Matrix exponential : Lecture 2.

Recall: $\exp : M_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$

$\exp : M_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$,

defined as $X \longmapsto \sum_{k=0}^{\infty} \frac{X^k}{k!} = e^X$.

For $A \in M_n(\mathbb{C})$ $\frac{d}{dt} e^{tA} = A e^{tX}$.

. $e^{PXP^{-1}} = P e^X P^{-1}$.

. If $AB = BA$, then $e^A \cdot e^B = e^{A+B}$.

Eg: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, compute e^A.

$A^2 - 4A + 3 = 0$.

$A^2 = 4A - 3$

$A^3 = \cdots$

$A^2 - 4A + 3 = (A-3)(A-1)$

$\therefore$ The eigenvalues of A are $3, 1$.

Solve: $Ax = 3x$,

Take $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$Ay = y$.

$y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore$ Take $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. $\therefore P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$PAP^{-1} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}3 & 1\\ 3 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}6 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$$

$$e^{PAP^{-1}} = e^{\begin{pmatrix}3 & \\ & 1\end{pmatrix}} = \begin{pmatrix}e^3 & \\ & e^1\end{pmatrix}$$

$$\|$$

$$\therefore\ Pe^{A}P^{-1} = \begin{pmatrix}e^3 & \\ & e\end{pmatrix}$$

$$\therefore\quad e^{A} = P^{-1}\begin{pmatrix}e^3 & \\ & e\end{pmatrix}P.$$

$$= \begin{pmatrix}\frac{e^3+e}{2} & \frac{e^3-e}{2}\\ \frac{e^3-e}{2} & \frac{e^3+e}{2}\end{pmatrix}.$$

- Suppose that $A = \lambda I + B$ where B is strictly upper Δ^{lr}, i.e, $B = \begin{pmatrix}0 & & * \\ & \ddots & \\ 0 & & 0\end{pmatrix}$.

Then $B^n = 0$. Also $\lambda I \cdot B = B \cdot \lambda I$.

$$\therefore\quad e^{A} = e^{\lambda I + B} = e^{\lambda I}\cdot e^{B}$$

$$= \begin{pmatrix}e^{\lambda} & & 0\\ & \ddots & \\ 0 & & e^{\lambda}\end{pmatrix}\cdot\left(I + B + \frac{B^2}{2!} + \cdots + \frac{B^{n-1}}{(n-1)!}\right)$$

$\exp : M_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$.

- The map exp is a <u>local diffeo</u> around $0 \in M_n(\mathbb{R})$.

  i.e $\exists$ open set $U \ni 0$ in $M_n(\mathbb{R})$

  $I \in V \subset GL_n(\mathbb{R})$, V open st.

$U \to V$, $\quad X \longmapsto e^X$ is 1-1, onto,

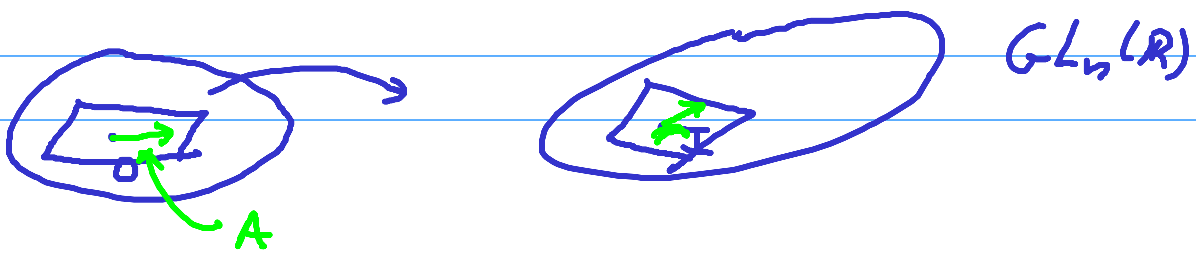
and has an inverse which is also smooth.

In fact, the inverse map $V \to U$ is denoted $\log : V \to U$.

$$\log(I + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \cdots$$

For the proof we need only show that the Jacobian matrix of exp at 0 is non-singular.

$$\begin{array}{cccc} \left.\dfrac{\partial}{\partial x_{ij}}\right|_0 & T_0 M_n(\mathbb{R}) & \xrightarrow{d\exp} & T_I GL_n(\mathbb{R}) \\ \downarrow & \cong & & \cong \\ e_{ij} \in & M_n(\mathbb{R}) & & M_n(\mathbb{R}) \end{array}$$



$$(-1,1) \to U \xrightarrow{\exp} \quad t \longmapsto e^{tA}.$$
$$t \longmapsto tA.$$

$$d\exp|_0(A) = \frac{d}{dt}\Big|_{t=0} (e^{tA})$$
$$= Ae^{tA}\Big|_{t=0}$$
$$= A.$$

$d\exp|_0$ is an iso. of (real) vector spaces.

$\Leftrightarrow$ $\mathrm{Jac}(\exp) \neq 0$.

$\Rightarrow$ exp is a loc. diffeo.

<u>Remark</u>: Exp is <u>NOT</u> 1-1.

$$\exp\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Put $t = 2\pi$. $\underline{\exp\begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix}} = I = \exp(0)$

- $(e^A)^t = e^{A^t}$.

- We want to analyse the exp map when it is restricted to following vector subspace of $M_n(\mathbb{C})$, $M_n(\mathbb{R})$.

