TOPOLOGY OF TORIC VARIETIES

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by

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DECLARATION

I declare that the thesis entitled Topology of Toric Varieties submitted by me for the Degree of Doctor of Philosophy is the record of work carried out by me during the period from August 1998 to December 2003 under the guidance of Prof. P. Sankaran and has not formed the basis for the award of any degree, diploma, associateship, fellowship, titles in this or any other University or other similar institution of Higher learning.

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Chennai
May 2004

CERTIFICATE FROM THE SUPERVISOR

I certify that the thesis entitled Topology of Toric Varieties submitted for the Degree of Doctor of Philosophy by Ms. V. Uma is the record of research work carried out by her during the period from August 1998 to December 2003 under my guidance and supervision and has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning.

It is further certified that the thesis represents independent work by the candidate and collaboration when existed was necessitated by the nature and scope of problems dealt with.

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At the feet of Mother

and

Sri Aurobindo

PREFACE

This thesis is devoted to a study of the cohomology and K-theory of complex toric bundles and the fundamental group of real toric varieties. A detailed description of the contents is given in the introduction.

Here I wish to thank the various people who have helped me in carrying out this work.

First and foremost I express my gratitude to my thesis supervisor Prof. Parameswaran Sankaran. It was his notes on toric varieties which initiated me into this subject. He encouraged and guided me at every step and patiently led me along the path of research mathematics. It was a great privilege and pleasure to work with him and learn this subject from him.

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Contents

Introduction

In those bright realms are Mind's first forward steps.
Ignorant of all but eager to know all,
Its curious slow enquiry there begins;
Ever searching it grasps at shapes around,
Ever it hopes to find out greater things.
... Yet all it does is on an infant scale.
Sri Aurobindo. Savitri, Book Two

A *toric variety* over $\mathbb{C}$ is an n-dimensional normal algebraic variety X containing the complex algebraic torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open set together with an action of T on X which extends the natural action of T on itself.

A toric variety is naturally associated with a combinatorial object called a *fan* and the geometric and topological concepts on the toric variety correspond to simple combinatorial notions in the associated fan. This correspondence helps us to address problems on toric varieties by translating them to the setting of fans.

Since the discovery of toric varieties in the early 1970's (see works of Demazure [**20**] and Knudsen, Kempf, Mumford and Saint Donat [**30**]), the subject has developed immensely. Some of the standard references on toric varieties are Fulton's book [**25**], the survey article by Danilov [**18**] and Oda's book [**35**]. The recent survey article by Cox [**17**] also gives a detailed account and update on the various developments in the subject.

In a slightly different context Davis and Januszkiewicz in their paper [**21**] have made a detailed study of toric manifolds which are a topological generalization of smooth projective toric varieties. What Davis and Januszkiewicz in [**21**] call a "toric manifold" is termed in recent literature a "quasitoric manifold". The theory of quasitoric manifolds and their generalization to unitary toric manifolds have later been developed by Masuda [32] and also by several other people (see [**14**]).

It is also of interest from the view-point of topology and geometry to study the real valued points of the complex toric variety X. These real toric manifolds have been studied earlier by Jurkiewicz in [**27**]. Davis and Januszkiewicz in [**21**] also consider the real part of the quasitoric manifold which they have termed a "small cover" and describe its topology. But unlike the complex case not much is known regarding the real toric manifold. Some other papers on real toric varieties and "small covers" are [**40**] and [**22**].

In this thesis we address two problems on toric varieties. We give a brief description of them below.

In Chapter 1, we consider a locally trivial fibre bundle $E(X) \to B$ with 'fibre type' a smooth projective complex toric variety X and base an arbitrary topological space B associated to a principal T-bundle. Our main purpose here is to describe (i) the singular cohomology ring of $E(X)$ as an $H^*(B;\mathbb{Z})$-algebra, and (ii) the topological K-ring $K^*(E(X))$ as a $K^*(B)$-algebra when B is compact and Hausdorff. Further, when B is an irreducible, nonsingular, noetherian scheme over $\mathbb{C}$ and $p : E \longrightarrow B$ is algebraic we describe (iii) the Chow ring $A^*(E(X))$ as an $A^*(B)$-algebra, and (iv) the Grothendieck ring $\mathcal{K}^0(E(X))$ of algebraic vector bundles on $E(X)$ as a $\mathcal{K}^0(B)$-algebra.

Here we mention that the parts (ii) and (iv) of the above results are new even in the case when the base B is a point since they give a complete description of the K-ring and the Grothendieck ring of the smooth projective complex toric variety X in terms of generators and relations. The analogous results for the singular cohomology ring and the Chow ring are classical, due to Danilov and Jurkiewicz.(§5.2, p.106 of [**25**]).

When X is the projective space, such a description of the K-ring is due to Adams [**1**]. The case when X is a weighted projective space is more recent, due to Al Amrani [**3**]. We refer the reader to [**12**], [**33**] for other descriptions of the K-ring as well as the equivariant K-ring of a toric variety.

We now briefly explain the method of proof of the results mentioned above:

Let $\mathcal{H}^*$ denote either the singular cohomology ring, the K-ring, the Chow ring or the Grothendieck ring depending on the context.

For the first three parts of the results, we use a Leray-Hirsch type theorem to obtain the structure of $\mathcal{H}^*(E(X))$ as a *module* over $\mathcal{H}^*(B)$. Then we construct a $\mathcal{H}^*(B)$-algebra homomorphism from the expected $\mathcal{H}^*(B)$-algebra to $\mathcal{H}^*(E(X))$ and verify that this algebra homomorphism is an isomorphism of $\mathcal{H}^*(B)$ *-modules*. The "Leray-Hirsch" theorem we need in the context of Chow rings is due to D.Edidin and W.Graham [**23**]. However we give a proof which is more suited to our specific situation. (See also [**24**].) The "Leray-Hirsch" in the context of K-theory of complex vector bundles that we need is Theorem 1.3, p.181, Chap. IV of [**29**] (also see Theorem 2.7.8, [**5**]). For part (iv) we use a result of Grothendieck [**9**] to prove the analogue of the Leray-Hirsch theorem.

We were motivated by the work of Al Amrani [**2**] who has computed the singular cohomology ring of a weighted projective space bundle. Another motivation for us was the work of H.Pittie and A.Ram [**38**] who established the Pieri-Chevalley formula in K-theory in the context of an algebraic G/B bundle associated to a principal B bundle where G is a complex simple algebraic group and B a Borel subgroup.

The first chapter of this thesis is devoted to the study of this problem where we state and prove the above mentioned results. The above work has appeared in [**41**].

In Chapter 2, we consider $X_{\mathbb{R}} = X_{\mathbb{R}}(\Delta)$ to be a smooth real toric manifold associated to the fan Δ. A real toric manifold is for us the real part of a smooth complex toric variety. The main problems dealt with here are (i) the determination of the fundamental group and the universal cover of $X_{\mathbb{R}}$, (ii) giving necessary and sufficient conditions on Δ under which $\pi_1(X_{\mathbb{R}})$ is abelian, (iii) giving necessary and sufficient conditions on Δ under which $X_{\mathbb{R}}$ is aspherical and (iv) giving necessary and sufficient conditions for $\mathcal{C}_\Delta$ to be a $K(\pi,1)$ space where $\mathcal{C}_\Delta$ is the complement of a real subspace arrangement associated to Δ.

In §2 we describe the fundamental group and the universal cover of $X_{\mathbb{R}}(\Delta)$. We were motivated by the paper [**21**] of Davis and Januszkiewicz (see Cor.4.5, p.415 of [**21**]), where they prove the corresponding results for a "small cover" (or the real part of a toric manifold). We show that the same results can be obtained for a real toric variety $X_{\mathbb{R}}(\Delta)$ associated to any smooth fan Δ not necessarily complete. The basic tool for us is the theory of *developments of complexes of groups* in Chapter II.12 of [**11**].

In §6 we further give necessary and sufficient conditions on Δ for $X_{\mathbb{R}}(\Delta)$ to be aspherical. This too was motivated by the recent papers of Davis, Januszkiewicz and Scott (see Theorem 2.2.5, p.27 of [**22**]) where they prove similar results for a small-cover. For this purpose, we rely primarily on the results of Davis [**19**]. However in many places we give different proofs using the technique of *development*, consistent with the theme of our work.

Here we wish to remark that the structure of the fundamental group of a complex toric variety is well known and is relatively simple (see §3.2 of [**25**]). But the methods used to determine the fundamental group of a complex toric variety cannot be applied directly to a real toric manifold, primarily because $\mathbb{R}^*$ is not connected. The theory of *development of complexes of groups* seems more naturally applicable in the setting of real toric manifolds.

Besides generalizing the previous results to the setting of a smooth real toric manifold $X_{\mathbb{R}}(\Delta)$, we give a presentation for the fundamental group $\pi_1(X_{\mathbb{R}}(\Delta))$ completely in terms of the fan (see §3). Furthermore, in §5 we give necessary and sufficient conditions on Δ under which $\pi_1(X_{\mathbb{R}}(\Delta))$ is abelian. We also show that the torsion elements are always of order 2.

In §7, we relate $X_{\mathbb{R}}(\Delta)$ to the complement of a certain real subspace arrangement (which we denote by $\mathcal{C}_\Delta$) and give necessary and sufficient conditions for $\mathcal{C}_\Delta$ to be a $K(\pi, 1)$ space. Finding $K(\pi, 1)$ arrangements seems to be an interesting problem in topology (see [**36**] and [**28**]). We get many such examples.

We mention here that Jurkiewicz in [**27**] has given a complete description of the cohomology ring of a real toric manifold with $\mathbb{Z}_2$-coefficients and also its first integer homology group.

The second chapter of this thesis is devoted to the study of this problem where we state and prove the above mentioned results. The above work has appeared in [**42**].

CHAPTER 1

Cohomology of toric bundles

1. Basic definitions and statement of results

Let $T \cong (\mathbb{C}^*)^n$ denote the complex algebraic torus. Let $M = Hom_{alg}(T, \mathbb{C}^*) \cong \mathbb{Z}^n$ and $N = Hom_{alg}(\mathbb{C}^*, T) \cong \mathbb{Z}^n$ denote the group of characters and the group of 1-parameter subgroups of T respectively. Note that $M = N^\vee := Hom(N; \mathbb{Z})$ under the natural pairing $\langle\ ,\ \rangle : M \times N \longrightarrow \mathbb{Z}$, given by $\chi^u \circ \lambda_v(z) = z^{\langle u,v\rangle}$ for all $z \in \mathbb{C}^*$. (Here $\chi^u \in \mathbb{C}(T) = \mathbb{C}[M]$ denotes the character corresponding to $u \in M$ and λ_v the 1-parameter subgroup corresponding to $v \in N$.)

Let Δ be a fan in N such that the T-toric variety $X := X(\Delta)$ is complete and non-singular. Let $p : E \longrightarrow B$ be a principal bundle with structure group and fibre the torus T over an arbitrary topological space B. When the bundle $p : E \longrightarrow B$ is algebraic, it is well-known that the bundle $E \longrightarrow B$ is Zariski locally trivial.

Consider the fibre bundle $\pi : E(X) \longrightarrow B$ with fibre the toric variety X, where $E(X)$ is the fibre product $E \times_T X$, and the projection map is defined as $\pi([e, x]) = p(e)$. Note that the bundle $\pi : E(X) \longrightarrow B$ is Zariski locally trivial when $p : E \longrightarrow B$ is algebraic. In this chapter, we describe the integral singular cohomology ring $H^*(E(X); \mathbb{Z})$, and the K-ring $K(E(X))$ when B is a compact Hausdorff topological space. Also, when $p : E \longrightarrow B$ is algebraic and B an irreducible nonsingular noetherian scheme over $\mathbb{C}$, we describe the Chow ring $A^*(E(X))$, and the Grothendieck ring $\mathcal{K}^0(E(X))$ of algebraic vector bundles of the complex variety $E(X)$.

Suppose that $q : V \longrightarrow X$ is a T-equivariant vector bundle over X, then we obtain a vector bundle $E(V)$ over $E(X)$ with total space $E \times_T V$ where the bundle projection is the map $[e, v] \mapsto [e, q(v)]$. In case V is a T-equivariant line bundle associated to a character $\chi^u : T \longrightarrow \mathbb{C}^*$, the bundle $E(V)$ is isomorphic to the pull-back bundle $\pi^*(\xi_u)$ where $\xi_u \longrightarrow B$ is the line bundle got from $E \longrightarrow B$ by 'extending' the structure group via χ^u. After fixing an isomorphism, $T \cong (\mathbb{C}^*)^n$, $u \in M$ corresponds to an element $(a_1, \cdots, a_n) \in \mathbb{Z}^n$. The bundle E is then the principal bundle associated to the Whitney sum of line bundles ξ_i, $1 \leq i \leq n$, and $E(V)$ can then be identified with the tensor product $\xi_1^{a_1} \otimes \cdots \otimes \xi_n^{a_n}$. (Here it is understood that, when $a < 0$, $\xi^a = (\xi^\vee)^{-a}$.)

When B is a non-singular variety any line bundle ξ over B is isomorphic to $\mathcal{O}(Y)$ for some divisor Y in B. The divisor class $[Y]$ is the first Chern class $c_1(\xi) \in A^1(B)$ of ξ.

We use the notations of [**25**] throughout the chapter.

For $k \geq 1$, $\Delta(k)$ will denote the set of k dimensional cones in Δ. We let $d = \#\Delta(1)$, and write $v_1, \cdots, v_d$ for the primitive elements of N along the edges in Δ. Let $\rho_j \in \Delta(1)$ be the edge $\mathbb{R}_{\geq 0} v_j$. Recall that our hypothesis that X is smooth is equivalent to the statement that the set of the primitive vectors along the edges of any cone in Δ is part of a $\mathbb{Z}$-basis for N.

For a cone $\sigma \in \Delta$, U_σ denotes the affine toric variety defined by σ and $V(\sigma)$ denotes the closure in X of the variety whose local equation in U_σ is $\chi^u = 0$ for all $u \notin \sigma^\perp$, $u \in \sigma^\vee$. The $V(\sigma), \sigma \in \Delta$, are the orbit closures for the action of T on X.

For $1 \leq j \leq d$ let L_j denote the T-equivariant line bundle over X which corresponds to the piecewise linear function ψ_j defined by $\psi_j(v_i) = -\delta_{i,j}$. The line bundle L_j admits a global T-equivariant section s_j whose zero locus is the variety $V(\rho_j)$.

Let $\sigma_1, \cdots, \sigma_m$ be an ordering of the cones in $\Delta(n)$. Let $\tau_i \in \Delta$ be the intersection with σ_i of those cones σ_j, $j > i$, such that $\dim(\sigma_i \cap \sigma_j) = n - 1$. Thus $\tau_1 = 0$, and $\tau_m = \sigma_m$. Consider the condition:

$$\tau_i < \sigma_j \Longrightarrow i \leq j. \qquad (*).$$

Set $\tau'_i < \sigma_i$ to be the cone such that $\tau_i \cap \tau'_i = 0, \dim(\tau_i) + \dim(\tau'_i) = n$, $1 \leq i \leq m$. Also consider the condition

$$\tau'_i < \sigma_j \Longrightarrow j \leq i. \qquad (*')$$

Note that τ'_i is the intersection with σ_i of those cones σ_j with $j < i$ and $\dim(\sigma_i) \cap \dim(\sigma_j) = n - 1$ and so condition $(*')$ is the same as $(*)$ for the reverse ordering on $\Delta(n)$. It is well known that when X is (nonsingular) projective, then there exists an ordering of the cones in $\Delta(n)$ such that both conditions $(*)$ and $(*')$ hold. We shall assume that there exists an ordering of $\Delta(n)$ such that property $(*)$ holds. (See [**25**], §5.2.)

However, there are complete nonsingular varieties $X(\Delta)$ which are *not* projective such that $\Delta(n)$ admits an ordering satisfying both $(*)$ and $(*')$. The following three dimensional fan that corresponds to a complete non-projective toric variety (see p. 84 [**35**]) is one such example.

EXAMPLE 1.1. Let $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$. Let $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, $v_0 = -e_1 - e_2 - e_3$, $v_1' = v_0 + v_1 = -e_2 - e_3$, $v_2' = v_0 + v_2 = -e_1 - e_3$, $v_3' = v_0 + v_3 = -e_1 - e_2$. Let Δ be the fan in N consisting of the faces of the following ten three dimensional cones:
$\sigma_1 = \langle v_1, v_2, v_3 \rangle$, $\sigma_2 = \langle v_1, v_2, v_1' \rangle$, $\sigma_3 = \langle v_2, v_3, v_2' \rangle$, $\sigma_4 = \langle v_3, v_1, v_3' \rangle$, $\sigma_5 = \langle v_1, v_1', v_3' \rangle$, $\sigma_6 = \langle v_2, v_2', v_1' \rangle$, $\sigma_7 = \langle v_3, v_3', v_2' \rangle$, $\sigma_8 = \langle v_0, v_1', v_2' \rangle$, $\sigma_9 = \langle v_0, v_2', v_3' \rangle$, $\sigma_{10} = \langle v_0, v_3', v_1' \rangle$.
It is easily seen that the ordering $\sigma_1, \sigma_2, \ldots, \sigma_{10}$ satisfies both $(*)$ and $(*')$.

By relabelling the v_j's if necessary, we assume that $v_1, \cdots, v_n \in N$ are primitive vectors along the edges of σ_m and let $u_1, \cdots, u_n$ be the dual basis of M.

DEFINITION 1.2. *Let S be a ring with 1. Let $r_i, 1 \le i \le n$, be in the centre of S. Consider the polynomial algebra $S[x_1, \cdots, x_d]$. We denote by I_S the two-sided ideal generated by the following two types of elements*

$$x_{j_1} \cdots x_{j_k},\ 1 \le j_p \le d, \qquad (i)$$

where $v_{j_1}, \cdots, v_{j_k}$ do not span a cone of Δ, and,

$$y_u := \sum_{1 \le j \le d} \langle u, v_j \rangle x_j - r_u \qquad (ii)$$

where $r_u = \sum_{1 \le i \le n} a_i r_i$, $\forall\ u = \sum_{1 \le i \le n} a_i u_i \in M$. Define $R = R(S, \Delta) = S[x_1, \cdots, x_d]/I_S$. Assume further that the elements $r_i \in S$, $1 \le i \le n$, are invertible. We denote by $\mathcal{I}_S$ the two-sided ideal generated by elements of type (i) above and the elements

$$z_u := \prod_{j, \langle u, v_j \rangle > 0} (1 - x_j)^{\langle u, v_j \rangle} - r_u \prod_{j, \langle u, v_j \rangle < 0} (1 - x_j)^{-\langle u, v_j \rangle} \qquad (ii')$$

where $r_u = \prod_{1 \le i \le n} r_i^{a_i}$, $\forall\ u = \sum_{1 \le i \le n} a_i u_i \in M$. Define $\mathcal{R} = \mathcal{R}(S, \Delta) = S[x_1, \cdots, x_d]/\mathcal{I}_S$.

