Studies in Loop Quantum Cosmology

By

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DECLARATION

I declare that the thesis entitled *Studies in Loop Quantum Cosmology*, submitted by me for the Degree of Doctor of Philosophy is the record of work carried out by me during the period from April 2003 to Jan 2006 under the guidance of Prof. S. Kalyana Rama and has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

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CERTIFICATE

I certify that the Ph. D. thesis titled "Studies in Loop Quantum Cosmology" submitted for the degree of Doctor of Philosophy by Mr. Golam Mortuza Hossain is the record of bona fide research work carried out by him during the period from April 2003 to Jan 2006 under my supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

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Abstract

In this thesis we have demonstrated that one can understand the consequences of the quantum dynamics implied by loop quantum cosmology in a more familiar classical spacetime picture in terms of an *effective* Hamiltonian. The effective Hamiltonian incorporates important non-perturbative modifications coming from loop quantum cosmology which imply modifications of Einstein dynamics when universe comes closer to Planck size. The effective dynamics goes over to the classical or Einstein dynamics for large volumes.

We show that the absence of isotropic or the *big bang* singularity in loop quantum cosmology can be understood in the effective description as the universe exhibiting a generic *big bounce*. The non-perturbative modifications coming from loop quantum cosmology to the scalar matter sector is known to imply inflation. We further prove that loop quantum cosmology modified scalar field generates near exponential inflation in the small scale factor regime, for all positive definite potentials, independent of initial conditions and independent of ambiguity parameters i.e. inflation is generic in loop quantum cosmology.

In the context of inflationary scenario, it is widely believed that quantum field fluctuations in an inflating background create the primeval seed perturbations which through subsequent evolution lead to the observed large scale structures of the universe. Using similar techniques in the context of effective framework, we show that loop quantum cosmology induced inflationary scenario can produce scale invariant power spectrum as well as *small amplitude* for the primordial density perturbations without any *fine tuning*. Further its power spectrum has a qualitatively distinct feature which is in principle falsifiable by observation and can distinguish it from the standard inflationary scenario.

In the effective framework of loop quantum cosmology, non-perturbatively modified dynamics of a minimally coupled scalar field violates weak, strong and dominant energy conditions when they are stated in terms of equation of state parameter. While violation of strong energy condition is desirable to permit a non-singular evolution, violation of weak and dominant energy conditions raises concern about

the causality and stability of the effective model, since in general relativity precisely these conditions ensure causality of the system and stability of vacuum via Hawking-Ellis conservation theorem. We show that although the non-perturbatively modified dynamics leads to violation of energy conditions but it still ensures positivity of energy density, as scalar matter Hamiltonian remains bounded from below. We also show that the modified dynamics restricts group velocity for inhomogeneous modes to remain sub-luminal thus ensuring causal propagation across spatial distances.

In conclusion, the effective dynamics implied by loop quantum cosmology gracefully modifies the Einstein dynamics retaining stability, causality, evading the big bang singularity, generically implying a desirable inflationary phase. Furthermore, the inflationary phase leads to a scale invariant power spectrum for primordial density perturbation with a *small amplitude* and with a *distinguishable signature*.

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Chapter 1

Introduction

1.1 Standard Model of Cosmology

When we look around us in day-to-day life, we find ourselves surrounded by objects which are arguably complex in structures. However, once we increase the length scale of observation, say to astronomical scale, the structures appears to be rather simple. In fact, on the largest scale of physical observations ($\sim 10^3$ mega parsec) our universe appears to be remarkably simple. In particular, our visible universe turns out to be nearly *isotropic* when observations are made from the earth and around it. In modern science, it is believed that we do not live in a privileged position of our universe and on the large scale average it would appear similar in nature when it is viewed from any other point. This postulate is formally referred as Cosmological Principle and it says that on large scale *there exists neither a preferred direction nor a preferred place* in our universe.

The standard model of cosmology is based on Einstein's relativistic theory of gravity, viz *general relativity*, together with an implementation of the cosmological principle. The space-time of the standard model is thus taken to be spatially homogeneous and isotropic. Such space-times are described by a single degree of freedom, namely the scale of the spatial geometry. The matter distribution also has to be consistent with spatial homogeneity and isotropy since matter and geometry both determine each other. The matter stress-energy tensor is taken to be of the so-called perfect fluid form. Such perfect fluids are described by two functions – the energy density and the pressure – together with an equation of state connecting the two. The cosmological principle, consistent with large scale observations, thus reduces the general field theoretical problem with infinitely many gravitational and matter degrees of freedom to a much simpler system involving only three variables. This is adequate to describe the large scale dynamics of space-time geometry and matter.

The standard model of cosmology is formulated using the Friedmann-Robertson-Walker (FRW) solution of general relativity. The structure of the FRW solution is dictated by the cosmological principle. Depending on spatial topology, the FRW solution can be classified into *three* classes namely spatially *flat, close and open*. In this thesis, we will be mainly dealing with spatially flat and close models. We begin with a spatially flat FRW spacetime. The invariant line element in such a spacetime (using *natural units i.e.* $c = \hbar = 1$) is given by the so-called spatially flat FRW metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)\, d\mathbf{x}^2 , \tag{1.1.1}$$

where $a(t)$ is the *scale factor*. The coordinate time t in the metric (1.1.1) is also referred as *synchronous* time. The perfect fluid form of the matter stress-energy tensor is given by,

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + g_{\mu\nu} P, \tag{1.1.2}$$

where u^μ denotes the 4-velocity of the isotropic observers. Here ρ and P denote energy density and pressure of the matter respectively. The equation of state ω is defined as the ratio of pressure and energy density *i.e.* $\omega := P/\rho$. The typical value of equation of state parameter for *radiation* fluid is $\frac{1}{3}$ whereas for *dust* matter it is 0. Einstein equation for FRW metric (1.1.1), commonly referred as Friedmann equation, is given by

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\, \rho(t) , \tag{1.1.3}$$

where G is Newton's constant of gravitation. Now Einstein equation implies that as long as energy density is positive, the universe will be either expanding or contracting. Along with conservation equation, this also imply that a currently expanding universe must have begun a finite time ago in a highly singular state, the *big bang*. The initial singularity in this context means that the scale factor (or size of the universe) vanishes a finite time ago if one considers its backward evolution. This vanishing size also implies that energy density, spacetime curvature diverge at this time. Due to this fact, standard model of cosmology is also referred as standard big bang cosmology. Thanks to *singularity theorems*, singularity within the context of homogeneous and isotropic expanding space-times is unavoidable, as long as the matter contents satisfy the so-called *strong energy condition*.

Ignoring the precise moment of creation, according to the standard model the history of the universe is as follows [1]. The expansion history of our universe begins from an extremely *hot* state. The subsequent geometrical expansion gradually makes the universe cooler. Around few seconds after the big bang, the temperature of our universe becomes of the order of $10^{10}K$. This brings cosmological dynamics into the realm of the standard model of particle physics. The subsequent era between the period of few seconds to few minutes is known as the era of *nucleosynthesis*. During this period the nuclei of light elements such as Deuterium, Helium and Lithium are 'cooked'. The matter and the radiation continue to interact with each other thermally through ionisation and recombination until the temperature falls down around 10^3K due to further expansion. Around this temperature formation of neutral atoms begins as the photons no longer have the sufficient energy to ionise the atoms. This leads the photons to decouple from the matter and to start moving freely. This cosmic event is referred as *decoupling*, sometimes also as *recombination*. After decoupling, the era of matter domination begins. The small inhomogeneities created in early universe then start growing under gravitational influences and give rise to the current large structures of the universe.

As mentioned earlier, the standard model of classical cosmology contains an initial singularity. Apart from the breakdown of theoretical framework, the existence of singularity also leads to various conceptual problems in classical cosmology. The most severe of them is the so called *horizon problem*. The horizon problem is directly related with the fact that the standard model of classical cosmology contains an initial singularity. The particle horizon with respect to a space-time point is defined by the maximum proper distance a particle could have travelled since the beginning of the universe. Due to the singular behaviour of the scale factor, this is a finite distance. It also means that any space-time point could have causal contact only with a finite patch of the space-time around it. By itself, existence of particle horizon need not be a problem. However, in conjunction with the thermal history of the universe, the finite horizon size implies that the last scattering surface of the cosmic microwave background photons has regions which could not have been in causal contact. Yet, there is remarkable isotropy (to within few parts of hundred thousand) in their angular distribution. This is the *horizon problem* of the big bang cosmology.

The most popular approach to solve this puzzle (along with few other puzzles) is to postulate a phase of *inflation* [2]. Phase of inflation generally refers to a period during which the universe goes through a rapid (generally exponential) expansion. This is generally achieved by introducing a scalar field with a self interaction potential. The scalar field is also known as the *inflaton field* . By now there are several versions of inflationary models [3]. Generically these models solve the horizon problem (and other traditional problems such as the flatness problem). In addition, quantum fluctuations in such inflating background quite generally produce *scale invariant* power spectrum of inhomogeneous density perturbations which is consistent with present observations. While these are attractive features of inflationary models, generally they need fine tuning of the potential and initial conditions for the inflaton to ensure a sufficient amount of inflation with graceful exit. In a sense, the isotropic singularity in Einsteinian gravity implies existence of particle horizon which leads to the horizon problem which needs an inflationary scenario to be invoked with its own set of problems of fine tuning and initial conditions.

The initial singularity, however is viewed as an attempt to extrapolate the classical theory beyond its natural domain of validity. In other words, the spacetime singularity signals breakdown of the theoretical framework of *classical* general relativity. It is widely expected that a quantum theory of gravity will provide a more accurate description which will hopefully be free of such breakdowns. Unfortunately we are yet to formulate a completely satisfactory quantum theory of gravity.

The Standard model of particle physics, a description of matter fields based on perturbative quantum field theory, is one of the most predictive physical theory ever constructed. A similar strategy based on techniques of *perturbative* quantum field theory was attempted to construct a quantum theory of gravity. Unfortunately, the techniques of perturbative quantum field theory when applied to theory of gravity, fail severely due to non-renormalizibility. Thus, in a quest for a quantum theory of gravity one is being compelled to explore different courses. Nevertheless, in last few decades, two promising candidate theories of quantum gravity have emerged. In the most popular approach, known as *string theory* [4], one postulates the basic constituents of the theory to be one dimensional objects. The other leading candidate quantum theory of gravity is known as *loop quantum gravity* (LQG) [5, 6, 7, 8, 9] where one attempts to formulate a *background independent, non-perturbative* theory of quantum gravity.

The thesis consists of studies of implications of a particular *quantum theory* addressing the issues of cosmological singularities. This quantum theory is a detailed adaptation of methods used in loop quantum gravity to the cosmological context and has come to be known as Loop Quantum Cosmology (LQC) [10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

1.2 Loop Quantum Cosmology

As mentioned, loop quantum gravity is an attempt to formulate a background independent, non-perturbative theory of quantum gravity. In this approach one follows the canonical quantization (with constraints) procedure. It should be emphasized here that canonical quantization is one of the approaches to construct a quantum theory. It is well known that general relativity admits a canonical formulation in terms of the so-called ADM (Arnowitt-Deser-Misner [20]) variables. In the canonical formulation one views the space-time as being foliated by 3 dimensional spacelike hypersurfaces. The ADM variables consist of a 3-metric and a symmetric rank 2 tensor on the 3-manifold. The later represents the extrinsic curvature of the slice. This leads to a constrained Hamiltonian system. The constraints reflect the diffeomorphism of the 3-manifold (diffeomorphism or vector constraint) and the normal (temporal) deformation of it (Hamiltonian or scalar constraint). However, these variables turn out to be unsuitable for the quantization. On the other hand, Ashtekar's discovery of an alternative choice of variables to describe the gravitational dynamics, namely a non-abelian gauge connection (1-form) and densitized triad (related to a 2-form) permits progress toward construction of a quantum theory. In this approach the space-time geometry is a *derived* quantity. This formulations also is a constrained Hamiltonian system. The construction of quantization method proceeds in two steps [5, 6, 7, 8, 9]. In the first step one constructs a Hilbert space disregarding the constraints. This is called the kinematical Hilbert space and is constructed as the representation space of a suitable commutative C^* algebra and the basic variables being 1 and 2 forms, allows one to do so without having to use any background metric. In this sense it is (metric) background independent. The kinematical Hilbert space is typically non-separable and is very different from the Fock space of usual quantization. In the second step one defines operators representing the constraints and proceeds to define a *physical* Hilbert space from the set of solutions of the constraints and also *physical* operators (also known as Dirac operators) which act invariantly on these states. Once these are defined, one can make predictions by computing physical matrix elements. This step is not yet adequately carried out, particularly with regards to the Hamiltonian constraint.

While understanding of dynamics implied by full theory of loop quantum gravity is far from complete, one has applied the methods used in the full theory in much more simpler context. In particular, the methods of loop quantum gravity has been closely followed in the cosmological (spatially homogeneous) context with notable success. In loop quantum cosmology, one uses the techniques of LQG for quantization of classically symmetry reduced phase space. As one considers finitely many degrees of freedom truncated by the classical symmetry, this approach of quantization is also referred as *minisuperspace* approach. It is worth mentioning here that similar approach of quantizing cosmological models has been attempted earlier as well and is known as Wheeler-DeWitt quantum cosmology. This uses the usual method of quantization (Schroedinger quantization) and is based on metric variable.

In the minisuperspace approach, one considers only finitely many degrees of freedom. These degrees of freedom are chosen after considering their relevance in the aspects of the physical system one is studying. The minisuperspace approximation nevertheless leaves several questions unanswered in the process. In particular, whether contributions from the ignored degrees of freedom can change the conclusion of minisuperspace approximation dramatically. In fact, there is example [21] where conclusion from minisuperspace quantum theory differs when some ignored degrees of freedom are included. However, such symmetry reduction turns out to be unstable even classically under inclusion of some of the ignored degrees of freedom. So the problem there cannot be correlated directly with the quantization procedure itself. The quantization of classically symmetry reduced system greatly reduces the complexity of analysis. Being simpler, it allows explicit calculations to be carried out. This in turns allows many clues into the dynamics expected from the full theory.

In loop quantum cosmology, one follows the spirit of minisuperspace approach but uses the quantization methods of LQG. Loop quantum cosmology framework differs from earlier approach of quantum cosmology namely Wheeler-DeWitt quantum cosmology in a fundamental way. In loop quantum cosmology, the quantum configuration space is constructed from the gravitational *holonomies* of gauge connection as opposed to gauge connection itself. Further, it can be rigorously shown that the quantization method used in loop quantum cosmology is *inequivalent* to Schroedinger quantization which is used in the Wheeler-DeWitt quantum cosmology. Naturally, many conclusions from loop quantum cosmology fundamentally differ from those of Wheeler-DeWitt quantum cosmology.

It has already been shown that loop quantum cosmology framework is free of singularity, in the isotropic context [22] as well as more generally for homogeneous diagonal models [23, 24]. There are two aspects of this singularity-free nature. The first aspect can be seen at the kinematical level itself. In loop quantum cosmology the operators corresponding to inverse powers of the scale factor have *bounded* spectrums [25]. This in turns leads to the matter energy density and curvatures remain finite at all sizes. Second aspect of singularity free nature of loop quantum cosmology can be understood at the dynamical level. The imposition of the Hamiltonian constraint ("Wheeler–DeWitt equation") in loop quantum cosmology leads to a *difference equation* with eigenvalues of the densitized triad variable serving as labels. These eigenvalues can take negative values corresponding to reversal of orientation and in this context classical singularity lies in the interior of the classical phase space instead of at a boundary. The difference equation, viewed as an evolution equation in these labels, allows solutions to evolve through the zero eigenvalue (zero size) [22]. Thus there is no breakdown of evolution equation at the classically indicated singularity at zero size. In other words, the basic equation of loop quantum cosmology remains well-defined at all values of the scale factor including zero. It is *unlike* to the classical situation in which the governing equations break down at zero size.

1.3 Effective Description of LQC

While quantum theory is well specified at the kinematical level, the complete understanding of issues like Dirac observables, physical inner product to have a bona-fide Hilbert space of solutions of the Hamiltonian constraint (the difference equation) are still in the early stages (but see [26, 27, 28]; also [29] for recent developments toward these issues). Consequently, the semi-classical behaviour of loop quantum cosmology in terms of expectation values of observables is not yet completely understood [1]. To relate implications of loop quantum cosmology which is based on discrete quantum geometry, to observable (and more familiar) quantities described in terms of the continuum geometrical framework of general relativity, the idea of an *effective Hamiltonian* has been proposed [15, 30]. This Hamiltonian contains the modifica-

[1]Recently in [29], a significant progress has been made in this directions. In the concluding chapter we will briefly discuss the methods and the results obtained therein.

tions implied by loop quantum cosmology to the usual classical Hamiltonian. This approach retains the kinematical framework of Robertson-Walker geometry but gives modification to the dynamics of the Friedmann equation.

The effective Hamiltonian in loop quantum cosmology is generally derived in two steps [30, 31]. In first step, one develops a *continuum approximation* to the fundamental difference equation to obtain a differential equation. For large volumes where one expects the manifestation of discrete geometry to be negligible, the differential equation matches with the usual Wheeler-DeWitt equation (with certain factor ordering). In the second step one looks for a WKB form for solution of the differential equation to derive the corresponding Hamilton-Jacobi equation and read-off the Hamiltonian. This is the effective Hamiltonian. The effective Hamiltonian differs from the classical Hamiltonian due to the modifications in the differential equation derived from the difference equation. There are two sources of modifications. In the matter sector, the modifications come from using the modified inverse triad operator which incorporates the small volume deviations. These involve inverse powers of the Planck length and thus are non-perturbative. One can also get modifications in the gravity sector for small volumes. These have been obtained in [31, 32], exploiting non-separable nature of the kinematical Hilbert space of loop quantum cosmology [17].

It turns out that the dynamics (evolution with respect to the synchronous time) implied by the effective Hamiltonian captures essential features of the difference equation, in particular the dynamics is *non-singular*. A universe beginning at some large volume will never reach zero volume when evolved backwards. Since the framework for effective dynamics is that of the usual pseudo-Riemannian geometry, the arguments leading to the singularity theorem are applicable and therefore a non-singular evolution should lead to a violation of the strong energy condition through effective matter energy density and pressure.

So one of the question we address in the thesis is whether the modifications in the matter sector imply violation of strong energy condition. In general, the strong energy condition requires $R_{\alpha\beta}\xi^\alpha\xi^\beta = 8\pi G(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)\xi^\alpha\xi^\beta \geq 0$, for all time-like vectors ξ^a. Within the context of homogeneous and isotropic geometries, the strong energy condition applied to the congruence of isotropic observers (or four velocity of the matter perfect fluid), becomes $R_{00} = 4\pi G(\rho + 3P) \geq 0$ where ρ is the total energy density and P is the total pressure of the matter fluid. Defining $\omega := P/\rho$ (with ρ assumed to be positive definite) as the equation of state variable, the violation of strong energy condition is conveniently stated as $\omega < -\frac{1}{3}$. Note that since $R_{00} \sim -\frac{\ddot{a}}{a}$, violation of the strong energy condition in this context also implies an *accelerated* evolution of the scale factor or in other words an *inflationary phase*.

The non-perturbative modification coming from loop quantum cosmology to the scalar matter sector is known to imply inflation [33, 34]. We further prove [35] that loop quantum cosmology modified scalar field generates near exponential inflation in the small scale factor regime, for all positive definite potentials, independent of initial conditions and independent of ambiguity parameters. In other words, in small scale factor regime, the non-perturbatively modified scalar matter leads to generic violation of the strong energy condition. This is consistent with non-singular evolution as singularity theorems are bypassed. We show next [36] that the absence of isotropic singularity in loop quantum cosmology can be understood in an effective classical description as the universe exhibiting a *big bounce*. We show that with scalar matter field, the big bounce is generic in the sense that it is independent of quantization ambiguities and details of scalar field dynamics. In conjunction with generic occurrence of bounce at small volumes, particle horizon is absent thus eliminating the *horizon problem* of the standard big bang cosmology.

As mentioned, the homogeneous and isotropic solution of general theory of relativity appears to be an extremely good description of large scale spacetime dynamics of our universe. Extreme simplicity of the FRW solution nevertheless ignores some crucial features of the universe namely it has certain sub-structure as well. On large scale the deviation from homogeneity and isotropy being small one can treat them as small perturbations around homogeneous and isotropic background. The classical theory of large scale structure formation in principle can be used to 'derive' the observed structures of current universe but these models need to know the initial *seed* perturbations. In this sense the classical description of our universe is incomplete as there is no mechanism of *generating* the seed perturbation within the theory itself.

On the other hand the quantum field fluctuations in an *inflating* background quite generically produce density perturbations with scale-invariant power spectrum [37] which is consistent with current observations. This is certainly an attractive feature of the standard inflationary scenario. However, one major problem that plagues almost all potential driven inflationary scenario that these models generically produce too large amplitude for density perturbation, typically $\frac{\delta\rho}{\rho} \sim 1 - 10^2$ at horizon *re-entry* [38, 39]. The cosmic microwave background (CMB) anisotropy measurements on the other hand indicates $\frac{\delta\rho}{\rho} \sim 10^{-5}$. Naturally to make these models viable it is necessary to *fine tune* the parameters of the field potential [40]. In the presence of quantum fluctuations it is rather difficult to justify or sustain these fine tuning of field theoretical parameters.

We have mentioned that density perturbations generated by quantum field fluctuations in an inflating background are believed to be the *seed* perturbations responsible for the current large scale structures of the universe. We have also mentioned that non-perturbative modification of matter Hamiltonian in loop quantum cosmology leads to a generic phase of inflation. Naturally it is an important question to ask whether the density perturbations generated by quantum fluctuations during loop quantum cosmology induced inflationary phase can satisfy the basic requirements of viability such as scale-invariant power spectrum. Further, it may leave some distinct imprint on the power spectrum which may be observationally detectable as well.

Being inhomogeneous in nature treatment of these density perturbations requires *inhomogeneous* models of loop quantum cosmology. However the technology required to deal with inhomogeneity at fundamental level within loop quantum cosmology is *not* yet available. Not having such technology, one needs to proceed rather intuitively. Let us recall that in the standard inflationary scenario for computing power spectrum of density perturbations due to quantum fluctuations, one uses the techniques which broadly can be classified as *Quantum Field Theory in Curved Background* [41, 42]. In this approach one treats the background geometry as classical object whereas matter fields living in it are treated as quantum entities. The main justification for using such techniques comes from the fact the energy scale associated with inflationary scenario is few order of magnitude lower than the Planck scale. So one expects the geometry to behave more or less classically in this regime. Using similar approach we compute the power spectrum of density perturbations in the effective framework of loop quantum cosmology. In particular, we show that loop quantum cosmology induced inflationary scenario can produce scale invariant power spectrum as well as *small amplitude* for the primordial density perturbation without any *fine tuning*. Further, its power spectrum has a qualitatively distinct feature which is in principle falsifiable by observation and can distinguish it from the standard inflationary scenario [43].

In general relativity, dynamics of a spacetime is influenced by matter stress-energy tensor. Naturally, several properties regarding spacetime evolution can be concluded assuming some general properties of the matter stress-energy tensor, without having to know the details of the individual contributions from different matter sources. These requirements on the matter stress-energy tensor, widely called *energy conditions*, have been used to prove several important theorems in classical general relativity. One such theorem, the Hawking-Ellis *conservation theorem* [1, 44] says that if the matter stress-energy tensor is conserved, satisfies *dominant energy condition* and vanishes on a closed, *achronal* (no two points can be connected by timelike curves) set S then it also vanishes in the *domain of dependence* (complete set of events for which all conditions are determined by specifying conditions on S) $D(S)$ of the set. Physically, this theorem ensures the stability of classical vacuum. As mentioned, the conservation theorem stands true provided the matter stress-energy tensor satisfies dominant energy condition. This condition requires local energy density to be *non-negative* for all time-like observer and the energy-momentum 4-current to be *non-spacelike i.e.* the speed of energy-flow should not be exceeding the speed of light. Naturally, any violation of dominant energy condition raises concern about the causality and the stability of the system. However, above theorem does *not* have the *converse i.e.* although the dominant energy condition satisfying matter ensures causality and stability of the system but violation of this condition *does not* necessarily imply that the system violates causality or is unstable (see for example [45]). In such a situation, these issues should be considered for the specific context, as dominant energy condition violation and the Hawking-Ellis conservation theorem no longer vouch for the causality and the stability of the system.

In the effective framework of loop quantum cosmology, non-perturbatively modified dynamics of a minimally coupled scalar field violates weak, strong and dominant energy conditions when they are stated in terms of equation of state parameter. On the other hand several important features of loop quantum cosmology, that have been shown in literature, crucially rely on the effective classical description. Naturally, in the effective loop quantum cosmology, the violation of dominant energy condition raises concern. In particular, whether such effective classical description respects causality. In the cosmological context, any communication across spatial distances introduces inhomogeneity. So it is a natural concern to check whether the propagation of inhomogeneous modes respects causality. Also, whether such dominant energy condition violating effective description can still maintains stability of the vacuum, as the Hawking-Ellis conservation theorem no longer guarantees the same.

Studying the properties of the effective scalar matter Hamiltonian, we show [46] that the kinetic term due to the modified dynamics, contributes negative pressure even though it contributes positive energy density. This crucial feature essentially leads to violation of dominant energy condition in terms of the equation of state parameter but it also ensures a bounded (from below) scalar matter Hamiltonian. To study the causality issue, we consider the propagation of inhomogeneous modes in the modified background. In particular, we derive a modified dispersion relation for the inhomogeneous modes due to the modified dynamics. Then we show that the group velocity for the relevant inhomogeneous modes remains sub-luminal thus ensuring causal propagation across spatial distances.

In brief, one can develop an effective classical description using the effective Hamiltonian. Most of the implications of loop quantum cosmology can be obtained rather easily from the effective Hamiltonian. Furthermore, viewed by itself, the effective description is free from potential pathologies such as stability, a-causality, contradiction with observation.

The contents of this thesis are organised as follows. Apart from the introduction and the discussion chapters, the contents of the remaining chapters of the thesis have been published as independent articles. Nevertheless, to make the thesis coherent and to avoid repeating the contents of these chapters have been suitably modified. However, each chapter mostly retains the independent characters in its subject matters and the corresponding problems being studied therein.

In chapter 2, we derive the effective classical Hamiltonian for isotropic loop quantum cosmology. The consequences of the dynamics implied by the effective Hamiltonian are studied in details in remaining chapters. In particular, in chapter 3, we show that loop quantum cosmology modified scalar field generates near exponential inflation in the small scale factor regime, for all positive definite potentials, independent of initial conditions and independent of ambiguity parameters. In chapter 4, we show that the absence of isotropic singularity in loop quantum cosmology can be understood in the effective classical description as the universe exhibiting a *big bounce*. We show that with scalar matter field, the big bounce is generic in the sense that it is independent of quantization ambiguities and details of scalar field dynamics. In chapter 5, we study the properties of density perturbation generated during the loopy phase of inflation *i.e.* during the phase of inflation which is driven by non-perturbative modification. In chapter 6, we study the stability and causality issues that arise due the violations of dominant energy condition in the small scale factor regime of the effective dynamics.

In the concluding chapter, we summarize the basic results shown in the thesis. We also briefly discuss the applications of loop quantum gravity methods to more general homogeneous but *anisotropic* universes. In particular, implications of loop quantization in the context of Bianchi-IX model is briefly discussed. Further, we discuss the current open issues in loop quantum cosmology and recent attempts made in those directions.

Chapter 2

Effective Hamiltonian

2.1 Introduction

The singularities of classical general relativity, when specialized to homogeneous, isotropic models, manifest as reaching zero physical volume at finite synchronous time in the past. This in turn imply unbounded growth of space-time curvature and of matter densities etc and signals break down of the evolution equations at finite time in the past. It is widely believed that this feature of the classical theory will be modified in a quantum theory of gravity and recent development of *Loop Quantum Cosmology* (LQC) [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] corroborate this expectation [22, 25]. Loop quantum cosmology is a detailed adaptation of Loop Quantum Gravity (LQG) methods to the cosmological context.

The mechanism of 'singularity avoidance' [22] involves replacement of the classical evolution equation by a quantum one which is a *difference equation* [13] thanks to the necessity of using holonomy operators in the quantization of the Hamiltonian constraint in LQC. This equation exhibits the property that the quantum wave function can be evolved through zero volume unambiguously. In addition, the discreteness of the triad operator (having zero eigenvalue) necessitates defining inverse triad operator (or inverse scale factor operator) [25] indirectly. Thanks to the loop representation on the (non-separable) kinematical Hilbert space of LQC [17], these operators get so defined as to have a bounded spectrum implying only a bounded growth of curvatures/matter densities. This is true for all allowed values of the ambiguity parameters [47, 48].

The non-separable structure of the Kinematical Hilbert space of LQC however, also implies a huge set of solutions of the Hamiltonian constraint (a continuous infinity in the gravitational sector alone). Presumably, a suitable choice of physical inner product can be made to cut down the size of the admissible solutions of the Hamiltonian constraint. A choice of inner product and physical observables however are not yet available. The general issue of whether or not the non-separable kinematical Hilbert space is mandatory, is currently an open issue [49, 50]. In the present work however we assume the current framework of LQC [17].

Despite the open issues, it is possible to develop a WKB type semi-classical approximation from which an *effective continuum Hamiltonian constraint* can be deduced [15, 30]. This at once gives access to the usual classical Hamiltonian methods to construct and analyze the quantum modified space-time. This method relies on a *continuum approximation* [51] of the underlying difference equation to the Wheeler-DeWitt differential equation followed by the WKB ansatz for its solutions. For large volume corresponding to classical regime, the continuum approximation is always available, in fact as a requirement on quantization of the Hamiltonian operator. In this regime, the WKB ansatz naturally reproduces the classical Hamiltonian as the leading $o(\hbar^0)$ term. We would like to extend this method also to small volumes.

The large freedom offered by the non-separable structure of the kinematical Hilbert space can be exploited to propose a *restriction* to those solutions of the Hamiltonian constraint for which a continuum approximation is valid for *all* volumes. One can then develop the effective classical Hamiltonian constraint for all volumes and explore its consequences.

In this chapter we develop such a picture and in comparison with the usual FRW equations identify the effective density and pressure which includes the contributions of quantum fluctuations of the geometry. Some elementary consequences are also noted. Further implications for phenomenology are studied in the next chapters.

In section 2.2, we detail the effective Hamiltonian constraint for isotropic models. In section 2.3, we discuss the qualitative features of the corresponding dynamics namely, the possibility of 'bounce' solutions as well as solutions that could attempt to 'pass through' the zero volume and connect to the oppositely oriented isotropic universe. We discuss what features of the quantum evolution are captured by the effective classical evolution. In section 2.4, we summarize our conclusions and outlook.

2.2 Effective Hamiltonian Constraint

The kinematical Hilbert space [17] is conveniently described in terms of the eigenstates of the densitized triad operator,

$$\hat{p}|\mu\rangle \;=\; \tfrac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu|\mu\rangle \quad ; \quad \langle\mu|\mu'\rangle = \delta_{\mu\mu'} \; , \tag{2.2.1}$$

where $\mu \in \mathbb{R}$. It should be noted here that the eigenstates $|\mu\rangle$'s are normalized with Kronecker delta rather than Dirac delta even though μ takes values on the real line. It follows from the procedures followed in the full theory to construct a well-defined kinematical Hilbert space for loop quantum cosmology.

