eta^\pm),\, r(\xi^+))}} W^{\xi^+}_{\eta^+\cdot\theta}\, \overline{W^{\xi^-}_{\eta^-\cdot\theta}}\, \frac{\mu(r(\xi^+))}{\delta\,\mu(r(\eta^+))} (\eta^+, \eta^-) \\
&= \sum_{\substack{\eta^\pm\,\in\, P^H_k(*_1,\,-)\,:\, r(\eta^+) = r(\eta^-), \\ \vartheta,\theta\,\in\, P^V_1(r(\eta^+),\, r(\xi^+))}} W^{\xi^+}_{\eta^+\cdot\theta}\, \overline{W^{\xi^-}_{\eta^-\cdot\theta}}\, \frac{\mu(r(\xi^+))}{\delta\,\mu(r(\eta^+))} (\eta^+\cdot\vartheta, \eta^-\cdot\vartheta) \\
&= \sum_{\substack{\eta^\pm\,\in\, P^H_k(*_1,\,-)\,:\, r(\eta^+) = r(\eta^-), \\ \vartheta,\theta\,\in\, P^V_1(r(\eta^+),\, r(\xi^+))}} W^{\xi^+}_{\eta^+\cdot\theta}\, \overline{W^{\xi^-}_{\eta^-\cdot\theta}}\, \frac{\mu(r(\xi^+))}{\delta\,\mu(r(\eta^+))} \quad \times \\
&\qquad\qquad \sum_{\zeta^\pm\in P^{VH}_{1\,k}(*_1, r(\vartheta))} \overline{W^{\zeta^+}_{\eta^+\cdot\vartheta}}\, W^{\zeta^-}_{\eta^-\cdot\vartheta}\, (\zeta^+, \zeta^-) \qquad \text{(by A.14)} \\
&= \sum_{\substack{\eta^\pm\,\in\, P^H_k(*_1,\,-)\,:\, r(\eta^+) = r(\eta^-), \\ \vartheta,\theta\,\in\, P^V_1(r(\eta^+),\, r(\xi^+)), \\ \zeta^\pm\,\in\, P^{VH}_{1\,k}(*_1,\, r(\vartheta))}} \frac{\mu(r(\xi^+))}{\delta\,\mu(r(\eta^+))} \quad \begin{array}{ccc} *_1 & \xrightarrow{\eta^+} & \cdot \\ \xi^+_1 \downarrow & & \downarrow \theta \\ \cdot & \xrightarrow{\xi^+_{[2,\ k+1]}} & r(\xi^+) \end{array} \quad \times
\end{aligned}$$

$$\sqrt{\frac{\mu(r(\eta^-))\mu(r(\xi_1^-))}{\mu(r(\xi^-))}}\;\begin{array}{ccc} \cdot & \xrightarrow{\xi^-_{[2,\ k+1]}} & r(\xi^-) \\ \widetilde{\xi_1^-}\downarrow & & \downarrow\widetilde{\theta} \\ *_1 & \xrightarrow{\eta^-} & \cdot \end{array}\;\sqrt{\frac{\mu(r(\eta^+))\mu(r(\zeta_1^+))}{\mu(r(\zeta^+))}}\;\times$$

$$\begin{array}{ccc} \cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & r(\zeta^+) \\ \widetilde{\zeta_1^+}\downarrow & & \downarrow\widetilde{\vartheta} \\ *_1 & \xrightarrow{\eta^+} & \cdot \end{array}\quad\begin{array}{ccc} *_1 & \xrightarrow{\eta^-} & \cdot \\ \zeta_1^-\downarrow & & \downarrow\vartheta \\ \cdot & \xrightarrow{\zeta^-_{[2,\ k+1]}} & r(\zeta^-) \end{array}\quad(\zeta^+,\zeta^-)$$