Note that the S-algebras R and $\mathcal{R}$ depend not only on the fan Δ, but also on the the isomorphism $N \cong \mathbb{Z}^n$ resulting from the choice of $\sigma_m \in \Delta$ and the elements $r_i \in S$. The only non-commutative ring S we need to consider is the integral cohomology ring of B.

Note that for any cohomology theory $\mathcal{H}$, $\mathcal{H}^*(E(X))$ is an $\mathcal{H}^*(B)$-algebra via the induced map $\pi^* : \mathcal{H}^*(B) \longrightarrow \mathcal{H}^*(E(X))$. The following is our main theorem:

THEOREM 1.3. *Let $\pi : E \longrightarrow B$ be a principal T-bundle over an arbitrary topological space B. Assume that X is a smooth complete T-toric variety and that $\Delta(n)$ has been ordered so that $(*)$ holds. With above notations,*
(i) The singular cohomology ring of $E(X)$ is isomorphic as an $H^*(B;\mathbb{Z})$-algebra to $R(H^*(B;\mathbb{Z}),\Delta)$, with $r_i = c_1(\xi_i^\vee) \in H^2(B;\mathbb{Z})$.
(ii) When B is compact and Hausdorff, the K-ring $K^*(E(X))$ of complex vector bundles over $E(X)$ is isomorphic as a $K^*(B)$-algebra to $\mathcal{R}(K^*(B);\Delta)$ where $r_i = [\xi_i^\vee] \in K(B)$, $1 \leq i \leq n$. Suppose $p : E \longrightarrow B$ is algebraic where B irreducible, non-singular and noetherian over $\mathbb{C}$. Furthermore, assume that $(*')$ also holds. Then:
(iii) The Chow ring $A^*(E(X))$ of $E(X)$ is isomorphic as an $A^*(B)$-algebra to $R(A^*(B),\Delta)$ where $r_i = c_1(\xi_i^\vee) \in A^1(B)$, $1 \leq i \leq n$.
(iv) The ring $\mathcal{K}(E(X))$ is isomorphic as a $\mathcal{K}(B)$-algebra to $\mathcal{R}(\mathcal{K}(B),\Delta)$ where $r_i = [\xi_i^\vee] \in \mathcal{K}(B)$.

We do not know if parts (iii) and (iv) of the main theorem remain valid without the hypothesis that $(*')$ hold. Neither do we know of an example where $\Delta(n)$ admits an ordering satisfying $(*)$ but no ordering that satisfies both $(*)$ and $(*')$.

REMARK 1.4. Examples of algebraic bundles $E(X) \longrightarrow B$ that we consider include as special cases the toric fibre bundles considered on p.41, [**25**].

2. The rings R and $\mathcal{R}$

In this section we prove certain facts about the rings R and $\mathcal{R}$ which will be needed in the proof of the main theorem.

We keep the notations of §1. We assume that $\Delta(n)$ has been so ordered that property $(*)$ holds. Recall that $v_1, \cdots, v_d$ are the primitive vectors along the edges of Δ, that $v_1, \cdots, v_n$ are in σ_m, and that $u_1, \cdots, u_n$ is the dual basis of M.

For any cone $\gamma \in \Delta$, denote by $x(\gamma)$ the monomial $x_{j_1} \cdots x_{j_r} \in S[x_1, \cdots, x_d]$ where $v_{j_1} \cdots, v_{j_r}$ are the primitive vectors along the edges of γ.

Recall from §1 the definition of the S-algebras R and $\mathcal{R}$. We shall denote by the same symbol $x(\gamma)$, in R and $\mathcal{R}$, the image of the monomial $x(\gamma) \in S[x_1, \cdots, x_d]$ under the canonical quotient map.

LEMMA 2.1. *(i) If $\gamma \in \Delta(r)$ is spanned by $v_{j_1}, v_{j_2}, \cdots, v_{j_r}$, then*

$$x_{j_1} x(\gamma) = -\sum_k \langle u, v_k \rangle x(\gamma_k) + r_u x(\gamma)$$

for some $u \in M$, where the sum on the right is over those cones γ_k in $\Delta(r+1)$ which are spanned by primitive vectors $v_k, v_{j_1}, v_{j_2}, \cdots, v_{j_r}$.
(ii) If $\alpha < \gamma \leq \beta$ are cones in Δ then there exist cones $\gamma_1, \cdots, \gamma_s \in \Delta$ with $\alpha < \gamma_k$ such that the γ_k are not contained in β, and

$$x(\gamma) = \sum_k c_k x(\gamma_k) + c x(\alpha)$$

for some $c, c_k \in S$.
(iii) The monomials $x(\tau_i)$, $1 \leq i \leq m$, span R as an S-module.

Proof: (i) Suppose $\gamma \leq \sigma$ where σ is n-dimensional. Let $v_{j_1}, \cdots, v_{j_n}$ be the primitive vectors which span σ such that the first r elements span γ, with $j_1 = j$. Let $u \in M$ be the dual basis element such that $\langle u, v_{j_k} \rangle = \delta_{j j_k}$. One has the relation:

$$x_{j_1} + \sum_{k \neq j_1} \langle u, v_k \rangle x_k - r_u = 0.$$

Multiplying both sides by $x(\gamma)$, and using the type (i) relations, we get

$$x_{j_1} x(\gamma) = -\sum_k \langle u, v_k \rangle x(\gamma_k) + r_u x(\gamma)$$

where the sum on the right is over those cones γ_k in $\Delta(r+1)$ which are spanned by primitive the vectors $v_k, v_{j_1} v_{j_2}, \cdots, v_{j_r}$ where $k \neq j_1, \cdots, j_n$. This proves (i).

(ii) Suppose $v_{j_1}, \cdots, v_{j_l}$ spans $\beta \in \Delta(l)$ such that the first r of these span α and the first p of these span γ, $p > r$. Without loss of generality we may assume that β is an n dimensional cone so that $l = n$, and $v_{j_1}, \cdots, v_{j_n}$ is a basis for N. Now let $u \in M$ be the dual basis element so that $\langle u, v_{j_q} \rangle = \delta_{p,q}$. Then we have

$$x_{j_p} + \sum_{k \neq j_p} \langle u, v_k \rangle x_k - r_u = 0. \tag{1}$$

Multiplying by $x_{j_1} \cdots x_{j_{p-1}}$ and observing that the coefficient of x_k in the sum is zero for $k \in \{j_1, \cdots, j_n\}$ and $k \neq j_p$, we get $x(\gamma) + \sum \langle u, v_k \rangle x(\gamma_k) - r_u x(\gamma') = 0$ where γ' is the cone spanned by $v_{j_1}, \cdots, v_{j_{p-1}}$ and the sum is over those cones $\gamma_k \in \Delta(p)$ which are spanned by $v_{j_1}, v_{j_2}, \cdots, v_{j_{p-1}}, v_k$, $k \neq j_1, \cdots, j_n$. Note that each of these γ_k contains α but is not contained in β. If $\gamma' = \alpha$, we are done. Otherwise, by an induction on the dimension of γ the statement is true for γ'. Substituting this expression for $x(\gamma')$ in (1), we see that (ii) holds.

(iii) We now prove that the $x(\tau_i)$ span R. In view of (ii), it suffices to prove that for any γ, $x(\gamma)$ is in the S-submodule spanned by the $x(\tau_i)$. Property $(*)$ implies that given any $\gamma \in \Delta$, there exists a unique i such that $\tau_i \leq \gamma \leq \sigma_i$; indeed it is the smallest i for which $\gamma \leq \sigma_i$. (See [**25**], §5.2.) We prove, by a downward induction on this i, that $x(\gamma)$ is in the S-span of $x(\tau_j)$, $j \geq i$. If $i = m$, then $\gamma = \sigma_m = \tau_m$ and there is nothing to prove.

Let $\tau_i \leq \gamma \leq \sigma_i$ for some $i < m$. Now, using (ii), we can write $x(\gamma)$ as an S-linear combination of $x(\tau_i)$ and $x(\gamma_j)$ where $\tau_i < \gamma_j$, and γ_j is not contained in σ_i. It follows that each γ_j is such that $\tau_r \leq \gamma_j \leq \sigma_r$ for some r (depending on j) with $r > i$. By inductive hypothesis, each of the $x(\gamma_j)$ is in the S-span of $x(\tau_q)$, $q \geq r$. It follows that $x(\gamma)$ is in the S-span of $x(\tau_r), r \geq i$, completing the proof. □

Concerning the structure of $\mathcal{R}$ we have the following.

LEMMA 2.2. *(i) Let $\alpha < \gamma \leq \beta$ be cones in Δ. Suppose that γ is spanned by $v_{j_1}, \cdots, v_{j_k}$, then*

$$x_{j_1} x(\gamma) = (1 - r_u) x(\gamma) + \sum_p a_p x(\gamma_p)$$

for some $u \in M$, where $a_p \in S$, and $\gamma_p \in \Delta$ are such that $\alpha < \gamma_p$, γ_p are not contained in β and $\dim(\gamma_p) > \dim(\gamma)$.

(ii) Let $\alpha < \gamma \leq \beta$ be cones in Δ. Then

$$x(\gamma) = \sum_p b_p x(\gamma_p) + bx(\alpha)$$

for some $b_p, b \in S$ and suitable cones $\gamma_p \in \Delta$ which contain α and are not contained in β.
(iii) The monomials $x(\tau_i), 1 \leq i \leq m$ span $\mathcal{R}$ as an S-module.

Proof: (i) Without loss of generality, we may assume that β is an n-dimensional cone. We prove this by descending induction on the dimension of γ. Suppose that $v_{j_1}, \cdots, v_{j_n}$ span β, and that $v_{j_1} \notin \alpha$. Let $u \in M$ be the dual basis element such that $\langle u, v_{j_r}\rangle = \delta_{1,r}$. The relation $z_u = 0$ can be rewritten as

$$(1 - x_{j_1}) \prod_{j,\langle u,v_j\rangle > 0} (1 - x_j)^{\langle u,v_j\rangle} = r_u \prod_{j,\langle u,v_j\rangle < 0} (1 - x_j)^{-\langle u,v_j\rangle}$$

Note that none of the $x_{j_r}, 2 \leq r \leq n$ occur in the above relation. Multiplying both sides by $x(\gamma)$,

$$(x(\gamma) - x_{j_1}x(\gamma)) \prod_p (1 - x_p)^{\langle u,v_p\rangle} = r_u x(\gamma) \prod_q (1 - x_q)^{-\langle u,v_q\rangle} \qquad (2)$$

where the product is over those p, (resp. q) such that v_p, (resp. v_q), $v_{j_1} \cdots, v_{j_k}$ span a cone of Δ, $\langle u, v_p\rangle > 0$, (resp. $\langle u, v_q\rangle < 0$). In particular, if γ is n-dimensional, then the above equation reads $x_{j_1}x(\gamma) = (1 - r_u)x(\gamma)$, which proves the lemma in this case. Assume that $k < n$ and that the statement holds for all higher dimensional cones. Then from equation (2), we see that the lemma follows by repeated application of the inductive hypothesis and by the observation that if $\gamma' < \gamma''$ and if γ' is not contained in β, then neither is γ''.
Parts (ii) and (iii) follow from (i) just as in the proof of Lemma 2.1. □

REMARK 2.3. One can show that if $r_i = 0$ for all $i, 1 \leq i \leq n$, then $x_j^{n+1} = 0,\ 1 \leq j \leq d$ in R and that $x(\tau_i), 1 \leq i \leq m$, form a basis for R as a module over S. Similarly, if $r_i = 1$ for all $1 \leq i \leq n$, then $x_j^{n+1} = 0$ for $1 \leq j \leq d$ and $x(\tau_i), 1 \leq i \leq m$, form a basis for $\mathcal{R}$ as an S-module.

3. Singular cohomology and Chow ring

In this section we shall prove parts (i) and (iii) of the Main Theorem 1.3.

Let Δ be a complete nonsingular fan in N. We assume that $\sigma_1, \cdots, \sigma_m$ is an ordering of $\Delta(n)$ such that property $(*)$ holds. (See §1.) This implies that the toric variety $X = X(\Delta)$ has an algebraic cell decomposition, namely, there exists closed subvarieties $X = Z_1 \supset \cdots \supset Z_m$ of X such that $Z_i \setminus Z_{i+1} =: Y_i \cong \mathbb{C}^{k_i}$ for some integers k_i. In fact, with τ_i as in §1, the closure of Y_i is just the variety $V(\tau_i)$. See **[25]**. This yields the structure of a (finite) CW complex on X with cells only in even dimensions.

Notation: We shall denote $V(\tau_i)$ by V_i. If $(*')$ also holds, then we set $V_i' = V(\tau_i')$.

Assume that $p : E \longrightarrow B$ is complex algebraic and B irreducible, nonsingular, and noetherian over $\mathbb{C}$. Now since the varieties V_i are stable under the T-action, one has the associated bundles $\pi_i : E(V_i) \longrightarrow B$ with fibre V_i. Note that $E(V_i)$ is a smooth closed subvariety of $E(X)$. For any closed subvariety Z in an algebraic variety Y we denote by $[Z]$ its rational equivalence class in $A_*(Y)$. If Z and Y are smooth, we denote by $[Z]$ the cohomology class dual to Z in $H^{2r}(Y;\mathbb{Z})$ as well as the element in the Chow cohomology group $A^r(Y)$ where r is the codimension of Z in Y.

In case property $(*')$ also holds, then $[V_i].[V_j'] = 0$ if $j < i$, and $[V_i][V_i'] \in H^{2n}(X;\mathbb{Z}) \cong \mathbb{Z}$ is the positive generator with respect to the standard orientation coming from the complex structure on X. Also, in the Chow ring, $[V_i][V_i'] \in A^n(X) = A_0(X) \cong \mathbb{Z}$ denotes the class of the point $[V(\sigma_i)]$ which generates $A_0(X)$.

Lemma 3.1. *Let X be a complete nonsingular T-toric variety and suppose that property $(*)$ holds for an ordering of $\Delta(n)$. Let $\pi : E \longrightarrow B$ be a principal T-bundle over any topological space. Then:*

(i) The bundle $\pi : E(X) \longrightarrow B$ admits a cohomology extension of the fibre in singular cohomology with integer coefficients. $H^*(E(X);\mathbb{Z})$ is isomorphic to $H^*(B;\mathbb{Z}) \otimes H^*(X;\mathbb{Z})$ as an $H^*(B;\mathbb{Z})$-module.

Assume $\pi : E \longrightarrow B$ is complex algebraic where B an irreducible, nonsingular noetherian variety over $\mathbb{C}$. Suppose that properties $(*), (*')$ hold. Then:

(ii) The Chow group $A^*(E(X))$ is isomorphic as an $A^*(B)$-module to $A^*(B) \otimes A^*(X)$.

Proof: We shall fix a base point $b_0 \in B$ and identify X with the fibre $\pi^{-1}(b_0) \subset E(X)$. (i). Since the X has a CW decomposition with only even dimensional cells, its integral cohomology is isomorphic to the free abelian group with basis labelled by its cells. Indeed the dual cohomology classes $[V_i] \in H^{2l_i}(X;\mathbb{Z})$, $l_i = \dim(\tau_i)$, form a $\mathbb{Z}$-basis for $H^*(X;\mathbb{Z})$.

Let $\eta \in \Delta(r)$ spanned by the primitive vectors along $v_{j_1}, \cdots, v_{j_r}$. Denote by $L(\eta)$ the T-equivariant bundle $L_{j_1} \oplus \cdots \oplus L_{j_r}$, where the L_j are as defined in §1. The class $[V(\eta)] \in H^{2r}(X;\mathbb{Z})$ equals the the Chern class $c_r(L(\eta)) = c_1(L_{j_1}) \cdots c_1(L_{j_r})$. The bundle $\mathcal{L}(\eta) = E(L(\eta))$ over $E(X)$ restricts to $L(\eta)$ over X. By the naturality of Chern classes, $c_r(\mathcal{L}(\eta)) \in H^{2r}(E(X);\mathbb{Z})$ restricts to $c_r(L(\eta)) = [V(\eta)] \in H^{2r}(X;\mathbb{Z})$. In particular, it follows that $[V(\tau_i)], 1 \le i \le m$, are in the image of the restriction homomorphism $H^*(E(X);\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z})$. The lemma follows by Leray-Hirsch theorem ([**39**], p.258).

(ii) Our proof follows that of Lemma 2.8 [**24**] closely. (See also Lemma 6, [**23**].) Clearly the classes $[E(V_i)] \in A^*(E(V_i)), 1 \le i \le m$, restrict to elements of a $\mathbb{Z}$ basis (namely $[V_i] \in A^*(X)$). Consider the $A^*(B)$-linear map $\Phi : A^*(B) \otimes A^*(X) \longrightarrow A^*(E(X))$, defined as

$$\Phi(\sum_{1\le i\le m} b_i \otimes [V_i]) = \sum_{1\le i\le m} \pi^*(b_i).[E(V_i)].$$

To prove (ii) we show that Φ is an isomorphism. Suppose $\Phi(\sum b_i \otimes [V_j]) = 0 \in A^*(E(X))$. Assume that k is the smallest integer such that $b_k \neq 0$. Since for $j \ge k$, V_j and V'_k are disjoint unless $j = k$ in which case they intersect transversally and $V_j \cap V'_j = V(\sigma_j)$ scheme theoretically. We see that $[E(V_j)].[E(V'_k)] = 0$ if $j > k$, and, $E(V_j), E(V'_j)$ intersect transversally and so $E(V_j) \cap E(V'_j) = E(V(\sigma_j))$ scheme theoretically (where the subvarieties are given the reduced scheme structure). Therefore, $[E(V_j)].[E(V'_j)] = [E(V(\sigma_j))]$. Note that since $V(\sigma_j)$ is a T-fixed point, $E(V(\sigma_j)) \cong E/T = B$. Denote by π_j the restriction of $\pi : E(X) \longrightarrow B$ to $E(V(\sigma_j))$. Also let ι_j be the inclusion $E(V(\sigma_j)) \subset E(X)$. Note that $[E(V(\sigma_j))] = \iota_{j*}\pi_j^*([B]) \in A_*(E(X))$.