The action of the volume operator on the triad basis states are given by

$$\hat{V}\;|\mu\rangle \;=\; \left|\frac{1}{6}\gamma l_p^2\mu\right|^{\frac{3}{2}}\;|\mu\rangle \;:=\; V_\mu\;|\mu\rangle \; . \tag{2.2.2}$$

In the isotropic context we have two classes to consider, namely spatially flat and close models. The quantization of the corresponding Hamiltonian operators is given in [14]. By introducing a parameter η we can deal both classes together. The values $\eta = 0$ and $\eta = 1$ will give the flat and the close models respectively.

The action of the gravitational Hamiltonian on the triad basis states is then given by

$$\begin{aligned}\hat{H}^{(\mu_0)}_{\mathrm{grav}}|\mu\rangle \;&=\; \left(\frac{3}{4\kappa}\right)(\gamma^3\mu_0^3 l_p^2)^{-1}(V_{\mu+\mu_0} - V_{\mu-\mu_0}) \\ &\quad \left(\, e^{-i\mu_0\eta}|\mu + 4\mu_0\rangle \;-\; (2 + 4\mu_0^2\gamma^2\eta)|\mu\rangle \;+\; e^{i\mu_0\eta}|\mu - 4\mu_0\rangle\,\right) \, .\end{aligned} \tag{2.2.3}$$

Here, μ_0 is a quantization ambiguity parameter which enters through the fiducial length of the loops used in defining the holonomies. It is a real number of the order of 1. Notice that the Hamiltonian connects states differing in their labels by $\pm 4\mu_0$. This is a direct consequence of the use of holonomy operators which have to be used in the quantization of the Hamiltonian operator and is responsible for leading to a difference equation below.

A general kinematical state $|s\rangle$, in the triad basis has the form

$$|s\rangle \;=\; \sum_{\mu\in\mathbb{R}} s_\mu\,|\mu\rangle \qquad (\text{ sum over countable subsets})\,. \tag{2.2.4}$$

The Hamiltonian constraint of the classical theory is promoted as a condition to define physical states, i.e.,

$$(\hat{H}^{(\mu_0)}_{\text{grav}} + \hat{H}^{(\mu_0)}_{\text{matter}})|s\rangle \;=\; 0\,. \tag{2.2.5}$$

In terms of $\tilde{s}_\mu := e^{\frac{i\mu}{4}\eta}s_\mu$ and The Hamiltonian constraint (2.2.5) translates into a difference equation,

$$\begin{aligned} 0 \;&=\; A_{\mu+4\mu_0}\tilde{s}_{\mu+4\mu_0} \;-\; (2+4\mu_0^2\gamma^2\eta)A_\mu\tilde{s}_\mu \;+\; A_{\mu-4\mu_0}\tilde{s}_{\mu-4\mu_0} \\ &\quad + 8\kappa\gamma^2{\mu_0}^3\left(\frac{1}{6}\gamma l_p^2\right)^{-\frac{1}{2}} H_m(\mu)\tilde{s}_\mu \quad , \qquad \forall\,\mu\,\in\mathbb{R} \qquad (2.2.6) \\ A_\mu \;&:=\; |\mu+\mu_0|^{\frac{3}{2}} - |\mu-\mu_0|^{\frac{3}{2}}\;, \qquad \hat{H}^{(\mu_0)}_{\text{matter}}|\mu\rangle \;:=\; H_m(\mu)|\mu\rangle\;. \end{aligned}$$

$H_m(\mu)$ is a symbolic eigenvalue and we have assumed that the matter couples to the gravity via the metric component and *not* through the curvature component. In particular, $H_m(\mu = 0) = 0$. There are a few points about the above equation worth noting explicitly.

Although μ takes all possible real values, the equation connects the $\tilde{s}_\mu$ coefficients only in steps of $4\mu_0$ making it a difference equation for the coefficients. By putting $\mu := \nu + (4\mu_0)n$, $n \in \mathbb{Z}$, $\nu \in [0, 4\mu_0)$, one can see that one has a continuous infinity of independent solutions of the difference equation, labelled by ν, $S^\nu_n := \tilde{s}_{\nu+4\mu_0 n}$. For each ν an infinity of coefficients, S^ν_n, are determined by 2 'initial conditions' since the order of the difference equation in terms of these coefficients is 2. Coefficients belonging to different ν are mutually decoupled. Since the coefficients A_μ and the symbolic eigenvalues $H_m(\mu)$, both vanish for $\mu = 0$, the coefficient $\tilde{s}_0$ *decouples* from all other coefficients.

For large values of $\mu \gg 4\mu_0$ ($n \gg 1$), which correspond to large volume, the coefficients A_μ become almost constant (up to a common factor of $\sqrt{n}$) and the matter contribution is also expected similarly to be almost constant. One then expects the coefficients to vary slowly as n is varied. This suggests interpolating these slowly varying *sequences* of coefficients by slowly varying *functions* of the continuous variable $p(n) := \frac{1}{6}\gamma\ell_{\mathrm{P}}^2 n$ [15]. The difference equation satisfied by the coefficients then implies a *differential* equation for the interpolating functions which turns out to be *independent* of γ and matches with the usual Wheeler–DeWitt equation of quantum cosmology. This is referred to as a continuum approximation [51]. This is of course what one expects if LQC dynamics is to exhibit a semi-classical behaviour. While admissibility of continuum approximation is well motivated for large volume, one also expects it to be a poor approximation for smaller Planck scale volumes.

This logic is valid when applied to any *one* of the solutions S_n^ν. Thanks to the non-separable structure of the Hilbert space, we have an infinity of solutions of the Hamiltonian constraint. Although S_n^ν are uncorrelated for different ν, nothing prevents us from *choosing* them to be suitably correlated. In effect this amounts to viewing $\tilde{s}_\mu$ themselves as *functions* of the continuous variable μ and stipulating some properties for them. In the absence of a physical inner product, we don't have any criteria to select the class of solution. It is then useful to study properties of classes of solutions of the Hamiltonian constraint.

The class that we will concentrate on is the class of *slowly varying* functions. For these we will be able to have a continuum approximation leading to a differential equation. Making a WKB approximation for this differential equation, we will read-off the effective classical Hamiltonian constraint. In anticipation of making contact with a classical description, we will use the dimensionful variable $p(\mu) := \frac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu$ as the continuous variable. Correspondingly, we define $p_0 := \frac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu_0$ which provides a convenient *scale* to demarcate different regimes in p. We also use the notation: $\psi(p(\mu)) := \tilde{s}_\mu$. Now the definition of a slowly varying function is simple:

$\psi(p)$ is *locally slowly varying around* q if $\psi(q+\delta q) \approx \psi(q) + \delta q \frac{d\psi}{dq} + \frac{1}{2}\delta q^2 \frac{d^2\psi}{dq^2} + \cdots$ with successive terms smaller than the preceding terms, for $\delta q \lesssim 4p_0$. It is slowly varying if it is locally slowly varying around every $q \in \mathbb{R}$ [51]. Note that even an exponentially rising function can be locally slowly varying if the exponent is sufficiently small.

To explore the possibility of slowly varying solutions of the difference equation (2.2.6), consider the difference equation more explicitly. For $\nu \in (0, 4\mu_0)$, putting $S_n(\nu) := \tilde{s}_{\nu+4\mu_0 n}$, $A_n(\nu) := A_{\nu+4\mu_0 n}$ and momentarily ignoring the matter term for notational simplicity, the difference equation (2.2.6) can be written as,

$$S_{n+2}(\nu) = \left[(2 + 4\mu_0^2\gamma^2\eta)\,\frac{A_{n+1}(\nu)}{A_{n+2}(\nu)}\right] S_{n+1}(\nu) + \left[\frac{-A_n(\nu)}{A_{n+2}(\nu)}\right] S_n(\nu) \qquad (2.2.7)$$

Its general solution can be written as $S_n(\nu) = S_0(\nu)\rho_n(\nu) + S_1(\nu)\sigma_n(\nu)$, where the ρ_n, σ_n are fixed functions of ν determined by the same difference equation (2.2.7) with the 'initial' conditions: $\rho_0(\nu) = 1$, $\rho_1(\nu) = 0$ and $\sigma_0(\nu) = 0$, $\sigma_1(\nu) = 1$ and S_0, S_1 are arbitrary functions of $\nu \in (0, 4\mu_0)$. (The linearity of the equation means that only the ratio $\lambda(\nu) := S_1(\nu)/S_0(\nu)$ (say) parameterizes the general solution.)

It is clear that the arbitrary functions allow us to control the variation of $\tilde{s}_\mu$ within an interval of width $4\mu_0$. At the integral values of $\mu/(4\mu_0)$ corresponding to $\nu = 0$, there is a consistency condition coming from vanishing of the highest (lowest) order coefficient which fixes the ratio of $S_0(0)$ and $S_1(0)$. The values of $\tilde{s}_{\mu=4\mu_0 n}$ are fixed (up to overall scaling). The slowly varying class of functions will be assumed to approximate these exact values. The continuum approximation developed below may not be a good approximation at a *finite* subset of these values corresponding to smaller n.

With these remarks, we now proceed to derive consequences from the assumption of (every where) slowly varying, approximate solutions of the difference equation (2.2.6).

Defining $A(p) := (\frac{1}{6}\gamma\ell_P^2)^{\frac{3}{2}} A_\mu$ and substituting $\tilde{s}_\mu$ in terms of slowly varying $\psi(p)$

in the difference equation (2.2.6), leads to the differential equation,

$$0 = B_0(p,p_0)\psi(p) + 4p_0 B_-(p,p_0)\psi'(p) + 8p_0^2 B_+(p,p_0)\psi''(p) \qquad (2.2.8)$$

where,

$$B_\pm(p,p_0) := A(p+4p_0) \pm A(p-4p_0)\ , \qquad \text{and}$$

$$B_0(p,p_0) := A(p+4p_0) - \left(2 + 144\frac{p_0^2}{\ell_{\mathrm{P}}^4}\eta\right)A(p) + A(p-4p_0) + \left(288\kappa\frac{{p_0}^3}{\ell_{\mathrm{P}}^4}\right)H_m(\mu)$$

In the above equation, terms involving higher derivatives of $\psi(p)$ have been neglected as being sub-leading in the context of slowly varying solutions. This is not quite the continuum approximation referred to earlier since there is γ dependence hidden inside p_0 appearing explicitly in the coefficients of the differential equation. This is also not quite the Wheeler–DeWitt equation since this equation is valid over the entire real line (since p can take negative values corresponding to oppositely oriented triad) while the Wheeler–DeWitt equation using the scale factor as independent variable is defined only for half real line.

From the definitions of the coefficients $A, B_{\pm,0}$, it is obvious that under $p \to -p$ (change of orientation of the triad), A, B_+ and the gravitational part of B_0 are all *odd* while B_- is *even*. For notational convenience we restrict to $p \geq 0$ while writing the limiting expressions, the expressions for negative p can be obtained from the odd/even properties noted above. There are two obvious regimes to explore which are conveniently demarcated by the scale p_0, namely, $p \gg p_0$ and $0 \leq p \ll p_0$. The corresponding limiting forms for the coefficients $B_0, B_\pm$ are easily obtained. One gets,

$$
\begin{aligned}
p \gg p_0 \quad &: \quad A(p,p_0) \; \approx \; 3p_0\sqrt{p} - \frac{1}{8}p_0^3\, p^{-\frac{3}{2}} \\
&: \quad B_+(p,p_0) \; \approx \; 6p_0\sqrt{p} - \frac{49}{4}p_0^3\, p^{-\frac{3}{2}} \\
&: \quad B_-(p,p_0) \; \approx \; 12p_0^2\, p^{-\frac{1}{2}} + o(p_0^4\, p^{-\frac{5}{2}}) \\
&: \quad B_0(p,p_0) \; \approx \; -12p_0^3 p^{-\frac{3}{2}} - 432\frac{p_0^3\sqrt{p}}{\ell_{\mathrm{P}}^4}\eta + 288\kappa\frac{{p_0}^3}{\ell_{\mathrm{P}}^4}H_m(\mu)
\end{aligned}
\tag{2.2.9}
$$

$$
\begin{aligned}
p \ll p_0 \quad &: \quad A(p,p_0) \; \approx \; 3pp_0^{\frac{1}{2}} - \frac{1}{8}p^3{p_0}^{-\frac{3}{2}} \\
&: \quad B_+(p,p_0) \; \approx \; 3(5^{\frac{1}{2}} - 3^{\frac{1}{2}})pp_0^{\frac{1}{2}} \\
&: \quad B_-(p,p_0) \; \approx \; 2p_0^{\frac{3}{2}}(5^{\frac{3}{2}} - 3^{\frac{3}{2}}) \\
&: \quad B_0(p,p_0) \; \approx \; (3pp_0^{\frac{1}{2}})(5^{\frac{1}{2}} - 3^{\frac{1}{2}} - 2) - 432\frac{p_0^{\frac{5}{2}}\,p}{\ell_{\mathrm{P}}^4}\eta + 288\kappa\frac{{p_0}^3}{\ell_{\mathrm{P}}^4}H_m(\mu)
\end{aligned}
\tag{2.2.10}
$$

Notice that for large volume the explicit p_0 dependence cancels out. This equation corresponds to the usual continuum approximation which has no dependence on γ and matches with the Wheeler–DeWitt equation in a particular factor ordering. For small volume, the p_0 dependence survives, is non-trivial and the coefficient of the first derivative terms is non-zero. In view of the even/odd properties of the coefficients, it follows that the first derivative of $\psi(p)$ must vanish at $p = 0$. Further more, even for the flat model without matter, the B_0 coefficient is non-zero. Had we extrapolated the Wheeler–DeWitt equation from the large volume form, we would not have gotten these terms. Thus the quantum differential equation (2.2.8) agrees with the Wheeler–DeWitt for large volume but differs significantly for small volume.

The small volume form of the equation in fact shows that there are two possible behaviours namely $\psi(p) \sim$ constant or $\psi(p)$ diverges as an inverse power of p. Neither of the indicial roots depend on the matter Hamiltonian. The latter solution is not slowly varying and the former one implies that the wave function has a non-zero value at $p = 0$ and the wave function can obviously be continued to negative p. Thus the differential equation derived for slowly varying functions is both consistent at zero volume and mimics main features of the difference equations namely passing through zero volume and matching with Wheeler–DeWitt for large volume.

In the earlier quantization of isotropic models [14] based on point holonomies taking values in $U(1)$ representations (separable Hilbert space), the decoupling of s_0 coefficient also implied a consistency condition which helped select a unique solution [52] from solutions of the Wheeler–DeWitt equation valid at large volume. With the non-separable Hilbert space, such a condition can only result from S_n^0 family of coefficients. Nevertheless, one has gotten a *unique* solution (up to normalization) thanks to the slowly varying nature of the solutions. It is crucial here that for small $p \to 0$, the B_- coefficient has a non-vanishing limit which forces the first derivative to vanish at $p = 0$. (If the single derivative term has been dropped, both solutions would have been slowly varying near $p = 0$.)

In summary, with the restriction to slowly varying solutions, we have a continuum approximation (differential equation) valid for all values of the triad. Further more the differential equation permits a unique solution (for each matter state) passing through $p = 0$.

For future reference we also note that the differential equation admits a 'conserved current'. Taking imaginary part of ψ^* times the differential equation leads to,

$$2p_0 B_+ \tilde{J}' + B_- \tilde{J} = 0 \quad , \quad \tilde{J} := \psi^* \psi' - \psi(\psi^*)' \tag{2.2.11}$$

Defining $J(p) := f^{-1}(p)\tilde{J}(p)$ such that $J' = 0$ determines the function f. Explicitly,

$$\begin{aligned} J(p) &= \text{constant} \left(e^{\int \frac{B_-}{2p_0 B_+} dp} \right) \{\psi^* \psi' - \psi(\psi^*)'\} \quad , \quad J' = 0 \; ; & (2.2.12) \\ &\to \text{constant} \ (p) \ \{\psi^* \psi' - \psi(\psi^*)'\} & (p \gg p_0) \\ &\to \text{constant} \left(p^{\frac{8+\sqrt{15}}{3}} \right) \{\psi^* \psi' - \psi(\psi^*)'\} & (p \ll p_0) \,. \end{aligned}$$

We will now go ahead with a WKB type solution and infer an effective classical Hamiltonian.

Let $\psi(p) = C(p) e^{\frac{i}{\hbar}\Phi(p)}$. Substitution of this ansatz in (2.2.8) leads to a complex differential equation involving $C(p), \Phi(p)$. The real and imaginary parts lead to,

$$0 = B_0(p,p_0) + 4p_0 B_-(p,p_0)\left\{(\ell n C)'\right\} + 8p_0^2 B_+(p,p_0)\left\{-\frac{\Phi'^2}{\hbar^2} + (\ell n C)'^{\,2} + (\ell n C)''\right\} \tag{2.2.13}$$

$$0 = 4p_0 B_-(p,p_0)\left\{\Phi'\right\} + 8p_0^2 B_+(p,p_0)\left\{\Phi'' + 2\Phi'(\ell n C)'\right\} \tag{2.2.14}$$

The WKB *approximation* consists in assuming that the amplitude C is essentially constant and the double derivatives of the phase are small compared to the single derivatives. Consider the eq.(2.2.13) under the assumption of almost constant amplitude $C(p)$. Then this equation is a Hamilton-Jacobi partial differential equation. These are generally, partial differential equations involving only first derivatives with respect to time and position and they always have an associated Hamiltonian mechanics [53].

$$B_0(p,p_0) - 8p_0^2 B_+(p,p_0)\frac{\Phi'^2}{\hbar^2} = 0 \tag{2.2.15}$$

Noting that the Poisson bracket between the triad variable p and the extrinsic curvature variable K is $\frac{\kappa}{3}$, we identify $\Phi' := \frac{3}{\kappa}K$ and arrive at the *effective Hamiltonian* as,

$$\begin{aligned} H^{\text{eff}}(p,K,\ \phi,\ p_\phi) &:= -\frac{1}{\kappa}\left[\frac{B_+(p,p_0)}{4p_0}K^2 + \eta\frac{A(p)}{2p_0}\right] \\ &+ \frac{1}{\kappa}\left[\left(\frac{\ell_{\rm P}^4}{288p_0^3}\right)\left\{B_+(p,p_0) - 2A(p)\right\}\right] + H_m(p,\phi,p_\phi) \end{aligned} \tag{2.2.16}$$

We have multiplied by certain factors so as to get the matter Hamiltonian term appear without any pre-factors as in the classical case. The equation (2.2.13) of course implies $H^{\text{eff}} = 0$ and we will interpret this as the modified Hamiltonian constraint equation. The effective Hamiltonian is also *odd* under $p \to -p$ modulo the matter term.

Note that the K^2 and the η dependent terms are $o(\ell_{\rm P}^0)$. For large volume, the terms enclosed within the braces are vanishingly small and the effective Hamiltonian is indeed *classical* (the matter Hamiltonian receiving corrections from the inverse triad operator also goes to the classical form without any $\ell_{\rm P}$ dependence). For smaller volumes, the quantum modifications are present with explicit dependence on ℓ_P, p_0. The approximation used does not quite lead to a 'classical limit' due to explicit appearance of ℓ_P.

For small volumes, the equation (2.2.10) shows that B_0, B_+, A all vanish linearly with p while B_- goes to a positive constant. The real and the imaginary parts of the equation, equations(2.2.13, 2.2.14), then imply that Φ' and C' both vanish, which is consistent with $\psi \sim$ constant, as deduced directly from the differential equation (2.2.8).

To interpret the effective Hamiltonian constraint as generating the space-time dynamics, let us use the identification $|p| = a^2$. One can then obtain the extrinsic curvature K from the Hamilton's equation of motion of p as,

$$\begin{aligned} \dot{p} := \frac{dp}{dt} &= \frac{\kappa}{3}\frac{\partial H^{\mathrm{eff}}}{\partial K} = -\frac{B_+(p,p_0)}{6p_0}K \qquad \text{or,} \\ K &= -12\left(\frac{ap_0}{B_+(p,p_0)}\right)\dot{a} \end{aligned} \tag{2.2.17}$$

The large and small volume expressions for the effective Hamiltonian and the extrinsic curvature are,

$$H^{\mathrm{eff}} \rightarrow -\frac{3}{2\kappa}\sqrt{p}\left(K^2+\eta\right)+H_m \tag{2.2.18}$$

$$K \rightarrow -2\dot{a} \qquad (p \gg p_0) \tag{2.2.19}$$

$$H^{\mathrm{eff}} \rightarrow -\frac{3}{2\kappa}\frac{p}{\sqrt{p_0}}\left[\left(\frac{5^{\frac{1}{2}}-3^{\frac{1}{2}}}{2}\right)K^2+\eta+\frac{1}{144}\left(2-5^{\frac{1}{2}}+3^{\frac{1}{2}}\right)\left(\frac{\ell_P^4}{p_0^2}\right)\right]+H_m \tag{2.2.20}$$

$$K \rightarrow -\left(\frac{4\sqrt{p_0}}{5^{\frac{1}{2}}-3^{\frac{1}{2}}}\right)\frac{\dot{a}}{a} \qquad (p \ll p_0) \tag{2.2.21}$$

The large volume expressions are the same as for the classical Hamiltonian as expected. The small volume expressions are useful in exploring the behaviour of the effective dynamics close to the classical singularity.

It is worth expanding on the identification $|p| = a^2$. The basic variables of LQC are first obtained for a general homogeneous Bianchi class A models with the Maurer-Cartan forms normalized in the usual manner. Comparing with the metric ansatz then leads to the relations $|p_1| = a_2 a_3$ and cyclic. The basic variables of the *isotropic* models are obtained from the above Bianchi ansatz, by putting $p_I = p,\ a_I = a\ \forall I$

leading to the identification above. Let us denote this scale factor as a_{Bianchi}. The corresponding spatial Ricci scalar is ${}^3R(a_{\text{Bianchi}}) = \frac{3}{2a^2_{\text{Bianchi}}}$. On the other hand, the standard FRW metric ansatz is so chosen that the spatial curvature is given by ${}^3R(a_{\text{FRW}}) = \frac{6}{a^2_{\text{FRW}}}$. These two normalizations match provided $a_{\text{Bianchi}} = \frac{a_{\text{FRW}}}{2}$. To avoid writing the suffixes, we just note that while comparing the large volume Hamiltonian constraint with the standard Friedmann equation, one should use the replacement $a \to \frac{a}{2}$. This of course is relevant only for the close model.

2.3 Qualitative Features of Effective Dynamics

In the previous section we derived the effective Hamiltonian constraint (2.2.16), using a continuum approximation keeping terms up to *second* derivatives and using the WKB approximation. If we include higher derivative terms that these would give *perturbative* corrections in the large volume. Our focus is however on the small volume regime and leading corrections which for the matter sector include non-perturbative corrections. For our purposes the truncation to second derivatives suffices.

For large volume, we already see that the effective Hamiltonian reduces to the classical one to within terms of the order of $p^{-\frac{3}{2}}$. We are interested in checking if the effective dynamics is non-singular and precisely in what sense. For this it is sufficient to focus on the small volume expressions. We will compare the classical Hamiltonian (2.2.18) extrapolated to small volume and the effective Hamiltonian (2.2.20).

Consider first the classical case. As the scale factor goes to zero, the matter density diverges either as a^{-3} for pressure-less matter or as a^{-4} for radiation. Correspondingly, H_m either goes to a non-negative constant or diverges as a^{-1}. The Hamiltonian constraint then implies that K necessarily diverges. As is well known, in both cases the scale factor vanishes at a *finite* value of synchronous time and this of course is the big bang singularity. This also suggests a necessary condition for singular evolution: $p = 0$ should be reachable in finite time. Equivalently, if $p = 0$

is *not* reachable in finite time, the evolution is non-singular.

Momentarily, let us *assume* that for some reason, the matter Hamiltonian *vanishes* as the scale factor goes to zero, then K must remain finite and $p = 0$ is indeed on the constraint surface. Further more $\dot{p}$ evaluated at $p = 0$ is also zero implying that $p = 0$ is a *fixed point* (rather a fixed 'sub-manifold' of the phase space of gravity and matter). The $p = 0$ trajectories of the dynamics are then not accessible in finite time and the evolution is non-singular. Clearly (non-) divergence of matter Hamiltonian dictates (non-) singular evolution.

For generic (non-singular) trajectories, there are two possibilities now. Either (a) $p = 0$ is approached asymptotically as $t \to -\infty$ or (b) the trajectory exhibits a bounce, $K = 0$ at a finite, non-zero p. For example, in the case of scalar matter, with LQC modifications included, the former is realized for flat models ($\eta = 0$) while the latter is realized for close models ($\eta = 1$) [54].

Consider now the quantum case. The matter Hamiltonian is guaranteed to vanish due to the inverse volume operator definition. The arguments for $p = 0$ being fixed point apply. However, due to the presence of ℓ_{P}^4 term in eq. (2.2.20), there is a bounce *independent* of η [36]. The $p = 0$ is completely decoupled from *all* other trajectories. This is exactly the same feature exhibited by the fundamental difference equation. The exact solution $\tilde{s}_\mu = s_0\delta_{\mu,0}$ completely decouples from all other solutions. The bounce is then a completely generic feature of isotropic LQC. But a bounce also provides *a minimum volume* for the isotropic universe whose value is dependent on matter Hamiltonian. Such a natural, generic scale has implications for phenomenology as well [36].

Notice that while interpreting the effective Hamiltonian as a (modified) constraint equation, we are keeping the kinematics of space-time (a pseudo-Riemannian manifold) intact. The modifications imply modification of the dynamical aspects or equivalently of Einstein equations. To see the modifications conveniently, let us write the Hamilton's equations in a form similar to the usual Raychaudhuri and Friedmann equations in terms of the FRW scale factor.

Write the effective Hamiltonian (2.2.16) in the form,

$$H^{\mathrm{eff}} = -\frac{1}{\kappa}(\alpha K^2 + \eta\beta) + \frac{1}{\kappa}\nu + H_m \qquad \text{where,} \tag{2.3.1}$$

$$\alpha := \frac{B_+}{4p_0} \ , \ \beta := \frac{A}{2p_0} \ , \ \nu := \left(\frac{\ell_{\mathrm{P}}^4}{288p_0^3}\right)(B_+ - 2A) \tag{2.3.2}$$

Putting $p := \frac{a^2}{4}$ (a is the FRW scale factor and $\kappa = 16\pi G$) and denoting $\frac{d}{da}$ by $'$ leads to,

$$3\frac{\dot{a}^2}{a^2} + 3\frac{\eta}{a^2} = \frac{16}{3}\frac{\alpha\nu}{a^4} + 3\frac{\eta}{a^2}\left(1 - \frac{16}{9}\frac{\alpha\beta}{a^2}\right) + \frac{16\kappa}{3}\frac{\alpha H_m}{a^4} \tag{2.3.3}$$

$$\begin{aligned} 3\frac{\ddot{a}}{a} &= \frac{8\alpha}{3a^4}\left[-\left\{\left(2 - \frac{a\alpha'}{\alpha}\right)\nu - a\nu'\right\} + \eta\left\{\left(2 - \frac{a\alpha'}{\alpha}\right)\beta - a\beta'\right\}\right. \\ &\quad \left. -\kappa\left\{\left(2 - \frac{a\alpha'}{\alpha}\right)H_m - aH_m'\right\}\right] \end{aligned} \tag{2.3.4}$$

Comparing with the usual FRW equations, we *identify* effective perfect fluid density and pressure as,

$$\rho^{\mathrm{eff}} := \frac{32}{3}\frac{\alpha H_m}{a^4} + \frac{32}{3\kappa}\frac{\alpha\nu}{a^4} + \frac{6}{\kappa}\frac{\eta}{a^2}\left(1 - \frac{16}{9}\frac{\alpha\beta}{a^2}\right) \tag{2.3.5}$$

$$\begin{aligned} P^{\mathrm{eff}} &:= +\frac{32}{9}\frac{\alpha}{a^4}\left\{\left(1 - \frac{a\alpha'}{\alpha}\right)H_m - aH_m'\right\} + \frac{32}{9}\frac{\alpha}{\kappa a^4}\left\{\left(1 - \frac{a\alpha'}{\alpha}\right)\nu - a\nu'\right\} \\ &\quad -\frac{32}{9}\frac{\alpha}{\kappa a^4}\eta\left\{\left(1 - \frac{a\alpha'}{\alpha}\right)\beta - a\beta' + \frac{9}{16}\frac{a^2}{\alpha}\right\} \end{aligned} \tag{2.3.6}$$

The large and small volume expressions for the effective density and pressure are,

$$\text{for } p \gg p_0 \ : \ \alpha \to \frac{3}{4}a \ , \ \beta \to \frac{3}{4}a \ , \ \nu \to -\frac{1}{3}\ell_{\mathrm{P}}^4 a^{-3};$$

$$\rho^{\mathrm{eff}} \to +8\, a^{-3}H_m - \frac{8}{3\kappa}\ell_{\mathrm{P}}^4 a^{-6} \ , \tag{2.3.7}$$

$$P^{\mathrm{eff}} \to -\frac{8}{3}a^{-3}(aH_m') - \frac{8}{3\kappa}\ell_{\mathrm{P}}^4 a^{-6} \ ; \tag{2.3.8}$$

$$\text{for } p \ll p_0 \ : \ \alpha \to \frac{3}{16}bp_0^{-1/2}a^2 \ , \ \beta \to \frac{3}{8}p_0^{-1/2}a^2 \ , \ \nu \to \frac{b-2}{384}\ell_{\mathrm{P}}^4 p_0^{-5/2}a^2 \ , \text{where}$$

$$b := \sqrt{5} - \sqrt{3} \ ;$$

$$\rho^{\mathrm{eff}} \to 2\frac{b}{\sqrt{p_0}}a^{-2}H_m - \frac{b(2-b)}{192\,\kappa}\ell_{\mathrm{P}}^4 p_0^{-3} + \eta\frac{6}{\kappa}a^{-2}\left(1 - \frac{b}{8}\frac{a^2}{p_0}\right) \ , \tag{2.3.9}$$

$$P^{\mathrm{eff}} \to -\frac{2}{3}\frac{b}{\sqrt{p_0}}a^{-2}\left(H_m + aH_m'\right) + \frac{b(2-b)}{192\,\kappa}\ell_{\mathrm{P}}^4 p_0^{-3} + \eta\frac{2}{\kappa}a^{-2}\left(1 - \frac{3b}{8}\frac{a^2}{p_0}\right) \ . \tag{2.3.10}$$

The effective density and pressure receive contributions from the matter sector and also the spatial curvature (η-dependent terms). Apart from these, the homogeneous and isotropic quantum fluctuations of the geometry also contribute an effective density and pressure (ℓ_{P}^4 terms). While tiny, these are non-zero even for large volume. Notice also that while for large volume the spatial curvature does not contribute to the density and pressure, for small volume it does. As a by product, we have also obtained the density and pressure for matter, *directly in terms of the matter Hamiltonian*. This is useful because currently LQC modifications to the matter sector are incorporated at the level of the Hamiltonian and not at the level of an action. Consequently, usual prescription for construction of the stress tensor and reading off the density and pressure is not available. These definitions of course automatically satisfy the conservation equation: $a\rho' = -3(P + \rho)$.

Let us consider the vacuum sector, $H_m = 0$. The more general case of presence of scalar field matter is discussed in [35]. Even for the flat model ($\eta = 0$), the $o(\ell_{\mathrm{P}}^4)$ terms contributes to the effective density and pressure. This term in the effective Hamiltonian, $W_{qg} := \frac{\nu(p,p_0)}{\kappa}$, is a 'potential' term and will be referred to as the *quantum geometry potential.* It is actually of order $\sqrt{\frac{\hbar}{\kappa}}$ after expressing p, p_0 in terms of the μ, μ_0. It is easy to see that the quantum geometry potential is *odd* under $p \to -p$ (since B_+ and A are odd) and for $p > 0$ it is *negative-definite.* Its plot is shown in the figure 2.1. This immediately implies that in the absence of matter (and cosmological constant), all the three terms in the effective Hamiltonian must be individually zero which is not possible for the quantum geometry potential. In other words, there is no solution space-time. This is in contrast to the purely classical Hamiltonian which does give the Minkowski space-time [1] as a solution for the flat case ($\eta = 0$). This is also apparent from the non-zero value of the effective density which prevents the Minkowski solution.