$$= \sum_{\substack{\vartheta,\theta\,\in\,P_1^V(-,\ r(\xi^+))\,:\ s(\vartheta)=s(\theta),\\ \zeta^\pm\,\in\,P^{VH}_{1\,k}(*_1,\ r(\vartheta))}} \frac{\sqrt{\mu(r(\xi_1^-))\mu(r(\zeta_1^+))}}{\delta}\times$$

$$\begin{array}{ccc} \cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & r(\zeta^+) \\ \widetilde{\zeta_1^+}\downarrow & & \downarrow\widetilde{\vartheta} \\ *_1 & & s(\theta) \\ \zeta_1^+\downarrow & & \downarrow\theta \\ \cdot & \xrightarrow{\xi^+_{[2,\ k+1]}} & r(\xi^+) \end{array}\quad\begin{array}{ccc} \cdot & \xrightarrow{\xi^-_{[2,\ k+1]}} & r(\xi^-) \\ \widetilde{\xi_1^-}\downarrow & & \downarrow\widetilde{\theta} \\ *_1 & & s(\vartheta) \\ \zeta_1^-\downarrow & & \downarrow\vartheta \\ \cdot & \xrightarrow{\zeta^-_{[2,\ k+1]}} & r(\zeta^-) \end{array}\quad(\zeta^+,\zeta^-)$$

$$= \sum_{\substack{\vartheta,\ \theta\ \in\ P_1^V(-,\ r(\xi^+))\,:\ s(\vartheta)=s(\theta),\\ \zeta^\pm\ \in\ P^{VH}_{1\,k}(*_1,\ r(\vartheta))}} \frac{\sqrt{\mu(r(\xi_1^-))\mu(r(\zeta_1^+))}}{\delta}\times$$

$$\begin{array}{ccc} \cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & r(\zeta^+) \\ \widetilde{\zeta_1^+}\downarrow & & \downarrow\widetilde{\vartheta} \\ *_1 & & s(\theta) \\ \zeta_1^+\downarrow & & \downarrow\theta \\ \cdot & \xrightarrow{\xi^+_{[2,\ k+1]}} & r(\xi^+) \end{array}\quad\begin{array}{ccc} \cdot & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \widetilde{\vartheta}\downarrow & & \downarrow\widetilde{\zeta_1^-} \\ s(\theta) & & *_1 \\ \theta\downarrow & & \downarrow\xi_1^- \\ r(\xi^-) & \xrightarrow{\widetilde{\xi^-_{[2,\ k+1]}}} & \cdot \end{array}\quad(\zeta^+,\zeta^-)$$

$$= \sum_{\zeta^\pm\in P^{VH}_{1\,k}(*_1,r(\xi^+))} \frac{\sqrt{\mu(r(\xi_1^-))\mu(r(\zeta_1^+))}}{\delta}\times$$

$$\begin{array}{ccccc} \cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & r(\zeta^+) & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \widetilde{\zeta_1^+}\downarrow & & & & \downarrow\widetilde{\zeta_1^-} \\ *_1 & & & & *_1 \\ \xi_1^+\downarrow & & & & \downarrow\xi_1^- \\ \cdot & \xrightarrow{\xi^+_{[2,\ k+1]}} & r(\xi^+) & \xrightarrow{\widetilde{\xi^-_{[2,\ k+1]}}} & \cdot \end{array}\quad(\zeta^+,\zeta^-)$$

$$= \sum_{\zeta^\pm \in P^{VH}_{1\,k}(*_1,\, r(\xi^+)):\, \zeta_1^+=\zeta_1^-} \frac{\sqrt{\mu(r(\xi_1^-))\mu(r(\zeta_1^+))}}{\delta}\, \delta^{\xi_1^+}_{\xi_1^-} \times$$

$$\begin{array}{ccccc} \cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & r(\zeta^+) & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \downarrow & & & & \downarrow \\ *_1 & & & & *_1 \\ \downarrow & & & & \downarrow \\ \cdot & \xrightarrow{\xi^+_{[2,\ k+1]}} & r(\xi^+) & \xrightarrow{\widetilde{\xi^-_{[2,\ k+1]}}} & \cdot \end{array} (\zeta^+, \zeta^-)\ . \quad \text{(by 2*-flatness)}$$

This proves the assertion. □

Note that, by 2*-flatness, we can take arbitrary edges along the vertical paths (of same lengths).