Now since $\Phi(\sum_{1\le j\le m} b_j \otimes [V_j]) = 0$, we get
$0 = [E(V'_k)].\Phi(\sum_{1\le j\le m} b_j \otimes [V_j]) = \sum_{1\le j\le m} \pi^*(b_j)[E(V_j)].[E(V'_k)] = \pi^*(b_k)[E(V(\sigma_k))]$
$= \pi^*(b_k).\iota_{k*}\pi_k^*([B])$.

Applying π_* and using the projection formula we get
$0 = \pi_*(\pi^*(b_k).\iota_{k*}\pi_k^*([B])) = b_k.\pi_*\iota_{k*}\pi_k^*([B]) = b_k.\pi_{k*}\pi_k^*([B]) = b_k.[B] = b_k$.
This contradicts our hypothesis that $b_k \neq 0$. It follows that Φ is injective.

We now prove surjectivity of Φ. One has the filtration $B \cong E(Z_m) \subset \cdots \subset E(Z_1) = E(X)$. We claim that Φ defines surjections

$$\Phi_i : A^*(B) \otimes A^*(Z_i) \longrightarrow A^*(E(Z_i))$$

for each $i, 1 \leq i \leq m$. We prove this by downward induction on i. This is trivially true for $i = m$, since in this case $E(Z_m) \cong B$. Consider the diagram

$$\begin{array}{ccccccc} A^*(B) \otimes A^*(Z_{i+1}) & \longrightarrow & A^*(B) \otimes A^*(Z_i) & \longrightarrow & A^*(B) \otimes A^*(Y_i) & \longrightarrow 0 \\ \Phi_{i+1} \downarrow & & \Phi_i \downarrow & & \downarrow & \\ A^*(E(Z_{i+1})) & \longrightarrow & A^*(E(Z_i)) & \longrightarrow & A^*(E(Y_i)) & \longrightarrow 0 \end{array}$$

where the top horizontal row is obtained from tensoring with $A^*(B)$ the exact sequence $A^*(Z_{i+1}) \longrightarrow A^*(Z_i) \longrightarrow A^*(Y_i) \longrightarrow 0$. The homomorphism $A^*(B) \otimes A^*(Y_i) \longrightarrow A^*(E(Y_i))$ is an isomorphism by Prop. 1.9, ch. 1, **[26]**. Therefore the surjectivity of Φ_i follows by a diagram chase. □

REMARK 3.2. It follows from the proof of the above lemma that the classes $c_1(\mathcal{L}_j) \in H^2(E(X); \mathbb{Z}), 1 \leq j \leq d$, generate $H^*(E(X); \mathbb{Z})$ as an $H^*(B; \mathbb{Z})$-algebra. Similarly, when $p : E \longrightarrow B$ is algebraic and B a complete nonsingular variety, then $[E(V(\rho_j)] \in A^1(E(X))$, $1 \leq j \leq d$, generate $A^*(E(X))$ as an algebra over $A^*(B)$. Indeed for every $1 \leq i \leq m$, $E(V_i) = \cap_{k=1}^{l_i} E(V(\rho_{i_k}))$ (intersection is transversal and proper) where τ_i is spanned by $\rho_{i_1}, \rho_{i_2}, \cdots, \rho_{i_{l_i}}$ and $l_i = dim(\tau_i)$. (See p.100 of **[25]**)

We now turn to the ring structure of $H^*(E(X); \mathbb{Z})$ and $A^*(E(X))$.

Recall from §1 that the line bundle L_j over X admits a T-equivariant section $s_j : X \longrightarrow L_j$ whose zero locus is the divisor $V(\rho_j)$.

Suppose that $v_{j_1}, \cdots, v_{j_r}$ does not span a cone in Δ. Then $s = (s_{j_1}, \cdots, s_{j_r})$ is a nowhere vanishing T-invariant section of $L_{j_1} \oplus \cdots \oplus L_{j_r}$. By taking associated construction, we see that the bundle $\mathcal{L}_{j_1} \oplus \cdots \oplus \mathcal{L}_{j_r}$ admits a nowhere vanishing section. This implies that

$$c_1(\mathcal{L}_{j_1}) \cdots c_1(\mathcal{L}_{j_r}) = 0 \tag{3}$$

in $H^{2r}(E(X); \mathbb{Z})$.

When $p : E \longrightarrow B$ is algebraic with B nonsingular, we see that

$$[E(V(\rho_{j_1}))] \cdots [E(V(\rho_{j_r}))] = 0 \tag{4}$$

in the Chow ring $A^*(E(X))$.

Now, let $u \in M$ be any element. Consider the T-equivariant line bundle L_u on X corresponding to the principal divisor $\sum_{1\leq j\leq d}\langle u, v_j\rangle V(\rho_j) = div(\chi^{-u})$. Clearly L_u is isomorphic as a T-equivariant bundle to $\prod_{1\leq j\leq d} L_j^{\langle u,v_j\rangle}$ as both of these bundles correspond to the same piecewise linear function $-u : |\Delta| \longrightarrow \mathbb{R}$. (See [**25**].) Hence $E(L_u) \cong \prod_{1\leq j\leq d} \mathcal{L}_j^{\langle u,v_j\rangle}$. On the other hand the bundle $\mathcal{L}_u := E(L_u) = E(\chi^{-u})$ is isomorphic to $\pi^*(\xi_1)^{a_1} \cdots \pi^*(\xi_n)^{a_n}$, where $a_i = \langle -u, v_i\rangle$. This yields the following relations:

$$\sum_{1\leq j\leq d} \langle u, v_j\rangle c_1(\mathcal{L}_j) - \sum_{1\leq i\leq n} \langle u, v_i\rangle c_1(\pi^*(\xi_i^\vee)) = 0 \tag{5}$$

in $H^2(E(X);\mathbb{Z})$. In the case when $p : E \longrightarrow B$ is algebraic and B is nonsingular we obtain, in the Chow group $A^1(E(X))$,

$$\sum_{1\leq j\leq d} \langle u, v_j\rangle [E(V(\rho_j))] - \sum_{1\leq i\leq n} \langle u, v_i\rangle c_1(\pi^*(\xi_i^\vee)) = 0. \tag{6}$$

Proof of Theorem 1.3 (i), (iii): We first consider part (iii). In view of equations (4) and (6) above we see that we have a well defined homomorphism of algebras: $\psi : R(A^*(B), \Delta) \longrightarrow A^*(E(X))$ defined by $\psi(x_j) = [E(V(\rho_j))], 1 \leq j \leq d$.

Note that, by Remark 3.2, ψ is surjective. We need only prove that ψ is $1-1$. In view of Theorem 3.1, $A^*(E(X))$ is a free $A^*(B)$-module with basis $[E(V(\tau_i))]$, $1 \leq i \leq m$. It follows from Lemma 2.1(iv) that ψ is an isomorphism, completing the proof of 1.3(iii).

Proof of part (i) is similar. In view of equations (3), (5) above, $x_j \mapsto c_1(\mathcal{L}_j)$ defines a homomorphism $R(H^*(B), \Delta) \longrightarrow H^*(E(X);\mathbb{Z})$ which is indeed an isomorphism by 2.1(iv) and 3.1. □

Remark 3.3. If, instead of Δ being nonsingular, it is only assumed to be simplicial, then the toric variety X is only an orbifold. In this case Lemma 3.1 holds provided we replace integral homology by rational homology and the Chow group by the rational Chow group throughout. In this case we note that $[V_j].[V_j'] = q_j[V(\sigma_j)]$ for a rational number q_j and

$[V_j][V'_k] = 0$ for $j > k$. Computing the integral cohomology or Chow ring when the fibre X is only simplicial seems to be much more difficult. When X is a weighted projective space Al Amrani [2] has computed the integral cohomology of $E(X)$ in a more general setting.

4. K-theory

In this section we prove parts (ii) and (iv) of the main theorem.

In view of our assumption in 1.3 (iv) that both the base space B and the fibre X are smooth, the Grothendieck ring $\mathcal{K}^0(E(X))$ of algebraic vector bundles may be identified, via the duality isomorphism, with Grothendieck ring $\mathcal{K}_0(E(X))$ of coherent sheaves on $E(X)$. We shall denote either of them by $\mathcal{K}(E(X)$. Also if a smooth variety Y has an algebraic cell decomposition the forgetful map $\mathcal{K}^0(Y) \longrightarrow K(Y)$ is an isomorphism of rings. In particular, this holds when X is a complete nonsingular toric variety satisfying property $(*)$ (see §1).

We begin with the following lemma:

Lemma 4.1. *Suppose $\zeta_1, \cdots, \zeta_r$ are complex line bundles over a finite CW complex Y which has cells only in even dimensions such that $H^*(Y;\mathbb{Z})$ is generated by $c_1(\zeta_1), \cdots, c_1(\zeta_r) \in H^2(Y;\mathbb{Z})$. Then the ring $K^*(Y) = K^0(Y)$ is generated as a ring by $[\zeta_1], \cdots, [\zeta_r] \in K(Y)$.*

Proof: Let $f_i : Y \longrightarrow \mathbb{P}^N$ be a classifying map for the bundle ζ_i, $1 \leq i \leq r$ where $N > 1/2(\dim(Y))$. Consider the map $f : Y \longrightarrow (\mathbb{P}^N)^r$ which is defined as $f(y) = (f_1(y), \cdots, f_r(y))$. Then $f^* : H^*((\mathbb{P}^N)^r;\mathbb{Z}) \longrightarrow H^*(Y;\mathbb{Z})$ is easily seen to be a surjection. By the naturality of the Atiyah-Hirzebruch [7] spectral sequence it follows that f^* induces a surjection of K groups $K((\mathbb{P}^N)^r;\mathbb{Z}) \longrightarrow K(Y)$. Recall from [1] that $K(\mathbb{P}^N) = \mathbb{Z}[z]/\langle z^{N+1}\rangle$ where $z = [\omega] - 1$, ω being the class of the tautological line bundle on $\mathbb{P}^N$. Hence $K((\mathbb{P}^N)^r) = \mathbb{Z}[z_1, \cdots, z_r]/\langle z_i^{N+1},\ 1 \leq i \leq r\rangle$. Since f^* is *ring* homomorphism and since $f^*(z_i) = [\zeta_i] - 1$, the lemma follows. □

Lemma 4.2. *Suppose that Y is a complete nonsingular variety over $\mathbb{C}$ which has an algebraic cell decomposition and that $H^*(Y;\mathbb{Z})$ is generated as a ring by $H^2(Y;\mathbb{Z})$. Then there exist algebraic line bundles $\zeta_1, \cdots, \zeta_k$ over Y such that $\mathcal{K}(Y)$ is generated as a ring by $[\zeta_i], 1 \leq i \leq k$. In particular, the forgetful map $\theta : \mathcal{K}(Y) \longrightarrow K(Y)$ is an isomorphism.*

Proof: Since Y has an algebraic cell decomposition, the Chow ring is isomorphic to the cohomology ring $H^*(Y;\mathbb{Z})$, which is isomorphic as an abelian group to $\mathbb{Z}^m$ where m is the number of cells in Y. Since $A^*(Y)$ is free of rank m, it follows that $\mathcal{K}(Y)$ is also free of rank m. This can be seen as follows:

Recall that we have the "topological filtration" on $\mathcal{K}(Y)$ where $F^k(\mathcal{K}(Y)$ is the subgroup generated by $[\mathcal{O}_V]$ where V ranges over the subvarieties of codimension at least k. Let $Gr(\mathcal{K}(Y)$ denote the associated graded group for this filtration (see 15.1.5 of [**26**]). The map $\varphi : A^*(Y) \longrightarrow Gr(\mathcal{K}(Y))$ defined as $[V] \mapsto [\mathcal{O}_V]$, is a surjective homomorphism of groups. Therefore, since $A^*(Y)$ is free abelian of rank m it follows that $\mathcal{K}(Y)$ is a finitely generated abelian group of rank $\leq m$. Moreover, since the Chern character map $ch : \mathcal{K}(Y) \otimes \mathbb{Q} \longrightarrow A^*(Y) \otimes \mathbb{Q}$ is an isomorphism, it follows that rank $\mathcal{K}(Y)$= rank $A^*(Y)$. Thus we conclude that $\mathcal{K}(Y)$ is a free abelian group of rank m.

Further, since Y has a cell decomposition with cells only in even dimensions, we have $H^p(Y;\mathbb{Z}) = 0$ for p odd, and hence the Atiyah-Hirzebruch spectral sequence collapses. Moreover, since $H^*(Y;\mathbb{Z})$ is torsion free, it follows that $K(Y)$ is free abelian of rank m. (see p. 208 of [**7**]).

Let $a_1, \cdots, a_k$ be a $\mathbb{Z}$-basis for $H^2(Y;\mathbb{Z})$. Let $D_1, \cdots, D_k$ be divisors on Y such that $[D_i] \in A^1(Y)$ maps to $a_i \in H^2(Y;\mathbb{Z})$, $1 \leq i \leq k$. Since the first Chern class of $\mathcal{O}(D_i)$ is $a_i \in H^2(Y;\mathbb{Z})$, for $1 \leq i \leq m$, it follows that $[\mathcal{O}(D_i)] \in K(Y)$ generate $K(Y)$ as a ring. Thus, the forgetful homomorphism $\theta : \mathcal{K}(Y) \longrightarrow K(Y)$ is surjective. Since $K(Y)$ is free abelian of rank m it follows that θ is an isomorphism. In particular, $\mathcal{K}(Y)$ is generated as ring by $[\mathcal{O}(D_i)] \in \mathcal{K}(Y)$, $1 \leq i \leq k$. □

REMARK 4.3. Examples of varieties which satisfy the hypothesis of the above lemma are (complete nonsingular) toric varieties $X(\Delta)$ where Δ satisfies $(*)$ and $SL_n(\mathbb{C})/B$. However note that the conclusion of the lemma holds for all G/B where G is semi simple and $B \subset G$ a Borel subgroup and smooth Schubert varieties in G/B. Indeed $\mathcal{K}(G/B) = K(G_c/B_c) = R(G_c/B_c) = R(B_c) \otimes_{R(G_c)} \mathbb{Z}$ where, G_c denotes the maximal compact subgroup of G and $B_c = G_c \cap B$ is the maximal torus of G_c. Further, it is a free $\mathbb{Z}$ module of rank $|W(G)|$ where $W(G)$ denotes the Weyl group of G (see [**7**] and [**37**]).

Our next result gives a description of the K-ring of X. We keep the notations of the introduction.

Recall the definition of $\mathcal{R}$ from §1.

Proposition 4.4. *Let $X = X(\Delta)$ be a nonsingular complete toric variety where Δ satisfies the property $(*)$. The following relations hold in $\mathcal{K}(X)$ and $K(X)$:*
(i) $[\mathcal{O}_{V(\rho_{j_1})}]\cdots[\mathcal{O}_{V(\rho_{j_r})}] = 0$ if $v_{j_1},\cdots,v_{j_r}$ do not span a cone of Δ,
(ii) $\prod_{1\leq j\leq d}[L_j]^{\langle u,v_j\rangle} = 1$,
(iii) Set $r_i = 1 \in \mathbb{Z}$, $1 \leq i \leq n$. The homomorphism of rings $\theta : \mathcal{R} = \mathcal{R}(\mathbb{Z},\Delta)\longrightarrow\mathcal{K}(X) \cong K(X)$ defined by $x_i \mapsto [\mathcal{O}_{V(\rho_i)}] = (1 - [L_i^\vee])$ is an isomorphism.

Proof: Recall that $[\mathcal{O}_Y].[\mathcal{O}_Z] = [\mathcal{O}_{Y\cap Z}]$ if Y, Z are closed irreducible subvarieties of X which meet transversally. Relation (i) follows from the fact that $V(\rho_{j_1})\cap\cdots\cap V(\rho_{j_r}) = \emptyset$ if $v_{j_1},\cdots,v_{j_r}$ does not span a cone of Δ. Since for any $u \in M$ we have $\langle -u, v_j\rangle = \sum_{1\leq p\leq d}\langle u, v_p\rangle\psi_p(v_j)$, it follows that one has a T-equivariant isomorphism of bundles $\prod_{1\leq p\leq d} L_p^{\langle u,v_p\rangle} \cong L_u$, where L_u is the line bundle corresponding to the piecewise linear function $-u : |\Delta| = N_{\mathbb{R}}\longrightarrow\mathbb{R}$. But L_u is isomorphic to the trivial line bundle and so (ii) follows.

Now the section $s_j : X\longrightarrow L_j$ vanishes to order 1 on $V(\rho_j)$. Hence we have an exact sequence of coherent sheaves for $1 \leq j \leq d$: $0\longrightarrow L_j^\vee\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_{V(\rho_j)}\longrightarrow 0$. Thus $[\mathcal{O}_{V(\rho_j)}] = (1-[L_j^\vee])$ in $\mathcal{K}(X)$, i.e., $(1-[\mathcal{O}_{V(\rho_j)}]) = L_j^\vee$. Hence $x_j \mapsto [\mathcal{O}_{V(\rho_j}]$ defines a ring homomorphism $\theta : \mathcal{R}\longrightarrow\mathcal{K}(X)$. Since $\mathcal{K}(X)$ is free abelian of rank m and since by Lemma 2.2 $\mathcal{R}$ is generated by m elements $x(\tau_i), 1 \leq i \leq m$, it follows that θ is an isomorphism, completing the proof. □

Remark 4.5. Suppose $\eta_1,\cdots,\eta_k$ are line bundles over Y such that their Whitney sum $\eta := \oplus_{1\leq i\leq k}\eta_i$ admits a nowhere vanishing section, then, applying the γ^k-operation, we obtain $\gamma^k(\eta - k) = 0$. On the other hand, $\gamma^k([\eta] - k) = \gamma^k(\oplus_{1\leq i\leq k}([\eta_i] - 1)) = \prod([\eta_i] - 1)$. Hence, $\prod(1 - [\eta_i]) = 0$. Thus, one can avoid the use of the coherent sheaves in the proof of (i) above in the case of $K(X(\Delta))$ since we know that the section $s = (s_{j_1},\cdots,s_{j_r})$ of the bundle $\oplus_{1\leq p\leq r}L_{j_p}$ is nowhere vanishing whenever $v_{j_1},\cdots,v_{j_r}$ does not span a cone of Δ.