This feature can also be understood in the following manner. The differential equation has a unique solution (which is slowly varying every where). This solution

[1]The Minkowski space-time here refers to Riemann flat metric regardless of its global topology. In the cosmological context, the spatial slice is always compact.

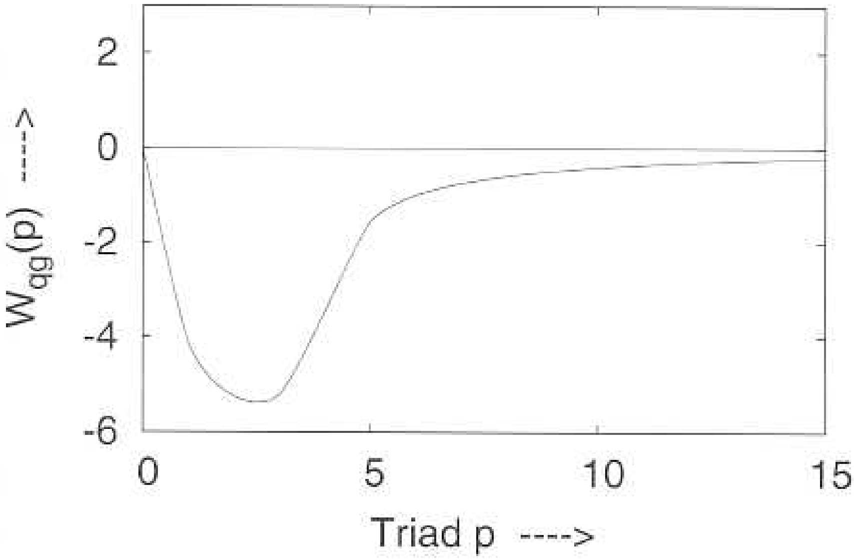

Figure 2.1: The quantum geometry potential. The triad variable p is in units of p_0 while the potential $W_{qg}(p)$ is in units of $\frac{1}{288\kappa}\ell_{\mathrm{P}}^4 p_0^{-3/2}$.

is however purely real and does not admit a WKB form. Consequently, the universe does not admit any 'classically allowed region' in the WKB sense and thus also does not exhibit a classical behaviour. Once matter is included, we have again a unique, real solution of the differential equation *for every matter state*. We can now have complex linear combinations admitting possibility of regimes of WKB form and corresponding classical behaviour.

It is straight forward to write down a Lagrangian from the H^{eff} as,

$$\begin{aligned} L^{\mathrm{eff}}(p,\dot{p}) &:= \frac{3}{\kappa}K\dot{p} - H^{\mathrm{eff}}(p,K) \\ &= -\frac{1}{\kappa}\left[\frac{9p_0}{B_+}\dot{p}^2 - \eta\frac{A(p)}{2p_0} + \left(\frac{\ell_{\mathrm{P}}^4}{288p_0^3}\right)\{B_+(p,p_0) - 2A(p)\}\right] \end{aligned} \quad (2.3.11)$$

It would be interesting to see if this Lagrangian can be obtained from a specialization of a generally covariant action to homogeneous, isotropic metric.

2.4 Discussions

The results of this chapter are based on two essential ingredients: the proposal of a continuum approximation for all volumes exploiting the non-separable nature of the kinematical Hilbert space and the derivation of the effective Hamiltonian via the WKB route.

The continuum approximation step leads to a differential equation for a (still) quantum wave function, $\psi(p)$. This equation matches with the Wheeler–DeWitt equation for large volume and has important deviations (the first derivative terms) from it at small volumes. These deviations allow continuation of the wave function through zero volume just as the fundamental difference equation does. For slowly varying solutions, it picks out the 'boundary' condition $\frac{d\psi}{dp}(0) = 0$. Again this is analogous to unique solution (per matter state) picked out by the difference equation obtained from the $U(1)$ point holonomies in the earlier work [52]. Thus the essential features of the fundamental difference equation namely non-singular quantum evolution with semi-classical limit are captured by the continuum differential equation.

The effective dynamics specified by the effective Hamiltonian deduced via the WKB approximation also reflects these features. The effective dynamics is non-singular, captures the decoupling of the $\tilde{s}_\mu = s_0\delta_{\mu,0}$ exact solution of the difference equation by making the classical $p = 0$ trajectory decouple and reduces to the usual classical dynamics of general relativity for large volumes. Since the essential features of quantum dynamics are now captured in classical geometrical terms, the effective dynamics is more reliable than the usual one and one can now simply work with the effective dynamics to do phenomenology. Already, at the qualitative level, one sees that all non-trivial evolutions necessarily show a bounce providing a natural scale for say, density perturbations and their power spectra.

Since the approach draws on the WKB method, a few remarks on the interpretational aspects are in order.

From the continuum quantum dynamics, with the WKB approximation, one obtains a Hamilton-Jacobi equation. As a mathematical result, any Hamilton-Jacobi differential equation has an associated Hamiltonian mechanics [53] and corresponding trajectories. In our context, we are interpreting these trajectories as possible evolutions of the isotropic universe. There is at least an implicit implication (or assumption) that a quantum system executing a WKB approximable quantum motion *physically* exhibits a classical motion governed by the Hamiltonian associated with the corresponding Hamilton-Jacobi equation. The justification for this comes from noting that for large volume we expect the universe to exhibit classical behaviour and there it is indeed governed by the associated Hamiltonian. For how small volumes can we assume this expectation? This question is naturally related to the domain of validity of the WKB approximation.

One expects the WKB approximation (slow variation of the phase and almost constancy of the amplitude of the wave function) to break down closer to the classically indicated singularity at zero volume. Noting that the differential equation is local in p and its solutions are also local solutions (valid in open intervals in p), we can begin with a WKB approximable solution valid in the larger volume regime and attempt to extrapolate it to smaller and smaller volumes. All through these extensions, one will have the effective Hamiltonian with its associated trajectories which can access the values of p in these intervals. The effective Hamiltonian constraint defines a sub-manifold of the phase space and all trajectories must lie on this. The range of configuration space variables (eg p in our case) allowed by the sub-manifold defines 'classically allowed region'. As is well known from the usual examples in non-relativistic quantum mechanics, the WKB approximation breaks down at the 'turning points'. These correspond to the boundary of classically allowed region which therefore demarcates the domain of validity of WKB approximation. Clearly, when such a boundary is reached by a trajectory, it must turn back. This is of course the bounce ($\dot{a} = 0$). The expectation that WKB breaks down at non-zero volume translates into the expectation of a bounce occurring at non-zero scale factor. The bounce can thus be understood as the smallest volume (or scale factor) down to which one may use the classical framework with some justification but below it one must use the quantum framework.

Further justification comes from other known examples. For example, solutions of the Maxwell equations in the eikonal approximation can be understood in terms of the normals to the wave fronts which follow null geodesic. The interpretation that this actually reflects rectilinear motion of light, may be justified by noting that the Poynting vector (energy flow) is also in the same direction as the normals. Likewise, in the context of usual Schroedinger equation of particle mechanics, the conserved probability current also points along the normals to the wave fronts giving credence to the interpretation that a quantum state of the WKB form realizes motion of a particle (or wave packet) governed by the associated Hamiltonian mechanics. In both these examples, further inputs other than the mathematical association between Hamilton-Jacobi differential equation and a Hamiltonian system, seem to be needed to understand the physical realization of the Hamiltonian system.

Interestingly, the equation (2.2.8) does admit a conserved current (2.2.12) which indeed is proportional to the gradient of the WKB phase. Whether this could be used for guessing physical inner product vis a vis a probability interpretation remains to be seen.

Chapter 3

Inflation is Generic

3.1 Introduction

The standard big bang model so far is the most successful large scale description of our universe. In this description, the evolution of our universe begins from a singularity. Within the context of homogeneous and isotropic expanding space-times, the singularity is unavoidable as long as the matter satisfies the so called strong energy condition. The singularity in this context means that the scale factor (or size of the universe) vanishes a finite time ago. This vanishing size also implies that the energy density diverges at this time. Furthermore, the scale factor vanishes slower than linearly with the synchronous time making the conformal time integral finite thus implying the existence of particle horizon.

The particle horizon with respect to a space-time point is defined by the maximum proper distance a particle could have travelled since the beginning of the universe. Due to the singular behaviour of the scale factor, this is a finite distance. It also means that any space-time point could have causal contact only with a finite patch of the space-time around it. By itself, existence of particle horizon need not be a problem. However, in conjunction with the thermal history of the universe, the finite horizon size implies that the last scattering surface of the cosmic microwave background photons has regions which could not have been in causal contact. Yet, there is remarkable isotropy (to within few parts of hundred thousand) in their angular distribution. This is the *horizon problem* of the big bang model.

The most popular approach to solve this puzzle (along with few other puzzles) is to introduce a phase of *inflation* [2]. Phase of inflation generally refers to a period during which the universe goes through a rapid (generally exponential) expansion. This is generally achieved by introducing a scalar field (an inflaton) with a self interaction potential. By now there are several versions of inflationary models [3]. Generically these solve the horizon problem (and other traditional problems such as the flatness problem) and in addition make specific predictions about the power spectra of inhomogeneous perturbations. While these are attractive features of inflationary models, generally they need fine tuning the potential and initial conditions for the inflaton to ensure a sufficient amount of inflation with graceful exit. In a sense, the isotropic singularity in Einsteinian gravity implies existence of particle horizon which leads to the horizon problem which needs an inflationary scenario to be postulated with its own set of problems of fine tuning and initial conditions.

The space-time singularity, however, signals breakdown of the theoretical framework of classical general relativity. It is widely expected that a quantum theory of gravity will provide a more accurate description which will hopefully be free of such breakdowns and recent developments of loop quantum cosmology [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] corroborate this expectation. It has already been shown that the LQC framework is free of singularity, both in the isotropic context [22] as well as more generally for homogeneous diagonal models [23, 24]. There are two aspects of this singularity-free property. The imposition of the Hamiltonian constraint ("Wheeler–DeWitt equation") of LQC leads to a difference equation with eigenvalues of the densitized triad variable serving as labels. These eigenvalues can take negative values corresponding to reversal of orientation. The difference equation, viewed as an evolution equation in these labels, allows solutions to evolve through the zero eigenvalue (zero size). Thus there is no breakdown of evolution equation at the classically indicated singularity at zero size. This is the first aspect of absence of singularity. The second aspect is that matter densities and curvatures remain finite at all sizes. The inverse scale factor operator that enters the definitions of these quantities turns out to have bounded spectrum. For an explanation and details see [17, 25].

To relate implications of LQC which is based on a discrete quantum geometry, to observable (and more familiar) quantities described in terms of the continuum geometrical framework of general relativity, the idea of an *effective Hamiltonian* has been proposed in [15, 31] and discussed in chapter 2. This Hamiltonian contains the modifications implied by LQC to the usual classical Hamiltonian. This approach retains the kinematical framework of Robertson-Walker geometry but gives modification of the dynamics of the Einstein equations.

3.2 Strong Energy Condition

As discussed before, the dynamics (evolution with respect to the synchronous time) implied by the effective Hamiltonian captures essential features of the difference equation, in particular the dynamics is *non-singular*. A universe beginning at some large volume will never reach zero volume when evolved backwards. Since the framework for effective dynamics is that of the usual pseudo-Riemannian geometry, the arguments leading to the singularity theorem are applicable and therefore non-singular evolution must imply violation of the strong energy condition on effective matter density and pressure. While the effective density and pressure [31] includes contributions from gravity sector also, in this chapter we concentrate on the matter sector modifications only.

The question we address is whether the modifications in the matter sector imply violation of strong energy condition. In general the strong energy condition requires $R_{\alpha\beta}\xi^\alpha\xi^\beta = 8\pi G(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)\xi^\alpha\xi^\beta \geq 0$, for all time-like vectors ξ^α. Within the context of homogeneous and isotropic geometries, the strong energy condition applied to the congruence of isotropic observers (or four velocity of the matter perfect fluid), becomes $R_{00} = 4\pi G(\rho + 3P) \geq 0$ where ρ is the total energy density and P is the total pressure of the matter fluid. Defining $\omega := P/\rho$ (with ρ assumed to be positive definite) as the equation of state variable, the violation of strong energy condition is conveniently stated as $\omega < -\frac{1}{3}$. Note that since $R_{00} \sim -\frac{\ddot{a}}{a}$, violation of the strong energy condition in this context also implies an *accelerated* evolution of the scale factor or in other words an *inflationary phase*.

For simplicity, let the matter sector consists of a single scalar field with a standard kinetic term and self interaction potential. For spatially homogeneous and isotropic fields, the density and pressure are read-off from the perfect fluid form as $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, $P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ while the classical Hamiltonian is $H_{cl} = \frac{1}{2}a^{-3}{p_\phi}^2 + a^3V(\phi)$, where p_ϕ is the conjugate field momentum. Note that in LQC, the scale factor has dimensions of length. (In LQC, densitized triad is redefined to absorb the coordinate volume integral, making the scale factor $a^2 =: |p|$ to have dimensions of length. This is to be kept in mind while making comparison with the standard scenarios which use a dimensionless scale factor.) The LQC modifications are incorporated by replacing the a^{-3} by a function coming from the definition of the inverse triad operator. The modified effective matter Hamiltonian is then given by $H^{\text{eff}} = \frac{1}{2}|\tilde{F}_{j,l}(a)|^{\frac{3}{2}}{p_\phi}^2 + a^3V(\phi)$, where $\tilde{F}_{j,l}(a) = (\frac{1}{3}\gamma\mu_0 j l_p^2)^{-1}F_l((\frac{1}{3}\gamma\mu_0 j l_p^2)^{-1}a^2)$ [33, 34]. The j and l are two quantization ambiguity parameters [47, 48]. The half integer j corresponds to the dimension of representation while writing holonomy as multiplicative operators while the real valued l $(0 < l < 1)$ labels different, classically equivalent ways of writing the inverse power of the scale factor in terms of Poisson bracket of the basic variables. A smooth approximation (except at one point) to the function $F_l(q)$ is given by [30]

$$
\begin{aligned}
F_l(q) \;:=\; & \Bigg[\frac{3}{2(l+2)(l+1)l}\Big(\,(l+1)\left\{(q+1)^{l+2}-\right. \\
& \left.|q-1|^{l+2}\right\} \;-\; (l+2)q\left\{(q+1)^{l+1}-\right. \\
& \left.\operatorname{sgn}(q-1)|q-1|^{l+1}\right\}\Big)\Bigg]^{\frac{1}{1-l}} \\
\rightarrow\; & q^{-1} \qquad\qquad (q \gg 1) \\
\rightarrow\; & \left[\frac{3q}{l+1}\right]^{\frac{1}{1-l}} \qquad (0 < q \ll 1)\,. \qquad (3.2.1)
\end{aligned}
$$

Thus, for the large values of the scale factor one has the expected classical behaviour for the inverse scale factor and the quantum behaviour is manifested for small values of the scale factor.

The density and pressure are usually defined from the perfect fluid form of the usual stress tensor which in turn is derived from an action principle for the matter field. Since the LQC modifications are incorporated at the level of the Hamiltonian, this route is not available. It is possible to define the density and pressure directly in terms of the matter Hamiltonian. This has been done generally in [31] (See also [55]). The relevant definitions, in terms of the notation in [31], are ($'$ denotes $\frac{d}{da}$)

$$\rho = \frac{32}{3}\frac{\alpha}{a^4}H \ , \ P = \frac{32}{9}\frac{\alpha}{a^4}\left\{\left(1 - \frac{a\alpha'}{\alpha}\right)H - aH'\right\} \tag{3.2.2}$$

In the above α is a specific function of a. For large a^2, α goes as $\frac{3}{4}a$ giving the familiar form of density as $8Ha^{-3}$ [31]. For the LQC modified scalar matter Hamiltonian, and for large a, these definitions of density and pressure match with those of [55]. The conservation equation $a\rho' = -3(\rho + P)$ is of course, automatically satisfied. It is straight forward to verify that for the classical Hamiltonian for the scalar field, these reduce to the usual definitions for large a.

Consider the two equation of state variables, P/ρ, defined by the effective Hamiltonian and by the classical Hamiltonian.

$$\begin{aligned} \omega^{\text{eff}} &= -\frac{\frac{1}{4}p_\phi^2 a^{-3}[\tilde{F}_{j,l}(a)]^{\frac{1}{2}}(a[\tilde{F}_{j,l}(a)]') + V(\phi)}{\frac{1}{2}p_\phi^2 a^{-3}[\tilde{F}_{j,l}(a)]^{\frac{3}{2}} + V(\phi)} \\ &\quad + \frac{1}{3}\left(1 - \frac{a\alpha'}{\alpha}\right) \qquad (3.2.3) \\ \omega &= \frac{\frac{1}{2}p_\phi^2 a^{-6} - V(\phi)}{\frac{1}{2}p_\phi^2 a^{-6} + V(\phi)} \qquad (3.2.4) \end{aligned}$$

The second term in eq.(3.2.3) vanishes for large volumes and goes to $-1/3$ for small volumes [31]. It is independent of the matter variables and will be suppressed below (see remarks on the scales at the end). The dynamical evolutions of these equations of state is of course governed by the corresponding Hamiltonians. It is however possible to derive qualitative behaviour of ω^{eff} for small scale factors *without having to know explicit time evolution*, as follows.

The equations (3.2.3, 3.2.4) can be thought of as two homogeneous algebraic equations for $p_\phi^2, V(\phi)$. For non-trivial values of these, the determinant must vanish which gives a relation between the two ω's as,

$$\omega^{\text{eff}} = -1 + \frac{(1+\omega)a^3[\tilde{F}_{j,l}(a)]^{\frac{3}{2}}\left(1 - \frac{a[\tilde{F}_{j,l}(a)]'}{2\tilde{F}_{j,l}(a)}\right)}{(1+\omega)a^3[\tilde{F}_{j,l}(a)]^{\frac{3}{2}} + (1-\omega)} . \tag{3.2.5}$$

Using the expression (3.2.1) it is easy to see that for the large values of the scale factor a, where one expects the quantum effects to be small, $\omega^{\text{eff}} = \omega$ and the dynamical evolution is controlled by the classical Hamiltonian. However for small values of a the ω^{eff} differs from the classical ω dramatically.

The numerator in the second terms of (3.2.5) vanishes as $a^{3+\frac{3}{1-\ell}}$. If $1-\omega$ in the denominator dominates, then clearly $\omega^{\text{eff}} \to -1$. This would happen either because $1-\omega \nrightarrow 0$ or it vanishes slower than $a^{3+\frac{3}{1-\ell}}$. In the former case we already have violation of strong energy condition. It is possible to get constraints on the behaviour of ω as the scale factor vanishes. For instance, the conservation equation expressed in terms of the scale factor implies that if $\omega \to 1$ then $\rho \sim a^{-6}$. This equation is independent of the LQC modification and applies also to effective density. Furthermore, from the definition it follows that $1-\omega = \frac{2V(\phi)}{\rho}$. Thus, the $1-\omega$ term in the denominator will dominate if $V(\phi(a))\ a^{-\frac{3\ell}{1-\ell}}$ diverges as $a \to 0$. This dominance is ensured if either (i) the potential never vanishes during the evolution or (ii) $V(\phi(a))$ vanishes at the most as a power law, a^{ξ}. In the former case, $\omega^{\text{eff}} \to -1$ will hold independent of the ambiguity parameter ℓ while in the latter case, for any given α we can always *choose* $\ell > \frac{\xi}{\xi+3}$ so that $\omega^{\text{eff}} \to -1$ is achieved. Note that this is *not* a fine tuning.

For the special case of identically vanishing potential, we get $\omega = 1$ and the expression for ω^{eff} simplifies to $-\frac{a[\tilde{F}_{j,l}(a)]'}{2\tilde{F}_{j,l}(a)}$. For small scale factor $\omega^{\text{eff}} \to -\frac{1}{1-\ell} < -1$ and violation of strong energy condition follows. Indeed, since $\omega^{\text{eff}} < -1$ holds, one has a phase of *super-inflation*. In fact this feature corresponds to situation considered in [33, 56]. However this feature is rather special because even a tiny but non-negative potential will force ω^{eff} to take the form (3.2.5) (see figure[3.1]).

The spikes in the figure correspond to the non-differentiability of the $F_l(q)$ at $q=1$. These can be removed by a local smoothing of the function around $q=1$ and thus have no physical significance.

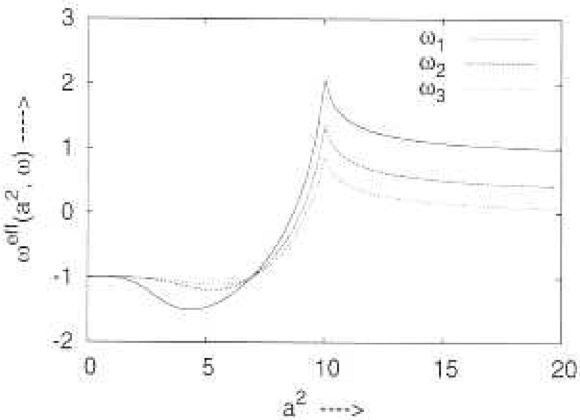

Figure 3.1: Plot of $\omega^{\rm eff}$ as a function of a^2 and ω for different constant values $\omega = 0.9, 0.33, 0.0001$. The ambiguity parameters are $j = 5, l = 0.5$ and a^2 is in units of $\frac{1}{6}\gamma\mu_0\ell_{\rm P}^2$. For small scale factor, $\omega^{\rm eff}$ always approaches -1 from below while for larger values it approaches ω.

3.3 Discussions

In summary, we find that if the scalar field potential satisfies $V(\phi) > 0$ then irrespective of what values we choose for the ambiguity parameters and irrespective of 'initial conditions' for the scalar field, there is always a violation of strong energy condition in the small volume regime and of course a corresponding inflationary phase. Furthermore since the effective equation of state variable approaches -1, we get to a phase of exponential inflation. If the potential has zeros which are approached as a power law for small scale factor, one can always choose a value of ℓ to get the same result. We emphasize that unlike the usual inflationary scenarios we do *not* need to invoke 'slow roll conditions' which constrain the potential as well as initial conditions for the scalar and effectively posit the equation of state variable to be -1. It is enough to have the evolution get to small volume regime to generate (exponential) inflation.

A couple of remarks are in order. Firstly, if LQC modifications from gravity sector (quantum geometry potential) are also included [31], then the results regarding behaviour of the effective equation of state as a function of the scale factor, are unchanged. These gravitational contributions to density and pressure violate the strong energy condition by themselves. Their effective equation of state parameter is +1 but both the density and pressure are negative.

A second remark concerns the scales. There are two basic scales available: (i) the 'quantum geometry scale', $L_{\mathrm{qg}}^2 := \frac{1}{6}\gamma\mu_0\ell_{\mathrm{P}}^2 = p_0$ [17] and (ii) the 'inverse scale factor scale', $L_{\mathrm{j}}^2 := \frac{1}{6}\gamma\mu_0\ell_{\mathrm{P}}^2(2j) = 2jp_0$. The former sets the scale for non-perturbative modifications in the gravitational sector while the latter does the same for the matter sector. Clearly, $L_{\mathrm{qg}} \leq L_{\mathrm{j}}$. It is easy to see [31] that the WKB approximation gets poorer close to L_{qg}. This is consistent with the physical expectation that below this scale one is in the deep quantum regime. Furthermore, the effective model, including the modifications in both gravity and matter sector, always shows a bounce i.e. a non-zero *minimum* scale factor at which $\dot{a}$ vanishes [36]. This introduces a *third* scale, L_{bounce}^2 which is smaller than L_j^2. Clearly, $L_{\mathrm{bounce}}^2 > L_{\mathrm{qg}}^2$ must hold to remain within the domain of validity of WKB approximation. In summary, $a \gg L_j$ is the classical regime, while for $a < L_{\mathrm{bounce}}$ one is strictly in the quantum domain in the WKB sense and the effective Hamiltonian is not valid. The semi-classical regime for the purposes of this paper has the scale factor between L_{bounce} and L_{j}. This implies that the suppressed term in the eq. (3.2.3), is vanishingly small in the semi-classical regime thus $\omega^{\mathrm{eff}} \to -1$.

The issue of whether the effective dynamics admits particle horizon or not, is a separate issue. In view of the generic bounce in the effective model [36], the universe would have existed for infinite time in the past. The evolution could have been oscillatory or there could have been just one bounce in the past. In the large volume regime, we have the usual decelerating evolution (modulo Λ-term) implying that the scale factor will diverge at the most as linear power of the synchronous time. For both possibilities, the conformal time integral would be infinite implying absence of particle horizon. However, *independent of the non-existence of particle horizon*, inflation comes built-in with the LQC modifications.

Chapter 4

Big Bounce is Generic

4.1 Introduction

It has been long expected that the existence of singularity in the classical general relativity which has been shown to be quite *generic* thanks to the *singularity theorems*, will be removed when classical framework of gravity is extended to a quantum framework of gravity. The issue of fate of classical cosmological singularities in the cosmological context has been addressed head-on within the loop quantum cosmology [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. It has been shown that the isotropic models (flat and closed)[22], and more generally the diagonal Bianchi class A models [23, 24], are free of singularity.

Some of consequences of loop quantum cosmology corrected cosmologies have already been noted. First, there is a natural mechanism for inflation [33, 34, 35] within the context of isotropic models. Secondly, for the Bianchi IX model there is a suppression of chaotic approach to singularity [30, 61]. Thirdly, there is indication of a bounce at the big crunch singularity as well [54, 57]. All of these have been explored within the framework of an *effective Hamiltonian* which incorporates the most significant non-perturbative corrections. These modifications stem from the non-trivial definition of the inverse triad operator in LQC [24, 25] which ensure that the matter density, spin connection components remain bounded as universe approaches zero volume.

4.2 Big Bounce

As discussed in chapter 2, the domain of validity of the continuum approximation has been extended in [31], by exploiting the non-separable structure of the kinematical Hilbert space of loop quantum cosmology [17] which has infinitely many solutions of the fundamental difference equation. Although each of these may not be slowly varying at smaller volumes, one can choose linear combinations to construct solutions which are slowly varying almost every where. This amounts to an ad-hoc restriction to a sub-class of solution. In the absence of any other criteria to limit the infinity of solutions, such as a physical inner product, this restriction is treated as exploratory. The extraction of effective Hamiltonian then follows the same method as before via a WKB approximation. The validity of the effective Hamiltonian is now limited only by the validity of WKB approximation i.e. to 'classically accessible regions'. The effective Hamiltonian is derived in [31] and is given by,

$$H^{\mathrm{eff}} = -\frac{1}{\kappa}\left[\frac{B_+(p)}{4p_0}K^2 + \eta\frac{A(p)}{2p_0}\right] + W_{qg} + H_m \qquad (4.2.1)$$

where $\kappa = 16\pi G$, $p_0 = \frac{1}{6}\gamma\ell_{\mathrm{P}}^2\mu_0$, K is the extrinsic curvature (conjugate variable of p), $A(p) = |p+p_0|^{\frac{3}{2}} - |p-p_0|^{\frac{3}{2}}$, η takes values $0, 1$ for spatially flat, closed models respectively, $B_+(p) = A(p+4p_0) + A(p-4p_0)$, $\ell_{\mathrm{P}}^2 := \kappa\hbar$ and $W_{qg} = \left(\frac{\ell_{\mathrm{P}}^4}{288\kappa p_0^3}\right)\{B_+(p) - 2A(p)\}$. Apart from the modification of the coefficients of the gravitational kinetic term and the spatial curvature term, the effective Hamiltonian (4.2.1) differs from the classical Hamiltonian by a non-trivial potential term referred to as *quantum geometry potential* and denoted as W_{qg}. It is odd under the reversal of orientation of the triad ($p \to -p$) and for $p > 0$, it is negative definite. The origin of this potential term is necessarily quantum gravitational as it explicitly involves ℓ_{P}. For large volume this potential falls-off as $p^{-3/2}$ while for small volume it vanishes as p. These small/large volume regions are delineated by the scale p_0.

For simplicity we consider a matter sector consisting of a single scalar field. Its classical Hamiltonian is LQC corrected in the usual manner [33, 58]. It is shown in [35] that for small volume ($p \ll 2jp_0$), with non-perturbative corrections, the scalar field effectively behaves like an *inflaton* field since the effective equation of state variable, $\omega^{\text{eff}} \to -1$ (or $-\frac{4}{3}$ if the triad variable p can get smaller than p_0). The matter Hamiltonian is related to the matter effective density [31, 35] as $H_m = \frac{3p_0}{8}\frac{a^4}{B_+}\rho^{\text{eff}}$ and the conservation equation implies that for a constant ω^{eff}, the effective density goes as $\sim a^{-3(1+\omega^{\text{eff}})}$. One can see that in either case of inflationary or super-inflationary regimes, the matter Hamiltonian always goes as $\sim p^{3/2}$.

Given the behaviours of the quantum geometry potential and the matter Hamiltonian during a (super) inflationary region and their opposite signs, it is clear that the quantum geometry potential will always dominate the matter Hamiltonian implying imaginary value for the extrinsic curvature i.e. existence of classically inaccessible scale factors. The two necessarily cancel each other at a finite, non-zero value of the scale factor. This would be so even after including the contribution of the spatial curvature (η). But this means that the extrinsic curvature vanishes at that value of the scale factor implying a bounce. Thus we see that a bounce is quite generic and the minimum scale factor defines a new length scale L_{bounce}. Below this scale, the effective classical picture fails. A graphical illustration of the existence of bounce can be seen in the figure (4.1).

