D-branes on Landau-Ginzburg orbifolds and Calabi-Yau spaces

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DECLARATION

I declare that the thesis entitled *D-branes on Landau-Ginzburg orbifolds and Calabi-Yau spaces*, submitted by me for the Degree of Doctor of Philosophy is the record of work carried out by me during the period from April 2002 to July 2005 under the guidance of Prof. T. Jayaraman and has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

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Abstract

In this thesis we have tried to identify the objects that correspond, at large volume, to the fractional 2-branes in $\mathbb{C}^n/\mathbb{Z}_N$ orbifolds.

We start with the fractional 2-branes in the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. One may easily construct such objects in the orbifold theory. The D-brane charges of these objects in the large volume basis can be computed by first determining the intersection of these fractional two-branes with the fractional zero-branes and with themselves. This can be easily done at the orbifold point. Subsequently, using these intersection numbers we can determine the large-volume charges of these fractional two-branes.

It turns out that these fractional two-branes have a 'fractional' first Chern class. Since the appearance of a 'fractional' first Chern class is somewhat surprising, we also show that these fractional two-branes have integer charges if we consider them as objects living in the ambient non-compact CY (the resolution of the orbifold), which is a line bundle over a projective base, rather than on the projective space itself.

In this thesis we also note a strong parallel between the relation of the fractional two-branes (and more generally fractional $2p$-branes) in the orbifold theory to the corresponding coherent sheaves at large volume and the notion of the quantum McKay correspondence due to Martinec and Moore [40]. While there is nothing quantum about the relation in our setting where space-time supersymmetry is not broken, nevertheless the fractional two-branes in $\mathbb{C}^n/\mathbb{Z}_N$ appear closely related to the fractional zero-branes of $\mathbb{C}^{n-1}/\mathbb{Z}_N$, the latter being the geometry associated with the non-supersymmetric B-type branes in the work of Martinec and Moore. In the full set of coherent sheaves that correspond to the quantum $\mathbb{Z}_N$ orbit of the fractional two-branes at large volume we are able to find the analogues of the so-called 'Coulomb branes' that they describe.

This fractional first Chern class ensures that in the case of fractional 2-branes of the $\mathbb{C}^5/\mathbb{Z}_5$ orbifold, when the large volume analogue of these objects are restricted to the CY hypersurface, one of them precisely becomes a zero-brane on the CY hypersurface. One may note that if we began with objects that have integer

first Chern class on the ambient projective space space then we would always obtain d zero-branes on the CY hypersurface, where d is the degree of the polynomial equation describing the CY.

Finally we also identify the CFT description of states of the LG model that correspond to fractional two-brane and four-brane states in the ambient non-compact orbifold that are restricted to the CY hypersurface. This turns out to be a sub-class of the B-type permutation branes of the Gepner models [41].

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Chapter 1

Introduction

Over the past two decades string theory [1][2] has emerged as the leading candidate for a quantum theory of gravity and possibly the sole framework for the unification of all the known fundamental forces. Strings are relativistic one dimensional objects with two distinct topological configurations - the closed loop (closed string) and the open interval (open string).

1.1 Closed strings

The action of a free string in flat space is a generalisation of the action of a point particle and is given by the invariant area of the world-sheet that the string sweeps out as it moves in spacetime. This is called the Nambu-Goto action.

$$S = \int d^2\sigma \sqrt{|-\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu|} \tag{1.1.1}$$

where the term inside the square root is the determinant of the induced metric on the world-sheet. Here σ^α are coordinates on the world-sheet and $X^\mu(\sigma^1, \sigma^2)$ denote the coordinates of spacetime. One usually does not work with this action since it is a non-polynomial action and is difficult to quantize. A better starting point is the Polyakov action,

$$S' = \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{1.1.2}$$

where one introduces an auxiliary field $g_{\alpha\beta}$ which is the metric on the world-sheet. The Polyakov action is a two dimensional action, and from the two dimensional point of view the spacetime coordinates are scalar fields with the Lorentz group acting like an internal symmetry group. One can get back the Nambu-Goto action from the Polyakov action by solving the equations of motion for $g_{\alpha\beta}$. Now this action is invariant under local Weyl transformations $g'_{\alpha'\beta'} = e^{\phi(\sigma)} g_{\alpha\beta}$ as well as under world-sheet coordinate reparametrization. As in any gauge theory one can fix these local invariances by choosing a gauge. One possible gauge choice is the conformal gauge in which $g_{\alpha\beta} = \delta_{\alpha\beta}$. In this gauge the Polyakov action becomes the action of D free scalar fields in two dimensions, where D is the spacetime dimension. The Weyl and reparametrization invariances do not get completely fixed and there remains a residual conformal invariance. Thus, in this gauge, we are left with a 2-dimensional conformal field theory of free scalar fields, which is most naturally written in complex coordinates,

$$S = \int d^2z \partial_z X^\mu \partial_{\bar{z}} X^\nu \eta_{\mu\nu} \tag{1.1.3}$$

where $z, \bar{z}$ are complex coordinates on the world-sheet.

Conformal invariance is the invariance under any holomorphic change of coordinates $z' = f(z)$ and $\bar{z}' = \bar{f}(\bar{z})$. These transformations are generated by the vector fields $L_n = z^{n+1}\partial_z$ and similarly for the anti-holomorphic transformations $\bar{L}_n = \bar{z}^{n+1}\partial_{\bar{z}}$. They satisfy the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} \tag{1.1.4}$$

$$\text{with} \qquad [L_n, \bar{L}_m] = 0 \tag{1.1.5}$$

The left moving and right moving modes of the strings correspond to two independent sectors. In the quantum theory, typically the above algebra gets modified by a central extension which is of the form

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0} \tag{1.1.6}$$

c is a number called the central charge of the conformal field theory and the presence of this central extension term implies that the classical symmetry is anomalous in

the quantum theory. Just as in any gauge theory the local invariances of the theory are crucial for the decoupling of the negative norm states, that one has due to the indefinite signature of the Lorentzian metric in target space, and therefore to have a unitary quantum theory.

The value of the central charge depends on the particular theory. For instance each scalar field theory upon quantization will give rise to a central charge $c = 1$. Further, a proper Fadeev-Popov quantization leads to a ghost term, which corresponds to a CFT of central charge $c = -26$. So together we get a CFT of net central charge $c = D - 26$. Then the condition for conformal invariance implies that we restrict to $D = 26$. Thus string theory quantized on flat space exists as unitary quantum theory only when the total spacetime dimension is 26.

One can generalise this to situations where the background metric is curved. In this case the action in the conformal gauge turns out to be

$$S = \int d^2z \partial_z X^\mu \partial_{\bar{z}} X^\nu G_{\mu\nu}(X) \tag{1.1.7}$$

The first point to notice about the action is that it corresponds to a 2-dimensional interacting field theory called the non linear sigma model(NLSM), unlike the flat space case where the 2-dimensional theory was a free field theory . Like in any interacting quantum field theory the couplings of the theory, which from the target space point of view are spacetime fields, flow with the change of the world-sheet scale, and the flow is governed by the beta function equations. The theory is conformally invariant for values of the couplings for which the beta function vanishes. Remarkably these equations are the spacetime equations of motion for the corresponding fields and in particular for the background metric $G_{\mu\nu}$, $\beta_G = 0$ is the vacuum Einstein equation (where β_G is the beta function for the coupling $G_{\mu\nu}$). Thus, in the classical limit, string theory reproduces general relativity. More generally one can consider string propagation in the presence of other background fields. In this case one gets Einstein's equation sourced by the energy-momentum tensor $T_{\mu\nu}$ of these fields, and also the equations of motion of these other background fields.

The spectrum of string theory in flat space contains a massless spin two particle. Kinematically this has the right property to be a graviton, the quantum excitation of gravity. However to identify it with the graviton this is not enough and one has to look at the dynamical properties of the gravitons. If one computes the 3-point and 4-point scattering amplitudes of these massless spin two fields in string theory, then to lowest order it reproduces the interactions of gravitons in perturbative general relativity. Thus even dynamically these spin two fields have the properties of the gravitons. The other massless states of the spectrum are the scalar dilaton and the two form antisymmetric field $B_{\mu\nu}$.

The theory we have been considering till now is known as the bosonic string. It is not realistic since it does not include any fermions. Moreover an analysis of the spectrum reveals that the ground state of the bosonic string theory in $D = 26$ is tachyonic. Now in field theory the presence of a tachyonic excitation implies that the vacuum around which one is doing perturbation is unstable. Clearly free string theory on flat space being unstable is not a very desirable property.

Both these defects (absence of fermions and presence of tachyons in flat space) can be remedied by making the theory supersymmetric. We will not go into the details of the action except to note that because of supersymmetry we now have fermions in the world-sheet theory. This introduces different sectors, the Ramond (R) sector and the Neveu-Schwarz (NS) sector in the theory corresponding to the periodic and anti-periodic boundary conditions that can be imposed on the world-sheet fermions. The R sector ground state is a space time fermion. Therefore there are four sectors in the full theory obtained by taking a tensor product of states in the holomorphic and anti-holomorphic sectors. These are the RR, NSNS, NS-R and R-NS sectors. The RR and NSNS sector states are bosonic and the R-NS and NS-R states are spacetime fermions. Spacetime supersymmetry is obtained by imposing a (GSO) projection on the states which basically removes the tachyon and leads to matching of spacetime fermionic and bosonic states at each mass level. The ghost structure of the theory also changes, due to the introduction of the superghosts, the supersymmetric partners of the original ghosts. Each free fermion theory is

a CFT of central charge $\frac{1}{2}$. The ghosts and superghosts now contribute a total $c_{gh} = -15$. Then the condition of anomaly cancellation is $D + \frac{1}{2}D - 15 = 0$. So we have $D = 10$. The spectrum of superstring theories has, like the bosonic theory, a finite number of massless states and an infinite number of massive states. In the supersymmetric theory there are more massless states than the bosonic theory. In the bosonic theory the massless states were the scalar dilaton, the graviton and the two form antisymmetric field $B_{\mu\nu}$. All higher form fields were massive. In the supersymmetric case there are higher form massless states. These fields are sourced by higher dimensional objects called D-branes, which we will describe later.

Interactions in the theory are analogous to the higher loops in the point particle theory. In string theory the loops are replaced by higher genus Riemann surfaces with an extra handle for every additional loop. Unlike the point particle case where there are several diagrams at any given loop, in string theory due to conformal invariance there is essentially a unique diagram at each loop. Further the string coupling (g_s) which is the parameter which controls the perturbation expansion is not an independent parameter, but rather is given by the vacuum expectation value of the dilaton $g_s \sim e^{<\phi>}$. The scattering matrix for n external fields involves a sum over Riemann surfaces with different number of handles. To define the measure for the path-integral for Riemann surfaces with handles, one has to mod out by the diff $\times$ Weyl invariances. In the case of torus (1-loop) diagram the inequivalent metrics are labelled by a complex parameter τ called the modulus. The domain of integration for τ is the complex plane quotiented by the so-called modular transformations, $\tau' = \frac{a\tau+b}{c\tau+d}$, where $ad - bc = 1$. Invariance under modular transformation is a very important property of string amplitudes which has no point particle analogue, and it makes string amplitudes finite, removing the divergences that would arise in a point-particle theory. This is because, due to this invariance, the integration region in the parameter space is cutoff and the badly behaved points which results in divergences are removed. Similar considerations are applicable for all genus g Riemann surfaces.

So we see that not only does string theory reproduce classical gravity, it

also describes the dynamics of small quantum gravity fluctuations around a classical background giving finite results. Moreover it has also been shown that for certain class of black holes, string theory can correctly reproduce black hole entropy from a statistical counting of states[1]. These are strong pieces of evidence suggesting that string theory is a well-defined quantum theory of gravity.

Perturbatively there are five different types of consistent superstring theories in flat spacetime with $D = 10$. Depending on the amount of supersymmetry they preserve and their spectrum, they are type IIA ,type IIB, type I and the heterotic $SO(32)$ and $E_8 \times E_8$. We will not review the details of their construction here, but simply note some of their properties. The type II theories have $N = 2$ spacetime supersymmetry while the type I and the heterotic theories have $N = 1$ supersymmetry. The spectrum of these theories are also different. While the type IIA theory has non-chiral spacetime fermions all the others have chiral fermions. The type I strings have both open and closed strings and has a gauge group $SO(32)$, while the two heterotic theories have $SO(32)$ and $E_8 \times E_8$ gauge groups respectively.

Non-perturbatively these five different string theories turn out to be related to each other and to another eleven dimensional theory, whose low energy limit is eleven dimensional supergravity, via various dualities, including S-duality which is essentially a non-perturbative duality and T-duality which is a perturbative duality which we will review in the next section.

Till now we have been discussing the case of string propagation in 10 or 26 flat spacetime directions. A more realistic solution would be one where the target space is of the form $M_4 \times X_6$. M_4 being the four dimensional Minkowski spacetime and X_6 being a compact manifold of small size. Before describing such realistic compactifications, we will briefly discuss the string spectrum on a circle, to differentiate the stringy features from that of a point particle.

[1]We will not review these results here as they are not directly related to work done in this thesis

1.1.1 String theory on a circle

If one of the spacetime coordinates is compactified to a circle, then one has a free field theory with periodic boundary conditions on the scalar field. This is a $c = 1$ CFT for any value of the radius of the circle. Parameters such as this radius which, when changed, do not change the central charge of the CFT are called the moduli of the CFT (these are distinct from the moduli of the world-sheet Riemann surface referred to earlier). The momentum of a quantum particle on a circle is quantized due to the condition of single valuedness of the wavefunction and is given by $p = \frac{n}{R}$, where n is an integer and R is the radius of the circle. However a string can have potential energy stored in it by winding non trivially around the circle. This energy will be proportional to the length of the string. Finally if it winds w times around the circle then the energy which will contribute to the mass m of a string state

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{1.1.8}$$

$$\text{with} \quad nw + N - \tilde{N} = 0 \tag{1.1.9}$$

where α' is the string tension and N, $\tilde{N}$ are the excitation levels of the left and right moving sectors of the string. From the above formula it is clear that the spectrum of the string is invariant under interchange of $n \leftrightarrow m$ and simultaneously taking $R \to R' = \frac{\alpha'}{R}$. Therefore the spectrum of a string compactified on a circle is invariant under $R \leftrightarrow \frac{\alpha'}{R}$. This invariance is known as T-duality.

The massless state at a generic point of the moduli space i.e at a arbitrary value of the radius is given by the state $n = m = 0$ and $N = \tilde{N} = 1$. At an arbitrary point the theory has a $U(1) \times U(1)$ symmetry. However at $R = \sqrt{\alpha'}$ there are more massless states

$$n = w = \pm 1, N = 0, \tilde{N} = 1, \qquad n = -w = \pm 1, N = 1, \tilde{N} = 0 \tag{1.1.10}$$

$$n = \pm 2, w = N = \tilde{N} = 0, \qquad w = \pm 2, n = N = \tilde{N} = 0 \tag{1.1.11}$$

It can be shown that at this value of the radius the theory has an $SU(2) \times SU(2)$ symmetry. Therefore at the point $R = \sqrt{\alpha'}$ there is a symmetry enhancement. This discussion can be generalised to toroidal compactifications in higher dimension.

1.2 String theory on orbifolds

In this thesis we will be discussing issues related to string theory on orbifolds as well as Calabi-Yau manifolds. Unlike string theory with toroidal compactifications, these compactifications reduce the target space supersymmetries. This feature is phenomenologically very important. We will therefore review string theory on orbifolds as well as Calabi-Yau manifolds.

Consider the quotient space $\mathbb{C}/\mathbb{Z}_n$ where $\mathbb{C}$ is the complex plane and the elements of the group $\mathbb{Z}_n$ are $1, \omega, ..., \omega^{n-1}$, with ω being the n'th root of unity. This is a conical space with angle $\theta = \frac{2\pi}{n}$. To describe closed string theory in this target space, the quotienting by $\mathbb{Z}_n$ is implemented as a twisted boundary condition,

$$Z(0,\tau) = \omega^l Z(2\pi,\tau), \qquad (l = 0, 1, ...n-1) \tag{1.2.1}$$

where $l = 0$ is the usual periodic boundary condition and $l = 1, ...n-1$ are the twisted boundary conditions corresponding to the identifications after rotation by elements of $\mathbb{Z}_n$. Each boundary condition gives rise to a different sector of the theory. Now it can be shown that the requirement of modular invariance of the partition function in this case implies that all these different sectors (twisted + untwisted) be added. i.e., the full partition function is

$$Z = Z_{untwisted} + \sum_{i=1}^{n-1} Z_{i'th\ twisted\ sector} \tag{1.2.2}$$

In this thesis we will be interested primarily in the case of the $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds. The orbifold action on the coordinates is defined as follows,

$$Z^1 \to \omega^k Z^1,\ Z^2 \to \omega^l Z^2,\ Z^3 \to \omega^m Z^3 \tag{1.2.3}$$

where ω is the n'th root of unity, $Z^j = X^{2j} + iX^{2j-1}$ $j = 1$ to 3 are the complex coordinates. If $k + l + m = 0 \bmod(n)$ then the $\mathbb{Z}_n$ group is a discrete subgroup of SU(3) and the resulting four dimensional theory is supersymmetric. The spectrum of the theory includes the untwisted sector and the twisted sectors and in each sector one looks for GSO invariant as well as $\mathbb{Z}_n$ invariant states. Orbifolds of this kind can be thought of as singular limits of some smooth "Calabi-Yau" manifolds to which we turn to now.

1.3 String theory on Calabi-Yau Manifolds

It is well-known that if one wants to have $N = 1$ supersymmetry in $D = 4$ in heterotic string theory, then the compact manifold X_6 must be a "Calabi-Yau" manifold. In type IIB theories, compactification on Calabi-Yau manifolds leads to an $N = 2$ theory in $D = 4$. The supersymmetry can be further reduced to a $N = 1$ $D = 4$ world-volume theory on a D-brane extended along the four dimensional non-compact spacetime. Therefore the study of string theory on Calabi-Yau manifolds is of physical significance.

A $2d$ dimensional Calabi-Yau manifold is a complex, Kähler manifold which has $SU(d)$ holonomy. We are of course interested in the case of $d = 3$. To introduce[2] the notion of a Kähler manifold one has to first introduce the notion of a Hermitian metric on a complex manifold. The components of a metric on a complex manifold can be classified as $(g_{ij}, g_{\bar{i}\bar{j}}, g_{i\bar{j}}, g_{\bar{i}j})$. Here the (i, j) run over the d complex coordinates and likewise the $(\bar{i}, \bar{j})$ run over the complex conjugate coordinates. Symmetry and reality conditions imply $g_{ij} = g_{ji}$ and $\bar{g}_{ij} = g_{\bar{i}\bar{j}}$, $\bar{g}_{i\bar{j}} = g_{i\bar{j}}$ A metric is called Hermitian if $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Given a Hermitian metric one can define a form of type (1,1) as follows,

$$J = ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}. \tag{1.3.1}$$

If J is closed i.e $dJ = 0$ then J is called a Kähler form and the manifold is called a Kähler manifold. It can be shown that the condition of $SU(d)$ holonomy is equivalent to demanding that the manifold admits a Ricci flat metric. Calabi conjectured that a complex Kähler manifold with vanishing first Chern class c_1 admits a unique Ricci flat metric. This theorem was proved later by Yau. This theorem is very useful because while it is hard to explicitly check whether a metric is Ricci flat, it is easy to check whether a given manifold has a vanishing first Chern class.

A particularly useful class of CY manifolds is that of algebraic CY manifolds, where the manifold is the zero locus of r complex polynomial equations in d complex dimensional space (that is typically a weighted projective space) such that

[2]The discussion presented here is based on [7]

$d - r = 3$. The polynomials G_i must obey a suitable 'transversality condition'. The primary example of a Calabi-Yau manifold that we will be dealing with in the thesis is the quintic hyper-surface in $\mathbb{P}^4$, which is given by the equation $G = \sum_{i=1}^{5} z_i^5 = 0$, where the Z_i are the homogeneous coordinates of $\mathbb{P}^4$.

Given a Calabi-Yau manifold with Ricci flat metric g, one can ask, what metric perturbations δg will preserve the condition of Ricci flatness? i.e. if $R_{i\bar{j}}(g) = 0$ then what are the perturbations such that $R_{i\bar{j}}(g + \delta g) = 0$? These questions were studied in detail in [8]. There are two possible types of metric deformations - δg_{ij}, $\delta g_{i\bar{j}}$ and their complex conjugates. The first type preserves the Hermitian nature of the metric while the other deformation does not preserve the original Hermitian structure. Demanding $R_{ij}(g + \delta g) = 0$ imposes restrictions on the metric. It turns out that the two types of deformations are associated with the cohomology groups of the manifold. The first type of deformations are associated to elements of the cohomology group $H_{\bar{\partial}}^{1,1}$ while the second type are associated to elements of $H_{\bar{\partial}}^{2,1}$. The deformations δg_{ij} are called the Kähler moduli while the deformations $\delta g_{i\bar{j}}$ are called complex structure moduli, because it can be shown that the new metric after such deformations are Hermitian with respect to a new complex structure. This means that the new coordinates with respect to which the new metric is Kähler cannot be obtained by a holomorphic coordinate transformation from the old coordinates. The complex structure moduli are simply given in the algebraic case as all possible deformations of the original equation by monomials of the same degree as the original equation.

If we consider a NLSM with Calabi-Yau manifold as the target space in type IIB string theory, then at the fixed point the world-sheet theory has $\mathcal{N} = (2, 2)$ superconformal symmetry. The (2,2) supersymmetry condition is satisfied if the target space is complex Kähler, while the conformal invariance comes from the condition of Ricci flatness as reviewed before. Gepner [10] showed how to construct more string compactifications with world-sheet $\mathcal{N} = (2, 2)$ superconformal symmetry based on exact CFTs without invoking a NLSM construction.

In the next section we will briefly highlight some features of the construc-

tion of these models and their relation to the Calabi-Yau compactifications.

1.3.0.1 Gepner models, LG orbifolds and GLSM

The Gepner construction is based on a tensor product of n unitary superconformal CFTs known as the superconformal minimal models. These $N = (2,2)$ minimal models are classified by their central charges $c_i = \frac{3k_i}{k_i+2}$. The total central charge of the model is therefore given by

$$c = \sum_{i=1}^{n} \frac{3k_i}{k_{i+2}}$$

An $N = 2$ superconformal algebra is generated by the stress energy tensor $T(z)$, the superconformal generators $G^+(z)$, $G^-(z)$ and a $U(1)$ generator $J(z)$. The explicit algebra in terms of the modes of these operators and also a more detailed review of Gepner models is given chapter 7

For a compactification to four dimensions one has to have an internal conformal field theory of $c = 9$, since the four dimensional CFT corresponding to spacetime will contribute $c = 6$ and $c_{gh} = -15$. However to get a realistic string theory it is not enough to have the right central charge. One has to also have a total odd $U(1)$ charge separately for the right as well left movers. This means that the sum of the internal $U(1)$ charges and the $U(1)$ charges of the superfields associated with the non-compact directions must be odd. This imposes the generalised GSO projections, thus leading to a spacetime supersymmetric spectrum. Further a state in the Gepner model must be a product of NS states from each sub-sector or of Ramond (R) states of each sub-sector. Finally of course we also need to construct a modular invariant partition function.

The minimal models have a Lagrangian representation in terms of a Landau-Ginzburg theory. This is a (2,2) world-sheet theory involving a single chiral superfield with a superpotential term

$$W(\Phi) = \Phi^{k+2} \tag{1.3.2}$$

for the k'th minimal model with central charge $c = \frac{3k}{k+2}$. This massive theory flows

in the IR regime to a fixed point. The critical theory is to be identified with the minimal model at level k with central charge $c_k = \frac{3k}{k+2}$.

The related Lagrangian description for the Gepner model is the Landau-Ginzburg orbifold. This is obtained by considering n chiral superfields (notation as in refs. [43, 44, 18]) Φ_i with a $\mathbb{Z}_K$ action ($K \equiv \mathrm{lcm}(k_1, \ldots, k_n)$):

$$\mathbb{Z}_K : \quad \Phi_i \to \omega^{Q_i}\, \Phi_i \quad ,$$

where $\omega = \exp(i2\pi/K)$ is a K-th root of unity and $Q_i \equiv K/(k_i + 2)$. The model has a quasi-homogeneous superpotential given by

$$G(\Phi) = \sum_{i=1}^{n} \Phi^{k_i+2} \ . \tag{1.3.3}$$

We will be focusing mainly on the case when $n = m + 2$ for which $K = \sum_i Q_i$.

We have mentioned above two seemingly different compactifications with N=(2,2) world-sheet supersymmetry. The NLSM with a Calabi-Yau target space and an abstract conformal field theory, the Gepner models or equivalently the LG orbifolds. Gepner argued [10] that these seemingly different compactifications are actually related. He conjectured that these abstract conformal field theory compactifications were in the same moduli space as the geometric compactifications on Calabi-Yau manifolds. In particular the Gepner model given by a product of five minimal models each of central charge $c = \frac{9}{5}$ and $k = 3$ is in the same moduli space as the NLSM for the quintic hypersurface in $\mathbb{P}^4$.

A clear physical understanding of the correspondence between the Landau-Ginzburg theory and the Calabi-Yau sigma models may be obtained via Witten's construction of the gauged linear sigma model (GLSM) [43]. We will review the GLSM construction [43] below for the example of the quintic hyper-surface on $\mathbb{P}^4$.

1.3.1 GLSM for the quintic hyper-surface on $\mathbb{P}^4$

The GLSM is a $d = 2$, $N = 2$ supersymmetric $U(1)$ gauge theory coupled to a set of chiral superfields. The action is given by

$$S = S_{kinetic} + S_W + S_{gauge} + S_{r,\theta} \tag{1.3.4}$$

Where the first term is the kinetic term for the chiral fields and the S_{gauge} is the kinetic term for the gauge fields. We will be needing only the explicit forms of S_W and $S_{r,\theta}$. In superfield notation it is

$$S_W = \int d^2y d\theta^+ d\theta^- W(P, \Phi_1, ...\Phi_5) \tag{1.3.5}$$

where W is the superpotential term. The $U(1)$ charges of the Φ_i and P fields are (1,...,1,-5), and

$$W = P{\cdot}G(\Phi_1, ..., \Phi_5). \tag{1.3.6}$$

where $G \equiv \sum_{i=1}^{5} \Phi_i^5$. $S_{r,\theta}$ is the combination, Fayet-Iliopoulos D-term and θ term.

$$S_{r,\theta} = -r \int d^2y(-rD + \frac{\theta}{2\pi} v_{01}) \tag{1.3.7}$$

where D is a real auxiliary field appearing in the vector superfield and v_{01} is the antisymmetric gauge field strength. We will be interested in studying the structure of classical ground states of this theory.

