COURSEWORK AND SYLLABI FOR THE PH.D. AND INTEGRATED PH.D. PROGRAMS IN MATHEMATICS AT IMSC

All courses in mathematics carry 4 credits except for the research methodology course and the publication ethics course. The Ph.D. program at IMSc requires a total of 7 courses excluding the research methodology and publication ethics courses for a total of 28 credits. The I.Ph.D. program requires 12 courses excluding the research methodology and publication ethics courses along with a 24 credit master's thesis for a total of 72 credits.

The following four courses are compulsory for both the Ph.D. program and the I.Ph.D. program: Algebra-I, Analysis-I, Complex Analysis, Topology-I; and at least two out of these three: Algebra-II, Analysis-II, and Topology-II. The research methodology and publications ethics courses are also compulsory.

The other courses are chosen from among the courses listed below along with other courses offered from time to time either at IMSc or at other institutes with which HBNI has an MoU such as CMI. Topics courses in mathematics may be repeated for credit and will be shown on the transcript with suffixes such as I, II, III etc. A credit seminar on a suitable topic may substitute any of the non-compulsory courses.

1. Algebra I - 4 credits

Group theory

- Group actions: Orbits, stabilisers, transitivity
- Lagrange, Cauchy, Sylow theorems in the language of group actions
- Direct and semidirect products
- symmetric and alternating groups

Matrices, determinants and linear maps

- Linear maps and matrices, dual = transpose
- determinants
- Equality of row, column and determinantal rank over a commutative ring

Representations of a single endomorphism

- Minimal and characteristic polynomials, eigenvalues and eigenvectors
- Rational and Jordan canonical forms
- S-N decomposition

Bilinear forms and spectral theorems

- Preliminaries and quadratic maps
- Symmetric forms, orthogonal basis, Sylvester's theorem
- Hermitian forms, polarization, Cauchy-Schwarz inequality
- Spectral theorems, polar decomposition

Basic category theory

- Categories and functors
- Universal properties
- Sums, products and limits

Rings and modules over a PID

- Finitely generated abelian groups
- PID $\Rightarrow$ UFD, R UFD $\Rightarrow$ $R[X]$ UFD, Gauss' lemma
- Irreducibility criteria
- Modules over a PID

Tensor products

- Of vector spaces, modules over a ring, basic properties
- connection with Hom, of algebras
- tensor, symmetric and exterior algebras and connection with the determinant

2. Algebra II - 4 credits

Group theory

- simple, solvable and nilpotent groups
- Jordan-Holder theorem

Galois theory

- Finite extensions, algebraic extensions, algebraic closure
- Splitting fields and normal extensions
- separable extensions
- Finite fields
- Inseparable extensions
- Galois extensions
- Examples and applications
- Cyclotomic fields
- Independence of characters, norm and trace
- Cyclic extensions
- Solvable and radical extensions

Instructor's choice from among the following suggested topics or others.

Semisimplicity

- Schur's lemma and semisimple modules
- Jacobson density theorem, DCT
- Structure of semisimple rings
- Structure of simple rings

Representations of finite groups

- Maschke's theorem
- Characters
- Class functions
- Orthogonality relations

Commutative algebra and Dedekind domains

- Prime, maximal ideals, Zariski topology, CRT
- Localization and its properties
- Integral extensions
- Dedekind domains - characterizations
- Unique factorisation - failure and restoration

3. Analysis I - 4 credits

Measure Theory

- Measurable spaces, Caratheodory's theorem and construction of measures, Lebesgue measure, Riesz representation theorem for compact metric spaces
- Measurable mappings, various convergence concepts like almost sure, convergence in measure.
- Integration, MCT, DCT.
- Product measures, Fubini's theorem.
- Radon-Nikodym theorem, Lebesgue decomposition theorem.
- L^p spaces: Basic theory, Holder's inequality, Minkowsky inequality, completeness, their duality.
- (*) Analysis on $\mathbb{R}^n$; convolutions; approximate identity; approximation theorems; Fourier transform; Fourier inversion formula; Plancherel theorem.

