Lecture-3: Matrix exponential.

$$\exp : M_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$$

$$\exp : M_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$$

Q: Suppose that $V \subset M_n(\mathbb{R})$ (or $M_n(\mathbb{C})$) is a real (or complex) vector space. Is $\exp(V) \subset GL_n(\mathbb{R})$ (or $GL_n(\mathbb{C})$) a subgroup?

Ans: No, in general.

Yes, if V is a Lie subalgebra of $M_n(\mathbb{R})$ (or $M_n(\mathbb{C})$).

If $X, Y \in M_n(\mathbb{R})$, $[X,Y] := XY - YX$.

V is a Lie subalg of $M_n(\mathbb{R})$ if $X, Y \in V \Rightarrow [X,Y] \in V$.

Example: (1) $V = \{ X \in M_n(\mathbb{R}) \mid \operatorname{tr} X = 0 \}$.

If $X, Y \in V$, then $\operatorname{tr}(X) = 0 = \operatorname{tr}(Y)$.

$$\begin{aligned} \operatorname{tr}([X,Y]) &= \operatorname{tr}(XY - YX) \\ &= \operatorname{tr}(XY) - \operatorname{tr}(YX) \\ &= \operatorname{tr}(XY) - \operatorname{tr}(XY) = 0. \end{aligned}$$

$[X,Y] \in V$.

$\exp(V) = SL_n(\mathbb{R})$.

② $V = \{ X \in M_n(\mathbb{R}) \mid X^t = -X \}$.

Suppose $X, Y \in V$. $X^t = -X, Y^t = -Y$

$$[X,Y]^t = (XY - YX)^t$$
$$= (XY)^t - (YX)^t$$
$$= Y^t X^t - X^t Y^t$$
$$= YX - XY = -[X,Y]$$

V is a Lie alg under $[\ ,\ ]$.

• $\exp(V) = SO(n)$.

$$e^X \cdot e^{X^t} = e^{X+X^t} = e^0 = I$$

Suppose $A \in SO(n)$.

Then $\exists P \in SO(n)$ s.t.

$$PAP^{-1} = \begin{pmatrix} R(t_1) & & & \\ & \ddots & & \\ & & R(t_m) & \\ & & & 1 \end{pmatrix} \qquad n = \begin{cases} 2m \\ 2m+1 \end{cases}$$

$$X = \begin{pmatrix} \begin{matrix} & t_1 \\ -t_1 & \end{matrix} & & & \\ & \ddots & & \\ & & \begin{matrix} & t_n \\ -t_n & \end{matrix} & \\ & & & 0 \end{pmatrix} \in V.$$

$$e^X = PAP^{-1}$$

$$e^{\underbrace{P X P^{-1}}_{\in V = \text{skew}}} = A.$$

$PXP^{-1} = PXP^t$.

$(PXP^{-1})^t = PX^tP^t$
$= -PXP^{-1}$

$\begin{pmatrix} 0 & * \\ -* & 0 \end{pmatrix}$

Cor: $\dim SO(n) = \dim_{\mathbb{R}} Sk_n$.
$= \binom{n}{2}$

$\dim SL_n(\mathbb{R}) = \dim M_n^0(\mathbb{R})$
$= n^2 - 1$

$\dim U(n) = n^2 = \dim_{\mathbb{R}} Sh_n$

skew-Herm. matrices.

$\begin{pmatrix} ix_1 & * \\ -\bar{*} & \ddots & \\ & & ix_n \end{pmatrix}$ $\in \mathbb{C}$

$2\binom{n}{2} = \cancel{2}\frac{n(n-1)}{\cancel{2}} = n(n-1)$

$\vdots$

What about symm matrices $/ \mathbb{R}$?

$Sy_n \subset M_n(\mathbb{R})$.

$A, B \in Sy_n$. $A^t = A, B^t = B$.

$[A,B]^t = (AB - BA)^t$

$= B^t A^t - A^t B^t = BA - AB$.

symmetric?

$= -[A,B]$.

NOT symm if $AB \neq BA$.