The bounce scale is obviously determined by the conditions $H^{\text{eff}} = 0 = K$ with $H_m = hp^{3/2}$, h being a constant of proportionality. This is a transcendental equation in p and the root(s) depend on the constant h. We expect the bounce value to be less than $2jp_0$ (above which we are in the classical regime). In this chapter we use *geometrized* units instead of *natural* units. In geometrized units, $\kappa = 1 = c$, we will refer to all lengths to the scale $\sqrt{p_0}$. Thus, putting $p := qp_0$ the region of interest is $0 \ll q \ll 2j$. This could be further divided into (i) $0 \ll q \ll 1$ and (ii) $1 \ll q \ll 2j$. Since the equation of state variable is a dimensionless function of the scale factor, it is in fact a function of q. The conservation equation can then be solved as: $\rho^{\text{eff}}(q) = \bar{\rho}\exp\{-\frac{3}{2}\int_{2j}^{q}(1+\omega^{\text{eff}}(q))\frac{dq}{q}\}$. The constant h is proportional to $\bar{\rho} = \rho^{\text{eff}}(2j)$. In summary, the equations determining the (non-zero) bounce scale, q_{bounce}, are,

$$
\begin{aligned}
H_m(q) &= 6\frac{p_0^3}{B_+}q^2\bar{\rho}\exp\left\{-\tfrac{3}{2}\int_{2j}^{q}(1+\omega^{\text{eff}}(q))\tfrac{dq}{q}\right\} && (4.2.2)\\
\| & \\
h\ (p_0q)^{\frac{3}{2}} &= \eta\tfrac{A(q)}{2p_0} - \tfrac{\ell_{\rm P}^4}{288p_0^3}(B_+(q)-2A(q)) && (4.2.3)
\end{aligned}
$$

In the two regions (i) and (ii), the equations simplify. In the region (i) one has, $A \to 3p_0^{3/2}q$, $B_+ \to 3(\sqrt{5}-\sqrt{3})p_0^{3/2}q$, $\omega^{\text{eff}}(q) \to -\frac{4}{3}$ and we get,

$$
\begin{aligned}
H_m(q) &\to \left[\sqrt{\frac{2}{j}}\frac{\bar{\rho}}{(\sqrt{5}-\sqrt{3})}\right](p_0q)^{\frac{3}{2}} && (4.2.4)\\
\sqrt{q}_{\text{bounce}} &= \frac{1}{2p_0h}\left[3\eta + \frac{(2-\sqrt{5}+\sqrt{3})}{48}\frac{\ell_{\rm P}^4}{p_0^2}\right]. && (4.2.5)
\end{aligned}
$$

In region (ii) one has $A \to p_0^{3/2}(3q^{1/2} - \frac{1}{8}q^{-3/2})$, $B_+ \to p_0^{3/2}(6q^{1/2} - \frac{49}{4}q^{-3/2})$, $\omega^{\text{eff}}(q) \to -1$ and we get,

$$
H_m(q) \to \bar{\rho}(p_0q)^{\frac{3}{2}}, \qquad (4.2.6)
$$

and the q_{bounce} is determined as a root of the cubic equation,

$$
(2p_0\bar{\rho})q^3 - 3\eta q^2 - \frac{1}{12}\left(\frac{\ell_{\rm P}^4}{p_0^2}\right) = 0 \qquad (4.2.7)
$$

Note that $p_0\bar{\rho}$ is dimensionless. It is easy to see that there is exactly one real root of this equation and in fact, get a close form expression for it.

In the region (i), $h \sim \bar{\rho}/\sqrt{j}$ and the $q_{\text{bounce}} \sim (p_0h)^{-2}$. The solution has explicit j dependence. The inequality for region (i), $q_{\text{bounce}} \ll 1$, implies that the effective density at $p = 2jp_0$ (roughly where the inverse scale factor function attains its maximum) must be larger than $\sqrt{j}$ times the Planck density $\sim \ell_{\rm P}^{-2}$. Also the bounce scale will be smaller than the Planck scale. This is indicative of the effective continuum model becoming a poor approximation.

In region (ii), $h = \bar{\rho}$. For the flat model ($\eta = 0$), $q_{\text{bounce}} \sim (p_0\bar{\rho})^{-1/3}$. For the close model, such a simple dependence does not occur. The inequalities, $1 \ll q \ll 2j$, translate into a window for $p_0\bar{\rho}$. Region (ii) bounce scale has *no explicit* dependence on the ambiguity parameter j, the implicit dependence being subsumed in the value of $\bar{\rho}$ which can be treated as a free parameter. The bounce scale is larger than the

Planck scale and the density $\bar{\rho}$ is smaller than the Planck density. The relation between h and the bounce scale is displayed in figure (4.2).

Clearly, as $h \to 0$, the scale $p_0 \to 0$ and so does the bounce scale. It also vanishes as $\bar{\rho} \to \infty$. However in the non-singular evolution implied by isotropic LQC (inverse scale factor having a bounded spectrum) implies that there is a maximum energy density attainable and the density $\bar{\rho}$ can be thought of as this maximum energy density. Correspondingly there is a minimum scale factor or minimum *proper* volume since in LQC the fiducial coordinate volume is absorbed in the definition of the triad. For volumes smaller than the minimum volume, the WKB approximation fails and the effective classical picture cannot be trusted. The quantum geometry potential plays a crucial role in this result.

A remark about the physical justification for the approximations used is in order. The results use the effective Hamiltonian picture which is based on a continuum approximation followed by the WKB approximation. The physical justification thus hinges on the physical justification for these two approximations. The continuum approximation for the geometry is physically expected to be a good approximation for length scales larger than the discreteness scale set by $4p_0 \propto 4\mu_0$, the step size in the fundamental difference equation. The WKB approximation is valid in a sub-domain of the continuum approximation, determined by slow variation of amplitude and phase. Mathematically, the amplitude variation begins to get stronger around $2p_0$ [31] while the phase variation is stronger at the turning point which determines the bounce scale. Thus, physically, the effective Hamiltonian (including the quantum geometry potential) is trustworthy for the bounce scale larger than $\sim p_0$. As shown by the figure 2, there is a range of $\bar{\rho} < l_p^{-2}$ such that the bounce scale is consistent with the physical domain of validity of the effective Hamiltonian. Note also that the behaviour of the matter Hamiltonian as $p^{\frac{3}{2}}$ is dependent on $p \ll 2jp_0$ and hence the bounce scale is also smaller than $2jp_0$.

4.3 Discussions

We will discuss now a possible implication of the minimum proper length on the inflationary cosmology. The standard inflationary scenario is considered a successful *paradigm* not only because it can effectively solve the traditional problems of standard classical cosmology, but also because it provides a natural mechanism of generating classical *seed* perturbations from quantum fluctuations. These seed perturbations are essential in a theory of large scale structure formation but there is no mechanism of generating the initial perturbation within the classical setup. A quantum field living in an inflating background quite generically produces *scale-invariant* power spectrum of primordial density perturbations which is consistent with the current observations.

However, one major problem that plagues almost all potential driven inflationary models is that these models generically predict too much *amplitude* for density perturbation [38, 39]. Considering the fluctuations of quantum scalar field on an inflating classical background, one can show that these models *naturally* predict density perturbations at *horizon re-entry* to be $\frac{\delta\rho}{\rho} \sim 1 - 10^2$. But CMB anisotropy measurements indicates $\frac{\delta\rho}{\rho} \sim 10^{-5}$. Thus it is very difficult to get desired amplitude for density perturbation from the standard inflationary scenarios unless one introduces some *fine tuning* in inflaton potential [40].

An interesting suggestion to get an acceptable amount of density perturbation from inflationary scenario was made by Padmanabhan [59, 60]. The basic idea of the suggestion is that any proper theory of quantum gravity should incorporate a *zero-point* proper length. This in turns damps the propagation of modes with proper wavelength smaller than the zero-point proper length. This mechanism reduces the amplitude of density perturbation by an exponential damping factor. The computations of [59] show that with the energy density (V_0) attainable during inflation to be order of the Planck energy density and the introduced cutoff (L) of the order of $(G\hbar/c^3)^{1/2}$, one can indeed get the necessary amount of damping. In the picture discussed above, we already have a correlation between $\bar{\rho}$ and the bounce scale.

As discussed above, the effective model derived from semi-classical LQC already shows the existence of a minimum proper length which can play the role of the zero-point length. Furthermore, this scale is *not* put in by hand but arises generically and naturally from the non-singularity of the effective model [31] and is correlated self-consistently with the maximum attainable energy density whose existence is also guaranteed in a non-singular evolution. It has no explicit dependence on the quantization ambiguity parameter j. With the genericness of (exponential) inflation shown in [35] one can expects that the effective LQC model has the potential to produce an acceptable primordial power spectrum as well as an acceptable amplitude for density perturbations. A detailed analysis for primordial density perturbations incorporating LQC modifications has been carried out [43] and will be described in the chapter 5.

Apart from the possible phenomenological implications of the existence of a bounce, there are some theoretical implications as well. Within the WKB approximation used in deriving the effective Hamiltonian, existence of bounce corresponds to existence of classically in-accessible regions (volumes). This can also be interpreted as limiting the domain of validity of continuum geometry or the kinematical framework of general relativity. Since the exact quantum wave functions do connect the two regions of the triad variable, there is also the possibility of tunnelling to and from the oppositely oriented universe ($p < 0$) through these regions. Because of this, the bounce can be expected to be 'fuzzy'. If and how the tunnelling possibility between oppositely oriented universe affects 'discrete symmetries' needs to be explored.

Finally, the bounce result has been derived using genericness of inflationary regime ($p \ll 2jp_0$). It is reasonable to assume that the maximum energy density would be comparable or less than the Planck density. In such a case the bounce scale will be greater than p_0. Thus, both the results regarding genericness of bounce and genericness of inflation would follow even if the underlying assumption of slowly varying wave functions is valid only down to the bounce scale.

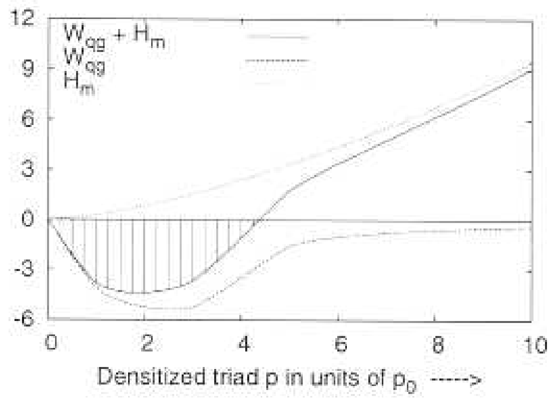

Figure 4.1: For the small volume with non-perturbative corrections, the scalar matter Hamiltonian along a trajectory is given by $H_m \approx hp^{\frac{3}{2}}$. Thus non-perturbatively corrected scalar matter Hamiltonian vanishes at $p = 0$. If there were no quantum geometry potential term, W_{qg}, in the effective Hamiltonian then for the spatially flat case $(\eta = 0)$ $p = 0$ point would have been accessible through the evolution. Although being a non-singular evolution it would have taken infinite coordinate time to reach $p = 0$ nevertheless there would have been no *minimum* proper length for the given space-time. But once we incorporate the effect of quantum geometry potential W_{qg} then we can see that the combined effect of H_m and W_{qg} will lead the extrinsic curvature to become zero at a non-zero value of p. Since for small p, $W_{qg} \sim -p$ and $H_m \sim p^{\frac{3}{2}}$ then there will always be a region where W_{qg} dominates over H_m. Naturally there will always exist a classically inaccessible region leading to a generic *big bounce*. The shaded region in the figure represents the classically forbidden region.

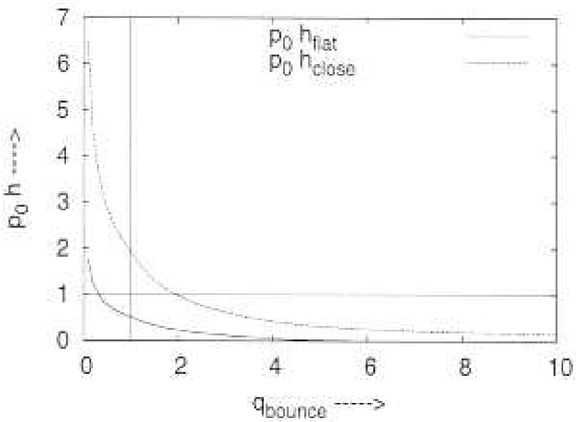

Figure 4.2: The plot shows how the dimensionless parameter hp_0 varies as a function of the bounce scale q_{bounce} as determined by the equations (4.2.2, 4.2.3). $p_0 = \frac{1}{6}\ell_{\text{P}}^2$ has been chosen for the plot.

Chapter 5

Primordial Density Perturbations

5.1 Introduction

The homogeneous and isotropic solution of general theory of relativity, namely the Friedmann-Robertson-Walker (FRW) solution appears to be an extremely good description of large scale spacetime dynamics of our universe. Extreme simplicity of the FRW solution nevertheless ignores some crucial features of the universe namely that it has certain sub-structure as well. On large scale the deviation from homogeneity and isotropy being small one can treat them as small perturbations around homogeneous and isotropic background. The classical theory of large scale structure formation in principle can be used to 'derive' the observed structures of current universe but these models need to know the initial *seed* perturbations. In this sense the classical description of our universe is incomplete as there is no mechanism of *generating* the seed perturbations within the theory itself.

On the other hand quantum field fluctuations in an *inflating* background quite generically produce density perturbations with scale-invariant power spectrum [37] which is consistent with current observations. This is certainly an attractive feature of the standard inflationary scenario. However, one major problem that plagues almost all potential driven inflationary scenario that these models generically produce too large amplitude for density perturbations, typically $\frac{\delta\rho}{\rho} \sim 1 - 10^2$ at horizon *re-entry* [38, 39]. The cosmic microwave background (CMB) anisotropy measurements on the other hand indicates $\frac{\delta\rho}{\rho} \sim 10^{-5}$. Naturally to make these models viable it is necessary to *fine tune* the parameters of the field potential [40]. In the presence of quantum fluctuations it is rather difficult to justify or sustain those fine tuning of field theoretical parameters.

It is worthwhile to mention that inflation was invented to solve some crucial problems of the standard big bang cosmology. The most important of them is the so called *particle horizon* problem. The horizon problem is directly related with the fact the standard model of cosmology contains an initial singularity where physical quantities like energy density, spacetime curvature blow up leading to a breakdown of classical description. The initial singularity, however is viewed as an attempt to extrapolate the classical theory beyond its natural domain of validity. Near the classical singularity one expects the evolution of the universe to be governed by a quantum theory of gravity rather than the classical one.

We have mentioned earlier that density perturbations generated by quantum field fluctuations in an inflating background are believed to be the *seed* perturbations responsible for the current large scale structures of the universe. Further in chapter 3, we have discussed that non-perturbative modification of a minimally coupled scalar matter Hamiltonian in loop quantum cosmology leads to a generic phase of inflation [33, 35] (see [110] for related discussions on other kinds of matter). Naturally it is an important question to ask whether the density perturbations generated by quantum fluctuations during loop quantum cosmology induced inflationary phase can satisfy the basic requirements of viability like scale-invariant power spectrum. Further, it may leave some distinct imprint on the power spectrum which may be observationally detectable as well.

Being inhomogeneous in nature treatment of these density perturbations requires *inhomogeneous* models of loop quantum cosmology. However the technology required to deal with inhomogeneity at fundamental level within loop quantum cosmology is *not* yet available. Not having such technology, one needs to proceed rather intuitively. Let us recall that in the standard inflationary scenario for computing power spectrum of density perturbations due to quantum fluctuations, one uses the techniques which broadly can be classified as *Quantum Field Theory in Curved Background* [41, 42]. In this approach one treats the background geometry as classical object whereas matter fields living in it are treated as quantum entities. The main justification for using such techniques comes from the fact the energy scale associated with inflationary scenario is few order of magnitude lower than the Planck scale. So one expects the geometry to behave more or less classically in this regime.

In loop quantum cosmology in principle one can think of using physical observables and physical inner product to evaluate the physical expectation values to find out the behaviour of at least the *homogeneous* part of the geometrical quantities. Unfortunately developments of physical observables, physical inner product and 'time' evolution in loop quantum cosmology are still in early stage [13, 26, 27]. Nevertheless, one can construct an *effective* but *classical* description of loop quantum cosmology using WKB techniques. As discussed in chapter 2, effective loop quantum cosmology [31] incorporates important non-perturbative modifications and has been shown to be generically non-singular as well [36, 54, 57].

In the effective loop quantum cosmology it has been shown [31] that in the region of interest (exponential inflationary phase) gravitational part of Hamiltonian constraint becomes same as the classical one with small quantum corrections. However the scalar matter part of the effective Hamiltonian remains non-perturbatively modified during this phase. In fact non-perturbative modification of scalar matter Hamiltonian is what drives inflation in loop quantum cosmology. Having a modified scalar matter Hamiltonian the scalar field satisfies a *modified* Klein-Gordon equation instead of standard Klein-Gordon equation. Naturally the mode functions of the scalar field which contain the necessary information about background geometry evolution and are essential to compute power spectrum of the density perturbations, are expected to be different from the standard mode-functions. Thus, although it may be justified to employ similar techniques to compute the power spectrum in effective loop quantum cosmology but certainly one *cannot* borrow the same mode functions used in the standard inflationary scenario.

Here we will compute power spectrum of density perturbations using the *direct* *method* [59]. In this method one directly uses operator expression of 'time-time' component of stress-energy tensor (which is classically energy density) to compute two-point *density correlation function* and then evaluates its Fourier transform to compute power spectrum of density perturbations. In the standard inflationary scenario one generally avoids this direct computation as the two point density correlation function in a pure classical background diverges badly for small *coordinate length* separation (*i.e. ultra-violet divergence*). There usually one first computes the power spectrum of field fluctuations. Using this one *reconstructs* inhomogeneous but *classical* field configuration which is then used to compute corresponding density perturbations. However it is important to understand that this divergence is rather *unphysical* because it arises when one tries to resolve any two spatial points with arbitrary precision.

In a quantum system, the expectation value of an operator which is classically a phase space function in general is not equal to the same function of the expectation values of basic phase space operators. Thus the use of direct method is preferable over the standard method as observational aspects deals with energy density directly rather than the field configuration. Further, in the standard method to relate the power spectrum of field fluctuations with that of density perturbations, one needs to know the *general* expression of the stress-energy tensor. In this context, it is not yet a settled issue; how to obtain an effective action from a quantum theory of gravity based on canonical quantization.

In the context of standard inflationary scenario it was outlined and explicitly shown [59, 60] that one can in fact regularize this field theoretical divergence by using the notion of *zero-point* proper length. Although it was used as an *ad-hoc* assertion but it was argued that the notion of *zero-point* proper length is expected from a proper theory of quantum gravity. The power spectrums of density perturbations computed using these two different method in the *relevant energy scale* however are not very different. Nevertheless there one can avoid rather cumbersome *indirect method* of computing power spectrum of density perturbations.

In effective loop quantum cosmology, it has been shown that the universe exhibits a generic *big bounce* with a non-zero minimum *proper* volume [36]. This in turns implies a *zero-point* proper length for the isotropic spacetime. Since the regularization technique is *naturally* available in the effective loop quantum cosmology scenario then it is quite appealing to use directly the operator expression of 'time-time' component of the stress-energy tensor to obtain the power spectrum of density perturbations due to quantum field fluctuations. In this sense this exercise can also be seen as an explicit example of quantum gravity motivated regularization technique to cure the *ultra-violet* divergence of standard quantum field theory [62].

In section 5.2 and 5.3, we briefly recall the standard scenario of quantum field living in a DeSitter background and then obtain corresponding two point density correlation function. In the next section we review the basic infrastructures required to describe the inflationary phase in effective loop quantum cosmology. In particular we discuss about the properties of the *effective equation of state* for the scalar matter field. The effective equation of state essentially summarizes the evolution of the background geometry. In the next section we derive the modified Klein-Gordon equation which leads to a modified mode function equation. We obtain an analytic solution for the mode function equation. This modified mode function reduces to the standard mode function in the appropriate limit. Using the mode functions in the next section we compute the power spectrum of the density perturbations. We discuss about the properties of the power spectrum and its observational implications.

5.2 Quantum field in a De-Sitter Background

In computing power spectrum of density perturbations in standard inflationary scenario, one considers background geometry to be homogeneous and isotropic. The invariant distance element in such spacetime (using *natural units i.e.* $c = \hbar = 1$) is given by famous Friedmann-Robertson-Walker metric $ds^2 = -dt^2 + a^2(t)\, d\mathbf{x}^2$, where $a(t)$ is the *scale factor*. During inflationary period the scale factor grows almost exponentially with coordinate time. The Hubble parameter defined as $H := \frac{\dot{a}}{a}$

remains almost *constant* during the period. For simplicity, in the intermediate period of calculation one treats Hubble parameter as constant *i.e.* the evolution of background geometry is considered to be De-Sitter like. One can approximately compute the effect of small variation of Hubble parameter on power spectrum, simply by considering the variation of the final expression of power spectrum. This is in fact a good approximation as the variation of Hubble parameter is rather very small.

In the standard scenario, inflation is driven by a scalar field known as *inflaton* field. We will consider here the most simple *single-field* inflationary scenario. The dynamics of a minimally coupled scalar field is governed by the action

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = \int d^4x\sqrt{-g}\mathcal{L}\,. \qquad (5.2.1)$$

We have mentioned earlier that we will be using the direct method to compute the power spectrum. So it will be quite useful to have the expression of the matter stress-energy tensor due to the scalar field. The stress energy tensor corresponding to the action (5.2.1) is given by

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}\,. \qquad (5.2.2)$$

Comparing with the *perfect fluid* ansatz *i.e.* $T_{\mu\nu} = (\rho+P)\,u_\mu u_\nu + g_{\mu\nu}P$, it is easy to see that T_{00} component represents energy density for the scalar field. In canonical quantization one treats Hamiltonian as a basic object. Thus, it is important here to have the expression of the matter Hamiltonian

$$H_\phi = \int d^3x\left[\frac{1}{2}a^{-3}\pi_\phi^2 + \frac{1}{2}a(\nabla\phi)^2 + a^3V(\phi)\right]\,, \qquad (5.2.3)$$

where $\pi_\phi = a^3\dot{\phi}$. In deriving expression (5.2.3) it is assumed that the background *geometry* is homogeneous and isotropic but *not* the scalar field itself. This *approximation* can be justified as long as the deviation from the homogeneity and isotropy remains small. To make it more explicit we rewrite the scalar matter Hamiltonian (5.2.3) as

$$H_\phi = a^{-3}\int d^3x\left[\frac{1}{2}\pi_\phi^2\right] + a\int d^3x\left[\frac{1}{2}(\nabla\phi)^2\right] + a^3\int d^3x\,[V(\phi)]\,. \qquad (5.2.4)$$

In loop quantum cosmology, the geometrical quantities like the scale factor a here are represented through corresponding quantum operators. While deriving effective classical Hamiltonian from loop quantum cosmology, these operator expression effectively get replaced by their corresponding eigenvalues. The kinetic term of the scalar matter Hamiltonian (5.2.4) involves inverse powers of the scale factor. In loop quantum cosmology the inverse scale factor operator has a bounded spectrum. Clearly one can see that the kinetic term of the effective scalar matter Hamiltonian will involve non-perturbative modifications due to loop quantization. Using the Hamilton's equations of motion for the scalar field *i.e.*

$$\dot{\phi} = \frac{\delta H_\phi}{\delta \pi_\phi} \quad ; \quad \dot{\pi}_\phi = -\frac{\delta H_\phi}{\delta \phi}, \tag{5.2.5}$$

one can derive the second order equation of motion for the scalar field, given by

$$\ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\phi} - \frac{\nabla^2 \phi}{a^2} + V'(\phi) = 0. \tag{5.2.6}$$

The equation of motion (5.2.6) for the scalar field is the standard Klein-Gordon equation. It is worthwhile to emphasize that one could have obtained the standard Klein-Gordon equation (5.2.6) simply by considering the variation of the scalar field action (5.2.1). However, one should remember that our ultimate aim is to compute power spectrum in effective loop quantum cosmology where non-perturbative modification in the matter sectors comes through its Hamiltonian.

To quantize the scalar field one proceeds in the standard way *i.e.* by decomposing scalar field operator in terms of annihilation and creation operators $\hat{a}$ and $\hat{a^\dagger}$ as follows

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\hat{a}_k \, f_k(t) \, e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a^\dagger}_k \, f_k^*(t) \, e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{5.2.7}$$

where $f_k(t)$ are the 'properly normalized' mode functions. Although one can quantize the scalar field analogous to that in Minkowski spacetime, one faces the well known problem of defining *vacuum state* in curved spacetime. In general, for curved background geometry there does not exist an unique choice for the vacuum state. Thus one needs to have some additional prescription to define it.

In the standard inflationary scenario one generally chooses the so called Bunch-Davies vacuum. It is defined as the state which gets annihilated by $\hat{a}$ where the mode functions f_k are so 'normalized' such that in 'Minkowskian limit' *i.e.*, $H \to 0$, the mode-function reduces to the flat space positive frequency mode function $\frac{1}{\sqrt{2\omega}}e^{-i\omega t}$. We will use analogous definition for the vacuum state for the calculation of power spectrum in effective loop quantum cosmology as well. For simplicity we consider the situation where the field potential is made of only the mass term (*i.e.* $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$). Then the mode functions are the solution of the equation

$$\ddot{f}_k + 3\left(\frac{\dot{a}}{a}\right)\dot{f}_k + \left(\frac{k^2}{a^2} + m_\phi^2\right) f_k = 0 . \tag{5.2.8}$$

The mode function equation (5.2.8) follows from the Klein-Gordon equation (5.2.6) and the expansion of the scalar field operator (5.2.7). For simplicity we consider the situation where the mass term can be neglected ($\frac{k}{a} >> m_\phi$) in the equation (5.2.8). The 'normalized' mode-function solutions are then given by

$$f_k = \frac{H}{\sqrt{2k^3}}\left(1 - i\frac{k}{Ha}\right)e^{i\frac{k}{Ha}} . \tag{5.2.9}$$

The mode function (5.2.9) in the 'Minkowskian limit' *i.e.* $H \to 0$ limit reduces (upto a constant phase) to flat space positive frequency mode function $\frac{1}{\sqrt{2\omega}}e^{-i\omega t}$. This defines the vacuum state $|0\rangle$ as $\hat{a}|0\rangle = 0$.

To compute the power spectrum of density perturbations using *indirect method*, one first computes the power spectrum of field fluctuations *i.e.* $\mathcal{P}_\phi(k) := \frac{k^3}{2\pi^2}|f_k|^2$. It is easy to see from the expression of the normalized mode function (5.2.9) that at the time of horizon crossing ($a(t) = \frac{k}{2\pi H}$) the corresponding power spectrum is *scale invariant*. In getting mode function solution (5.2.9), we have ignored the mass term of the scalar field. For the mass dominating case ($\frac{k}{a} << m_\phi$), the 'normalized' mode functions are $f_k = \frac{1}{\sqrt{2m_\phi}}a^{-\frac{3}{2}}e^{-im_\phi t\left(\sqrt{1-(\frac{3H}{2m_\phi})^2}\right)}$. It can be easily checked that for this case also the corresponding power spectrum is scale invariant at the time of horizon crossing. It is often argued that the scale invariance is mainly determined by the fact that during inflationary period the Hubble horizon H^{-1} remains almost constant. The details of particular model of inflation has rather small effect on this property of the power spectrum.

5.3 Two point density correlation function

Having specified the vacuum state one can proceed to evaluate vacuum expectation value of the two point density correlation function. Two point density correlation function can naturally be defined as

$$C(\mathbf{x}+\mathbf{l},\mathbf{x},t) \ := \ \langle 0|\ \hat{T}^0_0(\mathbf{x}+\mathbf{l},t)\ \hat{T}^0_0(\mathbf{x},t)\ |0\rangle \ . \tag{5.3.1}$$

Using the expression of the scalar field operator (5.2.7) and the expression of the stress-energy tensor (5.2.2) one can evaluate the two point density correlation function in terms of the mode-function, given by [59]

$$C(\mathbf{x}+\mathbf{l},\mathbf{x},t) \ = \ \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\ e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{l}} \left|\dot{f}_p\dot{f}_q \ - \ \left(\frac{\mathbf{p}\cdot\mathbf{q}}{a^2} - m_\phi^2\right) f_p f_q\right|^2 . \tag{5.3.2}$$

In evaluating two point density correlation function (5.3.2), one ignores a space independent (formally divergent) term as it would have contributed only to the $k=0$ mode while taking Fourier transform. Having known the normalized mode function solution f_k (5.2.9) one can explicitly calculate the two point density correlation function (5.3.2), given by [59]

$$C(l',t) \ := \ C(\mathbf{x}+\mathbf{l},\mathbf{x},t) \ = \ \frac{1}{4\pi^4}\left[\frac{2H^2}{(al')^6} + \frac{12}{(al')^8}\right] , \tag{5.3.3}$$

where $l' = |\mathbf{l}|$. The expression (5.3.3) of two-point density correlation function *expectedly* diverges near $l' = 0$. However, as shown in [59], one can regularize this divergence using the notion of zero-point proper length. The expression (5.3.3) in 'Minkowskian limit' *i.e.* $H \to 0$ limit reduces to the flat-space two point density correlation function.

5.4 Effective Isotropic Loop Quantum Cosmology

In isotropic loop quantum cosmology, the basic phase space variables are gauge connection c and densitized triad p. In loop quantum cosmology one redefines densitized triad to absorb the fiducial coordinate volume component. This makes the proper volume of the universe (1.1.1) to be $\int d^3x\sqrt{-g} \ = \ a^3V_0 \ = \ p^{\frac{3}{2}}$ [17].

The effective Hamiltonian for *spatially flat* isotropic loop quantum cosmology derived using WKB method [31], is given by

$$H^{\rm eff} = -\frac{1}{\kappa}\frac{B_+(p)}{4p_0}K^2 + W_{qg} + H^{\rm eff}_{\phi} \tag{5.4.1}$$

where $\kappa = 16\pi G$, $p_0 = \frac{1}{6}\gamma\ell_{\rm P}^2\mu_0$, γ is the Barbero-Immirzi parameter, K is the extrinsic curvature (conjugate variable of p), $A(p) = |p+p_0|^{\frac{3}{2}} - |p-p_0|^{\frac{3}{2}}$, $B_+(p) = A(p+4p_0) + A(p-4p_0)$, $\ell_{\rm P}^2 := \kappa\hbar$ and $W_{qg} = \left(\frac{\ell_{\rm P}^4}{288\kappa p_0^3}\right)\{B_+(p) - 2A(p)\}$. μ_0 here is viewed as a quantization ambiguity parameter and it is a order one number [17, 8]. Apart from the modifications of the gravitational kinetic term and scalar matter kinetic term, the effective Hamiltonian (5.4.1) differs from the classical Hamiltonian by a non-trivial potential term referred to as *quantum geometry potential* W_{qg}. In this chapter we will be interested in the regime ($p_0 << p$) where the quantum geometry potential has natural interpretation of being *perturbative* quantum corrections due to homogeneous quantum fluctuations around FRW background. The effective scalar matter Hamiltonian is given by

$$H^{\rm eff}_{\phi} = \frac{1}{2}|\tilde{F}_{j,l}(p)|^{\frac{3}{2}}p_\phi{}^2 \ + \ p^{\frac{3}{2}}V(\phi) \ , \tag{5.4.2}$$

where $p_\phi(= V_0\pi_\phi)$ is the field *momentum*, $\tilde{F}_{j,l}(p)$ is the eigenvalue of the inverse densitized triad operator $\hat{p^{-1}}$ and is given by $\tilde{F}_{j,l}(p) = (p_j)^{-1}F_l(p/p_j)$ where $p_j = \frac{1}{3}\gamma\mu_0 j l_{\rm p}^2$. The j and l are two quantization ambiguity parameters [47, 48]. The half integer j is related with the dimension of representation while writing holonomy as multiplicative operators. The real valued l ($0 < l < 1$) corresponds to different, classically equivalent ways of writing the inverse power of the densitized triad in terms of Poisson bracket of the basic variables. The function $F_l(q)$ is given by [30]

$$\begin{aligned} F_l(q) \ &:= \ \Bigg[\frac{3}{2(l+2)(l+1)l}\Big(\ (l+1)\left\{(q+1)^{l+2} - |q-1|^{l+2}\right\} \ - \\ &\qquad (l+2)q\left\{(q+1)^{l+1} - \ \mathrm{sgn}(q-1)|q-1|^{l+1}\right\} \ \Big) \ \Bigg]^{\frac{1}{1-l}} \\ &\rightarrow \ q^{-1} \qquad\qquad (q \gg 1) \\ &\rightarrow \ \left[\frac{3q}{l+1}\right]^{\frac{1}{1-l}} \qquad (0 < q \ll 1) \ . \end{aligned} \tag{5.4.3}$$

In expression (5.4.3) it should be noted that for large values of the densitized triad i.e. in the large volume one has the expected classical behaviour for the inverse densitized triad. The quantum behaviour is manifested for smaller values of the densitized triad. Here the meaning of large or small values of the triad p is determined necessarily by the values of p_j. The quantum mechanically allowed values for the ambiguity parameter l is $(0 < l < 1)$. Now one should also note that if one takes the ambiguity parameter value $l = 2$ then the small volume expression (5.4.3) becomes same as the large volume expression. In other words, here taking ambiguity parameter value $l = 2$ is equivalent of taking large volume limit *i.e.* classical limit. This observation will be very useful in fixing the choice of vacuum while computing two-point density correlation function in this effective background.