A.6 The Macro-Cells for Rotation

The only ingredient in definition (A.2.1) of rotation that we are yet to consider is V_k. Recall that, in the path algebra model, - see [JS97], [EK98] - a candidate for the Jones' projection e_k in $N' \cap M_k$ is given by

$$e_k = \begin{cases} \sum_{\xi^\pm \in P_1^V(*_1,-)} \frac{\sqrt{\mu(r(\xi^+))\mu(r(\xi^-))}}{\delta} (\xi^+ \cdot \widetilde{\xi^+},\ \xi^- \cdot \widetilde{\xi^-}), & k=1; \\ \sum_{\substack{\xi\ \in\ P^{VH}_{1\,k-2}(*_1,\ -), \\ \xi^\pm\ \in\ P_1^H(r(\xi),\ -)}} \frac{\sqrt{\mu(r(\xi^+))\mu(r(\xi^-))}}{\delta\ \mu(r(\xi))} (\xi \cdot \xi^+ \cdot \widetilde{\xi^+},\ \xi \cdot \xi^- \cdot \widetilde{\xi^-}), & k \geq 2, \end{cases} \tag{A.16}$$

where the expression for e_1 makes sense because we have a canonical identification between $P_1^V(*_1,-)$ and $P_1^H(*_0,-)$, both as paths as well as intertwiners. We have $E_i = \delta e_i,\ i \geq 1$, then by induction on k, it is readily seen that

$$V_{k+1} \quad = \sum_{\substack{\xi\ \in\ P^{VH}_{1\,k-1}(*_1,\ -), \\ \beta\ \in\ P_1^V(*_1,\ -), \\ \theta\ \in\ P_1^H(r(\xi),\ -)}} \sqrt{\frac{\mu(r(\theta))\mu(r(\beta))}{\mu(r(\xi))}} (\xi \cdot \theta \cdot \widetilde{\theta}, \beta \cdot \widetilde{\beta} \cdot \xi), \tag{A.17}$$

for all $k \geq 0$. Again $\widetilde{\beta}$ and the concatenation $\widetilde{\beta} \cdot \xi$ make sense because of the above identification between $P_1^V(*_1,-)$ and $P_1^H(*_0,-)$.

With all the ingredients at our disposal, we are ready to give the equation for the coefficients of the rotation map.

We get back to the tower (P), and identify the basis $\mathcal{B}_{1\,k-1}$, of $N' \cap M_{k-1}$, with its (horizontal) basis $\mathcal{B}_k := \{(\xi^+,\ \xi^-) : \xi^\pm \in P_k^H(*_0, -) \text{ with } r(\xi^+) = r(\xi^-)\}$. Then with respect to the orthonormal basis $\{\frac{x}{||x||} : x \in \mathcal{B}_k\}$ of $N' \cap M_{k-1}$, the coefficients of the rotation map are given by:

Theorem A.6.1

$$\begin{aligned} &\left\langle R_k\left(\frac{(\xi^+,\xi^-)}{||(\xi^+,\xi^-)||}\right),\ \frac{(\zeta^+,\zeta^-)}{||(\zeta^+,\zeta^-)||}\right\rangle \\ &= \quad \mu(r(\zeta_1^-))\mu(r(\xi_1^+)) \begin{array}{ccccccc} \cdot & \xrightarrow{\widetilde{\zeta_1^-}} & *_0 & \xrightarrow{\zeta^+} & \cdot & \xrightarrow{\widetilde{\xi^-_{[2,\ k]}}} & \cdot \\ \downarrow & & & & & & \downarrow \\ *_1 & & & & & & *_1 \\ \downarrow & & & & & & \downarrow \\ \cdot & \xrightarrow{\xi^+_{[2,\ k]}} & \cdot & \xrightarrow{\widetilde{\xi^-}} & *_0 & \xrightarrow{\xi_1^+} & \cdot \end{array}, \end{aligned} \tag{A.18}$$

for all $(\xi^+,\xi^-),\ (\zeta^+,\zeta^-) \in \mathcal{B}_k$.