COROLLARY 4.6. *(i) The elements* $[\mathcal{O}_{V(\tau_i)}] \in \mathcal{K}(X), 1 \leq i \leq m$, *form a* $\mathbb{Z}$*-basis for* $\mathcal{K}(X)$. *(ii) Let* $L(\tau_i) = \prod_{j, v_j \in \tau_i} L_j$ *for* $1 \leq i \leq m$. *Then* $[L(\tau_i)], 1 \leq i \leq m$, *form a* $\mathbb{Z}$*-basis for* $K(X)$.

Proof: This follows from the proof of 4.4 (iii). □

Recall that $\mathcal{L}_j = E(L_j)$ is the line bundle over $E(X)$ with total space $E \times_T L_j$. Denote by $\mathcal{L}(\tau_i)$ the line bundle $E(L(\tau_i)) = \mathcal{L}_{j_1} \otimes \cdots \otimes \mathcal{L}_{j_r}$, where $v_{j_1}, \cdots, v_{j_r}$ are the primitive vectors along the edges of τ_i. In view of proposition 4.6 (ii), the restriction of the bundles $\mathcal{L}(\tau_i)$, $1 \leq i \leq m$, to the fibre X form a $\mathbb{Z}$-basis for $K^*(X) = K^0(X)$. Further, since the base B is compact and Hausdorff (hence locally compact), we can find a finite closed cover $\{W_j\}$ of B such that $\pi |_{\pi^{-1}(W_j)}: \pi^{-1}(W_j) \to W_j$ is a trivial bundle for every j. Let Y be any closed subset of W_j. Then both Y and X are compact and $\pi^{-1}(Y) \simeq Y \times X$. Further, since $K^*(X)$ is free abelian, by the Künneth theorem for compact spaces it follows that the restrictions of $\mathcal{L}(\tau_i)$, $1 \leq i \leq m$, to $K^*(\pi^{-1}(Y))$ form a basis of $K^*(\pi^{-1}(Y))$ as a $K^*(Y)$ module (see [**10**]). Thus the hypotheses needed for applying Theorem 1.3, p.181, Chap. IV of [**29**] are satisfied. Hence it follows that $K^*(E(X))$ is a free $K^*(B)$-module with basis $\mathcal{L}(\tau_i)$, $1 \leq i \leq m$. (See Theorem 2.7.8 of [**5**] and also [**6**] for classical Künneth theorem for CW complexes).

Suppose $v_{j_1}, \cdots, v_{j_r}$ do not span a cone of Δ. The T-equivariant section $s = (s_{j_1}, \cdots, s_{j_r})$ of $L_{j_1} \oplus \cdots \oplus L_{j_r}$ is nowhere vanishing and extends to a nowhere vanishing section $E(s)$: $E(X) \longrightarrow \mathcal{L}_{i_1} \oplus \cdots \oplus \mathcal{L}_{i_r}$. Hence by Remark 4.5,

$$\prod_{1 \leq p \leq r} (1 - \mathcal{L}_{j_p}) = 0. \tag{7}$$

Now assume that $p : E \longrightarrow B$ is algebraic and B irreducible, nonsingular and noetherian over $\mathbb{C}$. Since the T-equivariant sections s_j are algebraic, equation (7) holds in $\mathcal{K}(E(X))$ as well.

For any $u \in M$, the T-equivariant isomorphism of bundles $\prod_{1 \leq j \leq d} L_j^{\langle u, v_j \rangle} \cong L_u$ yields an isomorphism of vector bundles $\prod_{1 \leq j \leq d} \mathcal{L}_j^{\langle u, v_j \rangle} \cong E(L_u)$. Since $E(L_u) = \prod_{1 \leq i \leq n} \xi_i^{-\langle u, v_i \rangle}$, we get

$$\prod_{1 \leq j \leq d} \mathcal{L}_j^{\langle u, v_j \rangle} \cong \pi^*(\xi_u^\vee) \tag{8}$$

where $\xi_u = \prod_{1\leq i\leq n} \xi_i^{\langle u, v_i\rangle}$.

We are now ready to prove the remaining parts of 1.3.

Proof of Theorem 1.3(ii), (iv): In view of equations (7) and (8), one has a well-defined homomorphism of $K(B)$-algebras $\Psi : \mathcal{R} = \mathcal{R}(K^*(B), \Delta) \longrightarrow K^*(E(X))$ defined by $x_j \longrightarrow (1 - [\mathcal{L}_j]), 1 \leq j \leq d$.

Since the $x(\tau_i), 1 \leq i \leq m$ span $\mathcal{R}$ by Lemma 2.2 (iv) and since $K^*(E(X))$ is a free $K^*(B)$ module of rank m, it follows that Ψ is an isomorphism, completing the proof of (ii).

Now let B be an irreducible, nonsingular, noetherian variety over $\mathbb{C}$ and let $p : E \longrightarrow B$ be algebraic. Equations (7) and (8) still hold in $\mathcal{K}(E(X))$ since the equivariant sections s_j are algebraic. Proceeding as above, we see that to complete the proof of 1.3 (iv), we need only show that $[\mathcal{O}_{E(V_i)}]$, $1 \leq i \leq m$, form a basis for $\mathcal{K}(E(X))$ as a $\mathcal{K}(B)$-module, where V_i stands for $V(\tau_i)$. Let $\Phi : \mathcal{K}(B) \otimes \mathcal{K}(X) \longrightarrow \mathcal{K}(E(X))$ be the $\mathcal{K}(B)$-linear map defined by $\sum_{1\leq i\leq m} b_i \otimes [\mathcal{O}_{V_i}] \mapsto \sum_{1\leq i\leq m} \pi^*(b_i)[\mathcal{O}_{E(V_i)}]$, $1 \leq i \leq m$. In view of 4.6(i), we need only show that Φ is an isomorphism.

We first prove surjectivity of Φ. This is proved by induction on the dimension of B, assuming only that B is noetherian over $\mathbb{C}$. Without loss of generality we may assume that B is irreducible. If B is a point, then the result is obvious. Suppose that $\dim(B) > 0$. Let U be an affine open set in B over which the T-bundle $p : E \longrightarrow B$ is trivial and let $Z = B \setminus U$ (with its reduced scheme structure). Note that Z may not be irreducible but $\dim(Z_k) < \dim(B)$ for each irreducible component Z_k of Z. By inductive hypothesis, $\mathcal{K}_0(Z) \otimes \mathcal{K}(X) \longrightarrow \mathcal{K}(\pi^{-1}(Z))$ is surjective homomorphism of abelian groups. Consider the commuting diagram of abelian groups and their homomorphisms:

$$\begin{array}{ccccccc}
\mathcal{K}_0(Z) \otimes \mathcal{K}(X) & \longrightarrow & \mathcal{K}_0(B) \otimes \mathcal{K}(X) & \longrightarrow & \mathcal{K}_0(U) \otimes \mathcal{K}(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{K}_0(\pi^{-1}(Z)) & \longrightarrow & \mathcal{K}_0(E(X)) & \longrightarrow & \mathcal{K}_0(\pi^{-1}(U)) & \longrightarrow & 0
\end{array}$$

where the horizontal rows are exact. The top horizontal row is got by tensoring with $\mathcal{K}(X)$ the exact sequence $\mathcal{K}_0(Z) \longrightarrow \mathcal{K}_0(B) \longrightarrow \mathcal{K}_0(U) \longrightarrow 0$. By Prop. 2.13, ch. II, (Exp. 0-App.), p.60, [9], the homomorphism $\mathcal{K}(U) \otimes \mathcal{K}(X) \longrightarrow \mathcal{K}(\pi^{-1}(U))$ is surjective. It follows that the homomorphism $\Phi : \mathcal{K}_0(B) \otimes \mathcal{K}(X) \longrightarrow \mathcal{K}_0(E(X))$ is a surjection.

Now we prove that Φ is a monomorphism. Suppose $\Phi(\sum_{1\leq i\leq m} b_i[\mathcal{O}_{V_i}]) = 0$ where b_i is non-zero for some i. Let $p \geq 1$ be the least so that $b_p \neq 0$. Then, writing V'_p for $V(\tau'_p)$, we have

$0 = [\mathcal{O}_{E(V'_p)}].(\sum_{1\leq i\leq m} \pi^*(b_i).[\mathcal{O}_{E(V_i)}]) = \sum_{p\leq i\leq m} \pi^*(b_i)[\mathcal{O}_{E(V'_p)\cap E(V_i))}]$
$= \pi^*(b_p)[\mathcal{O}_{E(V(\sigma_p))}]$,
since Δ satisfies property $(*')$.

Denote by π_p the restriction of π to $E(V(\sigma_p))$ and by ι_p the inclusion $E(V(\sigma_p)) \subset E(X)$. Then the homomorphism $\iota_{p*} : \mathcal{K}(E(V(\sigma_p))) \longrightarrow \mathcal{K}(E(X))$ maps $[\mathcal{O}_{E(V(\sigma_p))}] = 1 \in \mathcal{K}(E(V(\sigma_p)))$ to $[\mathcal{O}_{E(V(\sigma_p))}] \in \mathcal{K}(E(X))$. Also, π_p is an isomorphism of varieties. Therefore, applying π_* to the expression $0 = \pi^*(b_p)[\mathcal{O}_{E(V(\sigma_p))}]$ and using the projection formula (§15.1, [26]) we get

$$0 = \pi_*(\pi^*(b_p)[\mathcal{O}_{E(V(\sigma_p))}]) = b_p.\pi_*\iota_{p*}([\mathcal{O}_{E(V(\sigma_p))}]) = b_p.\pi_{p*}([\mathcal{O}_{E(V(\sigma_p))}]) = b_p[\mathcal{O}_B] = b_p.$$

This contradicts our choice of p. Hence we conclude that Φ is a monomorphism. □

Concluding remark 4.6. Parts (iii) and (iv) of the main theorem also hold when the base field $\mathbb{C}$ is replaced by any algebraically closed field k. Namely, let B be an irreducible nonsingular noetherian variety over k and let $\pi : E \longrightarrow B$ be a principal T bundle where $T = Spec(k[M])$. Since any toric variety is defined over the integers the fan Δ defines a nonsingular complete k-scheme $X = X(\Delta)$. Again $E(X) \longrightarrow B$ is a Zariski locally trivial bundle with fibre X. Then $\mathcal{K}(E(X))$ and $A^*(E(X))$ are isomorphic to $\mathcal{R}$ and R respectively.

CHAPTER 2

Topology of real toric varieties

1. Basic definitions and statement of results

1.0.1. Notations:

$N \cong \mathbb{Z}^n$, $M = Hom(N, \mathbb{Z})$ and $\langle , \rangle$= the dual pairing.

$N_{\mathbb{R}} = N \bigotimes_{\mathbb{Z}} \mathbb{R}$. Δ = smooth fan in $N_{\mathbb{R}}$; σ and τ denote cones in Δ.

Let σ be a cone in Δ. $S_\sigma = \sigma^\vee \cap M = \{u \in M : \langle u, v\rangle \geq 0 \ \forall\ v \in \sigma\}$.

$\Delta(k)$ = cones of dimension k. $\Delta(1)$ are the edges and $\#\Delta(1) = d$.

$\Delta(1) = \{\rho_1, \rho_2, \ldots, \rho_d\}$. Let v_j be the primitive vector along ρ_j then, $\langle v_{i_1}, \ldots, v_{i_k}\rangle$ denotes the cone spanned by $\{v_{i_1}, \ldots, v_{i_k}\}$.

$(U_\sigma)_{\mathbb{C}} = Hom_{sg}(S_\sigma, \mathbb{C})$, $U_\sigma = Hom_{sg}(S_\sigma, \mathbb{R})$ and $(U_\sigma)_+ = Hom_{sg}(S_\sigma, \mathbb{R}_+)$ $\forall\ \sigma \in \Delta$ where $\mathbb{R}_+ = \mathbb{R}^+ \cup \{0\}$. Here, Hom_{sg} denotes the semigroup homomorphisms which sends 0 to 1.

X = smooth real toric variety of dimension n associated to Δ.

$X_{\mathbb{C}}$ = the complex toric variety whose real part is X.

X_+ = the non-negative part of X.

$T_2 := Hom(M, \mathbb{Z}_2) \hookrightarrow T_{\mathbb{R}} := U_{\{0\}} = Hom(M, \mathbb{R}^*)$; $T_{\mathbb{C}} := Hom(M, \mathbb{C}^*)$; $T_+ := Hom(M, \mathbb{R}^+)$.

For every $\sigma \in \Delta$, $x_\sigma \in U_\sigma$ is a *distinguished point* defined as:

$$x_\sigma(u) = \begin{cases} 1 & \forall\ u \in \sigma^\perp \\ 0 & otherwise \end{cases}$$

O_τ = orbit of x_τ under the action of $T' \simeq (\mathbb{R}^*)^n$ and $V(\tau) = \overline{O_\tau}$.

$Stab(x_\tau)$= stabilizer of x_τ under the action of T_2.

$(O_\tau)_+$ = orbit of x_τ under the action of $(\mathbb{R}^+)^n$ and $V(\tau)_+ = \overline{(O_\tau)}_+$.

$W(\Delta) = \langle\ s_j : j = 1, 2 ..., d \ \mid\ s_j^2 : 1 \leq j \leq d,\ (s_i s_j)^2\ whenever\ \langle v_i, v_j\rangle \in \Delta\ \rangle$. Then, $W(\Delta)$ is a *right-angled Coxeter group* associated to Δ. In many places when the context is clear, we shall denote $W(\Delta)$ simply by W.

$S_N := (N_{\mathbb{R}} - \{0\})/\mathbb{R}_{>0}$ be the sphere in $N_{\mathbb{R}}$ and let $\pi : N_{\mathbb{R}} - \{0\} \longrightarrow S_N$ be the projection.

Let Δ be a smooth fan in the lattice $N \cong \mathbb{Z}^n$. Let $X_{\mathbb{C}}(\Delta)$ be the complex toric variety associated to Δ. Let $X_{\mathbb{R}}(\Delta)$ be the real valued points of $X_{\mathbb{C}}(\Delta)$ which we call the real toric manifold associated to Δ. For the definition and basic facts on real toric manifolds see Chapter 4. of [**25**] and §2 of [**27**]. We mainly follow [**25**] for notations and background material on toric varieties.

Before we state the main theorems let us fix the following terminology:

Let $\Delta(1)$ denote the edges of Δ, $d = \#\Delta(1)$, and let $\{v_1, v_2, \ldots\ldots, v_d\}$ denote the primitive vectors along the edges. We assume that $\{v_1, v_2, \ldots, v_n\}$ form a basis for the lattice N and let $\{u_1, \ldots, u_n\}$ be the dual basis in M.

Let $W(\Delta) = \langle s_{j_1}, \ldots s_{j_d} \mid s_j^2 : 1 \leq j \leq d, (s_i s_j)^2$ whenever the cone spanned by $\{v_i, v_j\}$ is in $\Delta\rangle$ be the right-angled Coxeter group associated to Δ.

We call the fan Δ *flag-like* if and only if the following condition holds for every collection of primitive edge vectors $\{v_{i_1}, \ldots, v_{i_r}\}$ in Δ: if for every $1 \leq k, l \leq r$, $\{v_{i_k}, v_{i_l}\}$ spans a cone in Δ, then $\{v_{i_1}, \ldots, v_{i_r}\}$ together spans a cone in Δ.

Let $\mathcal{C}_\Delta$ be the real valued points of the total space constructed by Cox (see [**16**]) for which the quotient under the action of $(\mathbb{C}^*)^{d-n}$ is the complex toric variety $X_{\mathbb{C}}(\Delta)$. It is also the complement of a coordinate subspace arrangement in $\mathbb{R}^d$.

Let Δ be a smooth fan and that the primitive vectors along $\Delta(1)$ span $N \otimes \mathbb{Z}_2$ so that $X(\Delta)$ is a smooth and connected real toric variety. (We shall see later that the condition that the the primitive vectors along $\Delta(1)$ span $N \otimes \mathbb{Z}_2$ is the necessary and sufficient condition for for $X(\Delta)$ to be connected.) We now state the main results.

THEOREM 1.1. *The fundamental group $\pi_1(X_{\mathbb{R}}(\Delta))$ is abelian if and only if one of the following holds in Δ.*

(i) *For every $1 \leq i, j \leq d$, $\{v_i, v_j\}$ spans a cone in Δ. In this case, $\pi_1(X_{\mathbb{R}}(\Delta))$ is isomorphic to $\mathbb{Z}_2^{d-n}$.*

(ii) *For each $1 \leq j \leq d$ there exists at most one $i = i_j$ with $1 \leq i_j \leq n$ such that, $\{v_{i_j}, v_j\}$ does not span a cone in Δ and $\langle u_{i_j}, v_j\rangle = 1 \bmod 2$. Further, for each $n+1 \leq k \leq d$ such that $k \neq j$ we have, $\langle u_{i_j}, v_k\rangle = 0 \bmod 2$.*

THEOREM 1.2. *The real toric manifold $X_{\mathbb{R}}(\Delta)$ is aspherical if and only if Δ is flag-like.*

THEOREM 1.3. *Let C_Δ be the complement of the subspace arrangement related to Δ as above. Then, $\pi_1(C_\Delta)$ is isomorphic to the commutator subgroup of $W(\Delta)$. Further, C_Δ is aspherical if and only if it is the complement of a union of codimension 2 subspaces.*

We prove Theorem 1.1 in §5, Theorem 1.2 in §6 and Theorem 1.3 in §7.

2. The Universal Cover of $X(\Delta)$

In this section we shall determine the universal cover and the fundamental group of X. For this purpose, we primarily apply the contents of pages 367-386 of Chapter II.12 of [**11**].

