Matrix exponential.

- $t \mapsto e^t$, $t \in \mathbb{R}$ or $t \in \mathbb{C}$.

$M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ or $M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$

$X \longmapsto e^X$, $X \in M_n(\mathbb{R})$ or $M_n(\mathbb{C})$.

$M_n(\mathbb{R})$ is a real v.s of dim n^2

$\cong \mathbb{R}^{n^2}$

If $A \in M_n(\mathbb{R})$, $\|A\|$ denotes

$$\|A\| = \max_{1 \le i,j \le n} |a_{ij}|, \quad A = (a_{ij}).$$

Recall $\sigma : \mathbb{R} \longrightarrow M_n(\mathbb{R})$

$t \longmapsto \sigma(t) = (\sigma_{ij}(t))$.

We say σ is smooth if $\sigma_{ij}(t)$ is a smooth fn. of $t \in \mathbb{R}$.

Recall, if $\sigma : \mathbb{R} \longrightarrow M_n(\mathbb{R})$ is smooth, then $\sigma'(t) = (\sigma'_{ij}(t))$

$$= \left(\frac{d}{dt}\sigma_{ij}(t)\right)$$

If $t \longmapsto A(t)$,

$t \longmapsto B(t)$

are smooth curves in $M_n(t)$, then

$t \longmapsto A(t) \cdot B(t)$ is also a smooth curve

$= C(t)$

Lemma: $\frac{d}{dt}(A(t) \cdot B(t)) = \frac{d}{dt} A(t) \cdot B(t) + A(t) \cdot \frac{dB(t)}{dt}$

i.e. $(A(t)B(t))' = C'(t) = A'(t) \cdot B(t) + A(t) \cdot B'(t)$

Fix ij. $c_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t) \cdot b_{kj}(t)$

$c'_{ij}(t) = \sum (a'_{ik}(t) \cdot b_{kj}(t) + a_{ik}(t) \cdot b'_{kj}(t))$

$= \sum a'_{ik}(t) \cdot b_{kj}(t) + \sum \cdots$

$\therefore \quad C'(t) = A'(t) \cdot B(t) + A(t) \cdot B'(t).$

$\exp : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$

$X \longmapsto \exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$

$\exp(X)_{ij} = \sum_{k=0}^{\infty} \frac{(X^k)_{ij}}{k!}$

Fact: $\|AB\| \leq n\|A\|\cdot\|B\|$. $\forall A,B \in M_n(\mathbb{R})$

Write $C = A\cdot B$

$$c_{ij} = \sum a_{ik} b_{kj}.$$

$$|c_{ij}| \leq \sum_k |a_{ik}|\cdot|b_{kj}|$$

$$\leq \|A\| \cdot \sum_{k=1}^{n} |b_{kj}|$$

$$\leq \|A\| \cdot n \cdot \|B\|.$$

$$\therefore \|A^k\| \leq n^{k-1}\cdot \|A\|^k \leq n^k \|A\|^k.$$

$$|(\exp(X))_{ij}| = \left|\sum \frac{(X^k)_{ij}}{k!}\right| \leq \sum_{k=0}^{\infty} \frac{\|X^k\|}{k!}$$

$$\leq \sum \frac{n^k \|X\|^k}{k!} = e^{n\|X\|}.$$

Hence $\displaystyle\sum_{k=0}^{\infty} \frac{X^k}{k!}$ converges absolutely and the convergence is uniform on compacts.

Hence $X \longmapsto \exp(X) = \displaystyle\sum_{k=0}^{\infty} \frac{X^k}{k!}$ is meaningful & is an analytic fn. of the x_{ij}, $X = (x_{ij})$.

Th: ① $X \mapsto e^X$ is absol. cgt & $= \sum \frac{X^k}{k!}$ uniformly so on compact subset of $M_n(\mathbb{R})$.

② Fix $A \in M_n(\mathbb{R})$, we have the curve $\mathbb{R} \longrightarrow M_n(\mathbb{R})$, $t \longmapsto e^{tA}$ is smooth with derivative Ae^{tA}.

ie $\frac{d}{dt} e^{tA} = A \cdot e^{tA}$.

③ If A and $B \in M_n(\mathbb{R})$ and st $AB = BA$, the $e^{A+B} = e^A \cdot e^B$.

④ $\exp : M_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$

In fact $(e^A)^{-1} = e^{-A}$.

Proof: ② $\frac{d}{dt} e^{tA} = A \cdot e^{tA}$.