For this reason we will be interested in the bosonic potential which is given by

$$U = |G(\phi_i)|^2 + |p|^2 \sum_i |\frac{\partial G}{\partial \phi_i}|^2 + \frac{1}{2e^2} D^2 + 2|\sigma|^2 (\sum_i |\phi_i^2| + 25|p|^2) \tag{1.3.8}$$

$$\text{with} \quad D = -e^2 (\sum_i |\phi_i|^2 - 5|p|^2 - r) \tag{1.3.9}$$

Here σ is a scalar field appearing in the twisted chiral multiplet, associated with the gauge sector. Note that fields with lower case letters are scalar components of the superfields appearing with corresponding capital letters.

For $r > 0$, the susy minimum $U = 0$ is given by the solution not all $\phi_i = 0$ and $p = \sigma = 0$ with the constraint

$$\sum_i |\phi_i|^2 = r \tag{1.3.10}$$

Also due to the $U(1)$ gauge invariance of the theory we have

$$(\phi_1, ..., \phi_5) \sim (e^{i\theta}\phi_1, ..., e^{i\theta}\phi_5) \tag{1.3.11}$$

The above two equations imply that the fields ϕ_i are coordinates of the quintic hyper-surface $G = 0$ in $\mathbb{P}^4$, which is a Calabi-Yau manifold. In the linear sigma model there are other fields not present in the Calabi-Yau NLSM. When we consider the quantum field theory about this classical ground state then these other fields will have masses depending on r. However if we consider the limit $r \gg 0$ then these other fields become very massive and drop out and so we recover the effective quantum field theory of the Calabi-Yau NLSM. As we flow to the IR, r will get renormalised and its value at the IR fixed point $\tilde{r}$ will be the actual Kähler modulus of the Calabi-Yau in the NLSM. If we flow to the IR point then any non zero mass dependence will drop out and the theory flows to a conformal fixed point of the NLSM for finite values of r as well.

We can similarly argue for the case of $r < 0$. In this case we find that for the ground state, all the $\phi_i = 0$ and $p = \sqrt{\frac{-r}{5}}$. Further there is an unbroken $\mathbb{Z}_5$ symmetry, because p has charge -5. Moreover from the form of the potential the fields ϕ_i are massless fluctuations around this unique vacuum. A theory with a unique vacuum and some massless states is a LG theory (actually an LG orbifold since the p-field which acquires a vev has a charge -5). As earlier the theory actually describes the effective quantum field theory of LG orbifolds for $|r| \gg 0$, because of suppression of the massive fluctuations. In the IR, the theory will flow to the fixed point of the LG orbifold for any value of r

Thus we see that the LG theory as well as the Calabi-Yau (CY) NLSM can be thought of as two 'phases' of the GLSM with the interpolation given by varying $r \gg 0$ to $r \ll 0$. Even though we talked only of the r moduli space, the presence of the antisymmetric tensor field in string theory makes the actual moduli space complex. In the GLSM this follows from the θ term with the complex modulus given by $t = \frac{\theta}{2\pi} + ir$. It can be shown that one can move from one 'phase' to another in this complexified moduli space smoothly. Even though in terms of r it seems that one has to go to negative values of r to reach the LG theory, it can be shown [9], [7] that the physical radius $\tilde{r}$ is always non-negative. Thus the LG theory can be thought of as the analytic continuation of the CY NLSM to small but positive

values of the physical Kähler parameter. We shall sometimes refer to the $r \gg 0$ 'phase' as the large volume phase and the $r \ll 0$ 'phase' as a non-geometric phase.

This analysis in terms of the GLSM makes the relationship between CY NLSM and the LG orbifold description evident. They belong to different phases of a larger theory -the GLSM and one can interpolate between them via non-conformal theories.

1.3.2 GLSM for the $\mathbb{C}^3/\mathbb{Z}_3$

One can construct a linear sigma model for orbifolds. For instance for the $\mathbb{C}^3/\mathbb{Z}_3$, the orbifold singularity can be resolved by blowing up the singular point by a compact $\mathbb{P}^2$ to give a non-compact Calabi-Yau, which is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$. Therefore the orbifold can be thought of as the limit in which the cycle $\mathbb{P}^2$ shrinks to zero size. This picture can be captured through a GLSM (a brief discussion of the GLSM in this case is given in [47]). In this case we will have a $U(1)$ sigma model with three chiral superfields Φ_i, $i = 1, 2, 3$ with $U(1)$ charge $+1$ each and a P field with charge -3 The bosonic potential for these scalar fields is given by

$$V = (|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - 3|p|^2 - r)^2 \tag{1.3.12}$$

Where r is again the Fayet-Iliopoulos parameter. The $r > 0$ phase corresponds to the blown up phase. In this phase not all ϕ_i can vanish. For $r < 0$ the $p \neq 0$. In this phase the exceptional divisor is blown down. In chapter 5 we will give the description for the orbifold as well as the resolved space in terms of toric geometry. The two descriptions, GLSM and toric geometry are related and one can read of the toric data from the GLSM and vice-versa.

1.4 D-branes and open strings

When we want to describe open strings we have to impose boundary conditions at the end points of the string. In flat space one usually imposes Neumann boundary conditions on $p+1$ directions and Dirichlet boundary conditions on the remaining

$d-p-1$ directions. This would mean that the open string is forced to move on a $p+1$ dimensional sub-manifold. This describes the world volume of a p dimensional sub-manifold along which the string moves-the Dp brane. The Dp-branes are dynamical objects and source the massless higher form (RR) fields of the closed superstring theory [11]. They are non-perturbative solitonic states of string theory with tension (mass per unit volume) $\sim \frac{1}{g_s}$. There are other non-perturbative states in string theory called the NS5 branes which carry the magnetic NS-NS $B_{\mu\nu}$ charge. The tension of these objects goes as $\frac{1}{g_s^2}$

A $p+1$ form potential C_{p+1} couples to the world volume of a Dp brane through the term $\int_{Dp} C_{p+1}$, where the integral is over the world-volume of the brane. This is a generalization of the electric coupling to a point particle in four dimensions. Therefore the Dp brane carries RR charge. Type IIA theory has odd dimensional potentials in its spectrum and therefore even dimensional branes and similarly type IIB theory has even dimensional potentials and odd dimensional branes.

Just as the quantum closed strings describe the metric fluctuations around the spacetime background, these open strings describe the fluctuations in geometry of the D-branes on which they end.

The world volume action for these $p+1$ dimensional objects is called the Born-Infield action and describes the dynamics of these branes.

The spectrum of an open string ending on a brane has a massless vector field along the direction of the brane and other massless scalars denoting the fluctuations along the directions transverse to the brane.

This describes a supersymmetric $U(1)$ gauge theory living on the brane. When one takes a configuration of N branes on top of each other then one gets N^2 massless gauge fields. This is because in this case the strings with endpoints on different branes also have massless fields in their spectrum. So in this case one obtains a $U(N)$ supersymmeric gauge theory. For a system of N parallel D-branes there are N massless gauge fields coming from the spectrum of open strings with endpoints on the same brane. In this case therefore one gets a $U(1)^N$ supersymmetric gauge theory. From the world-volume field theory point of view this reduction of

symmetry, when one takes the branes apart from each other, is really the Higgs mechanism where the scalar field describing the separation of the branes in the transverse space acquires a vacuum expectation value.

We had seen that closed string theory on a circle has a T-duality invariant spectrum. Under T-duality a $p+1$ dimensional brane becomes a p dimensional brane when the T-duality direction is the direction along the brane and a $p+2$ brane if the T-duality direction is transverse to the brane. The tree level open string diagram contributing to the S matrix calculations is the disk, and higher loops are introduced by adding more boundaries. For instance the one-loop diagram is an annulus.

While the dynamics of D-branes at weak coupling are described in terms of the scattering of open strings which end on them, the Dp branes also have a dual closed string description. For example the one loop vacuum to vacuum amplitude of a open string with endpoints on two branes can be viewed in the dual closed string picture as a closed string exchange between two D-branes. In the closed string picture D-branes are described as boundary states in CFT. For instance consider a Dp-brane in flat space. In the open string picture this corresponds to imposing Neumann boundary conditions along $p+1$ directions and Dirichlet boundary conditions along the remaining $9-p$ directions. The Dirichlet and Neumann boundary conditions are

$$\begin{aligned} \partial_t X^i|_{t=0} &= 0 \qquad \textit{Dirichlet} \\ \partial_\sigma X^i|_{t=0} &= 0 \qquad \textit{Neumann} \end{aligned} \tag{1.4.1}$$

Similarly there are boundary conditions on the world-sheet fermions λ^i,

$$\begin{aligned} \lambda^i - i\eta\bar{\lambda}^i|_{t=0} &= 0 \qquad \textit{Dirichlet} \\ \lambda^i + i\eta\bar{\lambda}^i|_{t=0} &= 0 \qquad \textit{Neumann} \end{aligned} \tag{1.4.2}$$

where $\eta = \pm 1$ and labels the spin structure. Then in the closed string channel the boundary states are obtained as solutions to the equations

$$\partial_t X^i(t=0,\sigma)|B;\eta\rangle = 0 \tag{1.4.3}$$

$$\partial_\sigma X^i(t=0,\sigma)|B;\eta\rangle = 0 \tag{1.4.4}$$

and the corresponding equations for the fermionic fields. From the world-sheet point of view the X^i (the spacetime coordinates) are free scalar fields and therefore can be Fourier expanded into oscillator modes α^i_n, $\tilde{\alpha}^i_n$, as in any free field theory. Here the two sets of oscillators correspond to the quantization of the left and right moving waves on the string. In terms of these modes equation 1.4.3 become,

$$\begin{aligned}(\alpha^i_n + \tilde{\alpha}^i_{-n})|B;\eta\rangle &= 0 \\ (\alpha^i_n - \tilde{\alpha}^i_{-n})|B;\eta\rangle &= 0\end{aligned} \tag{1.4.5}$$

respectively. There are similar equations in terms for the fermionic modes as well. These equations can be easily solved for $|B\rangle$, giving a solution in terms of coherent states of the oscillators.

Since the D-brane is a bosonic state, these equation have to be solved both in the NSNS as well as RR sectors. Thus one has four solutions for these equations, $|B;+\rangle_{NSNS}$, $|B;-\rangle_{NSNS}$, $|B;+\rangle_{RR}$, $|B;-\rangle_{RR}$. These boundary states must be in the spectrum of the closed string theory in which it is embedded. So one has to find the correct GSO invariant linear combination of these states in each sector, i.e. in the NSNS as well as RR sectors. As we discussed above a one loop open string partition function can be thought of as a closed string exchange between two D-branes. This condition imposes a further factorisation constraint on the boundary states. This can written as

$$\mathcal{Z} = \int \frac{dt}{2t}\left(\frac{1+(-)^F}{2}e^{-2tH_0}\right) = \int dl\langle B|e^{-lH_c}|B\rangle \tag{1.4.6}$$

where $\mathcal{Z}$ is the open string partition function, H_0, H_{cl} are the open and closed string Hamiltonians respectively, t,l are the open string time and closed string time respectively and are related by the transformation $t = \frac{1}{2l}$. The factor $\frac{1+(-)^F}{2}$ implements the GSO projection in the open string theory.

Solving these constraints one gets a GSO invariant boundary state which also has the right factorisation property.

One can readily generalize the above discussion to D-branes on orbifolds. In this case the closed string states must not only be GSO invariant but also be

invariant under the action of the orbifold group. Further as we mentioned earlier, the theory has new sectors called the twisted sectors corresponding to the twisted boundary conditions. Therefore one has equations like equation (1.4.5) in every sector. Further one has to specify a representation for the group action on the Chan-Paton factors. So one has boundary states in every sector $|B;\eta;m\rangle_a$, where m labels the various sectors, and a labels the representation of the group for its action on the Chan-Paton factors.

The open string partition function is now given by

$$\mathcal{Z}^a = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\int\frac{dt}{2t}tr(g\frac{1+(-)^F}{2}e^{-2tH_0}) \tag{1.4.7}$$

Here g labels the elements of the discrete group Γ and $|\gamma|$ is the dimensionality of the group. The sum over g normalised by $|\Gamma|$ ensures that one has only those states which are invariant under the action of Γ.

Therefore one is looking for the correct linear combination of the boundary states in each sector which will be GSO invariant as well as invariant under the group action and also have the right factorisation property.

$${}_a\langle B|e^{-lH_c}|B\rangle_a = \mathcal{Z}^a \tag{1.4.8}$$

In a general CFT (without other chiral algebras) the relevant boundary condition is

$$T_L(t,\sigma)|_{t=0} = T_R(t,\sigma)|_{t=0} \tag{1.4.9}$$

Here T_L and T_R are the stress energy tensors for the left and right moving sectors. This condition enforces that no momentum is transferred across the boundary of the 2d theory. In terms of the modes L_n, $\bar{L}_n$ of the stress energy tensors, the boundary state is the solution of the equation condition

$$(L_n - \bar{L}_{-n})|B\rangle = 0 \tag{1.4.10}$$

These solutions are called the Ishibashi states. As before the condition for correct factorisation of the cylinder amplitude between two boundary states in terms of the one loop annulus partition function picks out the correct linear combination of Ishibashi states. This state is called a Cardy state.

1.5 Boundary states in Gepner models and boundary conditions in LG orbifolds

These considerations readily generalize in the Gepner model to boundary states which preserve half of the supersymmetry of the bulk theory. In a general CFT with chiral algebra, apart from solving the equation (1.4.10), the boundary states must also obey the condition,

$$(W_n - (-1)^{h_w}\bar{W}_{-n})|B\rangle = 0 \tag{1.5.1}$$

where h_w is the conformal weight of the $W(z)$ operator. If the extended algebra has an automorphism group $W(z) \to \Omega W(z)$, then one can consider a 'twisted' boundary condition

$$(W_n - (-1)^{h_w}\Omega\bar{W}_{-n})|\Omega B\rangle = 0 \tag{1.5.2}$$

For instance in the case of the (2,2) superconformal algebra the automorphism group takes $J(z) \to -J(z)$, where $J(z)$ is the $U(1)$ current and so it turns out that there are two possible types of boundary states. The so called A branes correspond to the 'twisted boundary condition' and the B branes to the other condition.

However we are looking for boundary states in the full Gepner model and not in the individual minimal models. The condition is then

$$\sum_{i=1}^{r}(W_n^i - (-1)^{h_w}\bar{W}_{-n}^i)|B\rangle = 0 \tag{1.5.3}$$

Recknagel and Schomerus (RS) were the first to construct solutions to the above equation. They took the simplifying ansatz that the boundary state is a tensor product of the individual boundary state in each minimal model, $|B\rangle \sim \prod_{i=1}^{r} \otimes |B_i\rangle$, where the $|B_i\rangle$ are states in the individual minimal models. So the RS states are simply the product of boundary states in each minimal model.

The generators of symmetries in the Gepner model are given by the sum of the generators of the individual minimal models. Thus the model is invariant under the automorphism action generated by the permutation of the different component minimal models. For example one can consider the permutation group

(12)(3)(4)(5), where one is permuting the first and second minimal models. In terms of the boundary conditions this translates to looking for states for the equations

$$(W_n^1 - (-1)^{h_w}\bar{W}_{-n}^2)|B_1\rangle = 0,\ (W_n^2 - (-1)^{h_w}\bar{W}_{-n}^1)|B_2\rangle = 0 \qquad (1.5.4)$$

$$\text{and} \quad |B\rangle \sim |B_1\rangle \otimes |B_2\rangle \otimes \prod_{i=3}^{5} |B_i\rangle \qquad (1.5.5)$$

We will give more details of these construction in chapter 7.

One can similarly study the LG orbifold on a world-sheet with boundary preserving half of the bulk supersymmetry. Depending on which linear combination of the bulk supercharges are preserved one can have two choices, boundary conditions that preserve A-type supersymmetry and those preserving B-type supersymmetry. These correspond to the A-type and B-type branes in the Gepner models. Such boundary conditions were studied in [13], Here it was shown that the linear boundary conditions which preserve B-type supersymmetry are specified by a hermitian matrix B which squares to identity, $B^2 = 1$ and is block diagonal (it mixes fields with identical charges). The boundary conditions then take the form

$$\begin{aligned} (P_D)_i^{\ j}\ \phi_j = 0 \quad &, \quad (P_D)_i^{\ j}\ \tau_j = 0 \\ (P_N)_i^{\ j}\ \xi_j = 0 \quad &, \quad (P_N)_i^{\ j}\ \partial_n \phi_j = 0 \end{aligned} \qquad (1.5.6)$$

where $\xi_i \equiv (\psi_{+i} + \psi_{-i})/\sqrt{2}$ and $\tau_i \equiv (\psi_{+i} - \psi_{-i})/\sqrt{2}$ and the $+$ and $-$ signs label the left and right moving components of the fermion ψ. The matrices $P_N \equiv (1+B)/2$ and $P_D \equiv (1-B)/2$ project onto the Neumann and Dirichlet directions respectively. The matrix B which specifies the boundary conditions needs to satisfy an additional condition due to the presence of the superpotential in the LG model.

$$\frac{\partial G}{\partial \phi_i}\ (P_N)_i^{\ j} = 0\ . \qquad (1.5.7)$$

In simple models involving a single chiral field, the only possible condition is the Dirichlet one[3]. This carries over to the case of several chiral superfields when one imposes boundary conditions separately on each of the chiral superfields, i.e., the

[3]This assumes the absence of degrees of freedom other than those that come from the bulk LG theory.

matrix B is taken to be diagonal. For LG orbifolds associated with Gepner models, this implies that all the boundary states constructed by Recknagel and Schomerus in [15] must necessarily arise from Dirichlet conditions being imposed on all the chiral superfields. Further, when the superpotential is degenerate at $\phi_i = 0$, the condition $G = 0$ implies that the RS states arise from the boundary condition $\phi_i = 0$ for all i.

1.6 D-branes on Calabi-Yau manifolds

During the past five years, our understanding of the spectrum of D-branes (both A and B-type) that appear in type II compactifications on Calabi-Yau(CY) manifolds has significantly improved. In the Gepner models these correspond to the A branes and B branes that we discussed in the last section. Some of the progress has been achieved by relating geometric constructs such as branes wrapping geometric cycles of the Calabi-Yau space to boundary conditions in the LG orbifolds and boundary states in the associated Gepner models. The first step in this context appeared in [3] (see also [13] and [14] for a review).

In this thesis we will be interested in the B-type branes which in the simplest cases correspond to vector bundles on holomorphic cycle in the large volume Calabi-Yau. More generally B-type branes are coherent sheaves on the CY which can be thought of as bound states of those B-type branes which correspond to vector bundles on the whole CY. For instance a B-brane wrapping a 4-cycle on a CY corresponds to a D6-brane and an anti D6 brane with a 4-brane charge turned on inside. Such bound states are represented naturally as the cohomology of complexes of holomorphic vector bundles on the CY.

The method proposed to identify the large volume analogues for B-type D-branes at the orbifold end was to analytically continue periods (identified with the central charge of D-branes) from non-geometric regions to geometric regions and then use this information to add geometric insight into the story. This method is rather tedious but it lead the way to a simpler picture for the large-volume analogue of the Recknagel-Schomerus (RS) boundary states in the Gepner model [15]. From

this emerged a connection with the McKay correspondence [18, 24, 25, 26, 27].

Based on the results for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold [17], it was conjectured in [18] that the fractional zero-branes on $\mathbb{C}^n/\Gamma$ (with Γ a discrete abelian subgroup of $SU(n)$) correspond at large volume to (exceptional) coherent sheaves that provide a natural basis for bundles on the exceptional divisor of the (possibly partial) resolution of $\mathbb{C}^n/\Gamma$. A second conjecture in [18] was that the RS boundary states (to be precise, the $L_i = 0$ RS states) were given by the restriction of the fractional zero-branes to the Calabi-Yau hypersurface. Thus the RS boundary states turned out to be restriction of vector bundles or more generally coherent sheaves on the ambient (weighted) projective space to the CY hypersurface. Substantial evidence for this was provided in refs. [24, 25, 26].

However there are still several important aspects that are unclear and need to be clarified further. Among these is the question of an explicit description of B-type D-branes on Calabi-Yau manifolds that are not of this type. While the exceptional coherent sheaves obtained from fractional zero-branes do provide a basis for sheaves on the exceptional divisor of the resolution, this does not remain true on restriction to the Calabi-Yau threefold. For the case of the quintic, one finds that the bundles that are obtained by restriction span an index-25 sub-lattice of the lattice of RR charges of the quintic. In particular, the zero-brane and two-brane charges appear in multiples of 5 of the smallest possible value [18, 46]. Therefore the Recknagel-Schomerus construction of boundary states in the Gepner model does not have a state corresponding to the D0-brane on the CY manifold.

This suggests that one must generalise the Recknagel-Schomerus construction to obtain new boundary states in the Gepner model or equivalently consider more general boundary conditions in the LG orbifold. As was mentioned in the previous subsection, the RS boundary states correspond to imposing Dirichlet boundary conditions on all the fields in the LG orbifold. It is thus natural to consider boundary conditions that impose Neumann boundary conditions not on individual chiral fields (which is not possible), but on one or more linear combinations of fields as given in equation (1.5.6). For instance, in the LG orbifold for the superpotential

given by the Fermat quintic, one such boundary condition is (see sec. 3.3.3 of [13])

$$
\begin{aligned}
(\phi_1 + \phi_2) = 0 \quad &, \quad \phi_i = 0 \quad \text{for } i = 3, 4, 5 \ , \\
(\xi_1 - \xi_2) \ &= \ 0
\end{aligned}
\tag{1.6.1}
$$

It is easy to see that the above boundary conditions satisfy the constraint (1.5.7) or equivalently that $G = 0$ on the boundary. Such branes will be referred to as *fractional two-branes* (the term fractional will be justified in the subsequent chapters).

A significant recent development in this direction has been the study of B-type D-branes at the Landau-Ginzburg (LG) point in the Kähler moduli space of a Calabi-Yau manifold. In the LG theory, it is possible to provide a fairly explicit description of B-type branes using boundary fermions and the technique of matrix factorisation of polynomials [28]. This construction follows closely the conjecture of Kontsevich regarding the categorical description of B-type branes in the LG theory.

In an interesting development it was shown in [29] that a new class of fractional branes can in fact be defined in the LG theory. The D-brane charges of these objects in terms of the charge basis at the large-volume point in Kähler moduli space have been computed. Interestingly these fractional branes (in the LG description) include an object that corresponds at large volume to a single zero brane on the CY manifold. This is of particular interest since the Recknagel-Schomerus construction of boundary states in the Gepner model for CY manifolds appear to generically miss the D0-brane on the CY manifold. This D0-brane together with others that are related to it by the quantum symmetry at the LG point are of course only some examples of a large class of new branes that can be constructed using the technique of matrix factorisation of the world-sheet superpotential of the LG Lagrangian. This new approach to B-type branes in the LG theory is due to several authors [28, 29, 31, 32, 33][4].

We also note that, from a purely mathematical point of view, there have been further developments in the categorical description of these B-type branes at

[4]For completeness we may mention another major development in this new approach has been the computation of the world-volume superpotential of such branes [34]

the LG point and the derived equivalence of this category to the derived category of coherent sheaves on the same CY at large volume appears to have been established. This has been done in a series of papers by Orlov [35].

In this thesis we will investigate the new fractional branes, referred to above, due to ref [29], by different methods without using the technique of matrix factorisation and the introduction of boundary fermions. Our aim in particular, is to understand these states from a more transparently geometric point of view. The key observation is that the new fractional branes essentially arise from considering Neumann type boundary conditions on the fields of the LG theory. This is very similar to the particular matrix factorisation that ref. [29] use in order to construct their new fractional branes. Thus instead of the old fractional branes, which were fractional zero-branes, located at the singular point, we now consider fractional two-branes which are complex lines that pass through the fixed point.

One may easily construct such objects in the orbifold theory without the world-sheet superpotential. The D-brane charges of these objects in the large volume basis can be computed by first determining the intersection of these fractional two-branes with the fractional zero-branes and with themselves. This can be easily done at the orbifold point. Subsequently, using these intersection numbers we can determine the large-volume charges of these fractional two-branes.

It turns out that these fractional two-branes have a 'fractional' first Chern class. This fractional first Chern class ensures that when these objects are restricted to the CY hypersurface, one of them precisely becomes a zero-brane on the CY hypersurface. One may note that if we began with objects that have integer first Chern class on the ambient projective space then we would always obtain d zero-branes on the CY hypersurface, where d is the degree of the polynomial equation describing the CY.

Since the appearance of a 'fractional' first Chern class is somewhat surprising, we also show that these fractional two-branes have integer charges if we consider them as objects living in the ambient non-compact CY, which is a line bundle over a projective base, rather than on the projective space itself. In the

GLSM description, the true ambient space provided by the fields of the theory is in fact a non-compact CY. The CY itself comes from first restricting to the (weighted) projective space that is the base of the non-compact CY, and then restricting it to the appropriate hypersurface. We also show that all the boundary states in the Ramond sector corresponding to the fractional two-branes can be described using the boundary conditions on the *bulk* world-sheet fermions. It was observed in [26] (see also [37]) that this in fact could be done for the fractional zero-branes themselves and in this paper we extend this observation to the new fractional branes.

For definiteness, we illustrate our method in the case of the non-compact orbifold $\mathbb{C}^3/\mathbb{Z}_3$, the corresponding CY hypersurface being the elliptic curve given by a degree three equation in $\mathbb{P}^2$. The extension to the case of the quintic is straightforward. We note also that these fractional two-branes in non-compact orbifolds were earlier considered by Romelsberger[38] (for a discussion of the related boundary state construction see also [39]) and the appearance of a 'fractional' first Chern class was noted indirectly. This was explained there by the interplay of the relative homology of the ambient non-compact CY and the compact homology of the base projective space. Our description of the fractional two-branes in the ambient non-compact CY provides a clear toric description of the same phenomenon. In later chapters we also give a toric description of the fractional two-branes for $\mathbb{C}^3/\mathbb{Z}_5$ and $\mathbb{C}^3/\mathbb{Z}_7$ orbifolds.

In this thesis we also note a strong parallel between the relation of the fractional two-branes (and more generally fractional $2p$-branes) in the orbifold theory to the corresponding coherent sheaves at large volume and the notion of the quantum McKay correspondence due to Martinec and Moore [40]. While there is nothing quantum about the relation in our setting where space-time supersymmetry is not broken, nevertheless the fractional two-branes in $\mathbb{C}^n/\mathbb{Z}_N$ appear closely related to the fractional zero-branes of $\mathbb{C}^{n-1}/\mathbb{Z}_N$, the latter being the geometry associated with the non-supersymmetric B-type branes in the work of Martinec and Moore. In the full set of coherent sheaves that correspond to the quantum $\mathbb{Z}_N$ orbit of the fractional two-branes at large volume we are able to find the analogues of the

so-called 'Coulomb branes' that they describe.[5]

Finally we also identify the CFT description of states of the LG model that correspond to fractional two-brane and four-brane states in the ambient non-compact orbifold that are restricted to the CY hypersurface. This turns out to be a sub-class of the B-type permutation branes of the Gepner models [41].

1.7 Organization of the thesis

The organisation of the thesis is as follows:

In chapter 2 we describe the construction of fractional $2p$-branes in orbifolds of $\mathbb{C}^n/\mathbb{Z}_N$. We work out a *master* formula, equation. (2.1.15), for the intersection forms between $Dp - Dp'$ branes at the orbifold.