We begin with the elementary topological description of a real toric variety in the following proposition. The proof essentially follows from the proposition on p.79, Chapter 4. of [**25**] by replacing $X_{\mathbb{C}}$ by X and $\mathbb{S}^1$ by $\mathbb{S}^1 \cap \mathbb{R} \simeq \mathbb{Z}_2$. For details, also see p.36, §3 of [**27**].

PROPOSITION 2.1. *([**25**],[**27**]) There is a retraction, $X_+ \overset{i}{\hookrightarrow} X \overset{r}{\rightarrow} X_+$ given by the absolute value map, $x \mapsto |x|$ from $\mathbb{R}_+ \subset \mathbb{R} \rightarrow \mathbb{R}_+$ which identifies X_+ with the quotient space of X by the action of the compact real 2-torus, $T_2 = Hom(M, \mathbb{Z}_2)$. Further, there is a canonical mapping $T_2 \times X_+ \rightarrow X$ which realizes X as a quotient space, $T_2 \times X_+/\sim$ where, $(t,x) \sim (t',x')$ if and only if $x = x'$ and $t \cdot (t')^{-1} \in Stab(x_\tau)$ where, $x \in (O_\tau)_+$. The retraction $X \rightarrow X_+$ maps O_τ to $(O_\tau)_+$ and $V(\tau)$ to $V(\tau)_+$ and the fiber over $(O_\tau)_+$ is $T_\tau := Hom(\tau^\perp \cap M, \mathbb{Z}_2)$ which is a compact real 2-torus of dimension $n - dim(\tau)$.*

We now observe the following property of X_+.

LEMMA 2.2. *X_+ is contractible.*

Proof: Recall that $x_{\{0\}}$ is the distinguished point of $(U_{\{0\}})_+ \simeq (\mathbb{R}^+)^n$. We first show that for every $\sigma \in \Delta$, $(U_\sigma)_+ = Hom_{sg}(S_\sigma, \mathbb{R}_+)$ is contractible to the point $x_{\{0\}} \in (U_{\{0\}})_+ \subseteq (U_\sigma)_+$. This is because, $(1-t) \cdot x + t \cdot x_{\{0\}} \in Hom_{sg}(S_\sigma, \mathbb{R}_+)$ for every $t \in I = [0,1]$. The only thing we need to check here is that, if both $u, -u \in S_\sigma$ then $((1-t) \cdot x + t \cdot x_{\{0\}})(u) = (1-t) \cdot x(u) + t \cdot x_{\{0\}}(u) > 0$. This clearly holds since, $x(u) > 0$ and $\mathbb{R}^+$ is convex. Thus the map $H_\sigma : (U_\sigma)_+ \times I \rightarrow (U_\sigma)_+$ defined as $H_\sigma(x,t) = (1-t) \cdot x + t \cdot x_{\{0\}}$ is a strong deformation retraction of $(U_\sigma)_+$ to the point $x_{\{0\}}$.

Furthermore by definition, H_σ's for $\sigma \in \Delta$ are compatible with the inclusions $(U_\tau)_+ \subseteq (U_\sigma)_+$ whenever $\tau < \sigma$ in Δ. Therefore since X_+ is the union of $(U_\sigma)_+$'s for $\sigma \in \Delta$, we can glue together the maps $\{H_\sigma\}_{\sigma\in\Delta}$ to get a strong deformation retraction H of X_+ to $x_{\{0\}}$. Hence the lemma. □

PROPOSITION 2.3. *Let Δ be a smooth fan. We then have the following.*

(1) *$(X_+, (V(\tau)_+)_{\tau\in\Delta})$ is a stratified space with strata $\{V(\tau)_+\}_{\tau\in\Delta}$ indexed by the poset Δ.*
(2) *Associated to this stratified space we have a simple complex of groups $G(\Delta) = (G_\tau, \psi_{\sigma\tau})$ where the local group at the stratum $V(\tau)_+$ is $G_\tau = Stab(x_\tau)$ under the action of $T_2 = Hom(M, \mathbb{Z}_2)$ and $\psi_{\sigma\tau} : G_\tau \to G_\sigma$ (for $\tau < \sigma$ in Δ) are canonical inclusions and we have a simple morphism $\varphi = (\varphi_\tau) : G(\Delta) \to T_2 \simeq \mathbb{Z}_2^n$ injective at the local groups.*
(3) *For the above simple complex of groups $G(\Delta) = (G_\tau, \psi_{\sigma\tau})$, the direct limit $\widehat{G(\Delta)}$ is isomorphic to $W(\Delta)$. We therefore have a canonical simple morphism $\iota = (\iota_\tau) : G(\Delta) \to W(\Delta)$.*

Proof:

Proof of (1). Since the orbit space decomposition of X_+ under the action of T_+ is obtained by restriction of scalars from that of $X_\mathbb{C}$ under the action of $T_\mathbb{C}$, it follows that $(X_+, V(\tau)_+)$ is a stratified space with strata $V(\tau)_+$ indexed by Δ.

Proof of (2). Let $G_\tau = Stab(x_\tau) \subseteq T_2$. We then have canonical inclusions, $\psi_{\sigma\tau} : G_\tau \hookrightarrow G_\sigma$ whenever $\tau < \sigma$ in Δ and, $\varphi_\tau : G_\tau \hookrightarrow T_2$ for every $\tau \in \Delta$. Then $G(\Delta) = (G_\tau, \psi_{\sigma\tau})$ is a simple complex of groups over $(X_+, V(\tau)_+)$ where G_τ is the local group along the stratum $V(\tau)_+$. Further, $\varphi = (\varphi_\tau)_{\tau\in\Delta} : G(\Delta) \to T_2$ is a simple morphism injective at the local groups.

Proof of (3). $\widehat{G(\Delta)}$ is by definition the free product of G_τ with the relations $\psi_{\sigma\tau}(h) = h \; \forall \, h \in G_\tau$ whenever $\tau < \sigma$ in Δ. Thus, $\widehat{G(\Delta)}$ is simply the graph product of the vertex groups $G_{\rho_j} \simeq \mathbb{Z}_2$ over the graph $S_N \cap \Delta(2)$ where the vertices of the graph correspond to $\Delta(1)$ and the edges correspond to $\Delta(2)$. Therefore $\widehat{G(\Delta)} \simeq W(\Delta)$ and (3) follows. □

Let G be a group for which there exists a simple morphism $\varphi : G(\Delta) \to G$ injective at the local groups. Then $G \times X_+/\sim := \{(g,x) : g \in G,\ x \in X_+ \ :\ (g,x) \sim (g',x') \Leftrightarrow x = x'\ ;\ g\cdot(g')^{-1} \in G_\tau\}$ where τ is the unique cone such that $x \in O_\tau$. Let $D(\Delta,\varphi) = \sqcup_{\tau\in\mathcal{Q}} G/G_\tau$. Then $D(\Delta,\varphi)$ is a poset consisting of pairs $(g\cdot G_\tau, \tau)$ where $\tau \in \Delta$ and $g \cdot G_\tau$ is a coset of G_τ in G and has the partial order $(g\cdot G_\sigma,\sigma) < (g'\cdot G_\tau,\tau)$ if and only if $\sigma < \tau$ in Δ and $(g')^{-1}\cdot g \in G_\sigma$.

Lemma 2.4. *X is a stratified space over $D(\Delta,\varphi)$. Furthermore, the T_2 action on X is strata preserving, with X_+ as the strict fundamental domain.*

Proof : By definition, $(T_2 \times X_+/\sim)$ is a stratified space over $D(\Delta,\varphi)$ such that the action of T_2 on $T_2 \times X_+/\sim$ is strata preserving where, $t \in T_2$ takes the stratum $(t', V(\tau)_+)$ to the stratum $(tt', V(\tau)_+)$. A strict fundamental domain for this action is the copy $1 \times X_+$ corresponding to the identity element $1 \in T_2$. However, by Proposition 2.1 there is a canonical T_2-equivariant isomorphism from $(T_2 \times X_+)/\sim$ to X. Thus X gets a structure of a stratified space over $D(\Delta,\varphi)$ in such a way that the action of T_2 on X is strata preserving and the strict fundamental domain for this action is $X_+ \subseteq X$. □

Theorem 2.5.

(1) *Let $D(X_+,\varphi)$ and $D(X_+,\iota)$ denote the developments of X_+ with respect to φ and ι respectively. Then $D(X_+,\varphi) \simeq (T_2 \times X_+/\sim) \simeq X$ and $D(X_+,\iota) \simeq (W \times X_+/\sim) =: \widetilde{X}$. There are strata preserving actions of W on $D(X_+,\iota)$ and of T_2 on $D(X_+,\varphi)$ with strict fundamental domain X_+.*

(2) *X is connected if and only if the primitive vectors along the edges of Δ span $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$. In particular, X is connected whenever the primitive vectors contain a $\mathbb{Z}$ basis for N.*

(3) *$\widetilde{X} = W \times X_+/\sim$ is the universal cover of X and $\pi_1(X) \simeq ker(\widehat{\varphi})$ where, $\widehat{\varphi} : W \to T_2$ is the canonical homomorphism induced by φ.*

(4) *Let $h : W \to W_{ab} \simeq \mathbb{Z}_2^d$ be the surjective group homomorphism obtained by abelianisation. Associated to the map h, we have a simple morphism $\alpha : G(\Delta) \to \mathbb{Z}_2^d$ such that $\widehat{\alpha} = h$. Then $D(X_+,\alpha) \simeq \mathbb{Z}_2^d \times X_+/\sim$ is a covering space over X with deck transformation group $\mathbb{Z}_2^{d-n}$, it is a covering space of X and $\pi_1(D(X_+,\alpha)) = [W,W]$.*

Proof : To prove this theorem we use Prop. 12.20 of [**11**].

Proof of (1). By Prop.2.3, the development $D(X_+, \varphi)$ of X_+ with respect to the simple morphism φ from the simple complex of groups $G(\Delta)$ over X_+ to T_2 is a stratified space over $D(\Delta, \varphi)$ and is isomorphic to $T_2 \times X_+/\sim$ in such a way that the induced action of T_2 on $T_2 \times X_+/\sim$ is identical to that in Lemma 2.4. Hence by Lemma 2.4, $D(X_+, \varphi)$ is isomorphic to X as a stratified space and further the isomorphism is equivariant under the strata preserving action of T_2. Similarly the development $D(X_+, \iota)$ of X_+ with respect to the canonical simple morphism ι from $G(\Delta)$ to W is isomorphic to $(W \times X_+/\sim)$, which is a stratified space over the poset $D(\Delta, \iota)$ and further there is a strata preserving action of W on $D(X_+, \iota)$ with strict fundamental domain X_+.

Proof of (2). From Lemma 2.2 X_+ is contractible, in particular it is connected. Hence $D(X_+, \varphi)$ is connected if and only if $\widehat{\varphi}$ is surjective which is equivalent to the assumption that the image of the primitive edge vectors span $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$. This certainly happens if a part of the primitive vectors along $\Delta(1)$ form a $\mathbb{Z}$-basis for N.

Proof of (3). From Lemma 2.2 it follows that X_+ is simply connected and the strata of X_+ are arcwise connected. Further, from Prop.2.3 we know that $\widehat{G(\Delta)} \simeq W$. Therefore $(W \times X_+/\sim) \simeq D(X_+, \iota)$ is the universal cover of $X \simeq D(X_+, \varphi)$ and $ker(\widehat{\varphi}) \simeq \pi_1(X)$ where $\widehat{\varphi}$ is the canonical surjective homomorphism induced by φ.

Proof of (4). Since $\alpha = h \circ \iota$ and T_2 being abelian $\widehat{\varphi}$ factors through W_{ab}. Therefore the simple morphism α is injective at the local groups and the development $D(X_+, \alpha) \simeq \mathbb{Z}_2^d \times X_+/\sim$. The remaining claims of (4) follow simply by the direct application Prop 12.20 of [**11**]. □

REMARK 2.6. (*Connectedness of X*) If the primitive edge vectors of $\Delta(1)$ do not span $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$, then X is not connected and the number of connected components of X is equal to $[N \otimes_{\mathbb{Z}} \mathbb{Z}_2 : \varphi(W)]$. In fact Δ is supported on a smaller dimensional lattice and therefore X is isomorphic to $X' \times (\mathbb{R}^*)^{(k/2)}$ where $k = [N \otimes_{\mathbb{Z}} \mathbb{Z}_2 : \varphi(W)]$ and X' is a connected toric variety of dimension $n-k$. For example, the real toric variety associated to the fan $\Delta = \{e_1, -e_1, \{0\}\}$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^*$ and has two connected components $\mathbb{S}^1 \times \mathbb{R}^+$ and $\mathbb{S}^1 \times \mathbb{R}^-$. Indeed for X to be connected it is not necessary that the primitive edge vectors should span N. For example, the real toric variety associated to the fan $\Delta = \{\langle 2e_1 + 3e_2\rangle, \langle e_1\rangle, \{0\}\}$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ is smooth and connected but the edge vectors $\{2e_1 + 3e_2, e_1\}$ do not form a $\mathbb{Z}$-basis for N.

3. A presentation for $\pi_1(X)$

Let X be smooth and connected. In this section we shall give a presentation for $\pi_1(X)$ with generators and relations defined purely from the combinatorial structure of Δ.

Let $\{v_1, \dots, v_n\}$ be primitive vectors along $\Delta(1)$ which form a basis for $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and let $\{u_1, \dots, u_n\}$ be the dual basis. Let $a_{j,i} = \langle u_i, v_j\rangle \ mod\ \mathbb{Z}_2$ for $1 \leq j \leq d$ and $1 \leq i \leq n$. Then $A = (a_{j,i})$ is the *characteristic matrix of* Δ *with respect to* $\{v_1, \dots, v_n\}$.

For $\underline{t} = (t_1, \dots, t_n) \in \mathbb{Z}_2^n$, let $b_i^j = t_i + a_{j,i}$ for $1 \leq i \leq n, 1 \leq j \leq d$ and let $c_i^{p,q} = t_i + a_{p,i} + a_{q,i}$ for $1 \leq i \leq n$; $1 \leq p, q \leq d$. We shall denote the vector $(b_i^j)_{i=1,\dots,n}$ by $\underline{b}^j$ and the vector $(c_i^{p,q})_{i=1,\dots,n}$ by $\underline{c}^{p,q}$.

In the following proposition we will give a presentation for $\pi_1(X)$ using the above data.

PROPOSITION 3.1. *The fundamental group $\pi_1(X)$ has a presentation with generators*

$$\{y_{j,\underline{t}} : 1 \leq j \leq d \mid \underline{t} = (t_1, \dots, t_n) \in \mathbb{Z}_2^n\}$$

and relations

$$\bigcup_{\underline{t} \in \mathbb{Z}_2^n} \{y_{1,(0,\dots,0)}^{t_1} \cdot y_{2,(t_1,0,\dots,0)}^{t_2} \cdots y_{n,(t_1,\dots,t_{n-1},0)}^{t_n}\}$$

$$\bigcup_{\underline{t} \in \mathbb{Z}_2^n} \{y_{1,\underline{t}} \cdot y_{j,\underline{b}^j} \mid 1 \leq j \leq d\}$$

$$\bigcup_{\underline{t} \in \mathbb{Z}_2^n} \{y_{p,\underline{t}} \cdot y_{q,\underline{b}^p} \cdot y_{p,\underline{c}^{p,q}} \cdot y_{q,\underline{b}^q} \mid \langle v_p, v_q\rangle \in \Delta\}$$

Proof: We know from Theorem 2.5 that $\pi_1(X)$ is isomorphic to the kernel of the surjective homomorphism $\widehat{\varphi} : W \to T_2 \simeq \mathbb{Z}_2^n$ where W has the presentation $\langle S \mid R\rangle$ for $S = \{s_1, s_2, \dots, s_d\}$ and $R = \{s_1^2, s_2^2, \dots, s_d^2 ; (s_i s_j)^2$ whenever $\{v_i, v_j\}$ spans a cone in $\Delta\}$.

We further have the following commuting diagram:

$$\begin{array}{ccccccccc} 1 & \to & F' & \to & F(S) & \to & \mathbb{Z}_2^n & \to & 1 \\ & & \downarrow & & \downarrow \psi & & \| & & \\ 1 & \to & H & \to & W & \to & \mathbb{Z}_2^n & \to & 1 \end{array}$$

where $F(S)$ denotes the free group on S, ψ denotes the canonical surjection from $F(S)$ to W, H denotes $\pi_1(X)$ and $F' = \psi^{-1}(H)$.

Since $\mathcal{T} = \{s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \mid (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n\}$ is a *Schreier transversal* for F' in $F(S)$, we can apply the Reidemeister-Schreier theorem (see **[15]**,**[31]**) to obtain a presentation for $\pi_1(X)$ from that of W. Let

(3.1) $$S_H = \{y_{j,\underline{t}} : 1 \leq j \leq d; \underline{t} \in \mathbb{Z}_2^n\}$$

(3.2) $$R_H^1 = \{\alpha_0(u) \ \forall \ u \in \mathcal{T}\}$$

(3.3) $$R_{\underline{t}} = \{\alpha_{\underline{t}}(r) \ \forall \ r \in R \ ; \ \underline{t} \in \mathbb{Z}_2^n\}$$

where $0 := (0, 0, \ldots, 0) \in \mathbb{Z}_2^n$, $\alpha_{\underline{t}} : F(S) \to F(S_H)$ is defined recursively as follows:

$\alpha_{\underline{t}}(1) := 1$; $\alpha_{\underline{t}}(s_j) = y_{j,\underline{t}}$. Suppose that by induction we have defined $\alpha_{\underline{t}}(w)$ for $w \in F(S)$ then $\alpha_{\underline{t}}(w \cdot s_j) := \alpha_{\underline{t}}(w) \cdot \alpha_{\underline{t \cdot s_j}}(s_j)$ where $\underline{t \cdot s_j} \in \mathbb{Z}_2^n$ corresponds to the coset representative $\varphi(w') \in \mathcal{T}$ of $F' \cdot w'$ where $w' = s_1^{t_1} \cdots s_n^{t_n} \cdot s_j$.

