ASPECTS OF MATRIX MODELS

by

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CERTIFICATE

I certify that the Ph. D. thesis titled "Aspects of Matrix Models" submitted for the Degree of Doctor of Philosophy by Mr. Subrata Bal is the record of bonafide research work carried out by him during the period from February 1998 to December 2001 under my supervision, and that this work has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this University or any other University or Institution of Higher Learning. It is further certified that the thesis represents independent work by the candidate and collaboration was necessitated by the nature and scope of the problems dealt with.

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Abstract

The **Matrix Model** has proven to be an impressive candidate for M-theory (non-perturbative string theory). In $N \to \infty$ (N is the matrix size) limit the matrix model should give the string theory results. There are mainly two types of matrix models, Type IIA and Type IIB corresponding to IIA and IIB string theories respectively. The IIA matrix model is the (0+1) dimensional reduced model of 10 D SYM theory. In this theory the diagonal elements of the matrices are interpreted as the positions of the D0 branes and the off diagonal elements as the interactions between two such D0 branes. This system is described by the effective action for N D0-branes which is a particular $N \otimes N$ matrix quantum mechanics, to be taken in the $N \to \infty$ limit. Type IIB matrix model is (0+0) D reduced model of 10 D SYM theory. It has $\mathcal{N}$ = 2 Supersymmetry. The diagonal elements of the matrices in this matrix model is interpreted as the space time points and the off elements as the interactions between two such space time points. As the matrices in general do not commute under the multiplication, the noncommutative geometry of space time arises automatically. Non-commutative Yang Mills theory in flat back ground can be obtained from this reduced model.

In this thesis, we have investigated the following two important aspects of the physics of Matrix Model.

• Finite Temperature Matrix Model

The number density of the states in a relativistic string grows exponentially with energy *i.e.* with temperature. So, the canonical partition function diverges beyond a limiting temperature, Hagedorn Temperature (T_H). It is not clear whether the Hagedorn Temperature is really a limiting temperature. There were attempts to show it to be a phase transition temperature. In matrix model a string is obtained when D0 branes are arranged to form a membrane, which is then wrapped around a compact direction. At high temperatures the D0 branes prefer to cluster at point, thus the strings disappear. To get a clear answer to this puzzle we need to know the exact potential between the D0 branes. Thus the nonperturbative

formulation of string theory like the matrix model is essential. An attractive potential between two D0 branes (perturbatively up to one loop) is observed for high temperature at large N An attractive potential also been observed in a one loop calcution at low temperature.

With this motivation, we have attempted to study the high temperature behavior of a simple but non-trivial system - the system of two D-0-branes. This is essentially the BFSS matrix model with N=2. N is not big enough to describe M-theory. The D-0-brane action that we are interested in, is a quantum mechanical one (i.e. 0+1 dimensional). However after compactifying the Euclideanised time, it reduces to a 0+0 dimensional model, which has the similar structure to that of the IKKT matrix model.

We have derived the partition function for a system of two D0 brane (D-particle) kept at very high temperature, *up to the leading order in* β (inverse temperature) and *exact in coupling constant* g. We have calculated the leading non-trivial term of the partition function in short distance and large distance limit. From a scaling argument we have also determined the β and g dependence of the leading term for any N. Up to leading order the effective potential between two D-0 branes is proportional to $-\log l$ and $\log l$ for small and large l respectively (l - distance between two D0 branes). We can see that the potential increases at both l ends, though we can not clearly see the nature of the potential in the intermediate region but we can conclude that the potential is a confining potential and binds the D-0 branes. We find that $\langle l^2 \rangle \propto \sqrt{\frac{g}{\beta}}$ (true for any N), the finiteness of which also shows that there must be a potential between D-0 branes that binds them.

As a natural extension of our work on the 'High Temperature $N = 2$ matrix model', we have been trying to calculate the partition function for $N = 3$ matrix model (IIB). We have found out that this is a non-trivial exercise, so we would like to try a perturbative method of doing it, where the 5 parameters of $SU(3)/SU(2)$ are considered to be small. This system will give us the $SU(3)$ partition function as a perturbation from $SU(2)$ partition function.

• Fuzzy Sphere and The Matrix model:-

It is not yet clear how to study D-branes in a general curved background in the matrix model. It is important to address this issue. In the matrix model we are yet to understand the correspondence of the some of the symmetries of the string theory like the conformal symmetry, modular invariance, gauge symmetry and dualities. It is essential to look for a natural generalization of the matrix model to solve all such issues of matrix model.

Our recent work on the 'Interaction between two Fuzzy Spheres' is an attempt in this direction. We would like to understand the generalization of the matrix model to study D-branes in fuzzy sphere background, while it is not the most general curved background it is a nontrivial one.

Recently it has been discovered that the fuzzy sphere in finite matrix model (IIA) correspond to the spherical D2-brane wrapping on an S^3 in string theory which can be described by the $SU(2)$ WZW model. This gives us an interesting probe to study the D-branes in curved back ground in the string theory from the matrix model frame work. Earlier, D-branes in flat backgrounds have been explored within the framework of matrix models. Recently other non-commutative backgrounds, for *e.g.* the fuzzy sphere have also been studied. Non-commutative gauge theories on fuzzy spheres were obtained considering the supersymmetric three dimensional Type IIB matrix model action with a Chern Simons term. The fuzzy sphere in Type IIB matrix models may correspond to a spherical Euclidean D-brane in the string theory with a background linear B-field in S^3.

We have considered a general supersymmetric fuzzy sphere model in three dimensions, which allows a multi fuzzy sphere system with discretely arbitrary radii and arbitrary location in R^3. We have added a Chern Simons term to the reduced model of 3D SYM. In the original model the space points and objects (*e.g.* fuzzy spheres) are not separately distinguishable. We have artificially partitioned the matrices into multiple block diagonal form. In such case, the classical solution represents a system of space points and fuzzy spheres (branes). Classically these fuzzy spheres and space points are non-interacting. We have tried to calculate the interaction as the one loop quantum effect. We have calculated interaction of fuzzy spheres and space points. One loop interaction of fuzzy spheres in bosonic and supersymmetric case are studied. In particular, we have calculated the interaction between two fuzzy spheres with radii ($\rho_1 \sim \alpha n_1, \rho_2 \sim \alpha n_2$) ($n_1$ and n_2 is arbitrary) at distance ($r = \alpha c$). We have determined the one loop effective action for such system for both bosonic case and supersymmetric case for two fuzzy spheres for small ($c << 1$) and large distance ($c >> 1$) case. There is a partial cancellation between bosonic and fermionic part. In supersymmetric case, in 3 dimensions, there is an attractive force between two fuzzy sphere surfaces for both large and small distance case. This three dimensional fuzzy sphere model is a simple toy model in the context of string theory, also has relevance in other theories, such as the nonperturbative regularization of Quantum Field Theory using fuzzy space time and in studying quantum hall fluids of finite extent using finite matrix model. We have also studied an extension for such system in 10 dimensions, in the context of string theory. We have tried to include the other degrees of freedom. The fuzzy spheres we have considered, are separated in 1,2,3 directions and are on top of each other in other directions. We have got a repulsive force between two fuzzy spheres in short distance case and an attractive force in large distance case. Even though this model has a $\mathcal{N} = 2$ supersymmetry, the one loop contribution for the concentric case is non-zero for nearly equal n_1 and n_2. This is because of the fact the one loop approximation is not good approximation in this case.

Publications

Publications used in the Thesis

1. *Interaction between two Fuzzy Spheres*
   Subrata Bal, Hiroyuki Takata
   To be published in Int. J. Mod. Phys., hep-th/0108002
2. *High Temperature Limit of the $N = 2$ IIA Matrix Model*
   Subrata Bal, B. Sathiapalan
   Nucl.Phys.Proc.Suppl. **94** (2001) 693-696, hep-lat/0011039
3. *High Temperature Limit of the $N = 2$ Matrix Model*
   Subrata Bal, B. Sathiapalan
   Mod. Phys. Lett. A. **14** (1999) 2753-2766, hep-th/9902087

Other Publications

1. *The Character of the Exceptional Series of Representations of* $SU(1,1)$
   Debabrata Basu, Subrata Bal, K.V. Shajesh
   J. Math. Phy. **41** (2000) 461-467, hep-th/9906066
2. *Unified Treatment of Characters of SU(2) and SU(1,1)*
   Subrata Bal, K.V. Shajesh, Debabrata Basu
   J. Math. Phy. **38** (1997) 3209-3229, hep-th/9611236

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Chapter 1

Introduction

String theory has attracted a lot on interest in the past decade as a promising candidate for unified theory of all interactions in nature. In string theory the fundamental objects are one dimensional strings of length of the order 10^{-33} centimeter, instead of zero dimensional point particles. Interactions in string theory have a geometrical interpretation in terms of smooth Riemann surfaces. This way it is possible to get rid of the short distance divergences in gravity. Quantum Field Theory (QFT) gives a precise description of the natural interactions at our usual observable energy regime. Though QFT is essentially a theory of point particles, the standard methods of renormalization can handle the short distance or high energy divergences in interactions other than gravity. Gravity can be shown to be non-renormalisable using the usual renormalization methods. So, it is essential to search for an alternative quantum theory of gravity. In string theory the massless excited vibrational modes of strings, in principle, should give the elementary particle spectrum known in the standard model. String theory, in principle, also describes the abelian and non-abelian gauge interactions of the standard model. String theory contains a massless spin-2 state (graviton) whose low energy effective description is general relativity. This suggests string theory can be a natural and promising candidate for unified theory of all the forces including gravity.

Consistency of string theory requires supersymmetry, a symmetry between bosonic and fermionic degrees of freedom. According to string theory supersymmetry is a symmetry of nature at very high energy. At low energy this symmetry is spontaneously broken. Superstring theory is a ten dimensional theory, with six compact dimensions. Compactifications to lower dimensions give rise to a rich structure of Kaluza Klein fields. There are 5 known fully consistent string theory in ten dimensions. They are known as type IIA, type IIB, Type I, $E_8 \times E_8$ heterotic and

$SO(32)$ heterotic string theories. These theories are all related by duality transformations. Witten conjectured that the five string theories are in fact manifestations of one eleven-dimensional theory, namely *M-theory*. The existence of M-theory is not yet proven, but substantial evidence has been collected in the last several years. The two most important aspects of M-theory are :-

1) M-theory in the low energy limit gives eleven dimensional N = 1 supergravity.

2) M-theory compactified on a circle is dual to type IIA string theory. As the radius of this circle goes to infinity, the type IIA coupling also goes to infinity.

M-theory has many possible interpretations for the letter *'M'*. Matrix theory (model) is one proposal of M-theory. In the section 1.1, we will briefly introduce this model.

1.1 Matrix Model

The Matrix Model has proven to be an impressive candidate for M-theory. In $N \to \infty$ (N is the matrix size) limit, the matrix model should give the string theory results. There are mainly two types of matrix models, Type IIA and Type IIB corresponding to IIA and IIB string theories respectively.

1.1.1 BFSS Matrix Model

The IIA (BFSS) matrix model is proposed by Banks, Fischler, Shenker, Susskind [4]. They conjectured that M-theory in the infinite momentum frame (IMF) is a theory in which the only dynamical degrees of freedom are D0-branes each of which carries a minimal quantum of $p_{11} = 1/R$. In this theory the diagonal elements of the matrices are interpreted as the positions of the D0 branes and the off diagonal elements as the interactions between two such D0 branes. This system is described by the effective action for N D0-branes which is a particular $N \times N$ matrix quantum mechanics, to be taken in the $N \to \infty$ limit. The action of this model can be obtained by dimensionally reducing the ten dimensional $SU(N)$ Super Yang Mills theory to one (time, 0+1) dimension and is of the form

$$S = T_0 \int dt Tr \left(\frac{1}{2g_s} \left(D_0 X^i \right)^2 - \theta^T D_0 \psi + \frac{c^2}{4g_s} \left([X^i, X^j] \right)^2 + c\psi^T \Gamma^i [X_i, \psi] \right) \quad (1.1)$$

where $c = 1/(2\pi\alpha)$ and $T_0 = 1/\sqrt{\alpha} = 1/l_s$, D_0 is the time derivative. The indices i, j run from 1 to 9 over the nine transverse directions, the ψ are the sixteen component real spinors. The X^i and ψ are all in the adjoint representation of the gauge group $U(N)$, so they are hermitian $N \times N$ matrices.

The Hamiltonian can be obtained by fixing the gauge $X^0 = 0$,

$$H = Tr\left(\frac{1}{2}P^iP^i - \frac{1}{4}[X^i, X^j][X^i, X^j] - \frac{1}{2}\psi\Gamma^i[X^i, \psi]\right) \tag{1.2}$$

where the indices i, j run from 1 to 9 and the P^i are the canonical momenta for the X^i.

This model, at low energies, correctly reproduces the full Fock space of an arbitrary number of supergravitons (supergravity multiplet of 256 states) and also the (local) supergraviton interaction of supergravity.

The scattering in the IMF should be described by a nonrelativistic looking time-independent potential at vanishing p_{11}-transfer. This potential up to one loop calculation can be found out as

$$V_{eff}(r) = -\frac{15}{16}\frac{v^4}{r^7} + O\left(\frac{v^6}{r^{11}}\right) \tag{1.3}$$

This potential exactly matches with the corresponding result in eleven dimensional supergravity. Two loop calculation also maintain this matching.

The matrix model contains (super) membranes, and in the large N limit the matrix model dynamics goes over to the dynamics of the corresponding (super)membranes. The tension of these matrix model membranes agrees with the tension of the M-theory membranes.

1.1.2 IKKT Matrix Model

The IIB (IKKT) matrix model, conjectured by Ishibashi, Kawai, Kitazawa and Tsuchiya [5] is a large N 0+0 dimensional reduced model of ten dimensional $\mathcal{N} = 1$ maximally supersymmetric $SU(N)$ Yang-Mills theory.

$$S = -\frac{1}{g^2}Tr\left(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right). \tag{1.4}$$

A_μ, ψ_α are $N \times N$ traceless Hermitian matrices. A_μ is a 10-dimensional vector and ψ is 10-dimensional 16 component Majorana-Weyl spinor fields respectively.

This model can be related to the Green-Schwarz action for the superstring theory in the Schild gauge

$$S_{GS} = \int d^2\sigma \left[\sqrt{g}\alpha\left(\frac{1}{4}\{x^\mu, x^\nu\}^2_{PB} - \frac{i}{2}\bar{\psi}\Gamma^\mu\{x^\mu, \psi\}_{PB}\right) + \beta\sqrt{g}\right] \tag{1.5}$$

where σ are the two dimensional world sheet coordinates, $g = \det(g_{ab})$ is the determinant of the world sheet metric, and α, β are parameters and can be scaled out. The x^μ are target space coordinates. This theory can be regularised following method of Goldstone and Hoppe. In this, a function y on the world sheet is replaced by an $N \times N$ traceless hermitian matrix Y and correspondingly

$$\int d^2\sigma \sqrt{g} y \leftrightarrow \mathrm{Tr} Y \tag{1.6}$$

and

$$\{x, y\} \leftrightarrow -i[X, Y]. \tag{1.7}$$

Under this regularisation, equation 1.5 reduces to

$$S_{Schild} = -\alpha T r \left(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi] \right) + \beta N \tag{1.8}$$

The $\mathcal{N} = 2$ supersymmetry is manifest in S_{Schild}, which gets translated into the IIB matrix model. IIB matrix model has $\mathcal{N} = 2$ space time Supersymmetry.

$$\begin{aligned} \delta^{(1)}\psi &= \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon \\ \delta^{(1)} A_\mu &= i\bar{\epsilon}\Gamma^\mu\psi \end{aligned}$$

and

$$\begin{aligned} \delta^{(2)}\psi &= \xi, \\ \delta^{(2)} A_\mu &= 0. \end{aligned}$$

Taking the linear combination of $\delta^{(1)}$ and $\delta^{(2)}$ as,

$$\begin{aligned} \tilde{\delta}^{(1)} &= \delta^{(1)} + \delta^{(2)} \\ \tilde{\delta}^{(2)} &= i\left(\delta^{(1)} - \delta^{(2)}\right), \end{aligned}$$

we can get the $\mathcal{N} = 2$ supersymmetry algebra,

$$\begin{aligned} \left(\tilde{\delta}^{(i)}_\epsilon \tilde{\delta}^{(j}_\xi - \tilde{\delta}^{(j}_\xi \tilde{\delta}^{(i)}_\epsilon\right)\psi &= 0 \\ \left(\tilde{\delta}^{(i)}_\epsilon \tilde{\delta}^{(j}_\xi - \tilde{\delta}^{(j}_\xi \tilde{\delta}^{(i)}_\epsilon\right) A_\mu &= 2i\bar{\epsilon}\Gamma^\mu\xi\delta_{ij} \end{aligned}$$

As the maximal space time supersymmetry indicates the existence of gravitons and this theory contains massless particle in the spectrum, it votes for II B matrix model being a constructive definition of II B string theory. This is one reason for which the the eigenvalues of A_μ matrices in this model are interpreted as the space time points. In this model the space time consists of N discrete points. The offdiagonal elements denote interactions between two such space time points.

This model has no free parameter. The coupling constant g can be absorbed by the field redefinition, in a similar way as coupling constant can be absorbed in string theory to the dilaton vacuum expectation value.

As matrices in general do not commute under multiplication, noncommutative geometry arises automatically in matrix model. In [3], it was shown that Non-commutative Yang Mills theory with infinitely extended D-brane background can be obtained from large N twisted reduced model and provides a precise definition of noncommutative Yang-Mill theory. They have also studied the D-instanton interactions and have shown that these interactions when overlap, matches with the gauge theory description and for non-overlapping case satisfies the IIB supergravity description. A four dimensional gauge theory with D-brane background can also be obtained. It was also shown that the Maldachena conjecture [10] follows from IIB matrix model conjecture [5]. In [9] the effective string scale for noncommutative Yang Mills is identified with the noncommutative scale using its dual supergravity description. In [8] a bilocal representation of noncommutative field theory is proposed, which provides a simplified description for high momentum degrees of freedom. The bilocal fields can be identified to the zero modes of open strings and can be interpreted as the 'momentum' and 'winding'modes. Noncommutative Yang Mills can be associated with the von Neumann lattice by the bi-local representation and it is argued to be superstring theory on von Neumann lattice. Newton's force law may be obtained from 4 dimensional noncommutative Yang Mills theory with maximal supersymmetry. It suggests a nonperturbative compactification mechanism of IIB matrix model.

1.2 Thesis Plan

In this thesis, we have investigated the following two important aspects of the physics of matrix models namely :-

1) *Finite Temperature Matrix Model.*

2) *Matrix Model and Fuzzy Sphere.*

The thesis is organized in the following way.

In section 1.1, we have briefly introduced the matrix model. We will discuss about two types of Matrix Models,

1) Type IIA or BFSS matrix model and

2) Type IIB or IKKT matrix model.

In section 1.3 and 1.4, we will try to briefly introduce the motivation and nature of these two issues. In section 1.3, we will briefly review the finite temperature string theory and its importance and limitation. We will try to understand the motivation of studying finite temperature matrix model and the present background in this field. In 1.4, we will discuss about the importance of the fuzzy sphere in matrix model in studying the D-branes in string theory and introduce the present problem.

In chapter 2, we have considered a system of two D0-branes at high temperature. Such a system is described by $N = 2$ matrix model, time compactified on a circle of circumference β. We have calculated the partition function of such system nonperturbatively (exact in g) as a power series in β (inverse temperature). The leading term in the high temperature expression of the partition function and effective potential is calculated *exactly*. Physical quantities like the mean square separation can also be exactly determined in the high temperature limit.

In chapter 3, we have described the Chern-Simons-IKKT matrix model. Fuzzy sphere and Points are classical solutions of this action. We can describe a multi-fuzzy spheres system in same set up. We have calculated also calculated that two block and three blocks interactions. We have studied point-point interaction and point-fuzzy sphere interaction in detail.

In chapter 4, We have calculated interactions between two fuzzy spheres in 3 dimension. It depends on the distance r between the spheres and the radii ρ_1, ρ_2. There is no force between the spheres when they are far from each other (long distance case). We have also studied the interaction for $r = 0$ case. We find that an attractive force exists between two fuzzy sphere surfaces. We have also studied the extension of such system in 10 dimension.

1.3 High Temperature Limit of the $N = 2$ Matrix Model

In this subsection, we will briefly discuss about the finite temperature aspect of string theory and the problems related to it. We will also discuss the motivation for studying the finite temperature the matrix model.

1.3.1 Finite temperature String Theory

The study of string theory at finite temperature is a long standing field of interest. There are several reasons for this. String theory in and above the Plank scale regime is an issue of fundamental interests. In this regime we expect the stringy behavior to become more evident and might give new insights into its fundamental degrees of freedom and the structure of space time, thus providing a framework of unification of space time and matter. It is very poorly understood what is the correct configuration space and which states dominate the dynamics of string theory at that scale. One way to probe this region would be to study a very hot ensemble of strings.

Another motivation of studying such system is within the context of cosmology. The evolution of the early universe may be qualitatively altered by the effects due to fundamental strings or due to cosmic strings which have many of the same properties as fundamental strings.

It is well known that any string model has spectrum whose degeneracy grows exponentially as a function of the mass [10]. The number of states with mass between mass m and $m + dm$ is

$$\rho(m)dm = Am^{-B}Exp(\beta_0 m)dm \tag{1.9}$$

where A, B are constants and values depends on the particular string theory. As a result, the thermodynamical partition functions will diverge beyond the temperature $T_0 = \frac{1}{\beta_0}$ due to the competition between the entropy and the Boltzmann factor $Exp(-\beta m)$, indicating a existence of a limiting maximum temperature for a system in thermal equilibrium. This temperature is called Hagedorn Temperate (T_H) [11]. The consequences in the early universe due to such limiting temperature coming from the dual string model were studied by Huang and Weinberg [12]. What happens when we try to increase the temperature past this point is, despite much effort, merely a speculation.

In QCD, there are two phases: below the critical temperature T_c, the theory is in the confining phase where the spectrum is similar that of the string theory while beyond T_c the theory is in the deconfining phase. Drawing an analogy between string theory and QCD, it has been argued that the Hagedorn temperature is not really a limiting temperature, rather a phase transition temperature [13]. There are various interpretations for a weakly interacting ensemble of strings near the Hagedorn Temperature. It has been argued in [13], T_H is a first order transition temperature in which the structure of the string world sheet may break down. It has been shown that at Hagedorn temperature a certain 'winding' mode becomes massless, resulting in phase transition[14]. As the microcanonical specific heat is negative for high energies, in [15], it has been speculated that for high energies the string ensemble may be unphysical and the stability may set in above the Hagedorn Temperature [16] and all the energy will be taken away by a single energetic string

[17]. It is also argued that at intermediate energy density the energy may be shared by more than one very energetic strings [18]. In [17, 19], it was shown that the string ensemble makes sense all the way up to a canonical temperature equal to the Hagedorn temperature. It was also shown that if additional energy is added to the ensemble beyond this temperature in trying to raise the temperature, the extra energy leads to the formation of a long string; and analogy was drawn to an infinite latent heat [19, 20, 17]. In [21], a mixed phase structure was argued, which is characteristic of a system undergoing a phase transition. However, the system can never completely go into the 'high temperature' phase as it needs an infinite amount of energy. No instability is encountered in this scenario. This scenario may differ for interacting strings.