In loop quantum cosmology p_0 and p_j represent two important (square of) length scale. p_0 demarcate the *strong* quantum effect regime (*non-perturbative* regime) from *weak* quantum effect regime (*perturbative* regime) of the *gravity sector* whereas p_j demarcate the *same* for the *matter sector*. Since ambiguity parameter $j \geq \frac{1}{2}$, so it follows from their respective definition that $p_j \geq p_0$. Naturally, *non-perturbative modification of matter sector can survive longer than the same for the gravity sector* depending on the value of the ambiguity parameter j. We have mentioned earlier that to compute power spectrum of density perturbations we will use similar techniques used in the standard inflationary scenario. In this approach one treats geometry as a classical object whereas matter fields living in it are treated as quantum objects. Thus self-consistency of this framework requires that we should consider the regime where $p >> p_0$ in our calculation. In this regime the gravitational part of the Hamiltonian constraint becomes same as the classical Hamiltonian with small quantum correction. The reduced effective Hamiltonian in this regime is given by

$$H^{\text{eff}} = -\frac{3}{2\kappa}K^2\sqrt{p} - \frac{\ell_{\text{P}}^4}{24\kappa}p^{-\frac{3}{2}} + H_{\phi}^{\text{eff}} . \tag{5.4.4}$$

The loop quantum cosmology induced inflationary scenario persist as long as densitized triad p remains less than p_j. Thus we will be interested in computing power spectrum of density perturbations in the regime $(p_0 << p < p_j)$. In this regime the *effective* energy density and pressure are given by $\rho^{\text{eff}} = p^{-\frac{3}{2}}H_{\phi}^{\text{eff}}$ and $P^{\text{eff}} = -\frac{1}{3}p^{-\frac{3}{2}}(2p\ \frac{\partial H_{\phi}^{\text{eff}}}{\partial p})$ [31]. It can be checked easily using relation between scale factor and densitized triad that these definition satisfy standard *conservation* equa-

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tion $a\frac{d\rho^{\rm eff}}{da} = -3(\rho^{\rm eff} + P^{\rm eff})$. Furthermore one can recover standard expression of energy density and pressure using the standard scalar matter Hamiltonian H_ϕ in place of the modified scalar matter Hamiltonian $H_\phi^{\rm eff}$ in these definition. It is shown in [35] that the effective equation of state $\omega^{\rm eff} := P^{\rm eff}/\rho^{\rm eff}$ can be expressed as a function of standard equation of state ω and the densitized triad p

$$\omega^{\rm eff} = -1 + \frac{(1+\omega)p^{\frac{3}{2}}[\tilde{F}_{j,l}(p)]^{\frac{3}{2}}\left(1 - \frac{p}{\tilde{F}_{j,l}(p)}\frac{d\tilde{F}_{j,l}(p)}{dp}\right)}{(1+\omega)p^{\frac{3}{2}}[\tilde{F}_{j,l}(p)]^{\frac{3}{2}} + (1-\omega)} . \tag{5.4.5}$$

Using the expression (5.4.3) it is easy to see that for the large values of the densitized triad p, where one expects the quantum effects to be small, $\omega^{\rm eff} = \omega$ whereas for small values of p the $\omega^{\rm eff}$ differs from the classical ω dramatically. In this chapter we will be interested in the situation where $\omega^{\rm eff} \approx -1$ (for $p < p_j$). This requirement will automatically be satisfied if at the end of loop quantum cosmology induced inflation the radiation or matter domination or even another phase of classical acceleration ($i.e\ \omega = \frac{1}{3}, 0, < -\frac{1}{3}$) begins. Thus, during loop quantum cosmology induced inflationary period one can express the matter Hamiltonian as

$$H_\phi^{\rm eff} \approx \bar{\rho}\, p^{\frac{3}{2}} , \tag{5.4.6}$$

where $\bar{\rho}$ is a constant of integration. Physically $\bar{\rho}$ corresponds to the *maximum* energy density that can be 'stored' in the effective spacetime. This also defines the energy scale associated with the loop quantum cosmology induced inflationary scenario.

It has been shown in [36] that the effective loop quantum cosmology exhibits a generic bounce with non-zero minimum *proper* volume. It follows from the equation (5.4.4) and the equation (5.4.6) that the minimum value of the proper distance L_0, defined as $L_0^2 := p_{min} = p(H^{\rm eff} = 0; K = 0)$, is given by

$$L_0^6 = \frac{2\pi G}{3\,\bar{\rho}} . \tag{5.4.7}$$

Self-consistency of the expression (5.4.7) requires $p_0 << p_{min} < p_j$.

In standard cosmology one uses the scale factor as geometric variable. In isotropic loop quantum cosmology, the basic variable is densitized triad p defined as $p^{\frac{3}{2}} :=$

$\int d^3x\sqrt{-g} = a^3 V_0$, where V_0 is fiducial *coordinate* volume. Clearly the densitized triad p here is a dimensionful quantity whereas the scale factor a is dimensionless. Also, absolute value of the scale factor is physically irrelevant. Rather what matters is the ratio of scale factor at two different period. Naturally there is a freedom left in relating the scale factor with the densitized triad. We define the relation between the scale factor and the eigenvalues of densitized triad operator such that for small volume limit

$$\hat{p}\,\widehat{p^{-1}}\,|\mu\rangle \;:=\; a^{2(1+\frac{1}{1-l})}\,|\mu\rangle \,. \tag{5.4.8}$$

We have mentioned earlier that taking ambiguity parameter value $l = 2$ is equivalent of taking large volume limit of the inverse densitized triad spectrum. Clearly in our choice of definition the scale factor takes the value $a = 1$ at the transition point from non-perturbatively modified matter sector to the standard matter sector. For the regime $p < p_j$ one can approximate the effective equation of state (5.4.5) as

$$(1+\omega^{\rm eff}) \;\simeq\; C_\omega\left(\frac{n+2}{3}\right)\,a^{2(1-n)} \,, \tag{5.4.9}$$

where $C_\omega = 2\left(\frac{1+\omega}{1-\omega}\right)$ and $n = -\frac{1}{2}(1+\frac{3}{1-l})$. The last two terms in the effective Hamiltonian constraint (5.4.4) are comparable near bounce point. However once the densitized triad p starts increasing then it is clear from the equation (5.4.4) that the contribution from quantum geometry potential quickly drops out compared to the matter Hamiltonian (5.4.6). Naturally for the region away from the bounce point one can write down the Hamiltonian constraint ($H^{\rm eff} = 0$) in terms of the scale factor as

$$3\left(\frac{\dot{a}}{a}\right)^2 \;\simeq\; 8\pi G\,\bar{\rho} \,, \tag{5.4.10}$$

where we have used the Hamilton's equation of motion $\dot{p} = \frac{\kappa}{3}\frac{\partial H^{eff}}{\partial K}$. The equation (5.4.10) is nothing but the usual Friedmann equation. Using the equation (5.4.7) and the equation (5.4.10) we can define a dimensionless quantity

$$\sigma \;:=\; 2\pi H L_0 \;=\; 4\pi\left(\frac{2\pi}{3}\right)^{\frac{2}{3}}\left(\frac{\bar{\rho}}{M_p^4}\right)^{\frac{1}{3}} \,, \tag{5.4.11}$$

where $G = M_p^{-2}$. This will be a useful quantity in calculation of power spectrum. Since we consider the situation where $p_0 << L_0^2 < p_j$ *i.e.* bounce occurs at a time when proper volume of the universe is much larger than the Planck volume. Thus it is clear that σ is much smaller than unity ($\sigma << 1$) during the loop quantum cosmology induced exponential inflationary phase.

From the definition of FRW metric (1.1.1) it follows that the *proper* distance square say $d^2(a, l')$, between two points separated by *coordinate distance* l' on a given spatial slice ($dt = 0$) is simply $d^2(a, l') = (a\ l')^2$. In other word, in the classical geometry the proper distance between two points is simply 'coordinate distance times the scale factor'. In classical case one can choose coordinate distance separation arbitrarily small. Naturally the proper distance between two points can become arbitrarily small. In loop quantum cosmology the basic variable is a densitized triad instead of the usual metric variable. Further in loop quantum cosmology, one redefines the densitized triad by absorbing component of the fiducial coordinate volume. This makes the proper volume of the universe to be just $p^{\frac{3}{2}}$. In case of effective loop quantum cosmology, we have seen that there exist a *non-zero* minimum value for the densitized triad p. To incorporate such feature in the definition of the proper distance in effective loop quantum cosmology, we introduce the notion of *effective coordinate length* $l^{\text{eff}}(a, l')$. The proper distance between two points separated by coordinate distance l' is defined as

$$d^2(l', a) \ := \ (a\ l^{\text{eff}})^2 \ = \ L_0^2 + (a\ l')^2\ . \tag{5.4.12}$$

The effective coordinate length keeps the usual notion of proper distance *i.e.* 'coordinate distance times scale factor' intact and incorporate feature like zero-point proper length. Although, it is an *ad-hoc* notion but it allows one to use the standard machinery while computing power spectrum of density perturbations and acts as an ultra-violet regulator of standard quantum field theory. For large volume (*i.e.* (al') large) this definition is virtually equivalent to the standard definition of proper distance as L_0 is very small (a few Planck units).

5.5 Modified Klein-Gordon Equation

We have mentioned earlier that the kinetic term of the scalar matter Hamiltonian gets non-perturbative modification as its classical expression involve inverse powers of densitized triad. The effective scalar matter Hamiltonian obtained as outlined in the previous section is given by

$$H_\phi^{\mathrm{eff}} = V_0 |\bar{F}_{jl}(p)|^{\frac{3}{2}} \int d^3x \left[\frac{1}{2}\pi_\phi^2\right] + V_0^{-\frac{1}{3}} p^{\frac{1}{2}} \int d^3x \left[\frac{1}{2}(\nabla\phi)^2\right] + V_0^{-1} p^{3/2} \int d^3x \, [V(\phi)] \, . \tag{5.5.1}$$

It should be noted that we have now kept the gradient term in the effective Hamiltonian. Earlier while computing background evolution the gradient term was neglected as one assumes that the background evolution is mainly determined by the homogeneous and isotropic contribution of the matter Hamiltonian. In other words the inhomogeneity is assumed to be small. Using the Hamilton's equations of motion for the effective Hamiltonian (5.5.1) one can derive the corresponding *modified* Klein-Gordon equation, given by

$$\ddot{\phi} - 3\left(\frac{1}{1-l}\right)\left(\frac{\dot{a}}{a}\right)\dot{\phi} + a^{3+\frac{3}{1-l}}\left(-\frac{\nabla^2\phi}{a^2} + V'(\phi)\right) = 0 \, , \tag{5.5.2}$$

where we have substituted eigenvalue of the inverse triad operator by scale factor using the definition (5.4.8). It is easy to see that if one takes the value of the ambiguity parameter $l = 2$ then the modified Klein-Gordon equation (5.5.2) goes back to the standard Klein-Gordon equation (5.2.6).

The non-perturbative modification of the scalar matter Hamiltonian which is being studied here, comes from the bounded spectrum of the inverse scale factor operator. Since the modification affects the kinetic term of the scalar matter Hamiltonian, it essentially affects all the modes. It can be seen from the equation (5.5.2) as well. This modification is distinct from the other Planck scale effects studied in the literature. For example, in the context of trans-Planckian inflation [63, 64], one studies the possible effects of Planck scale modification of the dispersion relation or the possible effects of the space-time non-commutativity [65, 66].

5.6 Modified Mode Functions

Using the expression for the quantized scalar field (5.2.7) like in standard case one can derive the *modified* mode function equation for the scalar field

$$\ddot{f}_k - 3\left(\frac{1}{1-l}\right)\left(\frac{\dot{a}}{a}\right)\dot{f}_k + a^{3+\frac{3}{1-l}} H^2 \left(\frac{k^2}{H^2a^2} + \frac{m_\varphi^2}{H^2}\right) f_k = 0 . \qquad (5.6.1)$$

One can easily check that for $l = 2$ (*i.e. the classical case*) the modified mode function equation goes back to the standard mode-function equation (5.2.8). To compute the power spectrum it is essential to know the solutions of the mode function equation (5.6.1). For simplicity we will neglect the mass term *i.e.* we will assume $(\frac{k}{Ha} >> \frac{m_\varphi}{H})$. To simplify the mode function equation (5.6.1) further, we make change of variables as follows

$$f_k := a^{-n}\tilde{f}_k \ ; \ dt = a^n d\eta \qquad (5.6.2)$$

with the value of $n = -\frac{1}{2}(1+\frac{3}{1-l})$. In loop quantum cosmology allowed values for the l ambiguity parameter is $(0 < l < 1)$ whereas the classical situation can be obtained simply by taking $l = 2$. In terms the new parameter n, the classical situation corresponds to $n = 1$ and quantum situation is described for $(-\infty < n < -2)$. One may note here that for $n = 1$ the new variable $\eta = -\frac{1}{nHa^n}$ is nothing but *conformal* time. In terms of these new variables (5.6.2) the mode function equation (5.6.1) becomes

$$\frac{d^2\tilde{f}_k}{d\eta^2} - \left(2\,\frac{1}{2n} - 1\right)\frac{1}{\eta}\frac{d\tilde{f}_k}{d\eta} + \left(k^2 + \frac{(\frac{1}{2n})^2 - (1+\frac{1}{2n})^2}{\eta^2}\right)\tilde{f}_k = 0 . \qquad (5.6.3)$$

The equation (5.6.3) is a modified expression of Bessel differential equation and admits analytical solution of the form [67]

$$\tilde{f}_k = \eta^{\frac{1}{2n}}\left[A_{(k,n)}J_{-(1+\frac{1}{2n})}(k\eta) + B_{(k,n)}J_{(1+\frac{1}{2n})}(k\eta)\right] , \qquad (5.6.4)$$

where $A_{(k,n)}$ and $B_{(k,n)}$ are two constants of integration corresponding to *second order* differential equation of the mode-function.

To fix these constants of integration we require that for large volumes ($n = 1$) the modified mode function reduces to the standard 'normalized' mode function (5.2.9).

Since the standard mode function (5.2.9) are already 'normalized' to pick out the Bunch-Davies vacuum then this requirement will automatically fix the choice of vacuum in effective loop quantum cosmology. This fixes the mode function solution to be

$$f_k = \sqrt{\frac{n+2}{3}}(-nH)\sqrt{\frac{\pi}{4k^3}}\,(k\eta)^{1+\frac{1}{2n}}\left[J_{-(1+\frac{1}{2n})}(k\eta) \;+\; i\;J_{(1+\frac{1}{2n})}(k\eta)\right] . \qquad (5.6.5)$$

Using Bessel function identities $J_{n+1}(x)+J_{n-1}(x) = \frac{2n}{x}J_n(x)$, $J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\,cos(x)$ and $J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\,sin(x)$ one can easily check that for $n=1$ the modified mode function (5.6.5) reduces to the standard mode function (5.2.9).

It should be emphasized here that in the expression (5.6.5) we have specifically chosen the power of $(\frac{n+2}{3})$ to be $\frac{1}{2}$. However, it is clear that for any arbitrary power of $(\frac{n+2}{3})$, the mode function would reduce to the standard mode function. We have made this choice precisely to absorb similar term coming from the effective equation of state (5.4.9) that appears in the final expression of power spectrum. In other words we have chosen the vacuum state such that it satisfies Bunch-Davies prescription and the computed power spectrum is free from trivial *ambiguity parameter dependent* multiplicative factor.

5.7 Power Spectrum

Having known the exact solution of the mode function in principle one can evaluate the two-point density correlation function using the expression (5.3.2). Let us recall that we are mainly interested in finding out the power spectrum at the time of horizon crossing *i.e.* $\frac{k}{Ha} = 2\pi$. The argument of the Bessel function $k\eta = \frac{k}{Ha}\left(\frac{a^{1-n}}{-n}\right) << 1$ during horizon crossing. For super-horizon scale above inequality holds naturally; even for sub-horizon scale upto reasonable extent the same inequality will hold since for effective loop quantum cosmology $(-\infty < n < -2)$. Since the asymptotic form of the Bessel function is $J_m(x) \approx \frac{1}{\Gamma(1+m)}(\frac{x}{2})^m$ for $x << 1$ then clearly the dominating contribution in the mode function (5.6.5) comes from the first term. So we will approximate the mode function as

$$f_k \approx \sqrt{\frac{n+2}{3}}(-nH)\sqrt{\frac{\pi}{4k^3}}\,(k\eta)^{1+\frac{1}{2n}}\left[J_{-(1+\frac{1}{2n})}(k\eta)\right] , \tag{5.7.1}$$

for further evaluation and will use its asymptotic form for explicit calculation.

Using the expression of two-point density correlation function (5.3.2) and expression of mode function (5.7.1) one can simply follow the similar steps as in [59] to derive the expression of two point density correlation function

$$C(l',t) = \left(\frac{n+2}{3}\right)^2 \frac{a^{4(1-n)}}{2^{2-\frac{2}{n}}\,(2\pi)^2\,\Gamma(1-\frac{1}{2n})^4}\left[\frac{2H^2}{(al')^6}+\frac{a^{4(1-n)}}{(al')^8}\right] . \tag{5.7.2}$$

In deriving the above expression (5.7.2), it is quite helpful to use the Bessel function identity $\frac{d}{dx}[x^m J_{-m}(x)] = -x^m J_{1-m}(x)$ while evaluating time derivative of the mode function (5.7.1). Now it is easy to check that for $n=1$ (*i.e.* classical mode function) the expression becomes qualitatively same as (5.3.3). However, we may note that it is *quantitatively* slightly different than (5.3.3). The source of this difference can be traced back to the approximation that we have made. In the case of loop quantum cosmology the argument of the Bessel function $k\eta = \frac{k}{Ha}\left(\frac{a^{1-n}}{-n}\right) << 1$ at the time of horizon crossing. Clearly the same does *not* hold for the standard case ($n=1$).

The two point density correlation function (5.7.2) diverges as the "coordinate length" l' goes to zero. This feature is rather expected from a calculation based on standard quantum field theory. However as we have mentioned that one can regularize this expression using the notion of zero-point proper length which is naturally available in effective loop quantum cosmology. In section IV, we have introduced the notion of *effective coordinate length*. This basically allows one to use the available machinery used in the standard case. Essentially this step summarizes the ultraviolet regularization of two point density correlation function. We define *effective* two point density correlation function as the *regularized* form of the standard two point density correlation function as

$$C^{eff}(l',t) := C(l^{eff},t) . \tag{5.7.3}$$

Now we can evaluate the Fourier transform of the effective two point density correlation in usual way

$$\begin{aligned} |\rho_k(t)|^2 &:= \int d^3\mathbf{l}\, e^{i\mathbf{k}\cdot\mathbf{l}}\, C^{eff}(l',t) \\ &= \left(\frac{n+2}{3}\right)^2 \frac{a^{4(1-n)}}{2^{2-\frac{2}{n}}\,(2\pi)^2\,\Gamma(1-\frac{1}{2n})^4}\left[\frac{2H^2}{a^6}I_1 + \frac{a^{4(1-n)}}{a^8}I_2\right] \end{aligned} \tag{5.7.4}$$

where the integrals I_1 and I_2 can be evaluated using method of contour integration. They are given by

$$I_1 := \int \frac{d^3\mathbf{l}\, e^{i\mathbf{k}\cdot\mathbf{l}}}{\left(l'^2 + \frac{L_0^2}{a^2}\right)^3} = \frac{\pi^2 e^{-\frac{kL_0}{a}}}{4}\left(\frac{a}{L_0}\right)^3\left[1+\frac{kL_0}{a}\right], \tag{5.7.5}$$

and

$$I_2 := \int \frac{d^3\mathbf{l}\, e^{i\mathbf{k}\cdot\mathbf{l}}}{\left(l'^2 + \frac{L_0^2}{a^2}\right)^4} = \frac{\pi^2 e^{-\frac{kL_0}{a}}}{8}\left(\frac{a}{L_0}\right)^5\left[1+\frac{kL_0}{a}+\frac{1}{3}\left(\frac{kL_0}{a}\right)^2\right]. \tag{5.7.6}$$

The power spectrum of density perturbations generated during inflation however is not directly observable. Rather the observed power spectrum corresponds to the density perturbations at the time of horizon *re-entry* in the post-inflationary period. In the intermediate period between horizon exit and horizon re-entry the density contrast $\delta(:= \frac{\delta\rho}{\rho})$ remains almost constant for the super-horizon modes. Nevertheless due to the change in equation of state of the total matter field leads to a scaling of the amplitude of density perturbations. Super-horizon evolution in Bardeen's gauge invariant formalism [68] of density perturbations leads to a rather simple formula for the evolution of density contrast

$$\left|\frac{\delta_k}{1+\omega}\right|_{t=t_f} \approx \left|\frac{\delta_k}{1+\omega}\right|_{t=t_i}, \tag{5.7.7}$$

where $\delta_k := \frac{\rho_k(t)}{\bar{\rho}}$, t_i and t_f are initial and final time respectively. The power spectrum of density perturbations at the time of *re-entry* is given by

$$\mathcal{P}_\delta(k) = \frac{k^3}{2\pi^2}|\delta_k|^2_{\text{re-entry}} = \mathcal{A}^2\left[1 + c_0\left(\frac{k}{2\pi H}\right)^{4(1-n)}\right] \tag{5.7.8}$$

where $c_0 = \frac{\pi^2}{\sigma^2}[1+\frac{\sigma^2}{3(1+\sigma)}]$ and the $\mathcal{A}^2$ is given by

$$\mathcal{A}^2 = \frac{(1+\omega_{re})^2}{(2\pi)^2\, C_\omega^2}\,\frac{\sigma^3\,(1+\sigma)\,e^{-\sigma}}{2^{2-\frac{2}{n}}\,\Gamma(1-\frac{1}{2n})^4}. \tag{5.7.9}$$

The quantity σ, defined in (5.4.11), is given by $\sigma = 4\pi(\frac{2\pi}{3})^{\frac{2}{3}} \left(\frac{\bar{\rho}}{M_{\bar{P}}^4}\right)^{\frac{1}{3}}$. For the super-horizon modes ($\frac{k}{2\pi H} << 1$) the second term in the expression (5.7.8) is negligible compared to unity for the effective loop quantum cosmology ($-\infty < n < -2$). Thus it is clear from the expression (5.7.8) that power spectrum of density perturbations is broadly *scale-invariant* as during the inflationary period the Hubble parameter remains almost constant. The mode function solutions (5.6.5) being ambiguity parameter dependent, the expression of the power spectrum (5.7.8) depends on ambiguity parameter. However, it should be noted that this dependence is rather weak. The ambiguity parameter dependent term in the power spectrum $2^{-\frac{2}{n}} \Gamma(1 - \frac{1}{2n})^4$ varies only between 1 to 1.3499 for the range of ambiguity parameter value ($-\infty < n < -2$).

An important property of the power spectrum (5.7.8) is that $\mathcal{A}^2 \sim H^2$ (σ being small $(1+\sigma)\ e^{-\sigma} \approx 1$). This behaviour is exactly similar to the behaviour of the power spectrum in standard inflationary scenario. This property of the power spectrum will be very useful in comparison of spectral index between standard inflationary scenario and effective loop quantum cosmology scenario.

5.7.1 Amplitude of Density Perturbation

In section IV, we have shown that the self-consistency of the framework that we are using requires $\sigma << 1$. This requirement can be physically understood in the following way. The effective continuum (classical geometrical) description is an emergent description in the loop quantum cosmology framework in which underlying geometry is fundamentally discrete. Naturally the effective Hamiltonian description which has been used in the chapter has a restricted domain. In [36], it has been shown that the dynamics described by the effective Hamiltonian respects it own domain of validity provided the permissible values of $\bar{\rho}$ is chosen to be significantly smaller than unity when written in Planck units. In other words the requirement $\sigma << 1$, essentially defines the domain of validity of the effective Hamiltonian. On contrary, in a purely classical geometrical description (standard inflationary scenario) such restriction does not arises as the description itself a fundamental description of nature within the setup of general relativity. Thus it is clear from the expression (5.7.9) that in this scenario the amplitude for the power spectrum of density perturbations is *naturally* small. In other words, the small amplitude for the primordial density perturbations is a *prediction* of the framework of effective loop quantum cosmology.

It should be noted from the equation (5.7.7) that if the equation of state is very close to -1 during inflationary period then the amplitude of the density perturbations gets a large multiplicative factor at the time of horizon re-entry. In the case of loop quantum cosmology induced inflation the equation of state (5.4.9) indeed very close -1 as $a << 1$ during its inflationary period. However in this scenario, still one can produce small amplitude for the primordial density perturbations without fine tuning. One of the reasons behind this is the presence of the small factor $a^{4(1-n)}$ in the two-point density correlation function (5.7.2). The presence of this crucial small factor in the expression of the two-point density correlation function simply follows from the *modified* mode functions of the scalar field.

Another interesting property of the amplitude (5.7.9) is that it contains an exponential damping term $e^{-\sigma}$. The damping term is insignificant here as σ is required to be small. However if one naively takes the energy scale to be order of Planck scale even then amplitude of the density perturbations will remain small as the exponential term becomes significant in that scale. In fact this was the main motivation for the papers [59, 60].

Qualitatively the small amplitude of the primordial density perturbations is readily predicted but to have *quantitative* estimate one needs to choose some value for the associated energy scale. Assuming density perturbation is *adiabatic i.e.* it is same as curvature perturbation, one can relate amplitude of the power spectrum of the density perturbations to the CMB angular power spectrum as follows [69]

$$\mathcal{A}^2 = \left(\frac{3}{2}\right)^2 \frac{9}{(2\pi^2)} \frac{\tilde{l}(\tilde{l}+1)C_{\tilde{l}}^{AD}}{2\pi} . \qquad (5.7.10)$$

where $\tilde{l}$ is the multipole number of the angular power spectrum. We have also assumed that the relevant modes re-enter horizon during radiation dominated era. The COBE data implies that $\frac{l(l+1)C_l^{AD}}{2\pi} \simeq 10^{-10}$. Using the expression (5.7.9) one can easily deduce that $\sigma \approx 5.5 \times 10^{-3}$(we have assumed here that at the end of loop quantum cosmology induced inflation, the radiation domination begins *i.e.* $C_\omega = 4$; value of the ambiguity parameter l is chosen to be $\frac{1}{2}$). It follows from the expression of σ (5.4.11) that the corresponding energy density is $\bar{\rho} \approx (2.0 \times 10^{-3}\ M_p)^4 = (2.0 \times 10^{16}\mathrm{GeV})^4$. The associated energy scale comes down slightly if one assume that the end of loop quantum cosmology induced inflation is followed by a standard accelerating phase (For example, if one takes $C_\omega = \frac{2}{3}$ ($\omega = -\frac{1}{2}$) then the corresponding energy density is $\bar{\rho} \approx (0.8 \times 10^{-3}\ M_p)^4 = (0.8 \times 10^{16}\mathrm{GeV})^4$).

The energy scale required to produce observed amplitude of density perturbations in the effective loop quantum cosmology is *not* very different from the standard inflationary scenario where associated energy density is $\bar{\rho} \approx (2.0 \times 10^{16}\mathrm{GeV})^4$ [69]. Then it is quite important to understand why is it necessary to *fine tune* field theoretical parameters in standard scenario to produce small amplitude. Here we have considered a massive scalar field as the matter source. The energy density during standard inflationary period is $\bar{\rho} \approx \frac{1}{2}\ m_\phi^2\ \phi^2$. In standard inflationary scenario to produce sufficient amount of expansion (to solve horizon problem and others) one needs to choose the values of field to be $\phi^2 \approx 10M_p^2$ [69]. This in turns forces one to tune the mass parameter to be $m_\phi \approx 10^{-6}M_p$, in order to produce small amplitude for primordial density perturbations. These fine tunings of field strength and mass term are not only severe but also extremely difficult to sustain under standard quantum field theory. Specifically, sustaining such low mass parameter from loop corrections often requires new ingredients [70].

In other words, in the standard inflationary scenario to get correct amplitude of density perturbations, the required fine tunings are directly related with the method by which inflation is realized namely the imposition of *slow-roll* condition. One the other hand, in loop quantum cosmology the inflationary phase is realized generically [35] and *not* by imposing slow-roll condition. There is a physical understanding of it from the fundamental point of view. The famous singularity theorems tell us that in the cosmological set-up one cannot avoid initial singularity if the matter contents always satisfy the strong energy condition (equation of state $\omega \geq -\frac{1}{3}$). Naturally to avoid initial singularity the quantum gravity effects must drive the matter contents to effectively violate the strong energy condition, at least for small enough volume. It has been shown that in loop quantum cosmology one generically avoids singularity at the quantum level [22] as well as the effective classical level [36, 54, 57]. Using Raychaudhuri equation it is then easy to see that the violation of strong energy conditions i.e. $\omega < -\frac{1}{3}$ immediately implies an accelerating phase. In particular in [35], it has been proved that for *any* positive definite scalar potential loop quantum cosmology induced accelerating phase undergoes near exponential expansion. Thus, in loop quantum cosmology the realization of an early accelerating phase is intimately related with the removal of initial singularity. In this scenario, the inflation is essentially driven by the non-perturbative modification of the scalar field dynamics namely via the spectrum of the inverse scale factor operator. This inflationary phase begins as soon as the system gets into the small volume regime (non-perturbative domain). Therefore, one does *not* require to fine tune field strength to sufficiently uphill as done in the standard scenario. Consequently, one does not require to fine tune field theoretical parameters to produce small amplitude of density perturbations. Rather, as is shown in this chapter, the small amplitude is a prediction of the framework that has been used in the calculation.

Nevertheless, one can impose self-consistency requirement on the mass parameter in this calculation. Let us recall that in simplifying mode function equation (5.6.1) we have neglected the mass term $\frac{m_\phi}{H}$ compared to the term $\frac{k}{Ha}$. Since we are interested in calculating power spectrum at the time of horizon crossing *i.e.* $\frac{k}{Ha} = 2\pi$ then to be self-consistent we must require that $m_\phi < 2\pi H$. This in turns implies that the maximum value of the mass parameter to be $m_\phi \sim \frac{1}{27}(\bar{\rho})^{\frac{1}{4}}$. We may mention here that in effective loop quantum cosmology $\bar{\rho}$ in fact is the maximum energy density *i.e.* $(\bar{\rho})^{\frac{1}{4}}$ is precisely the cut-off scale. It should be emphasized that the restriction on mass parameter here is a self-consistency requirement of the calculation as solving the modified mode function equation (5.6.1) including the mass term turns out to be *not* so easy a task.