$\Rightarrow Sy_n$ is NOT a Lie algebra.

$\begin{pmatrix} 2 & \\ & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & \\ & 3 \end{pmatrix}$.

- Suppose that A is a symm. matrix in $M_n(\mathbb{R})$.

$B = e^A$ is symm. positive definite matrix.

Recall: we say that a symm. matrix$/\mathbb{R}$ B is pos. def. if $\forall\, x \in \mathbb{R}^n$

$$x^t B x > 0, \text{ if } x \neq 0.$$

- If $A \in M_n(\mathbb{R})$ is symm., then $\exists$ an orthogonal $P \in SO(n)$ st.

$$PAP^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in M_n(\mathbb{R}).$$

$$Pe^AP^{-1} = e^{PAP^{-1}} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} \longleftarrow \text{Positive definite}$$

$$Pe^AP^{-1} = PBP^{-1} = PBP^t.$$

If $x \in \mathbb{R}^n$. $\rightsquigarrow$ $y = P^t x$, $y^t = x^t P$

$${}^t y B y = {}^t x \underbrace{PBP^t}\, x = x^t \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} x.$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \rightsquigarrow \quad = \sum_{j=1}^{n} e^{\lambda_j} x_j^2 \geqslant 0$$

If A is symm., then $B = e^A$ is pos. def.

Conversely if B = symm & pos. def.

$\exists Q \in SO(n)$ $\quad QBQ^t = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$, $b_j > 0$

$$= \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}, \quad \lambda_j = \log b_j$$

$$B = Q^{-1} e^{\Lambda} Q$$

$$= e^{\underbrace{Q^T \Lambda Q}} \quad \text{symm.}$$

$$\{\text{Symm matrices}\} \xrightarrow[\exp]{\cong} \{\text{pos. def.}\} \subset GL_n$$

Th: Any $X \in SL_n(\mathbb{R})$ can be expressed uniquely as $X = k \cdot \exp(A)$, where $k \in SO(n)$ $A \in Sy_n$. This decomp. is unique.

In fact $\quad SL_n(\mathbb{R}) \longrightarrow SO(n) \times \exp(Sy_n)$.

$\quad X \longrightarrow (k \times \exp(A))$

is a diffeo.

Ex: $X \in M_n(\mathbb{R})$ can be expressed uniquely as $X = \left(\frac{X - X^t}{2}\right) + \left(\frac{X + X^t}{2}\right)$

Solving diff'l eqns:

Suppose $x(t) \in \mathbb{R}^n$, $t \in \mathbb{R}$.

$x(t) = (x_1(t), \ldots, x_n(t))^t$

$x'(t) = (x_1'(t), \ldots, x_n'(t))$

$$x'(t) = A \cdot x(t), \quad A \in M_n(\mathbb{R})$$

Eg: $x_1'(t) = 2x_1(t) + x_2(t)$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $x_2'(t) = x_1(t) + x_2(t).$

Th: The sys. of diff'l eqns

$$x'(t) = A x(t)$$

has solution space equal to the column span of the matrix e^{tA}.

Pf: Recall: $\boxed{\frac{d}{dt}(e^{tA}) = A \cdot e^{tA}}$.

Write $e^{tA} = (x_{ij}(t))$. The $\frac{d}{dt} e^{tA} = (x_{ij}'(t))$

$$v_j'(t) = \begin{pmatrix} x_{1j}'(t) \\ \vdots \\ x_{nj}'(t) \end{pmatrix} = A \cdot \begin{pmatrix} x_{1j}(t) \\ \vdots \\ x_{nj}(t) \end{pmatrix} \leftarrow v_j(t).$$

$\Rightarrow$ The j^{th} column satisfies

$$x'(t) = A \cdot x(t).$$

If $a(t)$, $b(t)$ are sol^{n}s

the so are $c\,a(t)$ & $a(t)+b(t)$

Any sol^{n} of $x'(t) = A\,x(t)$

is a lin. combination ~~of~~

$c_1 v_1(t) + \cdots + c_n v_n(t)$.