1. Space of all skew-symm matrices $Sk_n(\mathbb{R})$, $Sk_n(\mathbb{R})$

2. Space of skew-Herm. matrices
$sh_n \subset M_n(\mathbb{R})$.

3. {Symm. matrices}, {Herm. matrices}.

1. Recall: X is skew-symm if $X^t = -X$.
$Sk_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is a vector subspace.

Note that X^t & X commute if X is skew-symm.

$$\therefore (e^X)^t \cdot e^X = e^{X^t} \cdot e^X = e^{X^t + X}$$
$$= e^0 = I.$$

$$\Rightarrow e^X =: A \Longrightarrow A^t A = I \text{ i.e. } A \in O(n).$$

& we obtain $\exp: sk_n(\mathbb{R}) \longrightarrow O(n)$.

The image is precisely $SO(n) = \{A \in M_n(\mathbb{R}) \mid AA^t = I, \det A = 1\}$, which is group.

If $X = \begin{pmatrix} \begin{matrix} 0 & t_1 \\ -t_1 & 0 \end{matrix} & 0 \\ 0 & \ddots & \\ & & \begin{matrix} 0 & t_m \\ -t_m & 0 \end{matrix} \end{pmatrix}$, $n = 2m$, $2m+1$

$$\exp(X) = \begin{pmatrix} R(t_1) & & \\ & \ddots & \\ & & R(t_n) \end{pmatrix}_{\!\!+1}, \quad R(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in S^1$$
$$\in S^1 \times \cdots \times S^1$$

$$\begin{array}{c} M_n(\mathbb{C}) \\ \cup \\ X \in Sh_n \iff \bar{X}^t = -X \end{array}$$

In this case $\exp(X)$ is a unitary matrix. ie, $\bar{A}^t \cdot A = I$, where $A = \exp(X)$.

$U(n) = \{ A \mid \bar{A}^t \cdot A = I \}$.

This is a group.

Let $M_n^0(\mathbb{R}) = \{ A \mid \operatorname{tr} A = 0 \} \subset M_n(\mathbb{R})$
Vec Subspace.

$\dim M_n^0(\mathbb{R}) = n^2 - 1$.

$$\exp : M_n^0(\mathbb{R}) \longrightarrow SL_n(\mathbb{R})$$
$$SL_n(\mathbb{R}) = \{ A \mid \det A = 1 \}$$

Q: If $V \subset M_n(\mathbb{R})$ is a vector subspace, is it true that $\exp(V)$ onto a subgp of $GL_n(\mathbb{R})$?

Ans: NOT true in general.

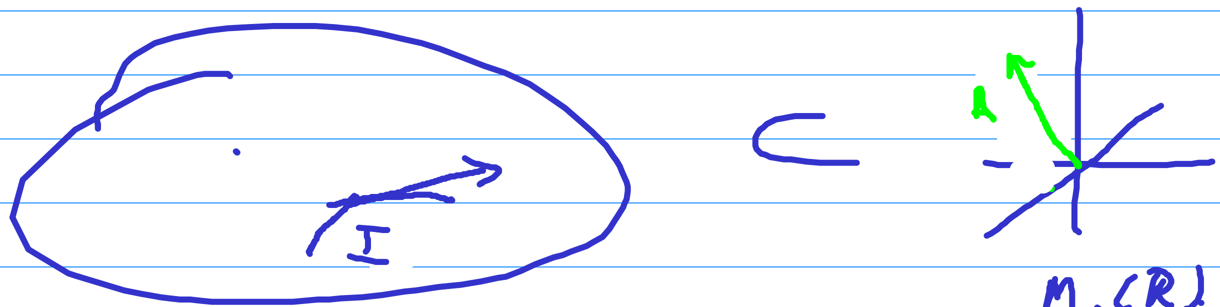
It is true if V has the following

* property: $\forall\, X, Y \in V \Rightarrow [X,Y] = XY - YX \in V$.

If V is 1-dim'l this is true.

If (*) holds we say that V is a Lie algebra.

Ex: Show that $Sk_n(\mathbb{R})$, $Sk_n(\mathbb{C})$, $M_n^0(\mathbb{R})$, $M_n^0(\mathbb{C})$ are Lie algebra.



$M_n(\mathbb{R})$

$GL_n(\mathbb{R})$

$$(-\varepsilon,\varepsilon) \to GL_n(\mathbb{R})$$

$$t \xmapsto{\sigma} I + tA \in GL_n(\mathbb{R}) \quad \forall\ |t| < \varepsilon.$$

$$\frac{d\,\sigma(t)}{dt} = \frac{d}{dt}(I) + \frac{d}{dt}(tA)$$
$$= 0 + A = A.$$

$$e^{A} \cdot e^{-A} = e^{A+(-A)} = e^{0} = I$$

since A commutes with $-A$.

$$t \longmapsto e^{tA} \qquad \frac{d}{dt} e^{tA} = A e^{tA}.$$

$$t \longmapsto I + tA.$$

$M_2(\mathbb{R})$ $\cancel{(\cdot)}$ I $A+I$ $\subset GL_n(\mathbb{R})$

$$X = (x_{ij}) \longleftrightarrow (x_{11}, x_{12}, x_{21}, x_{22}) \in \mathbb{R}^4$$

$$f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$$
$$f(x) = (f_{11}(x), \ldots, f_{nn}(x))$$

$$J_A(f) = \left(\frac{\partial f_{ij}(x)}{\partial x_{kl}} \Big|_A \right)_{n^2 \times n^2}$$

$$A = (a_{ij})$$