Note that $\forall \ \underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n$ we have

(i) $\alpha_0(s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n}) = (\alpha_0(s_1))^{t_1} \cdot (\alpha_{(t_1,0,0,\ldots,0)}(s_2))^{t_2} \cdots (\alpha_{(t_1,t_2,\ldots,t_{n-1},0)}(s_n))^{t_n}$,

(ii) $\alpha_{\underline{t}}(s_j^2) = \alpha_{\underline{t}}(s_j) \cdot \alpha_{\underline{b}^j}(s_j)$,

(iii) $\alpha_{\underline{t}}((s_p \cdot s_q)^2) = \alpha_{\underline{t}}(s_p) \cdot \alpha_{\underline{b}^p}(s_q) \cdot \alpha_{\underline{c}^{p,q}}(s_p) \cdot \alpha_{\underline{b}^q}(s_q) \ \forall \ 1 \leq j \leq d$.

It follows from the definition (3.2) and from the identity (i) above that

$$R_H^1 = \left\{ \ \alpha_0(s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n}) \quad | \ \underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n \ \right\}.$$
$$= \left\{ \ (y_{1,(0,0,\ldots,0)}^{t_1} \cdot y_{2,(t_1,0,\ldots,0)}^{t_2} \cdots y_{n,(t_1,t_2,\cdots,t_{n-1},0)}^{t_n}) \quad | \ \underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n \ \right\}$$

Also the definition (3.3) and the identities (ii) and (iii) above imply that

$$R_{\underline{t}} = \left\{ \alpha_{\underline{t}}(s_1^2), \ldots, \alpha_{\underline{t}}(s_d^2) \; ; \; (\alpha_{\underline{t}}(s_p s_q)^2) \text{ whenever } \{v_p, v_q\} \text{ spans a cone in } \Delta \right\}$$
$$= \left\{ \begin{array}{l} y_{1,\underline{t}} \cdot y_{1,\underline{b}^1}, \ldots\ldots, y_{d,\underline{t}} \cdot y_{d,\underline{b}^d} \; ; \\ y_{p,\underline{t}} \cdot y_{q,\underline{b}^p} \cdot y_{p,\underline{c}^{p,q}} \cdot y_{q,\underline{b}^q} \text{ whenever } \{v_p, v_q\} \text{ spans a cone in } \Delta \end{array} \right\}$$

Here $b_i^j = t_i + a_{ji} \; \forall \, 1 \le j \le d \; ; \; 1 \le i \le n$ and $c_i^{p,q} = t_i + a_{pi} + a_{qi} \; \forall \, 1 \le i \le n \; ; \; 1 \le p, q \le d$

Let $R_H^2 := \cup_{\underline{t} \in \mathbb{Z}_2^n} R_{\underline{t}}$. Then from the Reidemeister-Schreier theorem it follows that $\pi_1(X)$ has the presentation $\langle S_H \mid R_H^1 \; ; \; R_H^2 \rangle$. □

LEMMA 3.2. *The presentation can be simplified with lesser number of generators and relations by expressing the $y_{j,\underline{t}} \in S_H$ as words in S and using the relations in W if we make the further assumption on Δ that $\langle v_p, v_q \rangle \in \Delta$ for all $1 \le p, q \le n$.*

Proof: $\alpha_{\underline{t}}(s_j) = s_1^{t_1} \cdots s_n^{t_n} \cdot s_j \cdot \varphi(s_1^{t_1} \cdots s_n^{t_n} \cdot s_j)$ where $\varphi(w) \in \mathcal{T}$ is the coset representative of $F' \cdot w$ for $w \in W$. Notice further that in W the $\{s_1, \cdots, s_n\}$ commute among themselves by our assumption on $\{v_1, v_2, \ldots, v_n\}$. Hence we have $\varphi(s_1^{t_1} \cdots s_n^{t_n} \cdot s_j) = s_1^{t_1 + a_{j,1}} \cdot s_2^{t_2 + a_{j,2}} \cdots s_n^{t_n + a_{j,n}}$, $\forall \, (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n$. This implies that we have the following identities:

(i) $y_{j,\underline{t}} = s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \cdot (s_j) \cdot s_1^{t_1} \cdot s_2^{t_2} \cdots s_j^{t_j+1} \cdots s_n^{t_n} = 1$ for all $1 \le j \le n \; ; \underline{t} \in \mathbb{Z}_2^n$. (since, $a_{j,i} = \delta_{j,i}$ for all $1 \le j \le n$),

(ii) $y_{j,0} = \alpha_0(s_j) = s_j \cdot s_1^{a_{j,1}} \cdot s_2^{a_{j,2}} \cdots s_n^{a_{j,n}}$ for all $n+1 \le j \le d$,

(iii) $y_{j,\underline{t}} = \alpha_{\underline{t}}(s_j) = s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \cdot (\, s_j \cdot s_1^{a_{j,1}} \cdot s_2^{a_{j,2}} \cdots s_n^{a_{j,n}} \,) \cdot s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n}$ for all $n+1 \le j \le d$,

(iv) $y_{j,\underline{b}^j} = \alpha_{\underline{b}^j}(s_j) = s_1^{t_1 + a_{j,1}} \cdots s_n^{t_n + a_{j,n}} \cdot (s_j) \cdot s_1^{t_1} \cdots s_n^{t_n} = (s_1^{t_1} \cdots s_n^{t_n} \cdot (s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}) \cdot s_1^{t_1} \cdots s_n^{t_n})^{-1} = (y_{j,\underline{t}})^{-1}$.

Let $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ then from (ii) and (iii) above we see that the generators of $\pi_1(X)$ are $\{s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \cdot S_j \cdot s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \; \forall \, n+1 \le j \le d; \; \forall \, \underline{t} \in \mathbb{Z}_2^n\}$. Further, since R_H^1 consists of words in $\{y_{j,\underline{t}}$ for $1 \le j \le n$ and $\underline{t} \in \mathbb{Z}_2^n\}$, from (i) we see that $R_H^1 = \{1\}$. Furthermore, (iv) implies that the first set of relations in $R_{\underline{t}}$ are of the form$\{ \, y_{j,k} \cdot (y_{j,k})^{-1}\}$ therefore they trivially hold in the group $\pi_1(X)$. Thus finally the number of generators reduce to $(d-n) \cdot 2^n$

and they are: $\{y_{j,\underline{t}}$ for $n+1 \leq j \leq d$; $\underline{t} \in \mathbb{Z}_2^n\}$ and the non-trivial relations are of the form: $R_{\underline{t}} = \{\ y_{p\varphi(\underline{t})} \cdot y_{q\varphi(\underline{b}^p)} \cdot y_{p\varphi(\underline{c}^{p,q})} \cdot y_{q\varphi(\underline{b}^q)}\ whenever\ \{v_p, v_q\}\ spans\ a\ cone\ in\ \Delta\}\ \forall\ \underline{t} \in\ \mathbb{Z}_2^n$. Therefore the final presentation for $\pi_1(X)$ is: $\langle S_H, R_H\rangle$ where

$S_H = \{y_{j,\underline{t}}\ where\ n+1 \leq j \leq d\ and\ \underline{t} \in \mathbb{Z}_2^n\}$

$R_{\underline{t}} = \{\ y_{p,\varphi(\underline{t})} \cdot y_{q,\varphi(\underline{b}^p)} \cdot y_{p,\varphi(\underline{c}^{p,q})} \cdot y_{q,\varphi(\underline{b}^q)}\ whenever\ \{v_p, v_q\}\ spans\ a\ cone\ in\ \Delta\} \ \forall\ \underline{t} \in \mathbb{Z}_2^n$

$R_H = \cup_{\underline{t} \in \mathbb{Z}_2^n} R_{\underline{t}}$.

Further, $\pi_1(X)$ is generated as a subgroup of $W(\Delta)$ by $s_1^{t_1} \cdots s_n^{t_n} \cdot S_j \cdot s_1^{t_1} \cdots s_n^{t_n}$ where $\underline{t} \in \mathbb{Z}_2^n$ and $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ for $n+1 \leq j \leq d$. □

REMARK 3.3. Note that the fundamental group $\pi_1(X)$ and hence its presentation depends only on the 2-skeleton $\Delta(2)$ of Δ.

3.1. Real toric surfaces.

3.1.1. *Compact surfaces.* By the classification of two dimensional smooth complete fans (see p.42 of **[25]**) we observe that except the torus $\mathbb{S}^1 \times \mathbb{S}^1$ all other smooth complete real toric surfaces correspond bijectively to the two dimensional compact non-orientable manifolds. This can be seen as follows.

Let Δ be a smooth complete fan in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let $v_1, v_2, \ldots, v_d$ be the primitive vectors along the edges of Δ. We can assume without loss of generality that $v_1 = e_1$ and $v_2 = e_2$.

If $d = 3$ then X is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$. If $d = 4$ then X is the real part of the Hirzebruch surface $\mathbb{F}_a = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a))$. Hence it is isomorphic to the Klein-bottle if a is odd and $\mathbb{S}^1 \times \mathbb{S}^1$ if a is even. This is because, they are both $\mathbb{S}^1$ bundles over $\mathbb{S}^1$ and are determined upto homeomorphism by the first Stiefel-Whitney class of the line bundle $(\mathcal{O}(a))_{\mathbb{R}}$, which is 0 if a is even and 1 if a is odd.

We also observe here that the toric surface associated to the non-complete fan consisting of the faces of the cones $\langle e_1, e_2\rangle$ and $\langle e_2, -e_1 + ae_2\rangle$, is the total space of the line bundle $(\mathcal{O}(a))_{\mathbb{R}}$ over $\mathbb{S}^1$. It is therefore homeomorphic to the infinite Möbius strip if a is odd and to the infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$ if a is even.

If $d \geq 5$ then there exist a j, $1 \leq j \leq d$, such that v_{j-1} and v_{j+1} generate a strongly convex cone and $v_j = v_{j-1} + v_{j+1}$ (see p.43 of **[25]**). If $\sigma' = \langle v_{j-1}, v_j \rangle$ and $\sigma'' = \langle v_j, v_{j+1} \rangle$ then by the above observation we see that $U_{\sigma'} \cup U_{\sigma''}$ is an embedding of the Möbius strip in the surface. Therefore it follows that the toric surface is non-orientable. Let X' be the toric surface associated to the smooth complete fan Δ' obtained from Δ by removing the cones ρ_j, σ' and σ'', and adding the cone $\sigma = \langle v_{j-1}, v_{j+1} \rangle$. Then X is homeomorphic to $X' \# \mathbb{P}^2_{\mathbb{R}}$. Thus $\chi(X) = \chi(X') + \chi(\mathbb{P}^2_{\mathbb{R}}) - 2$ (where χ denotes the Euler -Poincaré characteristic). Note that if $d = 5$ then X' is the toric variety associated to a fan with 4 edges and is hence the Klein bottle or the torus. In this case since $\chi(X') = 0$, we have $\chi(X) = \chi(X') + 1 - 2 = -1$. Therefore by induction on d we can see that the Euler characteristic of X is $4 - d$. Hence the number of cross-caps is $d - 2$. Thus X is homeomorphic to $\mathbb{P}^2_{\mathbb{R}} \# \cdots \# \mathbb{P}^2_{\mathbb{R}}$ ($d-2$ copies) whenever it is non-orientable. And when it is orientable, it is homeomorphic to the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

3.1.2. *Non-Compact Surfaces.* Let Δ be a smooth non-complete fan in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let $v_1, v_2, \ldots, v_d$ be the primitive vectors along the edges of Δ. We can assume without loss of generality that $v_1 = e_1$ and $v_2 = e_2$. For simplicity we further assume that v_i and v_{i+1} *form a basis for* N, $\forall\ 1 \leq i \leq d$ (even if $\langle v_i, v_{i+1} \rangle$ does not belong to $\Delta(2)$), and that the support of Δ is *not* contained in any half space. Let Δ' be a fan in N with $\Delta'(1) = \Delta(1)$ and $\Delta'(2) = \Delta(2) \cup \{\langle v_i, v_{i+1} \rangle \mid \langle v_i, v_{i+1} \rangle \notin \Delta(2)\}$. Then we see that Δ' is a smooth complete fan and Δ is a sub-fan of Δ'. Therefore if X' is the toric surface associated to Δ', then X is obtained from the smooth compact toric surface X' by removing k points, where $k = \#(\Delta'(2) \setminus \Delta(2))$. This implies that X is orientable if and only if X' is orientable. Hence the Euler characteristic of X' and the number of punctures will determine X.

REMARK 3.4. The classical presentation for the fundamental group of the compact toric surfaces is apparently different from the presentation we have obtained, especially because it has only one relation. In the cases when $d = 3$ *and* 4, where the spaces are $\mathbb{P}^2_{\mathbb{R}}$, $\mathbb{S}^1 \times \mathbb{S}^1$ or the Klein -bottle $\simeq \mathbb{P}^2_{\mathbb{R}} \# \mathbb{P}^2_{\mathbb{R}}$ the presentations we give agrees with some of the classical ones. We hope to simplify the above presentation in each case to reduce the number of generators and relations, so that in general it agrees with one of the classical presentations.

In the following example of a smooth compact toric surface which is non-orientable with 3 cross-caps, we illustrate how the above presentation simplifies to a presentation with 3 generators and a single relation.

EXAMPLE 3.5. Let $N = \mathbb{Z}e_1 + \mathbb{Z}e_2$. Let $v_1 := e_1$, $v_2 := e_2$, $v_3 = -e_1 + e_2$, $v_4 = -e_1$ and $v_5 = -e_2$. Let Δ be the smooth complete fan in N consisting of the following cones: $\sigma_1 = \langle v_1, v_2\rangle$, $\sigma_2 = \langle v_2, v_3\rangle$,$\sigma_3 = \langle v_3, v_4\rangle$,$\sigma_4 = \langle v_4, v_5\rangle$,$\sigma_5 = \langle v_5, v_1\rangle$. Thus we have $n = 2$, $d = 5$,

$$W(\Delta) = \langle s_1, s_2, s_3, s_4, s_5 \mid s_1^2, s_2^2, s_3^2, s_4^2, s_5^2 \,;\, (s_1 \cdot s_2)^2, (s_2 \cdot s_3)^2, (s_3 \cdot s_4)^2, (s_4 \cdot s_5)^2, (s_5 \cdot s_1)^2\rangle$$

and $\mathbb{Z}_2^n = \{\alpha_1 = (0,0), \alpha_2 = (1,0), \alpha_3 = (0,1), \alpha_4 = (1,1)\}$. By applying Lemma 3.2 we get the following presentation for $\pi_1(X)$ where the generators S and the relations R are as listed below:

$$S = \{y_{3,\alpha_1} \,,\, y_{3,\alpha_2} \,,\, y_{4,\alpha_1} \,,\, y_{4,\alpha_3} \,,\, y_{5,\alpha_1} \,,\, y_{5,\alpha_2}\}$$

$$R_{\alpha_1} = \{y_{3,\alpha_1}y_{3,\alpha_2} \,,\, y_{5,\alpha_1}y_{5,\alpha_2}^{-1} \,,\, y_{3,\alpha_1}y_{4,\alpha_3}^{-1}y_{3,\alpha_2}^{-1}y_{4,\alpha_1}^{-1} \,,\, y_{4,\alpha_1}y_{5,\alpha_2}y_{4,\alpha_3}^{-1}y_{5,\alpha_1}^{-1}\}$$

$$R_{\alpha_2} = \{y_{3,\alpha_2}y_{3,\alpha_1} \,,\, y_{5,\alpha_2}y_{5,\alpha_1}^{-1} \,,\, y_{3,\alpha_2}y_{4,\alpha_3}y_{3,\alpha_1}^{-1}y_{4,\alpha_1} \,,\, y_{4,\alpha_1}^{-1}y_{5,\alpha_1}y_{4,\alpha_3}y_{5,\alpha_2}^{-1}\}$$

$$R_{\alpha_3} = \{y_{3,\alpha_2}^{-1}y_{3,\alpha_1}^{-1} \,,\, y_{5,\alpha_1}^{-1}y_{5,\alpha_2} \,,\, y_{3,\alpha_2}^{-1}y_{4,\alpha_1}^{-1}y_{3,\alpha_1}y_{4,\alpha_3}^{-1} \,,\, y_{4,\alpha_3}y_{5,\alpha_2}^{-1}y_{4,\alpha_1}^{-1}y_{5,\alpha_1}\}$$

$$R_{\alpha_4} = \{y_{3,\alpha_1}^{-1}y_{3,\alpha_2}^{-1} \,,\, y_{5,\alpha_2}^{-1}y_{5,\alpha_1} \,,\, y_{3,\alpha_1}^{-1}y_{4,\alpha_1}y_{3,\alpha_2}y_{4,\alpha_3} \,,\, y_{4,\alpha_3}^{-1}y_{5,\alpha_1}^{-1}y_{4,\alpha_1}y_{5,\alpha_2}\}$$

Let $R = \cup_{i=1}^4 R_{\alpha_i}$. Then $\langle S \mid R\rangle$ is the presentation obtained for $\pi_1(X)$. We can further simplify this presentation in the following way:

Let $a := y_{3,\alpha_1}$, $b := y_{4,\alpha_1}$, $c := y_{4,\alpha_3}$ and $d := y_{5,\alpha_1}$. The above relations can now be written as words in $a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}$ as follows: $ac^{-1}ab^{-1}$, $bdc^{-1}d^{-1}$, $a^{-1}ca^{-1}b$, $b^{-1}dcd^{-1}$, $ab^{-1}ac^{-1}$, $cd^{-1}b^{-1}d$, $a^{-1}ba^{-1}c$ and $c^{-1}d^{-1}bd$. Therefore the only non-trivial relations that remain are $ac^{-1}ab^{-1}$ and $b^{-1}dcd^{-1}$. If we let $A = ca^{-1}$; $B = b$; $D = d^{-1}$, then the above relations can be rewritten as a single relation $A^2BDB^{-1}D^{-1}$ in the generators A, B and D. Since X is homeomorphic to the connected sum of the $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{P}^2_{\mathbb{R}}$, the presentation $\langle A, B, D \mid A^2 \cdot [B, D]\rangle$ is in fact one of the classical presentations for $\pi_1(X)$ (see pp. 133-135 of [34]).

4. The Coxeter group $W(\Delta)$

In this section we prove some general results on right-angled Coxeter groups and in particular for $W(\Delta)$. Let $M = (m_{ij})$ denote the Coxeter matrix corresponding to W.