Let us first do some calculations related to the definition of the rotation map.

For $(\xi^+,\xi^-) \in \mathcal{B}_{1\,k-1}$, $||(\xi^+,\xi^-)|| = \frac{\sqrt{\mu(r(\xi^+))}}{\delta^{k/2}}$, so

$$\frac{(\xi^+,\xi^-)}{||(\xi^+,\xi^-)||}V_k \quad = \sum_{\beta\,\in\,P_1^V(*_1,-)} \delta^{k/2}\sqrt{\frac{\mu(r(\beta))}{\mu(r(\xi^-_{k-1}))}}\ (\xi^+\cdot\widetilde{\xi_k^-}, \beta\cdot\widetilde{\beta}\cdot\xi^-_{[1,\ k-1]}),$$

and, by Lemma (A.5.1),

$$\begin{aligned} &E_{M'\cap M_k}\left(\frac{(\xi^+,\xi^-)}{||(\xi^+,\xi^-)||}V_k\right) \\ &= \sum_{\beta\,\in\,P_1^V(*_1,-)} \delta^{k/2}\sqrt{\frac{\mu(r(\beta))}{\mu(r(\xi^-_{k-1}))}}\cdot \sum_{\zeta^\pm\,\in\,P_{1\,k}^{V\,H}(*_1,r(\xi^-_{k-1})):\ \zeta_1^+=\zeta_1^-} \\ &\qquad \frac{\sqrt{\mu(r(\zeta_1^+))\mu(r(\beta))}}{\delta}\,\delta_\beta^{\xi_1^+} \begin{array}{ccccc} \cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & \cdot & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \downarrow & & & & \downarrow \\ *_1 & & & & *_1 \\ \downarrow & & & & \downarrow \\ \cdot & \xrightarrow{\xi^+_{[2,\ k]}\cdot\widetilde{\xi_k^-}} & \cdot & \xrightarrow{\widetilde{\xi^-_{[1,\ k-1]}}\cdot\beta} & \cdot \end{array}\ (\zeta^+,\zeta^-) \end{aligned}$$

$$
= \sum_{\zeta^{\pm} \in P^{VH}_{1\,k}(*_1, r(\xi^-_{k-1}))\colon \zeta^+_1 = \zeta^-_1} \delta^{\frac{k}{2}-1}\ \mu(r(\xi^+_1))\sqrt{\frac{\mu(r(\zeta^+_1))}{\mu(r(\xi^-_{k-1}))}}\ \times
$$

$$
\begin{array}{ccccc}
\cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & \cdot & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\
\downarrow & & & & \downarrow \\
*_1 & & & & *_1 \\
\downarrow & & & & \downarrow \\
\cdot & \xrightarrow{\xi^+_{[2,\ k]}\cdot\widetilde{\xi^-_k}} & \cdot & \xrightarrow{\widetilde{\xi^-_{[1,\ k-1]}}\cdot\xi^+_1} & \cdot
\end{array}
\ (\zeta^+, \zeta^-)\ .
$$