According to the string uncertainly principle [22], when one increases collision energy, the energy becomes large enough to excite the oscillation modes of the the string and beyond this increased collision energy leads to poorer resolution. This may tend to indicate that distances shorter than the string scale and the corresponding high-frequency degrees of freedom do not exist. This can drastically change the fundamental degrees of freedom at high energy and this scenario of long string may also be changed.

So, we can see that it is not yet clear whether the Hagedorn Temperature is really a limiting temperature or a phase transition temperature. The string behavior near Hagedorn Temperature is not understood in perturbative string theory. Thus, a non-perturbative formulation like matrix models may be helpful in resolving these long standing issues.

1.3.2 Finite Temperature Matrix Model

It is clear that recent developments in non-perturbative string theory or M-theory [4, 5, 6, 7, 7] have some bearing on our understanding of the high temperature behavior of strings. Furthermore, study of high temperature behavior of a system is often a useful probe for the system and many calculations simplify at high temperature. We can thus hope to learn something about M-theory from its high temperature behavior.

In [23], there is an attempt to elucidate the nature of the Hagedorn temperature using the matrix model formalism. It was shown that this temperature is not a limiting temperature, rather a phase transition temperature. Hagedorn transition is shown to be similar with the deconfinement transition in gauge theories. The phase below the Hagedorn temperature corresponds to one where the D0-branes are arranged to form a membrane which can be wrapped around compact direction to give a string. By a one loop computation it was shown that at high temperatures the D0-branes have an attractive force between them and prefer to cluster as one point causing the string to disappear. This was also investigated in a subsequent paper using the AdS/CFT correspondence [24].

In [25], Ambjorn, Makeenko, Semenoff have studied a system of D0-branes with open strings between them in type IIA superstring theory using the matrices. They have computed the one loop effective potential between static D0-branes in the matrix theory at finite temperature and have shown that the result matches with string theory in the low temperature limit at leading order. The effective static potential between D0-branes is logarithmic and attractive in nature at short distances.

With this back ground, in chapter 2 of this thesis, we attempt to study the high temperature behavior of a simple but non-trivial system - the system of two D-0-branes. This is essentially the BFSS matrix model [4] with N=2. This N is not large enough to describe M-theory. In particular it would not include for instances processes involving pair-production of D-0 brane - anti-D-0 brane. Nevertheless it is already complicated enough. In particular the nature of the threshold and other bound states that have been studied [27, 28, 8, 30] are not fully understood. Furthermore we should keep in mind that while the matrix model reproduces string theory at short distances, the fact that it also does so at long distances seems to be entirely due to the super-symmetric non-renormalisation theorems. At finite temperature supersymmetry is broken and perhaps we should not expect this. For all these reasons the study of matrix models at high temperature is worthwhile.

A related model of D-instantons, the IKKT matrix model [5], which is 0+0 dimensional has been solved exactly for N=2 [26]. The partition function in such case was shown to be finite without any divergences.

The D-0-brane action that we are interested in, is a quantum mechanical one (i.e. 0+1 dimensional). However after compactifying the Euclideanised time, if one takes the high temperature limit, it reduces to a 0+0 dimensional model and have similar structure as that of IKKT matrix model. There is thus a hope of solving this model order by order in β but *to all orders in g* using the same techniques as [26]. One can then calculate physical quantities such as the mean square separation of the D-0-branes - a measure of the size of the bound states. This is what is attempted in this chapter. We obtain the leading behavior in β. We can also estimate, the corrections to the leading result. The noteworthy feature being that each term is *exact in its dependence on the string coupling constant .*

1.4 Matrix Model and Fuzzy Sphere

Fuzzy Physics is a field of long standing interests. Quantisation of the underlying space time (a fuzzy space time) is a well known method of introducing a short distance cut off. Attempt of nonperturbative regularisation of quantum field theory using such fuzzy space time have some important advantages over the lattice quantum field theory. Noncommutative matrix model description for fuzzy fields is successful in preserving the symmetries, topological features and overcoming the fermion doubling problem appears in lattice QFT.

Study of fuzzy physics and noncommutative geometry have drawn renewed attention, in recent days, for its relevance in better understanding of string theory. Specially the presence of the extended objects like D-branes in string theory makes it essential to think of noncommutativity of the space time, and demands better understanding of the fuzzy physics. Study of fuzzy sphere also have relevance in *Quantum Hall Effect.* Finite Chern - Simons matrix model on the plane is used as an effective description of fractional quantum Hall fluids of finite extent. Here, we will try to explore some unanswered questions related to the dynamics of the fuzzy sphere in the framework of Chern-Simons-IKKT matrix model. Our work is motivated from string theory.Though this model is a toy model from string theory point of view, it can be extended to higher dimensions to understand specific physical system in string theory. Before going into the details of the subject let us give a brief introduction about the fuzzy sphere.

1.4.1 Fuzzy Sphere

An ordinary sphere, as we know, is defined by the equation

$$x_1^2 + x_2^2 + x_3^2 = \rho^2 \tag{1.10}$$

where, x_1, x_2 and x_3 are real numbers. ρ is a constant. The coordinates x_1, x_2 and x_3 commute each other and the ρ can take any value from a continuous $\mathbf{R}^1$

The translational operator on such S^2 is not momentum, rather an angular momentum L_μ of three dimensional space. These angular momentum operators satisfy the $SU(2)$ algebra.

$$[L_\mu, X_\nu] = i\epsilon_{\mu\nu\lambda} X_\lambda$$

$$[L_\mu, L_\nu] = i\epsilon_{\mu\nu\lambda} L_\lambda$$

The casimir $L^2 = L_1^2 + L_2^2 + L_3^2$ has eigenvalue $l(l+1)$, where l takes only integral or half integral positive values.

To get the fuzzy version of the ordinary sphere, we identify the coordinate to translational operator on S^2. $X_\mu \equiv \alpha L_\mu$ α is a dimensionfull parameter. Hence, the coordinates are, now, noncommutating operators. They satisfy the $SU(2)$ algebra

$$[X_\mu, X_\nu] = i\alpha\epsilon_{\mu\nu\lambda}X_\lambda \tag{1.11}$$

Because of noncommutativity, all the X_μ's cannot be diagonalised simultaneously and eigenvalues are discretised. As a result the space time is no longer continuous and smooth, rather it is non-commutative and consist of discrete points.
In this present space time the equation 1.10,

$$X_1^2 + X_2^2 + X_3^2 = \rho^2$$

does not represent a smooth surface like earlier. Now, the surface is discretised and fuzzy. So, we call it a *fuzzy sphere.* We should also note here that the eigenvalues of $X_1^2 + X_2^2 + X_3^2 = \rho^2$, are not continuous. The radius of the sphere can only take discrete values $l(l+1)$ and depends on the representation chosen, where l is integer or half integer. For a fixed value l *i.e.* for radius of the sphere is $\frac{\rho^2}{\alpha^2} = l(l+1)$, we can choose a representation of X_μ as $2l+1 = N$ dimensional *irreducible* hermitian matrices.

We will not go into the details of the properties of the fuzzy space time, but let us give a table describing the analogy between fuzzy sphere and fuzzy plane(or torus)

	torus	sphere
translation operator	momentum p_μ	angular momentum l_μ
ordinarily	$[p_\mu, x_\nu] = i\delta_{\mu\nu}$ $[p_\mu, p_\nu] = 0$ $[x_\mu, x_\nu] = 0$	$[l_\mu, x_\nu] = i\epsilon_{\mu\nu\lambda}x_\lambda$ $[l_\mu, l_\nu] = i\epsilon_{\mu\nu\lambda}l_\lambda$ $[x_\mu, x_\nu] = 0$
$\Downarrow$ identify by	$x_\mu \equiv \theta_{\mu\nu}p_\nu$	$x_\mu \equiv \alpha l_\mu$
fuzzy	$[p_\mu, x_\nu] = i\delta_{\mu\nu}$ $[p_\mu, p_\nu] = -i\theta^{-1}_{\mu\nu}$ $[x_\mu, x_\nu] = i\theta_{\mu\nu}$	$[l_\mu, x_\nu] = i\epsilon_{\mu\nu\lambda}x_\lambda$ $[l_\mu, l_\nu] = i\epsilon_{\mu\nu\lambda}l_\lambda$ $[x_\mu, x_\nu] = i\alpha\epsilon_{\mu\nu\lambda}x_\lambda$
coordinates algebra	Heisenberg	SU(2)

Table : Analogy between fuzzy sphere and fuzzy plane(or torus)

As these fuzzy spheres are found to have correlation with the physical objects in string theory, we would like to study the dynamics of these fuzzy spheres. To study the dynamics of fuzzy sphere system, we look for an action such that the classical equations of motions is the algebra of coordinates:

$$[X_\mu, X_\nu] = i\alpha\epsilon_{\mu\nu\lambda}X_\lambda$$

It may give a model for *dynamics* of fuzzy spheres.

1.4.2 Matrix Model and Fuzzy Sphere

String theory in its present form, includes various types of extended objects like D-branes other than the fundamental string. The presence of D-branes brings the non-commutativity of space time. In fact we can derive non-commutative gauge theory on world volume of D-branes in string theory and this non-commutative gauge theory leads us to find the corresponding matrix model.

We have seen, in IIB matrix model the non-commutativity of space time and the interactions are described by the matrices. The non-commutative Yang-Mills theory in a flat back ground can be obtained by expanding the matrix model around a flat non-commutative background [2]. Such non-commutative gauge theory is obtained in string theory by introducing background constant B-field [3]. The non-commutative background is a D-brane-like background which is a solution of the equation of motion. So, we can study the D-branes in flat background within the framework of matrix model. We need to formulate a matrix model which as well gives us a scope to study the D-brane in curved background.

Recently other different non-commutative backgrounds, for *e.g.* a non-commutative sphere, or a fuzzy sphere, have also been studied [4, 5]. In [4], non-commutative gauge theories on fuzzy sphere were obtained expanding the supersymmetric three dimensional matrix model action with a Chern Simons term, around a classical solution. Although an ordinary matrix model has only a flat background as a classical solution, this matrix model can describe a curved background owing to this Chern Simons term.

Fuzzy sphere may correspond to spherical D2-brane in string theory with a background linear B-field in S^3 [6, 7, 8, 9, 10]. Specially, the 0 radius one correspond to the D0-brane. It is interesting to note that, in Matrix model, such objects correspond to the D2 or D0 brane in string theory. Similarly, BPS objects correspond to BPS D branes.

We have considered a supersymmetric fuzzy sphere model in three dimensions as in [4]. We get this model by adding a Chern Simons term to the three dimensional reduced model [1]. We try to provide a framework which allows us to study a multi fuzzy sphere system. We shall study the two fuzzy spheres system in detail and investigate the interaction between them.

We have also studied an extension of this toy model to 10 dimensions, in order to make contact with the string theory. We will try to include the other degrees of freedom. We will study a two fuzzy sphere configuration in the IIB matrix model with an added CS term, with an the action very similar to that of [12].

We have presented the model for the multi fuzzy sphere in background space. We have studied interaction of fuzzy spheres and space. We have also considered the dynamics of the fuzzy spheres. We have expanded the action around a classical background and studied the one loop interaction between fuzzy spheres for the bosonic and supersymmetric cases. In particular, we have calculated the interaction between two fuzzy spheres. We have calculated the potential for such system for both large and small distance cases. This potential is attractive for the supersymmetric case which vanishes for infinite distance. In section 4.2, we have studied an extension of this 3 dimensional model to 10 dimensions. Such a system may have corelation with D-brane in string theory and help in understanding of dynamics of D-branes in string theory from IIB matrix model aspect. In a recent work [31], there was an attempt to understand the dynamics of two fuzzy spheres in IIA matrix model aspect.

1.4.3 Space-time and brane from Matrix Model view point

In general, in matrix model, we deal with arbitrary Hermitian matrices. We can artificially partition these matrices into multiple blocks such that each diagonal block represents a part of space time. We call it a space time object or a brane. And the off diagonal blocks represent the interactions between such branes. The size of such brane depends on the size of the matrix-block representing the brane. For example, block of size 1, describes a space time point.

Though the overall matrix is traceless, the individual blocks need not be traceless and the values of traces of these blocks give the space time co-ordinates of the centers of the blocks (objects). In this paper we assume that the trace lies in $\mathbf{R}^{10}$.

We shall assume the 'dynamical compactification' of 10 dimensional space time to $M^7 \otimes \mathbf{R}^3(S^3)$, [11].

It is known that there are non-commutative solutions for classical equations of motion and non-commutative gauge theories on such space time objects (both plane and sphere case). Iso et al [4] and others have shown that a non-commutative gauge theory can be realized on a fuzzy sphere. For the flat case the gauge interaction can be explained as the open strings which ends on the object (brane). The force between two such branes at long distance can be understood as close strings exchange between them. It is interesting to understand whether this feature is valid for $M^7 \otimes \mathbf{R}^3(S^3)$ configuration of space time. However, here we treat the interaction of two fuzzy spheres in $\mathbf{R}^3$ only from matrix model point of view.

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Chapter 2

High Temperature Limit of the $N = 2$ Matrix Model

In this chapter, we study the high temperature behavior of a simple but non-trivial system - the system of two D-0-branes. This is essentially the BFSS matrix model [4] with N=2. The D-0-brane action that we are interested in, is a quantum mechanical one (i.e. 0+1 dimensional). However after compactifying the Euclideanised time, if one takes the high temperature limit, it reduces to a 0+0 dimensional model. There is thus a hope of solving this model order by order in β but *to all orders in g* using the same techniques as [3]. One can then calculate physical quantities such as the mean square separation of the D-0-branes - a measure of the size of the bound states. This is what is attempted in this chapter. We obtain the leading behavior in β. We also estimate, the corrections to the leading result. The noteworthy feature being that each term is *exact in its dependence on the string coupling constant* .

2.1 The action

The $(0 + 1)$ dimensional BFSS Lagrangian is

$$L = \frac{1}{2gl_s}\mathrm{tr}\left[\dot{X}^i\dot{X}^i + 2i\theta\dot{\theta} - \frac{1}{2l_s^4}[X^\mu, X^\nu]^2 - \frac{2}{l_s^2}\bar{\theta}\gamma_\mu[\theta, X^\mu] - \frac{i}{l_s^2}[X^0, X^i]\dot{X}^i\right] \quad (2.1)$$

where $i = 1, ..., 9; \mu = 0, ..., 9$; X^μ and θ are N x N hermitian matrices. X^μ is 10 a dimensional vector and θ is 16 component Majorana-Wyel spinor in 10 dimensional super-Yang Mills theory. For $N = 2$, we can re-write X^μ and θ as [3]

$$X^\mu = \sum_{a=1}^{3}\frac{1}{2}\sigma^a X_a^\mu \quad (2.2)$$

and

$$\theta = \sum_{a=1}^{3} \frac{1}{2}\sigma^a \theta_a \tag{2.3}$$

where, σ^a are the hermitian Pauli matrices and X^μ_a, θ_a are real fields in terms of which , the Lagrangian (2.1) is

$$\begin{aligned} L &= \frac{1}{4gl_s}\left\{\sum_{a,b=1}^{3} \dot{X^i}_a \dot{X}^i_a + 2i\sum_{a=1}^{3}\theta_a\dot{\theta}_a - \frac{2i}{l_s^2}\sum_{a,b,c=1}^{3}\theta_a\gamma_0\gamma_\mu\theta_b X^\mu_c\epsilon^{abc}\right. \\ &+ \left.\frac{\epsilon^{abc}}{4l_s^2}X^0_a X^i_b \dot{X}^i_c + \frac{1}{2l_s^4}\sum_{a,b=1}^{3} X^\mu_a X^\nu_b X^\mu_a X^\nu_b - \frac{1}{2l_s^4}\sum_{a,b=1}^{3} X^\mu_a X^\nu_b X^\mu_b X^\nu_a\right\} \end{aligned} \tag{2.4}$$

Now, we compactify the time on a circle of circumference of β. So, the action becomes

$$\begin{aligned} S &= \int_0^\beta L dt \\ &= \frac{1}{4gl_s}\int_0^\beta dt\left\{\sum_{a,b=1}^{3} \dot{X^\mu}_a \dot{X}^\mu_a + 2\sum_{a=1}^{3}\theta_a\gamma_0\dot{\theta}_a - \frac{2i}{l_s^2}\sum_{a,b,c=1}^{3}\theta_a\gamma_0\gamma_\mu\theta_b X^\mu_c\epsilon^{abc}\right. \\ &- \left.\frac{i}{2l_s^2}\sum_{a,b,c=1}^{3} X^0_a X^i_b \dot{X}^i_c\epsilon^{abc} + \frac{1}{2l_s^4}\sum_{a,b=1}^{3} X^\mu_a X^\nu_b X^\mu_a X^\nu_b - \frac{1}{2l_s^4}\sum_{a,b=1}^{3} X^\mu_a X^\nu_b X^\mu_b X^\nu_a\right\} \end{aligned}$$

As we are compactifying the time t on a circle of circumference β, the fields will satisfy the following boundary conditions,
1) X^μ will be periodic in time with periodicity β .

$$X^\mu(0) = X^\mu(\beta)$$

2) θ will be anti-periodic in time with anti-periodicity β .

$$\theta(0) = -\theta(\beta)$$

Considering these boundary conditions, we can expand the fields X^μ, θ in terms the modes as

$$X^\mu_a(t) = \sum_{n=-\infty}^{\infty} X^\mu_{a,n} e^{\frac{2\pi i n}{\beta}t} \tag{2.5}$$

and

$$\theta_a(t) = \sum_{r=-\infty}^{\infty} \theta_{a,r} e^{\frac{2\pi i r}{\beta}t} \tag{2.6}$$

where, n is an integer and r is a half-integer. We fix a convention that we will use : n, m, l, p will be used as as the indices for the modes of the vector fields $X^{\mu}_{a,n}$, and they are integers. The half-integers r, s will be used as the indices for the modes of the spinor fields $\theta_{a,r}$. In terms of these modes, the Lagrangian reduces to

$$\begin{aligned}
S &= \frac{\beta}{4gl_s}\Bigg\{\sum_{n=-\infty;n\neq 0}^{\infty}\frac{4\pi^2n^2}{\beta^2}X^{\mu}_{a,n}X^{\mu}_{a,-n} \\
&+ 2\sum_{r,s=-\infty,s+r=0}^{\infty}\frac{2\pi i}{\beta}s\theta_{a,r}\gamma_0\theta_{a,s} \\
&- \frac{2i}{l_s^2}\sum_{r,s,l=-\infty,s+r+l=0}^{\infty}\theta_{a,r}\gamma_0\gamma_\mu\theta_{b,s}X^{\mu}_{c,l}\epsilon^{abc} \\
&- \frac{i}{4l_s^2}\sum_{n+m+l=0,l\neq 0}2\pi i l X^0_{a,n}X^i_{b,m}X^i_{c,l}\epsilon^{abc}\beta \\
&+ \frac{1}{2l_s^2}\sum_{n,m,l,p=-\infty,n+m+l+p=0}^{\infty}X^{\mu}_{a,n}X^{\nu}_{b,m}X^{\mu}_{a,l}X^{\nu}_{b,p} \\
&- \frac{1}{2l_s^2}\sum_{n,m,l,p=-\infty,n+m+l+p=0}^{\infty}X^{\mu}_{a,n}X^{\nu}_{b,m}X^{\mu}_{b,l}X^{\nu}_{a,p}\Bigg\}
\end{aligned} \tag{2.7}$$

Now, we Eucledeanise the time, effectively the β here. Under this the action takes the form,

$$\begin{aligned}
S &= \frac{i\beta}{4gl_s}\Bigg\{-\sum_{n=-\infty;n\neq 0}^{\infty}\frac{4\pi^2n^2}{\beta^2}X^{\mu}_{a,n}X^{\mu}_{a,-n} \\
&+ 2\sum_{r,s=-\infty,s+r=0}^{\infty}\frac{2\pi}{\beta}s\theta_{a,r}\gamma_0\theta_{a,s} \\
&- \frac{2i}{l_s^2}\sum_{r,s,l=-\infty,s+r+l=0}^{\infty}\theta_{a,r}\gamma_0\gamma_\mu\theta_{b,s}X^{\mu}_{c,l}\epsilon^{abc} \\
&- \frac{i}{4l_s^2}\sum_{n+m+l=0,l\neq 0}2\pi i l X^0_{a,n}X^i_{b,m}X^i_{c,l}\epsilon^{abc}\beta \\
&+ \frac{1}{2l_s^2}\sum_{n,m,l,p=-\infty,n+m+l+p=0}^{\infty}X^{\mu}_{a,n}X^{\nu}_{b,m}X^{\mu}_{a,l}X^{\nu}_{b,p} \\
&- \frac{1}{2l_s^2}\sum_{n,m,l,p=-\infty,n+m+l+p=0}^{\infty}X^{\mu}_{a,n}X^{\nu}_{b,m}X^{\mu}_{b,l}X^{\nu}_{a,p}\Bigg\}
\end{aligned} \tag{2.8}$$

Here, $X^{\mu}_{a,n}, \beta$ have the dimension of length (L) and θ has the dimension $L^{\frac{1}{2}}$. We scale $X^{\mu}_{a,n}, \beta$ with a factor of $\frac{1}{l_s}$, and θ with a factor $l_s^{\frac{-1}{2}}$ to make them dimensionless, which is equivalent to replacing all the l_s in the action by 1.

The first and the second terms in the action give the masses of the modes of the vector and the spinor fields respectively, namely, $\frac{2\pi n}{\beta}$ and $\frac{2\pi s}{\beta}$.