Note: Topics marked with asterisk are optional.

4. Analysis II - 4 credits

Elementary functional analysis

- Topological vector spaces; Banach spaces; Hilbert spaces.
- bounded linear transformation;linear functionals and dual spaces.
- Hahn Banach theorem and it's geometric meaning.
- Category theoem and it's applications like open mapping theorem, uniform boundedness principle, closed graph theorem.
- Weak and Weak-* topologies, Banach-Alaoglu's theorem.

Instructor's choice from among the following suggested topics or others.

Distribution Theory

- The spaces $D(\Omega), E(\Omega)$, for Ω open in $I\!R^n$.
- $S(\mathbb{R}^n)$ and their duals, convolution, Fourier transform.
- Paley-Wiener theorems; fundamental solutions of constant coefficient partial differential operators.

Banach Algebras and Spectral Theory

- Banach algebras, spectrum of a Banach algebra element, Holomorphic functional calculus, Gelfand theory of commutative Banach algebras.
- Hilbert space operators, C^*-algebras of operators, commutative C^*-algebras.
- Spectral theorem for bounded self-adjoint and normal operators. formulation).
- Spectral theorem for compact operators,(*) application to Peter-Weyl theorem.

Note: Topics marked with asterisk are optional.

5. Topology I - 4 credits

Point-set topology

- Quotient topology including the construction of standard topological spaces such as surfaces and real and complex projective spaces as quotient spaces
- The notion of attachment of a cell to a topological space
- Group actions and orbit spaces
- Topologies on function spaces
- Baire category theorem
- Arzela-Ascoli theorem

Fundamental groups and covering spaces

- Fundamental groups, covering spaces and their relationship
- Free groups, free products of groups
- Seifert-van Kampen theorem - examples and applications

Introduction to homology

- Definition of homology groups
- Homotopy invariance of homology groups
- The first homology group as the abelianization of the fundamental group
- Review of homological algebra necessary to introduce the Mayer-Vietoris sequence
- Mayer-Vietoris sequence and its applications in computing homology groups of surfaces, complex projective spaces, real projective spaces etc.

Applications of fundamental groups and homology groups

Instructors choice among the following topics or other topics at this level.

- Jordan curve theorem
- Winding number of a closed curve
- Brouwer's fixed point theorem
- Fundamental theorem of algebra
- Nielsen-Schreier theorem

6. Topology II - 4 credits

Instructors choice among the following topics. It is suggested that one topic from the first two and basic notions of differential topology be covered in addition to some of the advanced topics.

Homology theory

- Quick review of homology theory
- Relative homology and the associated long exact sequence
- Excision theorem and its applications
- Characterisation of homology theory by the Eilenberg-Steenrod axioms
- Homology with coefficients

Cohomology theory and introduction to homotopy groups

- Basic notions of cohomology
- Universal coefficient theorem
- Künneth formula
- Cup product and the cohomology ring, Borsuk-Ulam theorem

Basic notions of differential topology

- Differentiable manifolds, tangent bundle, vector fields, flows
- Differential forms and de Rham cohomology
- Integration on manifolds
- Stokes theorem
- Poincaré duality using differential forms.

Advanced topics

- Higher homotopy groups and the Hurewicz theorem
- H-spaces, suspensions, fibre bundles
- Cap product and various forms of duality with integral coefficients
- Bott periodicity theorem
- Topics in differential geometry such as:
  - Smooth vector bundles
  - Notions of connection, curvature and parallel transport
  - Definition of Riemannian manifold
  - Gauss-Bonnet formula
  - Notion of geodesic and Hopf-Rinow theorem
- Obstruction theory and introduction to characteristic classes:
- Topics in Morse theory such as:
  - Definition and genericity of Morse functions
  - Lemma of Morse
  - Cell structure associated to a Morse function and Morse homology
  - Morse-Smale-Witten complex

