Lecture II.

Let G be a group. G is said to act on X if $\exists$ a map

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g \cdot x$$

Satisfying

i) $e \cdot x = x \quad \forall\, x \in X$

ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

—x—

<u>Example</u>: Let $X = \{1, \dots, n\}$, where n is a +ve integer

$$B(X) = \{ f : X \longrightarrow X \mid f \text{ is a bijection} \}$$

is a group under composition map and is denoted S_n

$$S_n \times \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$$
$$(\sigma, k) \longmapsto \sigma(k) =: \sigma.k.$$

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1) Let G acts on X and $x \in X$. Then

$$G_x = \{ g \in G \mid g \cdot x = x \}$$

is subgroup of G and is called the stabilizer of x in G

2) Let G acts on X and $x \in X$. Then

$$O(x) = \{ g \cdot x \mid g \in G \} \subset X$$

is known as the orbit of "x" under the action of G on X

Example:

1) $X = \{1, \dots, n\}$. $G = S_n$.

$x = n$.

$$G_x = \{\sigma \in S_n \mid \sigma(n) = n\} \simeq S_{n-1}$$

2) $X = \{1, \dots, n\}$, $G = S_n$

$x = n$

$O(x) = X$.

DFN: Let G be a group acting on a set X. If for some $x \in X$ $O(x) = X$ then we say that the group G acts transitively on X.

—x—x—

Example: Let $X = \{1,2\} \times \{1,2\}$

$G = S_2 = B(\{1,2\})$

$$S_2 \times X \longrightarrow X$$
$$(\sigma, (a,b)) \longmapsto (\sigma(a), \sigma(b))$$

This action is not transitive. (Verify).

Let G be gp acting on X.

$x \in X$.

$O(x)$ we can identify with G/G_x.

$gG_x \longmapsto g \cdot x$.

We see that the above map induces a bijection from G/G_x to $O(x)$.

If $y \in O(x)$ then is there a relation between the subgroups G_x and G_y and orbits $O(x)$ and $O(y)$

It is easy to Verify that $O(x) = O(y)$ and G_x and G_y are Conjugate Subgps of G:

If $y \in O(x)$ then $\exists\, g \in G$

$y = g \cdot x$

Claim: $G_y = g G_x g^{-1} = \{ g h g^{-1} \mid h \in G_x \}$.

if $t \in g G_x g^{-1}$ then $t = g h g^{-1}$ for some $h \in G_x$.

$t \cdot y = (g h g^{-1}) \cdot y = (g h g^{-1}) \cdot g \cdot x = g h (g^{-1} g) x$

$= g \cdot h \cdot x = g \cdot x = y. \implies t \in G_y$

i.e., $gG_xg^{-1} \subset G_y$ (Check that $G_y \subset gG_xg^{-1}$)

<u>Example</u>: Let G be a group. Then we can define an action G on G as follows:

$$G \times G \longrightarrow G$$
$$(g, g_1) \mapsto g \cdot g_1 g^{-1}.$$

(i.e., G acts on itself by conjugation.

If G is abelian then the conjugation action of G on itself is trivial:

An action of a group G on a set X is said to be trivial if $g \cdot x = x \ \forall g \in G$

and for all $x \in X$

i.e., The induced homomorphism

$$\rho : G \longrightarrow B(X) = \{ f : X \to X \mid f \text{ is bijection} \}$$

$$\rho(g) = Id_X \quad \forall\, g \in G.$$

—x—.

If G is acting trivially on X then

$O(x) = \{x\} \quad \forall\, x \in X$

and $G_x = G \quad \forall\, x \in X$

—x—.

Let p be prime number and G be a group of order p^n for some $n \geq 1$.

i.e., $\# G = p^n$.

If $H \subsetneq G$ is a sub gp then $\# H$ and $\# G/H$

are also of order p^m and p^{n-m} for some $0 \leq m < n$.

G acts on itself by Conjugation

Consider

$$C(G) = \{g \in G \mid ghg^{-1} = h \quad \forall\, h \in G\}. \text{ - Center of } G.$$
$$\Psi$$
$$e$$

If $g \in C(G) \Rightarrow g^{-1} \in C(G)$

If $g_1, g_2 \in C(G) \Rightarrow g_1 g_2 \in C(G)$

In other words $C(G)$ - Center of G is a Subgroup of G (If G is Abelian then $C(G) = G$)

If G is a finite gp acting on a finite set X

then we can write

$$X = \coprod_{x} O(x)$$

If $O(x) \cap O(y) \neq \phi \Rightarrow O(x) = O(y)$

if $z \in O(x) \cap O(y)$

then $O(x) = O(z) = O(y)$

Consider G a group of order p^n where

p - prime number and $n \geq 1$.

Then $C(G) \neq \{e\}$ i.e., $C(G)$ is a non-trivial subgroup of G.

Under the conjugation action of G on itself

$$G = \coprod_{\substack{g \in S \\ \cap \\ G}} O(g) = C(G) \amalg O(g_1) \amalg \cdots \amalg O(g_k)$$

If G is Abelian then $G = C(G)$. If G is not Abelian

$$p^m = \# C(G) + \# O(g_1) + \cdots + \# O(g_k)$$

$\#(g_1), \ldots, \# O(g_k)$ are divisible by p.

$\Rightarrow p \mid \# C(G)$ since $\# C(G) \geq 1$

$\Rightarrow C(G) \neq \{e\}$

—x—

Let p be a prime number and $n \geq 1$ be an integer

Consider the group

$$\left(\mathbb{Z}/p\mathbb{Z}\right)^n := \underbrace{\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}}_{n\text{- times}}.$$

Let r be integer $1 \leq r \leq n$

Consider $S_r^{(p)} = \left\{ M \mid M \subset \left(\mathbb{Z}/p\mathbb{Z}\right)^n,\ M \text{ is a Subgp and } M \cong \left(\mathbb{Z}/p\mathbb{Z}\right)^r \right\}$

We want find the cardinality of S_r

Example: if $r = n$ then

$\# S_n^{(p)} = 1$ as $M = \left(\mathbb{Z}/p\mathbb{Z}\right)^n$

$\# S_1^{(p)} = \dfrac{p^n - 1}{p - 1}$.

Let $p=2$ and $n=3$

$$\left(\mathbb{Z}/2\mathbb{Z}\right)^3$$

$$S_1^{(2)} = \left\{ M \subset \left(\mathbb{Z}/2\mathbb{Z}\right)^3 \;\middle|\; \underset{\text{As a gp}}{M \underset{|}{\simeq}} \mathbb{Z}/2\mathbb{Z} \right\}$$

$$\frac{2^3-1}{2-1} = 7.$$

If $(a,b,c) \in \left(\mathbb{Z}/2\mathbb{Z}\right)^3$ and $(a,b,c) \neq (0,0,0)$

then $\{(0,0,0),(a,b,c)\} \simeq \mathbb{Z}/2\mathbb{Z}$.

$$(a,b,c)+(a,b,c) = (0,0,0)$$