2.2 The Partition Function and the effective Potential.

In this section we will try to calculate this partition function. The partition function with this action is

$$Z = \int e^{-iS} \tag{2.9}$$

2.2.1 Pfaffian

In this subsection we will calculate the pfaffian for the above action, following the same method as in [3]. However, here we will consider the higher fermionic modes and due to which the supersymmetry is broken, unlike [3], where there were no higher modes and supersymmetry is preserved. So, though the method here will be same as [3], the details of the calculation will be different. The fermionic terms in the action are of the form $(\theta_{a,r}\gamma_0\gamma_\mu\theta_{b,s}X^\mu_{c,n})_{n+r+s=0}$ and $\theta_{a,r}\theta_{b,-r}$. In $\beta \to 0$ limit, the first term in the action (??) is dominant and $(X^\mu_{a,n})_{n\neq 0}$ is of the order $\sqrt{\beta}$ [1]. So, in large temperature limit the first term contributes to the partition function in the leading order of β only for $n = 0$ $(\theta_{a,r}\gamma_0\gamma_\mu\theta_{b,-r}X^\mu_{c,0})$. In this limit the action takes the form

$$S_f = \frac{i\beta}{4g}\sum_{r=\frac{1}{2}}^{\infty}\left\{\frac{4\pi r}{\beta}\theta_{a,r}\theta_{a,-r} - \frac{4\pi r}{\beta}\theta_{a,-r}\theta_{a,r} - 2i\theta_{a,r}\gamma_0\gamma_\mu\theta_{b,-r}X^\mu_{c,0}\epsilon^{abc} - 2i\theta_{a,-r}\gamma_0\gamma_\mu\theta_{b,r}X^\mu_{c,0}\epsilon^{abc}\right.$$

Now, we will try to find out the pfaffian

$$Z_f = \int \prod_r \prod_{a=1}^{3} d^{16}\theta_{a,r} d^{16}\theta_{a,-r} e^{-iS_f} \tag{2.10}$$

for this action

$$Z_f = \prod_{r=\frac{1}{2}}^{\infty} \int \prod_{a=1}^{3} d^{16}\theta_{a,r} d^{16}\theta_{a,-r} \exp\left[\frac{-\beta}{2g}\left\{\theta_{a,r}\left(\frac{2\pi r}{\beta}\delta^{ab} - iX^\mu_{c,0}\epsilon^{abc}\gamma_0\gamma_\mu\right)\theta_{b,-r}\right.\right.$$

[1] Strictly speaking it is of the order $\frac{\sqrt{\beta}}{n}$. As our aim is to see mainly the β dependence, we take it as $\sqrt{\beta}$, ie the case $n = 1$. . And for $n > 1$, $\frac{1}{n}$ dependence makes the argument for neglecting the higher order modes stronger.

$$-\theta_{a,-r}\left(\frac{2\pi r}{\beta}\delta^{ab} + iX^{\mu}_{c,0}\epsilon^{abc}\gamma_0\gamma_\mu\right)\theta_{b,r}\Big\}\Big] \tag{2.11}$$

Following [3], we rotate $X^{\mu}_{c,0}$ by a Lorentz transformation so that only $X^0_{c,0}$, $X^1_{c,0}$ and $X^2_{c,0}$ are nonzero. We take the representation of the Gamma matrices, in which

$$\gamma_0 = i\sigma_2 \otimes 1_8, \qquad \gamma_1 = \sigma_3 \otimes 1_8, \qquad \gamma_2 = -\sigma_1 \otimes 1_8 \tag{2.12}$$

With this choice of representation we can write

$$S_f = \frac{i\beta}{g}\sum_{r=-\infty}^{\infty}\left\{\theta_{a,r}\left(\frac{2\pi r}{\beta}\delta^{ab} + i\epsilon^{abc}\left(X^0_c 1_2 + X^1_c\sigma_1 + X^2_c\sigma_3\right)\right)\otimes 1_8\theta_{b,-r}\right\} \tag{2.13}$$

So, the pfaffian will be

$$Z_f = \left[\int\prod_{a,b=1}^{3} d\theta'_{a,r}d\theta'_{b,-r}\exp\left[\frac{\beta}{g}\left\{\theta'_{a,r}\left(\frac{2\pi r}{\beta}\delta^{ab} - i\epsilon^{abc}\left(X^0_{c,0}1_2 + X^1_{c,0}\sigma_1 + X^2_{c,0}\sigma_3\right)\right)\theta'_{b,-r}\right\}\right]\right]^8$$

where $\theta'_{a,r}$ and $\theta'_{b,-r}$ are the spinors in three dimension, and has two components. After doing the grassmann integral over $\theta'_{a,r}$, we can write

$$Z_F = \det\left[\frac{\beta}{g}\begin{pmatrix} \frac{2\pi r}{\beta} & -i\left(X^0_3 + X^1_3\sigma_1 + X^2_3\sigma_3\right) & i\left(X^0_2 + X^1_2\sigma_1 + X^2_2\sigma_3\right) \\ i\left(X^0_3 + X^1_3\sigma_1 + X^2_3\sigma_3\right) & \frac{2\pi r}{\beta} & -i\left(X^0_1 + X^1_1\sigma_1 + X^2_1\sigma_3\right) \\ -i\left(X^0_2 + X^1_2\sigma_1 + X^2_2\sigma_3\right) & i\left(X^0_1 + X^1_1\sigma_1 + X^2_1\sigma_3\right) & \frac{2\pi r}{\beta} \end{pmatrix}\right.$$

Calculate this determinant, to get

$$\begin{aligned} Z_f &= \frac{2^{16}\pi^{48}}{g^{24}}\prod_{r>0}\Big(16r^6 + \frac{8r^4\beta^2}{\pi^2}X^{\mu}_{a,0}X^{\mu}_{a,0} + \frac{\beta^4 r^2}{\pi^4}\left((X^{\mu}_{a,0}X^{\mu}_{a,0})^2 - 4(X^0_{a,0}X^0_{b,0}X^1_{a,0}X^1_{b,0})\right. \\ &\left. -4(X^0_{a,0}X^0_{b,0}X^2_{a,0}X^2_{b,0})\right) + \frac{\beta^6}{\pi^6}\left(\epsilon_{abc}\epsilon^{\mu\nu\gamma}X^{\mu}_{a,0}X^{\nu}_{b,0}X^{\gamma}_{c,0}\right)^2\Big)^8 \end{aligned} \tag{2.14}$$

We can see that the above expression has $SO(3)$ symmetry in spinor indices, a, b, c and $SO(2,1)$ symmetry in the vector indices, μ, ν, γ. The $16r^6$ term gives the free fermionic contribution. Note that it is temperature independent as the Hamiltonian is identically zero for free fermions in $0+1$ dimensions.

2.2.2 Free Bosonic sector

After doing the fermionic integral, the partition function is

$$\begin{aligned} Z &= \prod_{a=1}^{3}\int d^{10}X^{\mu}_{a,n}\left[\frac{2^{16}\pi^{24}}{g^{24}}\prod_{r}\left(4r^3+\frac{\beta^3}{\pi^3}\epsilon_{abc}\epsilon^{\mu\nu\gamma}X^{\mu}_{a,n}X^{\nu}_{b,n}X^{\gamma}_{c,n}+\frac{r\beta^2}{\pi^2}X^{\mu}_{a,n}X^{\mu}_{a,n}\right)^{16}\right] \\ &\exp\left[\frac{-\beta}{4g}\left\{-\sum_{n=-\infty}^{\infty}\frac{4\pi^2n^2}{\beta^2}X^{i}_{a,n}X^{i}_{a,-n}-\frac{n\pi\epsilon^{abc}}{2\beta}\sum_{n+l+p=0}X^{0}_{a,l}X^{i}_{b,m}X^{i}_{c,n}\right.\right. \\ &+\left.\left.\frac{1}{2}\sum_{\substack{n,m,l,p=-\infty,\\ n+m+l+p=0}}^{\infty}X^{\mu}_{a,n}X^{\nu}_{b,m}X^{\mu}_{a,l}X^{\nu}_{b,p}-\frac{1}{2}\sum_{\substack{n,m,l,p=-\infty,\\ n+m+l+p=0}}^{\infty}X^{\mu}_{a,n}X^{\nu}_{b,m}X^{\mu}_{b,l}X^{\nu}_{a,p}\right\}\right] \end{aligned} \qquad (2.15)$$

In infinite temperature limit *i.e.* $\beta \to 0$, the first term dominates. To see the comparative β dependence of the other terms;
1) we set $(X^{\mu}_{a,n})_{n\neq 0} \to \sqrt{\beta}X^{\mu}_{a,n}$
2) keep the terms contributing to the leading order of the partition function in $\beta \to 0$ limit (this is justified in Appendix A2).
3) transform back $\sqrt{\beta}X^{\mu}_{a,n} \to (X^{\mu}_{a,n})_{n\neq 0}$
and we can write the partition function up to a numerical factor

$$\begin{aligned} Z_{boson} &= \frac{1}{g^{24}}\int d^9X^{i}_{a,n}d^9X^{i}_{a,-n}\exp\left[\frac{\beta}{4g}\left\{-\sum_{n=-\infty}^{\infty}\frac{4\pi^2n^2}{\beta^2}(X^{i}_{a,n}X^{i}_{a,-n})\right\}\right] \\ &\int dX_0\prod_{s}d\theta_s\exp\left[\frac{\beta}{4g}\left\{\frac{1}{2}(X_{a,0}.X_{a,0})(X_{b,0}.X_{b,0})-\frac{1}{2}(X_{a,0}.X_{b,0})(X_{b,0}.X_{a,0})\right\}\right] \end{aligned} \qquad (2.16)$$

Thus $Z_{boson} = \int e^{-iS} = Z_{free}Z_0$ where the first part of the partition function Z_{free} is just the free bosonic particle partition function (per unit volume), which is

$$Z_{free} = \prod_{n}\prod_{a=1}^{3}\prod_{i=1}^{9}\int dX^{i}_{a,n}dX^{i}_{a,-n}\exp\left[\frac{\beta}{4g}\left\{-\sum_{\substack{n=-\infty;\\ n\neq 0}}^{\infty}\frac{4\pi^2n^2}{\beta^2}(X^{i}_{a,n}X^{i}_{a,-n})\right\}\right] = \prod_{n\neq 0}\left(\frac{\sqrt{\beta g\pi}}{\pi n}\right)^{54}$$

$$\begin{aligned} \log(Z_{free}) &= 54\sum_{n}\log\left(\sqrt{\frac{g\beta}{2\pi}}\frac{1}{n}\right) \\ &= 27\sum_{n}\log\left(\frac{g\beta}{2\pi}\right)-54\sum_{n}\log n \end{aligned}$$

We know Reinmann Zeta function is defined as

$$\zeta(s) = \sum_{n} n^{-s}$$

Hence,

$$\sum_n an^{-s} = a\sum_n n^{-s} = a\zeta(s)$$

putting $s = 0$, as $\zeta(0) = 1$, we can write

$$\sum_n a = a\zeta(0) = a.1 = a$$

Thus,

$$27\sum_n \log\left(\frac{g\beta}{\pi}\right) = 27\log\left(\frac{g\beta}{\pi}\right) \tag{2.17}$$

$$\frac{d}{ds}\left(n^{-s}\right) = -n^{-s}\log n$$

$$\frac{d}{ds}\zeta(s) = \sum_n \frac{d}{ds}\left(n^{-s}\right) = -\sum_n n^{-s}\log n$$

So,

$$\left[\frac{d}{ds}\zeta(s)\right]_{s=0} = -\sum_n \log n$$

and therefore,

$$-54\sum_n \log n = 54\zeta'(0) = 54(-\frac{1}{2}\log 2\pi) = -27\log 2\pi$$

$$(\zeta(0) = 1, \zeta'(0) = -\frac{1}{2}\log(2\pi))$$

$$\begin{aligned}
\log(Z_{free}) &= 27\sum_n \log\left(\frac{g\beta}{2\pi}\right) - 54\sum_n \log n \\
&= 27\log\left(\frac{g\beta}{\pi}\right) - 27\log 2\pi \\
&= 27\log\left(\frac{\beta g}{2\pi^2}\right) \\
&= \log\left(\frac{\beta g}{2\pi^2}\right)^{27}
\end{aligned}$$

$$Z_{free} = \left(\frac{2\pi^2}{\beta g}\right)^{-27} \tag{2.18}$$

The free fermionic contribution contribution was discussed in the previous section. Z_0 is calculated below.

2.2.3 Leading and Non-leading Interaction Terms.

Now we try to study the effect of the interactions.
As argued earlier the leading β dependence is given by the zero modes, so in the first approximation we drop the terms in the action involving the higher modes. Thus we get,

$$\begin{aligned} S_0 &= \frac{i\beta}{8g}\left\{X^\mu_{a,0}X^\nu_{b,0}X^\mu_{a,0}X^\nu_{b,0} - X^\mu_{a,0}X^\nu_{b,0}X^\mu_{b,0}X^\nu_{a,0}\right\} \\ &= \frac{i\beta}{8g}\left\{(X_{1,0})^2(X_{2,0})^2 + (X_{2,0})^2(X_{3,0})^2 + (X_{3,0})^2(X_{1,0})^2 \right. \\ &\qquad \left. - (X_{1,0}.X_{2,0})^2 - (X_{2,0}.X_{3,0})^2 - (X_{3,0}.X_{1,0})^2\right\} \end{aligned} \tag{2.19}$$

We would like to first calculate the leading order contribution to Z that is

$$Z_0 = \int dX^\mu_{a,0} e^{-iS_0} \tag{2.20}$$

The original action with all the modes is not Lorentz invariant as the terms $X^0_{a,l}X^i_{b,m}.X^i_{c,n}$ and $X^i_{a,n}X^i_{a,-n}$ in the original action are not Lorentz invariant. However, these terms do not contribute to the partition function in leading order . The action (2.19) with only the zero modes has Lorentz invariance. Hence, as long as we are interested in leading order contribution only, we can work with (2.19) and assume Lorentz invariance.
Now, using Lorentz invariance, consider the parametrisation

$$X_{1,0} = (x_1, \vec{r}_1), \qquad X_{2,0} = (x_2, \vec{r}_2), \qquad X_{3,0} = (l, 0) \tag{2.21}$$

Under this parametrisation, the action (2.19) takes the form

$$S_0 = \frac{i\beta}{8g}\left\{r_1^2 r_2^2 \sin^2\alpha + x_1^2 r_2^2 + r_1^2 x_2^2 + r_2^2 l^2 + l^2 r_1^2 - 2x_1.x_2 r_1 r_2 \cos\alpha\right\} \tag{2.22}$$

The partition function is

$$Z_0 = \int d^{10}X^\mu_{1,0} d^{10}X^\mu_{2,0} d^{10}X^\mu_{3,0} e^{-iS_0} \tag{2.23}$$

At this stage, we can find the temperature dependence of the partition function and the mean square separation of two D-0 branes from a simple scaling argument. This scaling argument in fact applies to $SU(N)$ also. To see this we note that the leading order *i.e.* the zero mode bosonic contribution of the partition function comes from the $[X^\mu, X^\nu]^2$ term in the Lagrangian (2.1), (which for the $SU(2)$ case is given in (2.19)). And this term is Lorentz invariant and hence just as in $SU(2)$ case can use Lorentz invariance while calculating the leading order contribution to the partition function from this term. Therefore in $SU(N)$ case also we can use a parametrisation similar to 2.21

$$X_{i,0} = (x_i, \vec{r}_i), \qquad 1 \leq i \leq N^2 - 2; \qquad\qquad X_{N^2-1,0} = (l, 0) \tag{2.24}$$

Under the above parametrisation $\frac{1}{2l_s^4}[X^\mu, X^\nu]^2$ will be homogeneous in l, r_i, x_i, $0 < i \leq N^2 - 2$ and of order 4. So, in general for any N , we need to scale these variables by $\beta^{-\frac{1}{4}} g^{\frac{1}{4}}$ to scale out the β from the exponent. And the temperature dependence of $\langle l^2 \rangle$ will be $\beta^{-\frac{1}{2}} g^{\frac{1}{2}}$ in the leading order. Under the above scaling the measure in Z_0 will pick up a $\beta^{-\frac{15}{2}} g^{\frac{15}{2}}$ factor for $SU(2)$, which comes from (3×10) $X^\mu_{a,n}$. In general for $SU(N)$ in D dimension there will be $D(N^2 - 1)$ $X^\mu_{a,n}$ in the measure. So, the partition function Z_0 has temperature dependence $\beta^{-\frac{D(N^2-1)}{4}} g^{\frac{D(N^2-1)}{4}}$. Z_{free} will be proportional to $(\beta g)^{(D-1)(N^2-1)}$.
Now we evaluate the partition function for this action.

$$\begin{aligned}
Z_0 &= \int dl d\Omega^{(9)} l^9 \int dx_1 dx_2 \int dr_1 dr_2 d\Omega_1^{(8)} d\Omega_2^{(7)} d\alpha r_1^8 r_2^8 \sin^7 \alpha e^{-iS} \\
&= \int dl d\Omega^{(9)} l^9 \int dr_1 dr_2 d\Omega_1^{(8)} d\Omega_2^{(7)} d\alpha r_1^8 r_2^8 \sin^7 \alpha \exp\left[-\frac{\beta}{8g} \left\{ r_1^2 r_2^2 \sin^2 \alpha + r_2^2 l^2 + l^2 r_1^2 + \right\} \right] \\
&\qquad \int dx_1 dx_2 \exp\left[-\frac{\beta}{8g} \left\{ x_1^2 r_2^2 + r_1^2 x_2^2 - 2x_1.x_2 r_1 r_2 \cos\alpha \right\} \right]
\end{aligned} \tag{2.25}$$

We know

$$d\Omega^{(n)} = \sin\alpha_1 d\alpha_1 \sin\alpha_2 d\alpha_2 \sin\alpha_3 d\alpha_3 .... \sin\alpha_{(n-3)} d\alpha_{(n-3)} \sin\alpha_{(n-2)} d\alpha_{(n-2)} d\alpha_{(n-1)}$$

So,

$$\int d\Omega^{(n)} = \int_0^\pi \sin\alpha_1 d\alpha_1 \int_0^\pi \sin\alpha_2 d\alpha_2 \int_0^\pi \sin\alpha_3 d\alpha_3 .... \int_0^\pi \sin\alpha_{(n-2)} d\alpha_{(n-2)} \int_0^{2\pi} d\alpha_{(n-1)}$$

Now, using $\int_0^\pi \sin\alpha_1 d\alpha_1 = -\cos\pi + \cos 0 = 2$, we can write
So,

$$\int d\Omega^{(n)} = 2^{(n-2)} . 2\pi = 2^{(n-1)} \pi \tag{2.26}$$

with this, the bosonic part reduces

$$\begin{aligned} Z_b &= 2^{21}\pi^3 \int_{-\infty}^{\infty} dll^9 dr_1 dr_2 d\alpha r_1^8 r_2^8 \sin^7\alpha \\ &\quad \exp\left[-\frac{\beta}{8g}\left\{r_1^2 r_2^2 \sin^2\alpha + r_2^2 l^2 + l^2 r_1^2 +\right\}\right] \\ &\quad \int_{-\infty}^{\infty} dx_1 dx_2 \exp\left[-\frac{\beta}{8g}\left\{x_1^2 r_2^2 + r_1^2 x_2^2 - 2x_1.x_2 r_1 r_2 \cos\alpha\right\}\right] \end{aligned} \tag{2.27}$$

Now, we integrate over x_1, x_2

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_1 dx_2 \exp\left[-\frac{\beta}{8g}\left\{x_1^2 r_2^2 + r_1^2 x_2^2 - 2x_1.x_2 r_1 r_2 \cos\alpha\right\}\right] \\ &= \int_{-\infty}^{\infty} dx_1 \exp\left[-\frac{\beta}{8g}\left\{x_1^2 r_2^2\right\}\right] \int_{-\infty}^{\infty} dx_2 \exp\left[-\frac{\beta}{8g}\left\{r_1^2 x_2^2 - 2x_1.x_2 r_1 r_2 \cos\alpha\right\}\right] \\ &= \int_{-\infty}^{\infty} dx_1 \exp\left[-\frac{\beta}{8g}\left\{x_1^2 r_2^2 (1-\cos^2\alpha)\right\}\right] \\ &\qquad \int_{-\infty}^{\infty} \frac{1}{r_1} d(r_1 x_2 - x_1 r_2 \cos\alpha) \exp\left[-\frac{\beta}{8g}\left\{(r_1 x_2 - x_1 r_2 \cos\alpha)^2\right\}\right] \\ &= \int_{-\infty}^{\infty} dx_1 \exp\left[-\frac{\beta}{8g}\left\{x_1^2 r_2^2 \sin^2\alpha\right\}\right] \frac{1}{r_1}\left(\frac{8g\pi}{\beta}\right)^{\frac{1}{2}} \\ &= \left(\frac{8g\pi}{\beta r_2^2 \sin^2\alpha}\right)^{\frac{1}{2}} \frac{1}{r_1}\left(\frac{8g\pi}{\beta}\right)^{\frac{1}{2}} \\ &= \frac{8g\pi}{\beta r_1 r_2 \sin\alpha} \end{aligned} \tag{2.28}$$

With this integral,

$$\begin{aligned} Z_b &= \frac{2^{24} g\pi^4}{\beta} \int_{-\infty}^{\infty} dll^9 \int_0^{\infty} dr_1 dr_2 d\alpha r_1^7 r_2^7 \sin^6\alpha \\ &\quad \exp\left[-\frac{\beta}{8g}\left\{r_1^2 r_2^2 \sin^2\alpha + r_2^2 l^2 + l^2 r_1^2 +\right\}\right] \\ &= \frac{2^{37} g^5 \pi^4 \Gamma(4)}{\beta^5} \int_{-\infty}^{\infty} dll^9 \int_0^{\infty} dr_1 r_1^7 \exp\left[-\frac{\beta}{8g}\left\{l^2 r_1^2\right\}\right] \\ &\quad \int_0^{\pi} d\alpha \sin^6\alpha \left\{(r_1^2 + l^2)\sin^2\alpha + l^2\cos^2\alpha\right\}^{-4} \end{aligned} \tag{2.29}$$

From Gradshteyn & Ryzhik (Fifth Edn., Page- 422, Problem no, 3.642.3), we get

$$\int_0^{\pi} \frac{\sin^{2n} x dx}{(a^2\cos^2 x + b^2 \sin^2 x)^{n+1}} = \frac{(2n-1)!!\pi}{2^n n! a b^{(2n+1)}} \tag{2.30}$$

here, we get

$$\begin{aligned}
&\int_0^\pi \frac{\sin^6 x dx}{\{l^2\cos^2 x + (r_1^2 + l^2)\sin^2 x\}^4} \\
&= \frac{5!!\pi}{2^3 3! l (r_1^2 + l^2)^{7/2}} \\
&= \frac{15\pi}{8.6 l (r_1^2 + l^2)^{7/2}} \\
&= \frac{15\pi}{48 l (r_1^2 + l^2)^{7/2}}
\end{aligned} \tag{2.31}$$

With this we can write the partition function as

$$\begin{aligned}
Z_b &= \frac{2^{37} g^5 \pi^4 \Gamma(4)}{\beta^5} \int_{-\infty}^{\infty} dl l^9 \int_0^\infty dr_1 r_1^7 \exp\left[-\frac{\beta}{8g}\left\{l^2 r_1^2\right\}\right] \frac{15\pi}{48 l (r_1^2 + l^2)^{7/2}} \\
&= \frac{2^{34} 15\, g^5 \pi^5}{\beta^5} \int_{-\infty}^{\infty} dl l^8 \int_0^\infty \frac{dr_1 r_1^7}{(r_1^2 + l^2)^{7/2}} \exp\left[-\left(\frac{\beta l^2}{8g}\right) r_1^2\right]
\end{aligned} \tag{2.32}$$