In chapter 3 the intersection forms involving fractional two-branes are independently computed in the large volume for the case of $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. In fact we will be able to identify the large volume analogue for these fractional two-branes. The case of $\mathbb{C}^3/\mathbb{Z}_3$ is worked out in some detail.

Chapter 4 discusses how, for instance, fractional two-branes in a supersymmetric orbifold $\mathbb{C}^n/\mathbb{Z}_N$ are related to fractional zero-branes on a related non-supersymmetric orbifold $\mathbb{C}^{n-1}/\mathbb{Z}_N$. We argue for the existence of a quantum McKay correspondence which relates sheaves associated with fractional $2p$-branes on $\mathbb{C}^n/\mathbb{Z}_N$ to the sheaves associated with tautological bundles for $(2n-2p)$-branes on the same orbifold.

In chapter 5 we continue with our discussion of the the large volume analogues of the fractional two-branes in the framework of toric geometry for the example of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, leaving the discussion for the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold and the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold for the appendices.

[5]We could also have referred to the analogue of the quantum McKay correspondence in the case at hand, of supersymmetric fractional-$2p$ branes, by a different name, possibly as an 'extended McKay correspondence'. However in order not to further increase jargon for what is a closely related geometric phenomenon we retain the nomenclature developed in [40].

Using the GLSM, chapter 6 gives a heuristic derivation of the large volume analogs of D-branes for a specific set of boundary conditions in the LG orbifold for the Fermat quintic. We provide evidence that these are indeed the new fractional branes obtained in reference [29].

Finally chapter 7 connects our results in the LG orbifold with boundary conformal field theory. We provide evidence that the fractional two-branes and a certain class of fractional four-branes on restriction to the compact Calabi-Yau hypersurface are given by a sub-class of the permutation branes constructed in the Gepner model [41].

We conclude in chapter 8 with a summary of our results and some comments on unresolved issues.

Chapter 2

Fractional branes in the $\mathbb{C}^n/\mathbb{Z}_N$ Orbifold

In this chapter we will be discussing the boundary state construction of fractional $2p$ branes in the $\mathbb{C}^n/\mathbb{Z}_N$ orbifold. We first begin with some general considerations of string theory and D-branes on orbifolds.

As was mentioned in the last chapter, the spectrum of closed strings on orbifolds of the type $\mathbb{C}^d/\mathbb{Z}_n$ has an untwisted sector where one has the usual periodic boundary conditions on the bosonic fields (which correspond to the coordinates of the orbifold), as well as $n-1$ twisted sectors. In this thesis we will be mainly interested in orbifolds of the type $\mathbb{C}^3/\mathbb{Z}_n$.

Let the complex coordinates of this space be Z^i, $(i = 1, 2, 3)$ and λ^i the corresponding world-sheet fermions. The action of the group on the fields is given by

$$g: \; Z^i \to \omega^{a_i} Z^i \quad \omega = e^{\frac{2\pi i}{n}} \tag{2.0.1}$$

$$g: \; \lambda^i \to \omega^{a_i} \lambda^i \tag{2.0.2}$$

with $a_1 + a_2 + a_3 = 0 \bmod (2n)$. This action ensures that the theory has a unbroken $N = 2$ supersymmetry in $d = 4$.

The NS-NS states in the twisted sectors of the theory have a direct geometric interpretation. There is a complex massless scalar field in each twisted sector

ϕ_k (where k labels the k'th twisted sector). Giving these fields a vev results in the blowing-up of the orbifold. Similarly the RR fields in the twisted sectors can be obtained from the Kaluza-Klein reduction of the RR fields in the resolved space over the various blow-up cycles [5].

In the case of open strings ending on branes which are point-like along the $\mathbb{C}^d/\Gamma$ orbifold ($\Gamma \subset SU(d)$), apart from specifying the action of the group on the complex coordinates along the $\mathbb{C}^d/\Gamma$ one has to specify the group action on the Chan-Paton indices as well [4, 5, 47]. The action of Γ on the bosonic gauge fields along the brane as well as on the scalar fields labelling the transverse complex coordinates Z^i along the $\mathbb{C}^d/\Gamma$ orbifold is given by

$$\gamma(g_a)A^\mu\gamma(g_a)^{-1} = A^\mu, \qquad a = (1, ..., |\Gamma|) \tag{2.0.3}$$

$$\gamma(g_a)Z^i\gamma(g_a)^{-1} = R(g_a)^i_j Z^j \tag{2.0.4}$$

Where $|\Gamma|$ is the dimension of the group, and $\gamma(g)$ is the representation of the group Γ defining it's action on the Chan-Paton indices (which have been suppressed in the above equations) and $R(g)$ is the d dimensional representation of the group defining its action on the coordinates of $\mathbb{C}^d/\Gamma$.

Consider the case when γ is an irreducible representation of Γ. For $\Gamma \equiv \mathbb{Z}_n$ the irreps are one dimensional representations, and there is a D-brane corresponding to every irreducible representation. The mass and RR charge of these branes can be obtained, at tree level, by computing the one-point function of the metric and the RR vertex operators respectively, on the disk with appropriate boundary conditions. The mass of the brane turns out to be $\frac{1}{|\Gamma| g_s}$, and the RR charge of the brane corresponding to the a'th representation is given by

$$Q_0^a = \frac{d_a}{\Gamma}, \qquad Q_m^a = \frac{\chi^a(g_m)}{\Gamma} \tag{2.0.5}$$

Where d_a is the dimension of the a'th irreducible representation, $\chi^a(g_m = \mathrm{Tr}(\gamma_a(g_m))$ is the character of the a'th representation and Q_0^a and Q_m^a are the RR charges of the brane corresponding to the untwisted and m'th twisted RR closed string fields respectively. Since the mass and charges are fractional these branes are called

fractional zero-branes. These branes are fixed at the orbifold singularity. This can be seen from the group action on the fields Z^i given in equation (2.0.3), by taking γ to be one-dimensional.

After the singularity is resolved these fractional zero-branes can be identified [5, 6] with the branes wrapping the various cycles of the resolved space.

One can construct boundary states for these fractional branes as has been mentioned in chapter 1. This is what we will discuss in the next section.

2.1 Fractional $2p$-branes in $\mathbb{C}^n/\mathbb{Z}_N$

We will now discuss the case of fractional $2p$-branes in $\mathbb{C}^n/\mathbb{Z}_N$. We will compute the intersection numbers between the fractional Dp and Dp' branes. We will present a *master formula* for the computation of the open string Witten-index for open strings connecting a fractional $2p$ brane and a fractional $2p'$ brane in the $\mathbb{C}^n/\mathbb{Z}_N$ orbifold. Though the computations are well-known and we will use some of those results, we will emphasise some non-trivial features of the calculation that have not attracted due attention earlier. Fractional $2p$-branes on orbifolds have been considered, for instance in refs. [47, 48, 39, 49, 50, 38].

We choose the orbifold action given by

$$\phi_i \longrightarrow e^{2\pi\nu_i}\,\phi_i\ ,$$

which we will compactly write as $[\nu_1, \nu_2, \ldots, \nu_n] \equiv \frac{1}{N}[a_1, a_2, \ldots, a_n]$ for some integers a_i. Further, the type II GSO projection will require us to choose $\sum_i \nu_i = 0 \mod 2$. In fact, we will require something a little bit more stringent in the sequel. The boundary states that we construct are similar in spirit to the ones constructed in ref. [51] (see in particular section 4.1 for the discussion on the GSO projection) for a single $\mathcal{N} = 2$ chiral multiplet. We will however not get into the details of the GSO projection because we do not include the spacetime part. They can be included in a straightforward fashion.

2.1.1 Fractional zero-branes in $\mathbb{C}^n/\mathbb{Z}_N$

The case of fractional zero-branes has been discussed in great detail in the paper by Diaconescu and Gomis[47].

We first write down the partition function for open strings stretched between two zero-branes. In the open-string channel, the amplitude in the m-th twisted sector is given by [47] (the notation is as in [47] as well)

$$\begin{aligned}
\mathcal{A}_m^{(0)} &= \mathrm{tr}\Big(g^m\,\frac{1+(-)^F}{2}\,e^{-2tH_o}\Big) \\
&= V_1\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-1/2}\times\prod_{j=1}^n\Big[\frac{2\sin(\pi m\nu_j)\eta(it)}{\theta_1(m\nu_j,it)}\Big] \\
&\quad\times\frac{1}{2}\Big[\prod_{j=1}^n\Big(\frac{\theta_3(m\nu_j,it)}{\eta(it)}\Big)\prod_{j=1}^n\Big(\frac{\theta_4(m\nu_j,it)}{\eta(it)}\Big)\prod_{j=1}^n\Big(\frac{\theta_2(m\nu_j,it)}{\eta(it)}\Big)\Big]
\end{aligned} \tag{2.1.1}$$

In the above expression, the second line is the contribution from the world-sheet bosons and the third line is the contribution from the world-sheet fermions. In the second line, $V_1(8\pi^2\alpha' t)^{-1/2}$ is from the bosonic zero-modes and the other term is from the non-zero modes (see equation. (B.6) of [39], for instance). Note that if either $m=0$ or some particular $\nu_i=0$, then we need to use the following identity:

$$\lim_{\nu\to 0}\Big[\frac{2\sin(\pi\nu)\eta(\tau)}{\theta_1(\nu,\tau)}\Big]=\frac{1}{\eta^2(\tau)}\,.$$

To go to the closed-string channel, we consider the modular transform of the above amplitude,i.e., $\tau=it\to -1/\tau\equiv 2il$.

$$\begin{aligned}
\mathcal{C}_m^{(0)} &= V_1(8\pi^2\alpha')^{-1/2}\int_0^\infty\frac{dl}{2l}\times l^{1/2}\Big[\frac{1}{2l\,\eta^2(2il)}\Big]^r\times\prod_{j=1}^{n-r}\Big[\frac{(-i)2\sin(\pi m\nu_j)\eta(2il)}{\theta_1(-2ilm\nu_j,2il)}\Big] \\
&\times\ \frac{1}{2}\Big[\prod_{j=1}^n\Big(\frac{\theta_3(-2ilm\nu_j,2il)}{\eta(2il)}\Big)\prod_{j=1}^n\Big(\frac{\theta_4(-2ilm\nu_j,2il)}{\eta(2il)}\Big)\prod_{j=1}^n\Big(\frac{\theta_2(-2ilm\nu_j,2il)}{\eta(2il)}\Big)\Big]
\end{aligned} \tag{2.1.2}$$

In the above expression, r is the number of directions for which $m\nu_i=0$. Thus, when $m=0$, one has $r=n$ and when $m\neq 0$, then r is the number of directions on which the orbifolding group has no action. One looks for a (GSO projected) state

$\|B0, m\rangle\rangle$ in the m-th twisted sector for which[1]

$$\mathcal{C}_m^{(0)} = \int_0^\infty dl \; \langle\langle B0, m\| \; e^{-lH_c} \; \|B0, m\rangle\rangle \tag{2.1.3}$$

Since we will not need much detail, the dedicated reader may obtain the precise form of the $\|B0, m\rangle\rangle$ from equations. (4.14-4.23) of [47] except for a small difference. We remove the character (for the irrep I of $\mathbb{Z}_N$) $\chi_I(g^m)$ (as given in equation (4.23) of [47]) since we wish to include as a part of the normalisation. It is useful to note that $\|B0, m = 0\rangle\rangle$ is the boundary state for a zero-brane in flat-space. The consistent boundary states are labelled by the irreps of $\mathbb{Z}_N$ (satisfying Cardy's consistency conditions) for the fractional zero branes are

$$\boxed{|B0 : I\rangle = \sum_{m=0}^{N-1} \psi_I^{(0)\ m} \; \|B0, m\rangle\rangle \quad I = 0, 1, \ldots, (N-1)} \tag{2.1.4}$$

where $\psi_I^{(0)\ m} = \chi_I(g^m)/\sqrt{N} = e^{2\pi i I m/N}/\sqrt{N}$ is the normalisation for the fractional zero-branes. The D0-brane charge comes from the RR charges in the untwisted sector and is $1/N$ the value in flat space – a $1/\sqrt{N}$ from the normalisation $\psi_I^{(0)\ 0}$ and another $1/\sqrt{N}$ from the "renormalisation" of the charge in the orbifolded space[39]. The RR charge from the m-th twisted sector is

$$Q_I^{(0)\ m} = \frac{\chi_I(g^m)}{N} \tag{2.1.5}$$

2.1.2 Fractional $2p$-branes

We will now consider the case where one imposes Neumann boundary conditions on one of the fields on which the orbifold group has a non-trivial action. In computing the annulus amplitude for open strings between two fractional $2p$ branes, the bosonic and fermionic non-zero mode contributions are unchanged from that of the zero-brane case. However, one has to treat the bosonic and fermionic zero-mode separately. The contribution from the bosonic zero-modes to the open-string partition function with a g^m insertion is given for a fractional two-brane, for instance,

[1]This is not quite the boundary state that satisfies Cardy's condition and hence we represent it by $\|B0, m\rangle\rangle$ rather than $|B0, m\rangle$ to avoid confusion.

by:

$$\text{bosonic zero-mode contribution} = \begin{cases} V_3(8\pi^2\alpha' t)^{-3/2} & m=0 \\ V_1(8\pi^2\alpha' t)^{-1/2}(4\sin^2\pi m\nu)^{-1} & m\neq 0 \end{cases} \tag{2.1.6}$$

The $m = 0$ case is the same as the one with no orbifolding. The $m \neq 0$ sectors are similar to the zero-brane case that we just considered except for an *additional factor* of $(4\sin^2\pi m\nu)^{-1}$. This factor arises from the orbifold action on the bosonic zero-mode in the open string partition function, which is given by [39]

$$\langle\vec{k}|g^m|\vec{k}\rangle = \delta^2((1-g^m)\vec{k}) = |1-g^m|^{-1}\delta^2(\vec{k}) = (4\sin^2(\pi m\nu))^{-1}\delta^2(\vec{k}) \tag{2.1.7}$$

where $g^m = e^{2i\pi m\nu}$ defines the group action, $\vec{k}$ is the two dimensional momentum vector along the orbifold plane with complex components k_z, $k_{\bar{z}}$ and the eigen-values of g^m action on $\vec{k}$ are $e^{2i\pi m\nu}$ and $e^{-2i\pi m\nu}$. Thus after integration over the $\vec{k}$ space one gets the factor $(4\sin^2\pi m\nu)^{-1}$.

Putting all this together, one obtains the following changes in the expressions for $\mathcal{A}_m^{(2p)}$ (for fractional $2p$-branes, the index $a = 1,\ldots,p$ runs over the Neumann directions) with respect to the zero-brane case given earlier, i.e., $\mathcal{A}_m$.

$$\begin{aligned} V_1(8\pi\alpha' t)^{-1/2} &\longrightarrow V_{2p+1}(8\pi\alpha' t)^{-(2p+1)/2} && m=0 \\ V_1 &\longrightarrow V_1\prod_{a=1}^{p}(4\sin^2\pi m\nu_a)^{-1} && m\neq 0 \end{aligned}$$

The annulus amplitude for the $2p$-brane in the untwisted ($m = 0$) sector is thus

$$\begin{aligned} \mathcal{A}_0^{(2p)} &= \mathrm{tr}\Big(g^m\,\frac{1+(-)^F}{2}\,e^{-2tH_o}\Big) \\ &= V_{2p+1}\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-(2p+1)/2}\times\prod_{j=1}^{n}\Big[\frac{2\sin(\pi m\nu_j)\eta(it)}{\theta_1(m\nu_j,it)}\Big] \\ &\quad\times\frac{1}{2}\Big[\prod_{j=1}^{n}\Big(\frac{\theta_3(m\nu_j,it)}{\eta(it)}\Big)\prod_{j=1}^{n}\Big(\frac{\theta_4(m\nu_j,it)}{\eta(it)}\Big)\prod_{j=1}^{n}\Big(\frac{\theta_2(m\nu_j,it)}{\eta(it)}\Big)\Big] \end{aligned} \tag{2.1.8}$$

and in the $m \neq 0$ sectors

$$\begin{aligned}
\mathcal{A}_m^{(2p)} &= \mathrm{tr}\Big(g^m \, \frac{1+(-)^F}{2} \, e^{-2tH_o}\Big) \\
&= V_1 \prod_{a=1}^{p} (4\sin^2 \pi m\nu_a)^{-1} \int_0^\infty \frac{dt}{2t} (8\pi^2\alpha' t)^{-1/2} \times \prod_{j=1}^{n} \Big[\frac{2\sin(\pi m\nu_j)\eta(it)}{\theta_1(m\nu_j, it)}\Big] \\
&\quad \times \frac{1}{2}\Big[\prod_{j=1}^{n}\Big(\frac{\theta_3(m\nu_j, it)}{\eta(it)}\Big) \prod_{j=1}^{n}\Big(\frac{\theta_4(m\nu_j, it)}{\eta(it)}\Big) \prod_{j=1}^{n}\Big(\frac{\theta_2(m\nu_j, it)}{\eta(it)}\Big)\Big] \qquad (2.1.9)
\end{aligned}$$

The boundary state is quite similar to the one for the fractional zero-branes as given in equation. (2.1.4) with the following replacements:

$$\begin{aligned}
\|B2p : m = 0\rangle\rangle &\equiv |B2p\rangle_{\text{flat space}} \qquad (2.1.10) \\
\|B2p : m \neq 0\rangle\rangle &\equiv \|\widetilde{B0, m}\rangle\rangle \,, \qquad (2.1.11)
\end{aligned}$$

where the tilde represents the operation which switches the signs on the non-zero modes in a manner suitable for a $2p$-brane. With this, we can write the boundary state for the fractional $2p$-branes:

$$\boxed{|B2p : I\rangle = \sum_{m=2p}^{N} \psi_I^{(0)\ m} \, \|B2p, m\rangle\rangle \quad I = 0, 1, \ldots, (N-1)} \qquad (2.1.12)$$

where

$$\psi_I^{(2p)\ m} = \frac{\chi_I(g^m)}{\sqrt{N}\ \prod_{a=1}^{p}(-2i\sin\pi m\nu_a)} = \frac{e^{2\pi i I m/N}}{\sqrt{N}\prod_{a=1}^{p}(-2i\sin\pi m\nu_a)} \,,$$

where we have included a constant phase factor of $(-i)$ along with the $2\sin\pi m\nu_a$ since it makes all intersection numbers being real. The above normalisation implies that the m-th twisted sector part of the boundary state for a fractional two-brane will be the same as the fractional branes with a multiplicative factor of $(-2i\sin\pi m\nu)^{-1}$ (for every Neumann boundary condition) and thus the RR-charge in the m-th twisted sector of the two-brane is given by

$$Q_I^{(2p)\ m} = \frac{\chi_I(g^m)}{N} \prod_{a=1}^{p} \frac{1}{2\sin\pi m\nu_a} \qquad (2.1.13)$$

where the index a runs over the Neumann directions and $I = 1, \ldots, N$ label the fractional $2p$-branes. Finally, the two-branes all carry $2p$-brane RR charge from the untwisted sector which is $1/N$ of the result in flat space.

2.1.3 Intersection numbers

From the appendix of [39], one can see that computation of the open-string Witten index is given by (equation.. (4.54) in [39])

$$\mathcal{I}_{IJ}^{p-p'} = \sum_{m\neq 0} \left(\psi_I^{(p)\ m}\right)^* \psi_J^{(p')\ m} \prod_{i=1}^{n} (-2i \sin \pi m \nu_i) \tag{2.1.14}$$

where ψ_I^m are the normalisations associated with the boundary condition I. For the fractional zero-branes, one has

$$\psi_I^{(0)\ m} = \frac{1}{\sqrt{N}}\, \chi_I(g^m)\ ,$$

where $\chi_I(g^m) = \exp(2\pi i m I)$ for $\mathbb{Z}_n$. Note that one omits the spacetime contribution to this since it multiplies the above result by zero.

We can also work out the general formulae for the open-string Witten index for open-strings that connect fractional $2p$-branes to fractional $2p'$-branes. This generalises the expression existent in the literature for the case of fractional zero-branes. A straightforward computation gives the following *master formula* for the $\mathbb{C}^n/\mathbb{Z}_N$ orbifold [39, 38]:

$$\boxed{\mathcal{I}_{I,J}^{2p,2p'} = -\frac{(-i)^{n+p-p'}}{N} \sum_{j=1}^{N}{}' e^{2\pi i (I-J) j/N} \times \frac{\prod_{k=1}^{n} (2 \sin \pi j \nu_k)}{\prod_{a=1}^{p} (2 \sin \pi j \nu_a) \prod_{b=1}^{p'} (2 \sin \pi j \nu_b)}} \tag{2.1.15}$$

where the product in the denominator of the RHS runs over the p (p') Neumann directions alone and the prime indicates that the sum does *not* include terms that have vanishing denominators – this happens when $j\nu_a$ become integers. One can see that the intersection between fractional $2p$-branes and fractional $2(n-p)$ branes (obtained by exchanging Neumann and Dirichlet boundary conditions) is the identity matrix.

2.1.4 Intersection numbers – examples

2.1.4.1 $\mathbb{C}^3/\mathbb{Z}_3$

The $\mathbb{Z}_3$ action is taken to be $\frac{1}{3}[-2,-2,-2]$ with the Neumann boundary condition chosen on the first field when fractional two-branes are considered[2]. There are three fractional zero and two-branes, which we will represent by S_I^0 and S_I^2 respectively. The quantum $\mathbb{Z}_3$ acts on these branes by shifting $I \to I+1$ mod 3. We will write the intersection numbers in terms of the generator g of the $\mathbb{Z}_3$ and is represented by the 3 dimensional shift matrix.

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.1.16}$$

The master formula, equation (2.1.15) gives on using $2\sin(\pi/3) = 2\sin(2\pi/3) = \sqrt{3}$

$$\begin{aligned} \mathcal{I}^{0,0} &= -(1-g)^3 \\ \mathcal{I}^{0,2} &= -g^2(1-g)^2 \\ \mathcal{I}^{2,2} &= g(1-g) \end{aligned} \tag{2.1.17}$$

Note that in the expression for the intersection form $\mathcal{I}^{0,2}$ between fractional zero and two-branes, the factor of g^2 can be gotten rid of by relabelling/shifting the I labels, of say, the fractional two-branes by two. Note that such a shift does not affect $\mathcal{I}^{2,2}$. This has to be kept in mind while comparing with the intersection form for the coherent sheaves that we propose as candidates for the large-volume analogs of the fractional two-branes in the next subsection.

2.1.4.2 $\mathbb{C}^5/\mathbb{Z}_5$

The $\mathbb{Z}_5$ action is taken to be $\frac{1}{5}[-4,-4,-4,-4,-4]$. We will consider the cases of fractional zero, two and four branes. Again, we will use the notation $S_I^{(2p)}$ with

[2]If we choose the Neumann direction to have $\nu = 1$ rather than $\nu = -2$, the intersection numbers are non-integral. This is related to the type II open-string GSO projection[51].

$I = 1, 2, 3, 4, 5 \bmod 5$ and the quantum $\mathbb{Z}_5$ being generated by g which takes $S_I^{(2p)} \to S_{I+1}^{(2p)}$. The various intersection matrices are given by:

$$
\begin{aligned}
\mathcal{I}^{0,0} &= -(1-g)^5 \\
\mathcal{I}^{0,2} &= -g^3(1-g)^4 \\
\mathcal{I}^{0,4} &= -g(1-g)^3 \\
\mathcal{I}^{2,2} &= g(1-g)^3 \\
\mathcal{I}^{2,4} &= g^4(1-g)^2 \\
\mathcal{I}^{4,4} &= -g^2(1-g)
\end{aligned}
\tag{2.1.18}
$$

Chapter 3

Coherent sheaves on the resolved space - I

3.1 B-type branes on Calabi-Yau manifolds

We will now discuss in greater detail the nature of the coherent sheaves that arise from the continuation to large-volume of the fractional p-branes that we have been studying at the orbifold point. For specificity we will focus on the case of fractional zero-branes and fractional two-branes in the case of the blow-up of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. In this case the manifold at large volume is the total space of the line bundle $\mathcal{O}(-3)$ on $\mathbb{P}^2$ which is a non-compact Calabi-Yau manifold.

As we already mentioned in the introduction, some obvious B-type branes on Calabi-Yau manifolds are branes wrapping holomorphic cycles in this Calabi-Yau manifold. The condition that the branes wrapping such holomorphic cycles preserve some supersymmetry imposes conditions on the gauge fields living on the branes. These conditions imply that these B-branes are described by holomorphic vector bundles [3, 12] (this means that the transition functions on the bundles are holomorphic functions) with support on the appropriate cycles.

Holomorphic vector bundles on the cycles of the Calabi-Yau manifold (apart from the six-cycle) can equivalently be considered as coherent sheaves on the Calabi-Yau. The charges of the B-type branes may be conveniently read off from

from the Chern character of the corresponding bundle or sheaf. Thus given the Chern character of a bundle (or sheaf)(E) for a 3-dimensional CY, the charges are given by [24]

$$n_6 = \mathrm{ch}_0(E) = \text{rank of the bundle}, \; n_4 = \mathrm{ch}_1(E) = c_1(E)$$
$$n_2 = \mathrm{ch}_2(E) = \frac{1}{2}c_1^2(E) - c_2(E) \text{ and } n_0 = \mathrm{ch}_3(E) \tag{3.1.1}$$

We note that correspondingly for a sheaf on $\mathbb{P}^2$,

$$\mathrm{ch}(E) = n_4 + n_2 J + n_0 J^2 \tag{3.1.2}$$

These charges are not identical to the RR charges but are an equivalent convenient basis.

We summarise in the table below how some simple examples of branes in the total space of $\mathcal{O}_{\mathbb{P}^2}(-3)$ can be represented by coherent sheaves, or equivalently the corresponding complexes.

Object	the associated sheaf	Chern ch.
A 4-brane wrapping $\mathbb{P}^2$	$\mathcal{O}_{\mathbb{P}^2}$	1
A 2-brane wrapping a $\mathbb{P}^1 \subset \mathbb{P}^2$	$\mathcal{O}_H \equiv [\mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}]$	$J - \frac{J^2}{2}$
A point on $\mathbb{P}^2$ by	$\mathcal{O}_{\mathrm{pt}} \equiv [\mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(-1) \to \mathcal{O}_{\mathbb{P}^2}]$	J^2

where J generates $H^2(\mathbb{P}^2, \mathbb{Z})$ and $\langle J^3 \rangle_{\mathbb{P}^2} = 1$. Note that we can always twist the 2-brane by tensoring it with $\mathcal{O}_{\mathbb{P}^2}(n)$. This changes the J^2 part in the Chern character for the 2-brane. To determine the Chern characters and hence the charges we need to use the fact that the alternating sum of the Chern characters of objects in a sequence is zero.