5.7.2 Spectral Index

So far in the calculation we have assumed that during inflationary period energy density is strictly constant. However, this was rather an approximation to simplify the calculation. We can in fact compute the effect of small variation of energy density. The small variation of Hubble parameter leads to a small deviation from the *scale-invariant* power spectrum. The scale dependent property of the power spectrum is conveniently described in terms of *spectral index*. From the conservation equation it follows that $\frac{d\ \ln\rho}{d\ \ln a} = -3(1+\omega^{\text{eff}}) = -C_\omega(n+2)a^{2(1-n)}$. Using this relation one can compute spectral index at horizon crossing

$$\begin{aligned} n_s - 1 \quad &:= \quad \frac{d\ \ln\mathcal{P}_\delta(k)}{d\ \ln\ \mathrm{k}} \\ &\approx \quad C_\omega(-n-2)\left(\frac{k}{2\pi H}\right)^{2(1-n)} \ + \ 4c_0(1-n)\left(\frac{k}{2\pi H}\right)^{4(1-n)} \end{aligned} \tag{5.7.11}$$

We may note here that the spectral index n_s is *extremely* close to unity and the difference $(n_s - 1)$ depends non-trivially on the ambiguity parameter. However, the most important property of the spectral index (5.7.11) is $(n_s - 1) > 0$ for all allowed values of the ambiguity parameter ($0 < l < 1$ *i.e.* $-\infty < n < -2$). This is in complete contrast to the standard single-field inflationary scenario. We have mentioned earlier that the power spectrum for both the scenarios varies as H^2. For single-field standard inflationary scenario the leading contribution to the spectral index deviation comes from the variation of Hubble parameter during inflationary period. So for the standard scenario the spectral index is given by

$$n_s - 1 \ := \ \frac{d\ \ln\mathcal{P}_\delta(k)}{d\ \ln\ \mathrm{k}} \ \approx \ -6\,\epsilon\,, \tag{5.7.12}$$

where $\epsilon = \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 4\pi G\frac{\dot{\phi}^2}{H^2}$ is the *slow roll* parameter of standard inflationary scenario. In fact due to the time variation of ϵ there will be additional contributions in (5.7.12). However, those will be *sub-leading* for *single-field* inflationary scenario. Thus for single-field standard inflationary scenario the spectral index satisfies $(n_s - 1) < 0$.

In the standard method (indirect method) of obtaining power spectrum of the density perturbations, one first computes the power spectrum of the field fluctuations. Using the expression (5.7.1), one can compute the power spectrum of the *field* fluctuations at the horizon exit

$$\mathcal{P}_\phi(k) := \frac{k^3}{2\pi^2}|f_k|^2 = \left(\frac{1}{2\pi}\right)\frac{|n+2|}{3}\,\frac{n^2\,2^{\frac{1}{n}}}{\Gamma(-\frac{1}{2n})^2}\,H^2, \tag{5.7.13}$$

which is *scale invariant*. In the standard inflationary scenario, this would have lead to a scale invariant power spectrum of *density* perturbations. However, we have already mentioned that this is not straightforward in the effective loop quantum cosmology. This is due to the fact that it is not yet well settled; how to obtain a general effective action, consequently an effective stress-energy tensor from a canonical quantum theory of gravity. This prevents one to compute the power spectrum of density perturbations using standard method, as one needs to know the expression of the general effective stress-energy tensor, to relate the power spectrums of field fluctuations to the density perturbations. Nevertheless, one may naively assume that the power spectrum of field fluctuations and that of density perturbations will have similar relation. In that case, the computed power spectrum of density perturbations using *direct* and *indirect* method would differ from each other by an extremely weak k dependence. This difference would then be similar in nature to the results of [59] in the context of the standard inflationary scenario. However this difference would at most affect the quantitative nature (that too by a vanishingly small amount) but *not* the qualitative nature of the spectral index (5.7.11) whatsoever.

Thus it is clear that for *effective loop quantum cosmology* induced inflationary scenario the spectral index has a qualitatively distinct feature compared to that of *single-field* driven *standard* inflationary scenario. In the next sub-section we discuss its observational consequences.

5.7.3 Observational Implications

The power spectrum of density perturbations generated during inflationary era is *not* directly observable. Rather the observed power spectrum corresponds to the density perturbations at the time of *re-entry* in the post-inflationary period. At the time of re-entry larger wavelength $(2\pi k^{-1})$ enters the horizon at later time compared to the smaller wavelength. It is clear from the expression (5.7.7) that if there is a change in the equation of state of the universe during re-entry then there will be an additional modification of the power spectrum. Since we are interested in estimating the original power spectrum generated during inflationary era then its quite important to avoid additional modification of the power spectrum coming from other possible sources.

The observed anisotropy in the CMB sky corresponds to the density perturbations on the *last scattering surface*. The last scattering surface broadly demarcate the end of radiation domination era to the beginning of matter domination era. Naturally during this period $(1+\omega)$ changes from $\frac{4}{3}$ to 1. While deriving the expression (5.7.11) of the spectral index we have assumed constant equation of state during re-entry. Thus for the purpose of comparison with observations, one must consider only those modes for which the equation of state was almost constant during re-entry. On last scattering surface they will corresponds to the modes which are well inside the horizon at the time of *decoupling*. Being smaller in wavelength these mode will subtend smaller angle in present day sky. Naturally these mode will corresponds to the higher multi-pole number. Also if one considers sufficiently narrow bands in these part of spectrum then one can avoid additional modification coming from the sub-horizon evolution of density perturbations in the period between their *re-entry* and the decoupling.

To infer the property of primordial density perturbations from the observed angular power spectrum of CMB, one needs to know the evolution of the universe for the period between the decoupling and the present day universe. Since major fraction of today's energy density is believed to be coming from mysterious *dark matter* and *dark energy* then it is quite obvious that there will be a considerable influence of them on the inferred primordial power spectrum. The current observational estimate of spectral index based on WMAP+SDSS data is $n_s = 0.98 \pm 0.02$ [71, 72, 73]. This estimate is based on the entire part of the observed angular power spectrum nevertheless this agrees (rather marginally) with the expression of the spectral index (5.7.11) which is strictly valid only for the part of the spectrum in the higher multipole region. For this purpose, it may be more convenient to *reconstruct* the primordial power spectrum using observed CMB angular power spectrum (for example as done in [74, 75, 76, 77]) and then consider the higher wavenumber part of the spectrum.

5.8 Discussions

In summary, we have computed the power spectrum of density perturbations generated during loop quantum cosmology induced inflationary phase. The resulting power spectrum is broadly *scale-invariant*. Further it is shown that the small amplitude for primordial density perturbations is a natural *prediction* of the framework of effective loop quantum cosmology. Unlike standard inflationary scenario, here one does not require to *fine tune* field theoretical parameters to produce small amplitude for density perturbations. The resulting power spectrum also has a qualitatively distinct feature compared to the standard single-field inflationary scenario. The spectral index in the effective loop quantum cosmology scenario satisfies $(n_s - 1) > 0$ whereas for the standard inflationary scenario it satisfies $(n_s - 1) < 0$.

Naturally, the spectral index of power spectrum for density perturbations generated during the loop quantum cosmology induced inflation and the standard inflation differs from each other in a non-trivial and non-overlapping way. This is a consequence of the fact that during loop quantum cosmology induced inflation the Hubble horizon *shrinks* marginally whereas in the standard inflationary scenario the Hubble horizon *expands*. This feature leads to the power spectrum for the corresponding density perturbations to be tilted in opposite directions to each other. We have argued that this feature is a *generic* property of the corresponding scenario and not a property of some particular model. We have also pointed out the part of the observed CMB angular power spectrum that may be better suited for testing this particular feature observationally, namely the part corresponding to the higher multipole numbers of the CMB angular power spectrum.

The computational techniques used here are analytic within the adopted framework and the approximations used here are mostly justified. Nevertheless one should keep it in mind that this calculation itself should *not* be considered as the *first principle* calculation of density perturbations within loop quantum cosmology. Rather this calculation is based on *effective* loop quantum cosmology. Here we have considered the non-perturbative modification of kinetic term of the scalar matter Hamiltonian. In a first principle calculation (using inhomogeneous model), one may naively expect to get corrections also in the gradient term of the matter Hamiltonian. This modification should depend on some ambiguity parameter similar to that of l. Here it is shown that the ambiguity parameter l dependence of the amplitude of the power spectrum is very weak. So the effect of such possible modifications on the amplitude of the power spectrum is expected to be rather small. The effect on spectral index is also expected to be small as it is mainly determined by the background evolution. Thus it is very likely that the calculation presented here should be a good approximation of what is expected from a first principle calculation in the energy scale concerned.

We have used direct method to compute the power spectrum of density perturbations. This method uses the techniques of standard quantum field theory. Naturally, one needs to have some kind of ultra-violet divergence regularization prescription. To regularize the ultra-violet divergence we have used the method outlined and explicitly shown in [59, 60]. This method relies on the assertion that a proper theory of quantum gravity should contain a *zero-point* proper length. In the effective loop quantum cosmology such length scale *is* naturally available. In this method, regularization is essentially carried out by adding a *zero-point* proper length to the standard definition of proper length. In this chapter the procedure was notationally simplified by using the notion of *effective coordinate length*. Nevertheless regularization procedure was carried out *by hand*. However, this was expected as the calculations here were done using standard quantum field theory. On the other hand one would expect that in a *first principle* calculation these regulator should come *built-in* as it has been argued in the context of full theory [62]. The crucial result of the chapter namely the properties of the spectral index are insensitive to the regularization method as it is dominantly determined by the nature of the effective equation of state during its inflationary phase.

In standard inflationary scenario one is also interested in computing power spectrum for tensor mode of the metric fluctuations *i.e.* gravitational waves. However, as we have mentioned that the technology required to deal with inhomogeneity at fundamental level in loop quantum cosmology is not yet available. In loop quantum gravity approach *geometry* is quantized in non-perturbative way. Thus it is not easy to 'guess' the structures of the quantum fluctuations of geometry until one carries out explicit computations within the framework. One would naively expect that the power spectrum for tensor mode perturbations should be similar to the standard scenario as the structure of the effective gravity sector Hamiltonian is similar to the classical Hamiltonian in the relevant length scale. Also, the energy scales of the corresponding inflationary periods are similar.

Now before we discuss the implications of possible outcomes of mentioned observational test, let us have a comparative study of standard inflationary scenario and loop quantum cosmology induced inflationary scenario. In order to have a *successful* inflation in the standard scenario, generally one requires multi-level of fine tuning of field parameters. In other words one faces several kind of *naturalness* problems to achieve a successful inflation.

The first one is to *start inflation*. In standard inflationary scenario it is needed to choose initial field velocity to be sufficiently small so that the equation of state $\frac{\frac{1}{2}\dot{\phi}^2-V(\phi)}{\frac{1}{2}\dot{\phi}^2+V(\phi)} \approx -1$. *The second* one is to *sustain inflation*. In standard scenario one requires to choose the field potential to be *flat* enough so that field does not gain momentum quickly. *The third* one is to *generate sufficient expansion* (to solve horizon problem and others). To achieve this in standard scenario, one requires to choose the initial field configuration sufficiently *uphill* in the potential. In other words, one requires to fine tune initial field configuration. *The fourth* one is to *end inflation*. In many cases this requires sort of *potential engineering* to have a *long flat plateau* and then a *fast fall-off* in the potential profile. *The fifth* one is to produce *small amplitude* for primordial density perturbations. To produce observed small amplitude of density perturbations one needs to fine tune parameters of the field potential. This fine tuning is basically required to *compensate* the 'third' fine tuning.

On the other hand, to achieve the first, second and fourth requirements in loop quantum cosmology induced inflationary scenario one *does* not require to fine tune the parameters. These requirements are naturally achieved as they simply follow from the spectrum of the inverse scale factor operator. The fifth requirement *i.e.* small amplitude, as shown in this chapter, is a natural prediction of effective loop quantum cosmology. The situation regarding the third problem also gets improved significantly. In the loop quantum cosmology, the generated amount of expansion is controlled by the ambiguity parameter j. Clearly to produce sufficiently large expansion, using loop quantum cosmology *alone*, one will require to choose the value of j to be large. Thus it is very likely that only the initial part of the inflation was driven by loop quantum cosmology modification. It has been argued in [34, 56, 78, 111, 112] that the loop quantum cosmology induced inflationary phase can lead to a secondary standard inflationary phase. This follows from the fact that the in-built inflationary period of loop quantum cosmology can produce favourable initial conditions for an additional standard inflationary phase. In [56], the authors have also studied the possible effects of the above mechanism on CMB angular power spectrum generated during the standard inflationary phase that follows the LQC induced inflationary phase and shown that it can lead to suppression of power in the low CMB multipoles. Since the observed part of CMB angular power spectrum generally corresponds to early period of inflation then it may well be the situation where the observed part of the CMB angular power spectrum corresponds to the loop quantum cosmology driven inflationary period.

It is worthwhile to emphasize that high amount of expansion in this scenario is required *not* to solve horizon problem (being non-singular this model avoids horizon problem [35]) rather to avoid a different kind of problem. We have seen that the 'initial size' of universe was typically order of Planck units and the corresponding energy scale was also typically order of Planck units. During relativistic particle (radiation) dominated era energy scale falls of typically with inverse power of the associated length scale. It is then difficult to understand why the universe is so large ($\sim 10^{60} L_p$) today but still it has relatively very high energy scale ($\sim 10^{-30} M_p$). During inflationary period, on the other hand, the energy scale remains almost constant whereas the length scale grows almost exponentially with coordinate time. It is clear that we can avoid this discrepancy between energy scale and length scale of the universe provided there existed an inflationary period with sufficiently long duration in early universe.

Now if the observed power spectrum turns out to be not in agreement with the computed power spectrum, then one should conclude that the phase of inflation corresponding to the observed window could not possibly be driven by loop quantum cosmology modifications alone. It may then restrict the allowed choices for the ambiguity parameter j. Consequently it will be an important issue to understand within the framework of *isotropic* loop quantum cosmology with *minimally coupled* scalar matter field, why the observed universe today is so large but still it has sufficiently high energy scale.

Chapter 6

Energy Conditions and Stability

6.1 Introduction

In general theory of relativity, dynamics of a spacetime is influenced by matter stress-energy tensor. Naturally, many properties regarding spacetime evolution can be concluded assuming some general properties of the matter stress-energy tensor, without having to know the details of the individual contributions from different matter sources. These requirements on the matter stress-energy tensor, widely called *energy conditions*, have been used to prove several important theorems in classical general relativity. One such theorem, the Hawking-Ellis *conservation theorem* [1, 44] says that if the matter stress-energy tensor is conserved, satisfies *dominant energy condition* and vanishes on a closed, *achronal* (no two points can be connected by timelike curves) set S then it also vanishes in the *domain of dependence* (complete set of events for which all conditions are determined by specifying conditions on S) $D(S)$ of the set. Physically, this theorem ensures the stability of classical vacuum. As mentioned, the conservation theorem stands true provided the matter stress-energy tensor satisfies the dominant energy condition. This condition requires local energy density to be *non-negative* for all time-like observer and the energy-momentum 4-current to be *non-spacelike i.e.* the speed of energy-flow should not be exceeding the speed of light. Naturally, the violation of dominant energy condition raises concern about the causality and the stability of the system. However, it is worth pointing out that the above theorem does *not* have the *converse i.e.* although the dominant energy condition satisfying matter ensures causality and stability of the system but violation of this condition *does not* necessarily imply that the system violates causality or is unstable (see for example [45]). In such a situation, these issues should be considered for the specific context, as dominant energy condition violation and the Hawking-Ellis conservation theorem no longer vouch for the causality and the stability of the system.

In the cosmological context, the issue of dominant energy condition violation has acquired significant importance in recent literature. The observational evidences [79, 80] seem to suggest that in our universe major fraction of the energy density is contributed by some kind of mysterious *dark energy* that exerts *negative* pressure. The experimental data in this context not only allows but often favours the values of the equation of state parameter to be less than -1 for the dark energy component [81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92]. Such values of the equation of state parameter require violation of dominant energy condition. This makes the problem of the dark energy even more severe which is otherwise itself a major theoretical challenge in the present day cosmology [93, 94, 95, 96, 97, 98]. A popular proposed model for dominant energy condition violating dark energy is so called *phantom matter* [81, 99, 100, 101, 102]. The phantom matter is essentially a minimally coupled scalar field model but with relatively *negative* kinetic term (but see [103, 104, 105, 106, 107] for other possibilities). Naturally, the classical Hamiltonian for the phantom matter becomes unbounded from below. Such unbounded Hamiltonian essentially leads to a classically unstable system, as ground state of such system gets pushed to negative infinity.

Apart from the mentioned observational indication of violation of energy condition, there are in fact theoretical reasons to argue that some of these energy conditions in general relativity, should be violated in appropriate regime. One such reason behind this, is the existence of another important set of theorems, so called *singularity theorems*. These theorems tell us that if the evolution of a globally hyperbolic spacetime satisfies Einstein equation and the matter stress-tensor satisfies so called *strong energy condition* then the backward evolution of such an expanding spacetime is necessarily singular, in a sense that the spacetime is geodesically incomplete. However, the appearance of singularity in a classical theory is generally considered as an attempt to extrapolate the classical theory beyond its natural domain of validity, rather than considering it as a *property of nature*. Near the classical singularity one expects the evolution of the spacetime to be governed by a quantum theory of gravity, as classical description signals its own breakdown. Further, one also believes that a proper theory of quantum gravity should resolve the singularity that appears in the classical general relativity. Naturally, one would naively expect that the quantum effects of such theory should force the matter contents to *effectively* violate the strong energy condition when its dynamics is viewed as an evolution of pseudo-Riemannian spacetime.

In effective loop quantum cosmology, non-perturbatively modified dynamics of a *minimally* coupled scalar field violates *weak*, *strong* and *dominant* energy conditions when they are stated in terms of equation of state parameter. The violation of strong energy condition helps to have non-singular evolution by evading singularity theorems thus leading to a generic inflationary phase. However, the violation of weak and dominant energy conditions raises concern, as in general relativity these conditions ensure causality of the system and stability of vacuum via Hawking-Ellis conservation theorem. In fact several important features of loop quantum cosmology, that have been shown in literature, crucially rely on the effective classical description. Naturally, in the effective loop quantum cosmology, the violation of dominant energy condition raises concern. In particular, whether such effective classical description respects causality. In the cosmological context, any communication across spatial distances introduces inhomogeneity. So it is a natural concern to check whether the propagation of inhomogeneous modes respects causality. Also, whether such dominant energy condition violating effective description can ensure stability of the vacuum, as the Hawking-Ellis conservation theorem no longer guarantees for the same (see also [113, 114] for related discussions).

In section 6.2, we briefly review the definitions of relevant energy conditions used in general relativity. In particular, for the cosmological context, we discuss the requirements on the equation of state parameter due to these energy conditions. In the next section, we discuss the properties of the equation of state parameters for a minimally coupled scalar field and also for the so-called phantom matter model of dark energy. In the section 6.4, we study the properties of the effective scalar matter Hamiltonian. In particular, we show that the kinetic term due to the modified dynamics, contributes negative pressure even though it contributes positive energy density. This crucial feature essentially leads to violation of dominant energy condition in terms of the equation of state parameter but it also ensures a bounded (from below) scalar matter Hamiltonian. In the next section, we derive a modified dispersion relation for the inhomogeneous modes due to the modified dynamics. Then we show that the group velocity for the relevant inhomogeneous modes remains subluminal thus ensuring causal propagation across spatial distances. We also compute the quantum corrections to the group velocity for a massless free scalar field at large volume.

6.2 Energy Conditions in General Relativity

The energy conditions, often regarded as sacred principles [115], were mostly postulated to prove several important theorems in classical general relativity. A few important among them are the so called *singularity theorems* and *conservation theorem*. In this section, we will briefly recall the definitions of some of these energy conditions. In the cosmological context, these energy conditions can be essentially stated in terms of the energy density and its relation to the pressure component *i.e.* the *equation of state* parameter. We will mainly follow the convention of Wald [1].

6.2.1 Weak Energy Condition

For a given matter stress-energy tensor $T_{\mu\nu}$, the quantity $T_{\mu\nu}\xi^\mu\xi^\nu$ physically represents local energy density for an observer whose 4-velocity is ξ^μ at a spacetime point. The *weak energy condition* is physically interpreted as the requirement of *non-negativity* for the classical energy density. Naturally, the weak energy condition is stated as

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \ , \tag{6.2.1}$$

for all time-like ξ^μ. Assuming that the stress-energy tensor can be diagonalized *i.e.* it can be written as $T_{\mu\nu} := \rho\ t_\mu t_\nu + P_1\ x_\mu x_\nu + P_2\ y_\mu y_\nu + P_3\ z_\mu z_\nu$ where $\{t^\mu, x^\mu, y^\mu, z^\mu\}$ is an orthogonal set of basis and t^μ is time-like, the weak energy condition requires $\rho \geq 0$ and $\rho + P_i \geq 0$ for $i = 1, 2, 3$ where P_i is the *principal* pressure. For the homogeneous and isotropic spacetime these requirements can be conveniently stated in terms of the equation of state parameter $\omega := P/\rho$ as $\omega \geq -1$ and the energy density $\rho \geq 0$.

6.2.2 Strong Energy Condition

A crucial requirement on the matter stress-energy tensor, for the singularity theorems to hold, is that it should satisfy so called *strong energy condition*. This energy condition requires matter stress-energy tensor to satisfy

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{1}{2}T \ , \tag{6.2.2}$$

for all *unit* time-like ξ^μ. Assuming diagonal form of the stress-energy tensor, the strong energy condition requires $\rho + \sum_{j=1}^{3} P_j \geq 0$ and $\rho + P_i \geq 0$ for $i = 1, 2, 3$. For the homogeneous and isotropic spacetime, these requirements in terms of the energy density and equation of state parameter can be stated as $\rho \geq 0$, $\omega \geq -\frac{1}{3}$. One may note here that the violation of strong energy condition which is *necessary* for non-singular cosmological evolution, implies an accelerating phase in its evolution via Raychaudhuri equation.

6.2.3 Dominant Energy Condition

The Hawking-Ellis conservation theorem requires matter stress-energy tensor to satisfy so called *dominant energy condition*. This condition requires local energy density to be *non-negative* for all time-like observer and local energy-momentum 4-current i.e $-T_{\mu\nu}\xi^{\mu}$ to be future directed, *non-spacelike* for all future directed, time-like ξ^{μ}. So the dominant energy condition is stated as

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0 \quad ; \quad T_{\mu\nu}\xi^{\nu}T^{\mu}_{\rho}\xi^{\rho} \leq 0 \ . \tag{6.2.3}$$

The second requirement can be physically interpreted as the requirement on matter stress-energy tensor such that the speed of energy-flow does not exceed the speed of light. Assuming diagonal form of the stress-energy tensor, the dominant energy condition requires $\rho \geq |P_i|$ for $i = 1, 2, 3$. In other words, the energy density is required to *dominate* the pressure components. For the homogeneous and isotropic spacetime, these requirements can be stated in terms of the equation of state parameter as $|\omega| \leq 1$ and energy density $\rho \geq 0$.

Apart from the above energy conditions, there are few more energy conditions that can be seen in the literature. For example, so called *null energy condition* requires matter stress-energy tensor to satisfy $T_{\mu\nu}n^{\mu}n^{\nu} \geq 0$, for all null vector n^{μ}.

6.3 Classical Scalar Matter Hamiltonian

In the cosmological scale, our universe appears to be spatially flat, homogeneous and isotropic with a very good precision. The invariant distance element in such spacetime (using *natural units i.e.* $c = \hbar = 1$) is given by Friedmann-Robertson-Walker metric (1.1.1). In this chapter we will consider a *minimally* coupled scalar field as the matter source. The dynamics of such scalar field is governed by the action (5.2.1). Let us recall that we are mainly interested in studying the effects on the scalar field dynamics, due to the non-perturbative modification coming from loop quantum cosmology. In the canonical quantization, as in loop quantum cosmology, one treats Hamiltonian as a basic object that governs the dynamics of the system. Thus, for our purpose it is necessary to have the expression for the scalar matter Hamiltonian

$$H_{\phi} = a^{-3} \int d^3x \left[\frac{1}{2}\pi_{\phi}^2\right] + a \int d^3x \left[\frac{1}{2}(\nabla\phi)^2\right] + a^3 \int d^3x \left[V(\phi)\right] \ , \tag{6.3.1}$$

where $a(t)$ is the *scale factor* and field *momentum density* $\pi_\phi = a^3\dot{\phi}$. In deriving expression (6.3.1), it is assumed that the background *geometry* is homogeneous, isotropic and described by the metric (1.1.1). However, we have assumed that the scalar field itself need *not* be homogeneous. This *approximation* greatly simplifies the analysis. Nevertheless, one should keep it in mind that it is trustworthy as long as the deviation from homogeneity and isotropy remains small.

In loop quantum cosmology, the geometrical quantities like the scale factor a here, are represented through corresponding quantum operators. While deriving effective classical Hamiltonian from loop quantum cosmology, these geometrical quantities effectively get replaced by the eigenvalues of their corresponding quantum operators. The kinetic term of the scalar matter Hamiltonian (6.3.1) involves inverse powers of the scale factor. In loop quantum cosmology, the inverse scale factor operator has a bounded spectrum. Clearly one can see that the kinetic term of the effective scalar matter Hamiltonian will involve non-perturbative modifications.

Given an arbitrary inhomogeneous scalar field in a spatially flat space, one can decompose it in terms its Fourier modes. In this case, the dynamics of the $k = 0$ mode *i.e.* the spatially homogeneous mode will essentially drive the evolution of the homogeneous background geometry, as the contribution from non-zero k modes are assumed to be small. So for the purpose of determining the background evolution, it is sufficient to consider only the homogeneous mode. In other words, we will neglect the contribution from the gradient term while evaluating the background evolution. Naturally, the scalar matter Hamiltonian (6.3.1) reduces to

$$H_\phi = p^{-\frac{3}{2}}\,\frac{1}{2}p_\phi^2 \;+\; p^{\frac{3}{2}}V(\phi)\,, \tag{6.3.2}$$

where $\int d^3x\sqrt{-g} := a^3V_0 := p^{\frac{3}{2}}$ and $p_\phi(= V_0\pi_\phi)$ is the field *momentum*. It is important to note here that we have absorbed the *fiducial* coordinate volume V_0 (of a given finite cell) in the definition of the variable p. In loop quantum cosmology, the variable p is known as redefined densitized triad and it is one of the basic phase space variables.

6.3.1 Classical Energy Density and Pressure

In the Lagrangian formulation, one can obtain the expression for the general stress-energy tensor by considering the variation of the action with respect to the spacetime metric. Naturally, one can use the general expression of the stress-energy tensor, to obtain the reduced expression for the energy density and the pressure component for the homogeneous and isotropic spacetime. On the other hand, in the Hamiltonian formulation such direct method is not available. However, one can define the expression for the energy density and the pressure component in terms of the classical Hamiltonian as

$$\rho := \frac{1}{2}\dot{\phi}^2 + V(\phi) = p^{-\frac{3}{2}} H_\phi \quad ; \quad P := \frac{1}{2}\dot{\phi}^2 - V(\phi) = -p^{-\frac{3}{2}}\left(\frac{2p}{3}\frac{\partial H_\phi}{\partial p}\right) . \tag{6.3.3}$$

It may be noted here that the definitions of the energy density and the pressure (6.3.3) in terms of the scalar matter Hamiltonian immediately ensure the matter conservation equation $\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)(\rho + P)$ along the classical trajectories.

6.3.2 Classical Equation of State

In the cosmological context, the equation of state parameter is defined as the ratio of the pressure component to the energy density as

$$w \ := \ \frac{P}{\rho} \ = \ \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{6.3.4}$$

For a minimally coupled scalar field, the values of the equation of state parameter (6.3.4) are restricted to be $|w| \leq 1$, as the scalar field ϕ takes values in the real line and the potential is required to satisfy $V(\phi) \geq 0$. In other words, the dynamics of a minimally coupled scalar field always respects the dominant energy condition. Thus, the Hawking-Ellis conservation theorem vouches for the stability of the ground state. The stability of the ground state can also be understood from the property of the scalar matter Hamiltonian (6.3.2). It is easy to see that the expression of the scalar matter Hamiltonian (6.3.2) ensures that it remains bounded from below. This property immediately implies a classically stable ground state for the system.

6.3.3 Phantom Matter Equation of State

As we have mentioned, in the phantom matter model of dark energy energy [81], one consider a minimally coupled scalar field but with relatively *negative* kinetic term. Thus, the energy density and the pressure component for the phantom field are given by

$$\rho_{\rm Phantom} := -\frac{1}{2}\dot{\phi}^2 + V(\phi) \ ; \ \ P_{\rm Phantom} := -\frac{1}{2}\dot{\phi}^2 - V(\phi) \ . \qquad (6.3.5)$$

Clearly, the equation of state for the phantom field $\omega_{\rm Phantom}$ $(:= P_{\rm Phantom}/\rho_{\rm Phantom})$ can take value less than -1. In other words, the phantom matter field violates the dominant energy condition. Naturally, the Hawking-Ellis conservation theorem does not guarantee for the stability of the ground state. In particular, using the corresponding Hamiltonian for the phantom matter, one can easily see that it no longer remains bounded from below. The unbounded (from below) Hamiltonian immediately implies that there does not exist a classically stable ground state for the system.

6.4 Effective Scalar Matter Hamiltonian

In isotropic loop quantum cosmology, the basic phase space variables are Ashtekar connection and densitized triad. The geometrical property of the space is encoded in the densitized triad p whereas the time variation of geometry is encoded in the connection. In loop quantum cosmology one redefines densitized triad to absorb the fiducial coordinate volume component. This makes the proper volume of the universe (1.1.1) to be $\int d^3x\sqrt{-g} = a^3V_0 = p^{\frac{3}{2}}$ [17]. The effective scalar matter Hamiltonian for the classical system whose dynamics is governed by the Hamiltonian (6.3.2), is given by [31]

$$H_{\phi}^{\rm eff} = \frac{1}{2}|\tilde{F}_{j,l}(p)|^{\frac{3}{2}}{p_{\phi}}^2 \ + \ p^{\frac{3}{2}}V(\phi) \ , \qquad (6.4.1)$$

where $\tilde{F}_{j,l}(p)$ is the eigenvalue of the inverse densitized triad operator $\widehat{p^{-1}}$ and is given by $\tilde{F}_{j,l}(p) = (p_j)^{-1}F_l(p/p_j)$ where $p_j = \frac{1}{3}\gamma\mu_0 j l_p^2$. The μ_0 is an order of unity parameter that appears while quantizing the Hamiltonian constraint operator in loop quantum cosmology [17]. The j and l are two quantization ambiguity parameters [47, 48]. The half integer j is related with the dimension of representation while writing holonomy as multiplicative operator. The real valued l ($0 < l < 1$) corresponds to different, classically equivalent ways of writing the inverse power of the densitized triad in terms of Poisson bracket of the basic variables. The function $F_l(q)$ is approximated as [30]

$$\begin{aligned} F_l(q) &= \Big[\frac{3}{2(l+2)(l+1)l}\,\big(\,(l+1)\left\{(q+1)^{l+2} - |q-1|^{l+2}\right\} - \\ &\qquad (l+2)q\left\{(q+1)^{l+1} - \operatorname{sgn}(q-1)|q-1|^{l+1}\right\}\,\big)\,\Big]^{\frac{1}{1-l}} \\ &\to q^{-1} \qquad\qquad (q \gg 1) \\ &\to \left[\frac{3q}{l+1}\right]^{\frac{1}{1-l}} \qquad (0 < q \ll 1)\,. \end{aligned} \tag{6.4.2}$$

It is clear from the expression (6.4.2) that for the large values of the densitized triad *i.e.* in large volume one recovers the expected classical behaviour for the inverse densitized triad. The quantum behaviour is manifested for smaller values of the densitized triad. Here the meaning of large or small values of the triad p is determined necessarily by the values of p_j. We will follow this convention throughout the chapter unless explicitly stated.