If we scale r_1 and l by a factor $\beta^{-\frac{1}{4}} g^{\frac{1}{4}}$, the integral reduces to

$$Z_0 = \frac{2^{34} 15\, g^{\frac{15}{2}} \pi^5}{\beta^{\frac{15}{2}}} \int_{-\infty}^{\infty} dl l^8 \int_0^\infty \frac{dr_1 r_1^7}{(r_1^2 + l^2)^{7/2}} \exp\left[-\left(\frac{l^2}{8}\right) r_1^2\right] \tag{2.33}$$

The integral over r_1 can be done to give

$$\begin{aligned}
Z_0 = \frac{2^{34} 15\, g^{\frac{15}{2}} \pi^5}{\beta^{\frac{15}{2}}} \int_{-\infty}^{\infty} dl l^8 \exp\left(l^4/8\right) &\left\{a^{-1}\Gamma\left(\frac{1}{2}, (l^4/8)\right) - 3l^2 a \Gamma\left(\frac{-1}{2}, (l^4/8)\right)\right. \\
&\left. + 3l^4 a^3 \Gamma\left(\frac{-3}{2}, (l^4/8)\right) - l^6 a^5 \Gamma\left(\frac{-5}{2}, (l^4/8)\right)\right\}
\end{aligned} \tag{2.34}$$

where $\Gamma(\alpha, x)$ is the incomplete Gamma function and $a = \frac{l}{\sqrt{8}}$.
In large l regime using the asymptotic expression for the incomplete Gamma function [9] [2]

[2] For large x from Gradshteyn & Ryzhik (Fifth Edn., Page- 949, Problem no, 8.355.1),

$$\Gamma(\alpha, x) = x^{(\alpha-1)} e^{-x} \left[\sum_{m=0}^{M-1} \frac{(-1)^m \Gamma(1-\alpha+m)}{x^m \Gamma(1-\alpha)} + O\left(|x|^{-M}\right)\right] \tag{2.35}$$

where,

$$\left[|x| \to \infty, -\frac{3\pi}{2} < arg x < \frac{3\pi}{2}, M = 1, 2, ...\right]$$

we can write this expression as

$$Z_0 = \frac{2^{34}15\ g^{\frac{15}{2}}\pi^5}{\beta^{\frac{15}{2}}} \int_{-\infty}^{\infty} dl l^8 I(l) \tag{2.36}$$

where

$$\begin{aligned} I(l) &= e^{-\left(\frac{l^4}{8}\right)} \left[\sum_{m=0}^{M-1} (-1)^m 8^{\left(\frac{m}{2}+1\right)} l^{(-2m-3)} \right. \\ & \left. \left(\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{1}{2})} - 3\frac{\Gamma(\frac{3}{2}+m)}{\Gamma(\frac{3}{2})} + 3\frac{\Gamma(\frac{5}{2}+m)}{\Gamma(\frac{5}{2})} - \frac{\Gamma(\frac{7}{2}+m)}{\Gamma(\frac{7}{2})} \right) + O\left(|(l^4/8)|^{-M} \right) \right] \end{aligned}$$

In large l approximation $I(l)$ boils down to

$$I(l) = \left(24576 l^{-15} - 2752512 l^{-19} + ... \right) \tag{2.37}$$

and in the small l regime the $I(l)$ function becomes [9] [3]

$$\begin{aligned} I(l) &= \sqrt{8} l^{-1} \exp\left(\frac{l^4}{8}\right) \left[\sqrt{\pi} \left\{ 1 - 24 l^4 + 256 l^8 + \frac{5120}{3} l^{12} \right\} \right. \\ & \left. + \left\{ 12 \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{8} l^{(2+2n)}}{n!} \left(\frac{8}{(4n^2 - 8n - 5)(4n^2 - 8n + 3)} \right) \right\} \right] \end{aligned} \tag{2.39}$$

and $l \to 0$ limit gives

$$I(l) = \left[\sqrt{8\pi} l^{-1} - \frac{256}{5} l + \left(\sqrt{\frac{\pi}{8}} - \frac{256}{3} \right) l^3 + ... \right] \tag{2.40}$$

The integrand $l^8 \int_0^{\infty} \frac{dr_1 r_1^7}{(r_1^2+l^2)^{7/2}} \exp\left[-\left(\frac{l^2}{8}\right) r_1^2 \right]$ has l^{-7} dependence for large l and l^7 dependence for small l. Hence, it converges for both large and small l and the integral is non-singular and independent of β. So, Z_0 has a temperature dependence of $T^{\frac{15}{2}}$.

We have already seen that the leading order fermionic contribution comes from the free fermionic terms. Here we will try to estimate the β dependence of the non-leading fermionic contributions.

[3] For small x

$$\Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{(\alpha+n)}}{n!(\alpha+n)} \tag{2.38}$$

In terms of the parametrisation in eqns. (2.21),

$$Z_f = \frac{2^{16}\pi^{24}}{g^{24}} \prod_r \left(4r^3 + \frac{2\beta^3}{\pi^3} r_1 r_2 l \sin\alpha + \frac{r\beta^2}{\pi^2}\left({r_1}^2 + {r_2}^2 + (x_1)^2 + (x_2)^2 + l^2\right)\right)^{16} \tag{2.41}$$

which, in $\beta \to 0$ can be written as

$$\begin{aligned} Z_f &= g^{-24} \prod_r \left(r^3 O(\beta^0 g^0) + O(\beta^{\frac{3}{2}} g^{\frac{1}{2}}) + rO(\beta^{\frac{9}{4}} g^{\frac{3}{4}}) + ...\right) \\ &= g^{-24}\left(O(1) + O(\beta^{\frac{3}{2}} g^{\frac{1}{2}}) + O(\beta^{\frac{9}{4}} g^{\frac{1}{4}}) + rO(\beta^{\frac{9}{4}} g^{\frac{3}{4}}) + ...\right) \end{aligned} \tag{2.42}$$

where the first term is the leading order fermionic partition function we have discussed in subsection (3.2).

So we can see that in the partition function at high temperature the contribution of the zero modes (bosonic) is dominant. We have earlier argued that the higher modes of the bosonic fields will also contribute in non-leading terms. The β dependence of these nonleading term is worked out is the Appendix (A2.8) (for each component of X_0^μ) and gives

$$Z_{boson} = g^{\frac{15}{2}} \beta^{-\frac{15}{2}} \left(O(1) + O(\beta^{\frac{3}{4}}) + O(\beta^{\frac{3}{2}}) + ...\right) \tag{2.43}$$

Combining the free, fermionic and the bosonic parts, we can write the β dependence as

$$Z = \beta^{\frac{39}{2}} \left(O(1) + O(\frac{3}{4}) + O(\beta^{\frac{9}{8}}) + O(\beta^{\frac{3}{2}}) + O(\beta^{\frac{9}{4}}) + rO(\beta^{\frac{27}{8}}) + ...\right) \tag{2.44}$$

2.2.4 Effective Potential & Mean-square Separation of the D-0 branes.

For high temperature we have evaluated the partition function both for large and small l (eqn.(2.37),(2.40)). Up to leading order in β the effective potential between two D-0 branes is proportional to $-\log l$ and $\log l$ for small and large l respectively. We can see that the potential increases at both l ends, though we can not clearly see the nature of the potential in the intermediate region but we can conclude that the potential is a confining potential and binds the D-0 branes.

As we are working in Euclidean metric and since for zero mode calculation we have Lorentz symmetry, we can identify l as one of the spatial components and hence as the separation between two D-0 branes.

Now, we try to see the temperature dependence of the expectation value of l^{2n}

$$\langle l^{2n} \rangle = \frac{\int e^{-iS} l^{2n}}{Z} \tag{2.45}$$

i.e.

$$\langle l^{2n} \rangle = (\beta^{-\frac{1}{4}} g^{\frac{1}{4}})^{2n} \frac{\int_{-\infty}^{\infty} dl l^8 l^{(2n)} \int_0^{\infty} \frac{dr_1 r_1^7}{(r_1^2+l^2)^{7/2}} \exp\left[-\left(\frac{l^2}{8}\right) r_1^2\right]}{\int_{-\infty}^{\infty} dl l^8 \int_0^{\infty} \frac{dr_1 r_1^7}{(r_1^2+l^2)^{7/2}} \exp\left[-\left(\frac{l^2}{8}\right) r_1^2\right]} \tag{2.46}$$

As l is the separation of two D-0 branes, we get the mean square separation of two D-0 branes from this, by putting $n = 1$ and doing the integral numerically, and restoring l_s, we get

$$\left\langle \left(\frac{l}{l_s}\right)^2 \right\rangle = 6.385 \left(\frac{\beta}{g l_s}\right)^{-\frac{1}{2}} \tag{2.47}$$

If we assume high temperature expression has a finite radius of convergence, we can conclude that the mean square separation is finite for finite temperature. This implies that there is a confining potential that binds the D-0 branes. As argued earlier the scaling argument that gives the β and g dependence in 2.47 is valid for *all* N . So we can conclude that $\langle l^2 \rangle \approx \sqrt{\frac{g}{\beta}}$ for all N.

2.3 Conclusion

In this chapter, we have attempted to solve the $N = 2$ matrix model in the high temperature limit . The leading nontrivial term of the partition function has been calculated exactly (eqn. 2.34). The non-leading terms can also be systematically calculated although we haven't attempted to work them out here. From a scaling argument we have also determined the β and g dependence of the leading term for any N. This complements the work of [2], where the one loop partition function was calculated with the entire β dependence. We find that $\langle l^2 \rangle \propto \sqrt{\frac{g}{\beta}}$ (eqn. 2.47) (true for any N), the finiteness of which shows that there must be a potential between D-0 branes that binds them. In [1, 2] also a logarithmic and attractive potential were found. The present calculation being exact in g is valid at all distances. Thus unlike in [1, 2], the (finite temperature) logarithmic potential found here is attractive at long distances and repulsive at short distances so it has a minimum at non-zero In [1] it was found that at high temperatures, the configuration with all the D-0-branes clustered at the origin *i.e.* with the zero separation, had lower free energy than the one where they were spread out. However, that was a large N calculation and also restricted to one loop. It is therefore possible that more exact calculation will resolve this issue.

As mentioned in the introduction, describing completely the dynamics of two D-0 branes in M-theory would require the infinite N model. Whether some high temperature expansion of that model within the $\frac{1}{N}$ approximation scheme can be attempted is an open question.

A2 Appendix : Comparative β dependence of the terms in action

We are interested in investigating the comparative β dependence of the terms in the bosonic part of the action in eqn. (??). For convenience, in this part we will suppress the isospin and vector indices. The action takes the form

$$S = \frac{\beta}{g}\left\{ -\sum_{n=-\infty}^{\infty} \frac{a_n}{\beta^2} X_n X_{-n} + \sum_{l+m+n=0} \frac{f_n}{\beta} X_l X_m X_n - c \sum_{\substack{n,m,l,p=-\infty,\\ n+m+l+p=0}}^{\infty} X_n X_m X_l X_p \right\} \tag{A2.1}$$

where a_n, c and f_n are constants and are given by $a_n = \pi^2 n^2$, $c = \frac{1}{8}$, and $f_n = \frac{n\pi}{4}$ Now, when we expand the sum over n, m, l and p in last two terms, we will get terms with all of these indices being 0, with two of the indices being 0 and with one of them being 0, so the action can be written in the form (taking one of each type, as the terms of the same type have same β dependence).

$$\begin{aligned} S &= -\frac{1}{\beta g} \sum_{\substack{n=-\infty;\\ n\neq 0}}^{\infty} a_n X_n X_{-n} - \frac{c\beta}{g} X_0^4 - \frac{c\beta}{g} \sum_{n\neq 0} X_0^2 X_n X_{-n} + \frac{1}{g} \sum_{n\neq 0} f_n X_0 X_{-n} X_n \\ &+ \frac{1}{g} \sum_{n+l+m=0} f_n X_l X_m X_n - \frac{c\beta}{g} \sum_{\substack{m,l,p\neq 0,\\ m+l+p=0}} X_0 X_m X_l X_p - \frac{c\beta}{g} \sum_{\substack{n,m,l,p\neq 0,\\ n+m+l+p=0}} X_n X_m X_l X_p \end{aligned} \tag{A2.2}$$

We can crudely write this as

$$S = -\frac{1}{\beta g} a_n {X_n}^2 - \frac{c\beta}{g} X_0^4 - \frac{c\beta}{g} X_0^2 {X_n}^2 + \frac{1}{g} f_n X_0 X_n^2 + \frac{1}{g} f_n X_n^3 - \frac{c\beta}{g} X_0 X_n^3 - \frac{c\beta}{g} X_n^4 \tag{A2.3}$$

where ${X_n}^2$, ${X_n}^3$, ${X_n}^4$ denote general quadratic, cubic and the quatic terms in (A2.2). Now, we set

$$(X_n)_{n\neq 0} \to \sqrt{g\beta}X_n; \qquad X_0 \to \left(\frac{g}{\beta}\right)^{\frac{1}{4}} X_0$$

Under this transformation, the action reduces to

$$S = -cX_0^4 - a_n {X_n}^2 - c\beta^{\frac{3}{2}}\sqrt{g}{X_n}^2X_0^2 + \beta^{\frac{3}{4}}g^{\frac{1}{4}}f_nX_0{X_n}^2 + \beta^{\frac{3}{2}}g^{\frac{1}{2}}f_n{X_n}^3 - c\beta^{\frac{9}{4}}g^{\frac{3}{4}}{X_n}^3X_0 - c\beta^{3}$$

So, the partition function for this action will be

$$Z = g^{\frac{1}{4}}\beta^{-\frac{1}{4}} \int_0^\infty dX_0 e^{-cX_0^4} \int_0^\infty dX_n f(X_n, b) e^{-a_n{X_n}^2} \tag{A2.5}$$

where, $f(X_n, b)$ includes the β dependent terms with X_n. The $g^{\frac{1}{4}}\beta^{-\frac{1}{4}}$ comes from the measure due to scaling of X_0. The scaling of X_n also give a overall β dependence out. We have taken that into consideration while calculating Z_{free}, hence we are not considering it here.

$$f(X_n, b) = exp(Ab^3 - Bb^6 + Cb^6 - Db^9 - Eb^{12})$$

and $b = \beta^{\frac{1}{4}}$, $A = f_n g^{\frac{1}{4}} X_0 {X_n}^2$, $B = \sqrt{g}{X_n}^2 X_0^2$, $C = f_n\sqrt{g}{X_n}^3$, $D = g^{\frac{3}{2}}cX_0{X_n}^3$, $E = cg{X_n}^4$

First, we try to evaluate the X_n integral. For convenience, we start with

$$\int_0^t dX_n f(X_n, b) e^{-a_n{X_n}^2} \tag{A2.6}$$

and evaluate it, take the limit $t \to \infty$. The Eb^4 term makes the integral convergent (Note that for any X_n, we have one term $(X_nX_{-n})^2$, which is positive definite.), so as $b \to 0$, we can use Taylor expansion and write $f(X_n, \beta)e^{-a_n{X_n}^2}$ as

$$f(X_n, \beta)e^{-a_n{X_n}^2} = e^{-a_n{X_n}^2}\left(1 + A\beta^{\frac{3}{4}} + (\frac{1}{2}A^2 - B + C)\beta^{\frac{3}{2}} + ...\right) \tag{A2.7}$$

Now, putting the above series in (A2.6), as the series is uniformly convergent we can integrate term by term, take the limit $\beta \to 0$ and sum. As the each integral has an $e^{-a_n{X_n}^2}$ term , all the integrals in the series will converge individually. Some of the terms in the series will cancel from symmetry while doing X_n and X_0 integrals. And the β dependence in the final bosonic partition function can be written as

$$Z = g^{\frac{1}{4}}\beta^{-\frac{1}{4}}\left(O(1) + O(\beta^{\frac{3}{4}}) + O(\beta^{\frac{3}{2}}) + ...\right) \tag{A2.8}$$

As we have suppressed the vector indices here in the appendix, this result is for one component of the X_0^μ. For original Z_{boson} there will be a a product over the vector indices and will reduce to eqn (2.43) . Now, we can see that in the limit $\beta \to 0$, the above function can be approximated by the leading term which is equivalent to writing the partition function in eqn. (A2.5) as.

$$Z = \int dX_0 \exp\left\{-\frac{c\beta}{g} X_0^4\right\} \prod_{n\neq 0} \int dX_n \exp(-a_n X_n^2)$$

and hence, the action becomes

$$S = \frac{a}{\beta} \sum_{n=-\infty; n\neq 0}^{\infty} X_n^2 - c\beta X_0^4 \qquad \text{(A2.9)}$$

which is also eqn.(2.19).

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Chapter 3

Matrix Model and Fuzzy Spheres

In this chapter of this thesis, we will consider a supersymmetric fuzzy sphere model in three dimensions as in [4]. We get this model by adding a Chern Simons term to the three dimensional reduced model [1]. We try to provide a framework which allows us to study a multi fuzzy sphere system. We shall study the two fuzzy spheres system in detail and investigate the interaction between them. This three dimensional fuzzy sphere model is just a simple toy model in the context of string theory, and has relevance in other theories, like the nonperturbative regularization of Quantum Field Theory using fuzzy space time and in studying quantum hall fluid of finite extent using finite matrix model.

We have presented the model for the multi fuzzy sphere in background space. We have calculated interaction of fuzzy spheres and space and considered the dynamics of the fuzzy spheres. We have expanded the action around a classical back ground and studied the one loop interaction between fuzzy spheres in bosonic and supersymmetric case. In particular, we have calculated the one loop effective action for two and three blocks. We have studied the interaction between two points and the interaction between a point and a fuzzy sphere in this section. We have also studied the interaction between two fuzzy spheres in next section.

3.1 Fuzzy Sphere Model

We start with $\mathcal{N} = 2$ SUSY Yang-Mills-Chern-Simons reduced model

$$S = \frac{1}{g^2} Tr\Big(-\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{2}{3} i\alpha \epsilon_{\mu\nu\lambda} A^\mu A^\nu A^\lambda + \frac{1}{2}\bar{\psi}\sigma^\mu[A_\mu, \psi]\Big). \qquad (3.1)$$

This is obtained by reducing the space-time volume of Yang-Mills-Chern-Simons theory to a single point [4] (c.f. Eguchi-Kawai and IKKT model [1]). The Chern-Simons term is added to the reduced model to have fuzzy sphere solutions as classical equations of motion. A_μ, ψ_α are $N \times N$ traceless Hermitian matrices. A_μ is a 3-dimensional vector and ψ is the two component Majorana spinor. $\sigma_\mu(\mu = 1, 2, 3)$ denote Pauli matrices. $\mu, \nu = 1 \sim 3$, $\alpha, \beta = 1, 2$. This action is also obtained as low energy effective action for spherical D2-brane in S^3, using $SU(2)$ WZW model as string theory in S^3 [6].

3.1.1 Symmetries

The action (3.1) has $SO(3)$ global symmetry, $A_\mu \to A_\mu + r_\mu \mathbf{1}$, translational symmetry and gauge symmetry $A_\mu \to U A_\mu U^\dagger$, $\psi \to U \psi U^\dagger$, where U is an unitary matrix. This action are also has $\mathcal{N} = 2$ supersymmetry

$$\begin{array}{l} \delta^{(1)} A_\mu = i\bar{\epsilon}\sigma_\mu \psi \\ \delta^{(1)}\psi = \frac{i}{2}([A_\mu, A_\nu] - i\alpha\epsilon_{\mu\nu\lambda}A_\lambda)\sigma^{\mu\nu}\epsilon \end{array} \tag{3.2}$$

and

$$\begin{array}{l} \delta A_\mu^{(2)} = 0 \\ \delta^{(2)}\psi = \xi. \end{array} \tag{3.3}$$

Let us check that these two transformations satisfies the $\mathcal{N} = 2$ supersymmetry algebra. The algebra is also modified due to the modification of supersymmetry. We have the following relations:

$$\begin{aligned} (\delta^{(1)}_{\epsilon^1}\delta^{(1)}_{\epsilon^2} - \delta^{(1)}_{\epsilon^2}\delta^{(1)}_{\epsilon^1})\psi &= i[\psi, \lambda] + i\theta_\alpha \frac{\sigma_\alpha}{2}\psi, \\ (\delta^{(1)}_{\epsilon^1}\delta^{(1)}_{\epsilon^2} - \delta^{(1)}_{\epsilon^2}\delta^{(1)}_{\epsilon^1})A_\mu &= i[A_\mu, \lambda] + \epsilon_{\mu\nu\lambda}\theta_\nu A_\lambda, \end{aligned} \tag{3.4}$$

where $\lambda = 2i(\bar{\epsilon}_2\sigma_\nu\epsilon_1)A_\nu$ and $\theta_\nu = 2i\alpha(\bar{\epsilon}_2\sigma_\nu\epsilon_1)$. The second term in the right hand side is a new term corresponding to $SO(3)$ rotation. Other commutation relations are calculated as follows,

$$\begin{aligned} (\delta^{(1)}_{\epsilon}\delta^{(2)}_{\xi} - \delta^{(2)}_{\xi}\delta^{(1)}_{\epsilon})\psi &= 0, \\ (\delta^{(1)}_{\epsilon}\delta^{(2)}_{\xi} - \delta^{(2)}_{\xi}\delta^{(1)}_{\epsilon})A_\mu &= -i\bar{\epsilon}\sigma_\mu\xi, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} (\delta^{(2)}_{\xi^1}\delta^{(2)}_{\xi^2} - \delta^{(2)}_{\xi^2}\delta^{(2)}_{\xi^1})\psi &= 0, \\ (\delta^{(2)}_{\xi^1}\delta^{(2)}_{\xi^2} - \delta^{(2)}_{\xi^2}\delta^{(2)}_{\xi^1})A_\mu &= 0. \end{aligned} \tag{3.6}$$

If we take a linear combination of $\delta^{(1)}$ and $\delta^{(2)}$ as

$$\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)},$$

$$\tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)}), \tag{3.7}$$

we can obtain the following commutation relations up to a gauge symmetry and $SO(3)$ symmetry,

$$\begin{aligned}
(\tilde{\delta}^{(1)}_\epsilon \tilde{\delta}^{(1)}_\xi - \tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\epsilon)\psi &= 0, \\
(\tilde{\delta}^{(1)}_\epsilon \tilde{\delta}^{(1)}_\xi - \tilde{\delta}^{(1)}_\xi \tilde{\delta}^{(1)}_\epsilon)A_\mu &= -2i\bar{\epsilon}\sigma_\mu\xi, \\
(\tilde{\delta}^{(2)}_\epsilon \tilde{\delta}^{(2)}_\xi - \tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\epsilon)\psi &= 0, \\
(\tilde{\delta}^{(2)}_\epsilon \tilde{\delta}^{(2)}_\xi - \tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(2)}_\epsilon)A_\mu &= -2i\bar{\epsilon}\sigma_\mu\xi, \\
(\tilde{\delta}^{(1)}_\epsilon \tilde{\delta}^{(2)}_\xi - \tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(1)}_\epsilon)\psi &= 0, \\
(\tilde{\delta}^{(1)}_\epsilon \tilde{\delta}^{(2)}_\xi - \tilde{\delta}^{(2)}_\xi \tilde{\delta}^{(1)}_\epsilon)A_\mu &= 0.
\end{aligned} \tag{3.8}$$

We find that these commutation relations indeed satisfy the $\mathcal{N} = 2$ supersymmetry algebra. A new feature is the appearance of $SO(3)$ rotational symmetry.