3.1.1 GLSM with boundary

The somewhat mathematical picture of B-branes that we just recalled briefly, can be explicitly realized in a physical description in terms of the GLSM. In chapter 1 we had reviewed the GLSM construction for the specific examples of the quintic

hyper-surface on $\mathbb{P}^4$ and the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. Specifically for the second example, the construction realized the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold and it's geometric resolution as different phases of the GLSM labelled by different values of r. By analytically continuing in the r space one could interpolate between the orbifold point and the large volume point. One can similarly study the analytic continuation of the D-branes between the orbifold point and the large volume point. For this one needs to introduce a GLSM with boundary. We will review the boundary GLSM construction of [44, 37] in this section. In later sections we will use this to identify the fractional zero-branes and fractional two-branes as coherent sheaves in the large volume. We will also need the boundary GLSM in chapter 6 for the identification of boundary states in the Gepner model of the quintic with the coherent sheaves in the smooth quintic Calabi-Yau.

For simplicity let us consider a GLSM without a superpotential. For the boundary GLSM we need to introduce boundary conditions which preserve half of the $\mathcal{N} = (2,2)$ supersymmetry. In the case of the B-type branes we must ensure that the boundary conditions on the $\mathcal{N} = (2,2)$ susy algebra in the matter sector is given by equations(1.5.6 and 1.5.7). By consistency this also imposes conditions on the gauge multiplet fields [44, 37]. For the analogue of the Chan-Paton factors one has to introduce new boundary fermionic multiplets into the action. The bulk chiral superfields (Φ) restricted to the boundary give rise to two boundary multiplets, a boundary chiral multiplet Φ' with components (ϕ, τ), where $\tau \equiv (\psi_- - \psi_+)/\sqrt{2}$ and a boundary fermionic multiplet with components $(\xi \equiv (\psi_- + \psi_+)/\sqrt{2}, -F)$, where ψ_-, ψ_+ are the left and right moving components of the bulk fermion(restricted to the boundary) and F is a auxiliary field. It must be ensured that the susy variations of these terms cancel the terms coming from the bulk variations.

The action for the new Fermi multiplets superfields is given by [37],

$$S_F = -\frac{1}{2}\int dx^0 d^2\theta \bar{\Pi}_a \Pi_a \tag{3.1.3}$$

where θ is the boundary superspace coordinates. The Fermi multiplet satisfies the following equation

$$\bar{\mathcal{D}}\Pi_a = \sqrt{2}E_a(\Phi') \tag{3.1.4}$$

where $E(\Phi')$ is a function of the Φ'. The component expansion of the Π_a is given by

$$\Pi_a = \pi_a + \sqrt{2}\theta l - \sqrt{2}\bar{\theta}E\Phi' + \theta\bar{\theta}(-iD_0\pi + \tau_i\frac{\partial E}{\partial \phi_i}).$$

This reduces to the usual chiral super field expansion for $E = 0$.

We can also add a further superpotential term,

$$S_J = -\frac{1}{\sqrt{2}}\int dx^0 d\theta(\Pi_a J^a)|_{\bar{\theta}=0} - \text{h.c} \tag{3.1.5}$$

where $J^a(\Phi)$ $(a = 1, \ldots, r)$ are r homogeneous polynomials of degree d, which satisfy the constraint $E_a J^a = 0$.

3.2 Fractional zero branes from Euler sequences

In this section we will use the boundary GLSM introduced in the previous section to identify the fractional zero-branes in the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold as bundles on $\mathbb{P}_2$ in the resolved space.

The simplest way to obtain the orbifold point[1] from the GLSM is to consider the limit $e^2 r \to -\infty$. In this limit, the fields in the vector multiplet behave as Lagrange multipliers. The D-field imposes the D-term constraints and the gauginos impose the constraint[44]:

$$\sum_i Q_i \phi_i \bar{\psi}_{\pm i} = 0 \ .$$

When one imposes *Dirichlet* boundary conditions on the fields, this equation imposes a condition on the combination $\bar{\xi}_i$, that is not set to zero by the boundary conditions. Thus, the gaugino constraint on the boundary is now

$$\sum_i Q_i \phi_i \bar{\xi}_i = 0 \tag{3.2.1}$$

It is important in what follows that these fermions ξ_i play the role of the boundary fermions that are used to construct coherent sheaves associated with B-type branes at large volume The argument for this is based on two observations. First, the trivial

[1]or the LG orbifold point in the quintic case

line bundle $\mathcal{O}_{\mathbb{P}^2}$ is one of the states obtained by a analytic continuation of fractional zero-branes to large volume [47] and hence arises from having Dirichlet boundary conditions on all the fields in the orbifold. This implies that in analytically continuing (in Kähler moduli space) the D-branes from the orbifold to the large volume limit, all Dirichlet boundary conditions become Neumann boundary conditions (see also the discussion in [44]). Second, in the GLSM construction that realises B-type branes as coherent sheaves, the boundary condition at large volume relates the ξ_i to the boundary fermions π_a via the boundary condition[37]

$$\bar{\xi}_i = i\frac{\partial J^a}{\partial \phi_i}\pi_a \tag{3.2.2}$$

where π_a are the lowest fermionic components of the Π_a. Thus for linear boundary conditions the ($J^a \propto \phi^a$), the $\bar{\xi}_i$ and π_a are identified.

This boundary condition appears for the coherent sheaf given by the following exact sequence

$$0 \to E \to \mathcal{O}^{\oplus r} \xrightarrow{J} \mathcal{O}(d) \to 0 \tag{3.2.3}$$

which corresponds to imposing the *holomorphic constraint* $J^a\pi_a = 0$ on r boundary fermions, π_a ($a = 1, \ldots, r$). considered as sections of $\mathcal{O}^{\oplus r}$. Thus we see that the sequences and complexes that we describe have a clear physical realisation.

In the case of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold one is left with *three* independent fermionic multiplets on the boundary. In this case the gaugino constraint equation is

$$\sum_{i=1}^{3} \phi_i\bar{\xi}_i = 0 \tag{3.2.4}$$

Treating this gaugino constraint as a degree one holomorphic constraint, that is setting $J^a \equiv \phi^a$ in (3.2.3), we get the Euler sequence on $\mathbb{P}^2$

$$0 \to \Omega(1) \to \mathcal{O}^{\oplus 3} \to \mathcal{O}(1) \to 0$$

with the boundary condition

$$\bar{\xi}_i = i\pi_i \ .$$

Thus, one can indeed bypass the introduction of boundary fermions by treating the $\bar{\xi}_i$ as boundary fermions and the gaugino constraint as a holomorphic constraint.

Of course, we have three boundary states associated with the orbifold. It turns out that the other two coherent sheaves are given by the following exact sequences that can be derived from the Euler sequence (given for $\mathbb{P}^n$ below though we only need the case of $n = 2$ here) associated with $\Omega^p(p) \equiv \wedge^p\Omega \otimes \mathcal{O}(p)$

$$0 \to \Omega^p(p) \to \mathcal{O}^{\oplus\binom{n+1}{p}} \to \Omega^{p-1}(p-1) \otimes \mathcal{O}(1) \to 0 \tag{3.2.5}$$

Note the appearance of the binomial coefficients $\binom{n+1}{p}$ in the above sequences. We define

$$\boxed{S^0_{i+1} \equiv (-)^i \Omega^i_{\mathbb{P}^2}(i) \quad , \; i = 0, 1, 2} \tag{3.2.6}$$

These branes S^0_i can be identified as the result of the analytic continuation to large-volume of the fractional branes in the orbifold limit. Under the quantum $\mathbb{Z}_3$ symmetry(generated by g), one has

$$g: \quad S^0_i \to S^0_{i+1} \mod 3$$

A basic test of this identification is based on the idea that we expect the intersections of these branes (which is a topological quantity[2] computed by the open-string Witten index) to be the same at the orbifold point and at large volume. At the orbifold point the intersections can be readily computed from CFT techniques. For coherent sheaves on a smooth CY we can compute the intersections using standard methods from differential geometry. The two must agree as they indeed do.

The boundary fermion construction naturally leads to the spinor bundle on E rather than the coherent sheaf E. In the GLSM construction to obtain just the coherent sheaf we restrict to one-particle states in the corresponding boundary state. It was observed in [37] (see sec. 5.3) that when E is the cotangent bundle, the spinor bundle decomposes at different fermion numbers[3] to the different fractional branes.

[2]It is the index of the Dirac operator counting the number of chiral fermions in the spectrum of open strings stretching between these branes.

[3]For the case of weighted projective spaces associated with one Kähler modulus Calabi-Yau manifolds, one replaces the fermion number by the $U(1)$ ($\mathbb{Z}_K$) charge.

Thus, the monodromy about the orbifold point is realised by suitably changing the restriction on the fermion number of the states. Thus, the three fractional branes for the orbifold are in one-to-one correspondence with the states: (the vacuum $|0\rangle$ satisfies $\bar{\xi}_i|0\rangle = 0$)

$$|0\rangle \quad , \quad \xi_i|0\rangle \quad , \quad \xi_i\xi_j|0\rangle \quad ,$$

subject to the condition $\sum_i \phi_i\bar{\xi}^i = 0$ being imposed [4].

3.2.1 Fractional two-branes in $\mathbb{C}^3/\mathbb{Z}_3$

Let us now consider the case of fractional two-branes in the $\mathbb{C}^3/\mathbb{Z}_3$ example. Let us impose Neumann boundary conditions on $(\phi_1 - \phi_2)$ and Dirichlet on $(\phi_1 + \phi_2)$ and ϕ_3[5] . Away from the orbifold point, thus there are two fermions (after eliminating $\bar{\xi}_2$ in favour of $\bar{\xi}_1$): $\bar{\xi}_1$ and $\bar{\xi}_3$. The gaugino constraint is

$$(\phi_1 + \phi_2)\bar{\xi}_1 + \phi_3\bar{\xi}_3 = 0 \ . \tag{3.2.7}$$

Thus, when $(\phi_1 + \phi_2, \phi_3) \neq (0,0)$, the constraint removes one fermion and when $(\phi_1 + \phi_2, \phi_3) = (0,0)$ the constraint is trivially satisfied. This is possible on $\mathbb{P}^2$, when $\phi_1 - \phi_2 \neq 0$. Note that these conditions specify a two-brane (denoted below by P) in the space $\mathcal{O}_{\mathbb{P}_2}(-3)$ which is the resolution of the $\mathbb{C}^3/\mathbb{Z}_3$ singularity.

This implies that the fermions are sections of the sheaf F_1 given by the following sequence

$$0 \to F_1 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1) \to \mathcal{X}_P \to 0 \tag{3.2.8}$$

The term involving $\mathcal{X}_P$ has been added to take care of the fact that (3.2.7) is trivially satisfied on P. The following comments are in order here:

1. $\mathcal{X}_P$, by definition, vanishes away from P. In particular, it vanishes on the $\mathbb{P}^1 \in \mathbb{P}^2$ where $(\phi_1 - \phi_2) = 0$.

[4] A related observation was made by Mayr in [26] where he referred to the fractional branes as providing a fermionic basis for branes

[5] We choose this boundary condition since this will be compatible to adding a superpotential $G(\phi) = \phi_1^3 + \phi_2^3 + \phi_3^3$.

2. Defining

$$0 \to F_0 \to \mathcal{O} \to \mathcal{X}_P \otimes \mathcal{O}(-1) \to 0 \,,$$

the sequence can be rewritten as

$$0 \to F_1 \to \mathcal{O}^{\oplus 2} \to F_0 \otimes \mathcal{O}(1) \to 0 \tag{3.2.9}$$

3. F_0 restricts to $\mathcal{O}_{\mathbb{P}^1}$ on the $\mathbb{P}^1$ not containing P.

4. The above sequence when restricted to the $\mathbb{P}^1$ not containing P becomes the Euler sequence. Thus $F_1|_{\mathbb{P}^1} = \Omega^1_{\mathbb{P}^1}(1)$.

5. Note that F_1 is however a sheaf on $\mathbb{P}^2$ even though the sequence which generates it is reminiscent of the Euler sequence for $\mathbb{P}^1$.

The above identification suggests that the remaining fractional branes can be given by exact sequences that on restriction to a $\mathbb{P}^1$ not containing P give the generalised Euler sequences of $\mathbb{P}^1$. The last sequence which generates F_2 is rather interesting. One can argue that $F_2 = -\mathcal{X}_P \otimes \mathcal{O}(1)$. That F_2 must at least have $\mathcal{X}_P$ as a factor is clear since the last sequence must restrict to zero on on the $\mathbb{P}^1 \in \mathbb{P}^2$ not containing P. This is because there is no corresponding generalized Euler sequence that appears on $\mathbb{P}^1$. The factor of $\mathcal{O}(1)$ can be deduced from the general pattern that we observe in these sequences. We refer to F_2 as the *Coulomb branch brane* because of this vanishing property on restriction to the $\mathbb{P}^1$ not containing P. The remaining branes (F_0, F_1) will be called as the *Higgs branch branes*. As will be explained later, this parallels the *missing branes* that one needed to make a correspondence (called the quantum McKay correspondence) between D-branes on a non-supersymmetric orbifold and D-branes on the Hirzebruch-Jung resolution as considered by Martinec and Moore [40]. This relationship will be made more precise in the next chapter.

Therefore finally we get three fractional two-branes given by the sequences

$$0 \to F_0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{X}_P \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to 0 \tag{3.2.10}$$

$$0 \to F_1 \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to F_0 \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0 \tag{3.2.11}$$

$$0 \to F_2 \to \mathcal{O}_{\mathbb{P}^2} \to F_1 \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0 \tag{3.2.12}$$

The main claim we make is that $(-)^I F_I$ can be identified with the analytic continuation of the fractional two-branes $S^{(2)}_{I+1}$ that we constructed at the orbifold end provided we take,

$$\mathrm{ch}(\mathcal{X}_P) = \frac{J}{3} \qquad \text{(up to } J^2 \text{ terms)} \tag{3.2.13}$$

The main evidence we provide is the matching of the open-string Witten index computed at the orbifold end with the intersection forms computed in the large volume using the candidate objects we present here. The calculation is presented in the next section.

The Euler form associated with these fractional two-branes, F_i have been computed in the next section and are generically fractional. However, the intersection form which is obtained by antisymmetrisation of the Euler form is integral [16]. The integrality of the intersection form implies that the charge quantisation condition is not violated.

The key point to note here is the assignment of a fractional 2-brane charge to $\mathcal{X}_P$. In later sections we will motivate the existence of such objects through a computation in cohomology as well as K-theory. Further in chapter 5 we will describe these objects in the framework of toric geometry.

3.3 Intersection matrices for $\mathbb{C}^3/\mathbb{Z}_3$

Before we begin the intersection computation, we can write down a more general form of the Chern character for $\mathcal{X}_P$. Clearly while we require its leading term to be of the form $J/3$, the J^2 term may depend on whether we have twisted the object by a line bundle $\mathcal{O}(n)$. Thus the general form will be

$$\begin{aligned} \mathrm{ch}(\mathcal{X}_m) &= \frac{1}{3}\mathrm{ch}[\mathcal{O}_{\mathbb{P}^2}(m-1) \to \mathcal{O}_{\mathbb{P}^2}(m)] \\ &= \frac{1}{3}\Big(J + \frac{(2m-1)}{2}J^2\Big) \end{aligned} \tag{3.3.1}$$

where, by construction, $\mathcal{X}_m$ carries 1/3 the charge of a two-brane on $\mathbb{P}^2$ (see table in section 3.1) after including a twist which we indicate by the subscript m. Note that $\mathcal{X}_{m+1} = \mathcal{X}_m \otimes \mathcal{O}_{\mathbb{P}^2}(1)$ and that $\mathrm{ch}(\mathcal{X}_{m+1}) = \mathrm{ch}(\mathcal{X}_m) + \frac{1}{3}\,\mathrm{ch}(\mathcal{O}_{\mathrm{pt}})$.

Finally using these objects $\mathcal{X}_m$ we rewrite the sequences that we wrote down in the previous section for the three fractional two-branes as,

$$0 \to F_0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{X}_{n-1} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0 \tag{3.3.2}$$

$$0 \to F_1 \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to F_0 \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0 \tag{3.3.3}$$

$$0 \to F_2 \to \mathcal{O}_{\mathbb{P}^2} \to F_1 \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0 \tag{3.3.4}$$

In the first line, we have included the fractional contribution by inserting $\mathcal{X}_{n-1}$ in the first line of the above equation to complete the sequence.

We then obtain the following identifications

$$\mathrm{ch}(F_0) = \mathrm{ch}\big[\mathcal{O}_{\mathbb{P}^2}\big] - \mathrm{ch}(\mathcal{X}_{n-1}) \tag{3.3.5}$$

$$-\mathrm{ch}(F_1) = \mathrm{ch}\big[\mathcal{O}_{\mathbb{P}^2}(1) - \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}\big] - \mathrm{ch}(\mathcal{X}_n) \tag{3.3.6}$$

$$\mathrm{ch}(F_2) = \mathrm{ch}\big[\mathcal{O}_{\mathbb{P}^2} - \mathcal{O}_{\mathbb{P}^2}^{\oplus 2}(1) + \mathcal{O}_{\mathbb{P}^2}(2)\big] - \mathrm{ch}(\mathcal{X}_{n+1}) \tag{3.3.7}$$

The objects in the square brackets are non-fractional objects and hence correspond to coherent sheaves on $\mathbb{P}^2$. Thus these terms must necessarily arise from the twisted sectors of the boundary state. The contribution of the untwisted sector is contained in the term containing the $\mathcal{X}$'s.

Now the Chern character add up as follows:

$$\mathrm{ch}(F_0 - F_1 + F_2) = \mathrm{ch}\big[\mathcal{O}(2) - \mathcal{O}(1)\big] - J + \frac{(2n+1)}{2} J^2$$

where we have kept the two contributions separate. Note that the $\mathcal{X}$'s have summed up to give an object that has the Chern class of a two-brane on $\mathbb{P}^2$.

We now present the computation of various intersection matrices for the coherent sheaves given by the sequences written out for fractional zero and two-branes. We present the Chern classes after *restriction* to the $\mathbb{P}^2$. The Chern classes for the fractional zero-branes are given from the Euler sequences for $\mathbb{P}^2$

$$\mathrm{ch}(S_1^{(0)}) = 1 \tag{3.3.8}$$

$$\mathrm{ch}(S_2^{(0)}) = -2 + J + \frac{J^2}{2} \tag{3.3.9}$$

$$\mathrm{ch}(S_3^{(0)}) = 1 - J + \frac{J^2}{2} \tag{3.3.10}$$

The Chern classes for the fractional two-branes are

$$\mathrm{ch}(S_1^{(2)}) = 1 - \frac{J}{3} - \frac{(2m+3)J^2}{6} \tag{3.3.11}$$

$$\mathrm{ch}(S_2^{(2)}) = -1 + \frac{2J}{3} - \frac{(m+1)J^2}{3} \tag{3.3.12}$$

$$\mathrm{ch}(S_3^{(2)}) = -\frac{J}{3} - \frac{(2m+1)J^2}{6} \tag{3.3.13}$$

The various Euler forms given below are obtained using the formula:

$$\chi(E,F) = \int_{\mathbb{P}^2} \mathrm{ch}(E)^* \, \mathrm{ch}(F) \, \mathrm{Td}(\mathbb{P}^2)$$

$$\chi(S^{(0)}, S^{(0)}) = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & -3 & 1 \end{pmatrix} \tag{3.3.14}$$

$$\chi(S^{(2)}, S^{(0)}) = \frac{1}{3} \begin{pmatrix} -m+3 & 2m+1 & -m-1 \\ -m-7 & 2m+6 & -m-2 \\ -m+1 & 2m-1 & -m \end{pmatrix} \tag{3.3.15}$$

$$\chi(S^{(0)}, S^{(2)}) = \frac{1}{3} \begin{pmatrix} -m & -1-m & -m-2 \\ 2m-2 & 2m+3 & 2m+5 \\ -m+5 & -m-5 & -m-3 \end{pmatrix} \tag{3.3.16}$$

$$\chi(S^{(2)}, S^{(2)}) = \frac{1}{9} \begin{pmatrix} -6m-1 & -1 & -3m-7 \\ -10 & 6m+11 & 3m+8 \\ -3m+2 & 3m-1 & -1 \end{pmatrix} \tag{3.3.17}$$

For bundles on a Calabi-Yau manifold these are antisymmetric. However, for bundles on divisors or more generally sheaves, the intersection form is obtained by explicitly antisymmetrising the above(as explained in [16]), i.e., let

$$I(E', E) \equiv \chi(E', E) - \chi(E, E')$$

for any two sheaves E and E'. We will write these in terms of the elements (g) of the $\mathbb{Z}_3$ monodromy group, where as before g is represented by the 3 dimensional

shift matrix.

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.3.18}$$

In our case, we then get

$$\begin{aligned} \mathcal{I}^{0,0} \equiv I(S^{(0)}, S^{(0)}) &= -(1-g)^3 \\ \mathcal{I}^{0,2} \equiv I(S^{(0)}, S^{(2)}) &= -(1-g)^2 \\ \mathcal{I}^{2,2} \equiv I(S^{(2)}, S^{(2)}) &= g(1-g) \end{aligned} \tag{3.3.19}$$

These intersections match with the results of the Witten index computations at the orbifold end presented in equation 2.1.17 up to shifts of g^2 in $\mathcal{I}^{0,2}$ as is discussed there. Note that the dependence on m disappears in the intersection matrix and thus we cannot fix its value. However, this is to be expected since a change in m is obtained by twisting by $\mathcal{O}(1)$, which is the monodromy at large volume. Since these sheaves are obtained via analytic continuation, the ambiguity in m is related to the possibility of choosing paths from the orbifold point to large volume which differ by a path around the large-volume point. The fact that the above matrix has *integer* entries implies that the DSZ charge quantisation condition is satisfied. Thus it is correct to assume that the fractional two-branes do carry fractional two-brane RR charge and not integral as suggested in [49]

3.4 Cohomology and K-theory computations

In this section we will try to motivate the existence of objects with fractional two-brane charge by a computation in cohomology as well as in the context of K-theory.

3.4.1 Computation in cohomology

We shall now consider the basic two-cycles in $M \equiv \mathcal{O}_{\mathbb{P}^2}(-3)$ which is a non-compact Calabi-Yau and is the crepant resolution of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold.

- There is one *compact* two-cycle given by a $\mathbb{P}^1$ in $\mathbb{P}^2$.

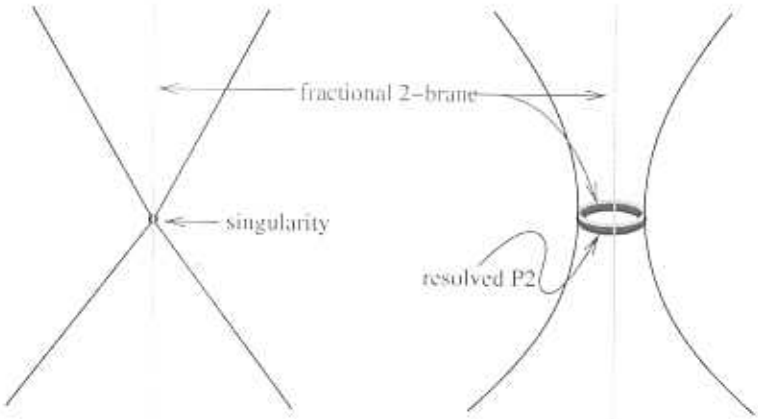

Figure 3.1: A schematic description of the fractional two-branes both before and after the singularity is resolved.

- In addition, there are three *non-compact* two-cycles corresponding to the fibre over $\mathbb{P}^2$. These two-cycles intersect the boundary at infinity, which is $S^5/\mathbb{Z}_3$, on a one-cycle $\gamma \in S^5/\mathbb{Z}_3$. ($\gamma$ is a element of $H_1(S^5/\mathbb{Z}_3, \mathbb{Z}) = \mathbb{Z}_3$.)

Consider a two-brane wrapping a non-compact two-cycle. What is the the representation for such branes? In particular, what is its D-brane charge as given by some appropriate Chern character?

In the case of non-compact manifolds the correct framework in which to discuss D-brane charges is compact cohomology or equivalently relative cohomology. For a non-compact manifold M with boundary N, the two-brane charges take values in $H^2_{\text{compact}}(M, \mathbb{Z}) \sim H^2(M, N, \mathbb{Z})$. A calculation (given in appendix A) shows that this is $\mathbb{Z}$ and since M has only one compact cycle, i.e., the $\mathbb{P}^2$, the basic two-brane is obtained by wrapping a $\mathbb{P}^1 \in \mathbb{P}^2$. Hence, J generates $H^2(M, N, \mathbb{Z})$. However objects that wrap the non-compact two-cycle of M will have a charge in $H^2(M, \mathbb{Z}) \sim \mathbb{Z}$. The two cohomologies are related by a long exact sequence in cohomology, the relevant part of which for our case reduces to the following (details are given in appendix A)

$$0 \to H^2(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to H^2(\mathcal{O}_{\mathbb{P}^2}(-3); \mathbb{Z}) \xrightarrow{j} H^2(S^5/\mathbb{Z}_3; \mathbb{Z}) \to 0$$

Putting in the known cohomology groups we get,

$$0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to \mathbb{Z}_3 \to 0 \tag{3.4.1}$$

Let J' generate $H^2(\mathcal{O}_{\mathbb{P}^2}(-3);\mathbb{Z})$. The exact sequence above indicates that $[J] \sim 3[J']$. Thus if $[J]$ is the charge basis of the 2-brane in the compact cohomology, then a charge +1 2-brane in the compact cohomology would correspond to a charge +3 2-brane in the $[J']$ basis. A single 2-brane wrapping the non-compact two-cycle would have charge +1 in the $[J']$ basis or charge 1/3 in the $[J]$ basis. Thus three two-branes wrapping a non-compact two-cycle of M can give an object which is equivalent[6] to a two-brane in $\mathbb{P}^2$ as elements of $H^2(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3;\mathbb{Z})$. In many ways, this is like the fractional zero-branes – the fractional zero-branes were localised at the singularity and couldn't be moved away from there unless three of them were taken to form a regular zero-brane.

Thus we have motivated the existence of objects with fractional two-brane charge measured in the charge basis associated with the compact cohomology. We can also perform an equivalent computation in the context of K-theory rather than cohomology, showing again the existence of fractional two-brane charges, but now obviously in the compact K-group associated with the $\mathbb{P}^2$. This computation is shown in the next section.

3.4.2 The K-theory computation

The K-theory computation in the relative case of a manifold with boundary that parallels the computation of A may also be carried out. In the case at had we are interested in computing the relative K-groups of the total space of the bundle $\mathcal{O}(-3)$ on $\mathbb{P}^2$ with respect to the boundary $S^5/\mathbb{Z}_3$. The computation is done by examining, as is typical in such situations, a six-term exact sequence (a consequence of collapsing a long exact sequence of K-groups using Bott periodicity) and using the known results on the K-group of $\mathbb{P}^2$ to compute the relative K-groups of interest. A useful reference for the computations of this section is [66].