Then

$$
V_k E_{M'\cap M_k}\left(\frac{(\xi^+, \xi^-)}{||(\xi^+, \xi^-)||} V_k\right)
$$

$$
= \sum_{\substack{\eta\ \in\ P^{VH}_{1\,k-2}(*_1,\ -),\\ \beta\ \in\ P^V_1(*_1,\ -),\\ \theta\ \in\ P^H_1(r(\eta),\ -)}} \sqrt{\frac{\mu(r(\theta))\mu(r(\beta))}{\mu(r(\eta))}}(\eta\cdot\theta\cdot\widetilde{\theta}, \beta\cdot\widetilde{\beta}\cdot\eta)\ \times
$$

$$
\sum_{\zeta^{\pm} \in P^{VH}_{1\,k}(*_1, r(\xi^-_{k-1}))\colon \zeta^+_1 = \zeta^-_1} \delta^{\frac{k}{2}-1}\ \mu(r(\xi^+_1))\sqrt{\frac{\mu(r(\zeta^+_1))}{\mu(r(\xi^-_{k-1}))}}\ \times
$$

$$
\begin{array}{ccccc}
\cdot & \xrightarrow{\zeta^+_{[2,\ k+1]}} & \cdot & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\
\downarrow & & & & \downarrow \\
*_1 & & & & *_1 \\
\downarrow & & & & \downarrow \\
\cdot & \xrightarrow{\xi^+_{[2,\ k]}\cdot\widetilde{\xi^-_k}} & \cdot & \xrightarrow{\widetilde{\xi^-_{[1,\ k-1]}}\cdot\xi^+_1} & \cdot
\end{array}
\ (\zeta^+, \zeta^-)
$$

$$
= \sum_{\substack{\eta\ \in\ P^{VH}_{1\,k-2}(*_1,\ r(\xi^-_{k-1})),\\ \theta\ \in\ P^H_1(r(\eta),\ -),\\ \zeta^-\ \in\ P^{VH}_{1\,k}(*_1,\ r(\xi^-_{k-1}))}} \delta^{\frac{k}{2}-1}\frac{\mu(r(\beta))\mu(r(\xi^+_1))\sqrt{\mu(r(\theta))}}{\mu(r(\xi^-_{k-1}))}\ \times
$$

$$\begin{array}{ccccc} \cdot & \xrightarrow{\widetilde{\zeta_1^-}\cdot\eta} & \cdot & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \downarrow & & & & \downarrow \\ *_1 & & & & *_1 \\ \downarrow & & & & \downarrow \\ \cdot & \xrightarrow{\xi^+_{[2,\ k]}\cdot\widetilde{\xi_k^-}} & \cdot & \xrightarrow{\widetilde{\xi^-_{[1,\ k-1]}}\cdot\xi_1^+} & \cdot \end{array} \ (\eta\cdot\theta\cdot\widetilde{\theta},\ \zeta^-).$$