3.1.2 Solution of Equation of Motion

The equations of motion corresponding to the action are

$$\begin{aligned}
[A_\nu, [A_\nu, A_\mu] + i\alpha\epsilon_{\mu\nu\lambda}A_\lambda] &= \frac{1}{2}\{\psi_\beta, \psi_\alpha\}(\sigma_0\sigma_\mu)_{\alpha\beta} \\
[\psi_\alpha, A_\mu](\sigma_0\sigma_\mu)_{\alpha\beta} &= 0, \quad \sigma_0 = i\sigma_2.
\end{aligned} \tag{3.9}$$

When $\psi = 0$, we can rewrite the equation of motion as

$$[A_\nu, [A_\nu, A_\mu]] = -i\alpha\epsilon_{\mu\nu\lambda}A_\lambda \tag{3.10}$$

The simplest solution of this equation of motion can be realized by commuting diagonal matrices

$$A_\mu = \mathrm{diag}(r^{(1)}_\mu, r^{(2)}_\mu, r^{(3)}_\mu, \cdots, \cdots, r^{(N)}_\mu) \tag{3.11}$$

another, typical solution for A_μ is, $A_\mu = X_\mu$, where

$$[X_\mu, X_\nu] = i\alpha\epsilon_{\mu\nu\lambda}(X_\lambda - R_\lambda), \quad [X_\mu, R_\nu] = 0 \tag{3.12}$$

represents an algebra of the fuzzy sphere configuration with $N \times N$ matrices R_λ. The commuting solution $X_\mu = \mathrm{diag}(r^{(1)}_\mu, r^{(2)}_\mu, r^{(3)}_\mu, \cdots, \cdots, r^{(N)}_\mu)$ is a special case of (3.12) in the $\alpha \to 0$ limit, and represents commuting coordinates of the space points in 3 dimensional space.

$$\begin{aligned}
x_1 &= \rho\sin\theta\cos\phi \\
x_2 &= \rho\sin\theta\sin\phi \\
x_3 &= \rho\cos\phi
\end{aligned}$$

where, ρ is the radius of the sphere. The noncommutative coordinates of (3.12) can be constructed from the generators of the N dimensional irreducible representation of $SU(2)$.

$$(X_\mu - R_\mu) = \alpha L'_\mu \tag{3.13}$$

where

$$\left[L'_\mu, L'_\nu\right] = i\epsilon_{\mu\nu\lambda} L'_\lambda \tag{3.14}$$

As the quadratic Casimir of $SU(2)$ in N dimensional irreducible representation is given by $\frac{N^2-1}{4}$, α and ρ are related by the relation

$$\rho^2 = \alpha^2 \frac{N^2-1}{4} \tag{3.15}$$

The Planck's constant, which represents the area occupied by an unit quantum of the fuzzy sphere, is given by

$$\frac{4\pi\rho^2}{N} = \frac{N^2-1}{N}\pi\alpha^2 \tag{3.16}$$

The commuting limit is realized by

$$\rho = \text{fixed}, \quad N \to \infty (\alpha \to 0) \tag{3.17}$$

In this model, the fuzzy sphere preserves half of the $\mathcal{N} = 2$ supersymmetries since (3.2) vanishes for the fuzzy sphere while (3.3) does not, and this solution corresponds to a 1/2 BPS background. Looking at the algebra (3.4), the remaining supersymmetry on the fuzzy sphere generates $SO(3)$ rotation instead of a constant shift of A_i. It is natural since translation on a sphere is generated by $SO(3)$ rotation. On the other hand, commuting matrices break all the supersymmetry.

3.1.3 Multiple Fuzzy Sphere

Remarkably, these equations of motion have solutions which represent *arbitrary number of points or/and fuzzy spheres* with *various radii* and *centers*. To see this, choose solution X_μ as a block diagonal type,

$$X_\mu = \begin{pmatrix} X_\mu^{(1)} & & & & \\ & X_\mu^{(2)} & & 0 & \\ & & X_\mu^{(3)} & & \\ & 0 & & \ddots & \\ & & & & X_\mu^{(l)} \end{pmatrix} \tag{3.18}$$

where the mth block $X_\mu^{(m)}$ is a $n_m \times n_m$ irreducible representation of $SU(2)$ ($\sum_{m=1}^{l} n_m = N$) and obeys

$$[X_\mu^{(m)}, X_\nu^{(m)}] = i\alpha\epsilon^{\mu\nu\lambda}(X_\lambda^{(m)} - R_\lambda^{(m)}) \tag{3.19}$$

where $R_\lambda^{(m)}$ is proportional to identity matrix $1_{n_m \times n_m}$. Though $Tr(X_\mu) = 0$, $X_\mu^{(m)}$ need not to be traceless. These relation are valid even when we assume the relation,

$$\sum_{\lambda=1}^{3}(X_\lambda^{(m)} - R_\lambda^{(m)})^2 = \rho_m^2 1_{n_m \times n_m} \tag{3.20}$$

where $\rho_m^2 = \alpha^2 \frac{n_m^2 - 1}{4}$. Because of equation (3.19, 3.20), we can think of this configuration as multi fuzzy spheres. Now $r_\mu^{(m)} = \frac{1}{n_m} Tr(X_\mu^{(m)}) = \frac{1}{n_m} Tr(R_\mu^{(m)})$ gives the co-ordinates of mth block (centers of mth fuzzy sphere or point) where ρ_m is the radius. At this point, we assume the elements of the matrix X_μ and the trace $r_\mu^{(m)}$ can be assumed to take any value from $\mathbf{R}$. Under this assumption the space configuration is $\mathbf{R}^3$.

Example 1. One Fuzzy Sphere Model

To construct a one fuzzy sphere model using this formalism, we assume X_μ to be of the form

$$X_\mu = \begin{pmatrix} X_\mu^{(1)} & & & 0 \\ & r_\mu^{(2)} & & \\ 0 & & \ddots & \\ & & & r_\mu^{(N-n_1+1)} \end{pmatrix} \tag{3.21}$$

here, $n_{m>1} = 1$, $r_\mu^{(m>1)} \in \mathbf{R}$ and

$$[X_\mu^{(1)}, X_\nu^{(1)}] = i\alpha\epsilon^{\mu\nu\lambda}(X_\lambda^{(1)} - R_\lambda^{(1)}). \tag{3.22}$$

This configuration represents a fuzzy sphere with center at $r_\mu^{(1)} = \frac{1}{n_1} Tr(X_\mu^{(1)})$ and $(N - n_1)$ points at co-ordinates $r_\mu^{(m>1)}$ as back-ground.

Example 2. Two Fuzzy Sphere Model

We can construct a multi-fuzzy sphere picture out of this model. To construct a k fuzzy sphere case, we consider the same configuration as equation(3.18) with k irreducible blocks and $N - \sum_{m=1}^{k} n_m$ points. For example, for two fuzzy sphere case we consider

$$X_\mu = \begin{pmatrix} X_\mu^{(1)} & & & & 0 \\ & X_\mu^{(2)} & & & \\ & & r_\mu^{(3)} & & \\ 0 & & & \ddots & \\ & & & & r_\mu^{(N-n_1-n_2+2)} \end{pmatrix} \tag{3.23}$$

here, $n_{m>2} = 1$, $r_\mu^{(m>2)} \in \mathbf{R}$ and

$$[X_\mu^{(1)}, X_\nu^{(1)}] = i\alpha\epsilon^{\mu\nu\lambda}(X_\lambda^{(1)} - R_\lambda^{(1)}), \quad [X_\mu^{(2)}, X_\nu^{(2)}] = i\alpha\epsilon^{\mu\nu\lambda}(X_\lambda^{(2)} - R_\lambda^{(2)}) \tag{3.24}$$

This configuration represents two fuzzy spheres with centers at $r_\mu^{(1)} = \frac{1}{n_1}Tr(X_\mu^{(1)}) = \frac{1}{n_1}Tr(R_\mu^{(1)})$, $r_\mu^{(2)} = \frac{1}{n_2}Tr(X_\mu^{(2)}) = \frac{1}{n_2}Tr(R_\mu^{(2)})$ and $(N-n_1-n_2)$ points at co-ordinates $r_\mu^{(m>2)}$ as back-ground.

3.2 One Loop Calculations for Blocks

We assume one loop correction is a good approximation for the interaction between fuzzy spheres.

To see the effect of the fluctuation for this model, we expand the original matrices around these back ground

$$A_\mu = X_\mu + \tilde{A}_\mu, \qquad \psi_\alpha = \chi_\alpha + \tilde{\varphi}_\alpha, \quad \text{and choose } \chi_\alpha = 0 \tag{3.25}$$

where $N \times N$ matrices $\tilde{A}$ and $\tilde{\varphi}$ are quantum fluctuation.
1-loop correction of effective action W is calculated as

$$W = -ln \int d\tilde{A}\, d\tilde{\varphi}\, e^{-S_2} \tag{3.26}$$

where S_2 arises out of the quadratic terms of fluctuations in action (3.1) and can be written as

$$\begin{aligned} S_2 &= \frac{1}{g^2}Tr\left(-\frac{1}{2}[\tilde{A}_\mu, X_\nu]^2 - \frac{1}{2}[X_\mu, X_\nu][\tilde{A}_\mu, \tilde{A}_\nu] - \frac{1}{2}[X_\mu, \tilde{A}_\nu][\tilde{A}_\mu, X_\nu] \right. \\ &\qquad \left. + 2i\epsilon_{\mu\nu\lambda}X_\mu\tilde{A}_\nu\tilde{A}_\lambda + \frac{1}{2}\bar{\tilde{\varphi}}\sigma^\mu[X_\mu, \tilde{\varphi}]\right). \end{aligned}$$

As, our original action has gauge symmetry, we add the gauge fixing term and the ghost term

$$S_{gf} = -\frac{1}{2g^2}Tr[X_\mu, \tilde{A}_\mu]^2, \qquad S_{gh} = -\frac{1}{g^2}Tr[X_\mu, B][X_\mu, C]$$

The whole expression can be simplified and re-written as the sum of three pieces.

$$S_{2,B} = \frac{1}{g^2}Tr\left(-\frac{1}{2}[\tilde{A}_\mu, X_\nu]^2 - [X_\mu, X_\nu][\tilde{A}_\mu, \tilde{A}_\nu] + 2i\epsilon_{\mu\nu\lambda}X_\mu\tilde{A}_\nu\tilde{A}_\lambda\right) \tag{3.27a}$$

$$S_{2,F} = \frac{1}{2g^2}Tr\left(\bar{\tilde{\varphi}}\sigma^\mu[X_\mu, \tilde{\varphi}]\right) \tag{3.27b}$$

$$S_{2,G} = -\frac{1}{g^2}Tr[X_\mu, B][X_\mu, C] \tag{3.27c}$$

where, $S_{2,B}$, $S_{2,F}$ and $S_{2,G}$ are the bosonic, fermionic and the ghost parts of S_2.

3.2.1 One Loop Calculation for Two Blocks

To study the interaction between two blocks we re-write the X_μ in equation (3.18), as

$$X_\mu = \begin{pmatrix} Y_\mu^{(1)} & 0 \\ 0 & Y_\mu^{(2)} \end{pmatrix}, \quad R_\mu = \begin{pmatrix} S_\mu^{(1)} & 0 \\ 0 & S_\mu^{(2)} \end{pmatrix} \tag{3.28}$$

where, $Y_\mu^{(1)}$ and $Y_\mu^{(2)}$ are $N_1 \otimes N_1$ and $N_2 \otimes N_2$ blocks respectively ($N_1 + N_2 = N$). We consider the fluctuations of the from

$$\tilde{A}_\mu = \begin{pmatrix} \tilde{A}_\mu^{(1)} & \tilde{B}_\mu \\ \tilde{B}_\mu^\dagger & \tilde{A}_\mu^{(2)} \end{pmatrix}, \tilde{\varphi}_a = \begin{pmatrix} \tilde{\varphi}_a^{(1)} & \tilde{\psi}_a \\ \tilde{\psi}_a^\dagger & \tilde{\varphi}_a^{(2)} \end{pmatrix}, B = \begin{pmatrix} \tilde{B}^{(1)} & \tilde{D} \\ \tilde{D}^\dagger & \tilde{B}^{(2)} \end{pmatrix}, C = \begin{pmatrix} \tilde{C}^{(1)} & \tilde{E} \\ \tilde{E}^\dagger & \tilde{C}^{(2)} \end{pmatrix}$$

in terms of the above components, we can re-write the bosonic, fermionic and ghost parts of action (up to the second order of the fluctuations) as

$$\begin{aligned} S_{2,B} &= \frac{1}{g^2}\sum_{p=1,2} Tr\left\{-\frac{1}{2}[\tilde{A}_\mu^{(p)}, Y_\nu^{(p)}]^2 - [Y_\mu^{(p)}, Y_\nu^{(p)}][\tilde{A}_\mu^{(p)}, \tilde{A}_\nu^{(p)}] + 2i\alpha\epsilon_{\mu\nu\lambda}Y_\mu^{(p)}\tilde{A}_\nu^{(p)}\tilde{A}_\lambda^{(p)}\right\} \\ &+\frac{1}{g^2}(\tilde{B}_\mu^\dagger)_{Ii}\Big[(\mathcal{H}^2)\delta_{\mu\nu} - 2\left[\mathcal{H}_\mu, \mathcal{H}_\nu\right] + 2i\alpha\epsilon_{\mu\nu\lambda}\mathcal{H}_\lambda\}\Big]_{ijIJ}(\tilde{B}_\nu)_{jJ} \end{aligned} \tag{3.29}$$

$$S_{2,F} = \frac{1}{2g^2}\left\{\sum_{p=1,2} Tr\bar{\varphi}^{(p)}\sigma^\mu[Y_\mu^{(p)}, \tilde{\varphi}^{(p)}]\right\} + \frac{1}{g^2}(\bar{\tilde{\psi}}^\dagger)_{Ii}\left[\sigma_\mu(\mathcal{H}_\mu)_{ijIJ}\right](\tilde{\psi})_{jJ} \tag{3.30}$$

$$\begin{aligned} S_{2,G} &= \frac{1}{g^2}\left\{\sum_{p=1,2} Tr[Y_\mu^{(p)}, \tilde{B}^{(p)}][Y_\nu^{(p)}, \tilde{C}^{(p)}]\right\} \\ &+\frac{1}{g^2}\left\{(\bar{\tilde{D}}^\dagger)_{Ii}(\mathcal{H}^2)_{ijIJ}(\tilde{E})_{jJ} - (\bar{\tilde{E}}^\dagger)_{Ii}(\mathcal{H}^2)_{ijIJ}(\tilde{D})_{jJ}\right\} \end{aligned} \tag{3.31}$$

where

$$(\mathcal{H}_\mu)_{ijIJ} = (Y_\mu^{(1)})_{ij} \otimes \mathbf{1}_{IJ} - \mathbf{1}_{ij} \otimes (Y_\mu^{(2)})_{IJ}{}^*$$

$i, j = 1\cdots N_1$, $I, J = 1\cdots N_2$ and "*" denote complex conjugate.

From this, we can get one-loop effective action for one fuzzy sphere or multi-fuzzy sphere system by considering one block (irreducible) or multi-block diagonal (reducible) form for $Y^{(1)}$. In equation (3.29-3.31) the first terms represent self interactions of blocks of equation (3.28) and the second terms represent interactions between two blocks.

Example : One Fuzzy Sphere case

For example, if we replace $Y^{(1)}$ by $X^{(1)}$ and $Y^{(2)}$ by the remaining diagonal blocks as in equation (3.21), we get the one loop effective action for one fuzzy sphere case. We can see, in such a case the first term of individual equations gives the self-interaction terms; $p = 1$ term is fuzzy sphere self interaction and $p = 2$ term gives the self interaction between the space points. The second part of the equations are the interactions between the fuzzy sphere and the space points.

If we take one fuzzy sphere at origin *ie* $R_\mu^{(1)} = 0$, the self interaction term for one fuzzy sphere is

$$\begin{aligned} S_{2,B}^{(self)} &= -\frac{1}{2g^2}Tr[\bar{A}_\mu^{(1)}, Y_\nu^{(1)}]^2 \\ S_{2,F}^{(self)} &= \frac{1}{2g^2}Tr\bar{\varphi}^{(1)}\sigma^\mu[Y_\mu^{(1)}, \bar{\varphi}^{(1)}] \\ S_{2,G}^{(self)} &= \frac{1}{g^2}Tr[Y_\mu^{(1)}, \bar{B}^{(1)}][Y_\nu^{(1)}, \bar{C}^{(1)}] \end{aligned}$$

we get similar expressions for self-interaction for the model in [4].

Similarly, if we replace $Y^{(1)}$ by a block diagonal form with two irreducible blocks (as equation (3.23)), we get the one loop effective action for two fuzzy sphere case. We will discuss this in detail in the following section.

3.2.2 One Loop Calculation for Three Blocks

To study the interaction between three blocks, we start with the Fuzzy Sphere model described in the earlier section. We start with the X and R to be block diagonal type with three blocks.

$$X_\mu = \begin{pmatrix} X_\mu^{(1)} & & 0 \\ & X_\mu^{(2)} & \\ 0 & & X_\mu^{(3)} \end{pmatrix} \tag{3.32}$$

We assume the corresponding fluctuations to be of the form

$$\bar{A}_\mu = \begin{pmatrix} a_\mu^{(1)} & b_\mu^{(1,2)} & b_\mu^{(1,3)} \\ (b_\mu^{(1,2)})^\dagger & a_\mu^{(2)} & b_\mu^{(2,3)} \\ (b_\mu^{(1,3)})^\dagger & (b_\mu^{(2,3)})^\dagger & a_\mu^{(3)} \end{pmatrix}; \quad \bar{\varphi} = \begin{pmatrix} \varphi^{(1)} & \varphi^{(1,2)} & \varphi^{(1,3)} \\ (\varphi^{(1,2)})^\dagger & \varphi^{(2)} & \varphi^{(2,3)} \\ (\varphi^{(1,3)})^\dagger & (\varphi^{(2,3)})^\dagger & \varphi^{(3)} \end{pmatrix} \tag{3.33}$$

$$B = \begin{pmatrix} b^{(1)} & d^{(1,2)} & d^{(1,3)} \\ (d^{(1,2)})^\dagger & b^{(2)} & d^{(2,3)} \\ (d^{(1,3)})^\dagger & (d^{(2,3)})^\dagger & b^{(3)} \end{pmatrix}; \quad C = \begin{pmatrix} c^{(1)} & e^{(1,2)} & e^{(1,3)} \\ (e^{(1,2)})^\dagger & c^{(2)} & e^{(2,3)} \\ (e^{(1,3)})^\dagger & (e^{(2,3)})^\dagger & c^{(3)} \end{pmatrix}; \tag{3.34}$$

Putting this in 3.27, we get the following equations

$$S_{2,B}^{(p)} = \frac{1}{g^2}Tr\left\{-\frac{1}{2}[a_\mu^{(p)}, X_\nu^{(p)}]^2 - [X_\mu^{(p)}, X_\nu^{(p)}][a_\mu^{(p)}, a_\nu^{(p)}] + 2i\alpha\epsilon_{\mu\nu\lambda}X_\mu^{(p)}a_\nu^{(p)}a_\lambda^{(p)}\right\}$$

$$
\begin{aligned}
S_{2,B}^{(p,q)} &= \frac{1}{g^2}(b_\mu^{(p,q)})^\dagger)_{Ii}\Big[(H^{(p,q)})^2\delta_{\mu\nu} - 2\left[H_\mu^{(p,q)}, H_\nu^{(p,q)}\right] + 2i\alpha\epsilon_{\mu\nu\lambda}H_\lambda^{(p,q)}\Big]_{ijIJ}(b_\nu^{(p,q)})_{jJ} \\
S_{2,F}^{(p)} &= \frac{1}{2g^2}\left\{\sum_{p=1,2} Tr\bar{\varphi}^{(p)}\sigma^\mu[X_\mu^{(p)}, \varphi^{(p)}]\right\} \\
S_{2,F}^{(p,q)} &= \frac{1}{g^2}(\varphi^{(\bar{p},q)})^\dagger{}_{Ii}\left[\sigma_\mu(H_\mu^{(p,q)})_{ijIJ}\right](\varphi^{(p,q)})_{jJ} \\
S_{2,G}^{(p)} &= \frac{1}{g^2}\left\{\sum_{p=1,2} Tr[X_\mu^{(p)}, b^{(p)}][X_\nu^{(p)}, c^{(p)}]\right\} \\
S_{2,G}^{(p,q)} &= +\frac{1}{g^2}\left\{(\bar{d}^{(p,q)})^\dagger_{Ii}(H^{(p,q)})^2_{ijIJ}(\bar{e})_{jJ} - (\bar{e}^{(p,q)})^\dagger_{Ii}(H^{(p,q)})^2_{ijIJ}(d)_{jJ}\right\}
\end{aligned}
$$

where

$$
(H_\mu^{(p,q)})_{ijIJ} = (X_\mu^{(p)})_{ij} \otimes \mathbf{1}_{IJ} - \mathbf{1}_{ij} \otimes (X_\mu^{(q)})_{IJ}{}^*
$$

$S^{(p)}$ and $S^{(p,q)}$ are the self terms corresponding to the pth block and the interaction terms of pth and qth blocks respectively.