We will take as the manifold M, the disc bundle $D(E)$ related to the

[6]The term equivalent can be made precise in the toric description of this geometry, where the equivalence is a consequence of the linear equivalence of divisors adapted to the non-compact setting.

bundle E defined by taking all vectors v in E such that their inner product (defined with respect to some appropriate Riemannian metric on E) $\langle v, v\rangle \leq 1$. The boundary of $D(E)$ will be sphere bundle $S(E)$ made of vectors v in E such that $\langle v, v\rangle = 1$. By a standard argument, the relative K-groups $K(D(E), S(E))$ may be identified with the K-groups $K(E, E_0)$ where E_0 is the complement of the zero-section of E.

To compute the K-groups $K(M, \partial M)$ for a compact manifold M with boundary ∂M we may use the following six-term exact sequence:

$$\begin{aligned} K^0(M) \to K^0(\partial M) \to K^1(M, \partial M) \to K^1(M) \to \\ \to K^1(\partial M) \to K^0(M, \partial M) \to K^0(M). \end{aligned} \tag{3.4.2}$$

In our case $M = D(E)$, $\partial M = S(E)$. For the case of the disc bundle since it is a deformation retract of E itself and the K-group of a bundle is isomorphic to the K-group of the base, we have $K(D(E)) = K(\mathbb{P}^2)$. To compute $K(S)$ we may compute it knowing the cohomology of $S(E) = S^5/\mathbb{Z}_3$. Though strictly speaking we should use the machinery of the Atiyah-Hirzebruch spectral sequence, it appears here to provide no surprises. We therefore get a $K^0(S(E))$ and $K^1(S(E))$ that is isomorphic to the cohomology.

We may now do the computation, using the data on the K-groups of $S(E)$ and $K(\mathbb{P}^2)$ (in particular, $K^1(\mathbb{P}^2) = 0$, since all the odd cohomologies of $\mathbb{P}^2$ are zero) to obtain the following shorter exact sequence:

$$0 \to K^1(S^5/\mathbb{Z}_3) \to K^0(\mathcal{O}(-3), S^5/\mathbb{Z}_3) \to K^0(\mathbb{P}^2) \to K^0(S^5/\mathbb{Z}_3) \to 0. \tag{3.4.3}$$

Using the data

$$K^0(\mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}; K^0(S^5/\mathbb{Z}_3) = \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3; K^1(S^5/\mathbb{Z}_3) = \mathbb{Z} \tag{3.4.4}$$

we obtain the result

$$K^0(M, \partial M) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \tag{3.4.5}$$

Note that one Z is in the kernel of the map from $K^0(M, \partial M)$ to $K^0(P^2)$. Two of the Z factors in $K^0(M, \partial M)$ form part of the sequence of the type

$$0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to \mathbb{Z}_3 \to 0. \tag{3.4.6}$$

What this shows is that if $[J']$ is the generator of $K^0(P^2)$ and $[J]$ is the generator of $K^0(M, \partial M)$ then $[J] \sim 3[J']$. If we take the standard unit of 2-brane charges to be given by $[J]$, then a $1/3$ charge in the $[J]$ basis becomes an integer charge in the $[J']$ basis.

Chapter 4

The quantum McKay correspondence

We now turn to the relationship between the fractional 2-branes of the susy orbifold $\mathbb{C}^n/\mathbb{Z}_N$ and the fractional zero-brane on a corresponding orbifold $\mathbb{C}^{n-1}/\mathbb{Z}_N$. We will also argue that this provides a new way to understand the quantum McKay correspondence, in a non-susy orbifold, due to Martinec and Moore by studying the fractional 2-brane in the higher-dimensional orbifold.

It is useful to review some aspects of the classical McKay correspondence [19] that are relevant for our considerations. A mathematical review of this correspondence as well as it's generalisation to higher dimensions is given in [23]. The discussion presented in the next section is based on [23] as well as the brief discussion in [39] and [59].

4.1 The classical McKay correspondence in two dimensions

The McKay correspondence is the statement of a one-one correspondence between the exceptional 2-cycles of the space Y obtained by the resolution of singularities of $X \equiv \mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, and the irreducible representations of Γ.

Let g be the elements of Γ and $(Z^1,\ Z^2)$, the complex coordinates of $\mathbb{C}^2$, and the action of g on the coordinates be defined by

$$g: \quad Z^i = \mathbb{Q}^i_j(g)Z^j, \quad i = 1, 2 \tag{4.1.1}$$

The tensor product of the two dimensional representation $\mathbb{Q}$ with any irreducible representations $\hat{R}$ can be expanded in terms of the irreducible representations as follows.

$$\mathbb{Q} \otimes \hat{R}_I = \oplus_J A^J_I \hat{R}_J \tag{4.1.2}$$

McKay observed that the $r \times r$ coefficient matrices A^J_I can be identified with the adjacency matrix of the extended Dynkin diagram associated with a simply laced Lie algebras G_Γ of rank $r-1$, where r is the number of irreducible representations of Γ. We will be interested in cases where $\Gamma \equiv \mathbb{Z}_k$, which are related to the A_{k-1} Dynkin diagrams.

The resolution of the $\mathbb{C}^2/\Gamma$ is a smooth ALE space Y, which has $r-1$ exceptional 2-cycles c_i. Let $c_0 = -\sum_i d_i c_i$, where the d_i are the dimensions of the various irreducible representations of Γ. Then the precise correspondence is that the extended Cartan matrix of G_Γ, given by

$$\hat{C}^I_J = 2\delta^I_J - A^I_J \tag{4.1.3}$$

is given by the negative of the intersection numbers between (c_I, c_J) where $I = 0, 1, ..., r-1$.

There is a version of the McKay correspondence due to Ito and Nakamura [21],[22], which will be of interest to us in identifying the large volume analogues of fractional zero branes in supersymmetric orbifolds. We are interested in the case of Γ abelian and Y a crepant $(c_1(Y) = 0)$ resolution of $\mathbb{C}^2/\Gamma$.

There exist line bundles(called the tautological bundles) R^i_s on Y which form a basis of $K(Y)$-the Grothendeick group of coherent sheaves on Y. The first Chern class $c_1(R^i_s)$ form a basis for the $H^2(Y, \mathbb{Z})$ dual to the homology group $H_2(Y, \mathbb{Z})$ generated by the cycles c_i that we discussed above. The McKay correspondence can then be stated as the one to one correspondence between the R^i_s and the characters of the irreducible representations of Γ, determined by the action of Γ on the

sections of R_s^i. Here the subscript s stands for supersymmetric, since the blow-up of supersymmetric orbifold singularities gives rise to a physical realization of the McKay correspondence as we review in the next section. For higher dimensional orbifolds of the type $\mathbb{C}^3/\Gamma$ there are several crepant resolutions possible, and one cannot define these line bundles in every such resolution. However it was shown that there exists a special crepant resolution called the G-Hilb M, where the orbifold is the quotient space $X = M/\Gamma$ [21] where it was possible to define these line bundles. This generalised McKay correspondence for the $\mathbb{C}^3/\Gamma$ orbifolds will be of use to us in the next chapter.

4.2 Review of $\mathbb{C}^2/\mathbb{Z}_{n(k)}$ orbifolds

This subsection is based on [40]. Consider the following orbifold action on $\mathbb{C}^2$ (with coordinates (ϕ_1, ϕ_2)):

$$(\phi_1, \phi_2) \to (\omega\phi_1, \omega^k\phi_2) \tag{4.2.1}$$

where $\omega = \exp(2\pi i/n)$. The case when $k = (n-1)$ is a supersymmetric orbifold and the orbifold is uniquely resolved by blowing up $(n-1)$ $\mathbb{P}^1$'s whose intersection matrix is -1 times the A_{n-1} Cartan matrix. For general non-supersymmetric $\mathbb{C}^2/\mathbb{Z}_{n(k)}$, there is a *minimal resolution* known as the Hirzebruch-Jung resolution. The resolution consists of r $\mathbb{P}^1$'s, where r is the number of terms in the continued fraction expansion of n/k:

$$\frac{n}{k} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\cdots 1/a_r}}} \equiv [a_1, a_2, \ldots, a_r] \tag{4.2.2}$$

where $a_\alpha \geq 2$. There are other resolutions with more $\mathbb{P}^1$'s for which some of the $a_\alpha = 1$. The supersymmetric case occurs when all $a_\alpha = 2$. One can check that $n/(n-1) = [2^{n-1}]$, $3/1 = [3]$ and $5/3 = [2, 3]$ are minimal. The intersection matrix

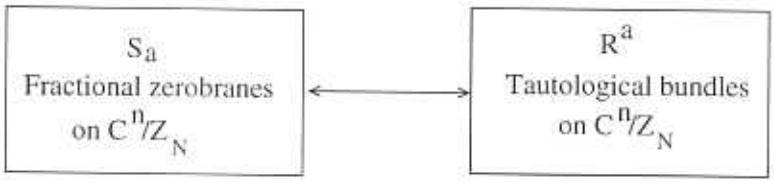

Figure 4.1: The McKay correspondence for $\mathbb{C}^n/\mathbb{Z}_N$

of the $\mathbb{P}^1$'s is given by the generalised Cartan matrix

$$C = \begin{pmatrix} a_1 & -1 & 0 & \cdots & 0 \\ -1 & a_2 & -1 & \cdots & 0 \\ 0 & -1 & a_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & & a_r \end{pmatrix} \tag{4.2.3}$$

4.2.1 Fractional zero-branes on $\mathbb{C}^2/\mathbb{Z}_{n(k)}$

One can construct boundary states for zero-branes on the $\mathbb{C}^2/\mathbb{Z}_{n(k)}$ orbifold. Standard methods (analogous to our earlier discussion) lead to n boundary states which we will call fractional zero-branes and label them $S_I^{\rm ns}$, $I = 1, 2, \ldots, n$ for the non-supersymmetric case. In the supersymmetric orbifold we will label the fractional zero branes as $S^{\rm s}$. The zero-brane which can move off the orbifold singularity is given by $\sum_I S_I^{\rm s}$ in the supersymmetric case and by $\sum_I S_I^{\rm ns}$ in the non-supersymmetric case.

These provide a basis for equivariant K-theory of the orbifold:

$$K_{\mathbb{Z}_n}(\mathbb{C}^2) = \mathbb{Z} \oplus \mathbb{Z}^{n-1} \tag{4.2.4}$$

where $\mathbb{Z}$ denotes the non-fractional zero-brane that can move off the singularity and $\mathbb{Z}^{n-1}$, the $(n-1)$ fractional branes that cannot move off the singularity.

4.2.2 The supersymmetric case: the McKay correspondence

The McKay correspondence arises when one considers a resolution X of the singularity in the supersymmetric orbifold $\mathbb{C}^2/\mathbb{Z}_{n(n-1)}$. In this case we have a unique

crepant (Calabi-Yau) resolution. One would like to know the precise objects, i.e., coherent sheaves that correspond to the continuation to large volume of the fractional zero-branes that we obtain at the orbifold point. We will focus on the cases where there is a description of the resolution via the GLSM or equivalently, that the resolution of the orbifold admits a toric description.

The GLSM for the resolved orbifold will be given by considering $(2+r)$ chiral superfields and r abelian vector multiplets[40], where r is the number of terms in the continued fraction representation of the Hirzebruch-Jung resolution. The orbifold limit is a special point in the Kähler moduli space. Another point of interest is the large-volume point, which corresponds to the point in the moduli space where all the $\mathbb{P}^1$'s that appear in the resolution have been blown-up to large volume.

In the supersymmetric case which happens when $r = (n-1)$, R^I_s turn out to be simple. $(n-1)$ of them are given by the line-bundles $\mathcal{O}(D_\alpha)$ (where D_α $(\alpha = 1, \ldots, r)$ represents the divisors associated with the r $\mathbb{P}^1$'s) and the last one is the trivial line-bundle $\mathcal{O}_X$. These line bundles are called the *tautological bundles* and provide a basis for $K(X)$, the Grothendeick group of coherent sheaves on X (which is a non-compact CY two-fold).

The fractional zero-branes furnish a basis for the equivariant K-group for the orbifold, i.e., $K_{\mathbb{Z}_n}(\mathbb{C}^2)$. In a similar fashion, it turns out that the the large-volume analogs of the fractional zero-branes S^s_I provide a basis for $K^c(X)$, the K-theory group with compact support. One expects the isomorphism

$$K_{\mathbb{Z}_n}(\mathbb{C}^2) \sim K^c(X) \ .$$

Further, there exists an isomorphism between $K^c(X)$ and $K(X)$ (see figure 4.1).

4.2.3 The non-supersymmetric case: the quantum McKay Correspondence

Martinec and Moore[40] considered the case of non-supersymmetric orbifolds and attempted to find the corresponding large-volume analogs of the R^I_{ns} in such cases.

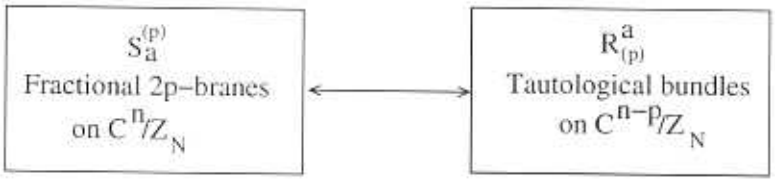

Figure 4.2: The proposed quantum McKay correspondence for $\mathbb{C}^n/\mathbb{Z}_N$

The *natural* candidates are the line-bundles $\mathcal{O}(D_\alpha)$ and the last one is the trivial line-bundle $\mathcal{O}_X$. There are $(r+1)$ of them as in the supersymmetric case with the only problem being that $r+1 < n$. So there are not enough line-bundles to complete the n $R^l_{\rm ns}$ at large-volume. The $r+1$ line-bundles are in one-to-one correspondence with the so-called *special* representations of $\mathbb{Z}_n$ in the mathematics literature[52].

We now review the resolution of this puzzle as given in [40]. We will propose another means of resolving this puzzle in the next subsection. The framework used is the GLSM that we discussed earlier where the Hirzebruch-Jung resolution appears in the Higgs branch of the GLSM. It is important to note that the Hirzebruch-Jung resolution is not a crepant one, since $c_1(X) < 0$. In the quantum GLSM, the world-sheet FI parameters flow under the world-sheet renormalisation group[43]. The singularities are resolved in the IR.

The resolution proposed in [40] is that one *must* include branes from all quantum vacua. In the IR, the theory has two branches – the Higgs and the Coulomb branches. The missing branes were identified with branes that appeared in the Coulomb branch and were dubbed the *Coulomb branch branes*. In analogous fashion, the tautological bundles in the Hirzebruch-Jung resolution were called the Higgs branch branes. Further aspects were discussed in a subsequent paper[53] (see also [54, 55]).

4.2.4 A different interpretation

We now consider a different resolution to the puzzle discussed in the previous subsection. Our idea is to embed the non-supersymmetric $\mathbb{C}^2/\mathbb{Z}_{n(k)}$ orbifold into a supersymmetric orbifold in one higher dimension, i.e., $\mathbb{C}^3/\mathbb{Z}_{n(k)}$, where we have added

a third coordinate, denoted by ϕ_3, with the following $\mathbb{Z}_n$ action:

$$\phi_3 \to \omega^l \, \phi_3, \quad \text{where } (l+k+1) = 0 \bmod 2n \ . \tag{4.2.5}$$

Next, we consider following fractional two-branes on $\mathbb{C}^3/\mathbb{Z}_n$: Impose Neumann boundary conditions on ϕ_3 and Dirichlet boundary conditions $\phi_1 = \phi_2 = 0$ on ϕ_1 and ϕ_2. There will be n such *fractional two-branes* and we will label them $S_I^{(2)}$.

Let $\widehat{X}$ be the crepant resolution of $\mathbb{C}^3/\mathbb{Z}_{n(k)}$. It is clear that the the projection $\pi : \widehat{X} \to X$ is obtained by setting $\phi_3 = 0$. As discussed in the previous section, there is a problem similar to the one seen with the non-supersymmetric orbifold resolution, the labels I corresponding to the *special representations* can be obtained using generalisations of the Euler sequences for the fractional zero-branes. In fact, one obtains the following when one restricts the $S_I^{(2)}$ to X:

$$S_I^{(2)}\big|_X = \begin{cases} S_I^{\text{ns}} & \text{when } I \text{ corresponds to special representations} \\ 0 & \text{otherwise} \end{cases} \ . \tag{4.2.6}$$

This is consistent with our observation in the $\mathbb{C}^3/\mathbb{Z}_3$ example where branes which disappeared on restriction are those with support on the complex line given by $\phi_3 \neq 0$ and $\phi_1 = \phi_2 = 0$.[1] Thus, the field ϕ_3 behaves like an order parameter with $\phi_3 = 0$ corresponding to the Higgs branch branes and $\phi_3 \neq 0$ giving rise to the Coulomb branch branes of [40].

4.2.4.1 An example – $\mathbb{C}^2/\mathbb{Z}_{3(1)}$

We have already worked out the large volume continuation of the fractional two-branes on $\widehat{X}$. In section 3.3, we have provided the Chern classes for these objects. We identified $S_3^{(2)}$ as the Coulomb branch brane. What are the candidates for the $R^I_{(2)}$'s? In $\widehat{X}$, the $\mathbb{P}^1 \in \mathbb{P}^2$ given by $\phi_3 = 0$ is to be identified with the $\mathbb{P}^1$ that appears in the Hirzebruch-Jung resolution X of $\mathbb{C}^2/\mathbb{Z}_{3(1)}$. The R^I_{ns} corresponding to

[1] The resolution of the more general supersymmetric $\mathbb{C}^3/\mathbb{Z}_{n(k)}$ orbifold requires one to add $(r-1)$ extra fields and abelian vector multiplets. The details of this and related issues require further work In this thesis, we will only provide details for the $\mathbb{C}^2/\mathbb{Z}_{3(1)}$ non-supersymmetric orbifold.

special representations are $\mathcal{O}_X(-1)$ and $\mathcal{O}_X$. They are "dual" to $S_1^{(2)}\big|_X$ and $S_2^{(2)}\big|_X$. The natural objects on $\widehat{X}$ are the push-forward of the R's on $\mathbb{C}^2/\mathbb{Z}_{3(1)}$:

$$R^1_{(2)} \sim j_*\left[\mathcal{O}_{\widehat{X}}(-1)\right] \text{ and } R^2_{(2)} \sim j_*\left[\mathcal{O}_{\widehat{X}}\right] \tag{4.2.7}$$

where j is the inclusion map from X to $\widehat{X}$. The last object, $R_3^{(2)}$ is a little bit more trickier to explain. Its Chern character as well as the those of the above $R^l_{(2)}$ can be worked out in the total space using the duality with the $S_l^{(2)}$. An important point to emphasise here is that due to the non-compactness of these D-branes, it is better to work in the space $\widehat{X}$, which is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$ in our case.

More generally, we conjecture that the general quantum McKay correspondence works as follows (see figure 4.2). For the case of fractional $2p$-branes on $\mathbb{C}^n/\mathbb{Z}_N$ (S_a^p), the corresponding dual objects are the tautological bundles on $\mathbb{C}^{n-p}/\mathbb{Z}_N$, the R_p^a. Establishing this, especially for general N with $n = 3$ or for $n > 3$ and general N would require further considerations beyond the preliminary considerations of this thesis.

Chapter 5

Coherent sheaves on the resolved space -II

In chapter 3 we discussed the large volume analogues of the fractional 2-branes in the context of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifolds. We argued there that they correspond to branes with support on some non-compact two cycle in the resolved space. However as noted earlier, while the discussion there provided an intuitive geometric picture supported by cohomological and k-theory arguments, the method was somewhat heuristic. In particular the sequences we wrote down there were imprecisely defined. In this chapter we will try to make these statements precise within the framework of toric geometry [59]. The discussion in the next section is based on [47, 18, 14].

5.1 Identifying the large volume branes

The problem of identifying the large volume analogues of the fractional zero branes after the resolution of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifolds was first solved by Diaconecsu and Gomis [47]. The resolved space after blow up is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$. It has a single Kähler parameter parametrising the size of the $\mathbb{P}^2$, which collapses to zero size at the orbifold point. The idea behind [47] is to compute and compare explicitly the BPS central charges of the large volume D-branes to the fractional branes at the orbifold limit (assuming that there exists a path in the moduli space

along which the objects are stable). This analytic continuation has to be done in the exact quantum moduli space of the world-sheet theory. To get the exact quantum moduli space one has to sum up the effect of world sheet instantons on the classical geometry. This can be done by using the techniques of local mirror symmetry and so one has to know the complex structure moduli space of the mirror geometry.

The complex structure moduli space of the mirror geometry has three singularities. The orbifold point where the theory is described by the orbifold conformal field theory, the 'large volume point', where the BPS central charge expression is given by $Z = \int_\Sigma e^{-F+B+i\omega}$, (where ω is the Kähler form and B, F are the usual B-field and the gauge field strengths respectively) and finally the conifold point where one of the BPS central charges vanish. At these singularities there is an associated monodromy action on the objects as one moves around these points. At the orbifold point this is the just the $\mathbb{Z}_3$ symmetry which permutes the fractional branes among themselves, while at the large volume end the effect of the monodromy is that the tensoring of the line bundles by a $\mathcal{O}(1)$. Using mirror symmetry techniques to find the $\mathbb{Z}_3$ monodromy in the large volume basis, one can identify the large volume analogues of the fractional zero branes. The bundles in the large volume corresponding to the fractional zero-branes, form a basis for the general construction of bundles on $\mathbb{P}^2$, which is the exceptional divisor of the resolution of $\mathbb{C}^3/\mathbb{Z}_3$ orbifold.

There is also another way of identifying the fractional branes after resolution for a general orbifold theory, using the ' generalised McKay correspondence' in the mathematical literature. A way to physically motivate this is discussed in ([18, 14]). For completeness, we will briefly review their argument here.

As is discussed in [18, 14] the main idea is to identify the large volume analogues of a dual set of extended fractional branes which fills the whole $\mathbb{C}^3/\Gamma$ orbifold. Then using these new objects one can identify the original fractional zero branes themselves.

If one takes the orbifold group Γ to be abelian and it's action on the Chan-Paton factors be defined by taking an irreducible representation of Γ, then the bundle associated with these extended fractional branes (called the tautological

bundles) turn out to be line bundles (R_0^i). So we now have two sets of objects, the R_0^i and the S_i^0, the latter being the original fractional zero branes themselves. Also from an analysis of the spectrum of strings stretching between a pair of branes (R_0^i, S_j^0), one sees that the open string Witten index computed at the orbifold point (which we denote by $\langle R_0^i, S_j^0 \rangle$) for these boundary conditions, is given by $\langle R_0^i, S_j^0 \rangle = \delta_j^i$.

So from the above arguments we see that the R_0^i are described by line bundles which after blow-up should have support over the full resolved non-compact space X. Also, the Witten index is a topological invariant and so must be equal to the intersection forms computed between the large-volume analogue of these R_0^i and S_i^0 given by the equation (5.1.1). Finally as seen in the $\mathbb{C}^3/\mathbb{Z}_3$ example using mirror symmetry methods [47], the S_i^0 form the basis for the bundles with support on lower dimensional cycles. This suggests that the S_i^0 corresponds to the classes of the compactly supported K theory group $\mathrm{K}_c(X)$.

In the mathematical literature it is known there exists a duality between $\mathrm{K}_c(X)$ and $K(X)$, which is the Grothendeick group for coherent sheaves on X. For abelian singularities $K(X)$ is generated by line bundles. Also there is a natural Poincare duality between $K(X)$ and $\mathrm{K}_c(X)$. If R^i are the generators of $K(X)$, then

$$\langle R^i, S_j^0 \rangle = \int \mathrm{ch}(R^{i*})\, \mathrm{ch}(S_j^0)\, \mathrm{Td}(X) = \delta_j^i \tag{5.1.1}$$

So it is natural to identify R_0^i with the R^i, which generate the $K(X)$.

The idea therefore is to compute the R_0^i [24, 26] and then compute $\mathrm{ch}(S_i^0)$ using the relation

$$\langle R_0^i, S_j^0 \rangle = \delta_j^i \tag{5.1.2}$$

We are of course interested in the corresponding picture for the fractional 2-branes. We denote the large volume analogues of the fractional two branes as S_i^2. Our aim here is to be able to give a toric description of these S_i^2. We will do this by taking an ansatz for the $\mathrm{ch}(S_i^2)$ in terms of the toric divisors following [59]. Then we compute the $\langle S_i^0, S_j^2 \rangle$ and $\langle S_i^2, S_j^2 \rangle$, which can be done using an integral similar to (5.1.1). Comparing the results with that of the intersection forms computed at the orbifold point, we will be able to precisely fit the charges of the fractional 2-branes.

This will enable us to justify the arguments of chapter 3, particularly the story of the 'fractional' first Chern class. We will also atleast partially justify the identification of the objects themselves (as distinct from matching just the charges).

We will start with a brief introduction to toric geometry where we will describe how the orbifold spaces are encoded in the toric data. The explicit construction of the fractional 2-branes in the toric framework is shown in the next section for the example of $\mathbb{C}^3/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_5$ and $\mathbb{C}^3/\mathbb{Z}_7$.

5.2 Toric Geometry - Basics

In this section, we will briefly review how to construct toric diagrams for orbifolds as well as to read off various information about the orbifold space from the toric data. We will discuss the specific examples of $\mathbb{C}^3/\mathbb{Z}_3$,$\mathbb{C}^3/\mathbb{Z}_5$ and $\mathbb{C}^3/\mathbb{Z}_7$ orbifolds.

5.2.1 The $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

First consider the case of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold with orbifold action $\frac{1}{3}[1,1,1]$. In the toric geometry picture this orbifold is represented by the cone spanned by the vertices

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (-1,-1,3) \tag{5.2.1}$$

To see that this cone describes the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, we first construct the dual cone. This is done by the following procedure. If (a,b,c) is a vector in the dual cone, then we look for those vectors such that the inner product of this with each of the above vertices is positive semidefinite. this gives the following inequalities.

$$a \geq 0,\ b \geq 0,\ 3c \geq b + a \tag{5.2.2}$$

Now we have to solve these inequalities to get the basis vectors of the dual cone. All other solutions to (a,b,c) can be written as a positive linear combination of these basis vectors and moreover no basis vector can be expressed as a positive linear combination of any others. For the case at hand the solutions are given by the

following 10 vectors.

$$v_1' = (0,0,1),\ v_2' = (3,0,1),\ v_3' = (0,3,1),\ v_4' = (2,1,1)$$
$$v_5' = (1,2,1),\ v_6' = (1,1,1),\ v_7' = (2,0,1),\ v_8' = (0,2,1)$$
$$v_9' = (1,0,1),\ v_{10}' = (0,1,1) \tag{5.2.3}$$

Each of these vectors is associated with a monomial. For example

$$v_1' \equiv Z,\ v_2' \equiv X^3 \tag{5.2.4}$$

Now we will digress a bit. Consider polynomials in two variables (U, V). Then the domain over which these arbitrary polynomials are well defined ,which we denote by $\mathbb{C}[U, V]$, is actually $\mathbb{C}^2$, so $\mathbb{C}[U, V]$ is the coordinate ring of $\mathbb{C}^2$. We will use the shorthand notation $\mathbb{C}[U, V] \equiv \mathbb{C}^2$. Similarly if we look at the domain over which polynomials of the variables (U, V, U^{-1}, V^{-1}) are well defined it describes the space $(\mathbb{C}^*)^2$, because the functions are not defined at $(U, V) = (0, 0)$. Similarly if we consider polynomials in three variables (U, V, W), then $\mathbb{C}[U, V, W] \equiv \mathbb{C}^3$. The orbifold $\mathbb{C}^3/\mathbb{Z}_3$ with orbifold action $\frac{1}{3}[1,1,1]$ on (U, V, W) can be described as the domain over which all polynomials constructed out of variables, which are single valued on the orbifold, is defined. Therefore,

$$\mathbb{C}^3/\mathbb{Z}_3 \equiv \mathbb{C}[U^3, V^3, W^3, UVW, UV^2, VU^2, VW^2, WV^2, UW^2, WU^2] \tag{5.2.5}$$

Now we can see how to read off the space from the data we obtained from the dual cone. Writing the monomial associated to each of the dual basis vectors we construct the domain over which polynomials with these monomials as the variables are well defined. This in our notation is written as
$\mathbb{C}[Z, X^3Z, Y^3Z, X^2YZ, XY^2Z, XYZ, X^2Z, Y^2Z, XZ, YZ]$. After changing variables to $X = \frac{U}{W}$, $Y = \frac{V}{W}$ and $Z = W^3$ we get
$\mathbb{C}[W^3, U^3, V^3, U^2V, V^2U, UVW, U^2W, V^2W, UW^2, VW^2,]$. This is the description of $\mathbb{C}^3/\mathbb{Z}_3$, that we saw earlier.