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

By ①, $\frac{d e^{tA}}{dt} = \sum_{k=0}^{\infty} \frac{d}{dt}\left(\frac{t^k A^k}{k!}\right)$

$= \sum k\, t^{k-1} \frac{A^k}{k!}$

$= A \cdot \sum_{k=1}^{\infty} t^{k-1} \frac{A^{k-1}}{(k-1)!}$

$= A \cdot e^{tA}.$

③ $\frac{d}{dt}\left(e^{-tA} \cdot e^{(tA+B)}\right)$

$= -A \cdot e^{-tA} \cdot e^{tA+B} + e^{-tA} \cdot \frac{d}{dt} e^{tA+B}$

$(A+B)^m = \sum \binom{m}{k} A^k B^{m-k}$ if A & B commute

$= -Ae^{-tA} \cdot e^{tA+B} + e^{-tA} \cdot Ae^{tA+B}$

$= 0$

$\Rightarrow \frac{d}{dt}\left(e^{-tA} e^{tA+B}\right) \equiv 0$

$\Rightarrow e^{-tA} \cdot e^{tA+B} = e^{B}$

Put $t=1$ $\quad e^{-A} e^{A+B} = e^{B}.$

$B=0 \Rightarrow e^{-A} = (e^{A})^{-1}$

$\therefore$ for any A, B, $AB=BA$, we have

$$e^{A+B} = e^{A} \cdot e^{B}.$$

Examples: (1) Let $A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$

a diagonal matrix.

$$A^k = \begin{bmatrix} a_1^k & & \\ & \ddots & \\ & & a_n^k \end{bmatrix}$$

$$e^A = \sum \frac{A^k}{k!} = \sum \begin{pmatrix} a_1^k & & \\ & \ddots & \\ & & a_n^k \end{pmatrix} / k!$$

$$= \begin{pmatrix} \sum a_1^k/k! & & 0 \\ & \ddots & \\ 0 & & a_n^k/k! \end{pmatrix}$$

$$= \begin{pmatrix} e^{a_1} & & 0 \\ & \ddots & \\ 0 & & e^{a_n} \end{pmatrix}.$$

(2) Take $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in M_2(\mathbb{R})$

$= a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} -a^2 & 0 \\ 0 & -a^2 \end{pmatrix} = -a^2 I.$$

$$A^3 = \begin{pmatrix} 0 & -a^3 \\ a^3 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} a^4 & 0 \\ 0 & a^4 \end{pmatrix}$$

$$= -a^2 A = -a^3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad = a^4 I$$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots$$

$$= I - \frac{a^2}{2!} I + \frac{a^4}{4!} I - \cdots$$

$$+ a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{a^3}{3!}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} + \frac{a^5}{5!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \cdots$$

$$= \cos a \cdot I + \left(a - \frac{a^3}{3!} + \frac{a^5}{5!} - \cdots\right)\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$= \cos a\, I + \sin a \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$= \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}$$

Put $a = \pi$, $\underline{\exp\begin{pmatrix} & \pi \\ -\pi & \end{pmatrix} = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}}$

<u>Exercise:</u> Compute $\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$.

<u>Properties of exp.</u>

Let $P \in GL_n(\mathbb{R})$, $A \in M_n(\mathbb{R})$.

Then

① $\exp(PAP^{-1}) = P\exp(A)P^{-1}$.

② $(\exp(A))^t = \exp(A^t)$.

③ Suppose $A \in M_n(\mathbb{C})$, Then

$$\exp(\bar{A}) = \overline{\exp(A)}.$$

④ $\det(\exp(A)) = \exp(tr(A))$.

Pf: $\exp(PAP^{-1})$

$$= \sum_{k=0}^{\infty} \frac{(PAP^{-1})^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{PA^kP^{-1}}{k!}$$

$$= P\left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)P^{-1}. = P\exp(A)P^{-1}.$$

Proof of 4:

Given $A \in M_n(\mathbb{C})$, there exists an invertible $P \in M_n(\mathbb{C})$ s.t.

$PAP^{-1} = B$ which is upper Δ^{lr}.

$= (b_{ij})$.

$$\det(\exp B)$$

$$\det \exp(PAP^{-1}) = \det(P(\exp A)\cdot P^{-1})$$

$$= \det(\exp A)$$

$$B = \begin{pmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix}, \quad B^k = \begin{pmatrix} b_{11}^k & & * \\ & \ddots & \\ & & b_{nn}^k \end{pmatrix}$$

$\exp(B) =$ upper Δ^{lr}, with diagnal entries $e^{b_{11}}, e^{b_{22}}, \cdots, e^{b_{nn}}$

$\Rightarrow$

$$\therefore \det(\exp B) = e^{b_{11} + \cdots + b_{nn}}$$

$$= e^{\operatorname{tr} B} = e^{\operatorname{tr}(PAP^{-1})}$$

$$= e^{\operatorname{tr}(A)}.$$

QED.