3.3 Interactions

Now in this section, we will try to calculate the one loop interaction corresponding to different interactions. For this we consider $X^{(1)}$ and $X^{(2)}$ in equation (3.33) as two fuzzy spheres and the $X^{(3)}$ a $2 \otimes 2$ diagonal matrix representing two space points. So $X^{(1)}$ and $X^{(2)}$ satisfies

$$
\left[X_\mu^{(p)}, X_\nu^{(p)}\right]_{(p=1,2)} = i\epsilon_{\mu\nu\lambda}\left(X_\lambda^{(p)} - R_\lambda^{(p)}\right)
$$

We also assume the $X^{(3)}$ with the corresponding fluctuations of the following forms,

$$
X_\mu^{(3)} = \begin{pmatrix} y^{(1)} & 0 \\ 0 & y^{(2)} \end{pmatrix}; \quad \tilde{A}_\mu^{(3)} = \begin{pmatrix} \tilde{a}_\mu^{(1)} & \tilde{b}_\mu \\ \tilde{b}_\mu^\dagger & \tilde{a}_\mu^{(2)} \end{pmatrix}, \quad \tilde{\varphi}_\alpha^{(3)} = \alpha s_\alpha, \quad s_\alpha = \begin{pmatrix} \tilde{s}_\alpha^{(1)} & \tilde{t}_\alpha \\ \tilde{t}_\alpha^\dagger & \tilde{s}_\alpha^{(2)} \end{pmatrix}
$$

$$
\tilde{B}^{(3)} = \begin{pmatrix} \tilde{b}^{(1)} & \tilde{d} \\ \tilde{d}^\dagger & \tilde{b}^{(2)} \end{pmatrix}, \quad \tilde{C}^{(3)} = \begin{pmatrix} \tilde{c}^{(1)} & \tilde{e} \\ \tilde{e}^\dagger & \tilde{c}^{(2)} \end{pmatrix}, \quad R_\lambda^{(3)} = \begin{pmatrix} \tilde{R}_\lambda^{(1)} & 0 \\ 0 & \tilde{R}_\lambda^{(2)} \end{pmatrix}
$$

3.3.1 Interaction between two points

To study the interaction between two points, we consider a system of only two points. To describe such a system of only two points, we assume that only X_3 in equation (3.33) is non-vanishining and both $X_\mu^{(1)}, X_\mu^{(2)}$ are zero. For this configuration S_2 in equation 3.35 takes the following form,

$$
\begin{aligned}
S_{2,B}^{(p,p)} &= \frac{\alpha^2}{g^2}\tilde{b}_\mu^\dagger\Big[(H^{(p,p)})^2\delta_{\mu\nu} - 2i\alpha\epsilon_{\mu\nu\lambda}c_\lambda^{(pp)}\Big]\tilde{b}_\nu \\
S_{2,F}^{(p,p)} &= \frac{\alpha^2}{g^2}\bar{\tilde{t}}_\alpha^\dagger\left(\sigma.H^{(pp)}\right)\tilde{t}_\alpha \\
S_{2,G}^{(p,p)} &= \frac{\alpha^2}{g^2}\left\{\tilde{d}^\dagger(H^{(pp)})^2\tilde{e} - \tilde{e}^\dagger(H^{(pp)})^2\tilde{d}\right\}
\end{aligned}
$$

where

$$
H_\mu^{(p,p)} = \frac{1}{\alpha}(y_\mu^{(1)}) - y_\mu^{(2)^*}) \quad \text{and} \quad c_\mu^{(p,p)} = \frac{1}{\alpha}(\bar{R}_\mu^{(1)}) - \bar{R}_\mu^{(2)^*})
$$

$c_\mu^{(p,p)}$ is the distance vector between two points.

The bosonic, fermionic and the ghost contributions to the one loop effective interaction for the system of two points can be written as,

$$
\begin{aligned}
W_B^{(pp)} &= -ln\int db\, db^\dagger e^{-b_\mu^\dagger\left[(H^{(pp)})^2\delta_{\mu\nu} - 2i\epsilon_{\mu\nu\lambda}c_\lambda\right]b_\nu} \\
&= -ln\left[det^{-\frac{1}{2}}\left((H^{(pp)})^2 - 2i\epsilon\cdot c\right)\right]^2 \\
W_F^{(pp)} &= -ln\int dt\, dt^\dagger e^{-\bar{t}^\dagger\left[\sigma_\mu H_\mu^{(pp)}\right]t} \\
&= -ln\left[det^{\frac{1}{2}}\sigma\cdot H^{(pp)}\right]^2 \\
W_G^{(pp)} &= -ln\int dd\, dd^\dagger\, dc\, dc^\dagger e^{-d^\dagger(H^{(pp)})^2 e + e^\dagger(H^{(pp)})^2 d} \\
&= -ln\left[det(H^{(pp)})^2\right]^2
\end{aligned}
$$

where,

$$
(H^{(pp)})^2 - 2i\epsilon\cdot c^{(pp)} = \begin{pmatrix} (H^{(pp)})^2 & -2ic_3^{(pp)} & 2ic_2^{(pp)} \\ 2ic_3^{(pp)} & (H^{(pp)})^2 & -2ic_1^{(pp)} \\ -2ic_2^{(pp)} & 2ic_1^{(pp)} & (H^{(pp)})^2 \end{pmatrix}
$$

and

$$
\det\left[(H^{(pp)})^2 - 2i\epsilon\cdot c^{(pp)}\right] = (H^{(pp)})^2\left[(H^{(pp)})^4 - 4(c^{(pp)})^2\right]
$$

where, $c^{(pp)} = \sqrt{(c_1^{(pp)})^2 + (c_2^{(pp)})^2 + (c_3^{(pp)})^2}$ is the distance between two points.

Summing up all these contributions, the total one loop effective action can be written as,

$$
W^{(pp)} = \frac{1}{2}ln\left[\det\left(\frac{\left((H^{(pp)})^2 - 2i\epsilon\cdot c^{(pp)}\right)}{(H^{(pp)})^4\left(\sigma\cdot H^{(pp)}\right)^2}\right)\right] = \frac{1}{2}ln\left[\left(\frac{\left((H^{(pp)})^4 - 4(c^{(pp)})^2\right)}{(H^{(pp)})^2(H^{(pp)})^2}\right)\right]
$$

We can write

$$
H_\mu^{(p,p)} = \frac{1}{\alpha}(y_\mu^{(1)}) - y_\mu^{(2)^*}) = c_\mu^{(pp)}
$$

So, with this, the one loop effective action for a two point system reduces to,

$$
W^{(pp)} = \frac{1}{2}ln\left|1 - \frac{4}{(c^{(pp)})^2}\right| \tag{3.35}
$$

We plot this effective potential as a function of the distance between two points (in figure 3.1).

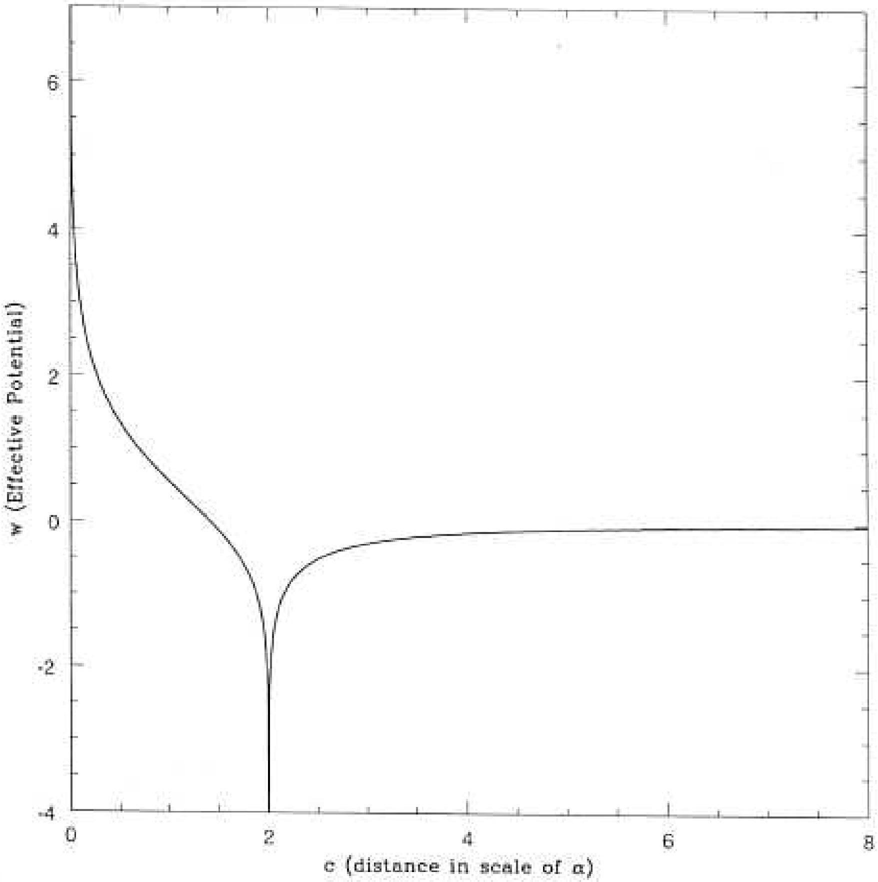

Figure 3.1: The effective potential between two points.

There is a repulsive force between two points when they are very close, and an attractive force when they are far. At $c^{(pp)} = 2$, the argument of the log in equation (3.35) is zero, so $W^{(pp)}$ has a minimum of $-\infty$. This suggests the existence of a minimum scale in space. In section 3.4, we consider the possibility of a lattice structure consisting of these points.

3.3.2 Interaction between a point and a Fuzzy sphere

To evaluate the interaction between a point and a fuzzy sphere, we consider a system consisting of a point and a fuzzy sphere. Such a configuration can be obtained by putting $X_\mu^{(2)} = 0$ and $y_\mu^{(2)} = 0$ in equation (3.33) and for such system, S_2 corresponding to the interaction between them can be obtained from equation (3.35) as follows,

$$\begin{aligned}
S^{(p,q)}_{2,B} &= \frac{1}{g^2}(b^{(f,p)}_\mu)^\dagger)_{Ii}\Big[(H^{(f,p)})^2\delta_{\mu\nu} - 2\left[H^{(f,p)}_\mu, H^{(f,p)}_\nu\right] + 2i\alpha\epsilon_{\mu\nu\lambda}H^{(f,p)}_\lambda\Big]_{ijIJ}(b^{(f,p)}_\nu)_{jJ} \\
S^{(f,p)}_{2,F} &= \frac{1}{g^2}(\varphi^{(\bar{f},p)})^\dagger{}_{Ii}\left[\sigma_\mu(H^{(f,p)}_\mu)_{ijIJ}\right](\varphi^{(f,p)})_{jJ} \\
S^{(f,p)}_{2,G} &= +\frac{1}{g^2}\left\{(\bar{d}^{(f,p)})^\dagger_{Ii}(H^{(f,p)})^2_{ijIJ}(\bar{e})_{jJ} - (\bar{e}^{(f,p)})^\dagger_{Ii}(H^{(f,p)})^2_{ijIJ}(d)_{jJ}\right\}
\end{aligned}$$

where

$$(H^{(f,p)}_\mu) = (X^{(1)}_\mu) - y^{(1)^*}_\mu\mathbf{1}$$

and

$$\left[H^{(f,p)}_\mu, H^{(f,p)}_\nu\right] = i\alpha\epsilon_{\mu\nu\lambda}\left(H^{(f,p)}_\lambda + y^{(1)}_\lambda - R^{(1)}_\lambda\right)$$

We assume that $R^{(1)}_\lambda$ and $C^{(f,p)}_\lambda$ are of the form,

$$R^{(1)}_\lambda = r^{(1)}_\lambda\mathbf{1}; \quad C^{(f,p)}_\lambda = (R^{(1)}_\lambda - y^{(1)}_\lambda\mathbf{1}) = c^{(f,p)}_\lambda\mathbf{1}$$

With these, the bosonic, fermionic and the ghost contribution to the one loop effective action is as follows:-

$$\begin{aligned}
W^{(fp)}_B &= -ln\left[det^{-\frac{1}{2}}\left((H^{(fp)})^2 - 2i\epsilon\cdot c\right)\right]^2 \\
W^{(fp)}_F &= -ln\left[det^{\frac{1}{2}}\sigma\cdot H^{(fp)}\right]^2 \\
W^{(fp)}_G &= -ln\left[det(H^{(fp)})^2\right]^2
\end{aligned}$$

where,

$$\det\left[(H^{(fp)})^2 - 2i\epsilon\cdot c^{(fp)}\right] = \det\begin{pmatrix} (H^{(fp)})^2 & -2ic^{(fp)}_3 & 2ic^{(fp)}_2 \\ 2ic^{(fp)}_3 & (H^{(fp)})^2 & -2ic^{(fp)}_1 \\ -2ic^{(fp)}_2 & 2ic^{(fp)}_1 & (H^{(fp)})^2 \end{pmatrix}$$

Each of the blocks in the right hand side matrix is of size $n_1 \times n_1$ matrices. To calculate the determinant, we use the $SO(3)$ symmetry to choose $c^{(fp)}_\mu = (0, 0, c^{(fp)})$, without any loss of generality. With this choice we can easily calculate the above determinant and the using that we get the one loop effective potential as

$$W^{(fp)} = \frac{1}{2}ln\left[\det\left(\frac{\left((H^{(fp)})^2 - 2c^{(fp)}\right)\left((H^{(fp)})^2 + 2c^{(fp)}\right)}{(H^{(fp)})^2\left(\sigma\cdot H^{(fp)}\right)^2}\right)\right] \tag{3.36}$$

A detailed calculation (similar to appendix A4) gives,

$$W^{(fp)} = \frac{1}{2}ln\left|w^{(fp)}_j\right| \tag{3.37}$$

where, $w_j^{(fp)} = \tilde{w_j^{(fp)}}(c^{(fp)})\tilde{w_j^{(fp)}}(-c^{(fp)})$ is the determinant of jth block and

$$\tilde{w_j^{(fp)}}(c^{(fp)}) = \frac{\left[(c^{(fp)}+j+1)\right]\left[(c^{(fp)})^2+2c^{(fp)}(j+1)+j(j+1)\right]}{(c^{(fp)}+j)\left[(c^{(fp)})^2+2c^{(fp)}j+j(j+1)\right]} \times \prod_{i=-j}^{j} \frac{\left[(c^{(fp)})^2+2c^{(fp)}(i+1)+j(j+1)\right]}{\left[(c^{(fp)})^2+c^{(fp)}(2i+1)+j(j+1)\right]}. \tag{3.38}$$

where

$$(2j+1)=n_1, \quad c^{(fp)}=\sqrt{(c_1^{(fp)})^2+(c_2^{(fp)})^2+(c_3^{(fp)})^2}$$

Though for $c=\infty$, the potential vanishes, for $c=0$ there is a non vanishing effective potential

$$(W^{(fp)})_{c=0} = \frac{1}{2}\log\left|\frac{n_1+1}{n_1-1}\right|$$

When the point is far from the fuzzy sphere ($c >> 1$), we can approximate the effective potential as the following power series,

$$\left(W^{(fp)}\right)_{c>>1} = -4n_1/c^2 - \frac{1}{8}n_1(49+15n_1^2)/c^4 + O\left(1/c^6\right) \tag{3.39}$$

In the small distance limit ($c \to 0$), we can approximate this effective potential by,

$$\left(W^{(1)(2)}\right)_{c<<1} = \frac{1}{2}\log\left|\frac{n_1+1}{n_1-1}\right| - \frac{48n_1}{n_1^2-1}c^2 + O\left(c^4\right) \tag{3.40}$$

To see the behavior of this one loop potential, let us consider two specific examples. Let us plot the the one loop effective potential $W^{(fp)}$ in the above equation as a function of distance c for $j = 1$ (*i.e.* $n_1 = 2, \rho = .43\alpha$) and $j = 10$ (*i.e.* $n_1 = 21, \rho = 10.488\alpha$) as shown in figure (3.2) and in figure (3.3).

We can clearly see, in the plot, that there exists an attractive force for large distance and a repulsive force at small distance.

3.4 One Dimensional Lattice of Points

Consider a one dimensional lattice structure consisting of points of spacing d with the nearest neighbor. We have seen that the potential between any two points can be written as,

$$W^{(pp)} = \frac{1}{2}\log\left|1-\frac{4}{c_{(pp)}^2}\right|$$

There exists an attractive force for $c_{(pp)} > 2$ and a repulsive force for $c_{(pp)} < 2$.

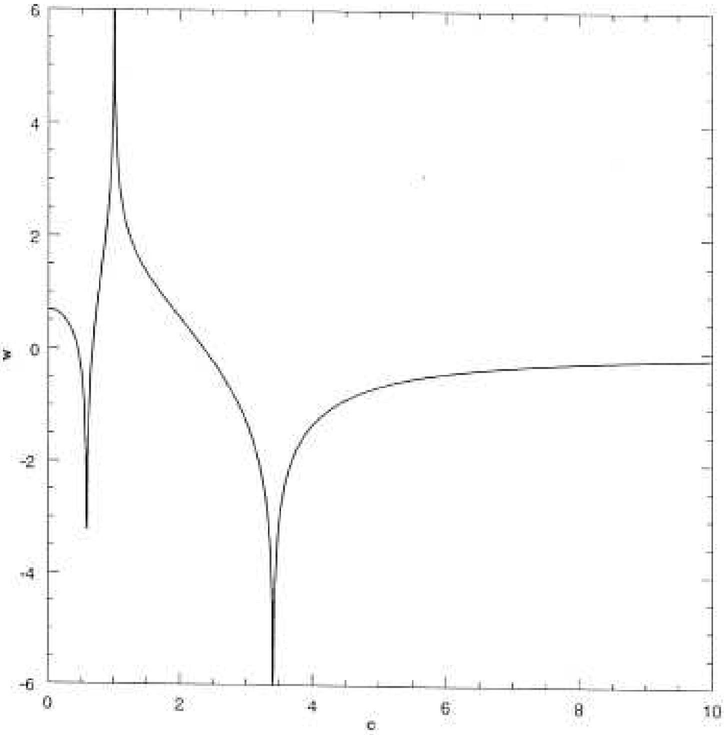

Figure 3.2: The effective potential between a point and a fuzzy sphere (j=1).

Now, when we have only two points,

$$\bullet \leftarrow 2 \rightarrow \bullet$$

naturally the points will adjust themselves such that the distance between them $c_{(pp)}$ is equal to 2. Now if we add a third point on the same line, the first point tries to pull it so that the distance from it is 2, but the second point will resist it coming inwards because of the repulsive potential for $d < 2$.

$$\cdots\cdots\cdots \bullet \leftarrow d \rightarrow \bullet \leftarrow d \rightarrow \bullet \leftarrow d \rightarrow \bullet \leftarrow d \rightarrow \bullet \leftarrow d \rightarrow \bullet \leftarrow d \rightarrow \bullet \cdots\cdots\cdots$$

Now, consider a case of an infinite lattice structure such that the distance between two adjacent points is d. In the following part we shall try to find out the value of d for which the system has minimum potential energy.

The total potential energy of such system is given by,

$$V = \frac{1}{2}\sum_{i,j;i\neq j} V_{i,j} = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2}\log\left|1-\frac{4}{(jd)^2}\right| = \sum_{i=0}^{\infty}\log\left|\frac{\sin\frac{2\pi}{d}}{\frac{2\pi}{d}}\right|$$

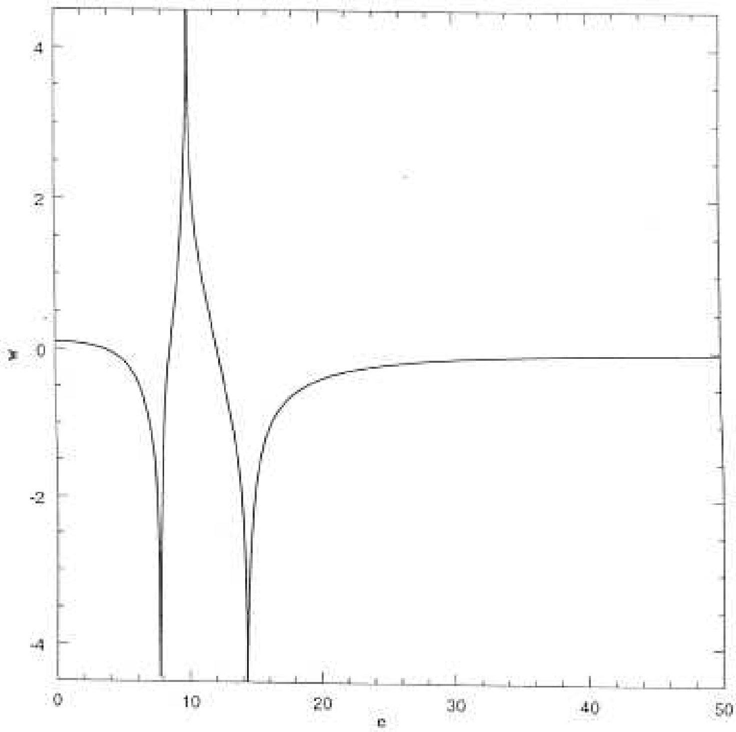

Figure 3.3: The effective potential between a point and a fuzzy sphere (j=10)

The sum over i runs over all the points. So for this infinite lattice the potential energy per point is (excluding the self energy) is

$$v_p = \log\left|\frac{\sin\frac{2\pi}{d}}{\frac{2\pi}{d}}\right| \tag{3.41}$$

$\sin n\pi$ for integer n is zero, so this potential is $-\infty$ value for the distances given by,

$$\frac{2\pi}{d} = n\pi \quad \rightarrow \quad d = 2/n$$

where n is positive integer. For $n = 1$, *i.e.* for lattice separation $d = 2$, there is a stable lattice structure, where every point is trapped in an infinite potential well at distance 2α from its neighbor as shown in figure (3.1). There can be stable solutions for $n = 2, 3, ...$ also. The potential energy per point as a function of lattice separation in figure (3.4) also suggests a possibility of an infinite lattice structure for these solutions. We can see, for $d = 2$, there is a stable one dimensional lattice structure. There may be lattice for shorter lattice separation either, for example $d = 1$. The value of this potential is infinite at $d = 0$, so there may be a lattice structure with very small but discrete lattice separation (when n is very large), but

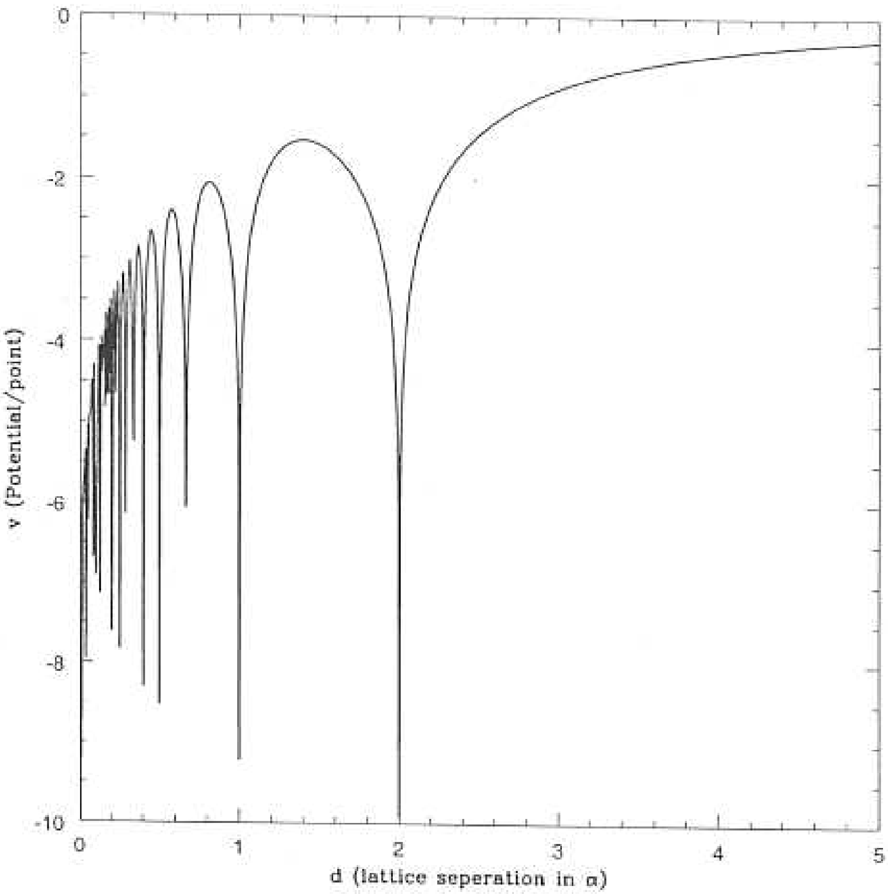

Figure 3.4: The potential / point as function of lattice separation

d can never be zero which will result in a collapse of all the points at one point. Even though there is a repulsive potential between the points (3.1) for small d, they are finite except at zero distance. The contribution due to the point in the potential trap *i.e.* ($d = 2$) dominates over the finite positive value, resulting in other minimum at various values of $d < 2$. However for $d = 0$, the potential is positive infinity, thus there is no stable solution for this case.