7. Complex Analysis - 4 credits

- Analytic function, Cauchy-Riemann equations, power series, exponential and logarithmic function
- Cauchy theorem on a disc, Integral formula, power series and Laurent series expansion Product development, Weierstrass theorem, Homotopy version of Cauchy's theorem, Liouville's theorem, residue theorem, Argument principle
- Maximum modulus principle, Schwarz lemma, Phragmen-Lindelof method
- Conformal mapping, Mobius transformation, Automorphisms of the disc and upper half plane, Riemann mapping theorem
- Harmonic functions, Dirichlet problem, Mean value property
- Analytic continuation, Monodromy theorem
- (optional) Introduction to Hyperbolic geometry
- (optional) Elliptic functions, Gamma and Zeta functions

8. Credit seminar - 4 credits

The topic of the seminar will be chosen by the student in consultation with an assigned faculty member. The student will read recent research papers as assigned by the faculty member, and present the results in a formal seminar.

9. Research Methodology - 2 credits

An introduction to the methods and techniques of academic research through a project and presentations - both oral and written.

10. Research and Publication Ethics - 1 credit

An introduction to the philosophy of science and ethics, research integrity and identification of research misconduct.

The following courses are advanced level research courses whose content will be decided by the instructor based on current research and the requirements of the individual students.

11. Topics in analytic number theory - 4 credits

Course content varies according to instructor's choice. Possibilities for topics are: introduction to arithmetic functions, convolution and Mobius inversion formula, basic asymptotic formulas for arithmetic functions, characters and Fourier analysis on finite abelian groups, theory of Dirichlet series, primes in arithmetic progression, Riemann zeta function, Poisson summation and functional equation, The prime number theorem, error term in prime number theorem, its oscillation and the Riemann hypothesis, equivalent formulations of Riemann hypothesis, zero-free regions, explicit formula and Siegel's theorem, introduction to sieve methods, Brun and Selberg sieve, large sieve and the Bombieri-Vinogradov theorem and Vinogradov's three prime Theorem.

12. Topics in algebraic number theory - 4 credits

Course content varies according to instructor's choice. Possibilities for topics are: Dedekind domains, ramification, different and discriminants, decomposition and inertia groups, quadratic fields and genus theory, classification of primitive quadratic characters, Gauss sums and quadratic reciprocity, geometry of numbers, finiteness of class number and explicit computations, regulators and Dirichlet's unit theorem, cyclotomic fields and inverse Galois problem for abelian number fields, Artin symbol and splitting in cyclotomic fields, Dedekind zeta function, the analytic class number formula and introduction to the Chebotarev density theorem

13. Topics in commutative algebra - 4 credits

Course content varies according to instructor's choice. One possibility is a course covering the second half of Matsumura's text including topics such as: regular sequences, Koszul complex, Cohen-Macaulay rings, Gorenstein rings, regular rings, UFDs, complete intersections, local flatness criterion, generic freeness, derivations and differentials, separability, I-smoothness, Cohen's structure theorems and applications of complete local rings.

14. Topics in modular forms - 4 credits

Course content varies according to instructor's choice. Possibilities include: Introduction to $SL_2(\mathbb{R})$ and its action on the Poincare upper half-plane $\mathcal{H}$, discrete subgroups Γ of $SL_2(\mathbb{R})$ and their cusps, the modular group $SL_2(\mathbb{Z})$, Topology, measure theory and complex structure on $\mathcal{H}/\Gamma$ and its compactification, Modular functions, modular forms and cusp forms on $SL_2(\mathbb{Z})$, examples : Eisenstein Series and the delta Function, finite dimensionality of space of modular forms, the Miller basis and the $\mathbb{Z}$-structure on the space of modular forms, growth of Fourier coefficients of cusp forms, introduction to Ramanujan's conjectures, theory of Hecke operators and Petersson inner-product on the space of cusp forms, application to Ramanujan's conjectures, the L-function of modular forms, congruence subgroups, modular forms and cusp forms on congruence subgroups, spectral theory of automorphic forms, introduction to Galois representations and Deligne's theorem, Lehmer's conjecture and the Atkin-Serre Conjecture.