6.4.1 Effective Energy Density and Pressure

In this chapter, we are interested in studying the effects on the *energy conditions* due to the non-perturbative modification coming from loop quantum cosmology and its further implications. In the cosmological context, the energy conditions are stated in terms of the energy density and its relation to the pressure *i.e.* the equation of state parameter. In loop quantum cosmology, one obtains non-perturbative modification at the level of the effective Hamiltonian but *not* at the level of an effective action. This prevents one to directly obtain the expression of the *effective* stress-energy tensor. On the other hand, in classical general relativity the energy conditions are defined in terms of the stress-energy tensor. Naturally, the issue of energy conditions violation in the effective dynamics, is crucially related to the definitions of the effective energy density and pressure. In the classical situation we have seen that it is possible to write down the reduced standard expressions of the energy density and the pressure (6.3.3) purely in terms of the reduced Hamiltonian. These definitions of the energy density and the pressure immediately ensure the matter conservation equation along the classical trajectories. Naturally, one can use the same definitions for the effective energy density and the pressure just replacing the standard Hamiltonian in terms of the effective Hamiltonian. So we define the *effective* energy density and the *effective* pressure, following the definitions of classical energy density and pressure (6.3.3), as

$$\rho^{\mathrm{eff}} := p^{-\frac{3}{2}} H_\phi^{\mathrm{eff}} \ ; \ \ P^{\mathrm{eff}} := -p^{-\frac{3}{2}} \left(\frac{2p}{3} \frac{\partial H_\phi^{\mathrm{eff}}}{\partial p} \right) . \tag{6.4.3}$$

It is worth pointing out that to define the effective energy density and the pressure, one could have proceeded as done in [31]. In this approach one first obtains the Hamilton's equations of motion for the matter degrees of freedom as well as the gravitational degrees of freedom. Then one rewrites these equations of motion, by suitable manipulations such that a part of these equations matches with the gravitational part of the standard Friedman equation and the Raychaudhuri equation. In the next step, one then reads off the expressions for effective energy density and the pressure by comparing with standard equations. These expressions of the energy density and the pressure agree with the definitions (6.4.3) when the contributions due to the non-perturbative modification of the *gravity sector* become negligible. Since the effective Hamiltonian description is strictly valid in the region where background geometry is essentially classical *i.e.* non-perturbative modification of geometry is negligible. Clearly, in such situation these two set of definitions agree with each other. It is important to emphasize here that although the non-perturbative modification of the *gravity sector* becomes negligible in the region of interest but the non-perturbative modification of the *matter sector* can still survive. In fact we are interested in studying the effects of non-perturbative modification of the scalar matter dynamics.

6.4.2 Effective Equation of State

Having known the expressions of the effective energy density and the pressure (6.4.3), one can easily define the effective equation of state parameter $\omega^{\text{eff}} := P^{\text{eff}}/\rho^{\text{eff}}$. The evolution of the effective equation of state parameter depends on the effective Hamiltonian. However, as shown in [35], one can eliminate the explicit appearance of the effective Hamiltonian and can express the effective equation of state parameter in terms of the classical equation of state parameter ω, as

$$\omega^{\text{eff}} = -1 + \frac{(1+\omega)p^{\frac{3}{2}}[\tilde{F}_{j,l}(p)]^{\frac{3}{2}}\left(1 - \frac{p}{\tilde{F}_{j,l}(p)}\frac{d\tilde{F}_{j,l}(p)}{dp}\right)}{(1+\omega)p^{\frac{3}{2}}[\tilde{F}_{j,l}(p)]^{\frac{3}{2}} + (1-\omega)} . \tag{6.4.4}$$

Using the expression (6.4.2), it is easy to see that for the large values of the densitized triad p, where one expects the quantum effects to be small, $\omega^{\text{eff}} \simeq \omega$. On the other hand, for small values of p, ω^{eff} differs from the classical ω dramatically. Using the small volume (small triad) expression of the inverse densitized triad (6.4.2), one may note that the effective equation of state satisfies $(\omega^{\text{eff}}+1) < 0$, for all allowed values of the ambiguity parameter l. Let's recall that in terms of equation of state parameter, the *weak* energy condition requires $(\omega + 1) \geq 0$, the *strong* energy condition requires $(\omega + \frac{1}{3}) \geq 0$ and the *dominant* energy conditions requires $|\omega| \leq 1$. So it is clear that in loop quantum cosmology, the effective equation of state parameter violates all of these energy conditions due to the non-perturbative modifications.

6.4.3 Kinetic Contribution to Pressure

The allowed values for the classical equation of state (6.3.4), are restricted to be $|\omega| \leq 1$. Naturally, it is an important question to ask how is it then possible for the effective equation of state to take values less than -1, instead of the facts that in both cases one begins with a *minimally* coupled scalar field and uses the same definition for the equation of state parameter in terms of their corresponding Hamiltonian. The answer to this question lies in the fact that in effective loop quantum cosmology, although one begins with a *standard minimally* coupled scalar field but for small volume this coupling gets altered dramatically. The effective coupling remains *minimal* in a sense that it couples only through the geometrical variables but *not* through curvatures. However, it is clear that the gravity coupling to the scalar matter no longer remains *standard* minimal coupling as the spectrum of the inverse triad operator differs from the classical expression dramatically for small volume. To understand this issue better, let us have a look at the contributions due to the kinetic term to the pressure component P_{KE} for both cases

$$P_{\mathrm{KE}} = -\,p^{-\frac{3}{2}}\left[\frac{p}{3}p_{\phi}^{2}\right]\frac{\partial}{\partial p}\left[p^{-\frac{3}{2}}\right] \quad ; \quad P_{\mathrm{KE}}^{\mathrm{eff}} = -\,p^{-\frac{3}{2}}\left[\frac{p}{3}p_{\phi}^{2}\right]\frac{\partial}{\partial p}\left[|\tilde{F}_{j,l}(p)|^{\frac{3}{2}}\right] \qquad (6.4.5)$$

It is evident from the equation (6.4.5) that in the standard case, the kinetic term contributes *positive* pressure. This is what one would intuitively expect from our understanding of ordinary thermo-dynamical system. However, in the effective loop quantum cosmology, using the expression of the inverse densitized triad (6.4.2), it is easy to see that the kinetic term contributes *negative* pressure for small volume even though for large volume it contributes positive pressure like in standard case. This crucial 'extra' negative pressure from the kinetic term is what essentially leads the effective equation of state to violate dominant energy condition. Clearly, the bounded spectrum of the inverse densitized triad plays a major role in this.

On the other hand, in the phantom matter model of dark energy, to obtain the values of the equation of state parameter to be less than -1, one makes the kinetic term relatively *negative* by hand. This step essentially forces the kinetic term to contribute negative pressure. However, it also leads the kinetic term to contribute negative energy density. This step essentially jeopardise energy density expression as its *positivity* is no longer remain guaranteed. Clearly, a relatively *negative* kinetic term in the scalar matter Hamiltonian, makes it unbounded from below. In other words, the ground state of such system gets pushed to negative infinity. Naturally, naive quantization of such system can lead to a catastrophic decay of vacuum [100]. On contrary, in the effective loop quantum cosmology scenario, the kinetic term gives negative contribution only in the pressure expression but not in the energy density expression. Thus, although the equation of state parameter in effective loop quantum cosmology violates dominant energy condition but it also necessarily ensures the *positivity* of the energy density. It is also evident from the expression of the scalar Hamiltonian (6.4.1) that it remains bounded from below signifying a stable ground state.

6.4.4 Example: Massive Scalar Field

Now we take an explicit example to illustrate the dynamics of the scalar field at small volume regime where non-perturbative modification plays a significant role. For simplicity, we consider the dynamics of a massive free scalar field. In other words, the scalar potential is consist of only the mass term *i.e.* $V(\phi) = \frac{1}{2}m_\phi^2\phi^2$. To simplify further, we choose the value of the ambiguity parameter to be $l \to 0+$. With these assumptions the effective matter Hamiltonian for small volume becomes

$$H_\phi^{\rm eff} \simeq p^{\frac{3}{2}} \left[\frac{1}{2}\left(3^{\frac{3}{2}} p_j^{-3}\right) p_\phi^2 + \frac{1}{2} m_\phi^2 \phi^2\right] . \qquad (6.4.6)$$

Using the Hamilton's equations of motion, one can obtain analytical solutions for the field equations, given by

$$\phi = \sqrt{\frac{2\bar{\rho}}{m_\phi^2}} \sin\left(\alpha p^{\frac{3}{2}} + c_1\right) \; ; \; p_\phi = \sqrt{\frac{2\bar{\rho}}{(3^{\frac{3}{2}} p_j^{-3})}} \cos\left(\alpha p^{\frac{3}{2}} + c_1\right) , \qquad (6.4.7)$$

where $\alpha = \sqrt{\frac{(3^{\frac{3}{2}} p_j^{-3})(m_\phi^2)}{24\pi G\bar{\rho}}}$, $\bar{\rho}$ and c_1 are two constants of integration. Using the field solutions (6.4.7), one can easily see that along any trajectory $H_\phi^{\rm eff} \simeq p^{\frac{3}{2}}\ \bar{\rho}$. One may note here that the energy density contribution due to the scalar field dynamics effectively looks a like contribution from a cosmological constant. The constant of integration $\bar{\rho}$ physically corresponds to the energy density during its evolution. This also implies an exponential inflationary phase. This is of course expected behaviour, as the effective equation of state parameter in loop quantum cosmology generically becomes $\omega^{\rm eff} \approx -1$ at small volume [35]. This simple example clearly shows that classical dynamics of the system is essentially stable, as we have argued for a general system with the modified scalar field dynamics.

6.5 Propagation of Inhomogeneous Modes

We have mentioned earlier that the second part of the dominant energy condition requires the speed of energy propagation not to exceed the speed of light. Naturally, the violation of dominant energy condition also raises the concern, whether such system can prohibit super-luminal flow of energy. In other words, whether such system can respect causality. In classical cosmology, one begins by postulating so called *cosmological principle i.e.* on large scale *there is neither a preferred direction nor a preferred place* in our universe. This principle is imposed by assuming that on cosmological scale our universe is spatially homogeneous and isotropic. The strict imposition of spatial homogeneity will prohibit any kind of spatial flow of energy as it will violate spatial homogeneity. However, this assumption undoubtedly is an idealisation and is made to rather simplify background dynamics. Naturally, if we want to allow some kind of spatial flow of energy then we must relax the spatial homogeneity. While relaxing this assumption nevertheless one should be careful so that we can still use the available machinery of the cosmological set-up. This is generally achieved by considering the deviation from spatial homogeneity to be small. Small spatial inhomogeneity in the matter field configuration will also lead to small inhomogeneity in the background geometry. For simplicity, however, we will treat the background geometry as homogeneous.

6.5.1 Modified Klein-Gordon Equation

We have seen earlier that the kinetic term of the scalar matter Hamiltonian gets non-perturbative modification, as its classical expression involves inverse powers of densitized triad. The effective scalar matter Hamiltonian, obtained as outlined in [43], is given by

$$H_\phi^{\rm eff} = V_0 |\tilde{F}_{j,l}(p)|^{\frac{3}{2}} \int d^3x \left[\frac{1}{2}\pi_\phi^2\right] + V_0^{-\frac{1}{3}} p^{\frac{1}{2}} \int d^3x \left[\frac{1}{2}(\nabla\phi)^2\right] + V_0^{-1} p^{3/2} \int d^3x \, [V(\phi)] \, . \tag{6.5.1}$$

One may note here that we have now kept the gradient term in the effective Hamiltonian. The gradient term was neglected earlier while computing background evolution, as one assumes that the background evolution is mainly determined by the homogeneous and isotropic contribution of the matter Hamiltonian. The gradient term of the equation (6.5.1) having the correct sign, the corresponding dynamics does not suffer from the so called gradient instability [116, 117], another pathological feature of the phantom matter models. Using the Hamilton's equations of motion for the effective Hamiltonian (6.5.1), one can derive the corresponding *modified* Klein-Gordon equation

$$\ddot{\phi} ~-~ 3\left(\frac{p~\tilde{F}'_{j,l}(p)}{\tilde{F}_{j,l}(p)}\right)\left(\frac{\dot{a}}{a}\right)\dot{\phi} ~+~ |\tilde{F}_{j,l}(p)|^{\frac{3}{2}}p^{\frac{3}{2}}\left(-\frac{\nabla^2\phi}{a^2}+V'(\phi)\right) ~=~ 0~, \qquad (6.5.2)$$

where $\tilde{F}'_{j,l}(p) \equiv \frac{d\tilde{F}_{j,l}(p)}{dp}$. Using the expression for the spectrum of the inverse triad (6.4.2), it is easy to see that the modified Klein-Gordon equation (6.5.2) reduces to the standard Klein-Gordon equation at large volume.

In a given spatially flat spacetime background, an inhomogeneous scalar field can be decomposed in terms of its Fourier modes. The dynamics of the $k=0$ mode *i.e.* the spatially homogeneous mode essentially drives the evolution of the background geometry as the contributions from non-zero k modes are assumed to be small. However, as we have argued that to study the energy propagation across spatial distance in the cosmological background, it is essential to consider the dynamics of inhomogeneous modes *i.e.* non-zero k modes. The Fourier decomposition of the inhomogeneous scalar field is defined as

$$\phi(\mathbf{x},t) ~=~ \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left[\phi_k(t)~e^{i\mathbf{k}\cdot\mathbf{x}}\right]~, \qquad (6.5.3)$$

where $\phi_k(t)$ are the Fourier components. For simplicity, we will consider the dynamics of a massless free scalar *i.e.* we will assume $V(\phi)=0$. Using the modified Klein-Gordon equation (6.5.2) and the equation (6.5.3), one can derive the modified equation for the Fourier modes

$$\ddot{\phi}_k(t) ~-~ 3\left(\frac{p~\tilde{F}'_{j,l}(p)}{\tilde{F}_{j,l}(p)}\right)\left(\frac{\dot{a}}{a}\right)\dot{\phi}_k(t) ~+~ |\tilde{F}_{j,l}(p)|^{\frac{3}{2}}p^{\frac{3}{2}}\left(\frac{k^2}{a^2}\right)\phi_k(t) ~=~ 0~. \qquad (6.5.4)$$

In the small volume regime where the spectrum of the inverse triad operator can be approximated as $\tilde{F}_{j,l}(p) \sim p^{\frac{1}{1-l}}$ and the effective equation of state parameter as

$w^{\rm eff} \approx -1$, one can obtain an analytical solution for the equation (6.5.4) [43], given by

$$\phi_k(t) \;=\; \eta^{(1+\frac{1}{2n})}\left[A_{(k,n)}J_{-(1+\frac{1}{2n})}(k\eta) + B_{(k,n)}J_{(1+\frac{1}{2n})}(k\eta)\right] \,, \tag{6.5.5}$$

where $J_n(x)$ are the Bessel functions, $A_{(k,n)}$ and $B_{(k,n)}$ are two constants of integration corresponding to *second* order differential equation. The variable η is defined as $d\eta := a^{-n}dt$, where the parameter $n = -\frac{1}{2}(1+\frac{3}{1-l})$. In loop quantum cosmology allowed values for the ambiguity parameter l is $(0 < l < 1)$. Naturally, the new parameter n takes values as $(-\infty < n < -2)$. The argument of the Bessel function $k\eta$ can be conveniently expressed in terms the scale factor as $k\eta = \frac{k}{Ha}\left(\frac{a^{1-n}}{-n}\right)$, where $H(\equiv \frac{\dot{a}}{a})$ is the Hubble parameter.

At first let's study the *large* wavelength $(k \to 0)$ behaviour of the general solution (6.5.5). For the general solution (6.5.5) when both constants of integration $A_{(k,n)}$ and $B_{(k,n)}$ are present then using the asymptotic form of the Bessel function $J_m(x) \approx \frac{1}{\Gamma(1+m)}(\frac{x}{2})^m$ for $x << 1$, it is easy to see that $\phi_k(t)$ becomes approximately constant and becomes proportional to $A_{(k,n)}$. For the special case when $A_{(k,n)}$ is identically zero then $\phi_k(t)$ remains time dependent but its time dependence is *non-oscillatory*. These features of the Fourier modes $\phi_k(t)$ can also be seen directly from the differential equation (6.5.4). For the larger wavelength modes the third term in the equation (6.5.4) can be neglected. The approximated second order differential equation then admits a *constant* solution and a *non-oscillatory* time-dependent solution, as expected. Clearly, the second term which is a (anti)friction term, plays a major role for the larger wavelength modes. Since, our main interest is to study the energy transmission across spatial distances then clearly the larger wavelength modes are not relevant for this purpose. On the other hand, for smaller wavelength $(k \to \infty)$ modes, the general solution become *oscillatory*, as the asymptotic form of the Bessel function is $J_m(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$ for $x >> 1$. Naturally, the smaller wavelength modes are the potential *carriers* for the energy transmission across spatial distances. For smaller wavelength modes, the effects of the (anti)friction term is negligible. Thus, for simplicity we will neglect the (anti)friction term in the equation (6.5.4) for further analysis. The information regarding assumed small inhomogeneity are encoded in the amplitudes of the mode functions $\phi_k(t)$. Since propagation speed of linear waves does not depend on their amplitudes, the causal properties of the propagating inhomogeneous modes are quite insensitive to the exact details of their amplitudes.

6.5.2 Modified Dispersion Relation

In the cosmological context, any spatial transmission of energy will introduce inhomogeneity. So to investigate causality of the system, it is natural to study the *group velocity* for the inhomogeneous modes. One may recall that in a medium where absorption (friction) or emission (anti-friction) is small, the *group velocity* essentially determine the speed of signal propagation [118]. To compute the group velocity it is convenient to find out the relation between its frequency and wave-number *i.e.* the *dispersion relation*. Using the governing equation for the inhomogeneous modes (6.5.4), neglecting the (anti)friction term, and making the ansatz $\phi_k(t) \sim e^{i\tilde{\omega}t}$, one can easily derive the *modified* dispersion relation in effective loop quantum cosmology as

$$\tilde{\omega}^2 \approx |\tilde{F}_{j,l}(p)|^{\frac{3}{2}} p^{\frac{3}{2}} \left(\frac{k^2}{a^2}\right) . \tag{6.5.6}$$

In the classical situation 'inverse triad' is just the inverse of triad *i.e* $p \times \tilde{F}_{j,l}(p) = 1$. The dispersion relation (6.5.6) then becomes same as the standard Minkowskian dispersion relation between frequency and *physical* wave number (k/a). In loop quantum cosmology the spectrum of the inverse triad operator is bounded. Hence the dispersion relation in effective loop quantum cosmology differs dramatically for small volume compared to the standard dispersion relation.

The modification in the dispersion relation that is being studied here, arises because of the bounded spectrum of the inverse triad. This modification is distinct from the different types of modification generally considered in the literature. For example, in the context of quantum gravity scenario [119, 120, 121, 122, 123] or in the context of trans-Planckian inflationary scenario [63, 64], one considers modification of standard dispersion relation by introducing appropriate *non-linearity*.

6.5.3 Group Velocity

The group velocity determines the speed of signal propagation only if the absorption or amplification of the signal remains small. In other words, 'signal transmission' makes sense only if the original signal reaches its target without major *distortion* while propagation (see [118] for related discussion). In the effective loop quantum cosmology scenario, we have argued that the relevant modes for energy transmission across spatial distances, are the smaller wavelength modes and for these modes the (anti)friction term plays very little role in their evolution. Using the dispersion relation (6.5.6), one can easily compute the *group velocity* for the inhomogeneous modes as

$$v_g := \frac{d\tilde{\omega}}{d(k/a)} = |\tilde{F}_{j,l}(p)|^{\frac{3}{4}} p^{\frac{3}{4}} \ . \qquad (6.5.7)$$

In the classical situation *right hand side* of the expression (6.5.7) is identically equal to unity. Physically, this implies that for the massless free scalar field, the inhomogeneous modes transmit signals at the speed of light. However, in the effective loop quantum cosmology it is no longer the case. Using the expression for the spectrum of the inverse triad (6.4.2), it is easy to see that in the small volume regime, the speed of signal propagation is in fact much *slower* than the speed of light (in classical vacuum). The group velocity for the inhomogeneous modes gradually increases and approaches the speed of light towards the end of the non-perturbatively modified dynamics.

It should be emphasized here that the actual spectrum of the inverse scale factor operator is fundamentally non-differentiable. However, to study the qualitative consequences of it within an effective analysis, one uses a peace-wise analytic function $F_l(q)$ (6.4.2) which approximates the spectrum of the inverse scale factor operator. This is a good approximation provided the scale p_j is sufficiently large. However, being peace-wise analytic this approximation is good as long as ($p << p_j$) or ($p >> p_j$) but not near the transition regime, as the approximation function $F_l(q)$ (6.4.2) is not analytic at $q = 1$ ($q = p/p_j$). So the governing equation of the mode functions (6.5.4) which involves $F_l(q)$ as well as its derivative, is not defined near $p = p_j$. Thus, the derivation and the subsequent expression of the group velocity (6.5.7) are valid as long as ($p << p_j$) or ($p >> p_j$) but not in the neighbouring regime of $p = p_j$. However, there still exist a significant small volume regime even excluding the regime near $p = p_j$, as the validity of approximation for the spectrum of the inverse scale factor operator, requires p_j to be large.

Thus, in effective loop quantum cosmology although non-perturbatively modified dynamics violates dominant energy condition in terms of the equation of state parameter but the underlying modified dynamics restricts the group velocity for the inhomogeneous modes to remain *sub-luminal*. In the cosmological context we have argued that any spatial transmission of energy will introduce spatial inhomogeneity. Here in the effective loop quantum cosmology, we have shown that the group velocity for the inhomogeneous modes remains sub-luminal due to the non-perturbative modification. Clearly, in effective loop quantum cosmology, non-perturbatively modified dynamics of a minimally coupled scalar field respects *causality*. The violation of dominant energy condition is essentially dictated by the $k = 0$ mode but this mode is not relevant for the purpose of signal transmission across spatial distances.

The 'speed of light' in this context is meant to imply the speed of electromagnetic wave propagation in the classical vacuum that determines the causal structure of the spacetime. This is important to emphasize because in loop quantum cosmology, one expects to get similar non-perturbative modification even to the electromagnetic wave propagation. Then the actual speed of light in the effective loop quantum cosmology itself may become slower compared to the speed of light in the classical vacuum. Intuitively, one may consider the small volume *effective* background geometry, coming from loop quantum cosmology, as a *refractive* medium with a value of the *group index* $n_g(\equiv c/v_g)$ is greater than unity. In this context, the group index is same as the *refractive index*, as the *phase velocity* is same as the group velocity.

6.6 Quantum Corrections To Group Velocity at Large Volume

Using the spectrum of the inverse triad operator (6.4.2), it is easy to see that although at large volume the leading term is just the inverse of triad but there are sub-leading terms also in its expression. Naturally, in the effective loop quantum cosmology, the group velocity for the inhomogeneous modes is not exactly equal to unity even at the large volume. Using the expression (6.4.2), for a massless free scalar field, one can compute the group velocity with quantum corrections as

$$v_g \simeq \left[1 + \frac{3(2-l)}{40}\left(\frac{p_j^2}{p^2}\right)\right] . \tag{6.6.1}$$

It is clear from the expression (6.6.1) that the corrections to the group velocity at large volume is *extremely* small but positive as ($0 < l < 1$). The group velocity becomes equal to unity as the volume of the system goes to infinity. To have some numerical estimate of this finite volume quantum correction, let us choose say $p_j \sim 10^5\ l_p^2$. The observed size of universe today is $\sqrt{p} \sim 10^{60}\ l_p$. Then the correction to the group velocity due to modified spectrum of the inverse triad operator, today is $\sim 10^{-231}$! It is extremely unlikely that such small correction will have any significant effect. Even for the cosmological context (time scale $\sim 10^{17}$ sec) such small deviation of group velocity, may be completely irrelevant.

6.7 Discussions

To summarize, in effective loop quantum cosmology, non-perturbatively modified dynamics of a minimally coupled scalar field violates *weak*, *strong* and *dominant* energy conditions when they are stated in terms of equation of state parameter. The violation of strong energy condition although helps to have non-singular evolution by evading the singularity theorems but the violation of weak and dominant energy conditions raises concern. In classical general relativity, these energy conditions are used to prohibit super-luminal flow of energy and to ensure the stability of classical vacuum via the Hawking-Ellis conservation theorem. Naturally, the violation of these energy conditions in terms of *effective* equation of state parameter, raises concern about the causality and the stability of the system. In this chapter, we have shown that although at face value these energy conditions are violated, underlying modified dynamics in effective loop quantum cosmology nevertheless ensures positivity of energy density, as scalar matter Hamiltonian remains bounded from below. Considering the modified dynamics for the inhomogeneous modes, we have shown that group velocity for the relevant modes remains sub-luminal in small volume regime, thus ensuring causal propagation across spatial distances. We have also computed the large volume quantum corrections to the group velocity of the inhomogeneous modes for the massless free scalar field.

Now, let us try to understand the physical phenomena behind this rather unusual feature of the non-perturbatively modified dynamics. In the case of classical dynamics of a minimally coupled scalar field, the values of the equation of state parameter are restricted to be $|\omega| \leq 1$. However, in the case of modified dynamics, the effective equation of state can take values less than -1. This is rather surprising given the facts that one begins with a minimally coupled scalar field and uses the same definition of equation of state for both the cases. This 'anomalous' behaviour follows from the fact that at the small volume, non-perturbatively modified gravity becomes *repulsive* although it remains *attractive* for the large volume. This feature can be easily seen by considering a classical trajectory of a massless free scalar field. The non-perturbatively modified scalar matter Hamiltonian, along any trajectory, increases with the increasing scale factor for small volume but decrease for large volume. Naturally, the gravitational Hamiltonian, to satisfy the Hamiltonian constraint ($H_\phi + H_{grav} = 0$), must decrease with increasing scale factor for small volume. Later, in the large volume it starts increasing with increasing scale factor. This immediately implies that modified gravitational interaction is repulsive for small volume whereas for large volume, as one expects, it is attractive. This repulsive nature of the gravitational interaction manifest itself through the non-standard gravity coupling to the scalar matter Hamiltonian via bounded spectrum of the inverse triad operator.

For a standard minimally coupled scalar field, the kinetic term contributes positive pressure. Of course, this is what one would intuitively expect from our understanding of ordinary thermo-dynamical system. However, in the effective loop quantum cosmology, the kinetic term contributes negative pressure for small volume even though for large volume it contributes positive pressure like in standard case. This crucial 'extra' negative pressure from the kinetic term is what essentially leads the effective equation of state to violate dominant energy condition. Clearly, the bounded spectrum of the inverse densitized triad plays a major role in this. On the other hand, in phantom matter model of dark energy, to obtain the values of the equation of state parameter to be less than -1, one makes the kinetic term relatively negative by hand. This change of sign essentially forces the kinetic term to contribute negative pressure. However, it also leads the kinetic term to contribute negative energy density. This step badly affects the energy density expression, as its positivity is no longer certain. In other words, a relatively negative kinetic term in the scalar matter Hamiltonian, makes it unbounded from below. This implies that the system does not have a stable classical ground state. On contrary, in the effective loop quantum cosmology scenario, the kinetic term gives negative contribution only in the pressure expression but not in the energy density expression. Thus, although the equation of state parameter in effective loop quantum cosmology violates dominant energy condition but it necessarily ensures the positivity of the energy density. It is also evident from the expression of the scalar Hamiltonian (6.4.1) that it remains bounded from below, signifying a stable classical ground state.

The bounded spectrum of the inverse triad (scale factor) operator plays the central role in violating the energy conditions. The violation of energy conditions although leads to a generic inflationary phase and allows to have a non-singular evolution but it also makes the causality and the stability of the system uncertain. However, as shown in this chapter, the same bounded spectrum in fact acts as a *saviour* to ensure the causality and the stability of the system. It should be emphasized here that the quantization of the inverse triad was *not* invented to obtain the bounded spectrum such that these physical features follow. Rather it was quantized following the techniques used in the full theory of loop quantum gravity. The quantization of the inverse triad involves ambiguities but these crucial features are insensitive to their precise values. It may be worth emphasising that although the exercise presented here is not directly related with the dark energy scenario, one may learn an important lesson from here that if one wants to construct a dominant energy condition violating yet well behaved scalar field model of dark energy then one should look beyond the standard minimal coupling.

It is now important to discuss some subtleties of the analysis presented here. In classical general relativity, the definitions of the energy conditions are generally covariant. However, in the cosmological context, the energy conditions are stated with respect to a preferred frame namely the so-called *comoving frame*. Thus, one must be careful while interpreting the results in more general context. Secondly, in the Lagrangian formulation one obtains the reduced expression of the energy density and pressure for the homogeneous and isotropic spacetime, using a generally covariant expression of the stress-energy tensor. In the Hamiltonian formulation such a spacetime covariant method is not available. Naturally, one needs to define the expression of energy density and pressure, in terms of the scalar matter Hamiltonian. In the classical situation although they are equivalent but with the non-trivial quantum corrections this issue is rather subtle. In the analysis presented here, we have assumed the background geometry as homogeneous although we have allowed the scalar field living in it to become inhomogeneous. This approximation is trustworthy as long as the deviation from the homogeneity remains sufficiently small. Further, we have considered the non-perturbative modification of the kinetic term only. Using slightly different quantization strategy, one could obtain a factor of 'triad times inverse triad' also in the gradient term. However, such modification would change only the *quantitative* nature of the results shown here but not the *qualitative* nature. Naturally, the features of the non-perturbatively modified dynamics shown here, are robust under this quantization ambiguity.

Chapter 7

Discussions

7.1 Summary

In this thesis we have demonstrated that it is possible to infer the consequences of the dynamics implied by loop quantum cosmology in a more familiar classical spacetime picture in terms of an *effective* Hamiltonian. It is derived for a class of solutions of the fundamental difference equation of isotropic loop quantum cosmology, using WKB techniques. The effective Hamiltonian incorporates important non-perturbative modification coming from loop quantum cosmology. The effective dynamics approximates the classical dynamics for large volumes.

The non-perturbative modification coming from loop quantum cosmology to the scalar matter sector is known to imply inflation. We further prove that loop quantum cosmology modified scalar field generates near exponential inflation in small scale factor regime, for all positive definite potentials, independent of initial conditions and independent of ambiguity parameters i.e. inflation is generic in loop quantum cosmology. Genericness of inflation also means that, in small scale factor regime, non-perturbatively modified scalar matter dynamics leads to a generic violation of strong energy condition. While violation of strong energy condition helps to bypass the singularity theorems but that does not necessarily *imply* a non-singular evolution. Nonetheless, we showed that the absence of isotropic singularity in loop quantum cosmology can be understood in the effective classical description as the universe exhibiting a big bounce and this is also generic. In particular, we show that with scalar matter field the big bounce is generic in the sense that it is independent of quantization ambiguities and details of scalar field dynamics.

In the context of inflationary scenario, it is widely believed that quantum field fluctuations in an inflating background create the primordial seed perturbations which through subsequent evolution lead to the observed large scale structures of the universe. In particular, quantum field fluctuations in the inflating background quite generically produce density perturbations with a scale-invariant power spectrum [37] which is consistent with current observations. Using similar techniques in the context of effective framework of loop quantum cosmology, it is shown that loop quantum cosmology induced inflationary scenario not only can produce scale invariant power spectrum but also small amplitude for the primordial density perturbations without any fine tuning. Further its power spectrum has a qualitatively distinct feature which is in principle falsifiable by observation and can distinguish it from the standard inflationary scenario.