3.5 Conclusion

In this chapter, we have presented a general fuzzy sphere model in three dimensions, which allows a multi fuzzy sphere system with discretely arbitrary radii and arbitrary locations in R^3. We have added a Chern Simons term to the reduced model of 3D SYM. In original model the space and objects *e.g.* fuzzy spheres are not separately distinguishable. We have artificially partitioned the matrices into multiple block-diagonal form. In such case, the classical solution represents a system of space and fuzzy spheres (branes). Classically these fuzzy spheres and space are non-interacting. We have tried to calculate the interaction as the one loop quantum effect. We have studied the one loop interaction between two and three blocks. In particular, we have calculated the interaction between two points and the interaction between a point and a fuzzy sphere. We have seen that in both the cases there is an attractive force for large separation and a repulsive force in small distance case. We have considered a one dimensional infinite lattice of points and found out that there are some stable one dimensional lattice configurations of points. This also suggests the existence of stable triangular lattice structure in two and stable tetrahedral structure in three dimensions.

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Chapter 4

Interaction between two Fuzzy Spheres

In this chapter, We have calculated one loop interaction between two fuzzy spheres in bosonic and supersymmetric case in 3 dimension. We have calculated the potential for such system for both large and small distance case. It depends on the distance r between the spheres and the radii ρ_1, ρ_2. There is no force between the spheres when they are far from each other (long distance case). We have also studied the interaction for $r = 0$ case. We find that an attractive force exists between two fuzzy sphere surfaces. We have also studied the extension of such system in 10 dimension.

4.1 One Loop Effective Action for Two Fuzzy Sphere system

As we have seen in earlier section, we can calculate the one loop effective action from equations(3.29-3.31). For this we consider the same configuration as equation(3.28), consider $Y_\mu^{(1)}$ as block-diagonal with two blocks (same as eqn. 3.23) , each block representing one fuzzy sphere. For two fuzzy sphere configuration, we assume the following form for the back-ground and fluctuation matrices.

$$Y_\mu^{(1)} = \begin{pmatrix} X_\mu^{(1)} & 0 \\ 0 & X_\mu^{(2)} \end{pmatrix}, \; \tilde{A}_\mu^{(1)} = \begin{pmatrix} a_\mu^{(1)} & b_\mu \\ b_\mu^\dagger & a_\mu^{(2)} \end{pmatrix}, \; \tilde{\varphi}_\alpha^{(1)} = \alpha s_\alpha, \; s_\alpha = \begin{pmatrix} s_\alpha^{(1)} & t_\alpha \\ t_\alpha^\dagger & s_\alpha^{(2)} \end{pmatrix}$$

$$\tilde{B}^{(1)} = \begin{pmatrix} b^{(1)} & d \\ d^\dagger & b^{(2)} \end{pmatrix}, \; \tilde{C}^{(1)} = \begin{pmatrix} c^{(1)} & e \\ e^\dagger & c^{(2)} \end{pmatrix}, \; S_\lambda^{(1)} = \begin{pmatrix} R_\lambda^{(1)} & 0 \\ 0 & R_\lambda^{(2)} \end{pmatrix} \tag{4.1}$$

where, in these matrices, the first diagonal block is $n_1 \times n_1$ matrix and the second is $n_2 \times n_2$. For calculational simplicity, we further assume $Y_\mu^{(2)} = 0, R_\lambda^{(2)} = 0$.

$R_\lambda^{(i=1,2)} = r_\lambda^{(i)} \mathbf{1}_{n_i \times n_i}$, $r_\lambda^{(1)} = \frac{n_2}{n_1+n_2} r_\lambda$, $r_\lambda^{(2)} = -\frac{n_1}{n_1+n_2} r_\lambda$, are the centers of two fuzzy spheres and r_λ is the distance vector between them.

$$\bar{D} = \begin{pmatrix} D^{(1)} \\ D^{(2)} \end{pmatrix},\ \bar{E} = \begin{pmatrix} E^{(1)} \\ E^{(2)} \end{pmatrix},\ \bar{B}_\mu = \begin{pmatrix} B_\mu^{(1)} \\ B_\mu^{(2)} \end{pmatrix},\ \bar{\psi}_a = \begin{pmatrix} \psi_a^{(1)} \\ \psi_a^{(2)} \end{pmatrix}$$

where, in these matrices, the upper block is $n_1 \times (N - n_1 - n_2)$ matrix and the second is $n_2 \times (N - n_1 - n_2)$.

Putting these in equation (3.29 - 3.31), the total bilinear terms can be written as the sum of following 9 terms.

$$S_{2,B}^{(self)} = \frac{1}{g^2} \sum_{i=1,2} Tr\left\{ -\frac{1}{2}[a_\mu^{(i)}, X_\nu^{(i)}]^2 + 2i\epsilon_{\mu\nu\lambda} R_\lambda^{(i)} a_\mu^{(i)} a_\nu^{(i)} \right\} \tag{4.2a}$$

$$S_{2,F}^{(self)} = \frac{\alpha^2}{2g^2} \sum_{i=1,2} Tr\left\{ s^{(i)} \sigma^\mu [L_\mu^{(i)}, s^{(i)}] \right\} \tag{4.2b}$$

$$S_{2,G}^{(self)} = \frac{\alpha^2}{g^2} \left\{ \sum_{i=1,2} Tr[L_\mu^{(i)}, b^{(i)}][L_\nu^{(i)}, c^{(i)}] \right\} \tag{4.2c}$$

$$S_{2,B}^{(back)} = \frac{\alpha^2}{g^2} \sum_{i=1,2} (\bar{B}_\mu^{(i)\dagger})_{Ik_i} \left[(L_\rho^{(i)} \otimes \mathbf{1})^2 \delta_{\mu\nu} - 2i\epsilon_{\mu\nu\lambda} c_\lambda^{(i)} \mathbf{1} \right]_{k_i l_i IJ} (\bar{B}_\nu^{(i)})_{l_i J} \tag{4.2d}$$

$$S_{2,F}^{(back)} = \frac{\alpha^2}{2g^2} \sum_{i=1,2} (\bar{\bar{T}})_{Ik_i} \left[\sigma_\mu (L_\mu^{(i)})_{k_i l_i IJ} \right] (\bar{T})_{l_i J} \tag{4.2e}$$

$$S_{2,G}^{(back)} = \frac{\alpha^2}{g^2} \sum_{i=1,2} \left\{ (\bar{\bar{D}}^\dagger)_{Ik_i} (L^{(i)} \otimes \mathbf{1})^2_{k_i l_i IJ} (\bar{E})_{l_i J} - (\bar{\bar{E}}^\dagger)_{Ik_i} (L^{(i)} \otimes \mathbf{1})^2_{k_i l_i IJ} (\bar{D})_{l_i J} \right\} \tag{4.2f}$$

$$S_{2,B}^{(1)(2)} = \frac{\alpha^2}{g^2} (b_\mu^\dagger)_{k_2 k_1} \left[(H^2)\delta_{\mu\nu} - 2i\epsilon_{\mu\nu\lambda} c_\lambda \otimes \mathbf{1} \right]_{k_1 l_1 k_2 l_2} (b_\nu)_{l_1 l_2} \tag{4.2g}$$

$$S_{2,F}^{(1)(2)} = \frac{2\alpha^2}{g^2} (\bar{t})_{k_2 k_1} \left[\sigma_\mu (H_\mu)_{k_1 l_1 k_2 l_2} \right] (t)_{l_1 l_2} \tag{4.2h}$$

$$S_{2,G}^{(1)(2)} = \frac{\alpha^2}{g^2} \left\{ (\bar{d})_{k_2 k_1} (H^2)_{k_1 l_1 k_2 l_2} (e)_{l_1 l_2} - (\bar{e})_{k_2 k_1} (H^2)_{k_1 l_1 k_2 l_2} (d)_{l_1 l_2} \right\} \tag{4.2i}$$

where

$$(H_\mu)_{k_1 l_1 k_2 l_2} = (L_\mu^{(1)})_{k_1 l_1} \otimes \mathbf{1}_{k_2 l_2} - \mathbf{1}_{k_1 l_1} \otimes (L_\mu^{(2)})_{k_2 l_2}^{*}$$

$k_i, l_i = 1 \cdots n_i$, $I, J = 1 \cdots (N - n_1 - n_2)$, $L_\mu^{(i)} = \frac{1}{\alpha} X_\mu^{(i)}$, $c_\mu = \frac{1}{\alpha} r_\mu$.

We can see each of bosonic, fermionic and ghost has three parts describing self-interaction (denoted by $S_2^{(self)}$), interaction between two fuzzy spheres (denoted by $S_2^{(1)(2)}$) and the extra piece coming from the interaction of each fuzzy sphere with the back ground (denoted by $S_2^{(back)}$ index i, for i-th one). We can as well say these extra piece as part of the self energy of the fuzzy spheres because they exist even for one fuzzy sphere case (section 3.2).

4.1.1 Interaction Between Two Fuzzy Spheres

We assume one loop correction is good approximation for the interaction between fuzzy spheres [1]. 1-loop correction of effective action W is calculated as

$$W = -ln \int da\, ds\, db\, dc\, e^{-S_2} \tag{4.3}$$

As the the total action S decouples into each sector, we can write

$$W = W^{(self)}_{(B+F+G)} + W^{(back)}_{(B+F+G)} + W^{(1)(2)}_{B} + W^{(1)(2)}_{F} + W^{(1)(2)}_{G} \tag{4.4}$$

where indices correspond to those of equations(4.2-4.2).
We are now interested in following parts those are from interactions between two fuzzy spheres.

$$\begin{aligned}
W^{(1)(2)}_{B} &= -ln \int db\, db^{\dagger} e^{-b^{\dagger}_{\mu}\left[H^2\delta_{\mu\nu} - 2i\epsilon_{\mu\nu\lambda}c_{\lambda}\right]b_{\nu}} \\
&= -ln \left[det^{-\frac{1}{2}}\left(H^2 - 2i\epsilon \cdot c\right)\right]^2 \\
W^{(1)(2)}_{F} &= -ln \int dt\, dt^{\dagger} e^{-t^{\dagger}[\sigma_{\mu}H_{\mu}]t} \\
&= -ln \left[det^{\frac{1}{2}}\sigma \cdot H\right]^2 \\
W^{(1)(2)}_{G} &= -ln \int dd\, dd^{\dagger}\, dc\, dc^{\dagger} e^{-d^{\dagger}H^2 e + e^{\dagger}H^2 d} \\
&= -ln\left[det H^2\right]^2
\end{aligned}$$

where squares of determinants come from two off-diagonal blocks of matrices and $\frac{1}{2}$ in $W^{(1)(2)}_{F}$ is because of Majorana spinor.

4.1.1.1 Bosonic Sector

Without loss of generality, two fuzzy spheres are assumed to be separated by r in 3rd direction

$$c_{\mu} = (0, 0, c), \quad r = \alpha c$$

Then diagonalise the operator in bosonic part

$$H^2 - 2i\epsilon \cdot c = \begin{pmatrix} H^2 & -2ic & 0 \\ 2ic & H^2 & 0 \\ 0 & 0 & H^2 \end{pmatrix} \sim \begin{pmatrix} H^2 - 2c & 0 & 0 \\ 0 & H^2 + 2c & 0 \\ 0 & 0 & H^2 \end{pmatrix}$$

So, we can write the bosonic contribution to the effective action (including the ghost part) as,

$$W^{(1)(2)}_{B} = \frac{1}{2}\mathrm{Ln}\left[\det\left(\frac{\left(H^2 - 2c\right)\left(H^2 + 2c\right)}{H^2}\right)\right] \tag{4.5}$$

[1]This is not a good approximation for two intersecting fuzzy spheres, for eg, $n_1 = n_2, c = 0$.

We define $J_\mu = H_\mu + c_\mu$ and $K_\mu = J_\mu + S_\mu$, where $S_\mu = \frac{\sigma_\mu}{2}$. Both J_μ and K_μ follow $SU(2)$ algebra. j, maximum eigen value of J_3, varies from $j_{min} = |\frac{n_1-n_2}{2}|$ to $j_{max} = (\frac{n_1+n_2}{2} - 1)$ and $k = j \pm \frac{1}{2}$. $H^2, H^2 \pm c$ or $H \cdot \sigma$) are block diagonal, each block representing a particular value of j. So, we can write

$$W^{(1)(2)}_{(B+G)} = \frac{1}{2} ln \left[\prod_{j=j_{min}}^{j_{max}} (w_B)_j \right] \tag{4.6}$$

where, $(w_B)_j$ is the determinant of jth block and can be calculated to be

$$(w_B)_j = \frac{(c^2 - 2c(j+1) + j(j+1))^2}{(c^2 + 2cj + j(j+1))^2} \prod_{i=-j}^{j} (c^2 + 2c(-i+1) + j(j+1))^2 \tag{4.7}$$

For $c << 1$, when the fuzzy spheres are concentric, there is a non-zero interaction

$$\begin{aligned} W^{(1)(2)}_{(B+G)} &= \sum_{j=j_{min}}^{j_{max}} (2j+1) ln[j(j+1)] \\ &- \frac{2}{3} \left[\frac{1}{n_1+n_2}\left(1 + \frac{24}{n_1+n_2}\right) - \frac{1}{n_1-n_2}\left(1 + \frac{24}{n_1-n_2}\right) + \sum_{j=j_{min}}^{j_{max}} \frac{1}{j} \right] c^2 + O\left(c^4\right). \end{aligned}$$

For $c >> 1$ *ie* when the fuzzy spheres are far apart,

$$W^{(1)(2)}_{(B+G)} = n_1 n_2 \log c^2 + \frac{n_1 n_2}{4}(n_1^2 + n_2^2 - 18)\frac{1}{c^2} + O\left(\frac{1}{c^4}\right). \tag{4.8}$$

4.1.1.2 Supersymmetric Case

Summing up all these contributions,

$$W^{(1)(2)} = \frac{1}{2} ln \left[\det \left(\frac{(H^2 - 2c)(H^2 + 2c)}{H^2 (\sigma \cdot H)^2} \right) \right] \tag{4.9}$$

A detailed calculation (the calculation is given in appendix A4) gives

$$W^{(1)(2)} = \frac{1}{2} ln \left[\prod_{j=j_{min}}^{j_{max}} w_j \right] \tag{4.10}$$

where, $w_j = \tilde{w}_j(c)\tilde{w}_j(-c)$ is the determinant of jth block and

$$\begin{aligned} \tilde{w}_j(c) = & \frac{[c^2 + c(2j+1) + j(j+1)]\,[c^2 + 2c(j+1) + j(j+1)]}{(c+j)^2\,[c^2 + 2cj + j(j+1)]} \\ & \times \prod_{i=-j}^{j} \frac{[c^2 + 2c(i+1) + j(j+1)]}{[c^2 + c(2i+1) + j(j+1)]}. \end{aligned} \tag{4.11}$$

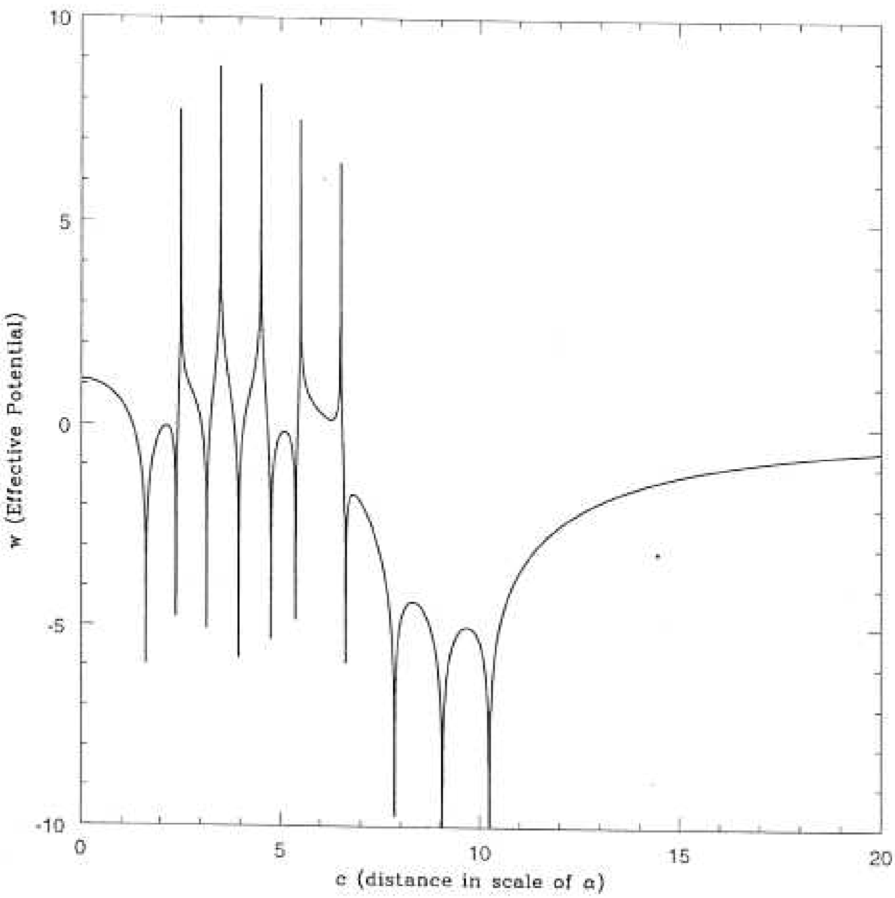

Figure 4.1: The effective potential between two fuzzy spheres ($n_1 = 10, n_2 = 5$).

When $c = \infty$, *ie* the fuzzy spheres are at infinite distance from each other, $W^{(1)(2)} = 0$, thus the two fuzzy spheres do not interact with each other. This feature is different from bosonic case. This is because of some cancellation between bosonic and the fermionic contributions.

Expanding $W^{(1)(2)}$ around $c = 0$ and $c = \infty$, we can get the potential between two fuzzy sphere for small and large distances respectively.
For small c,

$$W^{(1)(2)}_{c<<1} = \log\left|\frac{n_1 + n_2}{n_1 - n_2}\right| - 48\frac{n_1 n_2}{(n_1+n_2)^2(n_1-n_2)^2}c^2 + O\left(c^4\right). \tag{4.12}$$

We see an attractive force $(-(\partial W^{(1)(2)})/(\partial \alpha c))$ between the fuzzy sphere surfaces (repulsive between the centers), when the centers are close to each other. This can be seen from the figure(4.1 for the region of c very close to zero.

For large c,

$$W^{(1)(2)}_{c>>1} = -4n_1 n_2/c^2 + O\left(\frac{1}{c^4}\right). \tag{4.13}$$

So, there also exists an attractive force between the fuzzy spheres when they are at large distances. This result is consistent with the earlier observation that D2(D0)-branes form a bigger D2-brane in string theory [12, 8]. We can compare this to equation (4.8) and find some cancellation between bosonic and fermionic part, that is because of part of supersymmetry. We can see from figure (4.1), that there are some stable confined states corresponding to the minimum values of the effective potentials shown in the figure.

Spherical BPS D2-brane from string theory in S^2 does not interact when they are "parallel" ($c = 0$) to each other. However from our approach in matrix model the first term of equation (4.12) exists even for $c = 0$. It may be explained as the quadratic part of action S^2 (eqn. 4.2-4.2) itself is not supersymmetric under the susy transformations in eqn(??). But for large distance case ($c \to \infty$) this S^2 recovers $\mathcal{N} = 1$ supersymmetry and in such case $W^{(1)(2)}$ vanishes. Moreover in such case, we can not use only quadratic part of off-diagonal part in equation (4.1) when the size of n_1 and n_2 is not so different, that corresponds to overlapping of surface of two spheres.

4.2 Higher Dimensional Extension

So far, we have dealt with a three dimensional fuzzy sphere model. This model can be extended to higher dimensional matrix models like IKKT-type, where the fuzzy sphere may correspond to a D-brane in string theory.

As an example, we study one such particular extension in the context of IIB matrix model. We take the original 10 dimensional IIB matrix model with a 3 dimensional Chern-Simons term.

$$S = \frac{1}{g^2} Tr\Big(-\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{2}{3} i\alpha\epsilon_{abc}A^a A^b A^c + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\Big) \tag{4.14}$$

where μ, ν, λ are from 1 to 10 and a,b, c are from 1 to 3. The fermionic matrices are now 10 dimensional Majorana-Wyle spinors. The addition of the three dimensional Chern-Simons term breaks the global $SO(10)$ symmetry of the action into $SO(3) \otimes SO(7)$. In this present matrix model the matrices do not depend on any parameter like time, rather time is represented by one such Matrix . The degrees of freedom of space-time and brane are unified in matrices and we do not distinguish them in action.

This model has a typical classical solution

$$[X_a, X_b] = i\alpha\epsilon_{abc}(X_c - R_c), \quad X_i = 0, \quad [X_\mu, R_\nu] = 0, \quad \psi = 0, \tag{4.15}$$

which describes a system of two fuzzy spheres, separated only in 1, 2, 3 directions and at same point in other directions. In string theory picture, this system may correspond to a system with two D-branes lying in 1,2,3 directions.