5.2.1.1 Resolution of the orbifold

To resolve the orbifold, the strategy is to subdivide the cone into several smaller cones by inserting more vectors in the interior of the cone such that for each sub-cone the determinant of the generators of that particular cone, which is also the volume of the particular cone, is one. One can easily see that this criteria is not satisfied by the original cone itself. For the case of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, this is achieved by taking one more vector

$$v_4 = (0, 0, 1) \tag{5.2.6}$$

which subdivides the cone to three sub-cones, each of which have a unit determinant. The new cone so obtained is given in Figure 5.1 Now as before construct the dual cones for each of the cones, and as before we have the following inequalities.

cone 1	cone 2	cone 3
$c \geq 0,\ a \geq 0,\ 3c \geq a + b$	$c \geq 0,\ b \geq 0,\ 3c \geq a + b$	$c \geq 0,\ a \geq 0,\ b \geq 0$
$v_1' = (0, 3, 1), v_2' = (0, -1, 0)$	$v_1' = (3, 0, 1), v_2' = (-1, 0, 0)$	$v_1' = (1, 0, 0), v_2' = (0, 1, 0)$
and $v_3' = (1, -1, 0)$	and $v_3' = (-1, 1, 0)$	and $v_3' = (0, 0, 1)$
$\mathbb{C}[Y^3Z, Y^{-1}, XY^{-1}]$	$\mathbb{C}[X^3Z, X^{-1}, YX^{-1}]$	$\mathbb{C}[X, Y, Z]$

The divisor corresponding to v_4 is given by $Z = 0$ and is obtained by substituting $Z = 0$ in the above. Then one has the following spaces $\mathbb{C}[X, Y]$, $\mathbb{C}[X^{-1}, YX^{-1}]$, $\mathbb{C}[Y^{-1}, XY^{-1}]$. These are to be thought of as local coordinate patches of some space. What space do these patches describe? They describe the space $\mathbb{P}^2$. This can be seen by looking at the patches of $\mathbb{P}^2$. $\mathbb{P}^2$ is given by $(U, V, W) \sim (\lambda U, \lambda V, \lambda W)$. Then we have three patches given by the regions where U, V, W are individually non zero. In each of these patches the coordinates can be taken to be $(\frac{V}{U}, \frac{W}{U})$, $(\frac{U}{V}, \frac{W}{V})$, $(\frac{U}{W}, \frac{V}{W})$. Defining $X = \frac{U}{W}$ and $Y = \frac{V}{W}$, we have the following three patches (X, Y), (X^{-1}, YX^{-1}), (Y^{-1}, XY^{-1}), so comparing with what we got from the toric analysis we see that the space after resolution is indeed a $\mathbb{P}^2$ so we see that, $\mathbf{D_4} \equiv \mathbb{P}^2$

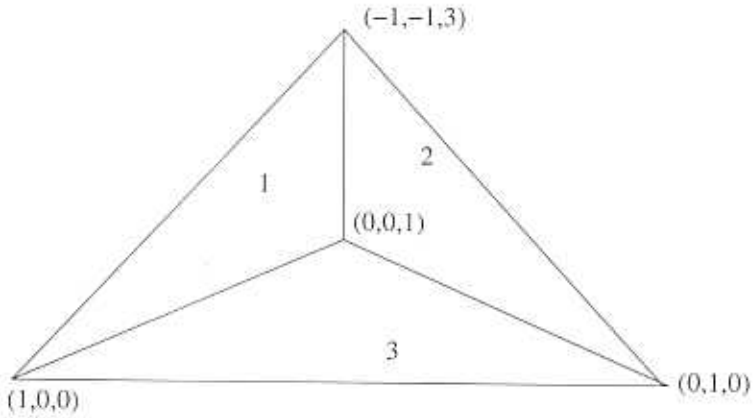

Figure 5.1: toric Diagram for the resolved $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

5.2.2 The $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

Now we consider the example of the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold with orbifold action $\frac{1}{5}[1,1,3]$. The vertices for the cone are given by

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (-1-3,5) \tag{5.2.7}$$

Using the same method outlined before one can check that this is indeed the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold.

5.2.2.1 The resolved $\mathbb{C}^3/\mathbb{Z}_5$

Following the process for resolution as described in the $\mathbb{C}^3/\mathbb{Z}_3$ example one finds that one has to insert two vertices

$$v_4 = (0,-1,2),\ v_5 = (0,0,1) \qquad \text{see Figure 5.2} \tag{5.2.8}$$

inside the cone to get the desired condition of unit determinant for the individual sub-cones. Now as before construct the dual fans for each of the cones, and as before we have the following inequalities.

$$\text{cone1: } c \geq 0,\ a \geq 0 \text{ and } b \geq 0 \tag{5.2.9}$$

The corresponding vertices are given by

$$v_1' = (1,0,0),\ v_2' = (0,1,0),\ v_3' = (0,0,1) \tag{5.2.10}$$

The space is given by $\mathbb{C}[X, Y, Z]$

$$\text{cone2: } c \geq 0,\ 5c \geq 3b + a \text{ and } b \geq 0 \tag{5.2.11}$$

The corresponding vertices are given by

$$v_1' = (-1, 0, 0),\ v_2' = (-3, 1, 0),\ v_3' = (5, 0, 1) \tag{5.2.12}$$

The space is given by $\mathbb{C}[X^{-1}, X^{-3}Y, X^5Z]$

$$\text{cone3:} c \geq 0,\ 2c \geq b \text{ and } 5c \geq 3b + a \tag{5.2.13}$$

The corresponding vertices are given by

$$v_1' = (3, -1, 0), v_2' = (-1, 2, 1), v_3' = (-1, 0, 0) \tag{5.2.14}$$

The space is given by $\mathbb{C}[X^3Y^{-1}, Y^2ZX^{-1}, X^{-1}]$

$$\text{cone4: } 2c \geq b,\ a \geq 0 \text{ and } 5c \geq 3b + a \tag{5.2.15}$$

The corresponding vertices are given by

$$v_1' = (0, 5, 3),\ v_2' = (1, -2, -1),\ v_3' = (0, -2, -1) \tag{5.2.16}$$

The space is given by $\mathbb{C}[Y^5Z^3, XY^{-2}Z^{-1}, Y^{-2}Z^{-1}]$

$$\text{cone5: } c \geq 0, 2c \geq b \text{ and } a \geq 0 \tag{5.2.17}$$

The corresponding vertices are given by

$$v_1' = (0, -1, 0),\ v_2' = (1, 0, 0),\ v_3' = (0, 2, 1) \tag{5.2.18}$$

The space is given by $\mathbb{C}[X, Y^{-1}, Y^2Z]$

Now the divisor D_4 corresponding to v_4 is given by $D_4 \equiv Z^2/Y = 0$. To find out what space this divisor corresponds to one has to analyse all the cones of which this is a common point. These will be the coordinate patches of the corresponding space. Since there are three cones surrounding this point, the corresponding space should be a $\mathbb{P}^2$. This can be checked rigorously, exactly as before.

To do this we substitute $D_4 = 0$ in the cones (3), (4) and (5). Take $Y^2Z \equiv A$ and $D_4 = 0$. We then get for the corresponding patches, $\mathbb{C}[AX^{-1}, X^{-1}]$, $\mathbb{C}[A, X]$ and $\mathbb{C}[XA^{-1}, A^{-1}]$ respectively. As noted earlier these are the patches of $\mathbb{P}^2$, so

$$D_4 \equiv \mathbb{P}^2 \tag{5.2.19}$$

To find the space corresponding to D_5 given by $Z = 0$ we similarly substitute $Z = 0$ in the patches (1),(2),(3),(5). We then get for the corresponding patches:

$$(1)\ \mathbb{C}[X, Y],\ (2)\ \mathbb{C}[X^{-1}, X^{-3}Y],\ (3)\ \mathbb{C}[X^3Y^{-1}, X^{-1}],\ (5)\ \mathbb{C}[Y^{-1}, X] \tag{5.2.20}$$

These are the coordinate patches associated with the space $\mathbb{F}_3$. In general the space $\mathbb{F}_a$ has the following four coordinate patches[36].

$$(X^{-1}, X^aY), (X, Y), (X^{-1}, X^{-a}Y^{-1}), (X, Y^{-1})$$

So we see that

$$D_5 \equiv \mathbb{F}_3 \tag{5.2.21}$$

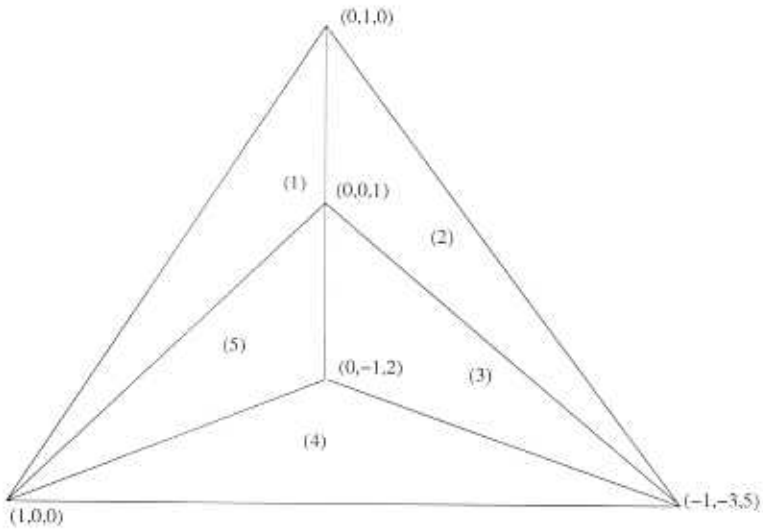

Figure 5.2: toric Diagram for the resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold with orbifold action $\frac{1}{5}[1,3,1]$

5.2.3 The $\mathbb{C}^3/\mathbb{Z}_7$ orbifold

Finally we will describe the Toric construction for the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold. We will consider the orbifold actions $\frac{1}{7}[5,1,1]$ and $\frac{1}{7}[2,4,1]$.

5.2.3.1 The $\mathbb{C}^3/Z_7$ orbifold with orbifold action $\frac{1}{7}[5,1,1]$

The orbifold is represented by the vertices:

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (-5,-1,7), \tag{5.2.22}$$

After resolution one has to add

$$v_4 = (0,0,1),\ v_5 = (-1,0,2),\ v_6 = (-2,0,3),\ \text{ see Figure 5.3}$$

The coordinate patches corresponding to the various cones are:

$$\begin{aligned}
&\text{cone1: } \mathbb{C}[X^7Y^5, X^{-3}Z^{-2}, X^{-3}YZ^{-2}],\ \text{cone2: } \mathbb{C}[Y, X^3Z^2, X^{-2}Z^{-1}]\\
&\text{cone3: } \mathbb{C}[Y^{-1}, X^3Y^{-1}Z^2, X^{-2}Y^3Z^{-1}],\ \text{cone4: } \mathbb{C}[Y, X^{-1}, X^2Z]\\
&\text{cone5: } \mathbb{C}[Y^{-1}, X^2Y^{-3}Z, X^{-1}Y^5],\ \text{cone6: } \mathbb{C}[X,Y,Z],\ \text{cone7: } \mathbb{C}[Y^{-1}, XY^{-5}, Y^7Z]
\end{aligned} \tag{5.2.23}$$

To find the Divisors D_4, D_5, D_6 one has to do a analyses of the various cones, similar to the one discussed above for the $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$ orbifolds. The final answer turns out to be:

$$D_4 \equiv \mathbb{F}_5,\ D_5 \equiv \mathbb{F}_3 \text{ and } D_6 \equiv \mathbb{P}^2 \tag{5.2.24}$$

5.2.3.2 The $\mathbb{C}^3/\mathbb{Z}_7$ orbifold with orbifold action $\frac{1}{7}[4,2,1]$

In this case also one has three divisors on resolution, however the structure of the diagram is quite different(see Figure 5.4) The vertices are :

$$\begin{aligned}
&v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (-4,-2,7)\\
&v_4 = (0,0,1),\ v_5 = (-1,0,2),\ v_6 = (-2,-1,4)
\end{aligned} \tag{5.2.25}$$

The coordinate patches corresponding to the various cones are:

$$\begin{aligned}
&\text{cone1: } \mathbb{C}[X,Y,Z],\ \text{cone2: } \mathbb{C}[Y, X^{-1}, X^2Z],\ \text{cone3: } \mathbb{C}[X^{-2}Z^{-1}, X^{-4}YZ^{-2}, X^7Z^4]\\
&\text{cone4: } \mathbb{C}[Y^2X^{-1}, X^4Y^{-1}Z^2, X^{-2}Z^{-1}], \text{cone5: } \mathbb{C}[XY^{-2}, Y^{-4}Z^{-1}, Y^7Z^2]\\
&\text{cone6: } \mathbb{C}[Y^{-1}, XY^{-2}, Y^4Z], \text{cone7: } \mathbb{C}[Y^{-1}, X^{-1}Y^2, X^2Z]
\end{aligned} \tag{5.2.26}$$

From the analyses of the cones, the divisors D_4, D_5, D_6 turn out to

$$D_4 \equiv \mathbb{F}_2,\ D_5 \equiv \mathbb{F}_2 \text{ and } D_6 \equiv \mathbb{F}_2 \tag{5.2.27}$$

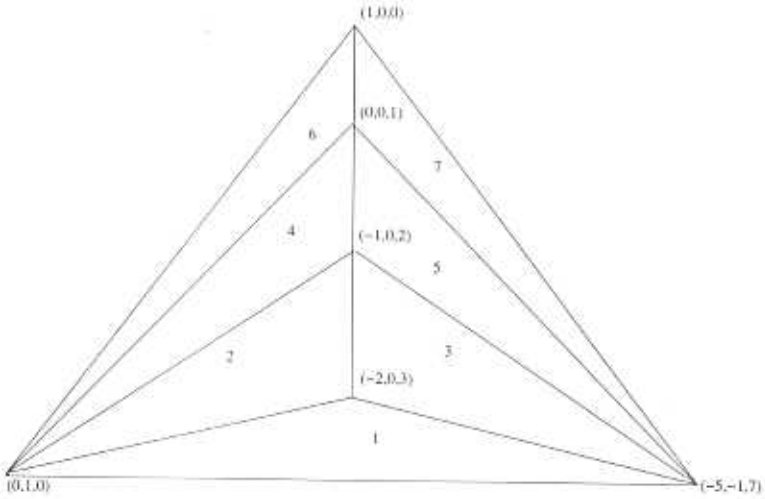

Figure 5.3: toric Diagram for the resolved $\mathbb{C}^3/\mathbb{Z}_7$ orbifold with orbifold action $\frac{1}{7}[5,1,1]$ orbifold

5.2.4 Triple intersections of divisors

In the last section we saw how the toric data encodes the information of the orbifold as well as it's resolution. We also saw how it encodes the information regarding the exceptional divisors which arise in the blow up of the orbifold, for example the divisor $D_4 \equiv \mathbb{P}^2$ in the case of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. In this section we will show how the toric data also encodes the information of all the triple intersection numbers of various divisors. Here we will give the working rules to compute the triple intersections of divisors in the simple example of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. One can do the same for the other examples considered in this chapter.

There are three sub-cones in the toric picture of the resolved $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, The triple intersections involving three different divisors is one if the vertices corresponding to the divisors span a cone. All other triple intersections with three

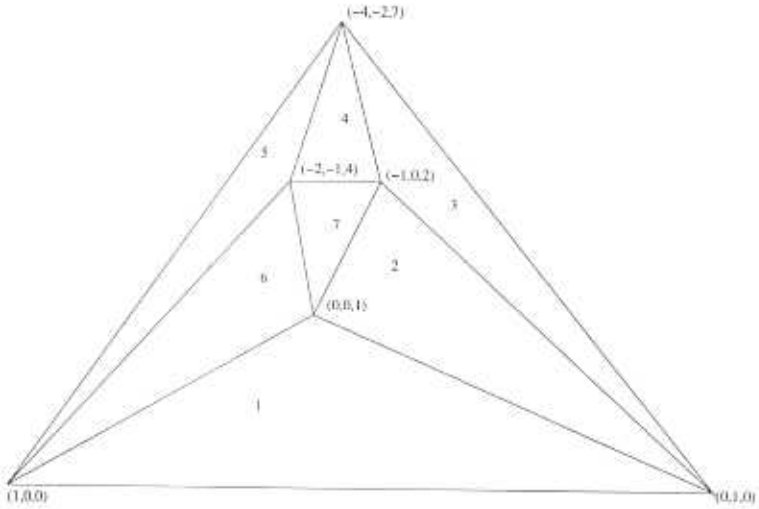

Figure 5.4: toric Diagram for the resolved $\mathbb{C}^3/\mathbb{Z}_7$ orbifold with orbifold action $\frac{1}{7}[4,2,1]$

different divisors which do not span a cone vanish. So for example from Figure 5.1 one sees that

$$D_1 \cdot D_2 \cdot D_4 = 1, \quad D_3 \cdot D_2 \cdot D_4 = 1, \quad D_1 \cdot D_3 \cdot D_4 = 1 \tag{5.2.28}$$

In this example of course these are the only triple intersections involving all different divisors. However in other examples one can have triple intersections of different divisors which do not span a cone which vanish. Now one has to compute the other triple intersections, which involve self-intersections of divisors. To compute these, one has to use the linear equivalence of divisors. There is an easy way to read off the linear equivalences given a toric diagram. We will give an outline how to work out the linear equivalence relations. In the example we are considering, the four basis vectors satisfy the relation,

$$v_1 + v_2 + v_3 - 3v_4 = 0 \tag{5.2.29}$$

The linear equivalence relations can be obtained from comparing the coefficients of the vertices. So we have

$$D_1 \sim D_2 \sim D_3 \sim -\frac{1}{3} D_4 \tag{5.2.30}$$

Now in other examples we will have more than one such relations like equation 5.2.29. In those cases we will have to find the common solutions to all such relations. The linear equivalences can be used to compute the triple intersections involving self intersections. For example in this case we have,

$$D_1^2 \cdot D_4 = D_2^2 \cdot D_4 = D_3^2 \cdot D_4 = 1, \tag{5.2.31}$$

$$D_4^2 \cdot D_1 = D_4^2 \cdot D_2 = D_4^2 \cdot D_3 = -3 \text{ and } D_4^3 = 9 \tag{5.2.32}$$

The triple intersections for the $\mathbb{C}^3/\mathbb{Z}_5$ are given in [59], while the triple intersections for the $\mathbb{C}^3/\mathbb{Z}_7$ example are worked out in appendixB.

5.3 Fractional branes in $\mathbb{C}^3/\mathbb{Z}_3$

In this section we will give a description of the fractional zero branes as well as fractional two branes in the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold with orbifold action $\frac{1}{3}[-2,-2,-2]$, in the toric set-up. The toric diagram is given in Figure 5.1 The idea, as already explained in the beginning of this chapter is that given the R_0^i one can find the $\mathrm{ch}(S_i^0)$ from solving the equation,

$$\int \mathrm{ch}(R_0^{i*})\, \mathrm{ch}(S_j^0)\, \mathrm{Td}(X) = \delta_j^i \tag{5.3.1}$$

Here X is the resolved space $\mathcal{O}_{\mathbb{P}^2}(-3)$, and $\mathrm{Td}(X)$ can be written down in terms of the toric divisors D_i.

$$\mathrm{Td}(X) = \prod_{i=1}^{4} \frac{D_i}{1-\exp(-D_i)} \tag{5.3.2}$$

Using the triple intersections computed in the last section, one can solve equation (5.3.1) to obtain $\mathrm{ch}(S_i^0)$ explicitly, in terms of the classes of the divisors and of other lower dimensional cycles. As a check one can also compute $\langle S_i^0, S_j^0 \rangle$ and compare the result with the intersection forms computed at the orbifold end. Now we can similarly find $\mathrm{ch}(S_i^2)$ from solving

$$\int \mathrm{ch}(S_i^{0*})\, \mathrm{ch}(S_j^2)\, \mathrm{Td}(X) = I_{0,2} \tag{5.3.3}$$

where $I_{0,2}$ is the intersection forms as computed in the orbifold end. We take an ansatz for the $\mathrm{ch}(S_i^2)$ of the form

$$\mathrm{ch}(S_i^2) = \sum_j a_j^i D_j + \sum_j b_j^i h_j + \sum_j b_j'^i h_j' + c^i p \tag{5.3.4}$$

Where we use the same symbol D_j to denote the class of the divisor D_j and similarly h_j denotes the class of the compact 2-cycle h_j, h_j' denotes the class of a non-compact cycle and similarly p denotes class of a point 'p'. We will now proceed with the explicit construction.

The intersection forms computed at the orbifold end are given below. The D0-D2 intersection has been multiplied by a factor of g^{-2}, to account for the change of basis in the large volume as compared to the orbifold end, as has been noted in earlier chapters.

$$D0 - D0 = (1-g)^3, D2 - D2 = -g(1-g) \text{ and } D0 - D2 = (1-g)^2 \tag{5.3.5}$$

Define:

$$\begin{aligned} I_{i,j} &= \langle S_i^0, S_j^2 \rangle \\ A_{i,j} &= \langle S_i^2, S_j^2 \rangle \end{aligned} \tag{5.3.6}$$

The 'tautological branes' R_0^i can be computed following the methods of [[24],[25],[26],[59]]. They are:

$$R_0^0 = \mathcal{O}, \quad R_0^1 = \mathcal{O}(D_1), \quad R_0^2 = \mathcal{O}(2D_1) \tag{5.3.7}$$

Given the R_0^i one can compute the $\mathrm{ch}(S_i^0)$, as has been outlined in the beginning of this section.

$$\begin{aligned} \mathrm{ch}S_0^0 &= D_4 + (3/2)h + \frac{3}{2}p \\ \mathrm{ch}S_1^0 &= -2D_4 - 2h - p \\ \mathrm{ch}S_2^0 &= D_4 + h/2 + \frac{1}{2}p \end{aligned} \tag{5.3.8}$$

Using these expressions for the $\mathrm{ch}(S_i^0)$, one can now compute the $\mathrm{ch}(S_i^2)$.

In chapter 3, we saw that one of the fractional 2-branes was purely along a non-compact 2-cycle. We will use this result to motivate an ansatz for the fractional 2-branes. We take the following ansatz for the fractional 2-branes,

$$\begin{aligned} \mathrm{ch}S_0^2 &= a_1 D_4 + a_2 D_1 \cdot D_2 + a_2' h + c_0 p \\ \mathrm{ch}S_1^2 &= b_1 D_4 + b_2 D_1 \cdot D_2 + b_2' h + c_1 p \\ \mathrm{ch}S_2^2 &= D_1 \cdot D_2 \end{aligned} \tag{5.3.9}$$

The form of equation (5.3.4) and subsequently equation (5.3.9) merits some detailed comment. In general, the Chern character is a map from $K_0(X)$ to the Chow group with rational coefficients $A^*(X)_{\mathbb{Q}}$ of the variety. By definition

$$A^*(X)_{\mathbb{Q}} = \oplus A^p(X)_{\mathbb{Q}} = \oplus A_{n-p}(X) \otimes \mathbb{Q}$$

In the toric setting, the Chow group $A_k(X)$ of $X = X(\Delta)$ (where Δ is the corresponding fan), is generated by all the classes of the orbit closures $V(\sigma)$ of $(n-k)$ dimensional cones σ of Δ (modulo relations). This allows a particularly simple algorithm to write the most general Chern character in terms of all the divisors D_i which correspond to one dimensional cones and the intersection of these divisors associated to orbit closures of higher dimensional cones spanned by these one-dimensional cones. These terms have co-efficients that we now try to fix.

The ansatz we make above is based on the fact that they solve the $I_{i,j}$ equations. For instance the rank term must be zero. Nor are any D_1, D_2, D_3 allowed. The computation involves some trivial algebra that we do not present here. Note, in particular, that the $D_1 \cdot D_2$ term corresponds to the fibre in the total space of the bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$. This can be shown following [36], (chap. 3, section 3.1). The S_2^2 would correspond to the 'Coloumb brane' of chapter 4. This form of S_2^2 automatically solves all the $S_{i,2}$ equations. Now using $A_{0,2} = 1$ we get $a_1 = 1$ Similarly,$A_{1,2} = -1$ gives $b_1 = -1$ Solving the remaining equations involving $S_{i,0}$ and $S_{i,1}$ we get

$$\begin{aligned} 2a_2 - 6a_2' &= -7 \\ 2b_2 - 6b_2' &= 5 \end{aligned} \tag{5.3.10}$$

The remaining equation $A_{1,0}$ does not give any new relation, but is consistent with the other equations. The c_i remain undetermined.

The above equations (5.3.10) imply that we cannot explicitly solve for all the variables and so we have to make a further choice.[1] The simplest choice we can make is to take the coefficient of the non-compact fibre to be 1. So we take $a_2 = b_2 = 1$. We then get,

$$\begin{aligned} \mathrm{ch}S_0^2 &= D_4 + D_1{\cdot}D_2 + \frac{3h}{2} + c_0 p \\ \mathrm{ch}S_1^2 &= -D_4 + D_1{\cdot}D_2 + \frac{-h}{2} + c_1 p \\ \mathrm{ch}S_2^2 &= D_1{\cdot}D_2 \end{aligned} \tag{5.3.11}$$

In chapter 3 we argued, that the fractional two branes when continued to large volume in the Kähler moduli space are described by sheaves which have support on a non-compact two cycle, unlike the case of the fractional zero branes, which are described by bundles with support on a compact two cycle. Further we saw that one of them was purely along a non-compact direction. In this section we have been able to realize these objects in the toric geometry framework. We have not constructed them independently, rather we have used the results in chapter 3 to motivate our ansatz. However, the fact that we have been able to find consistent solutions as in equation (5.3.11) for these objects in the toric framework ensures that the constructions are mathematically well defined, unlike the sequences that we wrote down in chapter 3.