LEMMA 4.1. *$[W, W]$ is abelian if and only if for all $1 \leq j \leq d$ there exists at most one i such that $\langle v_i, v_j \rangle \notin \Delta$.*

Proof: If there exist $i \neq k$ such that $\{v_i, v_j\}$ and $\{v_k, v_j\}$ does not span a cone in Δ then $[s_i, s_j] \cdot [s_k, s_j] \neq [s_k, s_j] \cdot [s_i, s_j]$ in $[W, W]$.

Conversely if for each $1 \leq j \leq d$ there exists at most one i such that $\langle v_i, v_j \rangle \notin \Delta$, then by using the relations in W it is easy to see that for any word $w \in W$, $w \cdot [s_i, s_j] \cdot w^{-1} = [s_i, s_j]$ *or* $[s_j, s_i]$. (It is $[s_j, s_i]$ iff either one of s_i or s_j but not both occurs in the reduced expression of w.) Now $[W, W]$ is the normal subgroup of W generated by the commutators $\{[s_i, s_j] \mid \langle v_i, v_j \rangle \notin \Delta\}$. Therefore under the above assumption, $\{[s_i, s_j] \mid \langle v_i, v_j \rangle \notin \Delta\}$ in fact generate $[W, W]$ as a subgroup of W. Further, since they commute among themselves $[W, W]$ is abelian. □

LEMMA 4.2. *A word $w \in W$ is of finite order if and only if it is of order 2. Moreover in this case, w is a conjugate in W to a word w' which is of the form $w' = s_{j_1} \cdots s_{j_l}$ with $s_{j_p} \cdot s_{j_q} = s_{j_q} \cdot s_{j_p}$ $\forall\ 1 \leq p, q \leq l$.*

Proof: Suppose $w = v \cdot w' \cdot v^{-1}$ where w' is as above and $v \in W$. Then w is clearly of order 2. On the other hand if w is not of the above form, then the reduced expression for w is of the form $w = s_{i_1} \cdots s_{i_k}$ where s_{i_p} and s_{i_q} do not commute for some $1 \leq p, q \leq k$. Indeed by repeatedly using the relation $s_i \cdot s_j = s_j \cdot s_i$ whenever $m_{ij} = 2$, we can assume without loss of generality that upto conjugation w is of the form $s_{i_1} \cdot s_{i_2} \cdots s_{i_k}$ where s_{i_1} and s_{i_k} do not commute. Then it follows that for any positive integer r, $w^r = (s_{i_1} \cdots s_{i_k}) \cdot (s_{i_1} \cdots s_{i_k}) \cdots (s_{i_1} \cdots s_{i_k})$ is in fact a reduced expression in W. Hence w is of infinite order. □

LEMMA 4.3. *Let $w = s_{i_1} \cdots s_{i_k} \in W$ where $\langle v_{i_1}, \ldots, v_{i_k} \rangle \in \Delta$ and let $w' = s_{j_1} \cdots s_{j_l}$ where $\langle v_{j_p}, v_{j_q} \rangle \in \Delta$ for all $1 \leq p, q \leq l$ but $\langle v_{j_1}, \ldots, v_{j_l} \rangle \notin \Delta$. Then $w \notin N(w')$ where $N(w')$ is the normal subgroup generated by w' in W.*

Proof: Suppose on the contrary that $w = v_1 \cdot w' \cdot v_1^{-1} \cdot v_2 \cdot w' \cdot v_2^{-1} \cdots v_r \cdot w' \cdot v_r^{-1}$ for some $v_1, v_2, \ldots v_r \in W$. By Lemma 4.2 we know that $(w')^2 = 1$. Hence the above expression can be rewritten as

(1) $w = [v_1, w'] \cdot [w', v_2] \cdot [v_3, w'] \cdots [w', v_r]$ if r is even
(2) $w = [v_1, w'] \cdot [w', v_2] \cdots [v_r, w'] \cdot w'$ if r is odd.

This implies that $w \in [W, W]$ in the case r is even and $w \cdot w' = w \cdot (w')^{-1} \in [W, W]$ in the case r is odd.

Now let $h : W \to \mathbb{Z}_2^d$ be the abelianisation map which takes s_j to the coordinate vector $e_j = (0, 0, \ldots, 1, \ldots 0)$ (*with* 1 *at the* j*th position*). Also by our choice of w and w' we observe that $\{s_{i_1}, \ldots, s_{i_k}\}$ and $\{s_{j_1}, \ldots, s_{j_l}\}$ pairwise commute in W and the tuples $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_l)$ are distinct.

Therefore $h(w) = \Sigma_{p=1}^{k} e_{i_p} \neq (0, \ldots, 0)$ when r is even and $h(w \cdot w') = \Sigma_{p=1}^{k} e_{i_p} + \Sigma_{q=1}^{l} e_{j_q} \neq (0, \ldots, 0)$ when r is odd. This is a contradiction since on the other hand, w & $w \cdot (w')^{-1} \in [W, W]$ when r is even and r is odd respectively. This proves the lemma. □

REMARK 4.4. The lemma 4.1 if phrased differently as, $[W, W]$ is abelian if and only if there exists at most one i for every j such that $m_{i,j} \neq 2$, holds not just for right-angled Coxeter groups but for more general class of Coxeter groups with $m_{i,j} = 2$ *or* $m_{i,j} \geq 5 \ \forall \ i, j$.

5. Criterion for $\pi_1(X)$ to be abelian

Let X be smooth and connected. In the following theorem we give conditions on Δ under which $\pi_1(X)$ is abelian. We shall follow the notations in §3 and further assume that $\langle v_p, v_q \rangle \in \Delta$ for every $1 \leq p, q \leq n$ as in Lemma 3.2.

THEOREM 5.1. *$\pi_1(X)$ is abelian if and only if one of the following holds in Δ.*

(1) *For every* $1 \le i,j \le d$, $\{v_i, v_j\}$ *spans a cone in* Δ. *In this case,* $\pi_1(X)$ *is isomorphic to* $\mathbb{Z}_2^{d-n}$.

(2) *For each* $1 \le j \le d$ *there exists at most one* $i = i_j$ *with* $1 \le i_j \le n$ *such that,* $\{v_{i_j}, v_j\}$ *does not span a cone in* Δ *and* $\langle u_{i_j}, v_j\rangle = 1 \ mod\ 2$. *Further, for each* $n+1 \le k \le d$ *such that* $k \ne j$ *we have,* $\langle u_{i_j}, v_k\rangle = 0 \ mod\ 2$.

Proof: Recall that we have an exact sequence $1 \to [W,W] \to \pi_1(X) \to \mathbb{Z}_2^{d-n} \to 1$ and further $[W,W]$ is generated as a normal subgroup of W by $[s_{i_p}, s_{i_q}]$ whenever $\{v_{i_p}, v_{i_q}\}$ does not span a cone in Δ.

Step 1. Since $[W,W]$ is a subgroup of $\pi_1(X)$, if $\pi_1(X)$ is abelian then $[W,W]$ must be abelian. By Lemma 4.1, $[W,W]$ is abelian if and only if for every v_j there exists at most one v_i such that $\{v_j, v_i\}$ does not span a cone in Δ.

Further, $[W,W] = \{1\}$ if and only if any two $\{v_j, v_i\}$ for $1 \le i,j \le d$ spans a cone in Δ which implies that, $W \simeq \mathbb{Z}_2^d$ and $\pi_1(X) \simeq \mathbb{Z}_2^{d-n}$.

Step 2. On the other hand if $[W,W] \ne \{1\}$, then there exists $\{v_j, v_i\}$ which does not span a cone in Δ. However since $[W,W]$ is abelian, this $i = i_j$ is must be unique for every such j. Thus in W, s_j and s_{i_j} do not commute but they both commute with s_k for every $1 \le k \le n$.

Step 3. Suppose now that for some $n+1 \le j \le d$ we have $n+1 \le i_j \le d$, then $\pi_1(X)$ is non-abelian for if S_j denotes the word $s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ in W then

$$S_j \cdot S_{i_j} = s_j \cdot s_{i_j} \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{i_j,1}} \cdots s_n^{a_{i_j,n}} \ne s_{i_j} \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{i_j,1}} \cdots s_n^{a_{i_j,n}} = S_{i_j} \cdot S_j.$$

Hence if $\pi_1(X)$ is abelian then for every $n+1 \le j \le d$ there is a unique index i_j such that $\langle v_j, v_{i_j}\rangle \notin \Delta$ and further $1 \le i_j \le n$.

Step 4. Now if for some $n+1 \le k \le d$ with $k \ne j$ we have $a_{k,i_j} = \langle u_i, v_{i_j}\rangle \ mod\ \mathbb{Z}_2 = 1$, then $\pi_1(X)$ is non-abelian. This is because, if $w = [S_k, S_j] \in \pi_1(X)$ then $w \ne 1$, which we can see by the following cases.

If $a_{k,i_k} = 0$ and $a_{j,i_k} = 0$ then $w = [s_{i_j}, s_j] \ne 1$.

If $a_{k,i_k} = 1$ and $a_{j,i_k} = 1$ then $w = [s_{i_j}, s_j] \ne 1$

If $a_{k,i_k} = 1$ and $a_{j,i_k} = 0$ then $w = [s_{i_j}, s_j] \cdot [s_k, s_{i_k}] \ne 1$

If $a_{k,i_k} = 0$ and $a_{j,i_k} = 1$ then $w = [s_{i_j}, s_j] \cdot [s_k, s_{i_k}] \neq 1$.

(Here we omit the proofs of the assertion that $w \neq 1$ in each case, as it follows easily from the relations in W).

Step 5. If $a_{j,i_j} = 0$ then again $\pi_1(X)$ is non-abelian since, the elements $s_{i_j} \cdot S_j \cdot s_{i_j}$ and S_j^{-1} do not commute in $\pi_1(X)$. This is because by Lemma 4.2, $[s_{i_j}, s_j]$ is an element of infinite order in W and hence $(s_{i_j} \cdot S_j \cdot s_{i_j}) \cdot S_j^{-1} = [s_{i_j}, s_j] \neq [s_j, s_{i_j}] = S_j^{-1} \cdot (s_{i_j} \cdot S_j \cdot s_{i_j})$.

Step 6. Therefore if $\pi_1(X)$ is abelian and $[W, W] \neq 1$ then it is necessary that the following conditions must hold:

For every $1 \leq j \leq d$, there exists a unique index i_j with $1 \leq i_j \leq n$ such that $\{v_j, v_{i_j}\}$ does not span a cone in Δ and $a_{j,i_j} = 1$. Further, for every $n+1 \leq k \leq d$ such that $k \neq j$, we have $a_{k,i_j} = 0$.

We shall now prove that these conditions are in fact sufficient for $\pi_1(X)$ to be abelian.

Claim:

(i) S_j and S_k commute for $n+1 \leq j, k \leq d$.

(ii) $w \cdot S_j \cdot w^{-1}$ and S_j commute where $w = s_1^{t_1} \cdots s_n^{t_n}$ for every $\underline{t} = (t_1, \ldots, t_n) \in \mathbb{Z}_2^n$ and $n+1 \leq j \leq d$.

Proof of the claim:

$$
\begin{aligned}
(i)\ S_j \cdot S_k &= s_j \cdot s_k \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{k,1}} \cdots s_n^{a_{k,n}} \{\textit{since } a_{j,i_k} = 0 \textit{ by assumption}\} \\
&= s_k \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{k,1}} \cdots s_n^{a_{k,n}} \{\textit{since } k \neq i_j\} \\
&= s_k \cdot s_1^{a_{k,1}} \cdots s_n^{a_{k,n}} \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \{\textit{since } a_{k,i_j} = 0 \textit{ by assumption}\} \\
&= S_k \cdot S_j.
\end{aligned}
$$

(ii) Let $w = s_1^{t_1} \cdots s_n^{t_n}$ such that $(t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n$.

$w \cdot S_j \cdot w^{-1} = (s_1^{t_1} \cdots s_n^{t_n}) \cdot (s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}) \cdot (s_1^{t_1} \cdots s_n^{t_n}) = z$

(a) If $t_{i_j} = 0$ then $z = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} = S_j$ {since $s_j \cdot w = w \cdot s_j$}. Thus $w \cdot S_j \cdot w^{-1} = S_j$.

(b) If $t_{i_j} = 1$ then $z = s_{i_j} \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_{i_j} = s_{i_j} \cdot S_j \cdot s_{i_j}$. Further,

$$\begin{aligned}(w \cdot S_j \cdot w^{-1}) \cdot (S_j) &= s_{i_j} \cdot S_j \cdot s_{i_j} \cdot S_j. \\ &= s_{i_j} \cdot s_j \cdot s_1^{a_{j,1}} \cdots \widehat{s_{i_j}} \cdots s_n^{a_{j,n}} \cdot S_j \{ \textit{ since } a_{j,i_j} = 1\} \\ &= s_{i_j} \cdot s_j^2 \cdot s_{i_j} = 1.\end{aligned}$$

This implies that, $w \cdot S_j \cdot w^{-1} = S_j^{-1}$.

Hence $w \cdot S_j \cdot w^{-1}$ commutes with S_k for all $n+1 \leq j,k \leq d$, since we have either $w \cdot S_j \cdot w^{-1} = S_j$ or S_j^{-1} in each of the cases. Therefore since the generators commute among themselves we conclude that $\pi_1(X)$ is abelian. □

REMARK 5.2. If Δ is complete then the condition $a_{j,i_j} = 1$ will be forced after *Step 4.* in which case we shall skip *Step 5.* However this is not true in general. For example in the non-complete fan $\Delta = \{\{0\}, \langle e_1, e_2\rangle, \langle -2e_1 + e_2\rangle\}$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$.

REMARK 5.3. *(Torsion elements)* By Lemma 4.2, since $\pi_1(X)$ is a subgroup of W the torsion elements in $\pi_1(X)$ are always of order 2. In particular when $\pi_1(X)$ is abelian, $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ for $n+1 \leq j \leq d$ is of order 2 iff $\langle v_j, v_i\rangle \in \Delta$ for all $1 \leq i \leq n$ and it is of infinite order iff there exists a unique $1 \leq i_j \leq n$ such that $\langle v_j, v_{i_j}\rangle \notin \Delta$ since in this case $a_{j,i_j} = 1$ and $S_j^2 = [s_j, s_{i_j}] \neq 1$ in $[W,W] \subset W$.

REMARK 5.4. If $\pi_1(X)$ is abelian then $\pi_1(X)$ is generated by $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ for $n+1 \leq j \leq d$. Let $\{j_1, j_2, \ldots, j_r\} = J = \{j \mid n+1 \leq j \leq d \textit{ and } \langle v_j, v_{i_j}\rangle \notin \Delta \textit{ for some } 1 \leq i_j \leq n\}$. Therefore if $j \notin J$ then $\langle v_j, v_i\rangle \in \Delta$ for every $1 \leq i \leq n$. Thus $\pi_1(X) \simeq \mathbb{Z}_2^{d-n-r} \oplus \mathbb{Z}^r$ where $\mathbb{Z}^r = \langle S_{j_p} = s_{j_p} \cdot s_1^{a_{j_p,1}} \cdots s_n^{a_{j_p,n}}$ for $1 \leq p \leq r\rangle$. Furthermore, $[W,W] = \langle [s_{j_p}, s_{i(j_p)}] = S_{j_p}^2$ for $1 \leq p \leq r\rangle \subset W$ is free abelian of rank r. We therefore have the following commuting diagram,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [W,W] & \longrightarrow & \pi_1(X) & \longrightarrow & \mathbb{Z}_2^{d-n} & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & \mathbb{Z}^r & \xrightarrow{\times 2} & \mathbb{Z}^r \oplus \mathbb{Z}_2^{d-n-r} & \longrightarrow & \mathbb{Z}_2^r \oplus \mathbb{Z}_2^{d-n-r} & \longrightarrow & 1 \end{array}$$

REMARK 5.5. If $\pi_1(X)$ is abelian then necessarily $d \leq 2n$, because to every $n+1 \leq j \leq d$ we associate a unique i_j with $1 \leq i_j \leq n$. Examples of toric varieties with abelian fundamental group are: (i) Products of real projective spaces. (ii) Toric bundles with base an aspherical toric variety with abelian fundamental group and fibre $\mathbb{P}^n_{\mathbb{R}}$ for $n \geq 2$. (However this is not true for a non-trivial bundle with fibre $\mathbb{P}^1_{\mathbb{R}}$ for example, $\pi_1((\mathbb{F}_1)_{\mathbb{R}})$ is non-abelian where $(\mathbb{F}_1)_{\mathbb{R}}$ denotes the real part of the Hirzebruch surface $\mathbb{F}_1$).

6. Asphericity of X

Let $S_N := (N_{\mathbb{R}} - \{0\})/\mathbb{R}_{>0}$ and let $\pi : N_{\mathbb{R}} - \{0\} \longrightarrow S_N$ be the projection. Let $\mathcal{S}_\Delta$ denote the simplicial complex associated to the smooth fan Δ, where each k-dimensional $\sigma \in \Delta$ corresponds to a $(k-1)$-dimensional spherical simplex $\pi(\sigma - \{0\})$. If further we assume Δ to be complete, then it gives rise to a triangulation of S_N. (see p. 52, [**35**]).

Recall that a simplicial complex $\mathcal{S}$ with vertices $\mathcal{V} = \{v_i\}$ is called a *flag complex* if the following condition holds for every finite subset $\{v_1, v_2, ...v_n\}$ of $\mathcal{V}$: If $\{v_i, v_j\}$ span a simplex in $\mathcal{S}$ for all $i, j \in \{1, 2, ...n\}$ then $\{v_1, v_2, ..., v_n\}$ span a simplex of $\mathcal{S}$.

Hence $\mathcal{S}_\Delta$ is a flag complex if and only if for every collection of primitive edge vectors $\{v_{i_1}, \ldots, v_{i_r}\}$, if $\{\langle v_{i_k}, v_{i_l}\rangle \in \Delta \ \forall\ 1 \leq k, l \leq r\}$ then $\langle v_{i_1}, \ldots, v_{i_r}\rangle \in \Delta$. We shall say that Δ is *flag-like* whenever $\mathcal{S}_\Delta$ is a flag complex.

THEOREM 6.1. *X is aspherical if and only if Δ is flag-like.*

Proof: If $\widetilde{X}$ is contractible then we claim that Δ is flag-like.