Proof of Theorem A.6.1: For $(\xi^+,\xi^-)\in\mathcal{B}_{1\,k-1}$, we have

$$\begin{aligned} & R_k\left(\frac{(\xi^+,\xi^-)}{||(\xi^+,\xi^-)||}\right) \\ = \ & \delta^2 E_{N'\cap M_{k-1}}\left(V_k E_{M'\cap M_k}\left(\frac{(\xi^+,\xi^-)}{||(\xi^+,\xi^-)||}V_k\right)\right) \\ = \ & \delta^2 \sum_{\substack{\eta\ \in\ P^{VH}_{1\,k-2}(*_1,\ r(\xi^-_{k-1})),\\ \theta\ \in\ P^{H}_{1}(r(\eta),\ -),\\ \zeta^-\ \in\ P^{VH}_{1\,k}(*_1,\ r(\xi^-_{k-1}))}} \delta^{\frac{k}{2}-1}\frac{\mu(r(\zeta_1^-))\mu(r(\xi_1^+))\sqrt{\mu(r(\theta))}}{\mu(r(\xi^-_{k-1}))} \ \times \\ & \begin{array}{ccccc} \cdot & \xrightarrow{\widetilde{\zeta_1^-}\cdot\eta} & \cdot & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \downarrow & & & & \downarrow \\ *_1 & & & & *_1 \\ \downarrow & & & & \downarrow \\ \cdot & \xrightarrow{\xi^+_{[2,\ k]}\cdot\widetilde{\xi_k^-}} & \cdot & \xrightarrow{\widetilde{\xi^-_{[1,\ k-1]}}\cdot\xi_1^+} & \cdot \end{array} \ \frac{\mu(r(\xi^-_{k-1}))}{\delta\mu(r(\theta))}\ \delta^{\widetilde{\theta}}_{\zeta^-_{k+1}}\ (\eta\cdot\theta,\ \zeta^-_{[1,\ k]}) \\ = \ & \sum_{\substack{\eta\ \in\ P^{VH}_{1\,k-2}(*_1,\ r(\xi^-_{k-1})),\\ \zeta^-\ \in\ P^{VH}_{1\,k}(*_1,\ r(\xi^-_{k-1}))}} \delta^{k/2}\frac{\mu(r(\zeta_1^-))\mu(r(\xi_1^+))}{\sqrt{\mu(r(\zeta_k^-))}} \ \times \\ & \begin{array}{ccccc} \cdot & \xrightarrow{\widetilde{\zeta_1^-}\cdot\eta} & \cdot & \xrightarrow{\widetilde{\zeta^-_{[2,\ k+1]}}} & \cdot \\ \downarrow & & & & \downarrow \\ *_1 & & & & *_1 \\ \downarrow & & & & \downarrow \\ \cdot & \xrightarrow{\xi^+_{[2,\ k]}\cdot\widetilde{\xi_k^-}} & \cdot & \xrightarrow{\widetilde{\xi^-_{[1,\ k-1]}}\cdot\xi_1^+} & \cdot \end{array} \ \frac{\sqrt{\mu(r(\zeta_k^-))}}{\delta^{k/2}}\ \frac{(\eta\cdot\widetilde{\zeta^-_{k+1}},\ \zeta^-_{[1,\ k]})}{||(\eta\cdot\widetilde{\zeta^-_{k+1}},\ \zeta^-_{[1,\ k]})||}. \end{aligned}$$

Thus, identifying the basis $\mathcal{B}_{1\,k-1}$ with $\mathcal{B}_k$, we obtain equation (A.18).

$\square$

Conjecture: For any two horizontal loops ξ and η on H_0, passing through $*_0$ and based at arbitrary vertices, say π_1 and π_2, respectively, and of same lengths , say k, we claim that:

$$\begin{array}{ccc} \pi_1 & \xrightarrow{\xi} & \pi_1 \\ \downarrow & & \downarrow \\ *_i & & *_i \\ \downarrow & & \downarrow \\ \pi_2 & \xrightarrow{\eta} & \pi_2 \end{array} = \frac{\mu(r(\xi_i))\,\mu(r(\eta_i))}{\mu(\pi_1)\,\mu(\pi_2)} \quad \begin{array}{ccccc} r(\xi_i) & \xrightarrow{\xi_{[i+1,\ k]}} & \pi_1 & \xrightarrow{\xi_{[1,\ i]}} & r(\xi_i) \\ \downarrow & & & & \downarrow \\ *_j & & & & *_j \\ \downarrow & & & & \downarrow \\ r(\eta_i) & \xrightarrow{\eta_{[i+1,\ k]}} & \pi_2 & \xrightarrow{\eta_{[1,\ i]}} & r(\eta_i) \end{array} \quad ,$$

for all $2 \leq i \leq k-1$, where $*_i, *_j$ are the $*$ vertices occurring in the vertical graphs passing through $s(\xi)$ and $s(\xi_{i+1})$, respectively.

It can be easily checked that this conjecture is true, at least in the case of the group subfactor $R^G \subset R$, and the dual-group subfactor $R \subset R \rtimes G$, for a finite group G acting outerly on the hyperfinite II_1-factor R.

One immediate consequence of the above conjecture is that, we can directly prove that R_k, as given in Theorem (A.6.1), is of order k.

Index

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