In chapter 3 we had further argued that the charge of such a two brane wrapping the non-compact two cycle would be fractional when measured in the basis of the two brane with compact cohomology. In the toric geometry description this is the statement of linear equivalence of divisors. As discussed in the previous section, $D_1 \sim \frac{-1}{3} D_4$. This implies that $D_1{\cdot}D_2 \sim \frac{-1}{3} h$, which is a statement that in any computation involving the intersection of $D_1{\cdot}D_2$ with any compact divisor one can instead use $\frac{-1}{3} h$. We re-emphasize however that linear equivalence can be used in these computations only in the presence of a compact divisor in one term,

[1]This is simply a consequence of the linear equivalence of the divisors.

in the triple intersections. Then in the other terms we may use linear equivalence. In appendix C we will similarly redo the analysis for $\mathbb{C}^3/\mathbb{Z}_5$ and $\mathbb{C}^3/\mathbb{Z}_7$ orbifolds.

Indeed we can go further and identify the objects themselves using the Chern character and the heuristic sequences we wrote down in chapter 3. If we ignore the $D_1 \cdot D_2$ term in the Chern character of S_1^2, S_2^2, then we can show that the rest of the terms correspond to the objects $i_*\mathcal{O}_{\mathbb{P}^2}$ and $i_*(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to \mathcal{O}(1))$. If we refer back to the sequences for S_1^2, S_2^2 that we wrote down in chapter 3 and the corresponding Chern characters on $\mathbb{P}^2$, we find that the objects F_0 and F_1 are precisely $\mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \to \mathcal{O}(1)$, if we ignore the fractional part. F_2 was argued to be purely of fractional Chern class and so corresponds to S_0^2. Thus despite the imprecisely defined sequences that we used, they appear to give the correct picture, not only for the charges (Chern characters) but also the objects themselves.

Chapter 6

Fractional two-branes: the quintic in $\mathbb{P}^4$

In this chapter, we will identify, the D-branes associated with the boundary conditions given in Eq. (1.6.1) for the LG orbifold, in the large volume after restriction to the quintic, following arguments similar to the ones used in chapter 3. These new D-branes will be shown to have the same intersection numbers as those of the new fractional branes proposed in [29], which were obtained by the techniques of matrix factorization of the superpotential.

Hence we identify the D0 brane on the quintic as one of the states obtained on restriction of the large volume analogue of the fractional 2-branes associated to the boundary condition 1.6.1, on the quintic hypersurface. This, we believe, provides *geometric insight* into the categorical construction of [29] in addition to identifying two apparently distinct approaches to D-branes on LG orbifolds. For purposes of illustration, we first consider the case of the fractional zero-branes.

6.1 Fractional zero-branes from Euler sequences

As has already been mentioned in chapter 1, the large volume analogue of the fractional zero-branes of $\mathbb{C}^5/\mathbb{Z}_5$ (to be identified with the Recknagel-Schomerus states on restriction to the quintic) are obtained by imposing Dirichlet boundary conditions on

all fields in the LG orbifold. Following the arguments of section 3.1.1 this leaves us with *five* independent fermionic multiplets on the boundary. We will use the GLSM to interpolate between the LG orbifold and the nonlinear sigma model(NLSM) as we did in chapter 3 for the orbifold theory.

As was mentioned in section 3.2, the simplest way to obtain the LG orbifold from the GLSM is to consider the limit $e^2 r \to -\infty$. The gaugino (of the GLSM) constraint [44] for this case is

$$\sum_{i=1}^{5} \phi_i \bar{\psi}_{\pm i} = 0 \ .$$

When one imposes *Dirichlet* boundary conditions on all fields, the gaugino constraint on the boundary becomes

$$\sum_i \phi_i \bar{\xi}_i = 0 \tag{6.1.1}$$

where as before $\bar{\xi}_i$ is the fermionic combination that is not set to zero by the boundary conditions.

Following the argument in chapter 3, setting $J^a \equiv \phi^a$ gives us the Euler sequence on $\mathbb{P}^4$

$$0 \to \Omega(1) \to \mathcal{O}^{\oplus 5} \to \mathcal{O}(1) \to 0$$

with the boundary condition [1].

$$\bar{\xi}_i = i\pi_i$$

We have five boundary states associated with the LG orbifold. Similar to the orbifold case, the other four coherent sheaves are given by the following exact sequences that can be derived from the Euler sequence associated with $\Omega^p(p) \equiv \wedge^p\Omega \otimes \mathcal{O}(p)$

$$0 \to \Omega^p(p) \to \mathcal{O}^{\oplus\binom{5}{p}} \to \Omega^{p-1}(p-1) \otimes \mathcal{O}(1) \to 0 \tag{6.1.2}$$

[1]The fact that one can bypass the introduction of boundary fermions by treating the $\bar{\xi}_i$ as boundary fermions and the gaugino constraint as a holomorphic constraint is also a hint why the matrix factorisation used in [29] must be equivalent to the boundary conditions considered in [13], at least for the case of linear factors

Finally as in the orbifold case, the five fractional branes for the quintic are in one-to-one correspondence with the states: (the vacuum $|0\rangle$ satisfies $\bar{\xi}_i|0\rangle = 0$)

$$|0\rangle \quad , \quad \xi_i|0\rangle \quad , \quad \xi_i\xi_j|0\rangle \quad , \quad \xi_i\xi_j\xi_k|0\rangle \quad , \quad \xi_i\xi_j\xi_k\xi_l|0\rangle$$

subject to the condition $\phi_i\bar{\xi}^i = 0$ being imposed.

Define following the notation of [29]

$$\boxed{\mathcal{M}_i \equiv (-)^i \Omega^1_{\mathrm{p4}}(i)\Big|_{\text{quintic}} \quad , \; i = 0, 1, 2, 3, 4} \tag{6.1.3}$$

These branes $\mathcal{M}_i$ can be identified as the result of the analytic continuation to large-volume of the RS states in the Gepner model for the quintic. Under the quantum $\mathbb{Z}_5$ symmetry(generated by g), one has

$$g: \quad \mathcal{M}_i \to \mathcal{M}_{i+1} \mod 5$$

A basic check for this identification is the matching of the intersections of the $\mathcal{M}_i$ to the CFT computations of the intersections for the $L_i \equiv 0$ states from among the RS states in the Gepner model.

6.2 Fractional two-branes from generalised Euler sequences

We are now ready to discuss the case of fractional two-branes. The discussion is very similar to the discussion of the fractional 2-branes in chapter 3. As we have already seen, the Neumann boundary condition on the combination $(\phi_1 - \phi_2)$ is obtained from the supersymmetric variation of the boundary condition $\xi_1 = \xi_2$. Thus, on the boundary, given these boundary conditions, we have effectively *four* independent fermionic multiplets. These fermionic multiplets are still subject to the condition imposed by the gaugino constraint of the GLSM. After eliminating $\bar{\xi}_2$ in favour of $\bar{\xi}_1$ using Eq. (1.6.1), the gaugino constraint can be re-written as the following condition

$$(\phi_1 + \phi_2)\bar{\xi}_1 + \sum_{i=3}^{5} \phi_i\bar{\xi}_i = 0 \tag{6.2.1}$$

Thus, this is equivalent to having four boundary fermions subject to the one condition above. Unlike the case of fractional zero-branes, we see that Eq. (6.2.1) is trivially satisfied when $\phi_1 + \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0$. This is possible on $\mathbb{P}^4$, when $\phi_1 - \phi_2 \neq 0$. These conditions specify a two-brane (denoted below by P) in the manifold which is the resolution of the $\mathbb{C}^5/\mathbb{Z}_5$ singularity. It is trivial to see that this two-brane restricts to a point on the quintic. Away from the two-brane P, Eq. (6.2.1) does reduce the number of fermions to three. This implies that the fermions are sections of the sheaf F_1 given by the following sequence

$$0 \to F_1 \to \mathcal{O}^{\oplus 4} \to \mathcal{O}(1) \to \mathcal{X}_P \to 0 \tag{6.2.2}$$

The term involving $\mathcal{X}_P$ has been added to take care of the fact that (6.2.1) is trivially satisfied on P. Then following the arguments for the fractional 2-branes in chapter 3 one can write down the sequences for all the fractional branes. Again one of them is given by a sequence which when restricted to the $\mathbb{P}^3$ not containing P becomes the Euler sequence. The remaining fractional branes are given by exact sequences that on restriction to a $\mathbb{P}^3$ not containing P give the generalised Euler sequences of $\mathbb{P}^3$. Finally we can write the explicit sequences as

$$\begin{aligned}
0 &\to F_0 \to \mathcal{O} \to \mathcal{X}_P \otimes \mathcal{O}(-1) \to 0 \\
0 &\to F_1 \to \mathcal{O}^{\oplus 4} \to F_0 \otimes \mathcal{O}(1) \to 0 \\
0 &\to F_2 \to \mathcal{O}^{\oplus 6} \to F_1 \otimes \mathcal{O}(1) \to 0 \\
0 &\to F_3 \to \mathcal{O}^{\oplus 4} \to F_2 \otimes \mathcal{O}(1) \to 0 \\
0 &\to F_4 \to \mathcal{O} \to F_3 \otimes \mathcal{O}(1) \to 0
\end{aligned} \tag{6.2.3}$$

One can argue similar to chapter 3 that $F_4 = -\mathcal{X}_P \otimes \mathcal{O}(3)$. That F_4 must at least have $\mathcal{X}_P$ as a factor is clear since the last sequence must restrict to zero on on the $\mathbb{P}^3 \in \mathbb{P}^4$ not containing P. This is because there is no corresponding generalized Euler sequence that appears on $\mathbb{P}^3$. The factor of $\mathcal{O}(3)$ can be deduced from the general pattern that we observe in these sequences. Keeping our previous notation, we will refer to F_4 as the *Coulomb branch brane* because of this vanishing property on restriction to the $\mathbb{P}^3$ not containing P and the remaining branes $(F_0, \ldots, F_3)$ will be called as the *Higgs branch branes*.

The main claim we wish to make is that the new fractional branes of [29] are to be identified with (a minus sign indicates an anti-brane)

$$\boxed{\mathcal{F}_i = (-)^i F_i\Big|_{\text{quintic}} \ , \quad i = 0, 1, 2, 3, 4} \tag{6.2.4}$$

provided we choose

$$\text{ch}\ (\mathcal{X}_P)\big|_{\text{quintic}} = J^3/5 \ . \tag{6.2.5}$$

where J generates the Kähler class on the quintic and is normalised such that $\langle J^3\rangle_{\text{quintic}} = 5\langle J^4\rangle_{\mathbb{P}^4} = 5$. As a first check, we have verified that the Chern classes of the $\mathcal{F}_i$ agree with those given by Ashok et. al. in [29]. (More details are provided in appendix D.) We also propose that under the quantum $\mathbb{Z}_5$ symmetry, one has

$$g: \quad \mathcal{F}_i \to \mathcal{F}_{i+1}\text{mod}5$$

In the orbifold case we motivated the existence of fractional charges by a K-theory computation. We were also able to see this in the toric geometry as a consequence of the linear equivalence of divisors. Here we will motivate the assignment of the fractional Chern character for the object $\mathcal{X}_P$ from another argument. It is clear from a simple argument that sheaves in the ambient projective space (obtained from the fractional branes by blowing up the orbifold singularity) when restricted to the Calabi-Yau would fail to give objects that have the charge of a single zero-brane on the CY. A two-brane wrapping a $\mathbb{P}^1 \in \mathbb{P}^4$ (which intersects the quintic on a point) will have Chern character $J^3 + aJ^4$ for some a. On restricting this to the quintic, we obtain an object with Chern character J^3 which has the charge of 5 zero-branes. Thus if we need to produce an object with the charge of a single zero-brane on the CY it appears that we must begin with a sheaf on $\mathbb{P}^4$ whose Chern character has leads off with a $J^3/5$ term.

The five new fractional two branes for the quintic are in one-to-one correspondence with the states: (the vacuum $|0\rangle$ satisfies $\bar{\xi}_i|0\rangle = 0$ and $i = 1, 3, 4, 5$ below)

$$F_0 \sim |0\rangle \ , \quad F_1 \sim \xi_i|0\rangle \ , \quad F_2 \sim \xi_i\xi_j|0\rangle \ , \quad F_3 \sim \xi_i\xi_j\xi_k|0\rangle \ , \quad F_4 \sim \xi_1\xi_3\xi_4\xi_5|0\rangle$$

subject to the modified gaugino constraint in Eq. (6.2.1) being imposed on them.

In the next chapter we will argue that in the Gepner model these new fractional branes are described by the permutation branes [41]

Chapter 7

Landau-Ginzburg description of permutation branes

7.1 Permutation branes

In the previous chapter we argued that the 'new fractional branes' of [29] correspond to imposing the following boundary conditions on the fields of the LG orbifold associated to the quintic CY.

$$\begin{aligned} &\phi_1 + \phi_2 = 0, \quad \phi_3 = \phi_3 = \phi_3 = 0 \\ &\xi_1 - \xi_2 = 0 \end{aligned} \tag{7.1.1}$$

where the ϕ_i are the fields of the LG orbifold. As we discussed in the introduction, the Recknagel-Schomerus states (with $L_i = 0$) in the Gepner model for the quintic correspond to imposing Dirichlet boundary conditions on every field of the LG orbifold. It is therefore of interest to ask what boundary states these new boundary condition of eq. (7.1.1) describe in the context of the Gepner models. In this chapter we will argue that the new boundary conditions correspond to a subset of the permutation branes of [41]. In the next section we will begin by recalling further details of the Gepner models as well as the Recknagel Schomerus (RS) branes and permutation branes.

7.1.1 Gepner models in the bulk

As we mentioned earlier, Gepner models are tensor products of r unitary $\mathcal{N} = (2,2)$ super-conformal theories(SCFT), called the minimal models, with a total central charge $c = \sum_{i=1}^{r} \frac{3k_i}{k_i+2}$.

The $N = 2$ super conformal algebra is

$$
\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
&[L_n, J_m] = -mJ_{n+m} \\
&[J_n, J_m] = \frac{c}{3}n\delta_{n+m,0} \\
&[L_n, G_r^{\pm}] = (\frac{n}{2} - r)G_{n+r}^{\pm} \\
&[J_n, G_r^{\pm}] = \pm G_{n+r}^{\pm} \\
&[G_r^{\pm}, G_s^{\mp}] = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}
\end{aligned}
\tag{7.1.2}
$$

where L_n are the Virasoro generators, the $G_s^{\pm}$ are the super-conformal generators and the J_n are the U(1) generators. Also n, m run over integers while r may be integer or half integer. The conformal weight and U(1) charges of the primary fields are given by

$$
\begin{aligned}
h_{m,s}^{l} &= \frac{l(l+2) - m^2}{4(k+2)} + \frac{s^2}{8}, \\
q_{m,s}^{l} &= \frac{m}{k+2} - \frac{s}{2}
\end{aligned}
\tag{7.1.3}
$$

where l, m, s are integers defined in the range, $0 \leq l \leq k$, $m \in Z_{2k+4}$, $s \in Z_2$, and $l + m + s =$ even, with the further identification on the characters $\chi_{m,s}^{l}$ that $\chi_{m,s}^{l} = \chi_{k+m+2,s+2}^{k-l}$. The states in the NS sector have $s = \pm 1$ while the states in the R sector have $s = 0, 2$. The minimal models have a $Z_{k+2} \times Z_2$ symmetry, with the action on the primary fields given by

$$
g: \ \phi_{m,s}^{l} = \exp \frac{2i\pi m}{k+2}\phi_{m,s}^{l}, \ h: \ \phi_{m,s}^{l} = (-1)^s \phi_{m,s}^{l} \tag{7.1.4}
$$

To be a consistent string compactification with four dimensional spacetime susy, one has to ensure that the total central charge $c = 9$. Further one has to ensure that the fields in the product theory are products of fields with same spin structure, i.e. all

NS or all Ramond states. The GSO projection is implemented by projecting onto states with odd total U(1) charge of the superconformal algebra. Gepner showed that a consistent modular invariant partition function can be constructed for such models. Following the notation of [3] this is given by (in the lightcone gauge)

$$Z = \sum_{(i,\bar{i}),\lambda,\mu} \sum_{b_0,b_j} \delta_\beta (-1)^{b_0} \chi_{i,\lambda,\mu}(q) \chi_{\bar{i},\lambda,b_0\beta_0+\sum_j b_j\beta_j}(\bar{q}) \tag{7.1.5}$$

where λ is the vector given by $\lambda = (l_1, ..., l_r)$, and $\mu = (m_1, ..., m_r; s_1, ...s_r)$ and β_j is a vector with value 2 at the position where μ has an entry s_j and all other values are zero. Similarly β_0 is a vector with all entries one, and $b_0 = 0, ...19$ and $b_j = (0, 1)$. $i, \bar{i}$ label the transverse space time coordinates. $\chi_{i,\lambda,\mu}$ is the product of the characters of the spacetime part as well as the individual minimal models of the Gepner model.

7.1.2 Boundary states in the Gepner models

In the context of B-type branes one is looking for those boundary states which preserve half of the supersymmetry of the bulk theory.

The super conformal algebra admit the following automorphism action on the generators

$$\Omega : J_n \to J_{-n}, \quad G_r^{\pm} \to G_r^{\mp} \tag{7.1.6}$$

Therefore there are two possible sets of Ishibashi conditions, and two possible types of boundary states. These are called the A-branes and B-branes.

$$\begin{aligned} (L_n - \bar{L}_{-n})|\text{A-brane}\rangle &= 0, \quad (J_n - \bar{J}_{-n})|\text{A-brane}\rangle = 0 \\ (G_r^+ + i\eta\overline{G}_{-r}^-)|\text{A-brane}\rangle &= 0, \quad (G_r^+ + i\eta\overline{G}_{-r}^-)|\text{A-brane}\rangle = 0, \end{aligned} \tag{7.1.7}$$

$$\begin{aligned} (L_n - \bar{L}_{-n})|\text{B-brane}\rangle &= 0, \quad (J_n + \bar{J}_{-n})|\text{B-brane}\rangle = 0 \\ (G_r^+ + i\eta\overline{G}_{-r}^+)|\text{B-brane}\rangle &= 0, \quad (G_r^- + i\eta\overline{G}_{-r}^-)|\text{B-brane}\rangle = 0, \end{aligned} \tag{7.1.8}$$

In what follows we will restrict our attention to the B-branes in the specific example of the 3^5 Gepner models, which is in the same Kähler moduli space as the quintic

hyper-surface in $\mathbb{P}^4$. The generators of the super conformal algebra of the product theory is the sum of the generators in the individual theories and therefore we have to solve the conditions

$$\sum_{i=1,5}(W^i_n + \alpha\overline{W}^i_{-n})|\text{B-branes}\rangle = 0 \tag{7.1.9}$$

where the W^i, are the generators of the symmetries, in this case $L_n, J_n, G^{\pm}_n$ and the index α can be ± 1 or $i\eta$, etc.

7.1.2.1 Recknagel-Schomerus states of the quintic

Recknagel and Schomerus provided the first solution of the above conditions by constructing boundary states [15] which were the product of boundary states in each of $N = (2,2)$ theories that make up the Gepner models. i.e,

$$\begin{aligned}
&(W^i_n + \alpha\overline{W}^i_{-n})|\text{B}_i\rangle = 0, \quad i = 1 \text{ to } 5 \\
&|\text{B}\rangle \sim \prod_{i=1}^{5}|B_i\rangle
\end{aligned} \tag{7.1.10}$$

In [15] Recknagel and Schomerus constructed the boundary states which had the correct modular transformation properties. Their ansatz (for the internal part alone)was

$$|A\rangle = \sum_{\lambda,\mu}\delta_\beta\delta_\Omega B^{\lambda,\mu}_A|\lambda,\mu\rangle \tag{7.1.11}$$

$$B^{\lambda,\mu}_A = \prod_i \frac{s(l_i, L_i)}{\sqrt{s(l_j, 0)}}\exp(i\frac{\pi M_i \cdot m_i}{k+2})\exp(-i\frac{\pi S_i \cdot s_i}{2}) \tag{7.1.12}$$

$$\text{and } (L, l) = \pi\frac{(L+1)(l+1)}{k+2} \tag{7.1.13}$$

Here δ_β imposes the U(1) integrality condition while the δ_Ω imposes the condition that the Ishibashi state $|\lambda,\mu\rangle$ is a part of the bulk spectrum. The label 'A' denotes all the Cardy labels (L_i, M_i, S_i). For B-type branes, the U(1) charge of the holomorphic and anti-holomorphic sector are opposite. This implies that all the m_i are the same modulo $(k+2 = 5)$. Also it is easily seen that the the action of the Z_5 and Z_2 symmetries on the boundary states takes $M_i \to M_i + 5$ and

$S_i \to S_i + 2$ respectively. From the δ_β condition it also follows that the physically inequivalent boundary states are labelled by the single variable $M = \sum_i M_i$. The $S = \sum_i S_i = 0$ or $2 \mod 4$. However because of the Z_2 symmetry it is sufficient to take $S = 0$. So in particular the Recknagel Schomerus states with $L_i = 0$ is labelled by a single variable M, $|0,0,0,0,0;M\rangle$,

7.1.2.2 Permutation branes of the quintic

The full super conformal algebra which is generated by the sum $\sum_{i=1}^{5} W^i(z)$, where as before the W^i denote the various symmetry generators, admits the following permutation group action on the generators.

$$
\begin{aligned}
&(12)(3)(4)(5) : [\, W_n^{(1)} \to W_n^{(2)} \to W_n^{(1)}],\ \ W_n^{(i)} \to W_n^{(i)},\ \ i = 3,4,5 \\
&(123)(4)(5) : [\, W_n^{(1)} \to W_n^{(2)} \to W_n^{(3)} \to W_n^{(1)}],\ \ W_n^{(i)} \to W_n^{(i)}, i = 4,5 \\
&(1234)(5) : [\, W_n^{(1)} \to W_n^{(2)} \to W_n^{(3)} \to W_n^{(4)} \to W_n^{(1)}],\ \ W_n^{(5)} \to W_n^{(5)}, \\
&(12345) : [\, W_n^{(1)} \to W_n^{(2)} \to W_n^{(3)} \to W_n^{(4)} \to W_n^{(5)} \to W_n^{(1)}]
\end{aligned}
\tag{7.1.14}
$$

and combinations of these. Recknagel [41] constructed more general boundary states called the permutation branes corresponding to 'twisted gluing' conditions where the 'twisting' implies the action of these permutation groups on the generators.

$$
\begin{aligned}
&(W_n^i + \alpha \overline{W}_{-n}^{\pi(i)})|B_i\rangle = 0 \\
&|B\rangle \sim \prod_{i=1}^{5} |B_i\rangle
\end{aligned}
\tag{7.1.15}
$$

where $\pi(i)$ can be any of the various permutation actions discussed above. In [41] Recknagel gave an ansatz for the boundary states of the permutation branes. For the case of the permutation $\pi : (12)(3)(4)(5)$ the ansatz is :

$$
|\pi(12)\rangle = \prod_{i=3}^{5} B_{\Lambda_\pi}^{l_i,m_i,s_i} s(l_1, L_1) \exp(i\frac{\pi M_1 m_1}{k+2}) \exp(i\frac{\pi M_2 m_2}{k+2}) \exp(i\frac{\pi S_2 s_2}{2}) \exp(i\frac{\pi S_2 s_2}{2}) |l, m, s\rangle
$$

In this case, unlike the (RS) case, it is easy to see that for the B branes, the condition that the holomorphic and anti-holomorphic sectors have opposite U(1) charges implies that $m_2 = -m_1 \mod (k+2)$ while for the other m_i are all

equal mod $(k+2)$ So unlike the (RS) states, the permutation branes are labelled by two variables $|0,0,0,0;M;\tilde{M}\rangle$, where $\tilde{M} = M_2 - M_1$.

7.2 Fractional two-branes and permutation branes

We now turn to the consideration of a specific class of permutation branes. We will argue that the boundary states for this are precisely the ones that should be associated to the fractional 2-branes on the blow-up of $\mathbb{C}^5/\mathbb{Z}_5$ that are then restricted to the quintic.

Consider the holomorphic involution π that permutes two fields [1]:

$$\pi : \quad \phi_1 \leftrightarrow \phi_2$$

The fixed point(s) of the action is $\phi_1 + \phi_2 = 0, \phi_3 = \phi_4 = \phi_5 = 0$ (this is a two-brane which we called P which restricts to a point on the quintic Q) as well as $\phi_1 - \phi_2 = 0$ (this is an six-brane which restricts to a four-brane on the quintic). Thus, we see the appearance of the boundary conditions of equation (7.1.1) as one of the fixed point sets of the holomorphic involution. This suggests that the permutation branes of ref. [41], corresponding to $\pi = (12)(3)(4)(5)$ may be the correct candidate for the boundary states in the Gepner model that correspond to the boundary conditions given in equation (7.1.1). This leads to the following conjecture:

Conjecture: The B-type permutation branes labelled[2]

$$|0,0,0,0,M,\widetilde{M}\rangle_\pi \quad , \pi = (12)(3)(4)(5)$$

of ref.[41] are the CFT boundary states for the boundary conditions given in equation. (1.6.1) associated with the fractional two-branes.

[1]This is not a symmetry of the Gepner model or the NLSM but can be made into one by combining it with world-sheet parity. Thus, this particular involution has been considered in the context of type IIB orientifolds.

[2]We choose $S = 0$ and M, $\widetilde{M}$ to be even.

7.2.1 Checks of the conjecture

A first check: Recall that we had obtained *five* fractional two-branes at large volume in section 3. But, we have 25 boundary states since both M and $\widetilde{M}$ each take 5 values. How can this make sense? In this regard it is useful to recall that there is a $(\mathbb{Z}_5)^5$ symmetry in the minimal models as well as their corresponding LG models which act as

$$g_i : \quad \phi_i \to \omega\, \phi_i \quad , \; g_i^5 = 1 \; ,$$

where ω is a non-trivial fifth-root of unity. Focusing on the $(\mathbb{Z}_5)^2$ which act on the fields ϕ_1 and ϕ_2, we see that the boundary condition $(\phi_1 + \phi_2) = 0$ is invariant only under the simultaneous action $g_1 g_2$ while g_1 or g_2 or the combination $g_1 g_2^{-1}$ act as *boundary condition changing operators*. Thus, the 25 boundary states in the CFT (of [41]) correspond to the five sets of boundary conditions:

$$\phi_1 + \omega^a \phi_2 = 0 \quad , \; a = 0, 1, 2, 3, 4. \tag{7.2.1}$$

Thus, the $\widetilde{M}$ index can be identified with $2a$.