15. Topics in elliptic curves - 4 credits

Course content varies according to instructor's choice. Possibilities for material to be covered include selected topics from Elliptic functions by Lang, The arithmetic of Elliptic curves by Silverman, Elliptic curves by Milne or Elliptic curves by Husemoller. Another possibility would be to prove Mazur's theorem which is a well-known and important result covering elliptic curves and abelian varieties, and the moduli of elliptic curves.

16. Topics in algebraic curves - 4 credits

Course content varies according to instructor's choice. Possibilities for material to be covered include selected topics from An invitation to arithmetic geometry by Lorenzini, Algebraic Curves by Fulton or lectures notes of Joseph Oesterle. Topics such as the basics of algebraic varieties over the complex numbers (with focus on dimension 1), singularities of curves (what are they and when is a curve nonsingular), desingularization of curves by normalization, the relationship between nonsingular algebraic curves and complex manifolds of dimension 1, nonsingular projective algebraic curves and function fields, the Riemann-Roch theorem, and also some of its applications.

17. Topics in diophantine geometry - 4 credits

Course content varies according to instructor's choice. Some possibilities are: introduction to Global fields, absolute values on global fields, theory of heights, rational points on conics. local-global principle and application to quadratic forms, affine and projective varieties, morphisms and rational maps, explicit arithmetic on function fields and their zeta-functions, divisors on curves, The Riemann-Roch theorem, elliptic curves over global fields, endomorphism rings of Elliptic curves, CM and non-CM curves, ehe Mordell-Weil group and rank of an elliptic curve and local-global principle on elliptic curves and the Tate-Shaferevich group.

18. Topics in transcendental number theory - 4 credits

Course content varies according to instructor's choice. Possibilities are: Liouville's theorem and Liouville Numbers, elements of rational approximation, transcendence of e and π, irrationality of $\zeta(3)$, introduction to algebraic independence, Lindemann-Weierstrass theorem, Schanuel's conjecture and Ax's theorem for formal power series, the Schneider-Lang Theorem, Hilbert's seventh problem and the Gelfond-Schneider theorem, Baker's Theorem and applications, six exponential theorem, introduction to heights and Roth's Theorem, the p-adic Baker theorem (by Brumer) and introduction to Leopoldt's conjecture and the p-adic subspace theorem and applications.

19. Topics in algebraic groups - 4 credits

Course content varies according to instructor's choice. One possibility is to cover the basic theory of linear algebraic groups over an algebraically closed field up to the classification of the reductive groups by means of root data, developing the necessary background from algebraic geometry as and when needed. Thus covering preliminaries from algebraic geometry, linear algebraic groups: definition and first properties, commutative algebraic groups, derivations, differentials, and Lie algebras, topological properties of morphisms applied to this context, Parabolic subgroups, Borel subgroups, and solvable subgroups, Weyl group, roots, and root datum and reductive groups and their classification: isomorphism and existence theorems.

20. Topics in infinite dimensional Lie algebras - 4 credits

Course content varies according to instructor's choice. Some possibilities are: generalized Cartan matrices and their associated Lie algebras, symmetrizability, the invariant bilinear form, the Weyl group, classification of indecomposable GCMs, finite, affine and indefinite types, affine Kac-Moody algebras, roots, the affine Weyl group, realizations of untwisted and twisted affine Kac-Moody algebras in terms of loop algebras, representation theory: integrable representations, category $\mathcal{O}$, proof of the Weyl-Kac character formula, highest weight integrable representations, weights, representations of affine Kac-Moody algebras.

21. Topics in functional analysis - 4 credits

Course content varies according to instructor's choice. Some possibilities are: analytic Fredholm theory, compact and Fredholm operators, Atkinson's theorem, Gelfand duality, properties of the analytic index, Toeplitz operators on Hardy spaces, Pseudo-differential operators and Elliptic regularity, Fourier transforms and Sobolev spaces on R^n, Symbol calculus and Pseudo-differential operators, Ellipticity and Pseudo-differential operators on smooth manifolds, construction of parametrices, Elliptic regularity theorem, Ellipticity and Fredholm property of Dirac operators on closed manifolds.