In the effective framework of loop quantum cosmology, non-perturbatively modified dynamics of a minimally coupled scalar field violates weak, strong and dominant energy conditions *when they are stated in terms of equation of state parameter*. While the violation of strong energy condition is desirable to permit a non-singular evolution, violation of weak and dominant energy conditions raises concern about the causality and stability of the effective model, since in general relativity precisely these conditions ensure causality of the system and stability of vacuum via Hawking-Ellis conservation theorem. We show that the kinetic term due to the non-perturbative modification, contributes negative pressure although it contributes positive energy density. This crucial feature leads to the violation of energy conditions in the effective loop quantum cosmology. In other words, although the non-perturbatively modified dynamics leads to violation of energy conditions but it still ensures positivity of energy density, as scalar matter Hamiltonian remains bounded from below. Further, considering small inhomogeneities around the homogeneous background, it is shown that the modified dynamics restricts group velocity for inhomogeneous modes to remain sub-luminal thus ensuring causal propagation across spatial distances.

7.2 Anisotropic models

Use of highly symmetric cosmological (homogeneous and isotropic) space in loop quantum cosmology makes the system much simpler to study and it allows explicit calculations to be carried out. This is certainly an attractive aspect of the symmetric system. However, such symmetry assumption ignores some crucial feature of our universe that it has inhomogeneities even on the large scale. Inhomogeneities on the large scale average are very small, nevertheless, their presence is qualitatively significant. Naturally, to make a more realistic model of our universe one must consider the inclusion of inhomogeneities at the fundamental level in loop quantum cosmology. While dealing with fully inhomogeneous system is very much an open issue at present, the techniques of loop quantum gravity has been applied to less symmetric homogeneous but *anisotropic diagonal* models. The basic conclusion about non-singular nature of loop quantum cosmology continues to hold for the these models as well.

As mentioned earlier that according to the singularity theorems, the space-time describing the backward evolution of an expanding universe is necessarily singular in the sense of geodesic incompleteness. While singularity theorems ensure the generality, it does not shade light on the specific nature of such a singularity for example in terms of curvature invariants. On the other hand, the so-called BKL (Belinskii-Khalatnikov-Lifshitz) approach to the issue of singularities asks [124]: In a neighbourhood of a presumed singularity, is there *a general* solution of the Einstein equation such that at least some curvature invariants diverge? In this formulation of the question, one also obtains information about the possible nature of the presumed singularity described in terms of the *approach to the singularity*. The conclusion of the long and detailed analysis is summarized in the BKL scenario: Generically, as the singularity is approached, the spatial geometry can be viewed as a collection of small patches each of which evolves essentially independently of the others according to the Bianchi IX evolution (for recent numerical evidence see [125]). The Bianchi IX evolution towards its singularity is described by an infinite succession of Kasner evolutions (two directions contracting while third one expanding) punctuated by permutations of the expanding/contracting directions as well as possible rotations of the three directions themselves. The qualitative analysis of these permutations and rotations ('oscillations') is done very conveniently in terms of a billiard ball bouncing off moving walls and has been analyzed for a possible *chaotic* behaviour [126]. It may be recalled that the Kasner solution is the solution of Einstein's equation for vacuum Bianchi-I model.

However, it has been well recognised that the conclusion of ad infinitum oscillations of the BKL singularity, a consequence of unbounded growth of the spatial curvature, cannot be trusted very close to the singularity where the classical Einstein equations themselves are expected to break down. Presumably the equations will be superseded by some quantum extensions. In the absence of any specific and detailed enough quantum theory the questions of the fate of the classical singularity vis a vis the behaviour of space-time near such a region, could not have been addressed. The focus therefore has been to include qualitatively expected quantum modification. For example, in the Kaluza-Klein picture and more recently the stringy picture, the minimal expected modification is the matter content – notably the dilaton and p-form fields. Higher derivative corrections in the effective action are also expected. Another qualitative implication of string theory namely the brane world scenarios is also a candidate to study implications of modified Einstein equations. All of these have been explored with varying conclusions regarding the BKL behaviour, e.g. [127, 128]. However, they *still contain the singularity* and hence all these modified equations must break down near the singularity raising questions about the validity of the conclusion.

Naturally, it is meaningful to ask if and how the BKL behaviour is modified when modifications coming from loop quantum cosmology are included. There is in fact a hint of what may be expected. For larger volumes, one can trust the classical picture.

Then, in the general inhomogeneous cosmological context, one can approximate smaller patches of the spatial geometry by the homogeneous Bianchi IX models. As the volume is decreased, the patches have to be made smaller to sustain the approximation. If the BKL behaviour were to continue for these individual patches, then the fragmentation into smaller patches must continue ad infinitum. But the underlying discrete structure cannot support such infinite fragmentation. Therefore, the quantum geometry which is responsible for singularity removal, must also ensure that Bianchi IX behaviour may at the most have finitely many oscillations. This is in fact a self consistency requirement on the procedures of loop quantum cosmology. It is shown through a detailed proof [30] that modifications coming from loop quantum cosmology lead to a non-chaotic effective behaviour [61]. On the other hand, it has been shown that the non-singular nature of effective dynamics of loop quantum cosmology holds also for Bianchi-I model [130, 129].

7.3 Open issues and future outlook

In ordinary quantum mechanics, generally one obtains physically relevant quantities in terms of physical expectation values of appropriate physical observables of a given system. We have mentioned earlier that loop quantum cosmology is a minisuperspace quantization *i.e.* it is essentially a quantum mechanical description. However, in loop quantum cosmology the issue of physical observables and physical inner product are still in early stages of development. In fact this was the main motivation to look for an alternative route to explore the consequences of the quantum modifications. This exploration using WKB methods has lead to the results which form the main contents of the thesis.

However, very recently [29], a significant progress has been made towards constructing physical observables and physical inner product in loop quantum cosmology. In this construction, one considers a free massless scalar field as matter degree of freedom. The classical evolution of such scalar field is monotonic with respect to the coordinate time. This motivates one to consider the scalar field itself as an *internal* time. With the scalar field taken as 'clock' variable, the difference equation of loop quantum cosmology can also be viewed as a Klein-Gordon equation in a static spacetime. This in turns allows the standard procedure to be carried out, for constructing appropriate physical observables and physical inner product for the system. Then one considers the evolution of a given semi-classical states under the dynamics of loop quantum cosmology. Using this method, it is shown that the semi-classical states exhibit its non-singular nature via a bounce in the small volume regime. One of the key results of this thesis is that the effective dynamics of loop quantum cosmology exhibits bounce in the small volume regime. The methods and details used in the recent approach differ fundamentally from the approach presented in the thesis. However the qualitative nature of results regarding the way non-singular nature is realized in loop quantum cosmology, agrees very much with each other.

In small volume regime, the modifications coming from loop quantum cosmology that play leading roles, are mainly *non-perturbative* in nature. This leads the effective dynamics to differ significantly from the classical dynamics in small volume regime. On the other hand, in large volume regime the effective dynamics approximates classical dynamics very well. Nevertheless, there are quantum corrections to it and it turns out these are *perturbative* in nature. It was noted [131] that one of these perturbative quantum corrections has striking resemblances with gravitational Casimir energy computed using techniques of perturbative quantum gravity. The perturbative approach of quantum gravity is known to lead to a *non-renormalizable* quantum field theory. So further explorations of this issue might shed some light on the issue of *renormalizibility* in the context of both perturbative and non-perturbative approaches of quantum gravity. Currently, further investigations in this direction is being carried out.

In the thesis, small inhomogeneities have been introduced around the effective background and their properties have been studied. However, the analysis presented is an effective one in nature. So it is quite reasonable to expect that the effective analysis may not capture all features of quantum dynamics specially the quantitative aspects. Naturally, one would like to carry out an analysis that includes inhomogeneities at the fundamental level in loop quantum cosmology. While there has been some restricted attempt [132] towards this direction full fledged analysis is still lacking. It would be quite interesting to explore the consequences of the inclusion of inhomogeneities at the fundamental level in loop quantum cosmology. Finally, we conclude this thesis with the quote; *"It is easier to perceive error than to find truth, for the former lies on the surface and is easily seen, while the latter lies in the depth, where few are willing to search for it."* – Johann Wolfgang von Goethe.

Bibliography

[1] Robert M. Wald, "General Relativity", University of Chicago Press, 1984.

[2] Guth A H, Phys. Rev. D **23**, 347 (1981).

[3] Linde A D, *Particle Physics and Inflationary Cosmology*, Harwood Academic (1990); Olive K A, Phys. Rep. **190**, 307 (1990); Turok N, Class. Quantum Grav. **19**, 3449 (2002).

[4] Green, M. B., Schwarz, J. H. and Witten, E., Superstring theory, vol. 1 and 2, Cambridge Univ. Press, (1987); Polchinski, J., String theory, vol. 1 and 2, Cambridge Univ. Press (1998).

[5] C. Rovelli, "Loop quantum gravity," Living Rev. Rel. **1**, 1 (1998) [arXiv:gr-qc/9710008].

[6] T. Thiemann, "Introduction to modern canonical quantum general relativity," [arXiv:gr-qc/0110034].

[7] T. Thiemann, "Lectures on loop quantum gravity," Lect. Notes Phys. **631**, 41 (2003) [arXiv:gr-qc/0210094].

[8] A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A status report," Class. Quant. Grav. **21**, R53 (2004) [arXiv:gr-qc/0404018].

[9] L. Smolin, "An invitation to loop quantum gravity," arXiv:hep-th/0408048.

[10] M. Bojowald, "Loop quantum cosmology. I: Kinematics," Class. Quant. Grav. **17**, 1489 (2000) [arXiv:gr-qc/9910103].

[11] M. Bojowald, "Loop quantum cosmology. II: Volume operators," Class. Quant. Grav. **17**, 1509 (2000) [arXiv:gr-qc/9910104].

[12] M. Bojowald, "Loop quantum cosmology. III: Wheeler-DeWitt operators," Class. Quant. Grav. **18**, 1055 (2001) [arXiv:gr-qc/0008052].

[13] M. Bojowald, "Loop quantum cosmology. IV: Discrete time evolution," Class. Quant. Grav. **18**, 1071 (2001) [arXiv:gr-qc/0008053].

[14] M. Bojowald, "Isotropic loop quantum cosmology," Class. Quant. Grav. **19**, 2717 (2002) [arXiv:gr-qc/0202077].

[15] M. Bojowald, "The semiclassical limit of loop quantum cosmology," Class. Quant. Grav. **18**, L109 (2001) [arXiv:gr-qc/0105113].

[16] M. Bojowald and H. A. Morales-Tecotl, "Cosmological applications of loop quantum gravity," Lect. Notes Phys. **646**, 421 (2004) [arXiv:gr-qc/0306008].

[17] A. Ashtekar, M. Bojowald and J. Lewandowski, "Mathematical structure of loop quantum cosmology," Adv. Theor. Math. Phys. **7**, 233 (2003) [arXiv:gr-qc/0304074].

[18] M. Bojowald, "The early universe in loop quantum cosmology," J. Phys. Conf. Ser. **24**, 77 (2005) [arXiv:gr-qc/0503020].

[19] M. Bojowald, "Loop quantum cosmology," Living Rev. Rel. **8**, 11 (2005) [arXiv:gr-qc/0601085].

[20] R. Arnowitt, S. Deser and C. W. Misner, "The Dynamics Of General Relativity," in *Gravitation: An introduction to current research*, ed. L. Witten (Wiley, New York, 1962). Also available as [arXiv:gr-qc/0405109].

[21] K. V. Kuchar and M. P. Ryan, "Is Minisuperspace Quantization Valid?: Taub In Mixmaster," Phys. Rev. D **40**, 3982 (1989).

[22] M. Bojowald, "Absence of singularity in loop quantum cosmology," Phys. Rev. Lett. **86**, 5227 (2001) [arXiv:gr-qc/0102069].

[23] M. Bojowald, "Homogeneous loop quantum cosmology," Class. Quant. Grav. **20**, 2595 (2003) [arXiv:gr-qc/0303073].

[24] M. Bojowald, G. Date and K. Vandersloot, "Homogeneous loop quantum cosmology: The role of the spin connection," Class. Quant. Grav. **21**, 1253 (2004) [arXiv:gr-qc/0311004].

[25] M. Bojowald, "The inverse scale factor in isotropic quantum geometry," Phys. Rev. D **64**, 084018 (2001) [arXiv:gr-qc/0105067].

[26] G. M. Hossain, "Hubble operator in isotropic loop quantum cosmology," Class. Quant. Grav. **21**, 179 (2004) [arXiv:gr-qc/0308014].

[27] M. Bojowald, P. Singh and A. Skirzewski, "Time dependence in quantum gravity," Phys. Rev. D **70**, 124022 (2004) [arXiv:gr-qc/0408094].

[28] K. Noui, A. Perez and K. Vandersloot, "On the physical Hilbert space of loop quantum cosmology," Phys. Rev. D **71**, 044025 (2005) [arXiv:gr-qc/0411039].

[29] A. Ashtekar, T. Pawlowski and P. Singh, "Quantum nature of the big bang," Phys. Rev. Lett. **96**, 141301 (2006) [arXiv:gr-qc/0602086].

[30] M. Bojowald, G. Date and G. M. Hossain, "The Bianchi IX model in loop quantum cosmology," Class. Quant. Grav. **21**, 3541 (2004) [arXiv:gr-qc/0404039].

[31] G. Date and G. M. Hossain, "Effective Hamiltonian for isotropic loop quantum cosmology," Class. Quant. Grav. **21**, 4941 (2004) [arXiv:gr-qc/0407073].

[32] K. Banerjee and G. Date, "Discreteness corrections to the effective Hamiltonian of isotropic loop quantum cosmology," Class. Quant. Grav. **22**, 2017 (2005) [arXiv:gr-qc/0501102].

[33] M. Bojowald, "Inflation from quantum geometry," Phys. Rev. Lett. **89**, 261301 (2002) [arXiv:gr-qc/0206054].

[34] M. Bojowald and K. Vandersloot, "Loop quantum cosmology, boundary proposals, and inflation," Phys. Rev. D **67**, 124023 (2003) [arXiv:gr-qc/0303072].

[35] G. Date and G. M. Hossain, "Genericness of inflation in isotropic loop quantum cosmology," Phys. Rev. Lett. **94**, 011301 (2005) [arXiv:gr-qc/0407069].

[36] G. Date and G. M. Hossain, "Genericness of big bounce in isotropic loop quantum cosmology," Phys. Rev. Lett. **94**, 011302 (2005) [arXiv:gr-qc/0407074].

[37] J. M. Bardeen, P. J. Steinhardt and M. S. Turner, "Spontaneous Creation Of Almost Scale - Free Density Perturbations In An Inflationary Universe," Phys. Rev. D **28**, 679 (1983).

[38] Brandenberger, R. H., "Inflationary cosmology: Progress and problems," International School on Cosmology, Kish Island, Iran, Jan. 22 - Feb. 4 1999, to be publ. in the proceedings (Kluwer, Dordrecht, 2000) [hep-ph/9910410].

[39] Narlikar, J. V. and Padmanabhan, T. Ann. Rev. Astron. Astrophys. **29**, 325 (1991).

[40] Adams, F. C., Freese, K. and Guth, A. H., "Constraints On The Scalar Field Potential In Inflationary Models," Phys. Rev. D **43**, 965 (1991).

[41] N.D. Birrell and P.C.W. Davies, "Quantum Fields in Curved Space", Cambridge University Press, 1982.

[42] L. H. Ford, "Quantum field theory in curved spacetime," arXiv:gr-qc/9707062.

[43] G. M. Hossain, "Primordial density perturbation in effective loop quantum cosmology," Class. Quant. Grav. **22**, 2511 (2005) [arXiv:gr-qc/0411012].

[44] S.W. Hawking and G.F.R. Ellis, "The Large Scale Structure of Space-time", Cambridge University Press, 1973.

[45] B. McInnes, "The dS/CFT correspondence and the big smash," JHEP **0208**, 029 (2002) [arXiv:hep-th/0112066].

[46] G. M. Hossain, "On energy conditions and stability in effective loop quantum cosmology," Class. Quant. Grav. **22**, 2653 (2005) [arXiv:gr-qc/0503065].

[47] M. Bojowald, "Quantization ambiguities in isotropic quantum geometry," Class. Quant. Grav. **19**, 5113 (2002) [arXiv:gr-qc/0206053].

[48] M. Bojowald, "Loop quantum cosmology: Recent progress," Pramana **63**, 765 (2004) [arXiv:gr-qc/0402053].

[49] Fairbairn W and Rovelli C 2004 Separable Hilbert space in Loop Quantum Gravity *Jour. Math. Phys.* **45** 2802-2814

[50] Velhinho J. M. 2004 Comments on the kinematical structure of loop quantum cosmology *Class. Quantum Grav.* **21** L109 [gr-qc/0406008]

[51] Bojowald M and Date G 2004 Consistency conditions for fundamentally discrete theories, *Class. Quantum Grav.* 21 (2004) 121–143, [gr-qc/0307083]

[52] Bojowald M 2001 Dynamical Initial Conditions in Quantum Cosmology *Phys. Rev. Lett.* **87** 121301

[53] Evans L C 1998 *Partial Differential Equations*, Graduate Studies in Mathematics, vol 19, American Mathematical Society, Providence, 1998

[54] P. Singh and A. Toporensky, "Big crunch avoidance in k = 1 loop quantum cosmology," Phys. Rev. D **69**, 104008 (2004) [arXiv:gr-qc/0312110].

[55] Lidsey J E, Mulryne D J, Nunes N J and Tavakol R (2004), Phys. Rev. **D70**, 063521 (2004) [gr-qc/0406042].

[56] S. Tsujikawa, P. Singh and R. Maartens, "Loop quantum gravity effects on inflation and the CMB," Class. Quant. Grav. **21**, 5767 (2004) [arXiv:astro-ph/0311015].

[57] G. V. Vereshchagin, "Qualitative approach to semi-classical loop quantum cosmology," JCAP **0407**, 013 (2004) [gr-qc/0406108].

[58] M. Bojowald, and F. Hinterleitner, Isotropic loop quantum cosmology with matter Phys. Rev. D **66**, 104003 (2002) [gr-qc/0207038].

[59] T. Padmanabhan, T. R. Seshadri and T. P. Singh, "Making Inflation Work: Damping Of Density Perturbations Due To Planck Energy Cutoff," Phys. Rev. D **39**, 2100 (1989).

[60] T. Padmanabhan, "Acceptable Density Perturbations From Inflation Due To Quantum Gravitational Damping," Phys. Rev. Lett. **60**, 2229 (1988).

[61] M. Bojowald and G. Date, "Quantum suppression of the generic chaotic behavior close to cosmological singularities," Phys. Rev. Lett. **92**, 071302 (2004) [arXiv:gr-qc/0311003].

[62] T. Thiemann, "QSD V: Quantum gravity as the natural regulator of matter quantum field theories," Class. Quant. Grav. **15**, 1281 (1998) [arXiv:gr-qc/9705019].

[63] J. Martin and R. H. Brandenberger, "The trans-Planckian problem of inflationary cosmology," Phys. Rev. D **63**, 123501 (2001) [arXiv:hep-th/0005209].

[64] J. C. Niemeyer, "Inflation with a high frequency cutoff," Phys. Rev. D **63**, 123502 (2001) [arXiv:astro-ph/0005533].

[65] S. Tsujikawa, R. Maartens and R. Brandenberger, "Non-commutative inflation and the CMB," Phys. Lett. B **574**, 141 (2003) [arXiv:astro-ph/0308169].

[66] R. Brandenberger and P. M. Ho, "Noncommutative spacetime, stringy spacetime uncertainty principle, and density fluctuations," Phys. Rev. D **66**, 023517 (2002) [AAPPS Bull. **12N1**, 10 (2002)] [arXiv:hep-th/0203119].

[67] Bowman F 1958, "Introduction to Bessel Functions," New York: Dover (p 117).

[68] J. M. Bardeen, "Gauge Invariant Cosmological Perturbations," Phys. Rev. D **22** (1980) 1882.

[69] A. Riotto, "Inflation and the theory of cosmological perturbations," Lectures given at ICTP Summer School on Astroparticle Physics and Cosmology, Trieste,

Italy, 17 Jun - 5 Jul 2002. Published in "Trieste 2002, Astroparticle physics and cosmology" 317-413, [arXiv:hep-ph/0210162].

[70] A.R. Liddle and D.H. Lyth, "Cosmological Inflation and Large-Scale Structure", Cambridge University Press, 2000.

[71] M. Tegmark, "What does inflation really predict?," arXiv:astro-ph/0410281.

[72] M. Tegmark *et al.* [SDSS Collaboration], "Cosmological parameters from SDSS and WMAP," Phys. Rev. D **69**, 103501 (2004) [arXiv:astro-ph/0310723].

[73] U. Seljak *et al.*, "Cosmological parameter analysis including SDSS Ly-alpha forest and galaxy bias: Constraints on the primordial spectrum of fluctuations, neutrino mass, and dark energy," arXiv:astro-ph/0407372.

[74] S. L. Bridle, A. M. Lewis, J. Weller and G. Efstathiou, "Reconstructing the primordial power spectrum," Mon. Not. Roy. Astron. Soc. **342**, L72 (2003) [arXiv:astro-ph/0302306].

[75] P. Mukherjee and Y. Wang, "Model-Independent Reconstruction of the Primordial Power Spectrum from WMAP Data," Astrophys. J. **599**, 1 (2003) [arXiv:astro-ph/0303211].

[76] S. Hannestad, "Reconstructing the primordial power spectrum - a new algorithm," JCAP **0404**, 002 (2004) [arXiv:astro-ph/0311491].

[77] A. Shafieloo and T. Souradeep, "Primordial power spectrum from WMAP," Phys. Rev. D **70**, 043523 (2004) [arXiv:astro-ph/0312174].

[78] M. Bojowald, J. E. Lidsey, D. J. Mulryne, P. Singh and R. Tavakol, "Inflationary cosmology and quantization ambiguities in semi-classical loop quantum gravity," Phys. Rev. D **70**, 043530 (2004) [arXiv:gr-qc/0403106].

[79] A. G. Riess *et al.* [Supernova Search Team Collaboration], "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant," Astron. J. **116**, 1009 (1998) [arXiv:astro-ph/9805201].

[80] S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], "Measurements of Omega and Lambda from 42 High-Redshift Supernovae," Astrophys. J. **517**, 565 (1999) [arXiv:astro-ph/9812133].

[81] R. R. Caldwell, "A Phantom Menace?," Phys. Lett. B **545**, 23 (2002) [arXiv:astro-ph/9908168].

[82] A. Melchiorri, L. Mersini-Houghton, C. J. Odman and M. Trodden, "The State of the Dark Energy Equation of State," Phys. Rev. D **68**, 043509 (2003) [arXiv:astro-ph/0211522].

[83] U. Alam, V. Sahni, T. D. Saini and A. A. Starobinsky, "Is there Supernova Evidence for Dark Energy Metamorphosis?," Mon. Not. Roy. Astron. Soc. **354**, 275 (2004) [arXiv:astro-ph/0311364].

[84] T. R. Choudhury and T. Padmanabhan, "A theoretician's analysis of the supernova data and the limitations in determining the nature of dark energy II: Results for latest data," Astron. Astrophys. **429**, 807 (2005) [arXiv:astro-ph/0311622].

[85] Y. Wang and P. Mukherjee, "Model-Independent Constraints on Dark Energy Density from Flux-averaging Analysis of Type Ia Supernova Data," Astrophys. J. **606**, 654 (2004) [arXiv:astro-ph/0312192].

[86] S. Nesseris and L. Perivolaropoulos, "A comparison of cosmological models using recent supernova data," Phys. Rev. D **70**, 043531 (2004) [arXiv:astro-ph/0401556].

[87] Y. G. Gong, "Supernova constraints on dark energy model," arXiv:astro-ph/0401207.

[88] U. Alam, V. Sahni and A. A. Starobinsky, "The case for dynamical dark energy revisited," JCAP **0406**, 008 (2004) [arXiv:astro-ph/0403687].

[89] Y. Wang and M. Tegmark, "New dark energy constraints from supernovae, microwave background and galaxy clustering," Phys. Rev. Lett. **92**, 241302 (2004) [arXiv:astro-ph/0403292].

[90] B. Feng, X. L. Wang and X. M. Zhang, "Dark Energy Constraints from the Cosmic Age and Supernova," Phys. Lett. B **607**, 35 (2005) [arXiv:astro-ph/0404224].

[91] H. K. Jassal, J. S. Bagla and T. Padmanabhan, "WMAP constraints on low redshift evolution of dark energy," Mon. Not. Roy. Astron. Soc. **356**, L11 (2005) [arXiv:astro-ph/0404378].

[92] P. S. Corasaniti, M. Kunz, D. Parkinson, E. J. Copeland and B. A. Bassett, "The foundations of observing dark energy dynamics with the Wilkinson Microwave Anisotropy Probe," Phys. Rev. D **70**, 083006 (2004) [arXiv:astro-ph/0406608].

[93] S. Weinberg, "The Cosmological Constant Problem," Rev. Mod. Phys. **61**, 1 (1989).

[94] S. M. Carroll, W. H. Press and E. L. Turner, "The Cosmological constant," Ann. Rev. Astron. Astrophys. **30**, 499 (1992).

[95] V. Sahni and A. A. Starobinsky, "The Case for a Positive Cosmological Lambda-term," Int. J. Mod. Phys. D **9**, 373 (2000) [arXiv:astro-ph/9904398].

[96] S. M. Carroll, "The cosmological constant," Living Rev. Rel. **4**, 1 (2001) [arXiv:astro-ph/0004075].

[97] P. J. E. Peebles and B. Ratra, "The cosmological constant and dark energy," Rev. Mod. Phys. **75**, 559 (2003) [arXiv:astro-ph/0207347].

[98] T. Padmanabhan, "Cosmological constant: The weight of the vacuum," Phys. Rept. **380**, 235 (2003) [arXiv:hep-th/0212290].

[99] A. E. Schulz and M. J. White, "The tensor to scalar ratio of phantom dark energy models," Phys. Rev. D **64**, 043514 (2001) [arXiv:astro-ph/0104112].

[100] S. M. Carroll, M. Hoffman and M. Trodden, "Can the dark energy equation-of-state parameter w be less than -1?," Phys. Rev. D **68**, 023509 (2003) [arXiv:astro-ph/0301273].

[101] P. Singh, M. Sami and N. Dadhich, "Cosmological dynamics of phantom field," Phys. Rev. D **68**, 023522 (2003) [arXiv:hep-th/0305110].

[102] M. Sami and A. Toporensky, "Phantom Field and the Fate of Universe," Mod. Phys. Lett. A **19**, 1509 (2004) [arXiv:gr-qc/0312009].

[103] V. K. Onemli and R. P. Woodard, "Super-acceleration from massless, minimally coupled ϕ^4," Class. Quant. Grav. **19**, 4607 (2002) [arXiv:gr-qc/0204065].

[104] S. Nojiri and S. D. Odintsov, "Quantum deSitter cosmology and phantom matter," Phys. Lett. B **562**, 147 (2003) [arXiv:hep-th/0303117].

[105] E. Elizalde, S. Nojiri and S. D. Odintsov, "Late-time cosmology in (phantom) scalar-tensor theory: Dark energy and the cosmic speed-up," Phys. Rev. D **70**, 043539 (2004) [arXiv:hep-th/0405034].

[106] V. K. Onemli and R. P. Woodard, "Quantum effects can render w < -1 on cosmological scales," Phys. Rev. D **70**, 107301 (2004) [arXiv:gr-qc/0406098].

[107] S. M. Carroll, A. De Felice and M. Trodden, "Can we be tricked into thinking that w is less than -1?," Phys. Rev. D **71**, 023525 (2005) [arXiv:astro-ph/0408081].

[108] K. Vandersloot, "On the Hamiltonian constraint of loop quantum cosmology," Phys. Rev. D **71**, 103506 (2005) [arXiv:gr-qc/0502082].

[109] D. J. Mulryne, R. Tavakol, J. E. Lidsey and G. F. R. Ellis, "An emergent universe from a loop," Phys. Rev. D **71**, 123512 (2005) [arXiv:astro-ph/0502589].

[110] P. Singh, "Effective state metamorphosis in semi-classical loop quantum cosmology," arXiv:gr-qc/0502086.

[111] J. E. Lidsey, D. J. Mulryne, N. J. Nunes and R. Tavakol, "Oscillatory universes in loop quantum cosmology and initial conditions for inflation," Phys. Rev. D **70**, 063521 (2004) [arXiv:gr-qc/0406042].

[112] D. J. Mulryne, N. J. Nunes, R. Tavakol and J. E. Lidsey, "Inflationary cosmology and oscillating universes in loop quantum cosmology," arXiv:gr-qc/0411125.

[113] D. H. Coule, "Contrasting quantum cosmologies," arXiv:gr-qc/0312045.

[114] D. H. Coule, "Quantum cosmological models," arXiv:gr-qc/0412026.

[115] B. Carter, "Energy dominance and the Hawking-Ellis vacuum conservation theorem," arXiv:gr-qc/0205010.

[116] S. D. H. Hsu, A. Jenkins and M. B. Wise, "Gradient instability for w¡-1," Phys. Lett. B **597**, 270 (2004) [arXiv:astro-ph/0406043].

[117] R. V. Buniy and S. D. H. Hsu, "Instabilities and the null energy condition," arXiv:hep-th/0502203.

[118] L. Brillouin, "Wave Propagation and Group Velocity," Academic Press, New York and London, 1960.

[119] M. Bojowald, H. A. Morales-Tecotl and H. Sahlmann, "On loop quantum gravity phenomenology and the issue of Lorentz invariance," arXiv:gr-qc/0411101.

[120] R. Gambini and J. Pullin, "Nonstandard optics from quantum spacetime," Phys. Rev. D **59**, 124021 (1999) [arXiv:gr-qc/9809038].

[121] J. Alfaro, H. A. Morales-Tecotl and L. F. Urrutia, "Quantum gravity corrections to neutrino propagation," Phys. Rev. Lett. **84**, 2318 (2000) [arXiv:gr-qc/9909079].

[122] J. Alfaro, H. A. Morales-Tecotl and L. F. Urrutia, "Loop quantum gravity and light propagation," Phys. Rev. D **65**, 103509 (2002) [arXiv:hep-th/0108061].

[123] H. Sahlmann and T. Thiemann, "Towards the QFT on curved spacetime limit of QGR. II: A concrete implementation," arXiv:gr-qc/0207031.

[124] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifschitz, "A general solution of the Einstein equations with a time singularity", Adv. Phys. **13** (1982) 639–667

[125] D. Garfinkle, "Numerical simulations of generic singuarities," Phys. Rev. Lett. **93**, 161101 (2004) [arXiv:gr-qc/0312117].

[126] T. Damour, M. Henneaux and H. Nicolai, "Cosmological billiards," Class. Quant. Grav. **20**, R145 (2003) [arXiv:hep-th/0212256].

[127] T. Damour and M. Henneaux, "Chaos in superstring cosmology," Phys. Rev. Lett. **85**, 920 (2000) [arXiv:hep-th/0003139].

[128] A. Coley, "No chaos in brane-world cosmology," Class. Quant. Grav. **19**, L45 (2002) [arXiv:hep-th/0110117].

[129] G. Date, "Pre-classical solutions of the vacuum Bianchi I loop quantum cosmology," Phys. Rev. D **72**, 067301 (2005) [arXiv:gr-qc/0505030].

[130] G. Date, "Absence of the Kasner singularity in the effective dynamics from loop quantum cosmology," Phys. Rev. D **71**, 127502 (2005) [arXiv:gr-qc/0505002].

[131] G. M. Hossain, "Large volume quantum correction in loop quantum cosmology: Graviton illusion?," arXiv:gr-qc/0504125.

[132] M. Bojowald, H. H. Hernandez and H. A. Morales-Tecotl, "Perturbative degrees of freedom in loop quantum gravity: Anisotropies," Class. Quant. Grav. **23**, 3491 (2006) [arXiv:gr-qc/0511058].