Suppose on the contrary that Δ is not flag-like. Then $\exists\ \{v_{j_1}, \ldots, v_{j_l}\}$ such that $\forall\ 1 \leq p, q \leq l$, $\langle v_{j_p}, v_{j_q}\rangle \in \Delta$ but $\langle v_{j_1}, \ldots, v_{j_l}\rangle \notin \Delta$.

Let $w' = s_{j_1} \cdots s_{j_l} \in W$ and let $N(w')$ be the normal subgroup of W generated by w' as in Lemma 4.3. Also let $\theta : W \to W/N(w')$ be the canonical surjection. Clearly, $\lambda = (\lambda_\tau = \theta \circ \iota_\tau)$ is a simple morphism from $G(\Delta) \to W/N(w')$. Further, Lemma 4.3. implies that, $\lambda_\tau : G_\tau \subseteq T_2 \to W/N(w')$ is injective $\forall\ \tau \in \Delta$. Hence λ is injective at the local groups. Now the development $D(X_+, \lambda)$ of X_+ with respect to λ has $D(X_+, \iota) \simeq \widetilde{X}$ as the universal cover and its fundamental group $\pi_1(D(X_+, \lambda) \simeq N(w')$ has w' as a torsion element. This is a contradiction since $D(X_+, \lambda)$ is a $K(\pi, 1)$ space, because of our assumption that $\widetilde{X}$ is contractible.

For proving the converse we apply Cor. 10.3 of the main result of [19] to the reflection system ($\Gamma = W, V = S$) on $M = \widetilde{X}$ with fundamental chamber $Q = X_+$ (which is contractible by Lemma 2.2). Here for every $T \subseteq S$, $Q_T = \cap_{s_j \in T} V(\rho_j)_+$. Let W_T be the subgroup generated by T in W. Then the following statements are equivalent:

(1) Q_T is acyclic for all $T \subseteq S$ with W_T finite.

(2) Δ is flag-like.

Proof of (1)⇒(2): Let $\rho_{j_1}, \ldots, \rho_{j_l}$ be edges such that $\{\rho_{j_p}, \rho_{j_q}\}$ spans a cone in Δ for all $1 \leq p, q \leq l$. Then (1) implies that $Q_T = \cap_{r=1}^{l} V(\rho_{j_r})_+ = V(\tau)_+$ is nonempty since by Lemma 4.2, $W_T = \langle s_{j_i} \ldots s_{j_l} \rangle$ is a finite subgroup of W. This implies that $\tau = \langle \rho_{j_1}, \ldots, \rho_{j_l} \rangle$ is a nonempty cone in Δ.

Proof of (2)⇒(1): Let $T = \{s_{j_1}, \ldots, s_{j_l}\} \subseteq S$ be such that W_T is finite. Then in particular, $w' = s_{j_1} \cdots s_{j_l}$ is an element of finite order in W. By Lemma 4.2, the edge vectors $v_{j_1} \ldots, v_{j_l}$ pairwise span cones in Δ. The assumption (2) further implies that $v_{j_1}, \ldots, v_{j_l}$ together span a cone τ in Δ. Thus $Q_T = \cap_{r=1}^{l} V(\rho_{j_r})_+ = V(\tau)_+$ is nonempty. Moreover $V(\tau)$ being a smooth toric variety, its non-negative part $V(\tau)_+$ is contractible by Lemma 2.2 and is hence acyclic if it is nonempty.

We therefore conclude from Cor. 10.3 of [19] that if Δ is a flag-like then $M = \widetilde{X}$ is contractible. □

REMARK 6.2. In fact since (1) ⇔ (2) above, it is clear that Cor 10.3 of [19] also proves the first implication of the above theorem. However, in our particular case (where W is a right -angled Coxeter group) the argument given above is self contained and is an application of the "method of development" which is consistent with the theme of this paper.

The following are some corollaries of the above theorem.

COROLLARY 6.3. *If X is aspherical then $V(\tau)$ is aspherical for every cone τ.*

Proof: This is immediate because $V(\tau)$ is the toric variety associated to the fan $Star(\tau)$ which by definition (see page 52 of [25]) is smooth and flag-like whenever Δ is smooth and flag-like. A proof for this is as follows: Let $\overline{\rho_{i_1}}, \ldots, \overline{\rho_{i_k}}$ be edge vectors which pairwise span cones in $Star(\tau)$. Therefore by the definition of $Star(\tau)$, the edges of τ and $\rho_{i_1}, \ldots \rho_{i_k}$ pairwise span cones in Δ. Since Δ is flag-like, this implies that $\gamma = \langle \tau, \rho_{i_1} \ldots, \rho_{i_k} \rangle$ is a cone in Δ and hence $\overline{\gamma} = \langle \overline{\rho_{i_1}}, \ldots, \overline{\rho_{i_k}} \rangle$ is a cone in $Star(\tau)$. Thus $Star(\tau)$ is flag-like. □

COROLLARY 6.4. *Let X be smooth and complete. We can blow up X along a number of T-stable subvarieties to get a smooth complete toric variety X' which is aspherical.*

Proof: Since Δ is a smooth and complete fan, $\mathcal{S}_{\Delta}$ is a simplicial decomposition of the sphere S_N. It is known that the barycentric subdivision of any simplicial complex is a flag complex (see [**11**] p. 210). Therefore if Δ' is the refinement of Δ obtained by taking the cones over the simplices in the barycentric subdivision of $\mathcal{S}_{\Delta}$ then Δ' is a flag-like fan. It is not difficult to see that Δ' is also smooth and complete. Hence the smooth complete toric variety $X(\Delta')$ which is obtained by blowing up X along certain T-stable subvarieties is aspherical. □

REMARK 6.5. However in some cases we need lesser number of blow ups to arrive at an aspherical space. For e.g (i) $\mathbb{P}^2_{\mathbb{R}}$ blown up at a T-fixed point is the Hirzebruch surface $(\mathbb{F}_1)_{\mathbb{R}}$ (the Klein-bottle) and $(\mathbb{F}_1)_{\mathbb{R}}$ is aspherical. (ii) $\mathbb{P}^2_{\mathbb{R}} \times \mathbb{S}^1$ needs to be blown up along a T-stable $\mathbb{P}^1_{\mathbb{R}}$ to get an aspherical space $(\mathbb{F}_1)_{\mathbb{R}} \times \mathbb{S}^1$.

7. Subspace arrangement related to Δ

Throughout this section we assume that Δ is a smooth and complete fan.

In this section we define a real subspace arrangement associated to Δ whose complement in $\mathbb{R}^d$ is denoted by $\mathcal{C}_{\Delta}$. Recall from [**16**] that, $X_{\mathbb{C}} \simeq X'_{\mathbb{C}}/(\mathbb{C}^*)^{d-n}$ where $X'_{\mathbb{C}}$ is the complement of a complex subspace arrangement in $\mathbb{C}^d$. By restricting scalars to $\mathbb{R}$ in the above quotient we show that $X \simeq \mathcal{C}_{\Delta}/(\mathbb{R}^*)^{d-n}$ where $\mathcal{C}_{\Delta} \simeq X'_{\mathbb{R}}$. We compute the fundamental group of $\mathcal{C}_{\Delta}$ and also give necessary and sufficient conditions for it to be a $K(\pi, 1)$ space.

DEFINITION 7.1. *A collection $\mathcal{P} = \{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}\}$ of edges in Δ is called a primitive collection if $\{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}\}$ together does not span a cone in Δ but every proper subcollection of $\mathcal{P}$ spans a cone in Δ. For the primitive collection $\mathcal{P}$ let $\mathcal{A}(P) =$ $\{(x_1 \ldots, x_d) \in \mathbb{R}^d \mid x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\ \}$.*

DEFINITION 7.2.

(i) *The coordinate subspace arrangement in* $\mathbb{R}^d$ *corresponding to a fan* Δ *denoted by* $\mathcal{A}_\Delta$ *is defined as follows:* $\mathcal{A}_\Delta = \cup_{\mathcal{P}} \mathcal{A}(\mathcal{P})$, *where the union is taken over all primitive collections* $\mathcal{P}$ *of edges in* Δ.

(ii) *Let* $\mathcal{C}_\Delta$ *denote the complement of* $\mathcal{A}_\Delta$ *in* $\mathbb{R}^d$, *i.e.* $\mathcal{C}_\Delta := \mathbb{R}^d - \mathcal{A}_\Delta$

Let $\{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_r\}$ be the set of all primitive collections in Δ consisting of two edges. Let $\mathcal{P}_i = \{\rho_{i_p}, \rho_{i_q}\}$ where $1 \leq i_p, i_q \leq d \ \forall\ 1 \leq i \leq r$.

The following lemma generalizes the description of a smooth complete complex toric variety as given in [**16**] and [**8**] to the corresponding real and non-negative parts. Although this follows almost immediately from the complex case, we give a proof for it since we have not seen the result mentioned anywhere explicitly.

LEMMA 7.3. *The real toric variety* X *corresponding to a smooth complete fan* Δ *is the geometric quotient of* $\mathcal{C}_\Delta$ *by the real algebraic torus* $(\mathbb{R}^*)^{d-n}$ *and we have a locally trivial principal bundle with total space* $\mathcal{C}_\Delta$, *base* X *and structure group* $(\mathbb{R}^*)^{d-n}$ *i.e.,* $\mathcal{C}_\Delta \to \mathcal{C}_\Delta/(\mathbb{R}^*)^{d-n} \simeq X$. *Similarly,* $X_+ \simeq (\mathcal{C}_\Delta)_+/(\mathbb{R}^+)^{d-n}$.

Proof: Let $\sigma = \langle v_1 \dots, v_n\rangle \in \Delta(n)$ be such that $\{v_1, \dots, v_n\}$ form a $\mathbb{Z}$ basis for N. Let $\{u_1, \dots, u_n\}$ be the dual basis. Let $N'' \simeq \mathbb{Z}^{d-n}$; $N' \simeq \mathbb{Z}^d$ and let $\{e'_j : 1 \leq j \leq d\}$, $\{e''_k : 1 \leq k \leq d-n\}$ denote the natural bases of N' and N'' respectively. Further, let $g : N' \to N$ map e'_j to v_j for every $1 \leq j \leq d$ and let $f : N'' \hookrightarrow N'$ be the map which takes e''_{d-j+1} to $e'_j - (\Sigma_{i=1}^n \langle u_i, v_j\rangle \cdot e'_i)$ for every $n+1 \leq j \leq d$. From the results of [**16**] we know that there is an exact sequence of fans:

$$0 \longrightarrow (\Delta'', N'') \overset{f}{\hookrightarrow} (\Delta', N') \overset{g}{\longrightarrow} (\Delta, N) \longrightarrow 0$$

where $\Delta'' = \{0\}$ and Δ' is the fan consisting of the cones $\tau' = \langle e'_{j_1}, \dots, e'_{j_k}\rangle$ corresponding to every $\tau = \langle v_{j_1}, \dots, v_{j_k}\rangle \in \Delta$. Observe that the real toric varieties corresponding to Δ'' and Δ' are $X(\Delta'') \simeq (\mathbb{R}^*)^{d-n}$ and $X(\Delta') \simeq \mathbb{R}^d - Z$ respectively, where Z is the zero locus in $\mathbb{R}^d$ of the monomials $x_{\hat{\sigma}} = \prod_{\rho \notin \sigma} x_\rho$ for every $\sigma \in \Delta$. Moreover it is easy to see that $\mathbb{R}^d - Z$ is also isomorphic to the complement of the subspace arrangement $\mathbb{R}^d - \mathcal{A}_\Delta = \mathcal{C}_\Delta$ defined above (see p. 130 of [**14**]). Hence from the above exact sequence of fans we see that, the smooth complete real toric variety X is the base space of a principal bundle with total space

$(\mathbb{R}^d - Z) \simeq (\mathbb{R}^d - \mathcal{A}_\Delta) \simeq \mathcal{C}_\Delta$ and structure group $(\mathbb{R}^*)^{d-n}$(see p. 59 of **[35]** and p. 27 of **[16]**). Similarly by restricting to the non-negative parts we see that X_+ is the base space of a principal fibre bundle with total space $\mathbb{R}^d_+ - Z_+$ and structure group $(\mathbb{R}^+)^{d-n}$.Thus we have the following:

$$X \simeq (\mathbb{R}^d - Z)/(\mathbb{R}^*)^{d-n} \simeq \mathcal{C}_\Delta/(\mathbb{R}^*)^{d-n}$$

$$X_+ \simeq \mathbb{R}^d_+ - Z_+/(\mathbb{R}^+)^{d-n} \simeq (\mathcal{C}_\Delta)_+/(\mathbb{R}^+)^{d-n}$$

REMARK 7.4. Note that the only property of a smooth and complete fan which we use in the above proof is that $\{v_1, \cdots, v_n\}$ form a $\mathbb{Z}$ basis of N. Thus Lemma 7.3 is true even for a smooth (not necessarily complete) fan Δ for which the primitive vectors along $\Delta(1)$ contains a $\mathbb{Z}$ basis for N.

LEMMA 7.5. *$\pi_1(\mathcal{C}_\Delta)$ is isomorphic to the commutator subgroup $[W, W]$ of the Coxeter group W defined in §2, which is generated as a normal subgroup of W by $[s_{i_p}, s_{i_q}]$ for $1 \leq i \leq r$ where $\mathcal{P}_i = \{\rho_{i_p}, \rho_{i_q}\}$ $\forall$ $1 \leq i \leq r$.*

Proof: From Lemma 7.3 we know that $X \simeq \mathcal{C}_\Delta/(\mathbb{R}^*)^{d-n}$. Moreover, since $(\mathbb{R}^*)^{d-n} \simeq (\mathbb{R}^+)^{d-n} \times \mathbb{Z}_2^{d-n}$, $X_1 = \mathcal{C}_\Delta/(\mathbb{R}^+)^{d-n}$ is a regular covering space over X with deck transformation group $\mathbb{Z}_2^{d-n}$. In fact it is the same covering space of X as in Theorem 2.5(4). Also observe that $\mathcal{C}_\Delta$ and X_1 are of the same homotopy type since $\mathcal{C}_\Delta$ is a fibre bundle over X_1 with contractible fibre $(\mathbb{R}^+)^{d-n}$. Therefore we have $\pi_1(\mathcal{C}_\Delta) \simeq [W, W]$. □

In the following theorem we shall find the necessary and sufficient conditions on Δ and hence on the arrangement $\mathcal{A}_\Delta$, under which $\mathcal{C}_\Delta$ is aspherical.

THEOREM 7.6. *$\mathcal{C}_\Delta$ is aspherical if and only if $\mathcal{A}_\Delta$ is a union of precisely codimension 2 subspaces.*

Proof: Since $\mathcal{C}_\Delta$ is of the homotopy type of a finite regular covering space over X, it follows that X is aspherical if and only if $\mathcal{C}_\Delta$ is aspherical. From Theorem 6.1 the necessary and sufficient condition for X to be aspherical is that Δ is flag-like. Therefore it suffices to show that Δ is flag-like if and only if $\mathcal{A}_\Delta$ is a union of precisely codimension two subspaces.

Now by Definition 7.1, the condition for Δ to be flag-like is equivalent to the condition that in Δ there are no primitive collections consisting of more than two edges. Also by Definition 7.2, $\mathcal{A}_\Delta = \cup_{\mathcal{P}} \mathcal{A}_{\mathcal{P}}$, where the union is over primitive collections $\mathcal{P}$ in Δ and where $\mathcal{A}_{\mathcal{P}}$ is a subspace in $\mathbb{R}^d$ of codimension precisely equal to the number of edges in $\mathcal{P}$. Thus Δ is flag-like if and only if $\mathcal{A}_\Delta = \cup_{\mathcal{P}_i} \mathcal{A}(\mathcal{P}_i)$, where the union runs over the primitive collections $\{\mathcal{P}_1, \cdots, \mathcal{P}_r\}$ consisting of two edges or equivalently, $\mathcal{A}_\Delta$ is a union of codimension two subspaces. Hence the theorem. □

REMARK 7.7. *($K(\pi,1)$-arrangements)* The barycentric subdivision of any simplicial complex is a flag complex. Hence given a smooth complete fan Δ, we can obtain several smooth complete flag-like fans whose cones are the cones over the simplices of the repeated barycentric subdivisions of $\mathcal{S}_\Delta$. We therefore get several examples of $K(\pi,1)$ arrangements finding which seems to be of interest in the topology of arrangements (see **[36]** and **[28]**). However note that even if we start with a flag-complex, an arbitrary subdivision need not result in a flag-complex. For example, let Δ be the fan consisting of the faces of $\sigma = \langle e_1, e_2, e_3 \rangle$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$. If we refine Δ by adding the edge vector through $v = e_1 + e_2 + e_3$, then the resulting fan Δ' is not flag-like since, e_1, e_2, e_3, v pairwise span cones in Δ' but together do not span any cone.

REMARK 7.8. Indeed both Lemma 7.5 and Theorem 7.6 follow directly from the fact that $\mathcal{C}_\Delta$ is the smooth non-complete toric variety associated to the fan $\Delta' = \{\langle e_{j_1}, \ldots, e_{j_k} \rangle \textit{ for every cone } \tau = \langle v_{j_1} \ldots, v_{j_k} \rangle \in \Delta\}$ in $N' = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_d$ (see Lemma 7.3) and applying Theorem 2.5 and Theorem 6.1. However since $\mathcal{C}_\Delta$ has been defined specifically as the complement of real coordinate subspace arrangement related to a smooth complete fan Δ, we therefore describe both its fundamental group and criterion for asphericity by using Δ.

REMARK 7.9. Since $\mathcal{C}_\Delta$ is the toric variety associated to the fan Δ', we can apply Theorem 3.1 and Theorem 4.1 respectively to give a presentation for $\pi_1(\mathcal{C}_\Delta)$ and give conditions on Δ' for it to be abelian. In particular it follows from Theorem 7.6 and Theorem 4.1 that $\mathcal{C}_\Delta$ is $K(\pi,1)$ with $\pi_1(\mathcal{C}_\Delta)$ abelian if and only if it is the complement of subspaces of codimension precisely 2 which pairwise intersect at $\{0\}$. Moreover it also follows from Lemma 4.2 that $\pi_1(\mathcal{C}_\Delta) = [W, W]$ is always torsion free.

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