22. Topics in non-commutative geometry - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Vector bundles, K-theory for topological spaces, Serre Swan theorem, K_0 and $K_1$for a C*-algebra, homotopy invariance, split exactness, half-exactness, stability of K-theory, inductive limits and K-theory, Bott periodicity, Six term exact sequences, computations with them, Pimsner-Voiculescu exact sequence, Thom isomorphism, Hilbert C*-modules, KK groups, Geometric index theory, Vector bundles, connections and curvature on Riemannian manifolds, structure equations of Cartan, invariant forms and characteristic classes in de Rham cohomology, Chern-Gauss-Bonnet theorem and idea of proof, topological index and statement of the Atiyah-Singer index theorem

23. Topics in Lie groups - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Introduction to Lie algebras, definitions, examples, abelian, nilpotent, solvable lie algebras, semisimple lie algebras, representation of Lie algebras, structure of general Lie algebras over characteristic zero field : statement of the Levi decomposition, statement of Ado's theorem, Introduction to real differentiable manifolds, and various standard objects associated with it, statement of the Frobenius theorem on integrability, definition of real Lie groups, examples, associated Lie algebra, the exponential map and its properties, closed subgroup theorem, continuous homomorphisms, definition of Lie subgroups and examples, association of lie subgroups and lie subalgebras, covering Lie groups, simply connected lie groups and association with real Lie algebras, the adjoint representation, the manifold structure of the left or right coset space with respect to a closed subgroup and the (subgroup)-principal bundle structure of the Lie group with respect to the projection to the coset space, Construction of left invariant Haar measure using left invariant differential forms, formula for modular function, compact Lie groups, Peter-Weyl theorem, embedding compact groups in linear Lie groups, Weyl group, conjugacy of maximal tori in connected compact Lie groups, Centralizers of tori, basic structure of semisimple Lie groups, existence of compact real forms of complex semi-simple Lie algebras, Cartan decomposition both at the Lie algebra and Lie group level, Iwasawa decomposition.

24. Topics in algebraic geometry - 4 credits

Course content varies according to instructor's choice. One possibility is an introduction to the language of schemes, properties of morphisms, and sheaf cohomology. So that the students gain an understanding of the basic notions and techniques of modern algebraic geometry.

25. Topics in differential geometry - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Definition of smooth manifolds, atlas, examples, tangent spaces, inverse and implicit functon theorems for manifolds, vector fields, flow, completeness of the flow function, integrability and Frobenius theorem, differential forms, pullback by functions, exterior derivative, orientations, manifolds with boundary, Stokes theorem, DeRham cohomology, computations using Mayer Vietoris, Riemannian metrics and geodesics.

26. Topics in partial differential equations - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Examples of partial differential equations, Strategies for studying PDE., Well posed problem, Brief introduction to classical solutions, weak solution and regularity, Transport equation, Laplace's equation, Heat equation and wave equation, Problems associated to these equations, notion of fundamental solution etc., Non-linear first order PDE, Hamilton Jacobi equations, calculus of variations, Hamilton's ode, Legendre transforms, etc., Theory of linear partial differential equations: Sobolev spaces, weak derivative, Sobolev inequalities, Elliptic equations, Weak solutions, the existence of weak solutions, regularity, maximum principles, eigenvalues and eigenfunctions of elliptic operators, compactness, etc.

27. Topics in mathematical physics - 4 credits

Course content varies according to instructor's choice. One possibility is to cover classical and quantum mechanics covering topics such as: review of Galilean group, mechanical system with one degree of freedom, mechanical system consisting of motion of a point in three dimensional space and motion of system of n points, review of calculus of variation, Lagrange's equation, Hamilton's equations, Liouville's theorem, Symplectic structures on phase spaces and Noether's theorem, D'Alembert's principle, Symplectic manifolds, Hamiltonian mechanics on symplectic manifolds, moment map, postulates of quantum mechanics, mathematical aspects of Schrödinger's equation, review of Lie group, Lie algebra and their representations with main focus on groups like $U(1)$, $SO(3)$, $SU(2)$, Spin groups in 3 and 4 dimensions, Spin $\frac{1}{2}$ particle in magnetic field, review of Fourier transforms, position and momentum space, Dirac notation, Heisenberg's uncertainty principle, Hydrogen atom, quantization, canonical quantization, The Groenewold-van Hove no-go theorem, canonical quantization in n–dimensions, quantization and symmetries.