A second check: Ref. [41] provides the intersection matrix between the permutation branes (though the normalisation of the boundary states was not fixed in that paper). The result which we quote here (after fixing the normalisation) is the following: the intersection form is *independent* of $\widetilde{M}$ and is given by $g(1-g)^3$. It is easy to see that the fractional two-branes that we propose at large volume also have the same intersection matrix (see appendix D). Since the intersection form does depends on only the Chern character (equivalently, RR charges) of the coherent sheaves, it will necessarily be independent of the $\widetilde{M}$ label. For instance, the D0-branes obtained from the boundary conditions, (7.2.1), are located at different points on the quintic. However, their charges are identical.

A third check: A last check is to compute the intersection matrix between the (12)(3)(4)(5) permutation boundary states and the RS boundary states and show that it equals $-(1-g)^4$ as obtained in the appendix D by computing the intersection matrix between vector bundles corresponding to the RS states and coherent sheaves corresponding to the fractional two-branes. This is a little bit

more subtle since it involves characters that are not present in the bulk/closed string partition function. This issue has been discussed in a recent paper [42], so we do not present any details but refer the reader to it for a more detailed discussion. As is finally shown in there, the intersection matrix does take the form $-(1-g)^4$ as predicted by the conjecture.

7.2.2 Other permutation branes

It is natural to see if other permutation branes can be obtained by considering other linear boundary conditions involving more fields. One obvious candidate for the LG boundary conditions associated with the permutation brane (12)(34)(5) is :

$$\phi_1 + \phi_2 = \phi_3 + \phi_4 = \phi_5 = 0 \ . \tag{7.2.2}$$

This will be a fractional four-brane and the open-string Witten index computation in section 2.1 for fractional four-branes can be compared with the intersection matrices for the (12)(34)(5) branes. The Coulomb branes have support on the hypersurface S corresponding to the conditions given in equation (7.2.2). This intersects the $\mathbb{P}^4$ on a $\mathbb{P}^1$. The Coulomb branes will be given by $\mathcal{X}_S \otimes \mathcal{O}(-1)$ and $\mathcal{X}_S \otimes \mathcal{O}$, where $\mathcal{X}_S$ is the sheaf with support on the hypersurface S (and Chern class $J^2/5$ reflecting the fractional charge) and $(\mathcal{O}, \mathcal{O}(1))$ are the tautological bundles on $\mathbb{P}^1$. Thus, their Chern classes will be

$$\frac{J^2}{5} - \frac{J^3}{5} \ , \text{ and } \frac{J^2}{5} \ .$$

These restrict to a two-brane (of minimal charge) on the quintic. This is consistent with the Chern classes of the permutation branes labelled V_3 and V_4 in equation. (6.13) of [42]. The Higgs branes will be related to Euler sequences on $\mathbb{P}^2$. The other branes given in the aforementioned reference also seem to fit this. So this also passes a similar set of checks. Thus, for the quintic, one is able to obtain objects which reproduce the full spectrum of charges – the fractional two-branes of $\mathbb{C}^5/\mathbb{Z}_5$ providing the zero-brane and the fractional four-brane providing the two-brane.

Another interesting boundary condition in the LG orbifold is the frac-

tional four-brane given by

$$\phi_1 + \phi_2 + \phi_3 = \phi_4 = \phi_5 = 0 \ .$$

This involves three fields and seems to be related to the permutation brane (123)(4)(5). However, this runs into trouble in the first check[3] that we used in the earlier cases. The permutation branes constructed in [41] do not have enough labels to account for the 125 boundary states that one anticipates by considering the action of the symmetries on this boundary condition. In this regard, we wish to point out that for cyclic orbifolds[61, 62], $\mathbb{Z}_\lambda$ with $\lambda > 2$, there is a fixed point resolution problem, related to the fact that the different primaries in the orbifold theory have the same character (see section 5 of [62]). This also suggests that the set of permutation branes given in [41] may not be minimal and their resolution will provide us with additional boundary states that may account for the 125 boundary states that we predict. A further discussion of these issues is outside the scope of this thesis.

[3]There is a subtlety even in the earlier examples with regard to other permutation branes with $L \neq 0$. It has been shown by the authors of [42] that the CFT states with $L = 1$ correspond to (in the LG) to quadratic factors chosen in a particular order. However, in the LG there doesn't seem to be any reason to prefer the ordering. These issues have been discussed in [42].

Chapter 8

Conclusion

8.1 Conclusion and some open questions

We first summarise the main results presented in this thesis.

- ☐ We have provided evidence that the new fractional branes obtained in [29] using the method of matrix factorisation and boundary fermions can be realized using the boundary conditions proposed in [13]. The methods used in this thesis also lead to sequences describing the new fractional branes which carry more information than Chern classes/RR charges. As a consequence, we show that fractional $2p$-branes constructed in the non-compact Calabi-Yau restrict to the compact Calabi-Yau hypersurface as well defined objects.

- ☐ We also provided evidence that a sub-class of the permutation branes proposed in [41] are related to the new fractional branes. This result has also been independently obtained by Brunner and Gaberdiel recently[42]. In particular, a detailed discussion of the permutation branes and the intersection matrices amongst them has been provided in their paper. More recently Enger etal [30] have further investigated the linear matrix factorisations associated to the B-type permutation branes. They propose a correspondence between a special class of linear matrix factorizations and arbitrary B-type permutation branes.

- ☐ By embedding non-supersymmetric orbifolds into supersymmetric orbifolds

in one higher dimension, we propose a quantum McKay correspondence that relates fractional $2p$-branes on $\mathbb{C}^n/\mathbb{Z}_N$ to (the push-forward of) tautological bundles on $\mathbb{C}^{n-p}/\mathbb{Z}_N$ branes in supersymmetric orbifolds. We also provide an alternate explanation to the "missing" branes discussed by Martinec and Moore in [40].

Another class of boundary conditions that can appear in these examples are those that are not linear. For instance, consider the boundary condition $(\phi_1 + \phi_2^2) = 0$ that is consistent with the $G = 0$ condition. How does one construct the boundary states corresponding to such a boundary condition? An added complication is that they do not seem to be minimal sub-manifolds. What are their intersection forms? In fact, it has been argued in [42] that in more complicated examples, the addition of the analogue of the fractional two-branes and four-branes that we have considered do not give rise to all possible RR charge vectors. Thus, one may be forced to deal with such boundary conditions. However the Gepner model for such examples typically involve minimal models with even level k. Here, even the RS boundary states are not minimal in the Cardy sense. So the issue is somewhat clouded by the need to 'resolve' the boundary states [61, 62].

Coming back to the case of the quintic – while it is indeed satisfying to find objects in the Gepner model/LG orbifold which provide all RR charges that appear on the quintic, it cannot be the end of the story. The RS boundary states were related to spherical objects (spherical in the sense that these objects have only $\mathrm{Hom}(E, E) \equiv \mathrm{Ext}^3(E, E) \neq 0$) on the quintic even though they did not span the lattice of RR charges. One would like to know, if there exists a basis of, say, four spherical objects on the quintic that give rise to all possible charges via bound state formation. In the framework of boundary fermions and matrix factorisation the super potentials on these branes have been computed and their relationship to obstruction theory has been discussed. It would be interesting to see if those computations agree with an extension of the superpotential computation carried out for the RS branes in the LG model in[63] to the case of the fractional two-branes and four-branes.

In this thesis, we have shown a parallel between fractional $2p$-branes on supersymmetric orbifolds and fractional zero-branes on non-supersymmetric orbifolds of dimensional lower by one. The Coulomb branes in the non-supersymmetric orbifolds have been identified with the minima of the quantum superpotential of twisted chiral superfields. The Coulomb branes, as we refer to them, among the fractional two-branes in our construction are associated with a chiral superfield. It appears possible to make the connection between the two situations precise by applying the Hori-Vafa map[64], which relates chiral superfields to twisted chiral superfields. This issue as well as an explanation for the change of basis proposed in [53] will be discussed in [58].

Our reinterpretation of the quantum McKay correspondence will be useful for non-supersymmetric orbifolds in higher dimensions such as $\mathbb{C}^3/\mathbb{Z}_n$ [54, 55] where there is no analogue of the Hirzebruch-Jung resolution via partial fractions. There is also the problem of terminal singularities that can appear. At least in cases where the higher dimensional supersymmetric orbifold admits a crepant resolution, one will be able to use the embedding to study aspects of the non-supersymmetric orbifold such as the possible end-points of tachyon condensation.

Appendix A

Some relevant cohomology groups for $\mathcal{O}_{\mathbb{P}^2}(-3)$

We consider a spacetime of the form $\mathbb{R} \times M$, where $\mathbb{R}$ represents the time direction and M is non-compact. Let N be the boundary of M. The D-brane charges take values in the relative cohomology $H^*(M, N; \mathbb{Z})$ [65]. These can be computed by considering the long-exact sequence in cohomology:

$$\cdots \to H^p(M;\mathbb{Z}) \xrightarrow{j} H^p(N;\mathbb{Z}) \to H^{p+1}(M,N;\mathbb{Z}) \xrightarrow{i} H^{p+1}(M;\mathbb{Z}) \to \cdots \qquad \text{(A.0.1)}$$

The map j corresponds to restricting p-forms on M to the boundary N. Let us choose M to be $\mathbb{C}^3/\mathbb{Z}_3$ (or equivalently $\mathcal{O}_{\mathbb{P}^2}(-3)$). Then, one has $N = \partial M = S^5/\mathbb{Z}_3$. One has the following non-vanishing cohomologies for N and M.

$$\begin{aligned} H^4(S^5/\mathbb{Z}_3;\mathbb{Z}) &= H^2(S^5/\mathbb{Z}_3;\mathbb{Z}) = \mathbb{Z}_3 \quad , \\ H^0(S^5/\mathbb{Z}_3;\mathbb{Z}) &= H^5(S^5/\mathbb{Z}_3;\mathbb{Z}) = \mathbb{Z} \quad , \\ H^{2p}(\mathcal{O}_{\mathbb{P}^2}(-3);\mathbb{Z}) &= H^{2p}(\mathbb{P}^2;\mathbb{Z}) = \mathbb{Z} \quad , \quad \text{for } p = 0,1,2 \quad , \end{aligned} \qquad \text{(A.0.2)}$$

Using the above data, we obtain the long-exact sequence breaks into four shorter sequences:

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to \mathbb{Z} \xrightarrow{j} \mathbb{Z} \to H^1(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to 0 \quad \text{(A.0.3)}$$

$$0 \to H^2(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to \mathbb{Z} \xrightarrow{j} \mathbb{Z}_3 \to H^3(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to 0 \quad \text{(A.0.4)}$$

$$0 \to H^4(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to \mathbb{Z} \xrightarrow{j} \mathbb{Z}_3 \to H^5(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to 0 \quad \text{(A.0.5)}$$

$$0 \to \mathbb{Z} \to H^6(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \to 0 \quad \text{(A.0.6)}$$

In the first equation above, the map j corresponds to restricting (constant) functions on $\mathcal{O}_{\mathbb{P}^2}(-3)$ to the boundary $S^5/\mathbb{Z}_3$. Clearly, this is an isomorphism and hence we obtain $H^0(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z})$ and $H^1(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z})$ both vanish. The last equation implies that $H^6(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) = \mathbb{Z}$. It can be shown that all odd (relative) cohomologies vanish. Thus, the second and third equations above reduce to

$$0 \to H^{2p}(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) \xrightarrow{a} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_3 \to 0 \quad (\text{for } p = 1, 2)$$

This implies that both above relative cohomologies are $\mathbb{Z}$ with the first map a representing multiplication by 3. Thus, one has the following result:

$$\boxed{H^{2p}(\mathcal{O}_{\mathbb{P}^2}(-3), S^5/\mathbb{Z}_3; \mathbb{Z}) = \mathbb{Z} \quad (\text{for } p = 1, 2, 3)} \quad \text{(A.0.7)}$$

Appendix B

Triple intersections for the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold

B.1 Triple intersections for the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold

In this appendix we will compute the triple intersections involving at least one compact divisor for the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold with orbifold action $\frac{1}{7}[4,2,1]$

The vertices of the toric diagram are given by (see Figure 6.4):

$$D_1 = (1,0,0), D_2 = (0,1,0), D_3 = (-4,-2,7),$$
$$D_4 = (0,0,1), D_5 = (-1,0,2), D_6 = (-2,-1,4) \tag{B.1.1}$$

The linear equivalences between the divisors are given by

$$D_1 \sim 2D_2 + D_5,$$
$$D_2 \sim 2D_3 + D_6,$$
$$D_3 \sim 2D_1 + D_4 \tag{B.1.2}$$

Using these linear equivalences one can now find the various triple intersections.

Triple intersections involving a single compact divisor

$$D_1 \cdot D_2 \cdot D_4 = 1, D_2 \cdot D_3 \cdot D_4 = 0, D_1 \cdot D_3 \cdot D_4 = 0$$
$$D_1 \cdot D_2 \cdot D_5 = 0, D_2 \cdot D_3 \cdot D_5 = 1, D_1 \cdot D_3 \cdot D_5 = 0$$
$$D_1 \cdot D_2 \cdot D_6 = 0, D_2 \cdot D_3 \cdot D_6 = 0, D_1 \cdot D_3 \cdot D_6 = 1 \tag{B.1.3}$$

$$D_4 \cdot D_1^2 = 2, D_4 \cdot D_2^2 = 0, D_4 \cdot D_3^2 = 0$$
$$D_5 \cdot D_1^2 = 0, D_5 \cdot D_2^2 = 2, D_5 \cdot D_3^2 = 0$$
$$D_6 \cdot D_1^2 = 0, D_6 \cdot D_2^2 = 0, D_6 \cdot D_3^2 = 2 \tag{B.1.4}$$

Triple intersections involving two compact divisors

$$D_4 \cdot D_6 \cdot D_1 = 1, D_4 \cdot D_5 \cdot D_1 = 0, D_5 \cdot D_6 \cdot D_1 = 0$$
$$D_4 \cdot D_6 \cdot D_2 = 0, D_4 \cdot D_5 \cdot D_2 = 1, D_5 \cdot D_6 \cdot D_2 = 0$$
$$D_4 \cdot D_6 \cdot D_3 = 0, D_4 \cdot D_5 \cdot D_3 = 0, D_5 \cdot D_6 \cdot D_3 = 1 \tag{B.1.5}$$

$$D_4^2 \cdot D_1 = -4, D_5^2 \cdot D_1 = 0, D_6^2 \cdot D_1 = -2$$
$$D_4^2 \cdot D_2 = -2, D_5^2 \cdot D_2 = -4, D_6^2 \cdot D_2 = 0$$
$$D_4^2 \cdot D_3 = 0, D_5^2 \cdot D_3 = -2, D_6^2 \cdot D_3 = -4 \tag{B.1.6}$$

Triple intersections involving only compact divisors

$$D_4^3 = 8, D_4^2 \cdot D_5 = 0, D_4^2 \cdot D_6 = -2,$$
$$D_5^2 \cdot D_4 = -2, D_5^3 = 8, D_5^2 \cdot D_6 = 0,$$
$$D_6^2 \cdot D_4 = 0, D_6^2 \cdot D_5 = -2, D_6^3 = 8, D_4 \cdot D_5 \cdot D_6 = 1 \tag{B.1.7}$$

Define (h, f, g) the cycles of the various $\mathbb{P}^2$'s. Then we can write down the intersections of various divisors in terms of (h, f, g),

$$D_4 \cdot D_2 = D_4 \cdot D_6 = h, D_5 \cdot D_3 = D_4 \cdot D_5 = f, D_6 \cdot D_1 = D_6 \cdot D_5 = g$$
$$D_4 \cdot D_1 = f + 2h, D_5 \cdot D_2 = 2f + g, D_6 \cdot D3 = h + 2g,$$
$$D_5 \cdot D_1 = 0, D_4 \cdot D_3 = 0, D_6 \cdot D_2 = 0$$
$$D_4^2 = -2f - 4h, D_5^2 = -2g - 4f, D_6^2 = -2h - 4g \tag{B.1.8}$$

The triple intersections involving the (h, f, g) are:

$$h{\cdot}D_1 = 1, h{\cdot}D_2 = 0, h{\cdot}D_3 = 0, h{\cdot}D_4 = -2, h{\cdot}D_5 = 1, h{\cdot}D_6 = 0$$
$$f{\cdot}D_1 = 0, f{\cdot}D_2 = 1, f{\cdot}D_3 = 0, f{\cdot}D_4 = 0, f{\cdot}D_5 = -2, f{\cdot}D_6 = 1$$
$$g{\cdot}D_1 = 0, g{\cdot}D_2 = 0, g{\cdot}D_3 = 1, g{\cdot}D_4 = 1, g{\cdot}D_5 = 0, g{\cdot}D_6 = -2 \qquad \text{(B.1.9)}$$

Appendix C

Fractional branes in $\mathbb{C}^3/\mathbb{Z}_5$ and $\mathbb{C}^3/\mathbb{Z}_7$ orbifolds

In this appendix we will study the toric description of the fractional D2 branes in the $\mathbb{C}^3/\mathbb{Z}_5$ and $\mathbb{C}^3/\mathbb{Z}_7$ orbifolds

C.1 Fractional branes in the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

Here we consider the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold with the orbifold action $\frac{1}{5}$[-4,-4,-2], with the orbifold action $Z \to \omega^{-4}Z$ along the brane in the 2-brane case, where ω is the fifth root of unity. The toric diagram is given in Figure 5.2. The intersection forms computed at the orbifold end are:

$$D_0 - D_0 = (1-g)^2(1-g^3), D_0 - D_2 = 1 + g - g^2 - g^4, \text{and}$$
$$D_2 - D_2 = g^4 - g \tag{C.1.1}$$

The tautological bundles R_i^0 are already given in [59]. We will not write them down here. The $\mathrm{ch}(S_i^0)$ are given by,

$$\begin{aligned}
\mathrm{ch}S_0^0 &= D_4 + D_5 + (3/2)h + (5/2)f + \frac{11}{6}p \\
\mathrm{ch}S_1^0 &= -2D_4 - D_5 - 2h - (3/2)f - \frac{4}{3}p \\
\mathrm{ch}S_2^0 &= D_4 + h/2 + \frac{1}{2}p \\
\mathrm{ch}S_3^0 &= -D_5 - (5/2)f - \frac{1}{3}p \\
\mathrm{ch}S_4^0 &= D_5 + (3/2)f + \frac{1}{3}p
\end{aligned} \tag{C.1.2}$$

Here h is a cycle of the exceptional divisor $\mathbb{P}^2$, while f is the fibre of $\mathbb{F}_3$ and p denotes the class of a point in $\mathbb{P}^2$. The various triple intersections of various divisors are given in [59] We take the following ansatz for the $\mathrm{ch}(S_i^2)$

$$\mathrm{ch}S_i^2 = a_1^i D_4 + a_1'^i D_5 + a_2^i D_1 \cdot D_2 + a_2'^i h + a_2''^i f + c_i p$$

As in the $\mathbb{C}^3/\mathbb{Z}_3$ case we take the coefficient of the noncompact fibre to be 1. So finally we get

$$\begin{aligned}
\mathrm{ch}S_0^2 &= D_1 \cdot D_2 + c_0 p \\
\mathrm{ch}S_1^2 &= D_4 + D_5 + D_1 \cdot D_2 + \frac{3h}{2} + \frac{5f}{2} + c_1 p \\
\mathrm{ch}S_2^2 &= -D_4 + D_1 \cdot D_2 - \frac{h}{2} + f + c_2 p \\
\mathrm{ch}S_3^2 &= D_1 \cdot D_2 + f + c_3 p \\
\mathrm{ch}S_4^2 &= -D_5 + D_1 \cdot D_2 - \frac{3f}{2} + c_4 p
\end{aligned} \tag{C.1.3}$$

The c_i remain undetermined as before. Comparison with the fractional 2-branes of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold suggests that the S_0^2 is purely along the noncompact fibre given by the intersection of the divisors, $D_1 \cdot D_2$.

C.2 Fractional branes in the $\mathbb{C}^3/\mathbb{Z}_7$ orbifold

We consider the example where the orbifold action is $\frac{1}{7}[4, 2, -6]$ and the orbifold action is given by $Z \to \omega^{-6} Z$, along the direction of the brane in the 2-brane case,

where $\omega^7 = 1$. The Toric diagram is given in Figure 5.4

$$\begin{aligned} D0 - D0 &= -(1-g)(1-g)^2(1-g^4) \\ D0 - D2 &= -g - g^2 + g^4 + g^6 \\ D2 - D2 &= g + g^2 - g^5 - g^6 \end{aligned} \tag{C.2.1}$$

The tautological bundles R_i for this example are,

$$\begin{aligned} &R_0 = \mathcal{O}, R_1 = \mathcal{O}(D_3), R_2 = \mathcal{O}(D_2) \\ &R_3 = \mathcal{O}(D_2 + D_3), R_4 = \mathcal{O}(D_1), R_5 = \mathcal{O}(D_1 + D_3), \\ &R_6 = \mathcal{O}(D_1 + D_2) \end{aligned} \tag{C.2.2}$$

Then as before, one can solve for the $\mathrm{ch}(S_i^0)$ by solving the equation,

$$\langle R_i, S_j^0 \rangle = \delta_{i,j} \tag{C.2.3}$$

Finally one has for the $\mathrm{ch}(S_i^0)$

$$\begin{aligned} \mathrm{ch}(S_0^0) &= D_4 + D_5 + D_6 + 2(h + f + g) + 2p \\ \mathrm{ch}(S_1^0) &= -D_5 - D_6 - f - 2g - \frac{2}{3}p \\ \mathrm{ch}(S_2^0) &= -D_5 - D_4 - h - 2f - \frac{2}{3}p \\ \mathrm{ch}(S_3^0) &= D_5 + f + \frac{1}{3}p \\ \mathrm{ch}(S_4^0) &= -D_6 - D_4 - 2h - g - \frac{2}{3}p \\ \mathrm{ch}(S_5^0) &= D_6 + g + \frac{1}{3}p \\ \mathrm{ch}(S_6^0) &= D_4 + h + \frac{1}{3}p \end{aligned} \tag{C.2.4}$$

where h, f and g are as defined in appendix B.

We take the following ansatz,

$$\mathrm{ch}(S_i^2) = a_1^i D_4 + a_1^{ri} D_5 + a_1^{mi} D_6 + b_1^i D_1.D_2 + b_1^{ri} h + b_2^{mi} f + b_1^{mi} g + c_1^i p$$

As before, we take the coefficient $b_1 = 1$ in all the equations. We can now solve for

the $\mathrm{ch}(S_i^2)$,

$$\begin{aligned}
\mathrm{ch}(S_0^2) &= -D_4 + D_1{\cdot}D_2 - h + c_0 p \\
\mathrm{ch}(S_1^2) &= D_1{\cdot}D_2 + c_1 p \\
\mathrm{ch}(S_2^2) &= D_4 + D_5 + D_6 + D_1{\cdot}D_2 + 2(h + f + g) + c_2 p \\
\mathrm{ch}(S_3^2) &= D_4 + D_1{\cdot}D_2 + 2h + f + c_3 p \\
\mathrm{ch}(S_4^2) &= -D_5 + D_1{\cdot}D_2 + h - f + c_4 p \\
\mathrm{ch}(S_5^2) &= D_1{\cdot}D_2 + h + c_5 p \\
\mathrm{ch}(S_6^2) &= -D_4 - D_6 + D_1{\cdot}D_2 - h - g + c_6 p \qquad \text{(C.2.5)}
\end{aligned}$$

The c_i's are undetermined as earlier.

Appendix D

Intersection matrices for the quintic

The Chern character of the *RS branes* on the Fermat quintic $Q \in \mathbb{P}^4$ are obtained from the generalised Euler sequences given in Eq. (6.1.2). One obtains

$$\begin{aligned}
\mathrm{ch}(\mathcal{M}_0) &= 1 \\
\mathrm{ch}(\mathcal{M}_1) &= -4 + J + \frac{J^2}{2} + \frac{J^3}{6} \\
\mathrm{ch}(\mathcal{M}_2) &= 6 - 3\,J - \frac{J^2}{2} + \frac{J^3}{2} \\
\mathrm{ch}(\mathcal{M}_3) &= -4 + 3\,J - \frac{J^2}{2} - \frac{J^3}{2} \\
\mathrm{ch}(\mathcal{M}_4) &= 1 - J + \frac{J^2}{2} - \frac{J^3}{6}
\end{aligned}$$

In the above J generates $H^2(Q, \mathbb{R})$ with the normalisation chosen to be $\langle J^3 \rangle_Q = 5\langle J^4 \rangle_{\mathbb{P}^4} = 5$. The Chern character for the *new fractional branes* are given in terms of the sequences that we proposed in eqs.. (6.2.3). The only additional input is our

proposal that $\mathrm{ch}(\mathcal{X}_P) = \frac{J^3}{5}$.

$$\begin{aligned}
\mathrm{ch}(\mathcal{F}_0) &= 1 - \frac{J^3}{5} \\
\mathrm{ch}(\mathcal{F}_1) &= -3 + J + \frac{J^2}{2} - \frac{J^3}{30} \\
\mathrm{ch}(\mathcal{F}_2) &= 3 - 2\,J + \frac{7\,J^3}{15} \\
\mathrm{ch}(\mathcal{F}_3) &= -1 + J - \frac{J^2}{2} - \frac{J^3}{30} \\
\mathrm{ch}(\mathcal{F}_4) &= \frac{-J^3}{5}
\end{aligned}$$

The Euler form on the quintic is defined as

$$\chi(E, F) = \int_Q \mathrm{ch}(E)^* \, \mathrm{ch}(F) \, \mathrm{Td}(Q) \, ,$$

The above formula leads to the following intersection matrices :

$$\chi(\mathcal{M}_\mu, \mathcal{M}_\nu) = \begin{pmatrix} 0 & 5 & -10 & 10 & -5 \\ -5 & 0 & 5 & -10 & 10 \\ 10 & -5 & 0 & 5 & -10 \\ -10 & 10 & -5 & 0 & 5 \\ 5 & -10 & 10 & -5 & 0 \end{pmatrix} \longleftrightarrow -(1-g)^5 \tag{D.0.1}$$

$$\chi(\mathcal{M}_\mu, \mathcal{F}_\nu) = \begin{pmatrix} -1 & 4 & -6 & 4 & -1 \\ -1 & -1 & 4 & -6 & 4 \\ 4 & -1 & -1 & 4 & -6 \\ -6 & 4 & -1 & -1 & 4 \\ 4 & -6 & 4 & -1 & -1 \end{pmatrix} \longleftrightarrow -(1-g)^4 \tag{D.0.2}$$

$$\chi(\mathcal{F}_\mu, \mathcal{F}_\nu) = \begin{pmatrix} 0 & 1 & -3 & 3 & -1 \\ -1 & 0 & 1 & -3 & 3 \\ 3 & -1 & 0 & 1 & -3 \\ -3 & 3 & -1 & 0 & 1 \\ 1 & -3 & 3 & -1 & 0 \end{pmatrix} \longleftrightarrow g(1-g)^3 \tag{D.0.3}$$

where we have rewritten the matrices in terms of the generator g of the quantum $\mathbb{Z}_5$ symmetry. The last formula coincides with the intersection matrix computed

between the $L = 0$ permutation branes for $\pi = (12)345$ given in the appendix of [41].

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