Classically, these fuzzy spheres do not interact with each other. We can calculate the one loop interaction following the similar method as in three dimension. However, in this case there will be some additional contribution from 4-10 th dimensions as quantum fluctuation. The calculation of the quantum correction of the potential is straight forward. There is no divergence again because of the cancellation between bosonic and fermionic contributions. For large separation ($c >> 1$), the effective potential does not change with the dimension D ($D = 3$ or 10).

$$\left(W_D^{(1)(2)}\right)_{c>>1} = -4n_1n_2/c^2 + O\left(\frac{1}{c^4}\right) \tag{4.16}$$

But, the situation changes for small distance case ($c << 1$), where we see an attractive force between the fuzzy spheres in three dimension ($D = 3$) but a repulsive force in ten dimension ($D = 10$).

$$\left(W_D^{(1)(2)}\right)_{c<<1} = (D-2)\log\left|\frac{n_1+n_2}{n_1-n_2}\right| + \frac{16(D-6)n_1n_2}{(n_1+n_2)^2(n_1-n_2)^2}c^2 + O\left(c^4\right) \tag{4.17}$$

4.3 Conclusion

In this chapter, we have calculated the interaction between two fuzzy spheres as the one loop quantum effect n bosonic and supersymmetric case. In particular, we have calculated the interaction between two fuzzy spheres with radii ($\rho_1 \sim \alpha n_1, \rho_2 \sim \alpha n_2$) ($n_1$ and n_2 is arbitrary) at distance ($r = \alpha c$). We have determined the one loop effective action for such system for both bosonic case and supersymmetric case for two fuzzy spheres for small ($c << 1$) and large distance ($c >> 1$) case. There is a cancellation between bosonic and fermionic part. In supersymmetric case, in 3 dimensions, there is an attractive force between two fuzzy sphere surfaces for both large and small distance case. We have also studied an extension of such system in 10 dimensions, where we have considered two fuzzy spheres are separated in

1,2,3 directions and at same point in other directions. We have got an repulsive force between two fuzzy spheres in short distance case and an attractive force in large distance case. Even this model has a $\mathcal{N}=2$ supersymmetry, the one loop contribution for concentric case is non-zero for nearly equal n_1 and n_2. This is probably because of the fact the one loop approximation is not good approximation in such case. It will be interesting to compare equations (4.12, 4.13) with those from spherical D2-brane interactions in $SU(2)$ WZW model.

A4 Appendix : Derivation of equation 4.10

In this appendix, we will calculate $\det(H^2+a)$ and $\det(\sigma.H)^2$, where H is given as

$$(H_\mu)_{k_1l_1k_2l_2} = (L_\mu^{(1)})_{k_1l_1} \otimes \mathbf{1}_{k_2l_2} - \mathbf{1}_{k_1l_1} \otimes (L_\mu^{(2)})_{k_2l_2}{}^*$$

$k_i, l_i = 1\cdots n_i$, $I, J = 1\cdots(N-n_1-n_2)$, $L_\mu^{(i)} = \frac{1}{\alpha}X_\mu^{(i)}$, $c_\mu = \frac{1}{\alpha}r_\mu$. we assume

$$J_\mu = H_\mu + C_\mu, \quad \text{and} \quad K_\mu = J_\mu + S_\mu$$

with this assumption, we can write

$$\begin{aligned}
J_\mu &= H_\mu + C_\mu \\
&= (L_\mu^{(1)}) \otimes \mathbf{1} - \mathbf{1} \otimes (L_\mu^{(2)})^* + C_\mu \\
&= (L_\mu^{(1)} + C_\mu^{(1)}) \otimes \mathbf{1} - \mathbf{1} \otimes ((L_\mu^{(2)})^* - C_\mu^{(2)}) \\
&= J_\mu^{(1)} \otimes \mathbf{1} - \mathbf{1} \otimes J_\mu^{(2)}
\end{aligned}$$

So,

$$[J_\mu, J_\nu] = i\epsilon_{\mu\nu\lambda}J_\lambda$$

$j = |j_1 - j_2|, \ldots\ldots\ldots, (j_1 + j_2), \quad 2j_i + 1 = n_i$

The commutating operators are K^2, K_z, J^2, $(J^{(1)})^2$, $(J^{(2)})^2$, S^2 and the corresponding eigenvalues are

$$\begin{aligned}
K^2 \Rightarrow k(k+1) \Rightarrow k = j + \frac{1}{2}, j - \frac{1}{2} \\
K_z \Rightarrow m = m_j \pm \frac{1}{2} \Rightarrow m_j = m_1 + m_2 \\
J^2 \Rightarrow j(j+1) \Rightarrow j_{min} = |j_1 - j_2|, j_{max} = j_1 + j_2 \\
(J^{(1)})^2 = j_1(j_1+1), \ (J^{(2)})^2 = j_2(j_2+1)
\end{aligned}$$

For, a fixed j_1 and j_2, we can either choose $|k, m\rangle$ basis or $|m_j, m_s\rangle$ basis to describe the vectors.

We can write,

$$(J.\sigma) = K^2 - J^2 - \frac{\sigma^2}{4}, \ (H.\sigma) = K^2 - J^2 - \frac{\sigma^2}{4} - c\sigma_3$$

and

$$H^2 = J^2 + C^2 + 2c_\mu J_\mu$$

Consider an operator A, which can be written as

$$A = D(K^2, K_z) + O(J_z, S_z)$$

where $D(K^2, K_z)$ is diagonal in $|k, m\rangle$ basis and $O(J_z, S_z)$ is diagonal in $|m_j, m_s\rangle$ basis.

$$\langle km|A|k'm'\rangle_j = \langle km|D(K^2, K_z)|k'm'\rangle_j + \langle km|O(K^2, K_z)|k'm'\rangle_j \qquad \text{(A4.1)}$$

Let us try to find out the $\langle km|D(K^2, K_z)|k'm'\rangle_j$, As $D(K^2, K_z)$ is diagonal in $|km\rangle$ basis, we can write

$$\langle km|D(K^2, K_z)|k'm'\rangle_j = \begin{pmatrix} \langle (j-\frac{1}{2})m_-|D(K^2, K_z)|(j-\frac{1}{2})m'_-\rangle_j & 0 \\ 0 & \langle (j+\frac{1}{2})m_+|D(K^2, K_z)|(j+\frac{1}{2})m'_+\rangle_j \end{pmatrix}$$

where

$$\begin{aligned} \langle (j-\frac{1}{2})m_-|D(K^2, K_z)|(j-\frac{1}{2})m'_-\rangle_j &= d\left((j-\frac{1}{2})(j+\frac{1}{2}), m_-\right)\delta_{m_-,m'_-} \\ \langle (j+\frac{1}{2})m_+|D(K^2, K_z)|(j+\frac{1}{2})m'_+\rangle_j &= d\left((j+\frac{1}{2})(j+\frac{3}{2}), m_+\right)\delta_{m_+,m'_+} \end{aligned}$$

and

$$-j-\frac{1}{2} \le m_+, m'_+ \le j+\frac{1}{2}, \quad -j+\frac{1}{2} \le m_-, m'_- \le j-\frac{1}{2}$$

Now, we try to calculate $\langle O(J^2, S^2, K^2, K_z)\rangle$. $\langle O(J^2, S^2, K^2, K_z)\rangle$ is not diagonal in $|km\rangle$ basis. So, we use

$$\begin{aligned} |j+\frac{1}{2}, m_+\rangle &= \sqrt{\frac{j+m_++\frac{1}{2}}{2j+1}}\,|m_+-\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{j-m_++\frac{1}{2}}{2j+1}}\,|m_+-\frac{1}{2}, -\frac{1}{2}\rangle \\ |j-\frac{1}{2}, m_-\rangle &= -\sqrt{\frac{j+m_-+\frac{1}{2}}{2j+1}}\,|m_--\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{j+m_-+\frac{1}{2}}{2j+1}}\,|m_-+\frac{1}{2}, -\frac{1}{2}\rangle \end{aligned}$$

Using these relations, we get the following expressions,

$$\langle km|O|k'm'\rangle$$
$$= \begin{pmatrix} \langle j_+,-j_+|O|j_+,-j_+\rangle & 0 & 0 & 0 \\ 0 & \langle j_+,n|O|j_+,n'\rangle & 0 & \langle j_+,n|O|j_-,n'\rangle \\ 0 & 0 & \langle j_+,j_+|O|j_+,j_+\rangle & 0 \\ 0 & \langle j_-,n|O|j_+,n'\rangle & 0 & \langle j_-,n|O|j_-,n'\rangle \end{pmatrix}$$

where $j_+ = j+\frac{1}{2}, j_- = j-\frac{1}{2}$
and

$$\langle j+\frac{1}{2},m_+|O\left(K^2,K_z\right)|j+\frac{1}{2},m'_+\rangle$$
$$= \frac{\delta_{m_+,m'_+}}{2j+1}\left[(j+m_++\frac{1}{2})o\left((m_+-\frac{1}{2}),\frac{1}{2}\right)+(j-m_++\frac{1}{2})o\left((m_++\frac{1}{2}),-\frac{1}{2}\right)\right]$$

$$\langle j+\frac{1}{2},m_+|O\left(K^2,K_z\right)|j-\frac{1}{2},m'_-\rangle$$
$$= \frac{\delta_{m_+,m'_-}}{2j+1}\sqrt{(j+m_++\frac{1}{2})(j-m_++\frac{1}{2})}\left[-o\left((m_+-\frac{1}{2}),\frac{1}{2}\right)+o\left((m_++\frac{1}{2}),-\frac{1}{2}\right)\right]$$

$$\langle j-\frac{1}{2},m_-|O\left(K^2,K_z\right)|j+\frac{1}{2},m'_+\rangle$$
$$= \frac{\delta_{m_-,m'_+}}{2j+1}\sqrt{(j-m_-+\frac{1}{2})(j+m_-+\frac{1}{2})}\left[o\left((m_-+\frac{1}{2}),-\frac{1}{2}\right)+-o\left((m_--\frac{1}{2}),\frac{1}{2}\right)\right]$$

$$\langle j-\frac{1}{2},m_-|O\left(K^2,K_z\right)|j-\frac{1}{2},m'_-\rangle$$
$$= \frac{\delta_{m_-,m'_-}}{2j+1}\left[(j-m_-+\frac{1}{2})o\left((m'_--\frac{1}{2}),\frac{1}{2}\right)+(j+m'_-+\frac{1}{2})o\left((m'_-+\frac{1}{2}),-\frac{1}{2}\right)\right]$$

Combining these together, we can write

$$\langle km|A|k'm'\rangle$$
$$= \begin{pmatrix} \langle j_+,-j_+|A|j_+,-j_+\rangle & 0 & 0 & 0 \\ 0 & \langle j_+,n|A|j_+,n'\rangle & 0 & \langle j_+,n|O|j_-,n'\rangle \\ 0 & 0 & \langle j_+,j_+|A|j_+,j_+\rangle & 0 \\ 0 & \langle j_-,n|O|j_+,n'\rangle & 0 & \langle j_-,n|A|j_-,n'\rangle \end{pmatrix}$$

Hence,

$$\det\langle km|A|k'm'\rangle$$

$$= \langle j_+, -j_+|A|j_+, -j_+\rangle\langle j_+, j_+|A|j_+, j_+\rangle \det\begin{pmatrix} \langle j_+, n|A|j_+, n'\rangle & \langle j_+, n|O|j_-, n'\rangle \\ \langle j_-, n|O|j_+, n'\rangle & \langle j_-, n|A|j_-, n'\rangle \end{pmatrix}$$

$$= \langle j_+, -j_+|A|j_+, -j_+\rangle\langle j_+, j_+|A|j_+, j_+\rangle \prod_{n=-j_-}^{j_-} \left[\langle j_+, n|A|j_+, n'\rangle\langle j_-, n|A|j_-, n'\rangle - (\langle j_+, n|O|\right.$$

$$= \left[d\left(K^2 = (j+\frac{1}{2})(j+\frac{3}{2}), m = -j-\frac{1}{2}\right) + o\left(J_z = -j, S_z = -\frac{1}{2}\right)\right]$$

$$\left[d\left(K^2 = (j+\frac{1}{2})(j+\frac{3}{2}), m = j+\frac{1}{2}\right) + o\left(J_z = -j, S_z = \frac{1}{2}\right)\right]$$

$$\prod_{n=-j+\frac{1}{2}}^{j-\frac{1}{2}} \left[\left\{d\left(K^2 = (j+\frac{1}{2})(j+\frac{3}{2}), m = n\right) + (j+n+\frac{1}{2})o\left(J_z = n-\frac{1}{2}, S_z = \frac{1}{2}\right)\right.\right.$$

$$\left. +(j-n+\frac{1}{2})o\left(J_z = n+\frac{1}{2}, S_z = -\frac{1}{2}\right)\right\}$$

$$\left\{d\left(K^2 = (j-\frac{1}{2})(j+\frac{1}{2}), m = n\right) + (j+n+\frac{1}{2})o\left(J_z = n-\frac{1}{2}, S_z = \frac{1}{2}\right)\right.$$

$$\left. +(j-n+\frac{1}{2})o\left(J_z = n+\frac{1}{2}, S_z = -\frac{1}{2}\right)\right\}$$

$$\left. -\frac{(j-n+\frac{1}{2})(j+n+\frac{1}{2})}{(2j+1)^2}\left(-o\left(J_z = n-\frac{1}{2}, S_z = \frac{1}{2}\right) + o\left(J_z = n+\frac{1}{2}, S_z = -\frac{1}{2}\right)\right)\right]$$

Calculating $\langle km|(H.\sigma)^2|k'm'\rangle_j$

$$(H.\sigma) = K^2 - J^2 - \frac{\sigma^2}{4} - 2c\sigma_z$$

so, we can assume, $D = K^2 - J^2 - \frac{\sigma^2}{4}$, $O = -2c\sigma_z$. Using the above formula we can easily see,

$$\det\langle km|((H.\sigma)|k'm'\rangle_j$$

$$= (j+c)(j-c)\prod_{n=j+\frac{1}{2}}^{j-\frac{1}{2}}\left[\left(j+\frac{-2nc}{2j+1}\right)\left(-j-1+\frac{2nc}{2j+1}\right) - \frac{4c^2(j-n+\frac{1}{2})(j+n+\frac{1}{2}}{(2j+1)^2}\right]$$

Simplifying this expression, we can write

$$\det\langle km|((H.\sigma)|k'm'\rangle_j$$

$$= (2c)(c+j)(c-j)\frac{\Gamma[-\frac{(-1+c-j)(c-j)}{2c}]}{\Gamma[-\frac{(-1+c)c+2cj+j^2}{2c}]}$$

$$= (c+j)(c-j)\prod_{i=-j+\frac{1}{2}}^{j-\frac{1}{2}}\left(-c^2 - 2ci - j(j+1)\right)$$

Calculating $\langle km|H^2|k'm'\rangle_j$

$$H^2 = J^2 + C^2 + 2c_\mu J_\mu$$

assuming $D = J^2 + C^2$ and $O \doteq 2c_\mu J_\mu$, we can calculate

$$\begin{aligned}
&\det\langle km|H^2|k'm'\rangle \\
&= (j^2 + j + c^2 - 2cj)(j^2 + j + c^2 + 2c) \\
&\prod_{n=-j+\frac{1}{2}}^{j-\frac{1}{2}} \left[\left\{j(j+1) + c^2 + \frac{4cnj}{(2j+1)}\right\}\left\{j(j+1) + c^2 + \frac{4cn(j+1)}{(2j+1)}\right\} - \frac{4c^2(j - n\frac{1}{2})(j + n + \frac{1}{2})}{(2j+1)^2}\right]
\end{aligned}$$

Simplifying this equation, we can write,

$$\begin{aligned}
\det\langle km|H^2|k'm'\rangle &= 4^{(1+2j)}c^{(2+4j)}\frac{\Gamma[\left(\frac{(c-j)^2+j}{2c} + 2j + 1\right)^2]}{\Gamma[\left(\frac{(c-j)^2+j}{2c}\right)^2]} \\
&= \prod_{i=-j}^{j}\left(c^2 + 2ic + j(j+1)\right)^2
\end{aligned}$$

Calculating $\langle km|H^2 \pm 2c|k'm'\rangle_j$

Similarly, we can also find out

$$\det\langle km|H^2 \pm 2c|k'm'\rangle = \prod_{i=-j}^{j}\left(c^2 + 2c(i \pm 1) + j(j+1)\right)^2$$

Calculating W_j

So, combining all this we can write

$$\begin{aligned}
w_j &= \frac{\left[c^2 + 2c(j + \frac{1}{2}) + j(j+1)\right]\left[c^2 - 2c(j + \frac{1}{2}) + j(j+1)\right]}{(c+j)^2(c-j)^2} \\
&\prod_{i=-j}{}^{j}\frac{[c^2 + 2c(i+1) + j(j+1)]^2[c^2 + 2c(i-1) + j(j+1)]^2}{[c^2 + c(2i+1) + j(j+1)][c^2 + c(2i-1) + j(j+1)][c^2 + 2ci + j(j+1)]} \\
&= \frac{\left[c^2 + 2c(j + \frac{1}{2}) + j(j+1)\right]\left[c^2 - 2c(j + \frac{1}{2}) + j(j+1)\right]}{(c+j)^2(j-c)^2}
\end{aligned}$$

$$\frac{[c^2+2c(j+1)+j(j+1)]^2[c^2-2c(j+1)+j(j+1)]^2}{[c^2+2cj+j(j+1)][c^2-2cj+j(j+1)]}$$
$$\prod_{i=-j}^{j}\frac{[c^2+2c(i+1)+j(j+1)]^2[c^2-2c(i+1)+j(j+1)]^2}{[c^2+c(2i+1)+j(j+1)][c^2-c(2i+1)+j(j+1)]}$$

So, we can write $w_j = \tilde{w}_j(c)\tilde{w}_j(-c)$, where

$$\tilde{w}_j(c) = \frac{[c+j+1]}{(c+j)}\frac{[c^2+2c(j+1)+j(j+1)]^2}{[c^2+2cj+j(j+1)]}\prod_{i=-j}^{j}\frac{[c^2+2c(i+1)+j(j+1)]^2}{[c^2+c(2i+1)+j(j+1)]} \tag{A4.2}$$

Chapter 5

Conclusion and Scope

In this thesis, we have studied the following issues in detail.

• High Temperature Matrix Model:-

It is not clear from the perturbative approach to string theory whether the Hagedorn Temperature is a limiting temperature or a phase transition temperature. Hence, a non-perturbative formulation is essential to understand this issue. We have studied the high temperature $SU(2)$ matrix model (a system of two D0-branes). In BFSS matrix model (0+1 d) we have calculated the partition function for this system. The leading nontrivial term of the partition function has been calculated exactly (eqn. 2.34). The non-leading terms can also be systematically calculated although we have not attempted to work them out here. From a scaling argument we have also determined the β and g dependence of the leading term for any N. We have calculated the temperature dependence of the mean square separation between two D0-branes, We find that $\langle l^2 \rangle \propto \sqrt{\frac{g}{\beta}}$ (eqn. 2.47) (true for any N), the finiteness of which shows that there must be a potential between D-0 branes that binds them. In [1, 2] also a logarithmic and attractive potential was found. The present calculation being exact in g is valid for all distances. Thus unlike in [1, 2], the (finite temperature) logarithmic potential found here is attractive at long distances and repulsive at short distances thus implying that it has a minimum at non-zero distance.

• Fuzzy Sphere and The Matrix model:-

Recently it has been shown that the fuzzy sphere in finite matrix model (IIA) corresponds to the spherical D2-brane wrapping on S^3 in string theory using the $SU(2)$ WZW model [3]. This gives us an interesting way to study the D-branes in string theory from the matrix model frame work. Earlier, D-branes in flat backgrounds have been explored within the this framework of Matrix Model. Recently other non-commutative backgrounds, for *e.g.* the fuzzy sphere have also been studied.

Non-commutative gauge theories on fuzzy spheres were obtained considering the supersymmetric three dimensional Type IIB matrix model action with a Chern Simons term [?]. The fuzzy sphere in Type IIB matrix models may correspond to the spherical D2-brane in the string theory with a background linear B-field in S^3. It is interesting to find out, in Type IIB matrix models, the object corresponding to the D2 or D0 brane in the string theory.

We have studied a general fuzzy sphere model in three dimensions, which allows a multi fuzzy sphere system with arbitrary discrete radii and arbitrary location in $\mathbf{R}^3$. [5]. We have studied the interaction as the one loop quantum effect. We have determined the one loop interaction between two and three blocks. We have calculated the potential between two points and the potential between a point and a fuzzy sphere. We have seen that for both the cases there is an attractive force for large separation and a repulsive force for small separation. We have considered a one dimensional infinite lattice of points and found out that there are stable one dimensional lattice configurations of points. A two fuzzy sphere system is also considered and the one loop potential for this system has been calculated. The potential is attractive for supersymmetric cases for small and large distance, and vanishes for infinite distances. We have also tried to see a 10 dimensional, more physical extension of this three dimensional model.

5.1 Scope for further work

• Matrix Model Partition function & Finite Temperature Matrix Model

As a natural extension of our work on the 'High Temperature $N = 2$ matrix model', it is interesting to calculate the partition function for $N = 3$ matrix model (IIB). It is found to be a non-trivial exercise, so a perturbative method may be easier, where the 5 parameters of $SU(3)/SU(2)$ are considered to be small. This system will give the $SU(3)$ partition function as a perturbation from $SU(2)$ partition function.

It is quite non-trivial to calculate the partition function for $SU(N)$ matrix model when $N > 2$ in the usual method. Moore *et al* [?] have rewritten the IIB matrix model action in the language of the Cohomological field theory and performed the path integral to get the partition function Z for a general N exactly. Similar method can be used to answer questions about the physical quantities related to the partition function of the $SU(N)$ matrix model for large N and the finite temperature $SU(N)$ matrix model.

• Fuzzy Sphere and Matrix Model

The unresolved issues in our recent work on the 'Interaction between two Fuzzy Spheres and points' like the SUSY and the BPS D2-branes can be addressed. In our model, one of the 2 SUSY transformations has a quadratic form. As a result the usual way of calculating the one loop correction is not supersymmetric. How to study the BPS D2-branes of string theory is not clear in our model. We would like to clarify this issue in the immediate future. Further generalization of this system to a more physical system in 10 dimensions may also be thought of. The issues of the stability and the effect of non-zero fermionic solutions and the addition of mass term in action in such case are also important. As an extension of our work the study the D-brane solutions in other non-trivial backgrounds for *e.g.* the $SU(3)$ fuzzy sphere and other curved back grounds such as the Kahler manifold may describe more physical situations from the point of string theory. In string theory, the $SU(3)$ model describes the dynamics of the D6-brane (as no. of casimir is 2) and the D0-brane in 8 dimension. In addition to these D-branes 7-branes also can be studied in this model. This is a non-trivial difference from the $SU(2)$ case, where only the D2-brane and D0-brane can be studied. We would like to find out the corresponding extension in the matrix model.

It is not yet clear how to study D-branes in a general curved background in the matrix model. It is important to address this issue. In the matrix model we are yet to understand the correspondence of the some of the symmetries of the string theory like the conformal symmetry, modular invariance, gauge symmetry and dualities. It is essential to look for a natural generalization of the matrix model to solve all such issues of matrix model.

The old type matrix model has a close relation to the non-critical string theory and the conformal field theory. We can expect the matrix model to relate also to some kind of conformal field theory. May be we are close to the unification of the old type matrix model and the present matrix model which may give a hint to natural generalization of the matrix model.

There is some hope for a background independent way of studying the matrix model following topological field theory and the K-theory. It will be interesting to address these issues some time in the future.

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