28. Topics in algebra - 4 credits

Course content varies according to instructor's choice. One possibility is a course in commutative algebra covering prime ideals and maximal ideals, nilradical and jacobson radical, prime avoidance and the Chinese remainder theorem, extension and contraction of ideals, modules, submodules and quotient modules, direct sum and direct product, finitely generated modules and Nakayama lemma, exact sequences, tensor products, restriction and extension of scalars, exactness properties of the tensor product, algebras, tensor product of algebras, localization, local properties, extended and contracted ideals in rings of fractions, primary decomposition, integral extensions, lying over, going-up theorems, integrally closed domains and the going-down theorem, valuation rings, Noetherian and Artinian modules, Noetherian rings, Hilbert basis theorem, primary decomposition in Noetherian rings, Artinian rings and their structure, discrete valuation rings and Dedekind domains, fractional ideals, completions, filtrations, topologies, and completions, graded rings and modules, associated graded ring, dimension theory, Hilbert functions, dimension theory of Noetherian local rings, regular local rings, transcendental dimension, relation to algebraic varieties and algebraic geometry.

29. Topics in operator algebras - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Banach algebras, spectrum, spectral radius formula, C*-algebras, Gelfand Naimark theorem, continuous functional calculus, GNS construction, positivity, measurable functional calculus, von Neumann algebras, Kaplansky density theorem, double commutant theorem, finite-dimensional C*-algebras, representation theory of the C*-algebra of compact operators, Toeplitz algebra, Coburn's theorem, group C*-algebras, crossed products, amenability, groupoid C*-algebras.

30. Topics in representation theory - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Lie algebras: definition and basic properties, ideals, subalgebras, homomorphisms, nilpotent and solvable Lie algebras, Lie's and Engel's theorems, semisimple Lie algebras, the Killing form, Cartan's criterion, abstract Jordan decomposition, classification of finite dimensional semisimple Lie algebras, Dynkin diagrams, the Weyl group, isomorphism and conjugacy theorems, representations, Verma modules, category $\mathcal{O}$, irreducible highest weight modules, complete reducibility, Weyl character formula, Freudenthal weight multiplicity formula, Kostant and Steinberg formulas.

31. Topics in algebraic combinatorics - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Partially ordered sets and Mobius inversion, generating functions, permutations and statistics, Robinson-Schensted correpondence, partitions, Young's lattice, hook-length formula, Representation theory of symmetric groups, similarity classes of matrices and orthogonal polynomials

32. Topics in topology - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Definitions and basic construction of homotopy groups, Whitehead's theorem, Hurewicz's theorem, stable homotopy groups, fibrations and obstruction theory, Bott's periodicity theorem, H-cobordism theorem, construction and applications of characteristic classes.

33. Topics in symplectic geometry - 4 credits

Course content varies according to instructor's choice. Possibilities include: Motivations of sympletic Geometry from Hamiltonian mechanics, neighbourhood theorems, compatible almost complex structure, and the contractibility of the space of almost complex structures, integrability of almost complex structures, Newlander-Nirenberg theorem, Hamiltonian circle actions on symplectic manifolds, moment maps, Fubini-Studi form on projective space, Kähler forms as Hessians of plurisubharmonic function on complex manifolds, introduction to pseudoholomorphic curves, outline of proof of Gromov's non-squeezing theorem.

34. Programming for mathematicians - 4 credits

Course content varies according to instructor's choice. Some possibilities are: Basic python syntax, Iterables and generators, Object oriented programming, introduction to Sage, the Numpy library, the Networkx library, graphics with Sage and Matplotlib and a Programming project.